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Motives of isogenous K3 surfaces

Daniel Huybrechts*

Abstract. We prove that isogenous K3 surfaces have isomorphic Chow motives. This provides a motivic interpretation of a long standing conjecture of Šafarevič which has been settled only recently by Buskin. The main step consists of a new proof of Šafarevič’s conjecture that circumvents the analytic parts in [2], avoiding twistor spaces and non-algebraic K3 surfaces.

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Two complex projective K3 surface S and S' are called *isogenous* if there exists a Hodge isometry $\varphi: H^2(S, \mathbb{Q}) \xrightarrow{\sim} H^2(S', \mathbb{Q})$, i.e. an isomorphism of \mathbb{Q} -vector spaces compatible with the intersection pairing as well as the Hodge structure on both sides. Via Poincaré duality and Künneth formula, φ corresponds to a Hodge class $[\varphi] \in H^{2,2}(S \times S', \mathbb{Q})$ on the product $S \times S'$ of the two surfaces.

In [19] Šafarevič asked whether any such $[\varphi]$ is algebraic, i.e. of the form $[\varphi] = \sum n_i [Z_i]$ for certain surfaces $Z_i \subset S \times S'$ and rational numbers n_i . Forty years later this was answered affirmatively by Buskin [2]. The result confirms the Hodge conjecture in a geometrically interesting situation and can be viewed as a generalization of the global Torelli theorem for K3 surfaces.

Indeed, the global Torelli theorem for K3 surfaces asserts that any effective integral Hodge isometry $\varphi: H^2(S, \mathbb{Z}) \xrightarrow{\sim} H^2(S', \mathbb{Z})$ can be lifted to an isomorphism $f: S \xrightarrow{\sim} S'$ and so $[\varphi] = [\Gamma_f]$, which is algebraic. Note that the global Torelli theorem not only answers Šafarevič’s question for (effective) integral Hodge isometries, it also provides a motivic reason for the class $[\varphi]$ being algebraic, namely that it is induced by an isomorphism between S and S' .

Examples of rational Hodge isometries can be produced by means of moduli spaces of sheaves, often leading to non-isomorphic but isogenous K3 surfaces.

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Assume $S' = M(v)$ is a fine moduli space of stable sheaves on S . Then the universal family \mathcal{E} on $S \times S'$, an analogue of the Poincaré bundle for abelian varieties, provides a class $\mathrm{ch}_2(\mathcal{E}) \in H^{2,2}(S \times S', \mathbb{Q})$. As shown by Mukai [17], a minor modification of this class yields indeed a Hodge isometry $H^2(S, \mathbb{Q}) \simeq H^2(S', \mathbb{Q})$. In fact, it defines an integral Hodge isometry $T(S) \simeq T(S')$ between the transcendental lattices of the two surfaces and a rational isometry $\mathrm{NS}(S) \otimes \mathbb{Q} \simeq \mathrm{NS}(S') \otimes \mathbb{Q}$ between their algebraic parts. The motivic nature of the rational Hodge isometry, beyond being induced by a universal sheaf, has been explained in [11]: For any fine moduli space $S' = M(v)$, the induced Hodge isometry $H^2(S, \mathbb{Q}) \simeq H^2(S', \mathbb{Q})$ can be lifted to an isomorphism $\mathfrak{h}(S) \simeq \mathfrak{h}(S')$ between the Chow motives of S and S' .

Mukai also constructs in [17] further classes that yield non-integral Hodge isometries between the transcendental parts $T(S) \otimes \mathbb{Q} \simeq T(S') \otimes \mathbb{Q}$ by allowing coarse moduli spaces, i.e. moduli spaces for which only a quasi-universal or a twisted universal family \mathcal{E} exists. This approach has led to the verification of Šafarevič's conjecture for Picard rank $\rho(S) \geq 5$, see [17, 18] and Remark 1.5.

Our first main result provides a moduli interpretation of isogenies between K3 surfaces:

Theorem 0.1. *Any Hodge isometry $H^2(S, \mathbb{Q}) \simeq H^2(S', \mathbb{Q})$ between two complex projective K3 surfaces can be written as a composition of Hodge isometries between projective K3 surfaces*

$$H^2(S = S_0, \mathbb{Q}) \simeq H^2(S_1, \mathbb{Q}) \simeq \cdots \simeq H^2(S_{n-1}, \mathbb{Q}) \simeq H^2(S_n = S', \mathbb{Q}),$$

with S_i isomorphic to a coarse moduli space of complexes of twisted coherent sheaves on S_{i-1} and the Hodge isometry $H^2(S_{i-1}, \mathbb{Q}) \simeq H^2(S_i, \mathbb{Q})$ induced (up to sign) by a twisted universal family \mathcal{E} of complexes of twisted sheaves on S_{i-1} .

