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Smooth zero-entropy diffeomorphisms with ergodic derivative extension

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Abstract. On any smooth compact and connected manifold of dimension 2 admitting a smooth non-trivial circle action we construct C^∞-diffeomorphisms of topological entropy zero whose differential is ergodic with respect to a smooth measure in the projectivization of the tangent bundle. The proof is based on a version of the "approximation-by-conjugation"-method.

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Introduction

Let M be a smooth compact and connected manifold of dimension $m \geq 2$ with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$, preserving a smooth volume μ. In their influential paper [1], D. V. Anosov and A. Katok introduced the so-called "approximation-by-conjugation"-method which enables the construction of smooth diffeomorphisms with specific ergodic properties (e.g. weakly mixing ones in [1, Section 5]) and spectral properties ([10]) or non-standard smooth realizations of measure-preserving systems (e.g. [1, Section 6], [2], [6]). These diffeomorphisms are constructed as limits of conjugates

$$f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1},$$

where

$$\alpha_{n+1} = \alpha_n + \frac{1}{k_n \cdot l_n \cdot q_n} \in \mathbb{Q}, \quad H_n = H_{n-1} \circ h_n,$$

and h_n is a measure-preserving diffeomorphism satisfying

$$S_{\frac{1}{q_n}} \circ h_n = h_n \circ S_{\frac{1}{q_n}}.$$

In each step the conjugation map h_n and the parameter k_n are chosen such that the diffeomorphism f_n imitates the desired property with a certain precision. Then the parameter l_n is chosen large enough to guarantee closeness of f_n to f_{n-1} in the C^∞-topology and so the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ to a limit diffeomorphism is provided. It is even possible to keep this limit diffeomorphism within any given C^∞-neighbourhood of the initial element S_{α_1} or, by applying a fixed diffeomorphism g first, of $g \circ S_{\alpha_1} \circ g^{-1}$. So the construction can be carried out in a neighbourhood of any diffeomorphism conjugate to an element of the action. Thus,

$$\mathcal{A}(M) = \overline{\{h \circ S_t \circ h^{-1} : t \in \mathbb{S}^1,\ h \in \mathrm{Diff}^\infty(M,\mu)\}}^{C^\infty}$$

is a natural space for the produced diffeomorphisms. Moreover, we will consider the restricted space

$$\mathcal{A}_\alpha(M) = \overline{\{h \circ S_\alpha \circ h^{-1} : h \in \mathrm{Diff}^\infty(M,\mu)\}}^{C^\infty}$$

for $\alpha \in \mathbb{S}^1$. See also the very interesting survey article [4] for more details and other results of this method.

As mentioned above Anosov and Katok proved that the set of weakly mixing diffeomorphisms is generic (i. e. it is a dense G_δ-set) in $\mathcal{A}(M)$ in the $C^\infty(M)$-topology. In extension of it R. Gunesch and A. Katok constructed weakly mixing diffeomorphisms preserving a measurable Riemannian metric in [7]. Actually, it follows from the respective proofs that both results are true in $\mathcal{A}_\alpha(M)$ for a G_δ-set of $\alpha \in \mathbb{S}^1$. However, both proofs do not give a full description of the set of $\alpha \in \mathbb{S}^1$ for which the particular result holds in $\mathcal{A}_\alpha(M)$. Such an investigation is started in [5]: B. Fayad and M. Saprykina showed that if $\alpha \in \mathbb{S}^1$ is Liouville, the set of weakly mixing diffeomorphisms is generic in the $C^\infty(M)$-topology in $\mathcal{A}_\alpha(M)$ in case of dimension 2. Generalising these results Gunesch and the author proved in [8] that if $\alpha \in \mathbb{R}$ is Liouville, the set of volume-preserving diffeomorphisms, that are weakly mixing and preserve a measurable Riemannian metric, is dense in the C^∞-topology in $\mathcal{A}_\alpha(M)$. Recently, it has been proven that for every $\rho > 0$ and $m \geq 2$ there exists a weakly mixing real-analytic diffeomorphism $f \in \mathrm{Diff}^\omega_\rho(\mathbb{T}^m,\mu)$ preserving a measurable Riemannian metric [11].

Such diffeomorphisms preserving an absolutely continuous probability measure and a measurable Riemannian metric are called IM-diffeomorphisms. In [7, Section 3] IM-diffeomorphisms and IM-group actions are discussed comprehensively. In particular, the existence of a measurable invariant metric for a diffeomorphism is equivalent to the existence of an invariant measure for the projectivized derivative extension which is absolutely continuous in the fibers. Hence, it is a natural question to study the ergodic properties of the projectivized derivative extension with respect to such a measure. Actually, the constructions in [7] as well as [8] are as non-ergodic as possible: Their projectivized derivative extensions are isomorphic to the direct product of the diffeomorphism in the base with the trivial action in the fibers so that each ergodic component intersects almost every fiber in a single point. In this paper we realise the other extreme possibility by constructing IM-diffeomorphisms whose differential is ergodic with respect to such a smooth measure in the projectivization of the tangent bundle:

Theorem 1. *Let M be a smooth compact and connected manifold of dimension 2 with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$, preserving a smooth volume μ. Then there exists a volume-preserving diffeomorphism in $\mathcal{A}(M)$, whose projectivized derivative extension is ergodic with respect to a measure in the projectivization of the tangent bundle which is absolutely continuous in the fibers.*

This construction provides the only known examples of volume-preserving diffeomorphisms whose differential is ergodic with respect to a smooth measure in the projectivization of the tangent bundle.

By the same approach as in [8] it is possible to obtain a weakly mixing diffeomorphism and to generalise this result to dimension $m \geq 2$. In order to alleviate notations and focus on the new parts of the construction we present a proof in case of dimension 2. It will be subject of future research to study further ergodic properties (e.g. weak mixing) of the projectivized derivative extension with respect to such a measure and to obtain real-analytic counterparts of these results.

1. Preliminaries

1.1. Definitions and notations. We refer to [8, Section 2.1] for useful definitions and notations. Additionally, we want to introduce the invariant measure for the projectivized derivative extension: Let $f: M \to M$ be a smooth diffeomorphism. On the tangent bundle TM we consider the derivative extension (f, df). Let $p \in M$. We can naturally identify the tangent space T_pM with $\mathbb{R}^2$. Next, we consider its projective space $\mathbb{P}\mathbb{R}^2$ that is diffeomorphic to the circle and introduce the notation $[a,b] \subset \mathbb{P}\mathbb{R}^2$ which describes the allowed values for the spherical coordinate $\varphi \in \mathbb{R}/\pi\mathbb{Z}$. This yields the projectivized tangent bundle which will be denoted by $\mathbb{P}TM$. In particular, we will use the notation $c \times [a,b] \subset \mathbb{P}TM$ with $c \subset M$ for the set in $\mathbb{P}TM$ with base points $x \in c$ and spherical coordinates $\varphi \in [a,b]$. On the projectivized tangent bundle we consider the projectivized derivative extension of a diffeomorphism $f: M \to M$. By misuse of notation we will denote it by (f, df) again.

Following the lines of [3, Chapter 5.1] we consider the cotangent bundle T^*M and the projection maps

$$\pi_1: TM \to M$$

as well as

$$\pi_2: TM^* \to M.$$

Then we define the canonical 1-form ω on TM^* by $\omega_{|\tau} = \pi_2^* \tau$, where $\omega_{|\tau}$ denotes the 1-form ω evaluated at $\tau \in TM^*$. Additionally we define the canonical

2-form Ω on TM^* by $\Omega = d\omega$, which is symplectic. In the next step, let M be a Riemannian manifold and $V\colon M \to \mathbb{R}$ be a function. Then we examine the Lagrangian $L\colon TM \to \mathbb{R}$ given by

$$L(\xi) = \frac{|\xi|}{2} - V \circ \pi_1(\xi),$$

where $|\xi|$ is computed with respect to the Riemannian metric. To this Lagrangian we associate a bundle map $FL\colon TM \to TM^*$ defined by

$$FL(\xi)(\eta) = \frac{\mathrm{d}}{\mathrm{d}t}(L(\xi + t\eta))_{|t=0}$$

for $p \in M$, $\xi, \eta \in T_pM$. Hereby, we define $\Theta = FL^*\Omega$ and $\nu = FL^*\omega$.

In [3, Chapter 5.1] the differential form $\nu \wedge \Theta$ on the unit tangent bundle SM is considered. It is proven that it is the local product, up to a constant multiple, of the Riemannian volume on M with the Lebesgue 1-form on the unit tangent spheres of M with respect to the Riemannian metric. In particular, for any $\nu \wedge \Theta$-integrable function g on SM we have "integrations over the fibers"

$$\int_{SM} g\, \nu \wedge \Theta = c \cdot \int_M \mathrm{dVol}(p) \int_{S_pM} g|_{S_pM}\, \mathrm{d}\mu_p,$$

where Vol is the volume form induced by the Riemannian metric and μ_p is the standard Borel measure on the tangent sphere S_pM with respect to the Riemannian metric.

By the same approach we can deduce the same formula for the constructed invariant measurable Riemannian metric ω_∞ and for any integrable function on $\mathbb{P}TM$. The corresponding measure will be denoted by $\bar{\mu}$. Moreover, we point out that in our constructions the measure induced by the measurable Riemannian metric ω_∞ coincides with the measure μ on M. Since ω_∞ is f-invariant, we conclude that $\bar{\mu}$ is (f, df)-invariant.

1.2. First steps of the proof. By the same arguments as in [8, Section 2.2] constructions on $M = \mathbb{S}^1 \times [0,1]$ equipped with Lebesgue measure μ and standard circle action $\mathcal{R} = \{R_\alpha\}_{\alpha\in\mathbb{S}^1}$ comprising of diffeomorphisms $R_\alpha(\theta, r) = (\theta + \alpha, r)$ can be transferred to a general 2-dimensional compact connected smooth manifold with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$.

