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Hyperbolic components of rational maps: Quantitative equidistribution and counting

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Abstract. Let Λ be a quasi-projective variety and assume that, either Λ is a subvariety of the moduli space $\mathcal{M}_d$ of degree d rational maps, or Λ parametrizes an algebraic family $(f_\lambda)_{\lambda\in\Lambda}$ of degree d rational maps on $\mathbb{P}^1$. We prove the equidistribution of parameters having p distinct neutral cycles towards the bifurcation current T^p_{bif} letting the periods of the cycles go to ∞, with an exponential speed of convergence. Several consequences of this result are:

– a precise asymptotic of the number of hyperbolic components of parameters admitting $2d-2$ distinct attracting cycles of exact periods $n_1,\dots,n_{2d-2}$ as $\min_j n_j\to\infty$ in term of the mass of the bifurcation measure and compute that mass in the case where $d=2$. In particular, in $\mathcal{M}_d$, the number of such components is asymptotic to $d^{n_1+\cdots+n_{2d-2}}$, provided that $\min_j n_j$ is large enough.
– in the moduli space $\mathcal{P}_d$ of polynomials of degree d, among hyperbolic components such that all (finite) critical points are in the immediate basins of (not necessarily distinct) attracting cycles of respective exact periods $n_1,\dots,n_{d-1}$, the proportion of those components, counted with multiplicity, having at least two critical points in the same basin of attraction is exponentially small.
– in $\mathcal{M}_d$, we prove the equidistribution of the centers of the hyperbolic components admitting $2d-2$ distinct attracting cycles of exact periods $n_1,\dots,n_{2d-2}$ towards the bifurcation measure μ_{bif} with an exponential speed of convergence.
– we have equidistribution, up to extraction, of the parameters having p distinct cycles of given multipliers towards the bifurcation current T^p_{bif} outside a pluripolar set of multipliers as the minimum of the periods of the cycles goes to ∞.

As a by-product, we also get the weak genericity of hyperbolic postcritically finiteness in the moduli space of rational maps. A key step of the proof is a locally uniform version of the quantitative approximation of the Lyapunov exponent of a rational map by the $\log^+$ of the moduli of the multipliers of periodic points.

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1. Introduction

For a holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ of degree $d > 1$ rational maps on the Riemann sphere $\mathbb{P}^1$ parametrized by a quasi-projective variety Λ, the *bifurcation locus* of $(f_\lambda)_{\lambda\in\Lambda}$ on Λ is the J-unstability locus in the sense of Mañe–Sad–Sullivan, i.e., the closure of the set of all parameters in Λ at which the Julia set $\mathcal{J}_\lambda$ of f_λ does not move continuously. It is now classical that this set is nowhere dense in Λ and admits several distinct topological descriptions, such as the closure of the set of parameters for which f_λ admits a non-persistent neutral cycle or the existence of an unstable critical dynamics (see e.g. [31, 33, 36]). From now on, pick any integer $d > 1$.

On the other hand, any (individual) rational map f of degree d on $\mathbb{P}^1$ admits a unique maximal entropy measure μ_f, whose support coincides with the Julia set $\mathcal{J}_f$ of f, and the Lyapunov exponent of f with respect to μ_f is defined by $L(f) := \int_{\mathbb{P}^1} \log|f'|\mu_f$ and satisfies $L(f) \geq \frac{1}{2}\log d > 0$. For a family $(f_\lambda)_{\lambda\in\Lambda}$, the induced Lyapunov function $L\colon \lambda \in \Lambda \longrightarrow L(f_\lambda) \in \mathbb{R}$ is *p.s.h* and continuous on the parameter space Λ. We can define the *bifurcation current* of $(f_\lambda)_{\lambda\in\Lambda}$ on Λ as the closed positive $(1,1)$-current

$$T_{\mathrm{bif}} := dd^c L.$$

By DeMarco [11], the support of $dd^c L$ coincides with the bifurcation locus of the family $(f_\lambda)_{\lambda\in\Lambda}$. For any integer $1 \leq p \leq \dim \Lambda$, Bassanelli and Berteloot also defined the *p-bifurcation current* T_{bif}^p as the p-th exterior product of T_{bif}. It is a positive closed current of bidegree (p,p) so the *bifurcation measure* $\mu_{\mathrm{bif}} := (dd^c L)^{\dim \Lambda}$ is a positive measure on Λ. If $p > 1$, the current T_{bif}^p detects, in a certain sense, stronger bifurcations than $T_{\mathrm{bif}} = T_{\mathrm{bif}}^1$ [1]. Indeed, its topological support admits several dynamical characterizations similar to that of the bifurcation locus: for example, it is the closure of parameters admitting p distinct neutral cycles or p critical points preperiodic to repelling cycles (see [14, 24]).

The group $\mathrm{PSL}_2(\mathbb{C})$ of Möbius transformations acts on the space Rat_d of degree d rational maps on $\mathbb{P}^1$, which is itself a holomorphic family of rational maps, by conjugacy. The *moduli space* $\mathcal{M}_d$ of degree d rational maps on $\mathbb{P}^1$ is the *orbit space* of $\mathrm{PSL}_2(\mathbb{C})$ in Rat_d, that is, the quotient of Rat_d resulting from this action of $\mathrm{PSL}_2(\mathbb{C})$. It is an irreducible affine variety of dimension $2d-2$, and is singular if and only if $d \geq 3$ (Silverman [44]). The Lyapunov function $f \mapsto L(f)$ on Rat_d descends to a continuous and psh function $\mathcal{L}\colon \mathcal{M}_d \to \mathbb{R}$. For any integer $1 \leq p \leq 2d-2$, the p-bifurcation current on $\mathcal{M}_d$ is thus given by $T_{\mathrm{bif}}^p := (dd^c\mathcal{L})^p$, and the *bifurcation measure* on $\mathcal{M}_d$ is by

$$\mu_{\mathrm{bif}} := T_{\mathrm{bif}}^{2d-2} = (dd^c\mathcal{L})^{2d-2},$$

which is a finite positive measure on $\mathcal{M}_d$ of strictly positive total mass (see [1]).