In the language of derived categories, the result says that there exist Brauer classes $\alpha \in \mathrm{Br}(S)$, $\alpha_1, \beta_1 \in \mathrm{Br}(S)$, \dots , $\alpha_{n-1}, \beta_{n-1} \in \mathrm{Br}(S_{n-1})$, and $\alpha' \in \mathrm{Br}(S')$ and exact linear equivalences between bounded derived categories of twisted coherent sheaves

$$\begin{aligned} \mathrm{D}^b(S, \alpha) &\simeq \mathrm{D}^b(S_1, \alpha_1), \\ \mathrm{D}^b(S_1, \beta_1) &\simeq \mathrm{D}^b(S_2, \alpha_2), \\ &\vdots \\ \mathrm{D}^b(S_{n-2}, \beta_{n-2}) &\simeq \mathrm{D}^b(S_{n-1}, \alpha_{n-1}), \\ &\mathrm{D}^b(S_{n-1}, \beta_{n-1}) \simeq \mathrm{D}^b(S', \alpha'). \end{aligned} \tag{0.1}$$

This usually does not mean that $\mathrm{D}^b(S, \alpha)$ and $\mathrm{D}^b(S', \alpha')$ are equivalent for appropriated choices of $\alpha \in \mathrm{Br}(S)$ and $\alpha' \in \mathrm{Br}(S')$, see Remark 1.3. It should not be too difficult to improve Theorem 0.1 such that the S_i are moduli spaces of twisted sheaves (and not complexes of those).

Combining Theorem 0.1 with the arguments in [11] generalized to the twisted case, one deduces a motivic interpretation of the notion of isogeneous K3 surfaces:

Theorem 0.2 (Motivic Šafarevič conjecture). *Any Hodge isometry $H^2(S, \mathbb{Q}) \simeq H^2(S', \mathbb{Q})$ between two complex projective K3 surfaces can be lifted to an isomorphism of Chow motives $\mathfrak{h}(S) \simeq \mathfrak{h}(S')$. In particular, two isogenous K3 surfaces have isomorphic Chow motives:*

$$H^2(S, \mathbb{Q}) \simeq H^2(S', \mathbb{Q}) \text{ (Hodge isometry)} \Rightarrow \mathfrak{h}(S) \simeq \mathfrak{h}(S').$$

Note that by Witt's theorem, there exists a Hodge isometry $H^2(S, \mathbb{Q}) \simeq H^2(S', \mathbb{Q})$ if and only if there exists a Hodge isometry $T(S) \otimes \mathbb{Q} \simeq T(S') \otimes \mathbb{Q}$. For integral coefficients this fails, which results in two global Torelli theorems, the classical and the derived,¹ see [8–10] for references.

The following strengthening of Theorem 0.2 is expected. It relaxes the assumption from the existence of a Hodge isometry to the existence of a simple isomorphism of Hodge structures, so one that is not necessarily compatible with the intersection pairing (cf. Section 3):

Conjecture 0.3 (Motivic global Torelli theorem). *For two complex projective K3 surfaces S and S' the following conditions are equivalent:*

- (i) $H^2(S, \mathbb{Q}) \simeq H^2(S', \mathbb{Q})$ (isomorphism of rational Hodge structures);
- (ii) $\mathfrak{h}(S) \simeq \mathfrak{h}(S')$ (isomorphism of Chow motives).

Theorem 0.1 has the following immediate consequences first proved in [2], see Proposition 1.2 and Remark 3.3.

Corollary 0.4 (Buskin). (i) *Any Hodge isometry*

$$\varphi: H^2(S, \mathbb{Q}) \xrightarrow{\sim} H^2(S', \mathbb{Q})$$

between complex projective K3 surfaces yields an algebraic class

$$[\varphi] \in H^{2,2}(S \times S', \mathbb{Q}).$$

- (ii) *If S is a complex projective K3 surface with complex multiplication, i.e. $\text{End}_{\text{Hdg}}(T(S) \otimes \mathbb{Q})$ is a CM-field, then the Hodge conjecture holds for $S \times S$, cf. Remark 3.3.*

The approach to Šafarevič's conjecture presented here differs from the one in [2]. It is more algebraic in spirit, which allows for the motivic interpretation of the conjecture as presented in Theorem 0.2. Central to his argument, Buskin proves “twistor path connectedness” of the moduli space of pairs of K3 surfaces together with an isogeny between them to reduce the situation to coarse moduli spaces of untwisted bundles. In our proof, cyclic isogenies are lifted directly to the level of

¹ (i) $H^2(S, \mathbb{Z}) \simeq H^2(S', \mathbb{Z})$ (Hodge isometry) $\Leftrightarrow S \simeq S'$ (isomorphism);

(ii) $T(S) \simeq T(S')$ (Hodge isometry) $\Leftrightarrow D^b(S) \simeq D^b(S')$ (exact linear equivalence).

derived categories of twisted K3 surfaces [13], thus avoiding analytic K3 surfaces and global moduli considerations. The notion of Hodge structures of twisted K3 surfaces introduced in [7, 12] provides an efficient tool to deal with the lattice theoretic parts and, in particular, replaces Buskin's κ -classes.

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1. Derived equivalence of isogenous K3 surfaces

This section is the technical heart of the paper. We show how to lift rational Hodge isometries to exact linear equivalences between bounded derived categories of twisted sheaves and use this to prove Šafarevič's conjecture. The first reduction step to cyclic Hodge isometries is taken from [2]. The rest of the argument uses twisted Chern characters and the main result of [13], instead of κ -classes and twistor space deformations. A brief comparison of the two approaches is included.