1.3. Outline of the proof. The constructions are based on the "approximation-by-conjugation"-method developed by D. V. Anosov and A. Katok in [1]. As indicated in the introduction, the desired diffeomorphism f with ergodic projectivized derivative extension is constructed as the limit of volume-preserving smooth diffeomorphisms f_n defined by

$$f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}.$$

Here, the rational numbers $\alpha_{n+1} \in \mathbb{S}^1$ and the conjugation maps $H_n \in \mathrm{Diff}^\infty(M, \mu)$ are constructed inductively:

$$\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \alpha_n + \frac{1}{k_n \cdot l_n \cdot q_n} \quad \text{and} \quad H_n = h_1 \circ \cdots \circ h_n,$$

where the conjugation map $h_n \in \mathrm{Diff}^\infty(M, \mu)$ has to satisfy $h_n \circ R_{\alpha_n} = R_{\alpha_n} \circ h_n$ and $k_n, l_n \in \mathbb{N}$ are parameters that have to be chosen appropriately. In particular, we will show convergence of the sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{A}(M)$ by choosing the parameter l_n sufficiently large in Section 3.

In our construction $h_n = i_n \circ \phi_n$ with two step-by-step defined smooth measure-preserving diffeomorphisms. As in [8] ϕ_n maps a strip of almost full vertical length to a set of small diameter on the one hand in order to get ergodicity of the map itself. On the other hand, ϕ_n acts as an isometry on large parts of the manifold. In comparison to [8], an additional map i_n is introduced in order to obtain ergodicity of the projectivized derivative extension. This map i_n acts as a composition of a translation and rotation on large parts of the domain where the angle of rotation is different from section to section.

Additionally, we will use a sequence of partial partitions ζ_n, which converges to the decomposition into points. On the partition elements of ζ_n the conjugation map h_n will act as an isometry and this will enable us to construct an f-invariant measurable Riemannian metric in Sections 4 and 5 by the same approach as in [8].

Finally, we will prove the ergodicity of the projectivized derivative extension. This proof bases upon estimates of Birkhoff sums for Lipschitz continuous observables $\rho: \mathbb{P}TM \to \mathbb{R}$. For this purpose, we introduce so-called "trapping regions" and "target sets" covering almost the whole space $\mathbb{P}TM$. Except for initial values in a set of very small measure the vast majority of iterates of the orbit under $R_{\alpha_{n+1}}$ is captured by the trapping regions. Under $(h_n, \mathrm{d}h_n)$ these iterates are mapped into the target sets almost uniformly distributed (see Lemma 7.3). At this juncture, we require the map i_n to act as a rotation by a different angle on different trapping regions. Since the diameter of these target sets is sufficiently small, we can approximate the value of the observable by the value of its integral on the particular target set. Hereby, we obtain the desired estimate on the Birkoff sum in Lemma 7.5.

2. Construction of the conjugation map

We fix an arbitrary countable set $\Xi = \{\rho_1, \rho_2, \dots\}$ of Lipschitz continuous functions $\rho_i: \mathbb{P}TM \to \mathbb{R}$, that is dense in $\mathrm{C}(\mathbb{P}TM; \mathbb{R})$. Since $\mathrm{C}(\mathbb{P}TM; \mathbb{R})$ is separable and Lipschitz continuous functions are dense in $\mathrm{C}(\mathbb{P}TM; \mathbb{R})$, this is possible. For any Lipschitz continuous function ρ on $\mathbb{P}TM$ we denote its Lipschitz constant by $\|\rho\|_{\mathrm{Lip}}$ and $\|\rho\|_0 = \max_{x \in \mathbb{P}TM} |\rho(x)|$.

We present step n in our inductive process of construction. We assume that we have already defined the rational numbers $\alpha_1, \dots, \alpha_n \in \mathbb{S}^1$ and the conjugation map

$$H_{n-1} = h_1 \circ \dots \circ h_{n-1} \in \mathrm{Diff}^\infty(M, \mu).$$

First of all, we put

$$\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \alpha_n + \frac{1}{k_n \cdot l_n \cdot q_n}$$

and choose the parameter $k_n \in \mathbb{N}$ large enough such that the following conditions are fulfilled:

$$k_n > n^2 \cdot \max_{i=1,\dots,n} \|\rho_i\|_{\mathrm{Lip}}, \tag{A}$$

$$k_n > 30 \cdot n^2 \cdot \max_{i=1,\dots,n} \|\rho_i\|_0 . \tag{B}$$

For every subset $c \subset \mathbb{P}TM$ of diameter $\mathrm{diam}(c) < \frac{3}{k_n}$ we have:

$$\mathrm{diam}((H_{n-1}, \mathrm{d}H_{n-1})(c)) < \frac{1}{n^2 \cdot \max_{i=1,\dots,n} \|\rho_i\|_{\mathrm{Lip}}}. \tag{C}$$

Moreover the sequence of parameters $(k_n)_{n\in\mathbb{N}}$ should satisfy

$$\sum_{j=n+1}^{\infty} \frac{1}{k_j} \leq \frac{1}{k_n}.$$

2.1. The conjugation map ϕ_n. In [8, Section 3.3] we constructed the smooth area-preserving diffeomorphism $\bar{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}$ on $\mathbb{S}^1 \times [0,1]$ satisfying the subsequent properties:

Proposition 2.1. *Let $\varepsilon, \varepsilon_2 \in (0, \frac{1}{4})$ and $\lambda, \mu \in \mathbb{N}$. Then there is a smooth area-preserving $\frac{1}{\lambda}$-periodic diffeomorphism $\bar{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2} \colon \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1]$ such that*

1. *Let $t_2 \in \mathbb{Z}$, $\lceil 2\varepsilon\mu \rceil \leq t_2 \leq \mu - \lceil 2\varepsilon\mu \rceil - 1$, $|u_2| \leq \varepsilon_2$, and $u_1 \in (2\varepsilon, \frac{1}{2})$ be of the form $\frac{t_1}{\mu}$ with $t_1 \in \mathbb{Z}$. Then we have*

$$\begin{aligned}\bar{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}&\left(\left[\frac{u_1}{\lambda}, \frac{1-u_1}{\lambda}\right] \times \left[\frac{t_2+u_2}{\mu}, \frac{t_2+1-u_2}{\mu}\right]\right) \\ &= \left[\frac{1}{\lambda} - \frac{t_2+1-u_2}{\mu\lambda}, \frac{1}{\lambda} - \frac{t_2+u_2}{\mu\lambda}\right] \times [u_1, 1-u_1].\end{aligned}$$

2. *$\bar{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}$ acts as an isometry on each cuboid*

$$\left[\frac{t_1+2\varepsilon_2}{\mu\cdot\lambda}, \frac{t_1+1-2\varepsilon_2}{\mu\cdot\lambda}\right] \times \left[\frac{t_2+2\varepsilon_2}{\mu}, \frac{t_2+1-2\varepsilon_2}{\mu}\right],$$

where $t_i \in \mathbb{Z}$, $\lceil 2\varepsilon\mu \rceil \leq t_i \leq \mu - \lceil 2\varepsilon\mu \rceil - 1$ for $i = 1, 2$.

The first property will enable us to prove in Lemma 7.2 that ϕ_n maps sets of almost full length in the r-coordinate to sets of small diameter. By the second property ϕ_n acts as an isometry on each partition element $\check{I}_n \in \zeta_n$ (see the proof of Lemma 5.2). In the construction of the map $\bar{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}$ one uses a map C_λ causing a stretch by λ in the first coordinate and a so-called "quasi-rotation" φ_ε constructed with the aid of "Moser's trick," which is the rotation by $\pi/2$ about the point $(\frac{1}{2}, \frac{1}{2})$ on $[2\varepsilon, 1-2\varepsilon]^2$ and coincides with the identity outside of $[\varepsilon, 1-\varepsilon]^2$. With these maps one also defines a family of "inner rotations of type A" ψ_{μ,ε_2} in order to get the second property stated above: A map of the form $C_\lambda^{-1} \circ \varphi_\varepsilon \circ C_\lambda$ would cause an expansion by λ in one coordinate and by λ^{-1} in another, so far away from being an isometry. The "inner rotations of type A" cause that C_λ and C_λ^{-1} act on the same coordinate on the elements $\check{I}_n \in \zeta_n$.

Proof of Proposition 2.1. As announced we will use the "quasi-rotations" introduced in [5] and [8, Lemma 3.7]:

Fact. *For every $\varepsilon \in (0, \frac{1}{4})$ there exists a smooth area-preserving diffeomorphism φ_ε on $\mathbb{R}^2$ which is the rotation by $\pi/2$ about the point $(\frac{1}{2}, \frac{1}{2})$ on $[2\varepsilon, 1-2\varepsilon]^2$ and coincides with the identity outside of $[\varepsilon, 1-\varepsilon]^2$.*

Furthermore, for $\lambda \in \mathbb{N}$ we define the maps

$$C_\lambda(x_1, x_2) = (\lambda \cdot x_1, x_2) \quad \text{and} \quad D_\lambda(x_1, x_2) = (\lambda \cdot x_1, \lambda \cdot x_2).$$

Moreover, let $\mu \in \mathbb{N}$. We construct a diffeomorphism ψ_{μ,ε_2} in the following way:

- Under the map D_μ any cube of the form

$$\left[\frac{l_1}{\mu}, \frac{l_1+1}{\mu}\right] \times \left[\frac{l_2}{\mu}, \frac{l_2+1}{\mu}\right]$$

  with $l_i \in \mathbb{N}$ is mapped onto $[l_1, l_1+1] \times [l_2, l_2+1]$.
- On $[0,1]^2$ we will use the diffeomorphism $\varphi_{\varepsilon_2}^{-1}$ from the above mentioned fact. Since this is the identity outside of $[\varepsilon_2, 1-\varepsilon_2]^2$, we can extend it to a diffeomorphism $\overline{\varphi}_{\varepsilon_2}^{-1}$ on $\mathbb{R}^2$ using the instruction

$$\overline{\varphi}_{\varepsilon_2}^{-1}(x_1 + l_1, x_2 + l_2) = (l_1, l_2) + \varphi_{\varepsilon_2}^{-1}(x_1, x_2),$$

  where $l_i \in \mathbb{Z}$ and $x_i \in [0,1]$.
- Now we define the smooth measure-preserving diffeomorphism

$$\psi_{\mu,\varepsilon_2} = D_\mu^{-1} \circ \overline{\varphi}_{\varepsilon_2}^{-1} \circ D_\mu.$$

  This is a smooth map because ψ_{μ,ε_2} is the identity in a neighbourhood of the boundary by construction.