One of the features of the bifurcation currents is to give measurable statements of the above density, or in general, accumulation properties. Let us be more precise. Let Λ be a quasi-projective variety such that, either $\Lambda \subset \mathcal{M}_d$, or parametrizing an algebraic family $(f_\lambda)_{\lambda\in\Lambda}$ of degree d rational maps on $\mathbb{P}^1$. For any $n \in \mathbb{N}^*$ and any $w \in \mathbb{C} \setminus \{1\}$, let $\mathrm{Per}_n(w)$ be the analytic hypersurface

$$\mathrm{Per}_n(w) := \{\lambda \in \Lambda : f_\lambda \text{ has a cycle of multiplier } w \text{ and the exact period } n\}$$

in Λ and denote by $[\mathrm{Per}_n(w)]$ the current of integration over $\mathrm{Per}_n(w)$ on Λ. Since Λ is quasi-projective, the hypersurfaces $\mathrm{Per}_n(w)$ are actually algebraic hypersurfaces of Λ (see e.g. [2]). By Bassanelli and Berteloot [2], the sequence $(d^{-n}[\mathrm{Per}_n(w)])$ weighted by the Lebesgue measure on the disk of center 0 and radius $|w|$ converges towards the bifurcation current T_{bif}. Similar dynamically significant equidistribution properties towards the bifurcation current have been recently established in various contexts, as general holomorphic families of rational maps [3,15,37] or moduli spaces of polynomials [7,25].

The proofs developed in op. cit. do not allow establishing equidistribution phenomena towards the bifurcation measure μ_{bif}. Indeed, any of the above convergences obtained is essentially L^1_{loc} convergence of the potentials of currents, which does not guarantee continuity of the intersection.

One of the main purposes of the article is to prove the equidistribution of parameters having p non-repelling cycles towards the bifurcation current T^p_{bif} as the minimum of the periods of those cycles goes to ∞, with an exponential speed of convergence. We will then deduce several important consequences, notably in counting hyperbolic components of disjoint types in $\mathcal{M}_d$. Notice that such counting results are of combinatorial and algebraic nature and have a priori no relation to bifurcation measures. Furthermore, they are the first general results in that direction so far.

Notations. Let $\mu\colon \mathbb{N}^* \to \{-1, 0, 1\}$ be the Möbius function. Define the sequence (d_n) in $\mathbb{N}^*$ by

$$d_n := \sum_{m|n} \mu\Big(\frac{n}{m}\Big)(d^m + 1) \in \mathbb{N}^*,$$

or equivalently $d^n + 1 = \sum_{m|n} d_m$ for any $n \in \mathbb{N}^*$, so that $d_n = d_n + O(d^{n/2})$ as $n \to \infty$. For any $p \in \mathbb{N}^*$, any $\underline{n} = (n_1, \dots, n_p) \in (\mathbb{N}^*)^p$, and any $\underline{\rho} = (\rho_1, \dots, \rho_p) \in]0, 1]^p$, we set $|\underline{n}| := \sum_{j=1}^p n_j$, so that $d^{|\underline{n}|} = \prod_{j=1}^p d^{n_j}$ and, in a similar way,

$$d_{|\underline{n}|} := \prod_{j=1}^{p} d_j.$$

For any $i \in \{0, 1, 2\}$ and any $n \in \mathbb{N}^*$, we also set $\sigma_i(n) := \sum_{m|n} m^i$, so in particular $\sigma_0 \le \sigma_1 \le \sigma_2$ on $\mathbb{N}^*$ (beware that $\sigma_2(n) \le Cn^2 \log\log n$ for some constant C).

For any $\underline{n} = (n_1, \dots, n_{2d-2}) \in (\mathbb{N}^*)^{2d-2}$ and any $\underline{w} = (w_1, \dots, w_{2d-2}) \in \mathbb{C}^{2d-2}$, let Stab$(\underline{n})$ (resp. Stab$(\underline{n}, \underline{w})$) be the set of all permutations of the indices $\{1, 2, \dots, 2d-2\}$ that do not change the ordered $(2d-2)$-tuple $(n_1, \dots, n_{2d-2}) \in (\mathbb{N}^*)^{2d-2}$ (resp. $((n_1, w_1), \dots, (n_{2d-2}, w_{2d-2})) \in (\mathbb{N}^* \times \mathbb{C})^{2d-2}$), so in particular $\#\mathrm{Stab}(\underline{n}, \underline{w}) \leq \#\mathrm{Stab}(\underline{n}) \leq (2d-2)!$.

For $r > 0$, we set $\mathbb{D}_r = \{|z| < r\}$, so that $\partial\mathbb{D}_r = \mathbb{S}_r = \{|z| = r\}$.

Statement of the main results. Let Λ be a quasi-projective variety either such that $\Lambda \subset \mathcal{M}_d$, or parametrizing an algebraic family $(f_\lambda)_{\lambda\in\Lambda}$ of degree d rational maps on $\mathbb{P}^1$. We refer to [9] for basics on positive closed currents and intersection theory on algebraic varieties.