1.1. As in [2], we apply the classical Cartan–Dieudonné theorem to reduce Šafarevič's conjecture to an easier case. Recall that for any lattice Λ and any rational isometry $\varphi: \Lambda_{\mathbb{Q}} \xrightarrow{\sim} \Lambda_{\mathbb{Q}}$, there exist $b_i \in \Lambda_{\mathbb{Q}}, i = 1, \dots, k$, with $(b_i)^2 \neq 0$, such that φ equals the composition

$$\varphi = s_{b_1} \circ \dots \circ s_{b_k}$$

of reflections $s_{b_i}: x \mapsto x - \frac{2(x, b_i)}{(b_i)^2} b_i$. Note that the number of reflections can be bounded by $k \leq \text{rk } \Lambda$. Clearly, we may assume that all $b_i \in \Lambda_{\mathbb{Q}}$ are contained in the lattice Λ and that they are actually primitive elements of Λ .²

Combining this with the surjectivity of the period map, one finds that any Hodge isometry $H^2(S, \mathbb{Q}) \xrightarrow{\sim} H^2(S', \mathbb{Q})$ can be written as a composition of Hodge isometries

$$H^2(S = S_0, \mathbb{Q}) \xrightarrow{\sim} H^2(S_1, \mathbb{Q}) \xrightarrow{\sim} \dots \xrightarrow{\sim} H^2(S_n = S', \mathbb{Q}), \quad (1.1)$$

such that after choosing markings $\Lambda \simeq H^2(S_i, \mathbb{Z})$ and $\Lambda \simeq H^2(S_{i+1}, \mathbb{Z})$ the Hodge isometry $H^2(S_i, \mathbb{Q}) \xrightarrow{\sim} H^2(S_{i+1}, \mathbb{Q})$ is of the form s_{b_i} . We call a Hodge isometry

²In [2], Buskin only uses the property of a reflection $\varphi = s_{\delta}$ to be *cyclic*, i.e. to have the property that $\varphi^{-1}(\Lambda) \cap \Lambda \subset \Lambda$ and $\varphi(\Lambda) \cap \Lambda \subset \Lambda$ have cyclic quotients. We shall really have to work with reflections.

of this type *reflective*. Thus, Theorem 0.1 is a consequence of the following result which will be proved in this section.

Theorem 1.1. *Assume $\varphi: H^2(S, \mathbb{Q}) \xrightarrow{\sim} H^2(S', \mathbb{Q})$ is a reflective Hodge isometry. Then S' is a coarse moduli space of complexes of twisted coherent sheaves on S and φ is (up to sign) induced by a twisted universal family of complexes of twisted sheaves.*

In other words, we claim that φ (up to sign) is induced by the Fourier–Mukai kernel \mathcal{E} of an exact linear equivalence $\Phi_{\mathcal{E}}: D^b(S, \alpha) \xrightarrow{\sim} D^b(S', \alpha')$ for suitable Brauer classes $\alpha \in \text{Br}(S)$ and $\alpha' \in \text{Br}(S')$. Here, \mathcal{E} is an object in the bounded derived category $D^b(S \times S', \alpha^{-1} \boxtimes \alpha')$ of $\alpha^{-1} \boxtimes \alpha'$ -twisted coherent sheaves on $S \times S'$, see below for details on the action on cohomology.

1.2. We begin with a few explicit lattice computations. Let $\varphi = s_b: \Lambda_{\mathbb{Q}} \xrightarrow{\sim} \Lambda_{\mathbb{Q}}$ be a reflection with $b \in \Lambda$ primitive. Then, for $x \in \Lambda$, the image $\varphi(x) \in \Lambda_{\mathbb{Q}}$ is contained in Λ if and only if (x, b) is divisible by $n := (b, b)^2/2$. So, if we let $B := \frac{1}{n}b \in \Lambda_{\mathbb{Q}}$, then φ induces an isometry of $\Lambda_B := \{x \in \Lambda \mid (x, B) \in \mathbb{Z}\} \subset \Lambda$. This is a finite index sublattice with a cyclic quotient of order n . Note that $\varphi(z) = z - (z, B)b$ and, hence, $\varphi(B) = -B$. Next consider

$$\exp(B): \Lambda_B \hookrightarrow \tilde{\Lambda} := \Lambda \oplus U, \quad x \mapsto x + (B, x)f, \quad (1.2)$$

which is a primitive embedding of lattices. Here, U is the hyperbolic plane with the standard isotropic basis e, f with $(e, f) = -1$. The sign is inserted to make U naturally isomorphic to $H^0(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}) \simeq \mathbb{Z} \cdot e \oplus \mathbb{Z} \cdot f$ endowed with the Mukai pairing. The orthogonal complement $(\exp(B)(\Lambda_B))^{\perp} \subset \tilde{\Lambda}$ is the lattice spanned by the isotropic vectors $b + ne + f$ and $-f$, which is thus isomorphic to the twisted hyperbolic plane $U(n)$. The isometry $\varphi: \Lambda_B \xrightarrow{\sim} \Lambda_B$ extends to an isometry $\tilde{\varphi}: \tilde{\Lambda} \xrightarrow{\sim} \tilde{\Lambda}$, i.e. there exists a commutative diagram of the form

$$\begin{array}{ccc} \Lambda_B & \xrightarrow{\exp(B)} & \tilde{\Lambda} \\ \varphi \downarrow \simeq & & \tilde{\varphi} \downarrow \simeq \\ \Lambda_B & \xrightarrow{\exp(B)} & \tilde{\Lambda}. \end{array} \quad (1.3)$$

The extension can be given explicitly as

$$\tilde{\varphi}(z + re + sf) := \varphi(z) + ((B, z) - s)b + n((B, z) - (r/n) - s)e - sf.$$

The compatibility with φ is easily shown using $(B, z) = -(B, \varphi(z))$. On $(\exp(B)(\Lambda_B))^{\perp} \simeq U(n)$, $\tilde{\varphi}$ interchanges the two basis vectors $b + ne + f$ and $-f$. This shows that $\tilde{\varphi}$ is indeed an isometry. An explicit computation shows that indeed $\tilde{\varphi}(\tilde{\Lambda}) = \tilde{\Lambda}$.

