Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 95 (2020)
Heft: 1

Artikel: Boundary rigidity of negatively-curved asymtotically hyperbolic surfaces
Autor: Lefeuvre, Thibault
DOI: https://doi.org/10.5169/seals-882407

Comment. Math. Helv. 95 (2020), 129–166
DOI 10.4171/CMH/483

Commentarii Mathematici Helvetici

Boundary rigidity of negatively-curved asymptotically hyperbolic surfaces

Thibault Lefeuvre

Abstract. In the spirit of Otal [17] and Croke [3], we prove that a negatively-curved asymptotically hyperbolic surface is *marked boundary distance rigid*, where the distance between two points on the boundary at infinity is defined by a renormalized quantity.

Mathematics Subject Classification (2010). 35R30, 37D40, 53C22.

Keywords. Inverse problems, boundary rigidity, asymptotically hyperbolic surfaces.

1. Introduction

1.1. Main result. We consider $\bar{M}$ a smooth compact connected $(n+1)$-dimensional manifold with boundary. We say that $\rho \colon \bar{M} \to \mathbb{R}_+$ is a *boundary defining function* on $\bar{M}$ if it is smooth and satisfies $\rho = 0$ on ∂M, $d\rho \neq 0$ on ∂M and $\rho > 0$ on M. Let us fix such a function ρ. A metric g on M is said to be *asymptotically hyperbolic* if

(1) the metric $\bar{g} = \rho^2 g$ extends to a smooth metric on $\bar{M}$,

(2) $|d\rho|_{\rho^2 g} = 1$ on ∂M.

The extension of the metric $\rho^2 g$ on the boundary, that is $\rho^2 g|_{T\partial M}$, is not independent of the choice of ρ but its conformal class is — it is called the *conformal infinity*.

Such a manifold admits a canonical product structure in a neighborhood of the boundary ∂M (see [7] for instance) that is, given a metric h_0 on ∂M (in the conformal class $[\rho^2 g|_{T\partial M}]$), there exists a smooth set of coordinates (ρ, y) on $\bar{M}$ (where ρ is a boundary defining function) such that $|d\rho|_{\rho^2 g} = 1$ in a neighborhood of ∂M and $\rho^2 g|_{T\partial M} = h_0$. The function ρ is uniquely determined by h_0 in a neighborhood of ∂M. Moreover, on a collar neighborhood near ∂M, the metric has the form

$$g = \frac{d\rho^2 + h_\rho}{\rho^2}, \quad \text{on } (0, \varepsilon) \times \partial M, \tag{1.1}$$

for some $\varepsilon > 0$ and where h_ρ is a smooth family of metrics on ∂M. From this expression, one can prove that the sectional curvatures of (M, g) all converge towards -1 as ρ goes to 0.

The manifold M is not compact and the length of a geodesic $\alpha(x,x')$ joining two points x and x' on the boundary at infinity is clearly not finite. However, in [8], a *renormalized length* $L(\alpha(x,x'))$ for a geodesic $\alpha(x,x')$ is introduced, which roughly consists in the constant term in the asymptotic development of the length of

$$\alpha_\varepsilon(x,x') := \alpha(x,x') \cap \{\rho \geq \varepsilon\}$$

as ε goes to 0. This yields a new object characterized by the asymptotically hyperbolic manifold (M,g) and one can actually wonder, as usual in inverse problem theory, up to what extent it conversely determines (M,g). Notice that the renormalized length is not independent of the choice of the boundary defining function ρ, and thus, neither of the choice of the conformal representative h_0 in the conformal infinity.

From now on, we further assume that M has dimension 2 and is negatively-curved. If M is simply connected, then it is a well-known fact that there exists a unique geodesic between any pair of points $(x,x') \in \partial M \times \partial M \setminus \mathrm{diag}$, where diag is the diagonal in $\partial M \times \partial M$. The *renormalized boundary distance* is defined as:

$$D\colon \partial M \times \partial M \setminus \mathrm{diag} \to \mathbb{R}, \quad D(x,x') = L(\alpha(x,x')),$$

where $L(\alpha(x,x'))$ denotes the renormalized length of the unique geodesic joining x to x'. In the terminology of [8], such surfaces are called *simple*: this definition naturally extends the notion of a simple manifold (compact manifold with boundary such that the exponential map is a diffeomorphism at each point) to the non-compact setting.

More generally, we will deal with the case of negatively-curved surfaces with topology. Then, the natural object one has to consider is the *renormalized marked boundary distance*. In this case, given two points $(x,x') \in \partial M \times \partial M \setminus \mathrm{diag}$, there exists a unique geodesic in each homotopy class $[\gamma] \in \mathcal{P}_{x,x'}$ of curves joining x to x' ($\mathcal{P}_{x,x'}$ being the set of homotopy classes). We define

$$\mathcal{D} := \big\{(x,x',[\gamma]), (x,x') \in \partial M \times \partial M \setminus \mathrm{diag}, [\gamma] \in \mathcal{P}_{x,x'}\big\},$$

and introduce the renormalized marked boundary distance D as:

$$D\colon \mathcal{D} \to \mathbb{R}, \quad D\big(x,x',[\gamma]\big) = L\big(\alpha\big(x,x',[\gamma]\big)\big), \tag{1.2}$$

where $\alpha(x,x',[\gamma])$ is the unique geodesic in $[\gamma]$ joining x to x' and L the renormalized length. Our main result is the following:

Theorem 1.1. *Assume (M,g_1) and (M,g_2) are two asymptotically hyperbolic surfaces with negative curvature. We suppose that g_1 and g_2 admit the same renormalized boundary distances, i.e. $D_1 = D_2$. Then, there exists a smooth diffeomorphism $\Phi\colon \bar{M} \to \bar{M}$ such that $\Phi^* g_2 = g_1$ on M and $\Phi|_{\partial M} = \mathrm{Id}$.*

Notice that if $\Phi\colon \bar{M} \to \bar{M}$ is a diffeomorphism preserving the boundary, then $L_g = L_{\Phi^* g}$, where both renormalized lengths are computed with respect to the same representative in the conformal infinity. In other words, the previous theorem asserts that the action of the group of diffeomorphisms preserving the boundary is the only obstruction to the injectivity of the map $g \mapsto L_g$.

This result can be seen as an analogue of [11, Theorem 2] for the case of asymptotically hyperbolic surfaces. It is new even in the simply connected case, where the marked boundary distance is simply the ordinary renormalized boundary distance. It is very likely that one can relax the assumption in Theorem 1.1 so that only one of the two metrics has negative curvature (but still a hyperbolic trapped set). In the usual terminology, Theorem 1.1 roughly says that an asymptotically hyperbolic surface with negative curvature is *marked boundary distance rigid* among the class of surfaces having negative curvature.

This result follows in spirit the ones proved independently by Otal [17] and Croke [3] establishing that two negatively-curved closed surfaces with same marked length spectrum are isometric. More recently, Guillarmou and Mazzucchelli [11] extended Otal's proof to the case of two surfaces with strictly convex boundary without conjugate points and a trapped set of zero Liouville measure, one being of negative curvature. In both cases, the central object of interest is the *Liouville current* η, which is the natural projection of the Liouville measure μ (initially defined on the unit tangent bundle SM) on the set of geodesics $\mathcal{G}$ of the manifold. Our arguments follow in principle the layout of proof of these articles, but we need to address new issues caused by the loss of the compactness assumption. The crucial step in our proof to deal with the infinite ends of the manifold is a version of Otal's lemma (see [17, Lemma 8]) with a stability estimate (Proposition 5.4). To the best of our knowledge, this bound had never been stated before in the literature.

As far as we know, this is also the first boundary rigidity result obtained in a non-compact setting. There is a long history of results regarding the boundary rigidity question on simple manifolds in the compact setting. We here mention the contributions of Gromov [9], for regions of $\mathbb{R}^n$, the original paper of Michel [16] for subdomains of the open hemisphere and the Besson–Courtois–Gallot theorem [2], which implies the boundary rigidity for regions of $\mathbb{H}^n$ (see also the survey of Croke [4]). In the case of a manifold with trapping, the first general results where obtained by Guillarmou–Mazzucchelli [11] for surfaces, where the local boundary rigidity was established under suitable assumptions. Global boundary rigidity theorems have also recently been obtained by Stefanov–Uhlmann–Vasy [21] for simply connected non-positively curved manifolds with strictly convex boundary. Let us eventually mention that boundary rigidity questions appear naturally in the physics literature concerning the AdS/CFT duality and holography (see [5, 19]).

1.2. Outline of the proof. In Section 2, we introduce the notion of renormalized length for a geodesic. We heavily rely on the cautious study made in [8] of the geodesic flow near the boundary at infinity. In Section 3, we recall the definition of the Liouville current η on the space of geodesics of the universal cover $\widetilde{M}$ and prove that if the renormalized marked lengths agree, then the Liouville currents agree, just as in the compact setting.

Section 4 is devoted to the construction of an application of deviation κ. Like in [17], we introduce *the angle of deviation* f between the two metrics on the universal cover $\widetilde{M}$. The idea is to make use of Gauss–Bonnet formula, in order to prove that this angle is the identity. This requires to introduce an *average angle of deviation.* Since we are in a non-compact setting, technical issues arise from the fact that the volume is infinite. In particular, we need to consider its average (denoted by Θ_ε) on compact domains $\{\rho \geq \varepsilon\}$ parametrized by ε and to study their limit as $\varepsilon \to 0$.

Because of the possible existence of a *trapped set*, we are unable to prove a priori that the averages Θ_ε are C^1 (or at least uniformly Lipschitz), which would truly simplify the proof. A cautious analysis of the derivative of the angle of deviation f is needed to deal with these technical complications. Combined with a version of Otal's lemma with an estimate (see Proposition 5.4), this allows to conclude that the average angle of deviation is the identity in the limit $\varepsilon \to 0$, which itself implies that the angle of deviation f is the identity. We then conclude the proof by constructing a natural application Φ which is an isometry between (M, g_1) and (M, g_2). Eventually, a last difficulty comes from the fact that it is not immediate that the isometry obtained is C^∞ down to the boundary of $\bar{M}$.

If the reader is familiar with Otal's proof [17], he will morally see the same features appear, but the novelty here is that we are able to deal with the asymptotic ends of the manifold. The price we have to pay is that this requires to compute tedious estimates in the limit $\varepsilon \to 0$.

Acknowledgements. We warmly thank Colin Guillarmou for suggesting this result and fruitful discussions. We also thank the anonymous referee for corrections and valuable suggestions. We acknowledge partial support from ERC funding COG IPFLOW No. 725967.

2. Geometric preliminaries

This section is not specific to the two-dimensional case, so we state it in full generality. (M, g) is only assumed to be an $(n+1)$-dimensional asymptotically hyperbolic manifold. In our setting, it will be more convenient to work on the unit cotangent bundle rather than on the unit tangent bundle, using the construction of Melrose [15] of b-bundles.

2.1. Geometry on the unit cotangent bundle.

2.1.1. The b-cotangent bundle. The unit cotangent bundle is defined by

$$S^*M := \{(x, \xi) \in T^*M \mid x \in M, \xi \in T_x^*M, |\xi|_g^2 = 1\}, \tag{2.1}$$

and we denote by $\pi\colon S^*M \to M$ the projection on the base. The geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ is induced by the Hamiltonian vector field X, obtained from the Hamiltonian $H(x,\xi) = \frac{1}{2}|\xi|_g^2$. We will denote by $\flat\colon TM \to T^*M$ the Legendre transform between these two vector bundles, that is $v \mapsto g(v,\cdot)$, and by $\sharp\colon T^*M \to TM$ its inverse. We stress that we will often drop the notation of these isomorphisms and identify (without mentioning it) a vector with its dual covector.

There exists a canonical splitting of $T(S^*M)$ according to:

$$T(S^*M) = \mathbb{H} \oplus \mathbb{V}, \tag{2.2}$$

where $\mathbb{V} := \ker d\pi$ is the vertical bundle and $\mathbb{H} := \ker \mathcal{K}$ is the horizontal bundle. $\mathcal{K}$ is the connection map, defined for $(x,\xi) \in S^*M$, $Z \in T_{(x,\xi)}(S^*M)$, by

$$\mathcal{K}(Z) = \nabla_{\dot{x}}\xi^\sharp(0) \in T_xM,$$

where $t \mapsto z(t) = (x(t),\xi(t)) \in S^*M$ is any curve such that $z(0) = (x,\xi)$ and $\dot{z}(0) = Z$ (see [18] for a reference). The metric g on M induces a natural metric G on S^*M, called the *Sasaki metric* and defined by:

$$G(Z,Z') := g(d\pi(Z), d\pi(Z')) + g(\mathcal{K}(Z),\mathcal{K}(Z')) \tag{2.3}$$

Recall from [15] that the *b-tangent bundle* ${}^bT\bar{M} \to \bar{M}$ is defined to be the smooth vector bundle whose sections are vectors fields tangent to ∂M. Let V be a smooth vector field on M. If $(\rho, y_1, \dots, y_n)$ denotes smooth local coordinates in a neighborhood of ∂M, we can write

$$V = a\partial_\rho + \sum_i b_i \partial_{y_i},$$

for some smooth functions a, b_i. If V vanishes on the boundary, then $a|_{\partial M} = 0$, and we can write $a = \rho\alpha$ for some smooth function α. In other words, in coordinates, $(\rho\partial_\rho, \partial_{y_i})$ is a local frame for ${}^bT\bar{M}$. Now, $\rho\partial_\rho$ is well defined on ∂M, independently of the choice of coordinates in a neighborhood of ∂M. Indeed, if (ρ', y') denotes another choice of coordinates, then one can write $\rho' = \rho A(\rho, y)$, $y_i' = Y_i(\rho, y)$ for some smooth functions (such that $A(0,0) > 0$) and one has

$$\rho\partial_\rho = \Big(1 + \frac{\rho}{A}\Big)\rho'\partial_{\rho'} + \frac{\rho'}{A}\sum_j \partial_\rho(Y_j)\partial_{y_j'},$$

that is, both elements $\rho\partial_\rho$ and $\rho'\partial_{\rho'}$ agree on the boundary as elements of ${}^bT\bar{M}|_{\partial M}$.

The *b-cotangent bundle* ${}^bT^*\bar{M}$ is the vector bundle of linear forms on ${}^bT\bar{M}$. In coordinates, $(\rho^{-1}d\rho, dy_i)$ is a local frame of ${}^bT^*\bar{M}$ and $\rho^{-1}d\rho$ on ∂M (the covector associated to $\rho\partial_\rho$) is independent of any choice of coordinates (and of the metric g). From the coordinates

$$\Big(\rho, y, \xi = \xi_0 d\rho + \sum_i \eta_i dy_i\Big)$$

on $T^*\bar{M}$, we introduce on ${}^bT^*\bar{M}$ the smooth coordinates

$$(x,\xi) = (\rho, y, \bar{\xi}_0, \eta),$$

where $\xi_0 = \bar{\xi}_0\rho^{-1}$, that is

$$\xi = \bar{\xi}_0\rho^{-1}d\rho + \sum_i \eta_i dy_i.$$

In particular, we see from the previous discussion that the function $\xi \mapsto \bar{\xi}_0$ on ${}^bT^*\bar{M}|_{\partial M}$ is intrinsic to the manifold, as well as the two subsets $\{\bar{\xi}_0 = \pm 1\}$ of ${}^bT^*\bar{M}|_{\partial M}$ (they do not depend on the choice of coordinate (ρ, y), not even on the metric g).

Note that given $\xi = \bar{\xi}_0\rho^{-1}d\rho + \sum_i \eta_i dy_i \in {}^bT^*\bar{M}$, one has:

$$|\xi|_g^2 = \bar{\xi}_0^2 + \rho^2|\eta|_{h_\rho}^2,$$

where, here, h_ρ actually denotes the dual metric on $T^*\partial M$. We denote by:

$$\overline{S^*M} = \{(x,\xi) \in {}^bT^*\bar{M},\ |\xi|_g^2 = 1\}.$$

One has for $x \in \bar{M}$:

$$\overline{S_x^*M} = \{(x,\xi) \in {}^bT^*\bar{M},\ \bar{\xi}_0^2 + \rho^2|\eta|_{h_\rho}^2 = 1\}.$$

As a consequence, there is a splitting:

$$\overline{S^*M} = S^*M \sqcup \partial_- S^*M \sqcup \partial_+ S^*M,$$

where $\partial_\pm S^*M = \{(x,\xi), x \in \partial M, \bar{\xi}_0 = \mp 1\}$ (which are independent of any choice). We see $\partial_- S^*M$ (resp. $\partial_+ S^*M$) as the *incoming* (resp. *outcoming*) boundary.

Lemma 2.1 ([8, Lemma 2.1]). *There exists a smooth vector field $\bar{X}$ on $\overline{S^*M}$ which is transverse to the boundary*

$$\partial\overline{S^*M} = \partial_- S^*M \sqcup \partial_+ S^*M$$

and satisfies $X = \rho\bar{X}$ on S^*M. Moreover, for $x \in \bar{M}$ sufficiently close to ∂M, in suitable local coordinates as before, we have $\bar{X} = \partial_\rho + \rho Y$, for some smooth vector field Y on $\overline{S^*M}$.