For any integer $1 \leq p \leq \min\{\dim\Lambda, 2d-2\}$, any $\underline{n} = (n_1, \dots, n_p) \in (\mathbb{N}^*)^p$, and any $\underline{\rho} = (\rho_1, \dots, \rho_p) \in]0, 1]^p$, the following positive closed current

$$T_{\underline{n}}^p(\underline{\rho}) := \frac{1}{d_{|\underline{n}|}} \int_{[0,2\pi]^p} \bigwedge_{j=1}^{p} \left[\mathrm{Per}_{n_j}(\rho_j e^{i\theta_j})\right] \frac{\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_p}{(2\pi)^p} \tag{1.1}$$

on Λ is well-defined, and coincides with $\bigwedge_{j=1}^p T_{n_j}^1(\rho_j)$ by the Fubini theorem (see e.g. [2]). We say a form Ψ on Λ is DSH if $dd^c\Psi = T^+ - T^-$ for some positive closed currents $T^\pm$ of finite masses on Λ. We refer to §2.1 for the precise definition of the semi-norm $\|\Psi\|^*_{\mathrm{DSH}}$.

One of our principal results is the following.

Theorem 1.1. *Let Λ be a quasi-projective variety which either is a subvariety in $\mathcal{M}_d$ or parametrizes an algebraic family $(f_\lambda)_{\lambda\in\Lambda}$ of degree d rational maps on $\mathbb{P}^1$. Then for any compact subset K in Λ, there exists a constant $C(K) > 0$ such that for any integer $1 \leq p \leq \min\{\dim\Lambda, 2d-2\}$, any $\underline{n} = (n_1, \dots, n_p) \in (\mathbb{N}^*)^p$, any $\underline{\rho} = (\rho_1, \dots, \rho_p) \in]0, 1]^p$, and any continuous* DSH*-form Ψ of bidegree $(m-p, m-p)$ supported in K, we have*

$$\left|\left\langle T_{\underline{n}}^p(\underline{\rho}) - T_{\mathrm{bif}}^p, \Psi\right\rangle\right| \leq C(K) \cdot \max_{1\leq j\leq p} \left(\left(1 + |\log\rho_j|\right)\frac{\sigma_2(n_j)}{d^{n_j}}\right) \cdot \|\Psi\|^*_{\mathrm{DSH}}.$$

We first prove this theorem in the case where $p = 1$. To do that, we show in Section 3 a locally uniform version of the second author's result [38] on the quantitative approximation of the Lyapunov exponent of an (individual) $f \in \mathrm{Rat}_d$ by the average of the logs of the moduli of the multipliers of all non-attracting n-periodic points of f (Lemma 3.3). This leads to an error term on the proximity between $f^n(c)$ and c for each critical point c of f and an error term on how close to 0 the multipliers of the periodic points of f are. To control those terms, we use a parametric version of a lemma of Przytycki [42, Lemma 1] proved by the first and third authors in [28]. Intersection of currents and integrations by parts lead to the result for any p.

Theorem 1.1 is proved in Section 4 and has the following consequence.

Corollary 1.2. *Let Λ be a quasi-projective variety either such that $\Lambda \subset \mathcal{M}_d$, or parametrizing an algebraic family $(f_\lambda)_{\lambda\in\Lambda}$ of degree d rational maps on $\mathbb{P}^1$. Pick an integer $1 \leq p \leq \min\{\dim \Lambda, 2d-2\}$. Then for any sequence $(\underline{n}_k)_{k\in\mathbb{N}^*}$ of p-tuples $\underline{n}_k = (n_{1,k}, \dots, n_{p,k})$ in $(\mathbb{N}^*)^p$ such that $\sum_k \max_j(n_{j,k}^{-1}) < \infty$, there exists a pluripolar subset $\mathcal{E}$ in $\mathbb{C}^p$ such that for any $\underline{w} = (w_1, \dots, w_p) \in \mathbb{C}^p \setminus \mathcal{E}$, $\bigcap_{i=1}^p \mathrm{Per}_{n_{i,k}}(w_i)$ is of pure codimension p in Λ for any $k \in \mathbb{N}^*$ and*

$$T_{\mathrm{bif}}^p = \lim_{k\to\infty} \frac{1}{d_{|\underline{n}_k|}} \bigwedge_{i=1}^p \left[\mathrm{Per}_{n_{i,k}}(w_i)\right]$$

in the weak sense of currents on Λ.

The techniques used in the proof of Corollary 1.2 also give that the current equidistributed on the set of parameters having p cycles of respective exact periods $n_{1,k}, \dots, n_{p,k}$ and multipliers $w_1, \dots, w_p$ distributed by a PB measure on $(\mathbb{P}^1)^p$ converges towards the bifurcation current T_{bif}^p when $k \to \infty$, with the best possible order estimate $O(\max_j(n_j^{-1}))$ as $\min_j(n_j) \to \infty$. (see Theorem 4.2 below).

Remark 1.3. Let us also observe that, as in [2], Theorem 1.1 gives another proof of Shishikura's upper bound $2d-2$ of the number of distinct cycles of Fatou components of a given rational map of degree d is sharp (see [43]). In fact, provided $\min_j n_j$ is large enough, we can construct a rational map having $2d-2$ distinct attracting periodic points of respective period n_j (we no longer need to take a subsequence and have no arithmetic restrictions on the periods).

Now let us focus on the moduli spaces of rational maps and hyperbolic components. Recall that the *hyperbolic locus* in $\mathcal{M}_d$ is the set of all conjugacy classes of hyperbolic maps that are uniformly expanding on their Julia sets. It is an open subset of $\mathcal{M}_d$ and a connected component of this hyperbolic locus is called a hyperbolic component in $\mathcal{M}_d$.