Let us now apply this to a reflective Hodge isometry $\varphi: H^2(S, \mathbb{Q}) \xrightarrow{\sim} H^2(S', \mathbb{Q})$. The analogue of $b \in \Lambda$ and $B \in \Lambda_{\mathbb{Q}}$ in the above setting are now classes $b \in H^2(S, \mathbb{Z})$ and $B = (1/n)b \in H^2(S, \mathbb{Q})$. We set $b' := -\varphi(b) \in H^2(S', \mathbb{Z})$ and $B' := -\varphi(B) \in H^2(S', \mathbb{Q})$. Then the Hodge isometry φ of rational Hodge structures induces a Hodge isometry of integral Hodge structures

$$\begin{aligned} \varphi: H^2(S, \mathbb{Z})_B &:= \{x \in H^2(S, \mathbb{Z}) \mid (x, B) \in \mathbb{Z}\} \\ &\xrightarrow{\sim} H^2(S', \mathbb{Z})_{B'} := \{x' \in H^2(S', \mathbb{Z}) \mid (x', B') \in \mathbb{Z}\}. \end{aligned}$$

Furthermore, (1.2) becomes the primitive embedding of lattices

$$\exp(B): H^2(S, \mathbb{Z})_B \hookrightarrow \tilde{H}(S, \mathbb{Z}), \quad x \mapsto x + x \wedge B,$$

where $\tilde{H}(S, \mathbb{Z})$ is the Mukai lattice, i.e. the lattice $H^*(S, \mathbb{Z})$ with a sign change in the pairing of H^0 and H^4 . The analogue of (1.3) is the commutative diagram

$$\begin{array}{ccc} H^2(S, \mathbb{Z})_B & \xhookrightarrow{\exp(B)} & \tilde{H}(S, \mathbb{Z}) \\ \varphi \downarrow \simeq & & \tilde{\varphi} \downarrow \simeq \\ H^2(S', \mathbb{Z})_{B'} & \xhookrightarrow{\exp(B')} & \tilde{H}(S', \mathbb{Z}). \end{array} \quad (1.4)$$

with $\tilde{\varphi}(r, z, s) = (n((B, z) - (r/n) - s), \varphi(z) + ((B, z) - s)b', -s)$.

The Hodge structure of $H^2(S, \mathbb{Z})_B$, inherited from $H^2(S, \mathbb{Z})$, induces a natural Hodge structure of weight two on the Mukai lattice $\tilde{H}(S, \mathbb{Z})$. The lattice $\tilde{H}(S, \mathbb{Z})$ endowed with this Hodge structure is denoted $\tilde{H}(S, B, \mathbb{Z})$. Explicitly, the $(2, 0)$ -part of $\tilde{H}(S, B, \mathbb{Z})$ is spanned by

$$\sigma + \sigma \wedge B \in H^2(S, \mathbb{C}) \oplus H^4(S, \mathbb{C})$$

for any $0 \neq \sigma \in H^{2,0}(S) \subset H^2(S, \mathbb{C})$ and the orthogonal complement $(\exp(B)(H^2(S, \mathbb{Z})_B))^{\perp} \subset \tilde{H}(S, B, \mathbb{Z})$ is of type $(1, 1)$. With the analogous convention for S' , the isometry $\tilde{\varphi}$ can be viewed as a Hodge isometry

$$\tilde{\varphi}: \tilde{H}(S, B, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(S', B', \mathbb{Z}) \quad (1.5)$$

that commutes with φ via $\exp(B)$ and $\exp(B')$.

If $\tilde{\varphi}$ does not preserve the natural orientation of the four positive directions in the Mukai lattice, then compose $\tilde{\varphi}$ with the Hodge isometry

$$\text{id}_{H^0} \oplus (-\text{id}_{H^2}) \oplus \text{id}_{H^4}: \tilde{H}(S', B', \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(S', -B', \mathbb{Z}).$$

This amounts to changing φ by a sign which does not affect our problem.

1.3. We are now ready to evoke the main result of [13] which asserts that any orientation preserving Hodge isometry (1.5) can be lifted to an exact equivalence

$$\Phi: D^b(S, \alpha) \xrightarrow{\sim} D^b(S', \alpha'). \quad (1.6)$$

Here, $\alpha \in \text{Br}(S)$ and $\alpha' \in \text{Br}(S')$ are the Brauer classes induced by B and B' via the exponential sequence

$$H^2(S, \mathbb{Q}) \longrightarrow H^2(S, \mathcal{O}_S) \longrightarrow H^2(S, \mathcal{O}_S^*).$$

The order of both classes divides n . However, although $H \subset H^2(S, \mathbb{Z})$ and $H' \subset H^2(S', \mathbb{Z})$ are subgroups of the same index n , in general $\text{ord}(\alpha) \neq \text{ord}(\alpha')$, e.g. for S' a non-fine moduli space of untwisted sheaves one has $\alpha = 1$ and $\text{ord}(\alpha') > 1$. Let us briefly recall what it means that “ Φ lifts $\tilde{\varphi}$ ” and what it implies for φ .