The flow on $\overline{S^*M}$ induced by $\bar{X}$ will be denoted by $\bar{\varphi}_\tau$. For $z \in S^*M$ and $\tau > 0$ such that $\bar{\varphi}_s(z)$ is defined for $s \in [0,\tau]$, one has $\varphi_t(z) = \bar{\varphi}_\tau(z)$, where

$$t(\tau, z) = \int_0^\tau \frac{1}{\rho(\bar{\varphi}_s(z))}\,ds. \tag{2.4}$$

2.1.2. Trapped set. The results of the following paragraph can be found in [8, Section 2.1]. We recall them for the sake of clarity. For $\varepsilon > 0$ small enough, the compact surfaces $M_\varepsilon := M \cap \{\rho \geq \varepsilon\}$ are strictly convex with respect to the geodesic flow.

Lemma 2.2 ([8, Lemma 2.3]). *There exists $\varepsilon > 0$ small enough so that for each $(x,\xi) \in S^*M$ with $\rho(x) < \varepsilon$,*

$$\xi = \xi_0 d\rho + \sum_{i=1}^{n-1} \xi_i \, dy_i$$

and $\xi_0 \leq 0$, the flow trajectory $\varphi_t(x,\xi)$ converges to some point $z_+ \in \partial_+ S^*M$ with rate $\mathcal{O}(e^{-t})$ as $t \to +\infty$ and $\rho(\varphi_t(x,\xi)) \leq \rho(x,\xi)$ for all $t \geq 0$. The same result holds with $\xi_0 \geq 0$ and negative time, with limit point $z_- \in \partial_- S^*M$.

We define the *tails* $\Gamma_\pm$: they consist of the points in S^*M which are respectively trapped in the past or in the future:

$$S^*M \setminus \Gamma_\mp := \{z \in S^*M,\ \rho(\varphi_t(z))_{t\to\pm\infty} \to 0\}. \tag{2.5}$$

The *trapped set* K is defined by:

$$K := \Gamma_+ \cap \Gamma_-. \tag{2.6}$$

In particular, in negative curvature, the trapped set has zero Liouville measure. We can define the exit and enter maps

$$B_\pm \colon S^*M \setminus \Gamma_\mp \to \partial_\pm S^*M$$

such that

$$B_\pm(z) := \lim_{t\to\pm\infty} \varphi_t(z). \tag{2.7}$$

These are smooth, well-defined maps and they extend smoothly to $\overline{S^*M} \setminus \overline{\Gamma_\mp}$, where $\overline{\Gamma_\mp}$ is the closure of $\Gamma_\mp$ in $\overline{S^*M}$ (see [8, Corollary 2.5]). There also exist smooth functions $\tau_\pm \colon \overline{S^*M} \setminus \overline{\Gamma_\mp} \to \mathbb{R}_\pm$ defined such that:

$$\overline{\varphi}_{\tau_\pm(z)}(z) = B_\pm(z) \in \partial_\pm S^*M. \tag{2.8}$$

Using the vector field $\overline{X}$, another way of describing the sets $\overline{\Gamma_\pm}$ is

$$\overline{\Gamma_\pm} = \{z \in \overline{S^*M},\ \tau_\mp(z) = \pm\infty\}. \tag{2.9}$$

The *scattering map* is the smooth map $\sigma \colon \partial_- S^*M \setminus \overline{\Gamma_-} \to \partial_+ S^*M \setminus \overline{\Gamma_+}$ defined by:

$$\sigma(z) := B_+(z) = \overline{\varphi}_{\tau_+(z)}(z). \tag{2.10}$$

2.1.3. Hyperbolic splitting in negative curvature. In this section, (M, g) has dimension 2 and negative curvature $\kappa < 0$. Since the curvature at infinity converges towards -1, we know that κ is pinched between two constants $-k_0^2 \le \kappa < -k_1^2 < 0$. It is a classical fact that the geodesic flow on such a surface is Anosov (see [6, 12]) in the sense that there exists some constants $C > 0$ and $\nu > 0$ (depending on the metric g) such that for all $z = (x, \xi) \in S^*M$, there is a continuous flow-invariant splitting

$$T_z(S^*M) = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z), \tag{2.11}$$

where $E_s(z)$ (resp. $E_u(z)$) is the *stable* (resp. *unstable*) vector space in z, which satisfy

$$\begin{aligned} |d\varphi_t(z) \cdot Z|_G &\le Ce^{-\nu t}|Z|_G, \quad \forall t > 0,\ Z \in E_s(z), \\ |d\varphi_t(z) \cdot Z|_G &\le Ce^{-\nu |t|}|Z|_G, \quad \forall t < 0,\ Z \in E_u(z). \end{aligned} \tag{2.12}$$

The norm, here, is given in terms of the Sasaki metric. The bundles $z \mapsto E_u(z)$, $E_s(z)$ are Hölder-continuous everywhere on S^*M. Moreover, the differential of the geodesic flow is governed uniformly by an exponential growth (see [20, Chapter 3]) in the sense that there exists (other) constants $C, k > 0$ such that:

$$|d\varphi_t(z) \cdot Z|_G \le Ce^{kt}|Z|_G, \quad \forall t > 0,\ \forall Z \in T_z(S^*M). \tag{2.13}$$

Let us now fix $\varepsilon > 0$ small enough and consider $M_\varepsilon := M \cap \{\rho \ge \varepsilon\}$. Like in [10], we define the *non-escaping mass function* $V_\varepsilon(T)$ on the domain M_ε by

$$V_\varepsilon(T) := \mu\big(\{z \in S^*M_\varepsilon \mid \forall s \in [0, T],\ \varphi_s(z) \in S^*M_\varepsilon\}\big).$$

Since the trapping set is hyperbolic, there exists a constant $Q < 0$ such that

$$Q := \limsup_{T \to +\infty} \log(V_\varepsilon(T))/T.$$

Note that this constant is independent of ε (see [10, Proposition 2.4]). In the rest of this paragraph, we fix some $\varepsilon_0 > 0$ small enough. For $0 < \varepsilon < \varepsilon_0$, we want to link explicitly the decay of the non-escaping mass function V_ε to V_{ε_0}.

Lemma 2.3. *Let $\delta \in (Q, 0)$. There exists a constant $C > 0$ and an integer $N_0 \in \mathbb{N} \setminus \{0\}$, such that for all $T \ge -N_0 \log(\varepsilon)$:*

$$V_\varepsilon(T) \le C\varepsilon^{-(1+4\delta)}e^{-\delta T}.$$

Proof. For $(x, \xi) \notin \Gamma_-$ we denote by $\ell_{\varepsilon,+}(x, \xi)$ the exit time of the manifold M_ε, that is the maximum time such that: $\forall t \in [0, \ell_{\varepsilon,+}(x, \xi)]$, $\varphi_t(x, \xi) \in S^*M_\varepsilon$. By Santaló's formula, we can express $V_\varepsilon(T)$ as:

$$V_\varepsilon(T) = \int_{\partial_- S^*M_\varepsilon} (\ell_{\varepsilon,+}(x, \xi) - T)_+ d\mu_{\nu,\varepsilon},$$

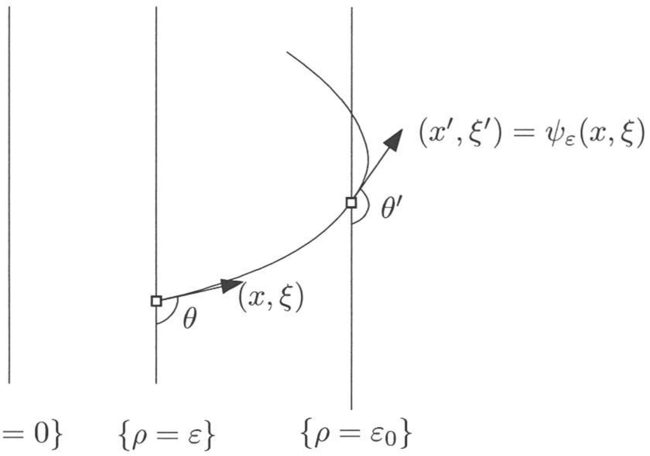

Figure 1. The diffeomorphism ψ_ε in the proof of Lemma 2.3.

where $x_+ = \sup(x, 0)$, $d\mu_{\nu,\varepsilon}(x, \xi) = |g(\xi, \nu)| i^*_{\partial S^* M_\varepsilon}(d\mu)$[1], ν is the unit covector conormal to the boundary, $i^*_{\partial S^* M_\varepsilon}(d\mu)$ is the restriction of the Liouville measure to the boundary (the measure induced by the Sasaki metric restricted to $\partial S^* M_\varepsilon$). There exists a maximum time T^*_ε, such that given any $(x, \xi) \in \partial_+ S^* M_{\varepsilon_0}$, $\varphi_{T_\varepsilon}(x, \xi)$ has exited the manifold M_ε. One can bound this time T^*_ε by $\log(C\varepsilon_0/\varepsilon)$, where $C > 0$ is some constant independent of (x, ξ) and ε (see the proof of [8, Lemma 2.3]). We introduce $T_\varepsilon := -2\log(\varepsilon) > T^*_\varepsilon$ for ε small enough. As a consequence, for $T \geq 2T_\varepsilon$, one has:

$$V_\varepsilon(T) \leq \int_{\partial_- S^* M_\varepsilon \cap D_\varepsilon} (\ell_{\varepsilon_0,+}(\psi_\varepsilon(x, \xi)) - (T - 2T_\varepsilon))_+ d\mu_{\nu,\varepsilon},$$

where

$$\psi_\varepsilon^{-1} \colon \partial_- S^* M_{\varepsilon_0} \to \psi_\varepsilon^{-1}(\partial_- S^* M_{\varepsilon_0}) =: D_\varepsilon \subset \partial_- S^* M_\varepsilon$$

is the diffeomorphism which flows backwards (by $\overline{\varphi}_\tau$) a point $(x, \xi) \in \partial_- S^* M_{\varepsilon_0}$ to the boundary $\partial_- S^* M_\varepsilon$ (see Figure 1).

The dependence of ψ_ε^{-1} on ε is smooth down to $\varepsilon = 0$: this follows from the implicit function theorem. In the local product coordinates (ρ, y), one can write

$$d\mu_{\nu,\varepsilon} = 1/\varepsilon \sin(\theta) h(\varepsilon, y)\, dy\, d\theta,$$

where $[0, \pi] \ni \theta \mapsto \xi(\theta)$ parametrizes the cosphere fiber, h is a smooth non-vanishing function down to $\varepsilon = 0$. The point (x, ξ) corresponds to (y, θ) in these coordinates and we write $(y', \theta') = \psi_\varepsilon(y, \theta)$. If T is large enough, for the integrand not to vanish, one has to require that the angle $\theta'(\psi_\varepsilon(y, \theta))$ is uniformly contained in a compact interval of $]0, \pi[$. In other words, if we fix some constant $c > 0$, there exists

[1]The metric g here actually denotes the dual metric to g which is usually written g^{-1}. As mentioned in the introduction, we do not employ this notation in order to keep the reading affordable.

an integer $N_0 \geq 2$ large enough (independent of ε) such that for $T \geq -N_0 \log(\varepsilon)$, if $\theta'(\psi_\varepsilon(y,\theta)) \in [0,c] \cup [\pi - c, \pi]$, it will satisfy

$$(\ell_{\varepsilon_0,+}(\psi_\varepsilon(y,\theta)) - (T - 2T_\varepsilon))_+ = 0.$$

We can now make a change of variable in the previous integral by setting $(y',\theta') = \psi_\varepsilon(y,\theta)$. Since the dependence of ψ_ε^{-1} is smooth in ε (down to $\varepsilon = 0$) and $[0,\varepsilon_0] \times \{\rho = \varepsilon_0\}$ is compact, $|\det(\psi_\varepsilon^{-1}(y',\theta'))|$ is bounded independently of (y',θ') and ε. We get for $T \geq -N_0 \log(\varepsilon)$:

$$\begin{aligned}
&\int_{\partial_- S^* M_\varepsilon \cap D_\varepsilon} (\ell_{\varepsilon_0,+}(\psi_\varepsilon(x,\xi)) - (T - 2T_\varepsilon))_+ d\mu_{\nu,\varepsilon} \\
&\quad = \int_{\partial_- S^* M_\varepsilon \cap D_\varepsilon} (\ell_{\varepsilon_0,+}(\psi_\varepsilon(y,\theta)) - (T - 2T_\varepsilon))_+ \sin(\theta) h(\varepsilon, y) \frac{dy\,d\theta}{\varepsilon} \\
&\quad = \int_{\partial_- S^* M_{\varepsilon_0}} (\ell_{\varepsilon_0,+}(y',\theta') - (T - 2T_\varepsilon))_+ \sin(\theta(\psi_\varepsilon^{-1}(y',\theta'))) \\
&\qquad\qquad \cdot h(\varepsilon, y(\psi_\varepsilon^{-1}(y',\theta')))|\det(\psi_\varepsilon^{-1}(y',\theta'))| \frac{d\theta' dy'}{\varepsilon} \\
&\quad \leq C \int_{\partial_- S^* M_{\varepsilon_0}} (\ell_{\varepsilon_0,+}(y',\theta') - (T - 2T_\varepsilon))_+ \frac{d\theta' dy'}{\varepsilon} \\
&\quad \leq C\varepsilon^{-1} \int_{\partial_- S^* M_{\varepsilon_0,+}} (\ell_{\varepsilon_0}(y',\theta') - (T - 2T_\varepsilon))_+ h(\varepsilon_0, y) \sin(\theta') \frac{d\theta' dy'}{\varepsilon_0} \\
&\quad \leq C\varepsilon^{-1} V_{\varepsilon_0}(T - 2T_\varepsilon),
\end{aligned}$$

for some constant $C > 0$ (which may be different from one line to another) and where the penultimate inequality follows from the uniform bound on the angle (i.e. $\sin(\theta') \in [\sin(c), 1]$). But we know that for any $\delta \in (Q, 0)$, there exists an(other) constant $C > 0$ such that for all $T \geq 0$, $V_{\varepsilon_0}(T) \leq Ce^{-\delta T}$. Thus, for $T \geq -N_0 \log(\varepsilon)$

$$V_\varepsilon(T) \leq C\varepsilon^{-1} e^{-\delta(T - 2T_\varepsilon)} \leq C\varepsilon^{-(1+4\delta)} e^{-\delta T}. \qquad \square$$

2.2. The renormalized length.

2.2.1. Definition. This paragraph provides the definition in [8, Section 4.1]. Let $\alpha(x,x')$ be a geodesic in M joining two distinct points at infinity $x, x' \in \partial M$. For the sake of simplicity, we will only write α in this paragraph, instead of $\alpha(x,x')$. The renormalized length of the geodesic α is the real number defined by:

$$L(\alpha) := \lim_{\varepsilon \to 0} \ell\big(\alpha \cap \{\rho \geq \varepsilon\}\big) + 2\log(\varepsilon), \tag{2.14}$$

where $\ell(\cdot)$ denotes the Riemannian length. This limit exists and is finite by [8, Lemma 4.1].

Note that there is *a priori* no canonical choice of the renormalized length L insofar as it depends on the choice of the boundary defining function ρ. One can actually prove that if $\widehat{\rho} = e^{\omega}\rho$ is another choice, then (see [8, Equation (4.2)]):

$$\widehat{L}(\alpha(x,x')) - L(\alpha(x,x')) = \omega(x) + \omega(x').$$

2.2.2. Action of isometries on the renormalized length. Let γ be an isometry on (M,g), then γ acts smoothly on the compactification $\overline{M}$ (see the arguments given in §6 for instance).

Lemma 2.4. *Let α be a geodesic joining two points $x, x' \in \partial M$. We have:*

$$L(\gamma \circ \alpha) = L(\alpha) + n^{-1}\log\left(|d\gamma_x||d\gamma_{x'}|\right),$$

where $|d\gamma_x|$ is the Jacobian of $\gamma|_{\partial M}$ in x with respect to the metric h, $n+1$ being the dimension of M.