Definition 1.4. A rational map $f \in \mathrm{Rat}_d$ is said to be *hyperbolic of type* $\underline{n} = (n_1, \dots, n_{2d-2}) \in (\mathbb{N}^*)^{2d-2}$ if f has $2d-2$ distinct attracting cycles of respective exact periods $n_1, \dots, n_{2d-2}$. A hyperbolic component Ω in $\mathcal{M}_d$ is said to be of *type* $\underline{n} \in (\mathbb{N}^*)^{2d-2}$ if, for any $[f] \in \Omega$, f is hyperbolic of type $\underline{n}$. A hyperbolic component in $\mathcal{M}_d$ is of *disjoint type* if it is of type $\underline{n}$ for some $\underline{n} \in (\mathbb{N}^*)^{2d-2}$.

Definition 1.5. For any $\underline{n} = (n_1, \dots, n_{2d-2}) \in (\mathbb{N}^*)^{2d-2}$, let $N(\underline{n})$ denote the number of hyperbolic components of type $\underline{n}$ in $\mathcal{M}_d$.

A striking application of Theorem 1.1 is the following asymptotic on the global counting of hyperbolic components of disjoint types.

Theorem 1.6. *As $\min_j n_j \to +\infty$,*

$$\#\mathrm{Stab}(\underline{n}) \cdot \frac{N(\underline{n})}{d_{|\underline{n}|}} = \int_{\mathcal{M}_d} \mu_{\mathrm{bif}} + O\Big(\max_j \Big(\frac{\sigma_2(n_j)}{d^{n_j}}\Big)\Big).$$

In particular, $N(\underline{n}) > 0$ if $\min_j n_j$ is large enough. Theorem 1.6 gives a combinatorial interpretation of the mass of the bifurcation measure. In the case $d = 2$, as a consequence of Theorem 1.6 together with the precise estimates of $N(n_1, n_2)$ by Kiwi and Rees [30], we can determine the (total) mass of the bifurcation measure on $\mathcal{M}_2$.

Corollary 1.7. *Let ϕ be the Euler totient function on $\mathbb{N}^*$. Then*

$$\int_{\mathcal{M}_2} \mu_{\mathrm{bif}} = \frac{1}{3} - \frac{1}{8} \sum_{n \geq 1} \frac{\phi(n)}{(2^n - 1)^2}.$$

In the proof of Theorem 1.6, it is crucial that the estimate in Theorem 1.1 involves *only* the DSH-semi-norm $\| \cdot \|^*_{\mathrm{DSH}}$ of the observable. Notice also that the mass of a limit of positive measures is not greater than the limit of the masses, so it could be possible that a proportion of components is lost passing to the limit as they would accumulate at the boundary of the moduli space. Theorem 1.6 says that it is not the case. The proof of Theorem 1.6 also relies crucially on the fact that the multipliers of attracting *cycles* parametrize the hyperbolic components of disjoint type of $\mathcal{M}_d$. Though this is essentially classical, there seems to be no available proofs in the literature so we include a proof of it in Section 6.2. The proof relies on the transversality of periodic *critical* orbit relations, which we show in Section 5, following the argument of Epstein [6, 20].

It is comparable to the common situation in dynamical systems where the existence of e.g. repelling periodic orbit of large period follows from an equidistribution property, see [4, 5] for holomorphic endomorphisms on $\mathbb{P}^k$.

As a consequence of Theorem 1.6, we also establish the weak genericity of hyperbolic postcritically finite maps in $\mathcal{M}_d$ (see Theorem 6.6 below), which is stronger than the Zariski density of such maps in $\mathcal{M}_d$.

We finally establish a quantitative equidistribution of parameters in hyperbolic components in $\mathcal{M}_d$ of disjoint type, having given multipliers. For any $\underline{n} = (n_1, \ldots, n_{2d-2}) \in (\mathbb{N}^*)^{2d-2}$ and any $\underline{w} = (w_1, \ldots, w_{2d-2}) \in \mathbb{D}^{2d-2}$, let $\mathrm{C}_{\underline{n},\underline{w}}$ denote the (finite) set of all conjugacy classes $[f] \in \mathcal{M}_d$ of hyperbolic rational maps $f \in \mathrm{Rat}_d$ of type $\underline{n}$ whose attracting cycle of exact period n_j has the multiplier w_j for any $1 \leq j \leq 2d - 2$, and set

$$\mu_{\underline{n},\underline{w}} := \frac{\#\mathrm{Stab}(\underline{n}, \underline{w})}{d_{|\underline{n}|}} \sum_{[f] \in \mathrm{C}_{\underline{n},\underline{w}}} \delta_{[f]}.$$

For simplicity, we denote $\mathrm{C}_{\underline{n},(0,\ldots,0)}$ and $\mu_{\underline{n},(0,\ldots,0)}$ by $\mathrm{C}_{\underline{n}}$ and $\mu_{\underline{n}}$, respectively, so that any element in $\mathrm{C}_{\underline{n}}$ is the center of a hyperbolic component in $\mathcal{M}_d$ of type $\underline{n}$.

The following in particular implies the weak convergence $\mu_{\underline{n},\underline{w}} \to \mu_{\mathrm{bif}}$ on $\mathcal{M}_d$, which is even new and it was one of our motivations to give a proof of this convergence.