Using these maps we build the following smooth area-preserving diffeomorphism:

$$\bar{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}\colon \left[0,\frac{1}{\lambda}\right] \times [0,1] \to \left[0,\frac{1}{\lambda}\right] \times [0,1], \quad \bar{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2} = C_\lambda^{-1} \circ \psi_{\mu,\varepsilon_2} \circ \varphi_\varepsilon \circ C_\lambda.$$

Afterwards, $\bar{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}$ is extended to a diffeomorphism on $\mathbb{S}^1 \times [0,1]$ by the description

$$\bar{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}\left(x_1 + \frac{1}{\lambda}, x_2\right) = \left(\frac{1}{\lambda}, 0\right) + \bar{\phi}_{\lambda,\varepsilon,\mu,\varepsilon_2}(x_1, x_2).$$

This map satisfies the properties stated in Proposition 2.1. □

Using these maps we define the diffeomorphism ϕ_n on $[0, \frac{1}{k_n \cdot q_n}] \times [0,1]$

$$\phi_n = \bar{\phi}_{k_n \cdot q_n, \frac{1}{2k_n^2}, k_n^2, \frac{1}{2 \cdot k_n^3 \cdot q_n}}.$$

Since ϕ_n coincides with the identity in a neighbourhood of the boundary of its domain, we can extend ϕ_n to a diffeomorphism on $\mathbb{S}^1 \times [0,1]$ using the description

$$\phi_n \circ R_{\frac{1}{k_n \cdot q_n}} = R_{\frac{1}{k_n \cdot q_n}} \circ \phi_n.$$

2.2. The conjugation map i_n. In this subsection we define the so-called "inner rotations of type B" i_n which will allow us to prove ergodicity of the projectivized derivative extension. In particular, we will exploit the different rotation angles on the particular sections in the proof of a "trapping property" in Lemma 7.3. This trapping property will be crucial in the estimates on Birkhoff sums in Lemma 7.5.

Proposition 2.2. *Let $a_n = k_n^6 \cdot q_n$, $c_n = k_n^2$ and $\varepsilon_n = \frac{1}{k_n^2}$. There is a smooth measure-preserving diffeomorphism*

$$i_n \colon \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1]$$

such that:

1. *Each square of the form*

$$\left[\frac{i}{a_n}, \frac{i+1}{a_n}\right] \times \left[\frac{j}{a_n}, \frac{j+1}{a_n}\right]$$

with $i, j \in \mathbb{Z}$ is mapped onto itself by i_n and i_n coincides with the identity on a $\frac{\varepsilon_n}{a_n}$-neighbourhood of its boundary.

2. *On every square*

$$\left[\frac{i}{a_n} + \frac{s_1+\varepsilon_n}{c_n \cdot a_n}, \frac{i}{a_n} + \frac{s_1+1-\varepsilon_n}{c_n \cdot a_n}\right] \times \left[\frac{j}{a_n} + \frac{s_2+\varepsilon_n}{c_n \cdot a_n}, \frac{j}{a_n} + \frac{s_2+1-\varepsilon_n}{c_n \cdot a_n}\right] \subset \left[\frac{i}{a_n}, \frac{i+1}{a_n}\right] \times [0,1],$$

where $s_1, s_2 \in \mathbb{Z}$, $1 \leq s_1, s_2 \leq c_n - 2$, i_n is a composition of a translation and a rotation by β_i, where $\beta_i = \frac{s \cdot \pi}{k_n}$ in case of $s \equiv i \bmod k_n$.

3. $i_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ i_n$.

For the construction we need the subsequent lemma:

Lemma 2.3. *Let $c \in \mathbb{N}$, $c \geq 3$, $\varepsilon \in (0, \frac{1}{5c}]$ and $\beta \in [0, \pi]$. Then there is a smooth measure-preserving diffeomorphism*

$$\psi_{c,\varepsilon,\beta}\colon [0,1]^2 \to [0,1]^2$$

satisfying the following properties:

- *$\psi_{c,\varepsilon,\beta}$ coincides with the identity on $[0,1]^2 \setminus [\varepsilon, 1-\varepsilon]^2$.*
- *On every square*

$$\Big[\frac{j+\varepsilon}{c}, \frac{j+1-\varepsilon}{c}\Big] \times \Big[\frac{k+\varepsilon}{c}, \frac{k+1-\varepsilon}{c}\Big]$$

with $1 \leq j,k \leq c-2$ the map $\psi_{c,\varepsilon,\beta}$ is equal to a composition of a translation and a rotation by arc β around a new center.

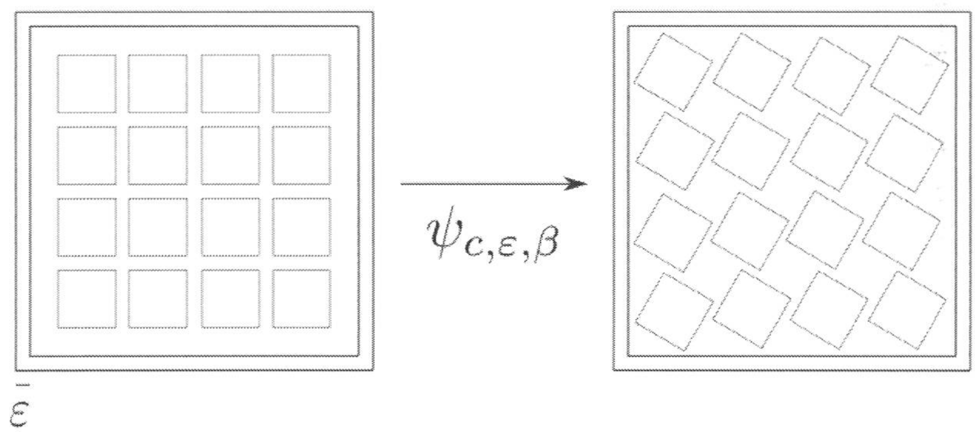

Figure 1. The action of $\psi_{c,\varepsilon,\beta}$ on $[0,1]^2$.

Proof. There is a rearrangement of these squares

$$\Big[\frac{j+\varepsilon}{c}, \frac{j+1-\varepsilon}{c}\Big] \times \Big[\frac{k+\varepsilon}{c}, \frac{k+1-\varepsilon}{c}\Big]$$

rotated by β in $[2\varepsilon, 1-2\varepsilon]^2$. Corresponding to this, each center $\big(\frac{j+0.5}{c}, \frac{k+0.5}{c}\big)$ of such a square is translated by $(a_{j,k}, b_{j,k})$. We will need these translations later. Moreover, we will use a smooth diffeomorphism $\psi_2\colon \mathbb{R}^2 \to \mathbb{R}^2$, which coincides with the identity on $\mathbb{R}^2 \setminus [\varepsilon, 1-\varepsilon]^2$ and with a dilation by $\frac{1}{5}$ in each coordinate about the center on each of the translated and rotated squares.

Now, let $\psi_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth diffeomorphism satisfying

$$\psi_1(x,y) = \begin{cases} (x,y) & \text{on } \mathbb{R}^2 \setminus [\varepsilon, 1-\varepsilon]^2, \\ \left(\frac{j+0.5}{c} + \frac{1}{5}\left(x - \frac{j+0.5}{c}\right), \frac{k+0.5}{c} + \frac{1}{5}\left(y - \frac{k+0.5}{c}\right)\right) & \\ \qquad \text{on each } \left[\frac{j+\frac{\varepsilon}{2}}{c}, \frac{j+1-\frac{\varepsilon}{2}}{c}\right] \times \left[\frac{k+\frac{\varepsilon}{2}}{c}, \frac{k+1-\frac{\varepsilon}{2}}{c}\right]. \end{cases}$$

Additionally, we choose a smooth diffeomorphism τ_1 that is the identity on $\mathbb{R}^2 \setminus [\varepsilon, 1-\varepsilon]^2$ and a rotation by β on each disc

$$\left\{\left(x - \frac{j+0.5}{c}\right)^2 + \left(y - \frac{k+0.5}{c}\right)^2 \leq \frac{1}{\sqrt{50}c}\right\}.$$

Furthermore, let τ_2 be a smooth diffeomorphism with the following properties

$$\tau_2(x,y) = \begin{cases} (x,y) & \text{on } \mathbb{R}^2 \setminus [\varepsilon, 1-\varepsilon]^2, \\ (x + a_{j,k}, y + b_{j,k}) & \\ \qquad \text{on each } \left\{\left(x - \frac{j+0.5}{c}\right)^2 + \left(y - \frac{k+0.5}{c}\right)^2 \leq \frac{1}{\sqrt{50}c}\right\}. \end{cases}$$

We define $\bar{\psi} := \psi_2^{-1} \circ \tau_2 \circ \tau_1 \circ \psi_1$. Then the diffeomorphism $\bar{\psi}$ coincides with the identity on $\mathbb{R}^2 \setminus [\varepsilon, 1-\varepsilon]^2$ and with a composition of a rotation by β and a translation on every square

$$\left[\frac{j+\varepsilon}{c}, \frac{j+1-\varepsilon}{c}\right] \times \left[\frac{k+\varepsilon}{c}, \frac{k+1-\varepsilon}{c}\right]$$

with $1 \leq j,k \leq c-2$. In particular, $\bar{\psi}$ is measure-preserving on the union of these sets. Hence, we can construct the desired measure-preserving diffeomorphism $\psi_{c,\varepsilon,\beta}$ with the aid of Moser's trick similarly to [8, Lemma 3.4]. □

Proof of Proposition 2.2. Using the dilation

$$D_a\colon \left[0, \frac{1}{a}\right]^2 \to [0,1]^2, \quad D_a(x_1,x_2) = (a \cdot x_1, a \cdot x_2)$$

for $a \in \mathbb{Z}$ we define the map

$$\psi_{a,c,\varepsilon,\beta}\colon \left[0, \frac{1}{a}\right]^2 \to \left[0, \frac{1}{a}\right]^2, \quad \psi_{a,c,\varepsilon,\beta} = D_a^{-1} \circ \psi_{c,\varepsilon,\beta} \circ D_a.$$

Since it coincides with the identity in a neighbourhood of the boundary, we can extend it to a smooth diffeomorphism on $\mathbb{S}^1 \times [0,1]$ equivariantly by the description

$$\psi_{a,c,\varepsilon,\beta}\left(x_1 + \frac{a_1}{a}, x_2 + \frac{a_2}{a}\right) = \left(\frac{a_1}{a}, \frac{a_2}{a}\right) + \psi_{a,c,\varepsilon,\beta}(x_1,x_2)$$

for $a_1, a_2 \in \mathbb{Z}$.