Proof. We denote by $z = (x,\xi)$ the point in $\partial_- S^*M$ generating α. Assume for the sake of simplicity that α is a half-line joining $x \in \partial M$ to a point in the interior M. Let $x_\varepsilon := \alpha \cap \{\rho = \varepsilon\}$ and $\alpha_\varepsilon := \alpha \cap \{\rho \geq \varepsilon\}$. We define $\varepsilon' := \rho(\gamma(x_\varepsilon))$. We have:

$$\ell(\alpha_\varepsilon) + \log(\varepsilon) = (\ell(\gamma(\alpha_\varepsilon)) + \log(\varepsilon')) - \log(\varepsilon'/\varepsilon). \tag{2.15}$$

As $\varepsilon \to 0$, the left-hand side converges to $L(\alpha)$ whereas the term between parenthesis on the right-hand side goes to $L(\gamma(\alpha))$, so all is left to compute is the limit of ε'/ε as $\varepsilon \to 0$. We write $\varepsilon' = \rho(\gamma(\pi(\overline{\varphi}_{\tau_\varepsilon}(z))))$, where τ_ε is defined to be the unique time such that $\rho(\overline{\varphi}_{\tau_\varepsilon}(z)) = \varepsilon$. By the implicit function theorem, $\varepsilon \mapsto \tau_\varepsilon$ is a smooth function of ε and it satisfies: $\rho(\overline{\varphi}_{\tau_\varepsilon}(z)) = \varepsilon = \tau_\varepsilon + \mathcal{O}(\tau_\varepsilon^2)$. Thus $\partial_\varepsilon \tau_\varepsilon|_{\varepsilon=0} = 1$ and:

$$\begin{aligned}
\lim_{\varepsilon\to 0} \varepsilon'/\varepsilon &= d\rho_{\gamma(x)}\Big(d\gamma_x\Big(d\pi_z\Big(\frac{d\tau_\varepsilon}{d\varepsilon}\frac{\partial\overline{\varphi}_{\tau_\varepsilon}}{\partial\tau_\varepsilon}(z)\Big|_{\varepsilon=0}\Big)\Big)\Big) \\
&= d\rho_{\gamma(x)}\big(d\gamma_x\big(d\pi_z(\overline{X}(z))\big)\big) \\
&= d\rho_{\gamma(x)}\big(d\gamma_x(\partial_\rho(x))\big).
\end{aligned}$$

Remark that at ∂M, $d\gamma_x(\partial_\rho(x)) = \lambda(x)\partial_\rho(\gamma(x))$ for some real number λ depending on x, since γ sends geodesics on geodesics. If $\eta_1, \dots, \eta_n \in T_x(\partial M)$ is an orthonormal frame for the metric h, one can prove that

$$h(d\gamma_x(\eta_i), d\gamma_x(\eta_j)) = \lambda^2(x)\delta_{ij}$$

by using the fact that $\gamma^* g = g$. As a consequence, the Jacobian of $\gamma|_{\partial M}$ at x with respect to the metric h is $\lambda^n(x)$. Thus:

$$\lim_{\varepsilon\to 0} \varepsilon'/\varepsilon = |d\gamma_x|^{\frac{1}{n}}.$$

Replacing this in (2.15), and adding the other part of the geodesic, we find the sought result. □

3. The Liouville current

We denote by $\widetilde{M}$ the universal cover of M: it is a topological disk on which we fix an orientation. All the objects $(g, \rho, X, \dots)$ lift to $\widetilde{M}$ and their corresponding object in the universal cover is invariant by the action of the fundamental group $\pi_1(M)$. Since we will only work on $\widetilde{M}$ in the following, for the reader's convenience, we will often drop the notation $\tilde{\cdot}$ when the context is clear, except for the universal cover itself $\widetilde{M}$. We define

$$\mathcal{G} := (\partial\widetilde{M} \times \partial\widetilde{M}) \setminus \text{diag},$$

which can be naturally identified with the set of untrapped geodesics (neither in the future nor the past) on $\widetilde{M}$. If $\mathcal{M}$ is the set of Borel measures on $\mathcal{G}$ which are invariant by the flip, then it is a classical fact from [17] that the Liouville measure induces a measure $\eta \in \mathcal{M}$ called the *Liouville current* (see also [11] for a proof).

3.1. Expression in coordinates. Given $x, x' \in \widetilde{M}$, we can parametrize α, the unique geodesic joining x to x', in the following way: if $z = (x, \xi) \in \partial_- S^*\widetilde{M}$ denotes the point generating α, then we parametrize the geodesic by $\alpha(t) = \varphi_t(m(z))$, where $m(z) = \bar{\varphi}_{\tau_+(z)/2}(z)$ is the middle point (this is a smooth map according to Section 2.1.2). We set $\gamma(t) := \pi(\alpha(t))$. We define

$$V := \{(\tau, \theta) \in \mathbb{R} \times (0, \pi), (\gamma(\tau),\ R_\theta \dot{\gamma}(\tau)) \notin \Gamma_- \cup \Gamma_+\}, \tag{3.1}$$

where R_θ is the rotation by a positive angle θ in the fibers of $S^*\widetilde{M}$. For $x, x' \in \widetilde{M}$, we denote by $\mathcal{F}(x, x') \subset \mathcal{G}$ the open subsets of points $(y, y') \in \mathcal{G}$ such that the geodesic joining y to y' has a transverse and positive (with respect to the orientation) intersection with the geodesic α in $\widetilde{M}$. If we further assume that $x, x' \in \partial\widetilde{M}$, we can consider the diffeomorphism $\phi: V \mapsto \mathcal{F}(x, x')$ defined by $\phi(\tau, \theta) = (y, y')$, the two points in $\partial\widetilde{M}$ such that the geodesic connecting them passes through the point $(\gamma(\tau), R_\theta\dot{\gamma}(\tau)) \in S^*\widetilde{M}$. The following lemma is a well-known fact (see [11, Lemma 3.1] for instance) and we do not provide its proof.

Lemma 3.1. $\phi^*\eta = \sin(\theta)\, d\theta\, d\tau$.

Remark 3.2. In negative curvature, the tails $\Gamma_- \cup \Gamma_+$ have zero Liouville measure. This implies that the set ${}^cV \subset \mathbb{R} \times (0, \pi)$ has zero measure in $\mathbb{R} \times (0, \pi)$ (for the measure $\sin(\theta)\, d\theta\, d\tau$). In particular, we will ignore trapped geodesics in the computations of the integrals of Section 4.4.

From the previous expression in coordinates, we recover the classical formula for $x, x' \in \widetilde{M}$ (see [17]):

$$\eta(\mathcal{F}(x, x')) = \int_0^\pi \int_0^{d(x,x')} \sin(\theta)\, d\theta\, d\tau = 2d(x, x'), \tag{3.2}$$

where $d(\cdot,\cdot)$ denotes the Riemannian distance between the two points. For $x, x' \in \partial\widetilde{M}$ and $\varepsilon > 0$ small enough, we denote by x_ε and x'_ε the two intersections of α (the geodesic joining x to x') with $\{\rho = \varepsilon\}$ in a respective neighborhood of x and x'. We have:

$$\begin{aligned}\eta(\mathcal{F}(x_\varepsilon, x'_\varepsilon)) + 4\log\varepsilon &= 2(d(x_\varepsilon, x'_\varepsilon) + 2\log\varepsilon)\\ &= 2(\ell(\alpha \cap \{\rho > \varepsilon\}) + 2\log\varepsilon)\\ &\to_{\varepsilon\to 0} 2L(\alpha)\end{aligned}$$

3.2. Liouville current and boundary distance. Let g_1 and g_2 be two negatively-curved metrics such that their renormalized lengths agree. We denote by η_1 and η_2 their respective Liouville currents.

Lemma 3.3. $\eta_1 = \eta_2$.

Proof. We recall that $\partial\widetilde{M}$ is a countable union of real lines embedded in the circle $\mathbb{S}^1$. The topology on $\partial\widetilde{M}$ is that naturally induced by the topology on $\mathbb{S}^1$. It is sufficient to prove that the two measures coincide on *rectangles*, namely on subsets $(x_1, x_2) \times (x_3, x_4)$, such that $(x_1, x_2), (x_3, x_4) \subset \partial\widetilde{M}$ are two intervals with disjoint closure, since they generate the Borel σ-algebra. We actually prove the:

Lemma 3.4. $\eta((x_1,x_2)\times(x_3,x_4)) = |L(x_1,x_3)+L(x_2,x_4)-L(x_2,x_3)-L(x_1,x_4)|$.

Note that that $\eta((x_1,x_2)\times(x_3,x_4)) = |[x_1,x_2,x_3,x_4]|$, the cross-ratio of the four points (see [14]). In particular, this proves that the right-hand side of Lemma 3.4 is a cross-ratio in the sense of [14], which may not be obvious at first sight. Actually, the properties of symmetry are immediate and the invariance by the diagonal action of the fundamental group follows from Lemma 2.4.

Given some $\varepsilon > 0$, we introduce the four horospheres $H_i(\varepsilon)$, $i \in \{1, \dots, 4\}$ such that $H_i(\varepsilon)$ intercepts x_i and the point defined as the intersection of the geodesic $\alpha(x_i, x_{i+2})$ ($i + 2$ is taken modulo 4) with $\{\rho = \varepsilon\}$ in a neighborhood of x_i.

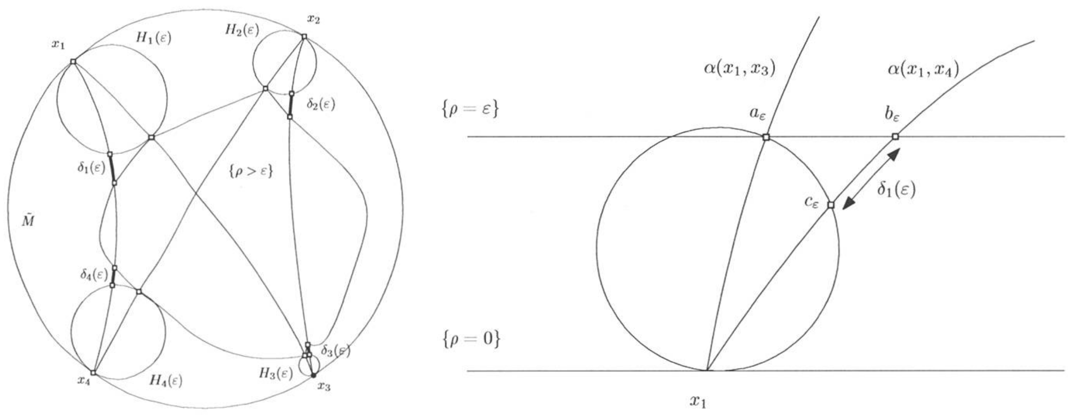

Figure 2. Left: The four horospheres and the lengths $\delta_i(\varepsilon)$. Right: The horosphere $H_1(\varepsilon)$.

We have:

$$
\begin{aligned}
&L(x_1, x_3) + L(x_2, x_4) - L(x_2, x_3) - L(x_1, x_4)\\
&= \lim_{\varepsilon\to 0} \ell(\alpha(x_1,x_3)\cap\{\rho>\varepsilon\}) + 2\log\varepsilon + \ell(\alpha(x_2,x_4)\cap\{\rho>\varepsilon\}) + 2\log\varepsilon\\
&\qquad - \ell(\alpha(x_2,x_3)\cap\{\rho>\varepsilon\}) - 2\log\varepsilon - \ell(\alpha(x_1,x_4)\cap\{\rho>\varepsilon\}) - 2\log\varepsilon\\
&= \lim_{\varepsilon\to 0} \ell(\alpha(x_1,x_3)\cap\{\rho>\varepsilon\}) + \ell(\alpha(x_2,x_4)\cap\{\rho>\varepsilon\}) - \ell(\alpha(x_2,x_3)\cap\{\rho>\varepsilon\})\\
&\qquad - \ell(\alpha(x_1,x_4)\cap\{\rho>\varepsilon\})\\
&= \lim_{\varepsilon\to 0} \ell(\alpha(x_1,x_3)\cap H_{\mathrm{ext}}(\varepsilon)) + \ell(\alpha(x_2,x_4)\cap H_{\mathrm{ext}}(\varepsilon)) - \ell(\alpha(x_2,x_3)\cap H_{\mathrm{ext}}(\varepsilon))\\
&\qquad - \ell(\alpha(x_1,x_4)\cap H_{\mathrm{ext}}(\varepsilon)) - \delta_1(\varepsilon) - \delta_2(\varepsilon) - \delta_3(\varepsilon) - \delta_4(\varepsilon),
\end{aligned}
$$

where $\delta_i(\varepsilon)$ is the algebraic distance on the geodesic between its intersection with $H_i(\varepsilon)$ and $\{\rho = \varepsilon\}$, positively counted from x_i, and $H_{\mathrm{ext}}(\varepsilon) := \widetilde{M} \setminus \cup_{i=1}^4 H_i(\varepsilon)$. Now, we know that the quantity

$$
\begin{aligned}
&|\ell(\alpha(x_1,x_3)\cap H_{\mathrm{ext}}(\varepsilon)) + \ell(\alpha(x_2,x_4)\cap H_{\mathrm{ext}}(\varepsilon))\\
&\qquad - \ell(\alpha(x_2,x_3)\cap H_{\mathrm{ext}}(\varepsilon)) - \ell(\alpha(x_1,x_4)\cap H_{\mathrm{ext}}(\varepsilon))|
\end{aligned}
$$

is actually independent of ε and equals $\eta([x_1,x_2]\times[x_3,x_4])$ (see [23] for instance). It is thus sufficient to prove that $\delta_i(\varepsilon)\to 0$ as $\varepsilon\to 0$. Let us consider $\delta_1(\varepsilon)$ and ε small enough so that we can work in the coordinates where the metric g can be written in the form $g = \rho^{-2}(d\rho^2 + h^2(\rho,y)dy^2)$ for some smooth positive function h^2 (down to the boundary).

We have:

$$
\delta_1(\varepsilon) = d(c_\varepsilon, b_\varepsilon) \le d(c_\varepsilon, a_\varepsilon) + d(a_\varepsilon, b_\varepsilon) \le d(c_\varepsilon, a_\varepsilon) + l\big([a_\varepsilon, b_\varepsilon]\big),
$$

where the points $a_\varepsilon, b_\varepsilon, c_\varepsilon, d_\varepsilon$ are introduced in Figure 2, $[a_\varepsilon, b_\varepsilon]$ denotes the Euclidean segment joining a_ε to b_ε. Note that by construction $d(c_\varepsilon, a_\varepsilon)\to 0$ as $\varepsilon\to 0$ (the points are on the same family of shrinking horospheres).

The two geodesics $\alpha(x_1,x_3)$ and $\alpha(x_1,x_4)$ with endpoint x_1, seen as curves in $\widetilde{M}$, can be locally parametrized by the respective smooth functions $(\rho, y_3(\rho))$ and $(\rho, y_4(\rho))$, according to the implicit function theorem since the geodesics intersect transversally the boundary (see Lemma 2.1). One has by derivating at $\rho = 0$ that

$$
\lambda_i \partial_\rho = \partial_\rho + y_i'(0)\partial_y
$$

for some constant λ_i, that is $y_i'(0) = 0$ and $\lambda_i = 1$. In other words, we can parametrize locally both geodesics by $(\rho, y_0 + \mathcal{O}(\rho^2))$, where y_0 is some constant depending on the choice of coordinates. Thus,

$$
|y(a_\varepsilon) - y(b_\varepsilon)| = \mathcal{O}(\varepsilon^2).
$$

If we choose a parametrization $\gamma(t) = (\varepsilon, y(a_\varepsilon) + t(y(b_\varepsilon) - y(a_\varepsilon)))$, for $t \in [0,1]$, of the euclidean segment $[a_\varepsilon, b_\varepsilon]$, then one has:

$$\ell\big([a_\varepsilon, b_\varepsilon]\big) = \int_0^1 g(\dot\gamma(t), \dot\gamma(t))^{1/2}\, dt = \varepsilon^{-1}|y(b_\varepsilon) - y(a_\varepsilon)| \int_0^1 h(\gamma(t))\, dt,$$

where the integral is uniformly bounded with respect to ε. Thus, by the previous remarks, $l([a_\varepsilon, b_\varepsilon]) = \mathcal{O}(\varepsilon)$, which concludes the proof. □

4. Construction of the deviation κ

In this section, for the sake of simplicity, we will sometimes write $A = \mathcal{O}(\varepsilon^\infty)$ in order to denote the fact that for all $n \in \mathbb{N} \setminus \{0\}$, there exists $C_n > 0, \varepsilon_n > 0$ such that: $\forall \varepsilon \leq \varepsilon_n, |A| \leq C_n \varepsilon^n$.

4.1. Reducing the problem. Suppose g_1 and g_2 are two asymptotically hyperbolic metrics like in the setting of Theorem 1.1 that is, they are both negatively-curved and their renormalized distances coincide for some choices of conformal representatives in the conformal infinities. In local coordinates (ρ, y), for $i \in \{1,2\}$, one can write $g_i = \rho^{-2}(d\rho^2 + h_{\rho,i})$, for some smooth metrics $h_{\rho,i}$ on ∂M (note that this is the same boundary defining function for both metrics, see [8, Section 4.2]). By [8, Theorem 2], there exists a smooth diffeomorphism $\psi\colon \overline{M} \to \overline{M}$ fixing the boundary such that $\psi^* g_1 - g_2 = \mathcal{O}(\rho^\infty)$ at ∂M (that is $h_{\rho,1} - h_{\rho,2} = \mathcal{O}(\rho^\infty)$). In the following, we will argue with this new metric $\psi^* g_1$ but we will still denote it g_1 for the sake of simplicity.