One knows that any exact linear equivalence (1.6) is of Fourier–Mukai type [3], i.e. of the form $\Phi \simeq \Phi_{\mathcal{E}}: E \mapsto p_*(q^*E \otimes \mathcal{E})$ for some \mathcal{E} in the bounded derived category $D^b(S \times S', \alpha^{-1} \boxtimes \alpha')$ of $\alpha^{-1} \boxtimes \alpha'$ -twisted coherent sheaves on $S \times S'$ and p, q the two projections. The induced action

$$\Phi_{\mathcal{E}}^{B, B'}: \tilde{H}(S, B, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(S', B', \mathbb{Z})$$

is the correspondence given by the class $\text{ch}^{-B, B'}(\mathcal{E}) \cdot \sqrt{\text{td}(S \times S')}$, where the twisted Chern character is determined by the property $\text{ch}^{-B, B'}(\mathcal{E})^n = \exp(-b, b') \cdot \text{ch}(\mathcal{E}^{\otimes n})$. As $b = nB$ and $b' = nB'$ are both integral classes, $\mathcal{E}^{\otimes n}$ is naturally untwisted and its Chern character is well defined.³

The fact that $\Phi \simeq \Phi_{\mathcal{E}}$ lifts $\tilde{\varphi}$ by definition simply means that $\Phi_{\mathcal{E}}^{B, B'} = \tilde{\varphi}$ and the commutativity of (1.4) becomes

$$\varphi(x) = \left(\sqrt[n]{\text{ch}(\mathcal{E}^{\otimes n})} \cdot \sqrt{\text{td}(S \times S')} \right)_*(x),$$

cf. Section 2.1. In other words,

$$[\varphi] = \left(\sqrt[n]{\text{ch}(\mathcal{E}^{\otimes n})} \cdot \sqrt{\text{td}(S \times S')} \right)_{(2,2)} \in H^2(S, \mathbb{Q}) \otimes H^2(S', \mathbb{Q}),$$

which is clearly an algebraic class.

The discussion above is summarized by the following reformulation of Theorem 1.1, also proving Corollary 0.4 (i).

Proposition 1.2. *Assume $\varphi: H^2(S, \mathbb{Q}) \xrightarrow{\sim} H^2(S', \mathbb{Q})$ is a cyclic Hodge isometry. Then there exists an exact equivalence $\Phi: D^b(S, \alpha) \xrightarrow{\sim} D^b(S', \alpha')$ which induces φ (up to sign) in the above sense. In particular, $[\varphi]$ is an algebraic class. \square*

³We refer to [9, 12, 13] and Section 2.1 for the technical details. For example, one actually has to choose cocycles $b = \{b_{ijk}\}$, $B = \{B_{ijk} := (1/n)b_{ijk}\}$, and $\alpha = \{\alpha_{ijk} = \exp(B_{ijk})\}$ to make \mathcal{E} naturally untwisted.

Remark 1.3. We do not a priori expect an arbitrary (non-cyclic) Hodge isometry $H^2(S, \mathbb{Q}) \simeq H^2(S', \mathbb{Q})$ to be induced by some equivalence $D^b(S, \alpha) \simeq D^b(S', \alpha')$. Although we have not worked out a concrete example, this seems unlikely for two reasons: First, although any equivalence $D^b(S) \simeq D^b(S')$ yields a natural isomorphism $\text{Br}(S) \simeq \text{Br}(S')$ one should not expect that for a Brauer class $\alpha \in \text{Br}(S)$ and its image $\alpha' \in \text{Br}(S')$, there always exists an equivalence $D^b(S, \alpha) \simeq D^b(S', \alpha')$ (simply because for very general choices all two-dimensional moduli spaces of objects in $D^b(S, \alpha)$ should be isomorphic to S). Second, an equivalence $D^b(S, \alpha) \simeq D^b(S', \alpha')$ only induces a natural isomorphism $T(S, \alpha) \simeq T(S', \alpha')$ but none between the untwisted transcendental lattices and hence none between the Brauer groups.

So, in order to turn Theorem 0.1 into an “if and only if”-statement, one could define S and S' to be *twisted derived equivalent* if there exists a diagram as in (0.1). Then one has

Corollary 1.4 (Twisted derived global Torelli theorem). *Two complex projective K3 surfaces S and S' are isogenous if and only if they are twisted derived equivalent.* \square

1.4. We conclude this section with a comparison to the earlier approaches by Buskin [2] and Mukai [17].

Remark 1.5. Mukai’s approach in [17] was rather similar. Instead of decomposing a given Hodge isometry $H^2(S, \mathbb{Q}) \simeq H^2(S', \mathbb{Q})$ into cyclic ones as in (1.1), he suggested to only decompose the induced Hodge isometry $T(S)_{\mathbb{Q}} \simeq T(S')_{\mathbb{Q}}$ into cyclic ones:

$$T(S)_{\mathbb{Q}} = T(S_0)_{\mathbb{Q}} \simeq T(S_1)_{\mathbb{Q}} \simeq \cdots \simeq T(S_n)_{\mathbb{Q}} = T(S')_{\mathbb{Q}}. \quad (1.7)$$