On $\left[\frac{i}{k_n^6 \cdot q_n}, \frac{i+1}{k_n^6 \cdot q_n}\right] \times [0,1]$ we define:

$$\beta_i = \frac{s \cdot \pi}{k_n} \quad \text{in case of } s \equiv i \mod k_n$$

as well as

$$i_n = \psi_{k_n^6 \cdot q_n, k_n^2, \frac{1}{k_n^2}, \beta_i}.$$

Since each map coincides with the identity in a neighbourhood of the boundary, we can piece them together in order to get a smooth diffeomorphism on $\mathbb{S}^1 \times [0,1]$. □

2.3. The conjugation map h_n. With the aid of the previous constructions we define the conjugation map $h_n = i_n \circ \phi_n$. By the observations in the previous subsections we have $h_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ h_n$.

3. Convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Diff}^\infty(M,\mu)$

In the following we show that the sequence of constructed measure-preserving smooth diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges. For this purpose, the next result, that can be found in [6, Lemma 4] is very useful.

Lemma 3.1. *Let $k \in \mathbb{N}_0$ and h be a C^∞-diffeomorphism on M. Then we get for every $\alpha, \beta \in \mathbb{R}$:*

$$d_k(h \circ R_\alpha \circ h^{-1}, h \circ R_\beta \circ h^{-1}) \leq C_k \cdot |||h|||_{k+1}^{k+1} \cdot |\alpha - \beta|,$$

where the constant C_k depends solely on k. In particular $C_0 = 1$.

Under some conditions on the proximity of α_n and α_{n+1} we can prove convergence:

Lemma 3.2. *There exists a sequence $\alpha_n = \frac{p_n}{q_n}$ of rational numbers such that our sequence of constructed diffeomorphisms f_n converges in the* $\mathrm{Diff}^\infty(M)$*-topology to a diffeomorphism $f \in \mathcal{A}(M)$. Additionally, we have for every $\rho \in \{\rho_1, \ldots, \rho_n\} \subset \Xi$*

$$\sup_{x \in \mathbb{P}TM} |\rho((f^m, \mathrm{d}f^m)(x)) - \rho((f_n^m, \mathrm{d}f_n^m)(x))| < \frac{1}{n^2}$$

for every natural number $m \leq q_{n+1}$ and $n \in \mathbb{N}$.

Proof. First of all, we recall the relations

$$\alpha_{n+1} - \alpha_n = \frac{1}{k_n \cdot l_n \cdot q_n} \quad \text{and} \quad h_n \circ R_{\alpha_n} = R_{\alpha_n} \circ h_n.$$

Hereby we observe for any $m \in \mathbb{N}$

$$\begin{aligned} f_n^m = H_n \circ R^m_{\alpha_{n+1}} \circ H_n^{-1} &= H_{n-1} \circ h_n \circ R^m_{\alpha_n} \circ R^m_{\frac{1}{k_n \cdot l_n \cdot q_n}} \circ h_n^{-1} \circ H_{n-1}^{-1} \\ &= H_{n-1} \circ R^m_{\alpha_n} \circ h_n \circ R^m_{\frac{1}{k_n \cdot l_n \cdot q_n}} \circ h_n^{-1} \circ H_{n-1}^{-1}. \end{aligned}$$

Since the construction of the conjugation map h_n does not involve l_n, we can obtain

$$\sup_{x \in \mathbb{P}TM} d\big((f_n^m, \mathrm{d}f_n^m)(x), (f_{n-1}^m, \mathrm{d}f_{n-1}^m)(x)\big) < \frac{1}{2 \cdot k_n}$$

for every natural number $m \leq q_n$ as well as

$$|\alpha_{n+1} - \alpha_n| \leq \frac{1}{2^n \cdot C_n \cdot k_n \cdot q_n \cdot |||H_n|||_{n+1}^{n+1}} \tag{3.1}$$

by choosing $l_n \in \mathbb{N}$ large enough.

We can apply Lemma 3.1 for every $k, n \in \mathbb{N}$:

$$\begin{aligned} d_k(f_n, f_{n-1}) &= d_k\big(H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}, H_n \circ R_{\alpha_n} \circ H_n^{-1}\big) \\ &\leq C_k \cdot |||H_n|||_{k+1}^{k+1} \cdot |\alpha_{n+1} - \alpha_n|. \end{aligned}$$

By assumption (3.1) it follows for every $k \leq n$:

$$\begin{aligned} d_k(f_n, f_{n-1}) &\leq d_n(f_n, f_{n-1}) \\ &\leq C_n \cdot |||H_n|||_{n+1}^{n+1} \cdot \frac{1}{2^n \cdot C_n \cdot q_n \cdot |||H_n|||_{n+1}^{n+1}} < \frac{1}{2^n}. \end{aligned} \tag{3.2}$$

In the next step we show that for arbitrary $k \in \mathbb{N}$ $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathrm{Diff}^k(M)$, i.e. $\lim_{n,m \to \infty} d_k(f_n, f_m) = 0$. For this purpose, we calculate:

$$\lim_{n \to \infty} d_k(f_n, f_m) \leq \lim_{n \to \infty} \sum_{i=m+1}^{n} d_k(f_i, f_{i-1}) = \sum_{i=m+1}^{\infty} d_k(f_i, f_{i-1}). \tag{3.3}$$

We consider the limit process $m \to \infty$, i.e. we can assume $k \leq m$ and obtain from equations (3.2) and (3.3):

$$\lim_{n,m \to \infty} d_k(f_n, f_m) \leq \lim_{m \to \infty} \sum_{i=m+1}^{\infty} \frac{1}{2^i} = 0.$$

Since $\mathrm{Diff}^k(M)$ is complete, the sequence $(f_n)_{n \in \mathbb{N}}$ converges consequently in $\mathrm{Diff}^k(M)$ for every $k \in \mathbb{N}$. Thus, the sequence converges in $\mathrm{Diff}^\infty(M)$ by definition.

Moreover, we estimate for every $m \le q_{n+1}$

$$\begin{aligned}\sup_{x\in\mathbb{P}TM} & d\left((f_n^m, \mathrm{d}f_n^m)(x), (f^m, \mathrm{d}f^m)(x)\right) \\ &\le \sum_{j=n+1}^{\infty} \sup_{x\in\mathbb{P}TM} d\left((f_j^m, \mathrm{d}f_j^m)(x), (f_{j-1}^m, \mathrm{d}f_{j-1}^m)(x)\right) \\ &< \sum_{j=n+1}^{\infty} \frac{1}{2\cdot k_j} \le \frac{1}{k_n}.\end{aligned}$$

By requirement (A) on the number k_n we obtain for every $\rho \in \{\rho_1, \dots, \rho_n\} \subset \Xi$

$$\sup_{x\in\mathbb{P}TM} |\rho((f^m, \mathrm{d}f^m)(x)) - \rho((f_n^m, \mathrm{d}f_n^m)(x))| < \frac{1}{n^2}$$

for every number $m \le q_{n+1}$ and $n \in \mathbb{N}$. □

4. Criterion for the existence of a f-invariant measurable Riemannian metric

Let ω_0 denote the standard Riemannian metric on $M = \mathbb{S}^1 \times [0,1]$. By the same approach as in [7, Section 4.8] we prove the subsequent criterion for the existence of a f-invariant measurable Riemannian metric:

Proposition 4.1 (Criterion for the existence of a f-invariant measurable Riemannian metric). *Let $(\zeta_n)_{n\in\mathbb{N}}$ be a sequence of partial partitions whose elements cover a set of measure at least $1 - \frac{1}{n^2}$ for every $n \in \mathbb{N}$. Suppose that for every $n \in \mathbb{N}$ the conjugation map h_n acts as an isometry on every element of the partition ζ_n. Then the limit diffeomorphism $f = \lim_{n\to\infty} f_n$ of the sequence $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ admits an invariant measurable Riemannian metric.*

Proof. The assumption implies that for every $\check{I}_n \in \zeta_n$ $h_n^{-1}|_{h_n(\check{I}_n)}$ is an isometry as well. In the following we construct the f-invariant measurable Riemannian metric. For it we put $\omega_n := (H_n^{-1})^*\omega_0$. Each ω_n is a smooth Riemannian metric because it is the pullback of a smooth metric via a $C^\infty(M)$-diffeomorphism. Since $R_{\alpha_{n+1}}^*\omega_0 = \omega_0$ the metric ω_n is f_n-invariant:

$$\begin{aligned} f_n^*\omega_n &= (H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1})^*(H_n^{-1})^*\omega_0 \\ &= (H_n^{-1})^* R_{\alpha_{n+1}}^* H_n^*(H_n^{-1})^*\omega_0 \\ &= (H_n^{-1})^* R_{\alpha_{n+1}}^* \omega_0 = (H_n^{-1})^*\omega_0 = \omega_n. \end{aligned}$$

With the following lemmas we show that the limit $\omega_\infty := \lim_{n\to\infty}\omega_n$ exists μ-almost everywhere and is the desired f-invariant Riemannian metric.