Theorem 1.8. *For any compact subset K in $\mathcal{M}_d$, there exists $C_K > 0$ such that:*
(1) *for any test function $\Psi \in \mathcal{C}^2(\mathcal{M}_d)$ with support in K and any $\underline{n} \in (\mathbb{N}^*)^{2d-2}$,*

$$\left|\langle \mu_{\underline{n}} - \mu_{\mathrm{bif}}, \Psi\rangle\right| \leq C_K \cdot \max_{1\leq j\leq 2d-2}\left(\frac{\sigma_2(n_j)}{d^{n_j}}\right)\cdot \|\Psi\|_{\mathcal{C}^2},$$

(2) *for any test function $\Psi \in \mathcal{C}^1(\mathcal{M}_d)$ with support in K, any $\underline{n} \in (\mathbb{N}^*)^{2d-2}$, and any $\underline{w} = (w_1, \dots, w_{2d-2}) \in \mathbb{D}^{2d-2}$,*

$$\left|\langle \mu_{\underline{n},\underline{w}} - \mu_{\mathrm{bif}}, \Psi\rangle\right| \leq C_K \cdot \max_{1\leq j\leq 2d-2}\left(\frac{-1}{d^{n_j}\log|w_j|}, \frac{\sigma_2(n_j)}{d^{n_j}}\right)^{1/2}\cdot \|\Psi\|_{\mathcal{C}^1}.$$

Observe that an interpolation between Banach spaces gives a speed of convergence for any $\mathcal{C}^\alpha$-observable with $0 < \alpha \leq 2$ in the case of centers and $0 < \alpha \leq 1$ in general.

Even though Theorem 1.8 looks very close to known qualitative/quantitative equidistribution results for holomorphic/anti-holomorphic polynomial families, e.g., [22, 23, 27, 28, 39], the compactness of the support of the bifurcation measure was a crucial point in those earlier works. Such a compactness is not the case for $\mathcal{M}_d$. It might also be worth stressing that in [28], the first and third authors considered the currents of bifurcation T_c of marked critical points c so were looking at unstable critical dynamics. Although it seems to be similar, here we study directly the bifurcation current T_{bif} so the unstability of cycles (see the introduction of [2]). A feature of that approach is that we don't need the n_j to be distinct, which was necessary in the above works.

Section 8 is devoted to the study of the moduli space $\mathcal{P}_d^{\mathrm{cm}}$ of critically marked degree d polynomials where we give various results similar to those previously proved.

To finish, let us mention that, as an application of our approximation formula of the Lyapunov exponent, we give a proof of the estimate of the degeneration of the Lyapunov exponent of f as $f \to \partial\,\mathrm{Rat}_d$ along a punctured analytic disk in the spirit of [21] (see Theorem 3.6).

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2. Preliminaries

2.1. Currents and DSH functions. We refer to [18, Appendix A] for more details on currents and DSH functions. Pick any quasi-projective variety Λ. Let β be the restriction of the ambient Fubini–Study form to Λ. For any positive closed current T

of bidimension (k,k) defined on Λ and any Borel set $A \subset \Lambda$, we denote by $\|T\|_A$ the number

$$\|T\|_A := \int_A T \wedge \beta^k.$$

This is the *mass* of the current T in A. We simply write $\|T\|$ for $\|T\|_\Lambda$.

Let Ψ be an (ℓ,ℓ)-form in Λ. We say that Ψ is DSH if we can write $dd^c\Psi = T^+ - T^-$ where $T^\pm$ are positive closed currents of finite mass in Λ. We also set

$$\|\Psi\|^*_{\mathrm{DSH}} := \inf_{T^\pm}\big(\|T^+\| + \|T^-\|\big),$$

where $T^\pm$ ranges over all closed positive currents such that $dd^c\Psi = T^+ - T^-$ (note that $\|T^+\| = \|T^-\|$ since they are cohomologous).

This is not exactly the usual DSH norm but just a semi-norm. Nevertheless, one has $\|\Psi\|^*_{\mathrm{DSH}} \leq \|\Psi\|_{\mathrm{DSH}}$, where $\|\Psi\|_{\mathrm{DSH}} := \|\Psi\|^*_{\mathrm{DSH}} + \|\Psi\|_{L^1}$. The interest of those DSH-norms lies in the fact that they behave nicely under change of coordinates. Furthermore, when Ψ is $\mathscr{C}^2$ with support in a compact set K, there is a constant $C > 0$ depending only on K such that $\|\Psi\|_{\mathrm{DSH}} \leq C\|\Psi\|_{\mathscr{C}^2}$.

2.2. Resultant and the space Rat_d. We refer to [1] and [44] for the content of this paragraph.

Notations. Let $\pi\colon \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ be the canonical projection, $\|\cdot\|$ be the Hermitian norm on $\mathbb{C}^2$, and set $(z_0,z_1)\wedge(w_0,w_1) := z_0w_1 - z_1w_0$ on $\mathbb{C}^2\times\mathbb{C}^2$.

Pick an integer $d > 1$. A pair $F = (F_1, F_2) \in \mathbb{C}[x,y]_d \times \mathbb{C}[x,y]_d \simeq \mathbb{C}^{2d+2}$ of homogeneous degree d polynomials can be identified with a degree d homogeneous polynomial endomorphism of $\mathbb{C}^2$. The homogeneous resultant $\mathrm{Res} = \mathrm{Res}_d$ is the unique homogeneous degree $2d$ polynomial over $\mathbb{Z}$ in $2d+2$ variables such that $\mathrm{Res}(F) = 0$ if and only if F is degenerate, i.e. $F^{-1}(\{0\}) \neq \{0\}$, and $\mathrm{Res}((x^d, y^d)) = 1$. We thus identify the space of all degree d non-degenerate homogeneous polynomial endomorphisms of $\mathbb{C}^2$ with $\mathbb{C}^{2d+2}\setminus\{\mathrm{Res}=0\}$.

A rational map f on $\mathbb{P}^1$ of degree d admits a (non-degenerate homogeneous polynomial) lift, i.e. there exists a degree d homogeneous polynomial endomorphism $F\colon\mathbb{C}^2\to\mathbb{C}^2$ such that $\mathrm{Res}(F)\neq 0$ and that $\pi\circ F = f\circ\pi$ on $\mathbb{C}^2\setminus\{0\}$. Moreover, any two homogeneous polynomial endomorphisms F, G of $\mathbb{C}^2$ are lifts of the same f if and only if there exists $\alpha\in\mathbb{C}^*$ such that $F = \alpha\cdot G$. Let us denote by Rat_d the set of all degree d rational maps on $\mathbb{P}^1$. Since Res is homogeneous, we can also identify Rat_d with $\mathbb{P}^{2d+1}\setminus\{\mathrm{Res}=0\}$. In particular, it is a quasi-projective variety of dimension $2d+1$.