Remark 4.1. In particular, this implies that the respective renormalized vector fields satisfy $\overline{X}_1 - \overline{X}_2 = \mathcal{O}(\rho^\infty)$ at ∂M, that is their C^∞-jet coincide on the boundary. By Duhamel's formula (see [22, Lemma 2.2] for instance) this implies that on the boundary $\partial_- S^* M$, for any $k \geq 0$, one has $\|\overline{\varphi}^1_\tau - \overline{\varphi}^2_\tau\|_{C^k} = \mathcal{O}(\tau^\infty)$.

4.2. The diffeomorphism κ. We denote by $M_\varepsilon := M \cap \{\rho \geq \varepsilon\}$ and by $\widetilde{M}_\varepsilon$ its lift to the universal cover. Like before, all the objects are lifted on the universal cover. Unless it is mentioned, we will drop the notation $\tilde{\cdot}$, except for the universal cover itself. $S^*\widetilde{M}_i$ will denote the unit cotangent bundle with respect to the metric g_i. $\mathcal{G}_1$ and $\mathcal{G}_2$ denote the set of geodesics connecting points on the ideal boundary $\partial\widetilde{M}$, with respect to the metrics g_1 and g_2. They will sometimes be identified with $\partial\widetilde{M} \times \partial\widetilde{M} \setminus \mathrm{diag}$.

Given $(x,\xi) \in S^*\widetilde{M}_1 \setminus \Gamma^1_- \cup \Gamma^1_+$, we denote by $(z,z') \in \partial\widetilde{M} \times \partial\widetilde{M}$ (resp. $(y,y') \in \partial\widetilde{M} \times \partial\widetilde{M}$) the two points on the ideal boundary induced by the geodesic carrying the point (x,ξ) (resp. $(x, R_\theta\xi)$ if $\theta \in (0,\pi)$ and $(x, R_\theta\xi) \in S^*\widetilde{M}_1 \setminus \Gamma^1_- \cup \Gamma^1_+$). This defines a map:

$$\kappa_1 \colon \widetilde{W}_1 \to \mathscr{G}_1 \times \mathscr{G}_1 \setminus \text{diag}, \quad \kappa_1(x,\xi,\theta) = (z,z',y,y'),$$

where

$$\widetilde{W}_1 := \{(x,\xi,\theta) \in S^*\widetilde{M}_1 \times (0,\pi) \mid (x,\xi),(x,R_\theta\xi) \notin (\Gamma^1_- \cup \Gamma^1_+)\}.$$

The map κ_1 is clearly bijective. It is smooth because each of the coordinates (z,z',y,y') is smooth. Indeed, one has for instance

$$z(x,\xi,\theta) = \pi\big(\overline{\varphi}^1_{\tau_-(x,\xi)}(x,\xi)\big),$$

and this is a smooth application according to Section 2.1.2.

The g_2-geodesics with endpoints (z,z') and (y,y') intersect in a single point $(\widetilde{\text{x}}(x,\xi,\theta), \widetilde{\Xi}(x,\xi,\theta))$ (where $\widetilde{\Xi}$ is the covector on the g_2-geodesic with endpoints (z,z')) and form an angle $\widetilde{f}(x,\xi,\theta)$, which we call the *angle of deviation*. This defines a map

$$\widetilde{\kappa} := \kappa_2^{-1} \circ \kappa_1 \colon \widetilde{W}_1 \to \widetilde{W}_2, \quad \widetilde{\kappa}(x,\xi,\theta) = \big(\widetilde{\text{x}}(x,\xi,\theta), \widetilde{\Xi}(x,\xi,\theta), \widetilde{f}(x,\xi,\theta)\big), \tag{4.1}$$

where $\widetilde{W}_2$ is defined in the same fashion as $\widetilde{W}_1$. By the implicit function theorem, one can prove that κ_2^{-1} is smooth and thus $\widetilde{\kappa}$ too. It is a bijective map whose inverse $\widetilde{\kappa}^{-1} = \kappa_1^{-1} \circ \kappa_2$ is smooth by the same arguments. As a consequence, $\widetilde{\kappa}$ is a smooth diffeomorphism. Moreover, it is invariant by the action of the fundamental group and thus descends to the base as a map $\kappa \colon (x,\xi,\theta) \mapsto (\text{x}, \Xi, f)$.

4.3. Scattering on the universal cover. On the universal cover $\widetilde{M}$, the renormalized distance can actually be extended outside the boundary, namely we can set for $p,q \in \widetilde{M}$:

$$D_i(p,q) := d_i(p,q) + \log(\rho(p)) + \log(\rho(q)),$$

where d_i, $i \in \{1,2\}$ stands for the Riemannian distance induced by the metric g_i. D_i is clearly smooth on $\widetilde{M} \times \widetilde{M} \setminus \text{diag}$ and using the fact that there exists a unique geodesic connecting two points, one can prove like in [8, Proposition 5.15], that the extension of D_i to $\widetilde{\widetilde{M}} \times \widetilde{\widetilde{M}} \setminus \text{diag}$ is smooth. Now, as established in [8, Proposition 5.16] the renormalized distance on the boundary actually determines the scattering map σ_i (defined in (2.10)), that is:

Proposition 4.2. *If $L_1 = L_2$, then $\sigma_1 = \sigma_2$.*

The proof also applies here, in the universal cover. It is a standard computation since we know that D_i is differentiable, which relies on the fact that the gradient of $q \mapsto L_i(\alpha(p,q))$ (for $p,q \in \partial\widetilde{M}$) is the projection on the tangent space $T_q\partial\widetilde{M}$ of the gradient of $q \mapsto D_i(p,q)$ and the latter corresponds to the direction of the geodesic joining p to q when it exits $\widetilde{M}$.

We fix $\varepsilon > 0$ and define $S^*\widetilde{M}^i_\varepsilon := S^*\widetilde{M}_i \cap \{\rho \geq \varepsilon\}$. For $i \in \{1,2\}$, given $(x,\xi) \in \partial_- S^*\widetilde{M}^i_\varepsilon$ we can represent the vector $\xi = \xi(\omega)$ by the angle $\omega \in [0,\pi]$ such that $\sin\omega = |g_i(\nu_i(x),\xi)|$, where ν_i stands for the unit covector conormal to $\{\rho = \varepsilon\}$ (with respect to the metric g_i).

Lemma 4.3. *There exists an angle ω_ε (only depending on ε), such that for all $(x,\xi(\omega)) \in \partial_- S^*\widetilde{M}^1_\varepsilon \setminus \Gamma^1_-$, given by an angle $\omega \in [\omega_\varepsilon, \pi - \omega_\varepsilon]$, if $\alpha_1(p,q)$ denotes the g_1-geodesic generated by (x,ξ), with endpoints $(p,q) \in \partial\widetilde{M} \times \partial\widetilde{M}$, then the g_2-geodesic $\alpha_2(p,q)$ with endpoints p and q intercepts the set $\{\rho > \varepsilon\}$. Moreover, for any $N \in \mathbb{N}\setminus\{0\}$, there exists $\varepsilon_N > 0$ such that for all $\varepsilon < \varepsilon_N$, we can take $\omega_\varepsilon = \varepsilon^N$.*

Proof. Let $(x,\xi) \in \partial_- S^*\widetilde{M}^1_\varepsilon$. We set ourselves in the coordinates (ρ, y) induced by the conformal representative h. The trajectory

$$t \mapsto (\rho(t), y(t), \xi_0(t), \eta(t)) \in S^*\widetilde{M}$$

of the point (x,ξ) under the flow X is given by Hamilton's equation (see [8, Equation (2.8)]). Flowing backwards in time with φ_t, we know that (x,ξ) converges exponentially fast towards a point $(p,\zeta) \in \partial_- S^*\widetilde{M}$ (see [8, Equation (2.11)]) in the sense that there exists a constant C (uniform in the choice of points) such that:

$$\forall t \leq 0, \quad \rho(t) \leq C\rho(0)e^{-|t|} = \varepsilon C e^{-|t|}.$$

In particular, the time $\tau_-(x,\xi)$ taken by the point (x,ξ) to reach (p,ζ) with the flow $\overline{\varphi}^1_\tau$ is (see (2.4)):

$$\tau_-(x,\xi) = \int_{-\infty}^0 \rho(t)\,dt \leq C\varepsilon.$$

We also know, according to Hamilton's equations (see [8, Equation (2.8)]) that

$$\dot{\rho}(0) = \rho^2(0)\xi_0(0) = \varepsilon\sin(\omega),$$

where ω satisfies $\overline{\xi}_0(0) = \rho\xi_0(0) = \sin(\omega) = |g_1(\xi,\nu_1(x))|$. Let us fix an integer $N > 0$ and assume that $\varepsilon^N \leq \omega \leq \pi - \varepsilon^N$. Then $\dot{\rho}(0) \geq 2/\pi \cdot \varepsilon^{N+1}$ so there exists an interval $[0,\delta]$ such that for $t \in [0,\delta]$:

$$\varepsilon + t/\pi \cdot \varepsilon^{N+1} \leq \varepsilon + t/2 \cdot \dot{\rho}(0) \leq \rho(t) \leq 2\varepsilon.$$

In particular, $\rho(\delta) \geq \varepsilon + \delta/\pi \cdot \varepsilon^{N+1}$.

We go back to the flow $\overline{\varphi}^1_\tau$. By our previous remark, we know that there exists a time:

$$\tau_0 \leq C\varepsilon + \int_0^\delta \rho(t)\,dt \leq C'\varepsilon,$$

such that $\rho(\overline{\varphi}^1_{\tau_0}(p,\zeta)) \geq \varepsilon + \delta/\pi \cdot \varepsilon^{N+2}$. But since $g_1 = g_2 + \mathcal{O}(\rho^\infty)$, we know that $X_1 = X_2 + \mathcal{O}(\rho^\infty)$ and $\overline{X}_1 = \overline{X}_2 + \mathcal{O}(\rho^\infty)$. Moreover, since the scattering maps agree according to Proposition 4.2, we know that the two geodesics $\alpha_1(p,q)$ and $\alpha_2(p,q)$ are both generated by (p,ζ). As a consequence, one has:

$$\rho(\overline{\varphi}^1_\tau(p,\zeta)) = \rho(\overline{\varphi}^2_\tau(p,\zeta)) + \mathcal{O}(\tau^\infty)$$

(the remainder being independent of (p,ζ)). In particular, since $\tau_0 \leq C'\varepsilon$, there exists a constant $C'' > 0$ such that

$$|\rho(\overline{\varphi}^1_{\tau_0}(p,\zeta)) - \rho(\overline{\varphi}^2_{\tau_0}(p,\zeta))| \leq C''\varepsilon^{N+2}.$$

Thus:

$$\rho(\overline{\varphi}^2_{\tau_0}(p,\zeta)) \geq \varepsilon + \frac{\delta}{\pi}\varepsilon^{N+1} - C''\varepsilon^{N+2} > \varepsilon,$$

if ε is small enough. □

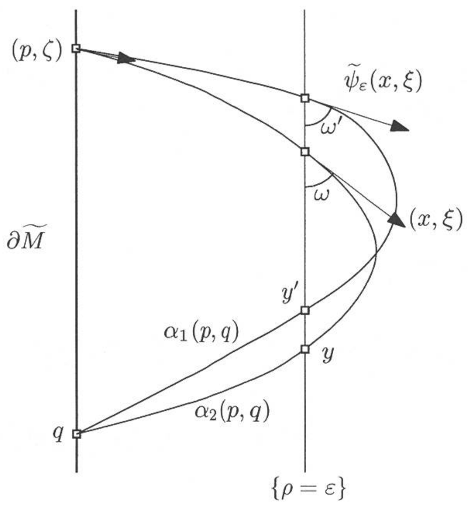

Figure 3. The diffeomorphism $\widetilde{\psi}_\varepsilon$.

In the following, we assume that such an integer N is fixed (and taken large enough) and we apply the previous lemma with $N+1$, that is $\omega_\varepsilon = \varepsilon^{N+1}$.

This allows us to define a map $\widetilde{\psi}$ on

$$\mathcal{U} := \{(x,\xi(\omega)) \in S^*\widetilde{M}^1,\ \overline{\xi}_0 \geq 0,\ \omega \in [\rho(x)^{N+1}, \pi - \rho(x)^{N+1}]\},$$

in the following way: to a point $(x,\xi) \in \mathcal{U}$, which we see as a boundary point $(x,\xi(\omega)) \in \partial_- S^*\widetilde{M}^1_\varepsilon$ for $\varepsilon = \rho(x)$, we associate the boundary point $(x',\xi') = \widetilde{\psi}(x,\xi)$ such that $\widetilde{\psi}(x,\xi) \in \partial_- S^*\widetilde{M}^2_\varepsilon$ is the point on the g_2-geodesic connecting p to q. A formal way to define $\widetilde{\psi}$ is to introduce another diffeomorphism $\widetilde{\psi}_1 \colon \mathcal{U} \to \partial_- S^*\widetilde{M} \times [0,\infty)$ such that $\widetilde{\psi}_1(x,\xi) = (\overline{\varphi}^1_{\tau_-(x,\xi)}(x,\xi), \rho(x))$ and to set

$$\widetilde{\psi}(x,\xi) = \widetilde{\psi}_2^{-1} \circ \widetilde{\psi}_1(x,\xi) = \overline{\varphi}^2_{\tau_\rho}(\overline{\varphi}^1_{\tau_-(x,\xi)}(x,\xi)), \tag{4.2}$$

where $\widetilde{\psi}_2$ is defined in the same fashion and τ_ρ is the time taken to reach the hypersurface $\{\rho = \rho(x)\}$. Note that $\widetilde{\psi}(x, \xi)$ exists according to the previous lemma and this point is well-defined (it is unique) according to Lemma 2.2. Moreover, it is smooth on $\mathcal{U}$ thanks to the results of Section 2.1.2 (this mainly follows from the implicit function theorem). Eventually, it is invariant by the action of the fundamental group and descends on the base as a map ψ. We write $\mathcal{U}_\varepsilon := \mathcal{U} \cap \{\rho = \varepsilon\}$. What we need, is to prove that $\widetilde{\psi}$ is the identity plus a small remainder.

Lemma 4.4. $\|\widetilde{\psi}_\varepsilon - \mathrm{Id}\,\|_{C^1} = \mathcal{O}(\varepsilon^\infty)$.

Proof. Since the two trajectories are $\mathcal{O}(\varepsilon^\infty)$ close, so will be the times τ_ρ and $-\tau_-(x, \xi)$ by which the g_1- and g_2-geodesics generated by (p, ζ) hit $\{\rho = \varepsilon\}$ (this can be proved by contradiction for instance, like in the proof of Lemma 4.3), which implies that $\widetilde{\psi}_\varepsilon(x, \xi) = (x, \xi) + \mathcal{O}(\varepsilon^\infty)$, where the remainder is uniform in (x, ξ). To obtain a bound on the derivatives, we see from the expression (4.2) and the fact that the two flows are $\mathcal{O}(\varepsilon^\infty)$ close in the C^1-topology (Remark 4.1), that it is sufficient to show that the times satisfy $\tau_\rho(x, \xi) = -\tau_-(x, \xi) + \mathcal{O}(\varepsilon^\infty)$ in the C^1-topology with a uniform remainder. Let $(p, \zeta) = \overline{\varphi}^1_{\tau_-(x,\xi)}(x, \xi)$. We have

$$\rho(\overline{\varphi}^1_{-\tau_-(x,\xi)}(p, \zeta)) = \varepsilon = \rho(\overline{\varphi}^2_{\tau_\rho}(p, \zeta)).$$

We are interested in the variations of x along $\{\rho = \varepsilon\}$ and of the angle $\xi(\omega)$. If we denote by z any of these two parameters, we get by derivating the previous equality:

$$-\frac{\partial \tau_-}{\partial z} d\rho(\bar{X}_1) + d\rho(d\overline{\varphi}^1_{-\tau_-}(d_z(p, \zeta))) = \frac{\partial \tau_\rho}{\partial z} d\rho(\bar{X}_2) + d\rho(d\overline{\varphi}^2_{\tau_\rho}(d_z(p, \zeta))).$$

The two terms containing the differential of the flow coincide to order $\mathcal{O}(\varepsilon^\infty)$ and we also have $d\rho(\bar{X}_2) = d\rho(\bar{X}_1) + \mathcal{O}(\varepsilon^\infty)$ by Remark 4.1. Thus:

$$\Big(-\frac{\partial \tau_-}{\partial z} - \frac{\partial \tau_\rho}{\partial z}\Big) d\rho(\bar{X}_1) = \mathcal{O}(\varepsilon^\infty).$$

But $d\rho(\bar{X}_1)$ is precisely the sine of the angle with which the geodesic generated by (p, ζ) enters the set $\{\rho \geq \varepsilon\}$ and this angle is contained in $[\varepsilon^N, \pi - \varepsilon^N]$ by construction of the set $\mathcal{U}$, so $d\rho(\bar{X}_1) \geq \varepsilon^N$. By dividing by $d\rho(\bar{X}_1)$, this term is swallowed in the $\mathcal{O}(\varepsilon^\infty)$, which provides the sought result. □

Given $(x, \xi) \in \partial_- S^*\widetilde{M}^i_\varepsilon$, we denote by $\ell^i_{\varepsilon,+}(x, \xi)$ the length of the geodesic generated by this point in $\widetilde{M}_\varepsilon$. Note that by strict convexity of the sets $\{\rho \geq \varepsilon\}$ the intersections of the geodesics (for both metrics) with $\widetilde{M}_\varepsilon$ have a single connected component, so this length is well-defined.