Lemma 4.2. *The sequence $(\omega_n)_{n\in\mathbb{N}}$ converges μ-a.e. to a limit ω_∞*

Proof. For every $N \in \mathbb{N}$ we have for every $k > 0$:

$$\begin{aligned}\omega_{N+k} = (H_{N+k}^{-1})^*\omega_0 &= (h_{N+k}^{-1} \circ \cdots \circ h_{N+1}^{-1} \circ H_N^{-1})^*\omega_0 \\ &= (H_N^{-1})^*(h_{N+k}^{-1} \circ \cdots \circ h_{N+1}^{-1})^*\omega_0.\end{aligned}$$

Since the elements of the partition ζ_n cover M except a set of measure at most $\frac{1}{n^2}$ and $h_n^{-1}|_{h_n(\check{I}_n)}$ is an isometry for every $\check{I}_n \in \zeta_n$, ω_{N+k} coincides with $\omega_N = (H_N^{-1})^*\omega_0$ on a set of measure at least $1 - \sum_{n=N+1}^{\infty}\frac{1}{n^2}$. As this measure approaches 1 for $N \to \infty$, the sequence $(\omega_n)_{n\in\mathbb{N}}$ converges on a set of full measure. □

Lemma 4.3. *The limit ω_∞ is a measurable Riemannian metric.*

Proof. The limit ω_∞ is a measurable map because it is the pointwise limit of the smooth metrics ω_n, which in particular are measurable. By the same reasoning $\omega_\infty|_p$ is symmetric for μ-almost every $p \in M$. Furthermore, ω_∞ is positive definite because ω_n is positive definite for every $n \in \mathbb{N}$ and ω_∞ coincides with ω_N on $T_1M \otimes T_1M$ minus a set of measure at most $\sum_{n=N+1}^{\infty}\frac{1}{n^2}$. Since this is true for every $N \in \mathbb{N}$, ω_∞ is positive definite on a set of full measure. □

Lemma 4.4. *ω_∞ is f-invariant, i.e. $f^*\omega_\infty = \omega_\infty$ μ-a.e.*

Proof. By Lemma 4.2 the sequence $(\omega_n)_{n\in\mathbb{N}}$ converges in the C^∞-topology pointwise almost everywhere. Hence, we obtain using Egoroff's theorem: For every $\delta > 0$ there is a set $C_\delta \subseteq M$ such that $\mu(M \setminus C_\delta) < \delta$ and the convergence $\omega_n \to \omega_\infty$ is uniform on C_δ.

The function f was constructed as the limit of the sequence $(f_n)_{n\in\mathbb{N}}$ in the C^∞-topology. Thus, $\widetilde{f_n} := f_n^{-1} \circ f \to \mathrm{id}$ in the C^∞-topology. Since M is compact, this convergence is uniform too.

Furthermore, the smoothness of f implies

$$f^*\omega_\infty = f^* \lim_{n\to\infty}\omega_n = \lim_{n\to\infty} f^*\omega_n.$$

Therewith, we compute on C_δ:

$$f^*\omega_\infty = \lim_{n\to\infty}((f_n\widetilde{f_n})^*\omega_n) = \lim_{n\to\infty}(\widetilde{f_n}^* f_n^*\omega_n) = \lim_{n\to\infty}\widetilde{f_n}^*\omega_n = \omega_\infty,$$

where we used the uniform convergence on C_δ in the last step. As this holds on every set C_δ with $\delta > 0$, it also holds on the set $\bigcup_{\delta>0} C_\delta$. This is a set of full measure and therefore the claim follows. □

Hence, the desired f-invariant measurable Riemannian metric ω_∞ is constructed and thus Proposition 4.1 is proven. □

5. Proof of existence of the f-invariant measurable Riemannian metric

In order to apply our criterion 4.1 for the existence of a f-invariant measurable Riemannian metric we define a partial partition ζ_n and check that the conjugation map h_n acts as an isometry on it.

5.1. Partial partition ζ_n. The partial partition ζ_n will be defined in such a way that it covers large parts of $M = \mathbb{S}^1 \times [0,1]$ and h_n acts as an isometry on it. For this purpose, the partition elements will be of the form

$$\left[\frac{t_1+\varepsilon_n}{c_n \cdot a_n}, \frac{t_1+1-\varepsilon_n}{c_n \cdot a_n}\right] \times \left[\frac{t_2+\varepsilon_n}{c_n \cdot a_n}, \frac{t_2+1-\varepsilon_n}{c_n \cdot a_n}\right]$$

(with the parameters a_n, c_n and ε_n in the construction of the conjugation map i_n) positioned in the domain, where ϕ_n acts as an isometry. To be precise the partial partition ζ_n consists of all multidimensional intervals of the following form:

$$\begin{aligned} &I_{u_0,u_1,u_2,u_3,u_4;v_1,v_2,v_3,v_4} \\ &\quad = \left[\frac{u_0}{k_n \cdot q_n} + \frac{u_1}{k_n^3 \cdot q_n} + \frac{u_2}{k_n^5 \cdot q_n} + \frac{u_3}{k_n^6 \cdot q_n} + \frac{u_4}{k_n^8 \cdot q_n} + \frac{1}{k_n^{10} \cdot q_n},\right. \\ &\qquad\qquad \left.\frac{u_0}{k_n \cdot q_n} + \frac{u_1}{k_n^3 \cdot q_n} + \frac{u_2}{k_n^5 \cdot q_n} + \frac{u_3}{k_n^6 \cdot q_n} + \frac{u_4+1}{k_n^8 \cdot q_n} - \frac{1}{k_n^{10} \cdot q_n}\right] \\ &\qquad \times \left[\frac{v_1}{k_n^2} + \frac{v_2}{k_n^5 \cdot q_n} + \frac{v_3}{k_n^6 \cdot q_n} + \frac{v_4}{k_n^8 \cdot q_n} + \frac{1}{k_n^{10} \cdot q_n},\right. \\ &\qquad\qquad\qquad \left.\frac{v_1}{k_n^2} + \frac{v_2}{k_n^5 \cdot q_n} + \frac{v_3}{k_n^6 \cdot q_n} + \frac{v_4+1}{k_n^8 \cdot q_n} - \frac{1}{k_n^{10} \cdot q_n}\right], \end{aligned}$$

where $u_0 \in \mathbb{Z}$ and $u_1, u_2, u_4, v_1, v_4 \in \{1, \ldots, k_n^2-2\}$ and $u_3, v_3 \in \{0, 1, \ldots, k_n-1\}$ and $v_2 \in \{k_n q_n, k_n q_n + 1, \ldots, k_n^3 q_n - k_n q_n - 1\}$.

Remark 5.1. For every $n \in \mathbb{N}$ the partial partition ζ_n consists of disjoint sets, covers a set of measure at least $(1-\frac{2}{k_n^2})^8 \geq 1-\frac{16}{k_n^2}$ and the sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges to the decomposition into points.

5.2. Application of the criterion. The following Lemma shows that the conjugation map $h_n = i_n \circ \phi_n$ constructed in Section 2 is an isometry with respect to ω_0 on the elements of the partial partition ζ_n.

Lemma 5.2. *Let $\check{I}_n \in \zeta_n$. Then $h_n|_{\check{I}_n}$ is an isometry with respect to ω_0.*

Proof. The proof is similar to the proof of [8, Lemma 7.1]. Let $\check{I}_n := I_{u_0,u_1,u_2,u_3,u_4;v_1,v_2,v_3,v_4} \in \zeta_n$ be a partition element. By Proposition 2.1 and our choice of parameters this element $\check{I}_n$ is positioned in such a way that ϕ_n acts as an isometry on it. In fact, $\phi_n(\check{I}_n)$ is equal to

$$\begin{aligned}&\left[\frac{u_0+1}{k_nq_n}-\frac{v_1+1}{k_n^3\cdot q_n}+\frac{u_2}{k_n^5\cdot q_n}+\frac{u_3}{k_n^6\cdot q_n}+\frac{u_4}{k_n^8\cdot q_n}+\frac{1}{k_n^{10}\cdot q_n},\right.\\ &\qquad\left.\frac{u_0+1}{k_nq_n}-\frac{v_1+1}{k_n^3\cdot q_n}+\frac{u_2}{k_n^5\cdot q_n}+\frac{u_3}{k_n^6\cdot q_n}+\frac{u_4+1}{k_n^8\cdot q_n}-\frac{1}{k_n^{10}\cdot q_n}\right]\\ &\qquad\times\left[\frac{u_1}{k_n^2}+\frac{v_2}{k_n^5\cdot q_n}+\frac{v_3}{k_n^6\cdot q_n}+\frac{v_4}{k_n^8\cdot q_n}+\frac{1}{k_n^{10}\cdot q_n},\right.\\ &\qquad\qquad\left.\frac{u_1}{k_n^2}+\frac{v_2}{k_n^5\cdot q_n}+\frac{v_3}{k_n^6\cdot q_n}+\frac{v_4+1}{k_n^8\cdot q_n}-\frac{1}{k_n^{10}\cdot q_n}\right]\\ &\qquad\qquad= I_{u_0,k_n^2-v_1-1,u_2,u_3,u_4;u_1,v_2,v_3,v_4}.\end{aligned}$$

On this set $i_n = \psi_{k_n^6\cdot q_n,k_n^2,\frac{1}{k_n^2},\beta_i}$ is equal to the composition of a translation and the respective rotations by the second statement in Proposition 2.2. □

Remark 5.3. As observed in Lemma 5.2 the map $h_n = i_n \circ \phi_n$ acts as the composition of the respective rotations and translations on every $\check{I}_n \in \zeta_n$. In the following $G_n := \bigcup_{\check{I}_n\in\zeta_n} \check{I}_n$ will be called the "good domain" of h_n. Its corresponding parts on the θ-axis are called the "good horizontal length" of h_n and are denoted by $G_{n,h}$. Analogously, its corresponding parts on the r-axis are called the "good vertical length" and are denoted by $G_{n,v}$. By the same arguments as in Remark 5.1 observe that for an interval $\left[\frac{l}{k_nq_n},\frac{l+1}{k_nq_n}\right]$ on the θ-axis the length

$$\left(1-\frac{2}{k_n^2}\right)^4\cdot\frac{1}{k_nq_n}\geq\left(1-\frac{8}{k_n^2}\right)\cdot\frac{1}{k_nq_n}$$

is part of the "good horizontal length". Similarly, the length

$$\left(1-\frac{2}{k_n^2}\right)^4\geq 1-\frac{8}{k_n^2}$$

is part of the "good vertical length" on the r-axis.