2.3. The dynamical Green function of a rational map on $\mathbb{P}^1$. In the whole text, we denote by ω_{FS} the Fubini–Study form on $\mathbb{P}^1$ normalized so that $\|\omega_{\mathrm{FS}}\| = 1$ and by $[\cdot,\cdot]$ the chordal metric on $\mathbb{P}^1$ given by

$$[z,w] = |z_0 w_1 - w_1 z_0| / \big(\sqrt{|z_0|^2 + |z_1|^2}\sqrt{|w_0|^2 + |w_1|^2}\big)$$

for any $z = [z_0 : z_1]$, $w = [w_0 : w_1] \in \mathbb{P}^1$, so that $\operatorname{diam}(\mathbb{P}^1) = 1$ and for any w, $dd^c_z \log[z,w] = \delta_w - \omega_{\mathrm{FS}}$.

For any ω_{FS}-psh function g on $\mathbb{P}^1$, i.e. such that

$$\omega_{\mathrm{FS}} + dd^c g =: \nu_g$$

is a probability measure on $\mathbb{P}^1$, we define the *g-kernel function* Φ_g by setting

$$\Phi_g(z,w) := \log[z,w] - g(z) - g(w) \tag{2.1}$$

on $\mathbb{P}^1 \times \mathbb{P}^1$. For a probability measure ν' on $\mathbb{P}^1$, set $U_{g,\nu'} := \int_{\mathbb{P}^1} \Phi_g(\cdot\,,w)d\nu'(w)$ on $\mathbb{P}^1$. Then $dd^c_z U_{g,\nu'} = \nu' - \nu_g$, so in the particular case where $\nu' = \nu_g$, we deduce that

$$U_{g,\nu_g} \equiv I_g := \int_{\mathbb{P}^1\times\mathbb{P}^1} \Phi_g(\nu_g \times \nu_g) \quad \text{on } \mathbb{P}^1.$$

Pick now $f \in \mathrm{Rat}_d$. For all (non-degenerate homogeneous polynomial) lift $F: \mathbb{C}^2 \to \mathbb{C}^2$ of f, there exists a Hölder continuous ω_{FS}-psh function $g_F: \mathbb{P}^1 \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{\log \|F^n\|}{d^n} - \log\|\cdot\| = g_F \circ \pi$$

uniformly on $\mathbb{C}^2 \setminus \{0\}$, which is called the *dynamical Green function* of F on $\mathbb{P}^1$. Since F is unique up to multiplication by $\alpha \in \mathbb{C}^*$ and $g_{\alpha\cdot F} = g_F + (\log|\alpha|)/(d-1)$ for any $\alpha \in \mathbb{C}^*$, the positive measure

$$\omega_{\mathrm{FS}} + dd^c g_F =: \mu_f$$

is independent of the choice of F, and is in fact the unique maximal entropy measure of f on $\mathbb{P}^1$. For later use, we point out the equality

$$I_{g_F} = -\frac{1}{d(d-1)} \log|\mathrm{Res}(F)|,$$

which is (a reformulation of) DeMarco's formula [10, Theorem 1.5] on the homogeneous capacity of the filled-in Julia set of F in $\mathbb{C}^2$.

Definition 2.1. The *dynamical Green function* g_f of f on $\mathbb{P}^1$ is the unique ω_{FS}-psh function on $\mathbb{P}^1$ such that $\mu_{g_f} = \mu_f$ on $\mathbb{P}^1$ and that $I_{g_f} = 0$.

Remark 2.2. In particular, $U_{g_f,\mu_f} \equiv I_{g_f} = 0$ on $\mathbb{P}^1$. Moreover, $g_F = g_f$ for some lift F of f, which is unique up to multiplication by a complex number of modulus one.

2.4. The dynatomic and multiplier polynomials. We refer to [45, §4.1] and to [2, 3, 34] (see also [22, §6]) for the details on the dynatomic and multiplier polynomials and the related topics.

Pick any $f \in \mathrm{Rat}_d$. For every $n \in \mathbb{N}^*$, let

- $\mathrm{Fix}(f^n)$ be the set of all fixed points of f^n in $\mathbb{P}^1$, and
- $\mathrm{Fix}^*(f^n)$ the set of all periodic points of f in $\mathbb{P}^1$ having exact period n.

The n-th *dynatomic polynomial* of a lift F of f is a *homogeneous* polynomial

$$\Phi_n^*(F,(z_0,z_1)) := \prod_{k|n}\big(F^k(z_0,z_1)\wedge(z_0,z_1)\big)^{\mu(n/k)}$$

in z_0, z_1 of degree d_n; there is a (finite) sequence $(P_j^{(n)})_{j\in\{1,\dots,d_n\}}$ in $\mathbb{C}^2\setminus\{0\}$ such that we have a factorization $\Phi_n^*(F,(z_0,z_1)) = \prod_{j=1}^{d_n}((z_0,z_1)\wedge P_j^{(n)})$, and setting $z_j^{(n)} := \pi(P_j^{(n)}) \in \mathbb{P}^1$ for each $j \in \{1,\dots,d_n\}$, the sequence $(z_j^{(n)})_{j=1}^{d_n}$ is independent of the choice of $(P_j^{(n)})_{j\in\{1,\dots,d_n\}}$ and that of F, up to permutation.