Lemma 4.5. $\|\ell^1_{\varepsilon,+} - \ell^2_{\varepsilon,+} \circ \widetilde{\psi}_\varepsilon\|_{C^0} = \mathcal{O}(\varepsilon^\infty)$, *where the sup is computed over* $\partial_- S^*\widetilde{M}^1_\varepsilon \setminus \Gamma^1_-$.

Proof. Recall that $(p,\zeta) \in \partial_- S^*\widetilde{M}$ is the point obtained by flowing backwards (x,ξ) down to the boundary. If D_i denotes the renormalized distance for both metrics, then we have:

$$D_1(p,x) = D_2(p,x'(x,\omega)) + \mathcal{O}(\varepsilon^\infty),$$

where the remainder is independent of (x,ξ). Indeed, considering $0 < \varepsilon' < \varepsilon$, and denoting by $\alpha_1(p,x)$ the g_1-geodesic joining p to x, one has:

$$\begin{aligned}\ell_1(\alpha_1(p,x) \cap \{\rho > \varepsilon'\}) + \log \varepsilon' &= \int_{\tau^1_{\varepsilon'}}^{\tau^1_\varepsilon} \frac{ds}{\rho(\overline{\varphi}^1_s(z))} + \log \varepsilon' \\ &= \int_{\varepsilon'}^{\varepsilon} \frac{(\psi_1^{-1})'(u)du}{u} + \log \varepsilon',\end{aligned}$$

where τ^1_ε and $\tau^1_{\varepsilon'}$ are defined such that

$$\rho(\overline{\varphi}^1_{\tau^1_\varepsilon}(z)) = \varepsilon, \quad \rho(\overline{\varphi}^1_{\tau^1_{\varepsilon'}}(z)) = \varepsilon',$$

and $\psi_1 \colon s \mapsto \rho(\overline{\varphi}^1_s(z))$ is a diffeomorphism. Note that $\psi_1(0) = 0, \psi_1'(0) = 1$. By assumption, the two metrics are close, thus $\psi_1(s) = \psi_2(s) + \mathcal{O}(s^\infty)$ and one can check (by induction) that this implies that $(\psi_1^{-1})^{(k)}(0) = (\psi_2^{-1})^{(k)}(0)$ for all $k \in \mathbb{N}$, that is $\psi_1^{-1}(u) = \psi_2^{-1}(u) + \mathcal{O}(u^\infty)$. Inserting this into the previous integral expression, we get the claimed result.

The same occurs for the other bits of the geodesics: namely, if y and y' denote the exit points of $\alpha_1(p,q)$ and $\alpha_2(p,q)$ in $\widetilde{M}_\varepsilon$, then $D_1(q,y) = D_2(q,y') + \mathcal{O}(\varepsilon^\infty)$. Now, using the fact that the renormalized lengths agree on the boundary, we obtain:

$$\begin{aligned}D_1(p,q) &= D_1(p,x) + d_1(x,y) + D_1(y,q) \\ &= D_1(p,x) + \ell^1_{\varepsilon,+}(x,\xi) + D_1(y,q) \\ &= D_2(p,q) \\ &= D_2(p,x') + \ell^2_{\varepsilon,+}(\widetilde{\psi}_\varepsilon(x,\xi)) + D_2(y',q).\end{aligned}$$

Thus: $\ell^1_{\varepsilon,+}(x,\xi) = \ell^2_{\varepsilon,+}(\widetilde{\psi}_\varepsilon(x,\xi)) + \mathcal{O}(\varepsilon^\infty)$. □

4.4. The average angle deviation. The angle of deviation $\widetilde{f}$ satisfies two elementary properties:

Lemma 4.6. (1) *It is π-symmetric, that is, for almost all $(x,\xi) \in S^*\widetilde{M}_1, \theta \in [0,\pi]$,*

$$\widetilde{f}(x,\xi,\theta) = \pi - \widetilde{f}(x, R_\theta\xi, \pi - \theta). \tag{4.3}$$

(2) *It is superadditive in the sense that, for almost all $(x,\xi) \in S^*\widetilde{M}_1, \theta_1, \theta_2 \in [0,\pi]$ such that $\theta_1 + \theta_2 \in [0,\pi]$,*

$$\widetilde{f}(x,\xi,\theta_1) + \widetilde{f}(x, R_{\theta_1}\xi, \theta_2) \le \widetilde{f}(x,\xi,\theta_1 + \theta_2). \tag{4.4}$$

We will denote by $\mathcal{H}: \mathcal{G}_1 \to \mathcal{G}_2$ the map that associates to a g_1-geodesic with endpoints $z, z' \in \widetilde{M}$ the g_2-geodesic with same endpoints. Note that when $\mathcal{G}_1$ and $\mathcal{G}_2$ are identified with $\partial \widetilde{M} \times \partial \widetilde{M}$, $\mathcal{H}$ is simply the identity, but we will rather see $\mathcal{G}_i$ as the set of geodesics connecting two boundary points.

Proof. The π-symmetry is obtained from the very definition of $\tilde{f}$. As to the superadditivity, it follows from Gauss–Bonnet formula in negative curvature. Indeed, consider the three geodesics $\alpha_1, \beta_1, \gamma_1$ of $\widetilde{M}_1$, respectively carried by the points $(x, \xi), (x, R_{\theta_1}\xi), (x, R_{\theta_1+\theta_2}\xi)$. Their image by $\mathcal{H}$ (that is the corresponding g_2-geodesics with same endpoints) are three geodesics

$$\alpha_2 = \mathcal{H}(\alpha_1), \quad \beta_2 = \mathcal{H}(\beta_2), \quad \gamma_2 = \mathcal{H}(\gamma_2),$$

forming a geodesic triangle which we denote by T, with angles

$$\tilde{f}(x, \xi, \theta_1), \quad \tilde{f}(x, R_{\theta_1}\xi, \theta_2), \quad \tilde{f}(x, R_{\theta_1+\theta_2}\xi, \pi - \theta_1 - \theta_2).$$

Now, we have by Gauss–Bonnet formula:

$$0 \geq \int_T \kappa \, d\mathrm{vol}_g = \tilde{f}(x, \xi, \theta_1) + \tilde{f}(x, R_{\theta_1}\xi, \theta_2) + \tilde{f}(x, R_{\theta_1+\theta_2}\xi, \pi - \theta_1 - \theta_2) - \pi. \tag{4.5}$$

Using π-symmetry, we obtain inequality (4.3). □

Note that the inequality (4.4) is saturated if and only if the geodesic triangle is degenerate, that is it is reduced to a single point, since the curvature is negative. As mentioned previously, $\tilde{f}$ descends on the base as a function f which also satisfies the properties of Lemma 4.6.

One of the ideas of Otal was to introduce the *average angle of deviation*. Since we work in a non-compact setting, we are forced to consider partial averages depending on ε. We define for fixed $\varepsilon > 0$:

$$\Theta_\varepsilon(\theta) := \frac{1}{\mathrm{vol}_{g_1}(S^* M_\varepsilon^1)} \int_{S^* M_\varepsilon^1} f(x, \xi, \theta) \, d\mu_1(x, \xi) \tag{4.6}$$

It also satisfies

$$\Theta_\varepsilon(0) = 0, \Theta_\varepsilon(\pi) = \pi. \tag{4.7}$$

Since the rotations R_θ preserve the Liouville measure, by integrating over $S^* M_\varepsilon^1$ the relations (4.3) and (4.4) given in Lemma 4.6, we see that Θ_ε also satisfies the π-symmetry:

$$\forall \theta \in [0, \pi], \quad \Theta_\varepsilon(\theta) = \pi - \Theta_\varepsilon(\pi - \theta), \tag{4.8}$$

and the superadditivity:

$$\forall \theta_1, \theta_2 \in [0, \pi], \text{ s.t. } \theta_1 + \theta_2 \in [0, \pi], \quad \Theta_\varepsilon(\theta_1) + \Theta_\varepsilon(\theta_2) \leq \Theta_\varepsilon(\theta_1 + \theta_2). \tag{4.9}$$

We now show that Θ_ε satisfies the following lemma.

Lemma 4.7. *Let $J\colon[0,\pi]\to\mathbb{R}$ be a convex continuous function. Then:*

$$\int_0^\pi J(\Theta_\varepsilon(\theta))\sin(\theta)\,d\theta \le \int_0^\pi J(\theta)\sin(\theta)\,d\theta + \|J\|_{L^\infty}\mathcal{O}(\varepsilon^N), \tag{4.10}$$

where the remainder only depends on ε, N is fixed by Lemma 4.3.

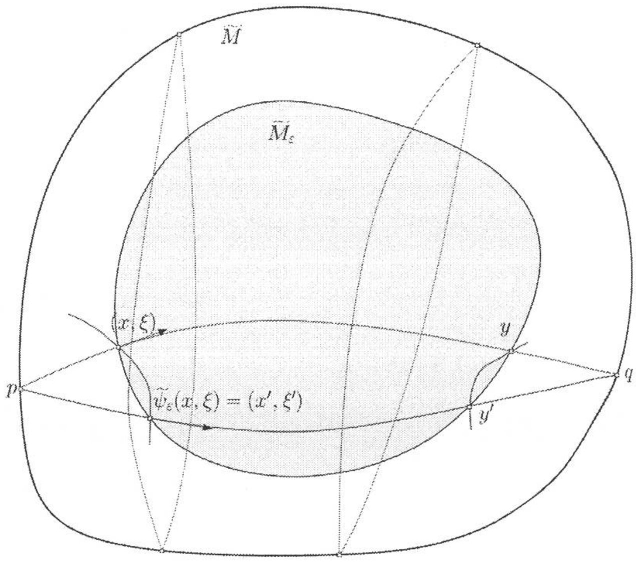

Figure 4. A picture of the situation: in dark grey, the g_2-geodesics, in light grey the g_1-geodesics.

The proof of this lemma relies on the use of Santaló's formula, together with the fact that the Liouville currents coincide. But let us make a preliminary remark. Consider $(x,\xi(\omega))\in\partial_-S^*\widetilde{M}^1_\varepsilon$ with $\omega\in[\omega_\varepsilon,\pi-\omega_\varepsilon]$. It generates the g_1-geodesic $\alpha_1(p,q)$ with endpoints $p,q\in\partial\widetilde{M}$ which enters (resp. exits) $\widetilde{M}_\varepsilon$ at x (resp. y). We denote by α_2 the g_2-geodesic joining p and q which enters (resp. exits) $\widetilde{M}_\varepsilon$ at $x'=x'(\widetilde{\psi}_\varepsilon(x,\xi))$ (resp. y'). Let us denote by $\mathcal{F}_1(x,y)\subset\mathcal{G}$ the g_1-geodesics which have a positive transverse intersection with the geodesic segment $\alpha^1_\varepsilon:=\alpha_1\cap\widetilde{M}_\varepsilon$. $\mathcal{F}_2(x',y')$ denotes its analogue for the second metric, that is the g_2-geodesics having a positive transverse intersection with $\alpha^2_\varepsilon:=\alpha_2\cap\widetilde{M}_\varepsilon$.

Since $\mathcal{H}$ preserves the Liouville measure (that is $\mathcal{H}_*\eta_1=\eta_2$), we have:

$$\eta_1(\mathcal{F}_1(x,y))=\eta_2(\mathcal{H}(\mathcal{F}_1(x,y))).$$

We could hope that $\mathcal{H}(\mathcal{F}_1(x,y))=\mathcal{F}_2(x',y')$ but this is not the case (see Figure 4), insofar as there is a slight defect due to the fact that we are not looking at points on the boundary, and this is where the arguments of Otal fail to apply immediately. However, we have:

Lemma 4.8. $\eta_1(\mathcal{F}_1(x,y))=\eta_2(\mathcal{F}_2(x',y'))+\mathcal{O}(\varepsilon^\infty)$, *where the remainder is independent of (x,ξ).*

Proof. It follows from Lemma 4.5, combined with equation (3.2). □

We can now establish the lemma on convexity. We will denote with a tilde $\tilde{\cdot}$ the objects on the universal cover.

Proof. $d\mu_1/\mathrm{vol}_{g_1}(S^*M_\varepsilon^1)$ is a probability measure on $S^*M_\varepsilon^1$ and by Jensen inequality, we have, for all $\theta \in [0,\pi]$:

$$J(\Theta_\varepsilon(\theta)) \leq \frac{1}{\mathrm{vol}_{g_1}(S^*M_\varepsilon^1)} \int_{S^*M_\varepsilon^1} J(f(x,\xi,\theta))d\mu_1(x,\xi).$$

Multiplying by $\sin(\theta)$, integrating over $[0,\pi]$ and applying Fubini's Theorem, we obtain:

$$\int_0^\pi J(\Theta_\varepsilon(\theta))\sin(\theta)\,d\theta \\ \leq \frac{1}{\mathrm{vol}_{g_1}(S^*M_\varepsilon^1)} \int_{S^*M_\varepsilon^1}\int_0^\pi J(f(x,\xi,\theta))\sin(\theta)\,d\theta\,d\mu_1(x,\xi).$$

Using Santaló's formula, we obtain for the last integral:

$$\int_{S^*M_\varepsilon^1}\int_0^\pi J(f(x,\xi,\theta))\sin(\theta)\,d\theta\,d\mu_1(x,\xi) \\ = \int_{\partial_- S^*M_\varepsilon^1}\int_0^{\ell^1_{\varepsilon,+}(x,\xi)}\int_0^\pi J(f(\varphi_\tau^1(x,\xi),\theta))\sin(\theta)\,d\theta\,d\tau\,d\mu_{1,\nu}(x,\xi),$$

where $d\mu_{1,\nu}(x,\xi) = |g_1(\xi,\nu_1)| i^*_{\partial S^*M_\varepsilon^1}(d\mu_1)$, ν_1 is unit covector conormal to the boundary, $i^*_{\partial S^*M_\varepsilon^1}(d\mu_1)$ is the restriction of the Liouville measure to the boundary (the measure induced by the Sasaki metric restricted to $\partial S^*M_\varepsilon^1$), and $\ell^1_{\varepsilon,+}(x,\xi)$ is the length of the geodesic starting from (x,ξ) in M_ε. Note that we would formally need to remove the set of trapped geodesics when applying Santaló's formula. However, as mentioned in Remark 3.2, they have zero measure and do not influence the computation, so we forget them in order not to complicate the notations. By parametrizing each fiber $\partial_- S_x^*M_\varepsilon^1$ with an angle $\omega \in [0,\pi]$, we can still disintegrate the measure $d\mu_{1,\nu} = \sin(\omega)\,d\omega\,dx$, where dx is the measure induced by the metric g_1 on ∂M_ε and $d\omega$ is the measure in the fiber $\partial_- S^*M_\varepsilon^1$, so that:

$$\int_{S^*M_\varepsilon^1}\int_0^\pi J(f(x,\xi,\theta))\sin(\theta)\,d\theta\,d\mu_1(x,\xi) \\ = \int_{\partial M_\varepsilon}\int_0^\pi\int_0^{\ell^1_{\varepsilon,+}(x,\xi)}\int_0^\pi J(f(\varphi_\tau^1(x,\xi),\theta))\sin(\theta)\,d\theta\,d\tau\,\sin(\omega)\,d\omega\,dx \\ = \int_{\partial M_\varepsilon}\int_{\omega_\varepsilon}^{\pi-\omega_\varepsilon}\int_0^{\ell^1_{\varepsilon,+}(x,\xi)}\int_0^\pi J(f(\varphi_\tau^1(x,\xi),\theta))\sin(\theta)\,d\theta\,d\tau\,\sin(\omega)\,d\omega\,dx \\ + \|J\|_{L^\infty}\mathcal{O}(\varepsilon^N).$$

Recall that we applied Lemma 4.3 with $\omega_\varepsilon = \mathcal{O}(\varepsilon^{N+1})$. The loss of 1 in the exponent is due to the fact that we have to swallow uniformly the lengths $\ell^1_{\varepsilon,+}(x,\xi) = \mathcal{O}(-\log\varepsilon)$ in the integral.