Since the elements of the partial partition ζ_n cover a set of M of measure at least $1-\frac{16}{k_n^2}$ (see Remark 5.1), we are able to apply the criterion in Proposition 4.1 and conclude the existence of a measurable f-invariant Riemannian metric.

6. Criterion for ergodicity of the derivative extension

A continuous transformation $f: X \to X$ on a compact metric space X preserving a Borel probability measure ν is ergodic with respect to ν if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \varphi(f^i(x)) = \int_X \varphi \, \mathrm{d}\nu \text{ for } \nu\text{-almost every } x \in X$$

for every $\varphi \in C(X;\mathbb{R})$ [12]. Since $C(X;\mathbb{R})$ is a separable metric space and Lipschitz continuous functions are dense in $C(X;\mathbb{R})$, we can choose a countable set

$$\Xi = \{\varphi_k : X \to \mathbb{R} \mid k \in \mathbb{N}\}$$

of Lipschitz continuous functions that is dense in $C(X;\mathbb{R})$. With the aid of the following lemma one can prove ergodicity in the general setup of the approximation-by-conjugation-method.

Lemma 6.1. *Consider a compact metric space (X,d), a Borel probability measure ν on X and a countable dense set*

$$\Xi = \{\varphi_k : X \to \mathbb{R} \mid k \in \mathbb{N}\} \subseteq C(X;\mathbb{R})$$

of continuous functions. Let $(q_n)_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers and $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous transformations, which converges uniformly to a map f. Moreover, let $(\varepsilon_n)_{n\in\mathbb{N}}$ a decreasing sequence of numbers converging to 0 and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ a sequence of subsets of X with $\sum_{n=1}^{\infty} \nu(X \setminus \mathcal{D}_n) < \infty$. Suppose that for each $k = 1,\dots,n$

$$d^{(q_{n+1})}(\varphi_k \circ f_n, \varphi_k \circ f) := \max_{x\in M} \max_{i=0,\dots,q_{n+1}-1} |\varphi_k(f_n^i(x)) - \varphi_k(f^i(x))| < \varepsilon_n \quad (6.1)$$

and

$$\left| \frac{1}{q_{n+1}} \sum_{j=0}^{q_{n+1}-1} \varphi_k(f_n^j(x)) - \int \varphi_k \, \mathrm{d}\nu \right| < \varepsilon_n \quad \textit{for every } x \in \mathcal{D}_n. \quad (6.2)$$

Then f is ergodic with respect to ν.

Since every continuous function on the compact metric space is uniformly continuous, we can fulfill requirement (6.1) if f and f_n are sufficiently close to each other.

Proof. By our assumption (6.1) we get:

$$\left\| \frac{1}{q_{n+1}} \sum_{j=0}^{q_{n+1}-1} \varphi_k \circ f^j - \frac{1}{q_{n+1}} \sum_{j=0}^{q_{n+1}-1} \varphi_k \circ f_n^j \right\|_0 < \varepsilon_n$$

for $k = 1, \dots, n$. Hence,

$$\left\| \frac{1}{q_{n+1}} \sum_{j=0}^{q_{n+1}-1} \varphi_k(f^j(x)) - \int \varphi_k \, \mathrm{d}\nu \right\| < 2\varepsilon_n$$

for every $x \in \mathcal{D}_n$ by assumption (6.2). By the Borel–Cantelli lemma

$$\nu\Big(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} X \setminus \mathcal{D}_k \Big) = 0.$$

Thus we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \varphi(f^i(x)) = \int_X \varphi \, \mathrm{d}\nu \text{ for } \nu\text{-almost every } x \in X$$

for every $\varphi \in \Xi$. By an approximation argument this equality holds true for every $\varphi \in C(X;\mathbb{R})$. □

Hereby, we deduce the following criterion for the ergodicity of the projectivized derivative extension.

Proposition 6.2 (Criterion for ergodicity of the projectivized derivative extension). *We consider a sequence of diffemorphisms $(f_n)_{n\in\mathbb{N}}$ constructed as above converging to $f = \lim_{n\to\infty} f_n$ in the C^∞-topology and its projectivized derivative extension $(f, \mathrm{d}f)$ on $\mathbb{P}TM$ with invariant measure $\bar{\mu}$. Let*

$$\Xi = \{\varphi_k : \mathbb{P}TM \to \mathbb{R} \mid k \in \mathbb{N}\} \subseteq C(\mathbb{P}TM;\mathbb{R})$$

be a countable dense set of continuous functions, $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing sequence of numbers converging to 0 and $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{P}TM$ with

$$\sum_{n=1}^{\infty} \bar{\mu}(\mathbb{P}TM \setminus \mathcal{D}_n) < \infty.$$

Suppose that for each $k = 1, \dots, n$

$$\max_{x\in\mathbb{P}TM} \max_{i=0,\dots,q_{n+1}-1} \left| \varphi_k((f_n^i, \mathrm{d}f_n^i)(x)) - \varphi_k((f^i, \mathrm{d}f^i)(x)) \right| < \varepsilon_n \tag{6.3}$$

and

$$\left| \frac{1}{q_{n+1}} \sum_{j=0}^{q_{n+1}-1} \varphi_k((f_n^j, \mathrm{d}f_n^j)(x)) - \int \varphi_k \, \mathrm{d}\bar{\mu} \right| < \varepsilon_n \quad \text{for every } x \in \mathcal{D}_n. \tag{6.4}$$

Then the projectivized derivative extension $(f, \mathrm{d}f)$ is ergodic with respect to $\bar{\mu}$.

Proof. This Proposition is Lemma 6.1 stated in the setting of our constructions. □

7. Proof of ergodicity of the derivative extension

In order to apply our criterion for ergodicity of the projectivized derivative extension in Proposition 6.2, we have to estimate the Birkhoff sums

$$\frac{1}{q_{n+1}} \sum_{j=0}^{q_{n+1}-1} \varphi_k((f_n^j, \mathrm{d}f_n^j)(x))$$

for an increasing set of $x \in \mathbb{P}TM$. For this purpose, we introduce the following "target sets" and "trapping regions."

7.1. Collection of targets sets. The collection $\mathcal{U}_n$ of "target sets" consists of all sets

$$\Delta_{t_1,t_2,t_3} = \left[\frac{t_1}{k_n \cdot q_n}, \frac{t_1+1}{k_n \cdot q_n}\right] \times \left[\frac{t_2}{k_n}, \frac{t_2+1}{k_n}\right] \times \left[\frac{t_3}{k_n}, \frac{t_3+1}{k_n}\right]$$

in $\mathbb{P}TM$ for $t_1 \in \mathbb{Z}$, $t_2 \in \{1, \ldots, k_n - 2\}$ and $t_3 \in \{0, 1, \ldots, k_n - 1\}$. We denote the union of target sets by U_n and note

$$\bar{\mu}(\mathbb{P}TM \setminus U_n) \leq \frac{2}{k_n}. \tag{7.1}$$

Remark 7.1. By condition (C) we have

$$\operatorname{diam}((H_{n-1}, \mathrm{d}H_{n-1})(\Delta)) < \frac{1}{n^2 \cdot \max_{i=1,\ldots,n} \|\rho_i\|_{\mathrm{Lip}}}$$

for every $\Delta \in \mathcal{U}_n$

7.2. Collection of trapping regions. In the next step, we introduce the family $\mathcal{T}_n$ of trapping regions

$$\begin{aligned}
&T_{u_0,u_1,u_2,u_3,u_4;v_1,v_2,v_3,v_4;j} \\
&\quad = I_{u_0,u_1,u_2,u_3,u_4;v_1,v_2,v_3,v_4} \times \left[\frac{j}{k_n}, \frac{j+1}{k_n}\right) \\
&\quad = \left[\frac{u_0}{k_n \cdot q_n} + \frac{u_1}{k_n^3 \cdot q_n} + \frac{u_2}{k_n^5 \cdot q_n} + \frac{u_3}{k_n^6 \cdot q_n} + \frac{u_4}{k_n^8 \cdot q_n} + \frac{1}{k_n^{10} \cdot q_n},\right. \\
&\qquad \left. \frac{u_0}{k_n \cdot q_n} + \frac{u_1}{k_n^3 \cdot q_n} + \frac{u_2}{k_n^5 \cdot q_n} + \frac{u_3}{k_n^6 \cdot q_n} + \frac{u_4+1}{k_n^8 \cdot q_n} - \frac{1}{k_n^{10} \cdot q_n}\right] \\
&\qquad \times \left[\frac{v_1}{k_n^2} + \frac{v_2}{k_n^5 \cdot q_n} + \frac{v_3}{k_n^6 \cdot q_n} + \frac{v_4}{k_n^8 \cdot q_n} + \frac{1}{k_n^{10} \cdot q_n},\right. \\
&\qquad \left. \frac{v_1}{k_n^2} + \frac{v_2}{k_n^5 \cdot q_n} + \frac{v_3}{k_n^6 \cdot q_n} + \frac{v_4+1}{k_n^8 \cdot q_n} - \frac{1}{k_n^{10} \cdot q_n}\right] \times \left[\frac{j}{k_n}, \frac{j+1}{k_n}\right)
\end{aligned}$$

in $\mathbb{P}TM$, where

$$u_0 \in \mathbb{Z}, \quad u_1, u_2, u_4, v_1, v_4 \in \{1, \dots, k_n^2 - 2\}, \quad u_3, v_3 \in \{0, 1, \dots, k_n - 1\},$$
$$\text{and} \quad v_2 \in \{k_n q_n, k_n q_n + 1, \dots, k_n^3 q_n - k_n q_n - 1\}.$$

We note that the M-factor $I_{u_0,u_1,u_2,u_3,u_4;v_1,v_2,v_3,v_4}$ belongs to the "good domain" of the conjugation map h_n for any $T_{u_0,u_1,u_2,u_3,u_4;v_1,v_2,v_3,v_4;j} \in \mathcal{T}_n$. Hence, we can describe the mapping behaviour of the projectivized derivative extension $(h_n, \mathrm{d}h_n)$ on the "trapping regions" explicitly.