We recall that the set $\{z_j^{(n)} : j \in \{1,\dots,d_n\}\}$ is the disjoint union of $\mathrm{Fix}^*(f^n)$ and the set of all periodic points z of f having exact period $m < n$ and dividing n and whose multiplier $(f^m)'(z)$ is a n/m-th primitive root of unity. In particular, $(f^n)'(z) = 1$ for every $z \in \{z_j^{(n)} : j \in \{1,\dots,d_n\}\} \setminus \mathrm{Fix}^*(f^n)$, and for every $z \in \mathrm{Fix}^*(f^n)$, we have $\#\{j \in \{1,\dots,d_n\} : z_j^{(n)} = z\} = 1$ if $(f^n)'(z) \neq 1$. For every $n \in \mathbb{N}^*$, the n-th *multiplier polynomial* of f is the polynomial

$$p_n(f,w) := \left(\prod_{j=1}^{d_n}\big((f^n)'(z_j^{(n)}) - w\big)\right)^{1/n} \tag{2.2}$$

in w of degree d_n/n, which is unique up to multiplication in n-th roots of unity.

Let Λ be a quasi-projective variety parametrizing an algebraic family $(f_\lambda)_{\lambda\in\Lambda}$ of degree d rational maps on $\mathbb{P}^1$. Then for any $n \in \mathbb{N}^*$, the *n-th multiplier polynomial $p_n\colon \Lambda\times\mathbb{C} \longrightarrow \mathbb{C}$ of $(f_\lambda)_{\lambda\in\Lambda}$* defined by

$$p_n(\lambda,w) := p_n(f_\lambda,w)$$

is holomorphic, and since Λ is a quasi-projective variety, this $p_n\colon \Lambda\times\mathbb{C}\to\mathbb{C}$ is actually a regular function with $\deg_w(p_n(\lambda,w)) = d_n/n$ for all $\lambda \in \Lambda$ and with $\deg_\lambda(p_n(\lambda,w)) \le C d_n$ for all $w \in \mathbb{C}$, where $C > 0$ depends only on the family $(f_\lambda)_{\lambda\in\Lambda}$, see e.g. [3, §2.2]. For any $n \in \mathbb{N}^*$ and any $w \in \mathbb{C}$, we set

$$\mathrm{Per}_n(w) := \{\lambda \in \Lambda\ ;\ p_n(\lambda,w) = 0\}$$

and denote by $[\mathrm{Per}_n(w)]$ the current of integration defined by the zeros of $p_n(\cdot,w)$

on Λ. Remark that for all $w \in \mathbb{C}$ and all $n \in \mathbb{N}^*$, since $\deg_\lambda(p_n(\lambda, w)) \leq Cd_n$, we have

$$\frac{1}{d_n}\|[\mathrm{Per}_n(w)]\| \leq C. \tag{2.3}$$

Beware also that, since the existence of a cycle of given period and multiplier is invariant under Möbius conjugacy, the n-th multiplier polynomial $p_n\colon \mathrm{Rat}_d \times \mathbb{C} \to \mathbb{C}$ of Rat_d also descends to a regular function $p_n\colon \mathcal{M}_d \times \mathbb{C} \to \mathbb{C}$, enjoying the same properties.

2.5. A parametric version of Przytycki lemma. For a $\mathscr{C}^1$ map $f\colon \mathbb{P}^1 \to \mathbb{P}^1$, the *chordal derivative* $f^\#$ of f is the non-negative real valued continuous function

$$f^\#(z) := \lim_{y \to z} \frac{[f(z), f(y)]}{[z, y]}$$

on $\mathbb{P}^1$. For any rational map $f \in \mathrm{Rat}_d$, we set

$$M(f) := \sup_{\mathbb{P}^1} (f^\#)^2 \in]1, +\infty[.$$

We shall use the following, which is a direct consequence of [28, Lemma 3.1] and of the fact that the spherical and the chordal distance are equivalent on $\mathbb{P}^1$.

Lemma 2.3. *There exists a universal constant $0 < \kappa < 1$ such that for any holomorphic family $(f_\lambda)_{\lambda \in \Lambda}$ of degree d rational maps with a marked critical point $c\colon \Lambda \to \mathbb{P}^1$ which does not lie persistently in a parabolic basin of f_λ and is not persistently periodic, the following holds: for any $n \in \mathbb{N}^*$ and any $\lambda \in \Lambda$, if $f_\lambda^n(c(\lambda)) \neq c(\lambda)$, then:*

- *either $[f_\lambda^n(c(\lambda)), c(\lambda)] \geq \kappa \cdot M(f_\lambda)^{-n}$,*
- *or $c(\lambda)$ lies in the immediate basin of an attracting periodic point $z(\lambda)$ of f_λ of period dividing n, $[c(\lambda), \mathcal{J}_\lambda] \geq \kappa M(f_\lambda)^{-n}$, and $2[f_\lambda^n(c(\lambda)), c(\lambda)] \geq [z(\lambda), c(\lambda)]$.*

2.6. A length-area estimate. The modulus of an annulus A conformally equivalent to $A' = \{z \in \mathbb{C}\ ;\ r < |z| < R\}$ with $0 < r < R < +\infty$ is defined by

$$\mathrm{mod}(A) = \mathrm{mod}(A') = \frac{1}{2\pi} \log\Big(\frac{R}{r}\Big).$$

We shall use the following classical estimate ([4, Appendix]).