Let us fix $(x,\xi(\omega)) \in \partial_- S^*M^1_\varepsilon \setminus \Gamma_-$ and consider one of its lift on the universal cover $(\widetilde{x},\widetilde{\xi}(\omega)) \in \partial_- S^*\widetilde{M}^1_\varepsilon \setminus \widetilde{\Gamma}^1_-$. It generates a geodesic with endpoints $(p,q) \in \partial\widetilde{M} \times \partial\widetilde{M}$. We can rewrite the integral

$$\int_0^{\ell^1_{\varepsilon,+}(x,\xi)} \int_0^\pi J(f(\varphi^1_\tau(x,\xi),\theta))\sin(\theta)\,d\theta\,d\tau = \int_0^{\widetilde{\ell}^1_{\varepsilon,+}(\widetilde{x},\widetilde{\xi})} \int_0^\pi J(\widetilde{f}(\widetilde{\varphi}^1_\tau(\widetilde{x},\widetilde{\xi}),\theta))\sin(\theta)\,d\theta\,d\tau.$$

We will now use the diffeomorphisms $\phi_i\colon V_i \to \mathcal{F}(p,q)$ (for $i = 1,2$) introduced in Section 3 (see Equation (3.1)). The $\widetilde{g}_1$-geodesic joining p to q is denoted by $\alpha_1(p,q)$: we choose a parametrization $\gamma\colon\mathbb{R} \to \alpha_1(p,q)$ by arc-length using the middle point (see Section 3). Remark that the composition $\phi_2^{-1}\circ\phi_1\colon V_1 \to V_2$ has the form $(\tau,\theta) \mapsto (\,\cdot\,, \widetilde{f}(\gamma(\tau),\dot\gamma(\tau),\theta))$ (the first coordinate is of no interest to us). Moreover,

$$(\phi_2^{-1}\circ\phi_1)^*\sin(\theta)\,d\theta\,d\tau = \phi_1^*\eta_2 = \phi_1^*\eta_1 = \sin(\theta)\,d\theta\,d\tau,$$

since the two Liouville currents agree according to Lemma 3.3. We have:

$$\begin{aligned}
&\int_0^{\widetilde{\ell}^1_{\varepsilon,+}(\widetilde{x},\widetilde{\xi})} \int_0^\pi J(\widetilde{f}(\widetilde{\varphi}^1_\tau(\widetilde{x},\widetilde{\xi}),\theta))\sin(\theta)\,d\theta\,d\tau \\
&\qquad = \phi_1^*\eta_1(J\circ\phi_2^{-1}\circ\phi_1\cdot\mathbf{1}_{[T,T+\widetilde{\ell}^1_{\varepsilon,+}(\widetilde{x},\widetilde{\xi})]\times[0,\pi]}) \\
&\qquad = \eta_1(J\circ\phi_2^{-1}\cdot\mathbf{1}_{\mathcal{F}_1(\widetilde{x},\widetilde{y})}) \\
&\qquad = \eta_2(J\circ\phi_2^{-1}\cdot\mathbf{1}_{\mathcal{H}(\mathcal{F}_1(\widetilde{x},\widetilde{y}))}) \\
&\qquad = \eta_2(J\circ\phi_2^{-1}\cdot\mathbf{1}_{\mathcal{F}_2(\widetilde{x}',\widetilde{y}')}) + \|J\|_{L^\infty}\mathcal{O}(\varepsilon^\infty) \\
&\qquad = \int_0^{\widetilde{\ell}^2_{\varepsilon,+}(\widetilde{x}',\widetilde{\xi}')} \int_0^\pi J(\theta)\sin(\theta)\,d\theta\,d\tau + \|J\|_{L^\infty}\mathcal{O}(\varepsilon^\infty) \\
&\qquad = \widetilde{\ell}^2_{\varepsilon,+}(\widetilde{x}',\widetilde{\xi}')\int_0^\pi J(\theta)\sin(\theta)\,d\theta + \|J\|_{L^\infty}\mathcal{O}(\varepsilon^\infty),
\end{aligned}$$

where the fourth equality follows from Lemma 4.8. The constant T on the second line is unknown and appears in the choice of parametrization of the geodesic segment $\alpha_1(\widetilde{x},\widetilde{y})$ but does not influence the computation. The point $(\widetilde{x}',\widetilde{\xi}') = \widetilde{\psi}_\varepsilon(\widetilde{x},\widetilde{\xi})$ is the image of $(\widetilde{x},\widetilde{\xi})$ by the diffeomorphism $\widetilde{\psi}_\varepsilon$ defined in Section 4.3. We recall that this diffeomorphism is invariant by the fundamental group and descends on the base as ψ_ε.

Inserting this into the previous integrals, we obtain:

$$\int_{S^*M_\varepsilon^1} \int_0^\pi J(f(x,\xi,\theta)) \sin(\theta)\, d\theta\, d\mu_1(x,\xi)$$
$$= \int_0^\pi J(\theta)\sin(\theta) d\theta \int_{\partial M_\varepsilon} \int_{\omega_\varepsilon}^{\pi-\omega_\varepsilon} \ell_{\varepsilon,+}^2(\psi_\varepsilon(x,\xi(\omega))) \sin(\omega)\, d\omega\, dx$$
$$+ \|J\|_{L^\infty} \mathcal{O}(\varepsilon^N).$$

According to Lemma 4.4, we know that $\psi_\varepsilon = \mathrm{Id} + \mathcal{O}(\varepsilon^\infty)$ in the C^1 topology. In particular, the Jacobian of ψ_ε is $1 + \mathcal{O}(\varepsilon^\infty)$ and by a change of variable:

$$\int_{\partial M_\varepsilon} \int_{\omega_\varepsilon}^{\pi-\omega_\varepsilon} \ell_{\varepsilon,+}^2(\psi_\varepsilon(x,\xi(\omega))) \sin(\omega)\, d\omega\, dx$$
$$= \int_{\partial M_\varepsilon} \int_0^\pi l_\varepsilon^2(x',\xi') \sin(\omega')\, d\omega'\, dx' + \mathcal{O}(\varepsilon^N)$$
$$= \mathrm{vol}_{g_2}(S^*M_\varepsilon^2) + \mathcal{O}(\varepsilon^N)$$
$$= \mathrm{vol}_{g_1}(S^*M_\varepsilon^1) + \mathcal{O}(\varepsilon^N),$$

where the two volumes agree to order $\mathcal{O}(\varepsilon^N)$ according to the same computation with $J \equiv 1$. Inserting this into the previous integrals, we obtain the sought result. □

Remark that we can actually consider in Lemma 4.7 a family of functions J_ε, instead of a single function. We can assume that $\|J_\varepsilon\|_{L^\infty} = \mathcal{O}(1/\varepsilon^\alpha)$, for some $\alpha > 0$ which we may take as large as we want. Then, we can always apply the lemma with $N' := N + \lfloor \alpha \rfloor + 1$, so that in the end, the sup norm $\|J_\varepsilon\|_{L^\infty}$ is swallowed in the term $\mathcal{O}(\varepsilon^N)$. We actually obtain for free a better version:

Lemma 4.9. *Let $N \in \mathbb{N} \setminus \{0\}$ be an integer and $\alpha > 0$. Let $J_\varepsilon \colon [0,\pi] \to \mathbb{R}$ be a family of convex continuous function such that $\|J_\varepsilon\|_{L^\infty} = \mathcal{O}(\varepsilon^{-\alpha})$. Then:*

$$\int_0^\pi J_\varepsilon(\Theta_\varepsilon(\theta)) \sin(\theta)\, d\theta \le \int_0^\pi J_\varepsilon(\theta)\sin(\theta)\, d\theta + \mathcal{O}(\varepsilon^N), \tag{4.11}$$

where the remainder only depends on ε.

5. Estimating the average angle of deviation

As mentioned previously, we are unable to prove a priori that the Θ_ε are uniformly Lipschitz. Nevertheless, we can show that they decompose as a sum $\Theta_\varepsilon^{(a)} + \Theta_\varepsilon^{(b)}$ where the $\Theta_\varepsilon^{(a)}$ are Lipschitz (and their Lipschitz constant is controlled) and the $\Theta_\varepsilon^{(b)}$ have a "small" C^0 norm. This will be sufficient to apply our version of Otal's estimate (see Proposition 5.4).

Note that we will sometimes drop the notation C for the different constants which may appear at each line of our estimates and rather use the symbol $\lesssim$. By $\|A\| \lesssim \|B\|$, we mean that there exists a constant $C > 0$, which is independent of the elements A and B considered and such that, $\|A\| \leq C\|B\|$.

5.1. Derivative of the angle of deviation. The purpose of this paragraph is to estimate the derivative (with respect to θ) of the angle of deviation f. We recall that

$$W_1 = \{(x,\xi,\theta) \in S^*M_1 \times (0,\pi) \mid (x,\xi),(x,R_\theta\xi) \notin (\Gamma_-^1 \cup \Gamma_+^1)\}.$$

Lemma 5.1. *There exist constants $C, k > 0$ (independent of ε) such that for all $(x,\xi,\theta) \in S^*M_\varepsilon^1 \cap W_1$:*

$$\left|\frac{\partial f}{\partial \theta}(x,\xi,\theta)\right| \leq C \exp\big(k\big(\ell_{\varepsilon,+}^1(x,R_\theta\xi) + |\ell_{\varepsilon,-}^1(x,R_\theta\xi)|\big)\big).$$

Proof. We can write the derivative of f as:

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial y'}\Big(\frac{\partial y'}{\partial \theta}\Big) + \frac{\partial f}{\partial y}\Big(\frac{\partial y}{\partial \theta}\Big), \tag{5.1}$$

where y and y' are defined in Section 4.2 and study the different terms separately.

The idea is to study the behaviour (and more precisely the growth) of Jacobi vector fields in a neighborhood of the boundary. Given a geodesic which enters the set $\{\rho \geq \varepsilon\}$, we will use the bounds (2.13) to estimate the Jacobi vector fields on the segment contained in $\{\rho \geq \varepsilon\}$. Then, by convexity, the geodesic exits $\{\rho \geq \varepsilon\}$ with a coordinate $\bar{\xi}_0 \leq 0$. On the set

$$\mathcal{C} = \{\rho < \delta\} \cap \{\bar{\xi}_0 \leq 0\}$$

(for some $\delta > 0$ small enough), we can study the behaviour of the geodesics more explicitly. Namely, given any point $(x,\xi) \in S^*M$ in $\mathcal{C}$, we know that it converges uniformly exponentially fast to the boundary in the sense that there exists $C > 0$ (uniform in (x,ξ)) such that if $\rho(t) := \rho(\varphi_t(x,\xi))$, then one has

$$\rho(0)e^{-t} \leq \rho(t) \leq C\rho(0)e^{-t}$$

for $t \geq 0$ (see [8, Lemma 2.3]). From the expression of the metric (1.1) in local coordinates, one can check that the curvature is given by $\kappa = -1 + \rho \cdot \mathcal{O}(1)$. As a consequence, if $\kappa(t) = \kappa(\pi(\varphi_t(x,\xi)))$ and $\delta > 0$ is chosen small enough at the beginning, one has that

$$-1 - \frac{1}{10}e^{-t} \leq \kappa(t) \leq -1 + \frac{1}{10}e^{-t},$$

for any such (x, ξ). If $t \mapsto \gamma(t)$ denotes the geodesic generated by this point and J is a normal Jacobi vector field along γ, we write

$$J(t) = j(t) R_{\pi/2}\dot{\gamma}(t),$$

where j satisfies the Jacobi equation $\ddot{j}(t) + \kappa(t) j(t) = 0$. Assume $j(0) = 0$, $\dot{j}(0) = 1$, then $j(t) > 0$ (there are no conjugate points) and thus

$$\ddot{j}(t) \leq \Big(1 + \frac{1}{10}e^{-t}\Big) j(t).$$

By a comparison argument, $j(t) \leq z(t)$ where z is the solution to $\ddot{z}(t) - (1 + \frac{1}{10}e^{-t})z(t) = 0$ with $z(0) = j(0), \dot{z}(0) = \dot{j}(0)$.

But making the change of variable $u = 2\sqrt{10}e^{-t/2}$, $\tilde{z}(u) = z(t)$, one can prove that $\tilde{z}$ solves the modified Bessel equation of parameter 2 that is

$$u^2 \frac{d^2\tilde{z}}{du^2} + u\frac{d\tilde{z}}{du} - (u^2 + 2^2)\tilde{z} = 0$$

and thus $\tilde{z}(u) = A \cdot I_2(u) + B \cdot K_2(u)$ for some parameters $A, B \in \mathbb{R}$ depending on $\tilde{z}(0)$, $\dot{\tilde{z}}(0)$, I_2 and K_2 being the modified Bessel functions of first and second kind. Thus:

$$z(t) = A \cdot I_2\big(2\sqrt{10}e^{-t/2}\big) + B \cdot K_2\big(2\sqrt{10}e^{-t/2}\big),$$

where

$$I_2\big(2\sqrt{10}e^{-t/2}\big) \sim_{t\to+\infty} Ce^{-t}, \quad K_2\big(2\sqrt{10}e^{-t/2}\big) \sim_{t\to+\infty} Ce^{t}$$

(see [1, 9.6.7–9.6.9]). For instance, if $j(0) = 0, \dot{j}(0) = 1$, which corresponds to a vertical variation of geodesics, then we obtain

$$|d\pi \circ d\varphi_t(V)| = |J(t)| \leq Ce^t$$

for some constant $C > 0$ independent of the point. Using this technique of comparison and decomposing any vector by its vertical and horizontal components, one obtains that

$$\|d\varphi_t(x, \xi)\| \leq Ce^t$$

for $(x, \xi) \in \mathcal{C}$, where the constant $C > 0$ is uniform in (x, ξ).

We fix (x_0, ξ_0, θ_0) and look at the variation $\theta \mapsto (x_0, R_{\theta_0+\theta}\xi_0)$. For each θ, we thus have a g_1-geodesic $t \mapsto \gamma_\theta(t)$ generated by this point and it hits the boundary in the future at $y'(\theta)$. We set $\gamma := \gamma_0$. We denote by $J(t) := \partial_\theta \gamma_\theta(t)$ the Jacobi vector field along γ. Writing in short $\ell^1_{+,\varepsilon} = \ell^1_{+,\varepsilon}(x_0, R_{\theta_0}\xi_0)$, $V = V(x_0, R_{\theta_0}\xi_0)$, we have for $t = s + l_\varepsilon, s \geq 0$:

$$|J(t)|_{g_1} = |d\pi \circ d\varphi_{s+\ell^1_{+,\varepsilon}}(V)| \leq Ce^s |d\pi \circ d\varphi_{l_\varepsilon}(V)| \leq Ce^s e^{k\ell^1_{+,\varepsilon}}.$$

The first inequality follows from our previous remarks whereas the second one is a consequence of (2.13). Now, we know that

$$\rho(\ell^1_{+,\varepsilon})e^{-s} = \varepsilon e^{-s} \leq \rho(t) \leq C\varepsilon e^{-s} = C\rho(\ell^1_{+,\varepsilon})e^{-s}.$$

As a consequence, for t large enough, we have:

$$|J(t)|_{\bar{g}_1} = \rho(t)|J(t)|_{g_1} \leq C \cdot \varepsilon e^{k\ell^1_{+,\varepsilon}}.$$

By making $t \to +\infty$, we obtain that $|\frac{\partial y'}{\partial \theta}|_h \leq C \cdot \varepsilon e^{k\ell^1_{+,\varepsilon}}$.

Conversely, we consider a family of points $y'(u)$ in a neighborhood of y'_0 on the boundary (such that $|\frac{\partial y'}{\partial u}|_h = 1$) and we look at the g_2-geodesics joining y to $y'(u)$. They intersect the g_2-geodesic joining z to z' (the endpoints of the geodesic genreated by (x, ξ)) at some point $\mathrm{x}(u)$, and we obtain $(\mathrm{x}(u), \Xi(u))$ and an angle $f(u)$. From another perspective, we have a family of points $(\mathrm{x}(u), R_{f(u)}\Xi(u))$ which generate geodesics joining $y'(u)$ (in the future) to y (in the past). Like before, we denote by γ the geodesic obtained for $u = 0$ and by J the Jacobi vector field along γ. Since the point y joined in the past by the geodesic is fixed (it does not depend on u), J (more precisely, its lift in TS^*M) lies in the unstable bundle. We write

$$\partial_u(\mathrm{x}(u), R_{f(u)}\Xi(u)) = d\pi^{-1}(J(0)) + \mathcal{K}^{-1}(\nabla_t J(0)) = \lambda \cdot \xi_u,$$

where ξ_u is one of the two unit vectors (with respect to the g_2-Sasaki metric) generating E_u. Note that the vertical component of this vector is precisely $\frac{\partial f}{\partial u} V$ and thus $|\lambda| \geq |\frac{\partial f}{\partial u}|$. We write $\ell^2_{+,\varepsilon} = \ell^2_{+,\varepsilon}(\mathrm{x}, R_f\Xi)$. For $t = s + \ell^2_{+,\varepsilon}, s \geq 0$:

$$\begin{aligned} |J(t)|_{g_2} &= |d\pi \circ d\varphi_t(\lambda\xi_u)| \\ &= |\lambda| \cdot |d\pi \circ d\varphi_s(d\varphi_{\ell^2_{+,\varepsilon}}(\xi_u))| \\ &\geq |\lambda| \cdot e^s |d\varphi_{\ell^2_{+,\varepsilon}}(\xi_u)| \\ &\geq C|\lambda| e^s e^{k\ell^2_{+,\varepsilon}} \geq C\left|\frac{\partial f}{\partial u}\right| e^s e^{k\ell^2_{+,\varepsilon}}. \end{aligned}$$

The term in $e^{k\ell^2_{+,\varepsilon}}$ follows from (2.13) whereas the term e^s is a consequence on the bounds of the curvature. More precisely, for fixed bounds, that is $-k_0^2 \leq \kappa \leq -k_1^2$, such a lower bound is obtained in [13, Theorem 3.2.17], and the same proof applies here, except that we have bounds $-1 - \frac{1}{10}e^{-t} \leq \kappa(t) \leq -1 + \frac{1}{10}e^{-t}$. But the argument of Klingenberg is based on Gronwall lemma and $t \mapsto e^{-t}$ is integrable, so we get the same result in the end. Multiplying by $\rho(t)$ and taking the limit as $t \to +\infty$, we eventually obtain that $|\frac{\partial y'}{\partial u}|_h = 1 \geq C\varepsilon e^{k\ell^2_{+,\varepsilon}}|\frac{\partial f}{\partial u}|$.