Lemma 7.2. *For any $T_{u_0,u_1,u_2,u_3,u_4;v_1,v_2,v_3,v_4;j} \in \mathcal{T}_n$ we have*

$$(h_n, \mathrm{d}h_n)(T_{u_0,u_1,u_2,u_3,u_4;v_1,v_2,v_3,v_4;j}) \subset \Delta_{u_0, \lfloor \frac{u_1}{k_n} \rfloor, (j+u_3) \bmod k_n}.$$

In particular, a strip $\bigcup_{v_1,v_2,v_3,v_4} I_{u_0,u_1,u_2,u_3,u_4;v_1,v_2,v_3,v_4}$ of almost full vertical length is mapped to a set of small diameter under h_n.

Proof. In the proof of Lemma 5.2 we computed the mapping behaviour of ϕ_n on $I_{u_0,u_1,u_2,u_3,u_4;v_1,v_2,v_3,v_4}$. In addition to this we note that $\mathrm{d}_p\phi_n = \mathrm{id}$ for base points $p \in I_{u_0,u_1,u_2,u_3,u_4;v_1,v_2,v_3}$. Altogether we get

$$(\phi_n, \mathrm{d}\phi_n)(T_{u_0,u_1,u_2,u_3,u_4;v_1,v_2,v_3,v_4;j}) = T_{u_0,k_n^2-v_1-1,u_2,u_3,u_4;u_1,v_2,v_3,v_4;j}.$$

By the second statement in Proposition 2.2 i_n is a composition of a translation and a rotation by $\frac{u_3 \cdot \pi}{k_n}$ on $I_{u_0,k_n^2-v_1-1,u_2,u_3,u_4;u_1,v_2,v_3,v_4}$. Moreover, the first statement of Proposition 2.2 yields that the image of $I_{u_0,k_n^2-v_1-1,u_2,u_3,u_4;u_1,v_2,v_3,v_4}$ under i_n stays contained in

$$\left[\frac{u_0+1}{k_n q_n} - \frac{v_1+1}{k_n^3 \cdot q_n} + \frac{u_2}{k_n^5 \cdot q_n} + \frac{u_3}{k_n^6 \cdot q_n}, \frac{u_0+1}{k_n q_n} - \frac{v_1+1}{k_n^3 \cdot q_n} + \frac{u_2}{k_n^5 \cdot q_n} + \frac{u_3+1}{k_n^6 \cdot q_n}\right]$$
$$\times \left[\frac{u_1}{k_n^2} + \frac{v_2}{k_n^5 \cdot q_n} + \frac{v_3}{k_n^6 \cdot q_n}, \frac{u_1}{k_n^2} + \frac{v_2}{k_n^5 \cdot q_n} + \frac{v_3}{k_n^6 \cdot q_n}\right].$$

Hence, we conclude for $h_n = i_n \circ \phi_n$:

$$(h_n, \mathrm{d}h_n)(T_{u_0,u_1,u_2,u_3,u_4;v_1,v_2,v_3,v_4;j}) \subset \Delta_{u_0, \lfloor \frac{u_1}{k_n} \rfloor, (j+u_3) \bmod k_n}. \qquad \square$$

With the aid of this understanding of the mapping behaviour under $(h_n, \mathrm{d}h_n)$ we can prove the following "trapping property."

Lemma 7.3. *Let $(\theta, r, v) \in \mathbb{P}TM$ with $r \in G_{n,v}$ and $\Delta_{t_1,t_2,t_3} \in \mathcal{U}_n$ be arbitrary. Then at least $\left(1 - \frac{3}{k_n^2}\right)^3 \cdot \frac{q_{n+1}}{k_n^3 \cdot q_n}$ and at most $\frac{q_{n+1}}{k_n^3 \cdot q_n}$ many of the iterates*

$$\left(h_n \circ R^i_{\alpha_{n+1}}, \mathrm{d}(h_n \circ R^i_{\alpha_{n+1}})\right)(\theta, r, v),$$

$0 \le i < q_{n+1}$, lie in Δ_{t_1,t_2,t_3}.

Proof. Let

$$r \in \left[\frac{v_1}{k_n^2} + \frac{v_2}{k_n^5 \cdot q_n} + \frac{v_3}{k_n^6 \cdot q_n} + \frac{v_4}{k_n^8 \cdot q_n} + \frac{1}{k_n^{10} \cdot q_n}, \right.$$
$$\left. \frac{v_1}{k_n^2} + \frac{v_2}{k_n^5 \cdot q_n} + \frac{v_3}{k_n^6 \cdot q_n} + \frac{v_4+1}{k_n^8 \cdot q_n} - \frac{1}{k_n^{10} \cdot q_n}\right]$$

and

$$v \in \left[\frac{j}{k_n}, \frac{j+1}{k_n}\right),$$

where $j \in \mathbb{Z}$, $0 \leq j < k_n$. We choose $u \in \{0, \dots, k_n - 1\}$ such that $j + u \equiv t_3 \bmod k_n$. By Lemma 7.2 only the trapping regions $T_{t_1,u_1,u_2,u,u_4;v_1,v_2,v_3,v_4;j}$ with $t_2 k_n \leq u_1 < (t_2+1)k_n$ (for all allowed values $u_2, u_4 \in \{1, \dots, k_n^2 - 2\}$) are mapped into Δ_{t_1,t_2,t_3} under $(h_n, \mathrm{d}h_n)$. Since the orbit $\{\theta + i \cdot \alpha_{n+1}\}_{i=0,\dots,q_{n+1}-1}$ is equidistributed on $\mathbb{S}^1$, there are at least

$$\left\lfloor \left(1 - \frac{2}{k_n^2}\right) \cdot \frac{1}{k_n^8 \cdot q_n} \cdot q_{n+1} \right\rfloor$$

and at most

$$\left\lfloor \frac{q_{n+1}}{k_n^8 \cdot q_n} \right\rfloor$$

many points of the orbit $\{R^i_{\alpha_{n+1}}(\theta, r)\}_{i=0,\dots,q_{n+1}-1}$ contained in a set of the form $I_{t_1,u_1,u_2,u,u_4;v_1,v_2,v_3,v_4}$. Hence, there are at least

$$k_n \cdot (k_n^2 - 2)^2 \cdot \left\lfloor \left(1 - \frac{2}{k_n^2}\right) \cdot \frac{1}{k_n^8 \cdot q_n} \cdot q_{n+1} \right\rfloor$$

and at most

$$k_n \cdot (k_n^2 - 2)^2 \cdot \left\lfloor \frac{q_{n+1}}{k_n^8 \cdot q_n} \right\rfloor$$

many iterates $(h_n \circ R^i_{\alpha_{n+1}}, \mathrm{d}(h_n \circ R^i_{\alpha_{n+1}}))(\theta, r, v)$, $0 \leq i < q_{n+1}$, in Δ_{t_1,t_2,t_3}. □

Remark 7.4. For any point $x = (\theta, r) \in M$ with $r \in G_{n,v}$ there are at most $\frac{9}{k_n^2} \cdot q_{n+1}$ many iterates $R^i_{\alpha_{n+1}}(x)$, $0 \leq i < q_{n+1}$, that are not contained in the "good domain" of h_n, i. e. in one one of the trapping regions, by Remark 5.3.

7.3. Estimates on Birkoff sums. Using the notation from Section 6 we introduce the sets

$$\mathcal{D}_n = \mathbb{S}^1 \times G_{n,v} \times [0, 1)$$

in $\mathcal{P}TM$. By Remark 5.3 we have $\overline{\mu}(\mathcal{D}_n) \geq 1 - \frac{8}{k_n^2}$. With the aid of the previous "trapping properties" we obtain the following estimate on Birkhoff sums for points in $\mathcal{D}_n$ and observables in our chosen family Ξ of Lipschitz continuous functions.