This reduces Šafarevič’s conjecture to a Hodge isometry $T(S)_{\mathbb{Q}} \simeq T(S')_{\mathbb{Q}}$ for which the intersection $T := T(S) \cap T(S')$ has finite cyclic quotients in $T(S)$ and in $T(S')$ and, using $\text{Br}(S) \simeq \text{Hom}(T(S), \mathbb{Q}/\mathbb{Z})$, can then be written as

$$T(S) \supset T(S, \alpha) \simeq T \simeq T(S', \alpha') \subset T(S')$$

for certain Brauer classes $\alpha \in \text{Br}(S)$ and $\alpha' \in \text{Br}(S')$. If, furthermore, $T \simeq T(\tilde{S})$ for some K3 surface \tilde{S} , then S and S' can both be viewed as coarse moduli spaces of sheaves on \tilde{S} and the inclusions $T(S) \supset T(\tilde{S}) \subset T(S)$ are both algebraic, induced by the twisted universal sheaves. Unfortunately, the existence of the surface \tilde{S} cannot be deduced from the surjectivity of period in general (in contrast to (1.1)) and, in fact, \tilde{S} may simply not exist. This limited Mukai’s approach [17] to the case $\rho(S) \geq 11$, later improved to $\rho(S) \geq 5$ by Nikulin [18].

The idea of the present approach is that \tilde{S} is not needed. Instead of viewing S and S' as coarse moduli spaces of untwisted sheaves on some auxiliary K3 surface \tilde{S} , one realizes S' directly as a coarse(!) moduli space of (complexes of) twisted(!) sheaves on S . This accounts for the two additional Brauer classes at once: $\alpha \in \text{Br}(S)$, as the twist with respect to which one considers the twisted sheaves on S , and $\alpha' \in \text{Br}(S')$, as the obstruction to the existence of a universal family on $S \times S'$ (of α -twisted sheaves in S).

Buskin starts with the case of a coarse moduli space of vector bundles $S' = M(v)$ with a twisted universal bundle $\mathcal{E} \in \text{Coh}(S \times S', 1 \boxtimes \alpha')$, where α' is the obstruction to the existence of a universal family. He then considers the graded(!) Hodge isometry

$$\tilde{H}(S, \mathbb{Q}) \xrightarrow{\sim} \tilde{H}(S', \mathbb{Q})$$

induced by the class $\kappa(\mathcal{E}) \cdot \sqrt{\text{td}(S \times S')}$, where $\kappa(\mathcal{E}) = \sqrt[n]{\text{ch}(\mathcal{E}^{\otimes n} \otimes \det(\mathcal{E}^*))}$ (up to dualizing \mathcal{E}). Here the crucial observation is that $\mathcal{E}^{\otimes n} \otimes \det(\mathcal{E}^*)$ is naturally untwisted for any representative of the Brauer classes. A straightforward computation shows that $\kappa(\mathcal{E})$ differs from $\text{ch}^{-B', B}(\mathcal{E})$ by the factor $\exp(-c_1(\mathcal{E}^{\otimes n})/n^3) \cdot \exp(B, -B')$ and so the difference between the action of $\kappa(\mathcal{E})$ and φ is caused by an additional factor $\exp(-c_1(\mathcal{E}^{\otimes n})/n^3)$ on the product. Note that by construction φ preserves the degree two part, which is not obvious from this comparison.

So, in the language of Remark 1.5, the starting point in [2] is of the form

$$T(S) \simeq T \simeq T(S', \alpha') \subset T(S').$$

In a next step, S and S' are deformed along a twistor space to K3 surfaces S_t and S'_t . This is a topologically trivial operation, so yields isometries

$$H^2(S_t, \mathbb{Q}) \simeq H^2(S, \mathbb{Q}) \simeq H^2(S', \mathbb{Q}) \simeq H^2(S'_t, \mathbb{Q})$$

and, for a suitable simultaneous choice of the twistor deformation, in fact a Hodge isometry $H^2(S_t, \mathbb{Q}) \simeq H^2(S'_t, \mathbb{Q})$. However, on the transcendental part it leads to a situation of the form

$$T(S_t) \supset T(S_t, \alpha_t) \simeq T_t \simeq T(S'_t, \alpha'_t) \subset T(S'_t),$$

which provides more flexibility. Then Buskin argues that although T_t may not be the transcendental part of a K3 surface, the correspondence is still algebraic. Indeed, the (partially) twisted bundle \mathcal{E} deforms to a (completely) twisted bundle \mathcal{E}_t on $S_t \times S'_t$, which uses the existence of Hermite–Einstein metrics on stable bundles. At this point it becomes important to work not with complexes of sheaves as in our approach but with vector bundles.

To conclude, Buskin has to show that any cyclic Hodge isometry $H^2(\tilde{S}, \mathbb{Q}) \simeq H^2(\tilde{S}', \mathbb{Q})$ can be reached by this procedure, applying several twistor deformations which requires to work with non-projective K3 surfaces.

It should be possible to build upon Buskin's work to prove Proposition 1.2. The deformation of \mathcal{E} to \mathcal{E}_t , along several twistor lines and involving non-projective K3 surface when changing from one to the next twistor line, should yield an equivalence. The approach presented here is more direct and more suitable to deal with K3 surfaces over other fields.

2. Motives of coarse moduli spaces of twisted sheaves

In this section we show the following result which generalizes [11] from the case of fine moduli spaces of (complexes of) untwisted sheaves to the case of coarse(!) moduli spaces of (complexes of) twisted(!) sheaves.

Theorem 2.1. *Any exact linear equivalence $D^b(S, \alpha) \simeq D^b(S', \alpha')$ between twisted projective K3 surfaces (S, α) and (S', α') over an arbitrary field k induces an isomorphism between their Chow motives*

$$h(S) \simeq h(S').$$

2.1. We shall need a few facts on Chern character of twisted sheaves. The arguments are all standard, but as there is no appropriate reference we sketch the relevant bits in a rather ad hoc manner.