Putting the previous bounds together, and using (5.1), we obtain the sought result. □

5.2. Derivative of the exit time. We set $T_\varepsilon = -N_0 \log \varepsilon$ for some integer N_0, like in the proof of Lemma 2.3.

Lemma 5.2. *There exist constants $C, k > 0$ (independent of ε) such that for all $(x, \xi, \theta) \in S^*M^1_\varepsilon \cap W_1$ such that*

$$T_\varepsilon \leq \ell^1_{\varepsilon,+}(x, R_\theta \xi) + |\ell^1_{\varepsilon,-}(x, R_\theta\xi)|,$$

one has:

$$\partial_\theta\big(\ell^1_{\varepsilon,+}(x, R_\theta\xi) + |\ell^1_{\varepsilon,-}(x, R_\theta\xi)|\big) \leq C \exp\big(k\big(\ell^1_{\varepsilon,+}(x, R_\theta\xi) + |\ell^1_{\varepsilon,-}(x, R_\theta\xi)|\big)\big).$$

Proof. Let us deal with the case of the exit time in the future, the other case being similar. The exit time is defined by the implicit equation:

$$\rho\big(\varphi^1_{\ell^1_{\varepsilon,+}(x,R_\theta\xi)}(x, R_\theta\xi)\big) = \varepsilon.$$

Differentiating with respect to θ, we obtain:

$$\begin{aligned}\partial_\theta\big(\ell^1_{\varepsilon,+}(x, R_\theta\xi)\big) d\rho\big(X_1\big(\varphi^1_{\ell^1_{\varepsilon,+}(x,R_\theta\xi)}(x, R_\theta\xi)\big)\big)& \\ + d\rho\big(d\big(\varphi^1_{\ell^1_{\varepsilon,+}(x,R_\theta\xi)}\big)_{(x,R_\theta\xi)} V(x, R_\theta\xi)\big) &= 0,\end{aligned}$$

where $V(x, \xi) \in \mathbb{V}$ is the vertical vector in (x, ξ) (it is unitary with respect to the Sasaki metric G_1). But:

$$\big|d\rho\big(X_1\big(\varphi^1_{\ell^1_{\varepsilon,+}(x,R_\theta\xi)}(x, R_\theta\xi)\big)\big)\big| = \varepsilon|d\rho(\bar{X}_1)|,$$

and $d\rho(\bar{X}_1)$ is the sine of the angle with which the geodesic exits the region $\{\rho \geq \varepsilon\}$. If this angle is less than $\frac{1}{10}$ (any small constant works as long as the geodesics concerned stay in a region where the metric still has the usual expression (1.1)), then the geodesic will spend at most a bounded (independently of ε) amount of time in the region $\{\rho \geq \varepsilon\}$, thus contradicting the condition:

$$T_\varepsilon = -N_0 \log(\varepsilon) \leq \ell^1_{\varepsilon,+}(x, R_\theta\xi) + |\ell^1_{\varepsilon,-}(x, R_\theta\xi)|.$$

This can be proved using the Hamilton's equations, similarly to the proof of Lemma 4.3 for instance. Thus $|d\rho(\bar{X}_1)| \geq \frac{1}{10}$.

As to the second term, using the fact that $d\rho/\rho$ is unitary (with respect to the dual metric of g_1 on the cotangent space), we obtain that:

$$\begin{aligned}\Big|\rho\frac{d\rho}{\rho}\big(d\big(\varphi^1_{\ell^1_{\varepsilon,+}(x,R_\theta\xi)}\big)_{(x,R_\theta\xi)} V(x, R_\theta\xi)\big)\Big| &\leq \varepsilon\big|d\big(\varphi^1_{\ell^1_{\varepsilon,+}(x,R_\theta\xi)}\big)_{(x,R_\theta\xi)} V(x, R_\theta\xi)\big|_{G_1} \\ &\leq \varepsilon e^{k\ell^1_{\varepsilon,+}(x,R_\theta\xi)},\end{aligned}$$

for some constant k, following (2.13). This provides the sought result. □

5.3. An inequality on the average angle of deviation. We know that f is almost everywhere continuous and bounded, so Θ_ε is continuous by Lebesgue theorem. We now prove that the homeomorphism Θ_ε satisfies the following estimate:

Lemma 5.3. *For any $\delta \in (Q, 0)$ (defined in Lemma 2.3), for all $\beta > 0$ small enough, there exists $\beta' > 0$ (depending on β and converging towards 0 as $\beta \to 0$) such that:*

$$\forall \theta_1, \theta_2 \in [0, \pi], \quad |\Theta_\varepsilon(\theta_1) - \Theta_\varepsilon(\theta_2)| \lesssim \varepsilon^{-\beta'} |\theta_1 - \theta_2|^\beta + \varepsilon^\delta.$$

Proof. First, remark that it is sufficient to prove the lemma for $\theta_1, \theta_2 \in [0, \pi/2]$, since the result will follow from the π-symmetry of the homeomorphism Θ_ε. We fix $\varepsilon > 0$. We introduce the smooth cutoff function χ_T (for some $T > 0$ which will be chosen to depend on ε later) such that $\chi_T(s) \equiv 1$ on $[0, T]$ and $\chi_T(s) \equiv 0$ on $[2T, +\infty)$. Note that we can always construct such a χ_T so that $\|\partial_s \chi_T\|_{L^\infty} \leq 1$ (as long as $T > 1$, which we can assume since it will be chosen growing to infinity as $\varepsilon \to 0$). We write $\Theta_\varepsilon = \Theta_\varepsilon^{(a),T} + \Theta_\varepsilon^{(b),T}$, where:

$$\begin{aligned}\Theta_\varepsilon^{(a),T}(\theta) &:= \frac{1}{\mathrm{vol}_{g_1}(S^*M_\varepsilon^1)} \int_{S^*M_\varepsilon^1} \chi_T\big(\ell_{\varepsilon,+}^1(x, R_\theta\xi) + |\ell_{\varepsilon,-}^1(x, R_\theta\xi)|\big) \\ &\qquad\qquad\qquad\qquad \cdot f(x,\xi,\theta)\, d\mu_1(x,\xi) \\ &= \frac{1}{\mathrm{vol}_{g_1}(S^*M_\varepsilon^1)} \int_{S^*M_\varepsilon^1} \psi_T(x,\xi,\theta),\end{aligned}$$

where ψ_T is defined to be the integrand and

$$\Theta_\varepsilon^{(b),T}(\theta) := \Theta_\varepsilon - \Theta_\varepsilon^{(a),T}.$$

Morally, the cutoff function means that we integrate over the compact region

$$\{\ell_{\varepsilon,+}^1(x, R_\theta\xi) + |\ell_{\varepsilon,-}^1(x, R_\theta\xi)| \leq T\}.$$

By the Lebesgue theorem, $\Theta_\varepsilon^{(a),T}$ is C^1 on $[0, \pi/2]$. For $\beta > 0$, $\theta_1, \theta_2 \in [0, \pi/2]$, one has:

$$|\Theta_\varepsilon^{(a),T}(\theta_1) - \Theta_\varepsilon^{(a),T}(\theta_2)| \lesssim \sup_{\theta \in [0,\pi/2]} |\partial_\theta \Theta_\varepsilon^{(a),T}(\theta)|^\beta |\theta_1 - \theta_2|^\beta.$$

Let us estimate the former derivative. We have:

$$\partial_\theta \Theta_\varepsilon^{(a),T}(\theta) = \frac{1}{\mathrm{vol}_{g_1}(S^*M_\varepsilon^1)} \int_{S^*M_\varepsilon^1} \partial_\theta \psi_T(x,\xi,\theta)\, d\mu_1(x,\xi),$$

and the derivative under the integral is composed of a sum of two terms which we now estimate separately.

(1) By Lemma 5.1, the first term is bounded by:

$$\begin{aligned}&\big|\chi_T\big(\ell_{\varepsilon,+}^1(x, R_\theta\xi) + |\ell_{\varepsilon,-}^1(x, R_\theta\xi)|\big)\partial_\theta f(x,\xi,\theta)\big| \\ &\qquad\qquad \lesssim \exp\big(k\big(\ell_{\varepsilon,+}^1(x, R_\theta\xi) + |\ell_{\varepsilon,-}^1(x, R_\theta\xi)|\big)\big) \lesssim e^{2kT}.\end{aligned}$$

(2) And the second term is bounded by Lemma 5.2:

$$\big|\partial_\theta\big(\ell^1_{\varepsilon,+}(x,R_\theta\xi)+|\ell^1_{\varepsilon,-}(x,R_\theta\xi)\big) \\ \cdot\partial_s\chi_T\big(\ell^1_{\varepsilon,+}(x,R_\theta\xi)+|\ell^1_{\varepsilon,-}(x,R_\theta\xi)|\big)f(x,\xi,\theta)\big|\lesssim e^{2kT}.$$

Note that the constant $k>0$ may be different from one line to another. Gathering everything, we obtain that for all $\theta\in[0,\pi/2]$, $|\partial_\theta\Theta^{(a),T}_\varepsilon(\theta)|\lesssim e^{2kT}$ and thus:

$$|\Theta^{(a),T}_\varepsilon(\theta_1)-\Theta^{(a),T}_\varepsilon(\theta_2)|\lesssim e^{2k\beta T}|\theta_1-\theta_2|^\beta.$$

As to $\Theta^{(b),T}_\varepsilon$, we can write:

$$\Theta^{(b),T}_\varepsilon(\theta)\le\frac{1}{\mathrm{vol}_{g_1}(S^*M^1_\varepsilon)}\Big(\int_{S^*M^1_\varepsilon\cap\{\ell^{1,+}_\varepsilon(x,R_\theta\xi)>T\}}f\,d\mu_1 \\ +\int_{S^*M^1_\varepsilon\cap\{|\ell^{1,-}_\varepsilon(x,R_\theta\xi)|>T\}}f\,d\mu_1\Big).$$

If $T\ge -N_0\log(\varepsilon)$ (N_0 is a large integer defined in Lemma 2.3, independent of ε), then the two integrals can be estimated by Lemma 2.3 (note that we here divide by the volume which is bounded by $\mathcal{O}(\varepsilon)$). We obtain:

$$|\Theta^{(b),T}_\varepsilon(\theta)|\lesssim e^{-\delta T}\varepsilon^{-4\delta}.$$

We choose $T:=T_\varepsilon=-N_0\log(\varepsilon)$ and set $\Theta^{(a)}_\varepsilon:=\Theta^{(a),T_\varepsilon}_\varepsilon$, $\Theta^{(b)}_\varepsilon:=\Theta^{(b),T_\varepsilon}_\varepsilon$. Since N_0 is taken large enough (greater than 5 at least to swallow the $\varepsilon^{-4\delta}$), we obtain $\|\Theta^{(b)}_\varepsilon\|_{L^\infty}\lesssim\varepsilon^\delta$. And :

$$|\Theta^{(a),T}_\varepsilon(\theta_1)-\Theta^{(a),T}_\varepsilon(\theta_2)|\le\varepsilon^{-2\beta kN_0}|\theta_1-\theta_2|^\beta,$$

which provides the sought result by going back to Θ_ε. □

5.4. Otal's lemma revisited. In the spirit of Otal's lemma (see [17, Lemma 8]), we prove:

Proposition 5.4. *Assume $\Theta_\varepsilon\colon[0,\pi]\to[0,\pi]$ is a family of increasing homeomorphisms for $\varepsilon\in(0,\delta)$ such that:*

(1) $\Theta_\varepsilon(0)=0,\Theta_\varepsilon(\pi)=\pi$;

(2) *For all $\theta\in[0,\pi]$, $\Theta_\varepsilon(\pi-\theta)=\pi-\Theta_\varepsilon(\theta)$;*

(3) *For all $\theta_1,\theta_2\in[0,\pi]$ such that $\theta_1+\theta_2\in[0,\pi]$,*

$$\Theta_\varepsilon(\theta_1)+\Theta_\varepsilon(\theta_2)\le\Theta_\varepsilon(\theta_1+\theta_2);$$

(4) *There exists constants* $C, \beta, \beta' > 0$ *and* $\delta > 0$ *(independent of* ε*), such that for all* $\theta_1, \theta_2 \in [0, \pi]$,

$$|\Theta_\varepsilon(\theta_1) - \Theta_\varepsilon(\theta_2)| \leq C\left(\varepsilon^\delta + \varepsilon^{-\beta'}|\theta_1 - \theta_2|^\beta\right);$$

(5) *There exists* $\alpha > 2\beta'/\beta - 1$ *such that for all family of continuous convex functions* $J_\varepsilon\colon [0,\pi] \to \mathbb{R}$ *such that* $\|J_\varepsilon\|_{L^\infty} = \mathcal{O}(1/\varepsilon^\alpha)$,

$$\int_0^\pi J_\varepsilon(\Theta_\varepsilon(\theta)) \sin(\theta)\, d\theta \leq \int_0^\pi J_\varepsilon(\theta) \sin(\theta)\, d\theta + \mathcal{O}(\varepsilon).$$

Then $\Theta_\varepsilon = \operatorname{Id} + \mathcal{O}(\varepsilon^\gamma)$*, where we can take any* γ *up to the critical exponent*

$$\widehat{\gamma} := \frac{1 + \alpha - 2\beta'/\beta}{1 + 2/\beta},$$

as long as $\gamma < \delta$.

Proof. We argue by contradiction. Assume there there exists a sequence $\varepsilon_n \to 0$ such that $\|\Theta_n - \operatorname{Id}\|_{L^\infty} > n\varepsilon_n^\gamma$ (where $\Theta_n := \Theta_{\varepsilon_n}$). By π-symmetry, there exists an interval $[a_n, A_n]$ such that for all $\theta \in (a_n, A_n)$,

$$\Theta_n(\theta) < \theta - n\varepsilon_n^\gamma$$

and we can choose

$$\Theta_n(a_n) = a_n - n\varepsilon_n^\gamma, \quad \Theta_n(A_n) = A_n - n\varepsilon_n^\gamma.$$

We also construct the largest interval $[b_n, B_n] \supset [a_n, A_n]$ such that for all $\theta \in (b_n, B_n)$,

$$\Theta_n(\theta) < \theta - \varepsilon_n^\gamma,$$

and

$$\Theta_n(b_n) = b_n - \varepsilon_n^\gamma, \quad \Theta_n(B_n) = B_n - \varepsilon_n^\gamma.$$

Eventually, we define the largest interval $[c_n, C_n] \supset [b_n, B_n]$ such that for all $\theta \in (c_n, C_n)$,

$$\Theta_n(\theta) < \theta,$$

and

$$\Theta_n(c_n) = c_n, \quad \Theta_n(C_n) = C_n.$$

The π-symmetry implies that $\Theta(\pi/2) = \pi/2$ and since $\Theta(0) = 0, \Theta(\pi) = \pi$, we know that the points $c_n < b_n < a_n < A_n < B_n < C_n$ all lie either in $[0, \pi/2]$ or in $[\pi/2, \pi]$.

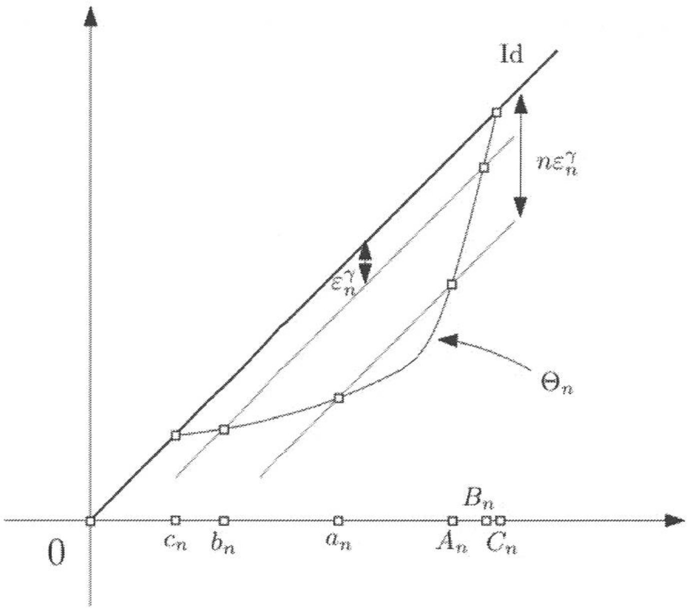

Figure 5. The points $c_n < b_n < a_n < A_n < B_n < C_n$.