Lemma 7.5. *Let $z = (\theta, r, v) \in \mathcal{D}_n$ and $\rho \in \{\rho_1, \dots, \rho_n\} \subset \Xi$. Then we have*

$$\Big| \frac{1}{q_{n+1}} \sum_{j=0}^{q_{n+1}-1} \rho((f_n, \mathrm{d} f_n)^j(z)) - \int_{\mathbb{P}TM} \rho \,\mathrm{d}\bar{\mu} \Big| < \frac{2}{n^2}.$$

Proof. Since $\rho \in \{\rho_1, \dots, \rho_n\} \subset \Xi$ is a Lipschitz continuous function on $\mathbb{P}TM$, we have

$$\begin{aligned} &\big|\rho((H_{n-1}, \mathrm{d}H_{n-1})(z_1)) - \rho((H_{n-1}, \mathrm{d}H_{n-1})(z_2))\big| \\ &\qquad \leq \|\rho\|_{\mathrm{Lip}} \cdot \operatorname{diam}((H_{n-1}, \mathrm{d}H_{n-1})(\Delta_{t_1,t_2,t_3})) < \frac{1}{n^2} \end{aligned}$$

for any $z_1, z_2 \in \Delta_{t_1,t_2,t_3} \in \mathcal{U}_n$ by Remark 7.1. Averaging over all $z_2 \in \Delta_{t_1,t_2,t_3}$ yields

$$\begin{aligned} &\Big|\rho((H_{n-1}, \mathrm{d}H_{n-1})(z_1)) \\ &\qquad - \frac{1}{\bar{\mu}((H_{n-1}, \mathrm{d}H_{n-1})(\Delta_{t_1,t_2,t_3}))} \int_{(H_{n-1},\mathrm{d}H_{n-1})(\Delta_{t_1,t_2,t_3})} \rho \,\mathrm{d}\bar{\mu}\Big| < \frac{1}{n^2}. \end{aligned} \tag{7.2}$$

Let $x \in \mathcal{D}_n$ be arbitrary. In the subsequent estimate we denote the set of iterates $j \in \{0, 1, \dots, q_{n+1} - 1\}$ such that $(h_n \circ R^j_{\alpha_{n+1}}, \mathrm{d}(h_n \circ R^j_{\alpha_{n+1}}))(x)$ is contained in $\Delta \in \mathcal{U}_n$ by I_Δ:

$$\begin{aligned} &\Big|\frac{1}{q_{n+1}} \sum_{j=0}^{q_{n+1}-1} \rho\big((H_{n-1} \circ h_n \circ R^j_{\alpha_{n+1}}, \mathrm{d}(H_{n-1} \circ h_n \circ R^j_{\alpha_{n+1}}))(x)\big) - \int \rho \,\mathrm{d}\bar{\mu}\Big| \\ &\quad = \Big|\frac{1}{q_{n+1}} \sum_{j=0}^{q_{n+1}-1} \rho\big((H_{n-1} \circ h_n \circ R^j_{\alpha_{n+1}}, \mathrm{d}(H_{n-1} \circ h_n \circ R^j_{\alpha_{n+1}}))(x)\big) \\ &\qquad\qquad - \sum_{\Delta \in \mathcal{U}_n} \int_{(H_{n-1},\mathrm{d}H_{n-1})(\Delta)} \rho \,\mathrm{d}\bar{\mu} - \int_{(H_{n-1},\mathrm{d}H_{n-1})(\mathbb{P}TM \setminus U_n)} \rho \,\mathrm{d}\bar{\mu}\Big| \\ &\quad \leq \Big| \sum_{\Delta \in \mathcal{U}_n} \Big(\frac{1}{q_{n+1}} \sum_{j \in I_\Delta} \rho\big((H_{n-1} \circ h_n \circ R^j_{\alpha_{n+1}}, \mathrm{d}(H_{n-1} \circ h_n \circ R^j_{\alpha_{n+1}}))(x)\big) \\ &\qquad\qquad - \int_{(H_{n-1},\mathrm{d}H_{n-1})(\Delta)} \rho \,\mathrm{d}\bar{\mu}\Big)\Big| \\ &\qquad\qquad + 2\bar{\mu}(\mathbb{P}TM \setminus U_n) \cdot \|\rho\|_0 + \frac{\frac{9}{k_n^2} \cdot q_{n+1}}{q_{n+1}} \|\rho\|_0, \end{aligned}$$

where the last summand follows from Remark 7.4. In order to estimate the first summand we exploit Lemma 7.3 and equation (7.2) to get

$$\frac{1}{q_{n+1}} \sum_{j\in I_\Delta} \rho\big((H_{n-1}\circ h_n \circ R^j_{\alpha_{n+1}}, \mathrm{d}(H_{n-1}\circ h_n \circ R^j_{\alpha_{n+1}}))(x)\big)$$
$$< \frac{1}{k_n^3\cdot q_n}\cdot\Big(\frac{1}{\bar\mu((H_{n-1},\mathrm{d}H_{n-1})(\Delta_{t_1,t_2,t_3}))}\int_{(H_{n-1},\mathrm{d}H_{n-1})(\Delta_{t_1,t_2,t_3})}\rho\,\mathrm{d}\bar\mu + \frac{1}{n^2}\Big)$$

on the one hand, and

$$\frac{1}{q_{n+1}} \sum_{j\in I_\Delta} \rho\big((H_{n-1}\circ h_n \circ R^j_{\alpha_{n+1}}, \mathrm{d}(H_{n-1}\circ h_n \circ R^j_{\alpha_{n+1}}))(x)\big)$$
$$> \frac{\big(1-\frac{3}{k_n^2}\big)^3}{k_n^3\cdot q_n}\cdot\Big(\frac{1}{\bar\mu((H_{n-1},\mathrm{d}H_{n-1})(\Delta_{t_1,t_2,t_3}))}\int_{(H_{n-1},\mathrm{d}H_{n-1})(\Delta_{t_1,t_2,t_3})}\rho\,\mathrm{d}\bar\mu - \frac{1}{n^2}\Big)$$

on the other hand. These both estimates yield

$$\Big|\frac{1}{q_{n+1}} \sum_{j\in I_\Delta} \rho\big((H_{n-1}\circ h_n \circ R^j_{\alpha_{n+1}}, \mathrm{d}(H_{n-1}\circ h_n \circ R^j_{\alpha_{n+1}}))(x)\big) - \int_{(H_{n-1},\mathrm{d}H_{n-1})(\Delta)}\rho\,\mathrm{d}\bar\mu\Big|$$
$$< \frac{10}{k_n}\cdot\int_{(H_{n-1},\mathrm{d}H_{n-1})(\Delta)}\rho\,\mathrm{d}\bar\mu + \frac{1}{k_n^3\cdot q_n}\cdot\frac{1}{n^2}.$$

We also recall $\bar\mu(\mathbb{P}TM\setminus U_n)\le \frac{2}{k_n}$ from equation (7.1). Altogether we conclude

$$\Big|\frac{1}{q_{n+1}} \sum_{j=0}^{q_{n+1}-1} \rho\big((H_{n-1}\circ h_n \circ R^j_{\alpha_{n+1}}, \mathrm{d}(H_{n-1}\circ h_n \circ R^j_{\alpha_{n+1}}))(x)\big) - \int\rho\,\mathrm{d}\bar\mu\Big|$$
$$< \frac{10}{k_n}\cdot\|\rho\|_0 + \frac{1}{n^2} + \frac{4}{k_n}\cdot\|\rho\|_0 + \frac{9}{k_n^2}\cdot\|\rho\|_0 < \frac{2}{n^2},$$

using requirement (B) on the number k_n in the last step. With $x=(H_n,\mathrm{d}H_n)^{-1}(z)$ we obtain the statement of the lemma. □

7.4. Application of the criterion. In order to check the requirements of Proposition 6.2 we consider the family $\Xi=\{\rho_1,\rho_2,\dots\}$ of Lipschitz continuous functions $\rho_i:\mathbb{P}TM\to\mathbb{R}$ chosen at the beginning and the sets

$$\mathcal{D}_n = \mathbb{S}^1\times G_{n,v}\times[0,1)\subset\mathbb{P}TM.$$

Since $\overline{\mu}(\mathcal{D}_n) \geq 1 - \frac{9}{k_n^2}$ we have

$$\sum_{n=1}^{\infty} \overline{\mu}(\mathbb{P}TM \setminus \mathcal{D}_n) < \infty.$$

In our successive construction the requirement (6.3) is fulfilled by Lemma 3.2 and condition (6.4) is satisfied by Lemma 7.5. Hence, we can apply Proposition 6.2 and obtain the ergodicity of the projectivized derivative extension $(f, \mathrm{d}f)$ with respect to the invariant measure $\overline{\mu}$.

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References

[1] D. V. Anosov and A. Katok, New examples in smooth ergodic theory. Ergodic diffeomorphisms, *Trudy Moskov. Mat. Obšč.*, **23** (1970), 3–36. Zbl 0255.58007 MR 370662

[2] M. Benhenda, Nonstandard smooth realization of translations on the torus, *J. Mod. Dyn.*, **7** (2013), no. 3, 329–367. Zbl 1290.37009 MR 3296558

[3] I. Chavel, *Riemannian geometry: a modern introduction*, Cambridge Tracts in Mathematics, 108, Cambridge University Press, Cambridge, 1993. Zbl 0810.53001 MR 1271141

[4] B. Fayad and A. Katok, Constructions in elliptic dynamics, *Ergodic Theory Dynam. Systems*, **24** (2004), no. 5, 1477–1520. Zbl 1089.37012 MR 2104594

[5] B. Fayad and M. Saprykina, Weak mixing disc and annulus diffeomorphisms with arbitrary Liouville rotation number on the boundary, *Ann. Sci. École Norm. Sup. (4)*, **38** (2005), no. 3, 339–364. Zbl 1090.37001 MR 2166337

[6] B. Fayad, M. Saprykina, and A. Windsor, Non-standard smooth realizations of Liouville rotations, *Ergodic Theory Dynam. Systems*, **27** (2007), no. 6, 1803–1818. Zbl 1127.37008 MR 2371596

[7] R. Gunesch and A. Katok, Construction of weakly mixing diffeomorphisms preserving measurable Riemannian metric and smooth measure. With an appendix by Alex Furman, *Discrete Contin. Dynam. Systems*, **6** (2000), no. 1, 61–88. Zbl 1009.37013 MR 1739594

[8] R. Gunesch and P. Kunde, Weakly mixing diffeomorphisms preserving a measurable Riemannian metric with prescribed Liouville rotation behavior, *Discrete Contin. Dyn. Syst.*, **38** (2018), no. 4, 1615–1655. Zbl 1394.37041 MR 3809009

[9] A. Katok, *Combinatorial constructions in ergodic theory and dynamics*, University Lecture Series, 30, American Mathematical Society, Providence, RI, 2003. Zbl 1030.37001 MR 2008435

[10] P. Kunde, Smooth diffeomorphisms with homogeneous spectrum and disjointness of convolutions, *J. Mod. Dyn.*, **10** (2016), 439–481. Zbl 1402.37032 MR 3565927

[11] P. Kunde, Real-analytic weak mixing diffeomorphism preserving a measurable Riemannian metric, *Ergodic Theory Dynam. Systems*, **37** (2017), no. 5, 1547–1569. Zbl 1378.37041 MR 3667999

[12] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, 79, Springer-Verlag, New York-Berlin, 1982. Zbl 0475.28009 MR 648108

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