Let $\alpha \in \text{Br}(X)$ be a Brauer class on a smooth projective variety X with a Čech representative (in the analytic or étale topology) $\alpha = \{\alpha_{ijk} \in \mathcal{O}^*(U_{ijk})\}$. We shall assume that $\alpha_{ijk}^n = 1$, which is stronger than just assuming $\alpha^n = 1$.

The abelian category of α -twisted coherent sheaves is incarnated by the category of $\{\alpha_{ijk}\}$ -twisted coherent sheaves $\text{Coh}(X, \{\alpha_{ijk}\})$, but we will use $\text{Coh}(X, \alpha)$ as a shorthand (see [13] for comments on the dependence of the choice). Now, observe that for any locally free $\{\alpha_{ijk}\}$ -twisted sheaf $E = \{E_i, \varphi_{ij}\}$ the tensor product $E^{\otimes n} = \{E_i^{\otimes n}, \varphi_{ij}^{\otimes n}\}$ is naturally untwisted, i.e. $\varphi_{ij}^{\otimes n} \circ \varphi_{jk}^{\otimes n} \circ \varphi_{ki}^{\otimes n} = \alpha_{ijk}^n = 1$, so that Chern classes of $E^{\otimes n}$ are well defined in $\text{CH}^*(X)$ (or in cohomology). Now define

$$\text{ch}(E) := \sqrt[n]{\text{ch}(E^{\otimes n})} \in \text{CH}^*(X)_{\mathbb{Q}}.$$

The n -th root is obtained by the usual purely formal operation, using that $\text{rk}(E^{\otimes n}) \neq 0$.⁴

We leave it to the reader to check the following facts:

(i) The definition is independent of n in the sense that

$$\text{ch}(E) = \sqrt[n]{\text{ch}(E^{\otimes n})} = \sqrt[mn]{\text{ch}(E^{\otimes mn})}.$$

⁴For K3 surfaces with a rational point, Chern characters of untwisted sheaves are integral. This does not hold for twisted sheaves, as taking the n -th root requires to work with rational coefficients.

(ii) For locally free $\{\alpha_{ijk}\}$ -twisted sheaves E and F we have

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F) \quad \text{and} \quad \mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F).$$

A similar formula holds for exact sequences.

(iii) As X is smooth projective, any twisted sheaf admits a locally free resolution and, hence, the Chern character is well-defined for all $E \in \mathrm{Coh}(X, \alpha)$ and even for objects in the bounded derived category $D^b(X, \alpha)$.⁵

(iv) For a morphism $f: Y \rightarrow X$ of smooth projective varieties, the Grothendieck–Riemann–Roch formula holds:

$$\mathrm{ch}(Rf_* \mathrm{ch}(E)) \cdot \mathrm{td}(X) = f_*(\mathrm{ch}(E) \cdot \mathrm{td}(Y))$$

in $\mathrm{CH}^*(X)_{\mathbb{Q}}$ for any $E \in D^b(Y, f^* \alpha)$. (This is easily reduced to the usual formula by tensoring both sides with $f^* G$ and G , respectively, for some locally free $\{\alpha_{ijk}^{-1}\}$ -twisted sheaf G on X .)

Once these facts are established, the yoga of Fourier–Mukai kernels \mathcal{E} , their action on the Chow ring, induced by $v(\mathcal{E}) := \mathrm{ch}(\mathcal{E}) \sqrt{\mathrm{td}(S \times S')} \in \mathrm{CH}^*(S \times S')_{\mathbb{Q}}$, and how they behave under convolutions, works as in the untwisted case, cf. [8]. The next result is an example. For this, we assume that $\alpha = \{\alpha_{ijk}\}$ and $\alpha' = \{\alpha'_{ijk}\}$ are Brauer classes on K3 surfaces S and S' , respectively, both satisfying $\alpha_{ijk}^n = 1$ and $\alpha'_{ijk} = 1$.

Corollary 2.2. *Let $\Phi_{\mathcal{E}}: D^b(S, \alpha) \xrightarrow{\sim} D^b(S', \alpha')$ be an exact equivalence with Fourier–Mukai kernel $\mathcal{E} \in D^b(S \times S', \alpha^{-1} \boxtimes \alpha')$. Then the induced action*

$$v(\mathcal{E})_*: \mathrm{CH}^*(S)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}^*(S')_{\mathbb{Q}} \tag{2.1}$$

is an isomorphism of ungraded \mathbb{Q} -vector spaces. □

2.2. The rest of the argument to prove Theorem 2.1 can be copied from [11]. Here is a rough outline: First, the motive of a K3 surface is decomposed into its algebraic and its transcendental part

$$\mathfrak{h}(S) \simeq \mathfrak{h}_{\mathrm{tr}}^2(S) \oplus \mathfrak{h}_{\mathrm{alg}}(S),$$

where $\mathfrak{h}_{\mathrm{alg}}(S) \simeq \mathbb{L}^0 \oplus \mathbb{L}^{\oplus \rho(S)} \oplus \mathbb{L}^2$ and the transcendental part $\mathfrak{h}_{\mathrm{tr}}^2(S)$, introduced in [15], has the property that $\mathrm{CH}^*(\mathfrak{h}_{\mathrm{tr}}^2(S)) = \mathrm{CH}^2(S)_0 \otimes \mathbb{Q}$. Now, derived equivalent K3 surfaces (S, α) and (S', α') have clearly the same Picard number and, therefore,

$$\mathfrak{h}_{\mathrm{alg}}(S) \simeq \mathfrak{h}_{\mathrm{alg}}(S').$$

⁵This also explains how to interpret $\sqrt[n]{\mathrm{ch}(\mathcal{E}^{\otimes n})}$ in Section 1.3.