Remark that $\Theta_n - \mathrm{Id}$ also satisfies the fifth item, namely:

$$
\begin{aligned}
|(\Theta_n - \mathrm{Id})(\theta_1) - (\Theta_n - \mathrm{Id})(\theta_2)| &\lesssim |\Theta_n(\theta_1) - \Theta_n(\theta_2)| + |\theta_1 - \theta_2| \\
&\lesssim \Big(\varepsilon_n^\delta + \frac{1}{\varepsilon_n^{\beta'}}|\theta_1 - \theta_2|^\beta\Big) + (2\pi)^{1-\beta}|\theta_1 - \theta_2|^\beta \\
&\lesssim \varepsilon_n^\delta + \frac{1}{\varepsilon_n^{\beta'}}|\theta_1 - \theta_2|^\beta .
\end{aligned}
$$

This implies that:

$$
|(\Theta_n - \mathrm{Id})(a_n) - (\Theta_n - \mathrm{Id})(b_n)| = (n-1)\varepsilon_n^\gamma \lesssim \varepsilon_n^\delta + \frac{1}{\varepsilon_n^{\beta'}}(a_n - b_n)^\beta .
$$

Thus:

$$
(a_n - b_n)^\beta \gtrsim (n-1)\varepsilon_n^{\gamma+\beta'} - \varepsilon_n^{\delta+\beta'} \gtrsim (n-1)\varepsilon_n^{\gamma+\beta'},
$$

for n large enough since $\delta > \gamma$. The same inequalities hold for the other points and we get, for n large enough:

$$
\begin{aligned}
a_n - b_n &\gtrsim (n-1)^{1/\beta}\varepsilon_n^{(\gamma+\beta')/\beta}, & B_n - A_n &\gtrsim (n-1)^{1/\beta}\varepsilon_n^{(\gamma+\beta')/\beta}, \\
b_n - c_n &\gtrsim \varepsilon_n^{(\gamma+\beta')/\beta}, & C_n - B_n &\gtrsim \varepsilon_n^{(\gamma+\beta')/\beta}.
\end{aligned}
$$

Now, for $h \in (0, C_n - c_n)$, by superadditivity:

$$
c_n + h > \Theta_n(c_n + h) \geq \Theta_n(c_n) + \Theta_n(h) = c_n + \Theta_n(h),
$$

that is $\Theta_n(h) < h$. In the same fashion, we have for $h \in (b_n - c_n, B_n - c_n)$, $\Theta_n(h) < h - \varepsilon_n^\gamma$.

Let us now consider the continuous convex functions

$$J_n(x) := \varepsilon_n^{-\alpha} \sup(C_n - c_n - x, 0) = \varepsilon_n^{-\alpha} \widetilde{J}_n(x)$$

on $[0, \pi]$. Using:

$$\int_0^{\pi} \widetilde{J}_n(\Theta_n(\theta)) \sin(\theta)\, d\theta \leq \int_0^{\pi} \widetilde{J}_n(\theta) \sin(\theta)\, d\theta + C\varepsilon_n^{1+\alpha},$$

where $C > 0$ is a constant independent of n, we obtain:

$$\begin{aligned} 0 &\leq \int_0^{C_n - c_n} (\Theta_n(\theta) - \theta) \sin(\theta)\, d\theta + C\varepsilon_n^{1+\alpha} \\ &= \int_0^{b_n - c_n} (\Theta(\theta) - \theta) \sin(\theta)\, d\theta + \int_{b_n - c_n}^{B_n - c_n} " + \int_{B_n - c_n}^{C_n - c_n} " + C\varepsilon_n^{1+\alpha} \\ &< C\varepsilon_n^{1+\alpha} - \varepsilon_n^{\gamma} \int_{b_n - c_n}^{B_n - c_n} \sin(\theta)\, d\theta, \end{aligned}$$

where we used the bounds stated above and the fact that both $b_n - c_n$ and $B_n - c_n$ are in $[0, \pi/2]$. But remark that:

$$\begin{aligned} \int_{b_n - c_n}^{B_n - c_n} \sin(\theta)\, d\theta &\geq ((B_n - c_n) - (b_n - c_n)) \sin(b_n - c_n) \\ &\geq C'(n-1)^{1/\beta} \varepsilon_n^{2(\gamma + \beta')/\beta}, \end{aligned}$$

for some constant $C' > 0$, by inserting the previous bounds and using the inequality $\sin(x) \geq 2x/\pi$ on $[0, \pi/2]$. Thus, we obtain:

$$0 < \varepsilon_n^{1+\alpha} \big(C - C'(n-1)^{1/\beta} \varepsilon_n^{(2/\beta + 1)\gamma + 2\beta'/\beta - 1 - \alpha}\big),$$

and $(2/\beta + 1)\gamma + 2\beta'/\beta - 1 - \alpha \leq 0$ by the definition of γ, so the right-hand side is negative as n goes to infinity. □

Remark 5.5. Let us mention that the result is still valid in the limit $\delta = +\infty$, $\beta = 1$, $\beta' = 0$ (the Θ_ε are uniformly Lipschitz) and $\alpha = 0$. It provides an exponent $\gamma = 1/3$. Had we been able to prove a priori that the family Θ_ε was uniformly Lipschitz, this would have been enough to conclude.

6. End of the proof

We can now conclude the proof.

Proof. Combining Lemmas 4.9, 5.3 and Proposition 5.4, we conclude that $\Theta_\varepsilon = \mathrm{Id} + \mathcal{O}(\varepsilon^N)$, for some N which we can choose large enough. Thus for $\theta_1, \theta_2 \in [0, \pi]$ such that $\theta_1 + \theta_2 \in [0, \pi]$:

$$\begin{aligned} 0 &\leq \frac{1}{\mathrm{vol}(S^*\widetilde{M}^1_\varepsilon)} \int_{S^*\widetilde{M}^1_\varepsilon} f(x,\xi,\theta_1+\theta_2) - f(x,\xi,\theta_1) - f(x, R_{\theta_1}\xi, \theta_2)\, d\mu_1(x,\xi) \\ &= \Theta_\varepsilon(\theta_1+\theta_2) - \Theta_\varepsilon(\theta_1) - \Theta_\varepsilon(\theta_2) \\ &= \mathcal{O}(\varepsilon^N). \end{aligned}$$

Since the integrand is positive and the inverse of the volume can be estimated by $\mathcal{O}(\varepsilon)$, this implies by taking $\varepsilon \to 0$ that

$$f(x,\xi,\theta_1+\theta_2) - f(x,\xi,\theta_1) - f(x, R_{\theta_1}\xi, \theta_2) = 0,$$

so the inequality is saturated in Gauss–Bonnet formula. As a consequence, three intersecting g_1-geodesics correspond to three intersecting g_2-geodesics with same endpoints.

We can now construct the isometry Φ between (M, g_1) and (M, g_2). We will use in this paragraph the notation $\tilde{\cdot}$ to refer to objects considered on the universal cover $\widetilde{M}$. Given $p \in \widetilde{M}$, we choose three g_1-geodesics α, β and γ passing through p with respective endpoints (x, x'), (y, y') and (z, z') in $\partial\widetilde{M} \times \partial\widetilde{M}$. By the previous section, we know that the g_2-geodesics with same endpoints meet in a single point which we define to be $\widetilde{\Phi}(p)$. Now, $\widetilde{\Phi}(p)$ is well-defined (it does not depend on the choice of the geodesics) and remark that for $(x,\xi) \notin \widetilde{\Gamma}_- \cup \widetilde{\Gamma}_+$ (such a covector always exists) and θ such that $(x, R_\theta\xi) \notin \widetilde{\Gamma}_- \cup \widetilde{\Gamma}_+$, we have $\widetilde{\Phi}(p) = \mathrm{x}(x,\xi,\theta)$, where x is defined in (4.1) (in other words, κ maps fibers to fibers). Thus $\widetilde{\Phi}$ is C^∞ in the interior (see Section 4.2) and extends continuously down to the boundary as $\widetilde{\Phi}|_{\partial\widetilde{M}} = \mathrm{Id}$.

Moreover, $\widetilde{\Phi}^*(\widetilde{g}_2) = \widetilde{g}_1$. Indeed, it is sufficient to prove that $\widetilde{\Phi}$ preserves the distance. Given $p, q \in \widetilde{M}$, we have

$$\widetilde{\mathcal{F}}_1(p,q) = \widetilde{\mathcal{F}}_2(\widetilde{\Phi}(p), \widetilde{\Phi}(q))$$

and thus:

$$d_{\widetilde{g}_1}(p,q) = \frac{1}{2}\eta_{\widetilde{g}_1}(\widetilde{\mathcal{F}}_1(p,q)) = \frac{1}{2}\eta_{\widetilde{g}_2}(\widetilde{\mathcal{F}}_2(\widetilde{\Phi}(p), \widetilde{\Phi}(q))) = d_{\widetilde{g}_2}(\widetilde{\Phi}(p), \widetilde{\Phi}(q)).$$

Now, observe that $\widetilde{\Phi}$ is invariant by the action of the fundamental group: it thus descends to a smooth diffeomorphism $\Phi\colon M \to M$ which extends continuously down to the boundary and satisfies $\Phi^* g_2 = g_1$.

We now conclude the argument by proving that Φ is actually smooth on $\bar{M}$. In the compact setting, it is a classical fact that an isometry which is at least differentiable is actually smooth and our argument somehow follows the idea of proof of this statement.

More precisely, we show that a smooth isometry on an asymptotically hyperbolic manifold actually extends as a smooth application on the compactification $\overline{M}$. The proof does not rely on the dimension two. Note that another proof could be given in this case using the fact that Φ is a conformal map.

Consider a fixed point $p \in M$ in a neighborhood of the boundary. For any point $q \in \overline{M}$ in a neighborhood of p, we denote by $\xi(q)$ the unique covector such that $w(q) := (p, \xi(q))$ generates the geodesic joining p to q. The map $q \mapsto \xi(q)$ is smooth down to the boundary by [8, Proposition 5.13]. Let us denote by $\tau_1(q)$ the time such that $q = \pi(\overline{\varphi}^1_{\tau_1(q)}(w(q)))$. It is smooth down to the boundary too. Since Φ conjugates the two geodesic flows, we can write:

$$\Phi(q) = \pi\big(\overline{\varphi}^2_{\tau_2(q)}(z(q))\big),$$

where $z(q) := (\Phi(p), d\Phi_p(\xi(q)))$, for some time $\tau_2(q)$. All is left to prove, is thus that τ_2 is smooth down to the boundary. If $t(q)$ denotes the g_1-geodesic distance between p and q (which is also that between $\Phi(p)$ and $\Phi(q)$ for g_2), one has:

$$t(q) = \int_0^{\tau_1(q)} \frac{ds}{\rho(\overline{\varphi}^1_s(p, \xi(q)))} = -\log\left(1 - \frac{\tau_1(q)}{\tau^1_+(w(q))}\right) + G(\tau_1(q), w(q)),$$

for some smooth function $(\tau, z) \mapsto G(\tau, z)$ down to the boundary (this is a computation similar to the one carried out in Section 2.2.1, see also [8, Lemma 2.7]). And:

$$\tau_2(q) = \tau^2_+(z(q)) - e^{-t(q)}\tau^2_+(z(q))H(e^{-t}, z(q)),$$

for some smooth positive function H on $[0, 1) \times \overline{S^*M} \setminus (\partial_- S^*M \cup \Gamma_-)$ (this stems from the previous equality, or see also [8, Lemma 2.7]). As a consequence:

$$\tau_2(q) = \tau^2_+(z(q)) - \left(1 - \frac{\tau_1(q)}{\tau^1_+(w(q))}\right) I(q),$$

for some smooth function I down to the boundary, which can be expressed in terms of H and G. This concludes the proof. □

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, 55. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964. Zbl 0171.38503 MR 167642

[2] G. Besson, G. Courtois, and S. Gallot, Entropies et rigidités des espaces localement symétriques de courbure strictement négative, *Geom. Funct. Anal.*, **5** (1995), no. 5, 731–799. Zbl 0851.53032 MR 1354289

[3] C. B. Croke, Rigidity for surfaces of non-positive curvature, *Comment. Math. Helv.*, **65** (1990), no. 1, 150–169. Zbl 0704.53035 MR 1036134

[4] C. B. Croke, Rigidity theorems in Riemannian geometry, in *Geometric methods in inverse problems and PDE control*, 47–72, IMA Vol. Math. Appl., 137, Springer, New York, 2004. Zbl 1080.53033 MR 2169902

[5] B. Czech, L. Lamprou, S. McCandlish, and J. Sully, Integral geometry and holography, *J. High Energy Phys.*, (2015), no. 10, Paper no. 175, 40pp. Zbl 1388.83217 MR 3435464

[6] P. Eberlain, Geodesic flow in certain manifolds without conjugate points, *Trans. Amer. Math. Soc.*, **167** (1972), 151–170. Zbl 0209.53304 MR 295387

[7] C. R. Graham, Volume and area renormalizations for conformally compact Einstein metrics. The Proceedings of the 19th Winter School "Geometry and Physics" (Srní, 1999), *Rend. Circ. Mat. Palermo (2)*, (2000), Suppl. No. 63, 31–42. Zbl 0984.53020 MR 1758076

[8] C. R. Graham, C. Guillarmou, P. Stefanov, and G. Uhlmann, X-ray transform and boundary rigidity for asymptotically hyperbolic manifolds, 2017. arXiv:1709.05053

[9] M. Gromov, Filling Riemannian manifolds, *J. Differential Geom.*, **18** (1983), no. 1, 1–147. Zbl 0515.53037 MR 697984

[10] C. Guillarmou, Lens rigidity for manifolds with hyperbolic trapped set, *J. Amer. Math. Soc.*, **30** (2017), no. 2, 561–599. Zbl 1377.53098 MR 3600043

[11] C. Guillarmou and M. Mazzucchelli, Marked boundary rigidity for surfaces, *Ergodic Theory Dynam. Systems*, **38** (2018), no. 4, 1459–1478. Zbl 1390.37054 MR 3789172

[12] W. Klingenberg, Riemannian manifolds with geodesic flow of Anosov type, *Ann. of Math. (2)*, **99** (1974), no. 1, 1–13. Zbl 0272.53025 MR 377980

[13] W. Klingenberg, *Riemannian geometry*, de Gruyter Studies in Mathematics, 1, Walter de Gruyter & Co., Berlin-New York, 1982. Zbl 0495.53036 MR 666697

[14] F. Ledrappier, Structure au bord des variétés à courbure négative, in *Séminaire de Théorie Spectrale et Géométrie. No. 13. Année 1994–1995*, 97–122, Sémin. Théor. Spectr. Géom., 13, Univ. Grenoble I, Saint-Martin-d'Hères, 1995. Zbl 0931.53005 MR 1715960

[15] R. Melrose, *The Atiyah–Patodi–Singer index theorem*, Research Notes in Mathematics, 4, A K Peters, Ltd., Wellesley, MA, 1993. Zbl 0796.58050 MR 1348401

[16] R. Michel, Sur la rigidité imposée par la longueur des géodésiques, *Invent. Math.*, **65** (1981/82), no. 1, 71–83. Zbl 0471.53030 MR 636880

[17] J-P. Otal, Le spectre marqué des longueurs des surfaces à courbure négative, *Ann. of Math. (2)*, **131** (1990), no. 1, 151–162. Zbl 0699.58018 MR 1038361

[18] G. P. Paternain, *Geodesic flows*, Progress in Mathematics, 180, Birkhäuser Boston, Inc., Boston, MA, 1999. Zbl 0930.53001 MR 1712465

[19] M. Porrati and R. Rabadan, Boundary rigidity and holography, *J. High Energy Phys.*, (2004), no. 1, Paper no. 034, 24pp. Zbl 1243.53120 MR 2045873

[20] R. O. Ruggiero, *Dynamics and global geometry of manifolds without conjugate points*, Ensaios Matemáticos [Mathematical Surveys], 12, Sociedade Brasileira de Matemática, Rio de Janeiro, 2007. Zbl 1133.37009 MR 2304843

[21] P. Stefanov, G. Uhlmann, and A. Vasy, Local and global boundary rigidity and the geodesic X-ray transform in the normal gauge, 2017. arXiv:1702.03638

[22] P. Stefanov, G. Uhlmann, and A. Vasy, Boundary rigidity with partial data, *J. Amer. Math. Soc.*, **29** (2013), no. 2, 299–332. Zbl 1335.53055 MR 3454376

[23] A. Wilkinson, Lectures on marked length spectrum rigidity, in *Geometric group theory*, 283–324, IAS/Park City Math. Ser., 21, Amer. Math. Soc., Providence, RI, 2014. Zbl 1306.20002 MR 3329731

Received May 21, 2018

T. Lefeuvre, Laboratoire de Mathématiques d'Orsay, Université Paris-Sud,
CNRS, Université Paris-Saclay, 91405 Orsay, France

E-mail: thibault.lefeuvre@u-psud.fr