Thus, it remains to find an isomorphism

$$h_{\text{tr}}^2(S) \simeq h_{\text{tr}}^2(S').$$

As morphisms $h_{\text{tr}}^2(S) \rightarrow h_{\text{tr}}^2(S')$ in $\text{Mot}(k)$ require degree two classes on $S \times S'$, instead of $v(\mathcal{E}) \in \text{CH}^*(S \times S')_{\mathbb{Q}}$, which induces the isomorphism (2.1), one has to consider the degree two component $v_2(\mathcal{E}) \in \text{CH}^2(S \times S')_{\mathbb{Q}}$. The induced action on $\text{CH}^*(h_{\text{tr}}^2(S)) = \text{CH}^2(S)_0 \otimes \mathbb{Q}$ coincides with the action of the full Mukai vector $v(\mathcal{E})$. Hence,

$$v_2(\mathcal{E})_*: h_{\text{tr}}^2(S) \rightarrow h_{\text{tr}}^2(S') \quad (2.2)$$

induces isomorphisms between the Chow groups of the motives. As this holds true after any base change, a version of Manin's identity principle then implies that (2.2) is an isomorphism, for details see [11].

3. Further comments

Let us briefly indicate the evidence for the motivic global Torelli theorem as formulated in Conjecture 0.3. According to the following remarks, Theorem 0.2, which provides evidence for the equivalence of (i) and (ii) in Conjecture 0.3, may also be seen as evidence for a much more general set of conservativity conjectures. Note that the following arguments apply to arbitrary surfaces (with trivial irregularity).

Proposition 3.1. *Assume the Hodge conjecture holds for the product $S \times S'$ of two complex projective K3 surfaces and assume that the motives $h(S)$ and $h(S')$ of both surfaces are Kimura finite-dimensional. Then any isomorphism of Hodge structures $H^2(S, \mathbb{Q}) \xrightarrow{\sim} H^2(S', \mathbb{Q})$ lifts to an isomorphism of motives $h(S) \simeq h(S')$.*

Proof. The argument is similar to the proof of [4, Thm. 21]. If the Hodge conjecture is assumed, the class $[\varphi] \in H^{2,2}(S \times S', \mathbb{Q})$ of any isomorphism of Hodge structures $\varphi: H^2(S, \mathbb{Q}) \xrightarrow{\sim} H^2(S', \mathbb{Q})$ is induced by a class $\gamma \in \text{CH}^2(S \times S')_{\mathbb{Q}}$, which defines a morphism $\gamma_*: h^2(S) \rightarrow h^2(S')$ of Chow motives. As Kimura finite-dimensionality implies conservativity, cf. [1, Cor. 3.16], γ_* is an isomorphism if and only if its numerical realization, which is nothing but φ , is an isomorphism. \square

Corollary 3.2. *The two conditions (i) and (ii) in Conjecture 0.3 are equivalent if the Hodge conjecture for $S \times S'$ and Kimura's finite-dimensionality conjecture for S and S' hold true.* \square

In an earlier version of this paper, Conjecture 0.3 included a third statement about the classes of $[S]$ and $[S']$ being equal in an appropriate localization of the Grothendieck ring of varieties $K_0(\text{Var}(\mathbb{C}))$. For example, if S and S' are isogenous, then according to (0.1), they are linked via a sequence of equivalences

$$D^b(S, \alpha) \simeq D^b(S_1, \alpha_1), \dots, D^b(S_{n-1}, \beta_{n-1}) \simeq D^b(S', \alpha').$$

We then speculated that maybe [16, Conj. 1.6] (with evidence provided by the examples studied in [6, 14, 16]) could also hold in the twisted case, so that $[S] - [S']$ in $K_0(\text{Var}(\mathbb{C}))$ is annihilated by some power of the Lefschetz motive $\ell := [\mathbb{A}^1]$, i.e. $[S] = [S']$ in $K_0(\text{Var}(\mathbb{C}))[\ell^{-1}]$. However, as shown by Efimov [5], this is false. There exist derived equivalent twisted(!) K3 surfaces that are not L-equivalent.

Remark 3.3. According to [21], the endomorphism field $\text{End}_{\text{Hdg}}(T(S) \otimes \mathbb{Q})$ of the rational Hodge structure $T(S) \otimes \mathbb{Q}$ is either totally real or has complex multiplication. The two cases can be distinguished by checking whether there exists a Hodge isometry other than $\pm \text{id}$, see [10, Ch. 3]. The Hodge conjecture for K3 surfaces with real multiplication has been verified in only very few cases, see [20].

In case of CM, the endomorphism field is spanned by Hodge isometries cf. [10, Thm. 3.3.7], which is enough to prove Corollary 0.4 (ii) and can also be used to prove the Hodge conjecture for products $S \times S'$ of K3 surfaces with complex multiplication for which there exists a Hodge isometry $T(S)_{\mathbb{Q}} \simeq T(S')_{\mathbb{Q}}$.

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