Lemma 2.4 (Briend–Duval). *There exists a universal constant $\tau > 0$ such that for any quasi-projective variety Λ, any Kähler metric ω on Λ and any pair of relatively compact holomorphic disks $D_1 \Subset D_2$ in Λ, so that $D_2 \setminus \overline{D_1}$ is an annulus, we have*

$$(\mathrm{diam}_\omega(D_1))^2 \leq \tau \cdot \frac{\mathrm{Area}_\omega(D_2)}{\min(1, \mathrm{mod}(D_2 \setminus \overline{D_1}))}.$$

3. Quantitative approximation of the Lyapunov exponent

Our precise result here can be stated as follows. This result relies on the combination of the arguments used in [38] as developed in Lemma 3.4 below and of the lemma "à la Przytycki" proved in [28]. The *locally uniform* speed of convergence obtained here is not as fast as the pointwise one obtained in [38]. This is due to our need to control the dependence of the constants on $f \in \mathrm{Rat}_d$ in the right-hand side. Here we obtain a continuous dependence.

Theorem 3.1. *There exists $A \geq 1$ depending only on d such that for any $r \in]0,1]$, any $f \in \mathrm{Rat}_d$, and any $n \in \mathbb{N}^*$, we have*

$$\left|\frac{1}{d_n}\int_0^{2\pi} \log\left|p_n\left(f, re^{i\theta}\right)\right|\frac{d\theta}{2\pi} - L(f)\right| \leq A\left(C([f]) + |\log r|\right)\frac{\sigma_2(n)}{d^n},$$

where $C([f]) = \inf\{\log(\sup_{\mathbb{P}^1} f_1^{\#}) + \sup_{\mathbb{P}^1}|g_{f_1}|\}$, where the infimum is taken over all $f_1 \in [f]$ and where g_{f_1} is the dynamical Green function of f_1 normalized as in §2.3.

Of course, as the left-hand side of the inequality is invariant under Möbius conjugacy, it is sufficient to prove that for any $0 < r \leq 1$, any $n \in \mathbb{N}^*$ and any $f \in \mathrm{Rat}_d$, we have

$$\left|\frac{1}{d_n}\int_0^{2\pi} \log\left|p_n\left(f, re^{i\theta}\right)\right|\frac{d\theta}{2\pi} - L(f)\right| \leq A\left(\log\left(\sup_{\mathbb{P}^1} f^{\#}\right) + \sup_{\mathbb{P}^1}|g_f| + |\log r|\right)\frac{\sigma_2(n)}{d^n}$$

for some constant A which depends only on d.

So we pick $f \in \mathrm{Rat}_d$. In the following, the sums over subsets in $\mathrm{Crit}(f)$, $\mathrm{Fix}(f^n)$, or $\mathrm{Fix}^*(f^n)$ take into account the multiplicities of their elements. For any $n \in \mathbb{N}^*$, the cardinality of $\mathrm{Fix}(f^n)$ and that of $\mathrm{Fix}^*(f^n)$ are $d^n + 1$ and d_n, respectively, taking into account of the multiplicity of each element of them as a fixed point of f^n.

A non-quantitative version of Theorem 3.1 can be shown using the equidistribution of repelling cycles towards μ_f and Pesin theory (both arguments being non-quantitative [3]). We instead use formula (3.1) to relate the multiplier of a n-periodic point with the distance between its orbit and the critical set (Lemma 3.2). Summing over all n-periodic points, we show we can control the left-hand side in Theorem 3.1, using Lemma 3.3, with the difference between the logarithm of $[f^n(c), c]$ for all critical points c and the logarithm of the multipliers. We then use Lemmas 2.3 and 3.5 to control that difference and Fatou's inequality to bound the cardinality of attracting periodic orbits.

3.1. Relating multipliers with the distance between cycles and critical points. Recall that, by [38, Lemma 2.4], we have

$$\log(f^{\#}) = L(f) + \sum_{c \in \mathrm{Crit}(f)} \Phi_{g_f}(\cdot, c) + 2(g_f \circ f - g_f) \quad \text{on } \mathbb{P}^1 . \tag{3.1}$$

This formula plays a key role in the proofs of Lemma 3.2 and 3.3.

Lemma 3.2. *Assume that f has no super-attracting cycles. Then for any $n \in \mathbb{N}^*$ and any $z \in \mathrm{Fix}(f^n)$, we have*

$$\frac{1}{n}\Big| \sum_{c \in \mathrm{Crit}(f)} \sum_{j=0}^{n-1} \log\big[f^j(z), c\big] - \log\big|(f^n)'(z)\big|\Big| \leq B_1(f),$$

where $B_1(f) := L(f) + 2(2d-2)\sup_{\mathbb{P}^1}|g_f|$.

Proof. By (3.1) applied to f^n, we have

$$\begin{aligned}\log\big((f^n)^{\#}\big) &= L(f^n) + \sum_{\tilde{c} \in \mathrm{Crit}(f^n)} \Phi_{g_{f^n}}(\cdot, \tilde{c}) + 2\big(g_{f^n} \circ f^n - g_{f^n}\big)\\ &= n \cdot L(f) + \sum_{c \in \mathrm{Crit}(f)} \Big(\sum_{j=0}^{n-1} \int_{\mathbb{P}^1} \Phi_{g_f}(\cdot, w)\,\big((f^j)^*\delta_c\big)(w)\Big)\\ &\qquad + 2\big(g_f \circ f^n - g_f\big)\end{aligned}$$

on $\mathbb{P}^1$. By [40, Lemma 3.4], for every $a \in \mathbb{P}^1$,

$$\int_{\mathbb{P}^1} \Phi_{g_f}(\cdot, w)\,(f^*\delta_a)(w) = \Phi_{g_f}(f(\cdot), a) \quad \text{on } \mathbb{P}^1.$$

In particular, for every $z \in \mathrm{Fix}(f^n)$, since $(f^n)^{\#}(z) = |(f^n)'(z)|$, we have

$$\frac{1}{n}\log\big|(f^n)'(z)\big| = L(f) + \sum_{c \in \mathrm{Crit}(f)} \frac{1}{n}\sum_{j=0}^{n-1} \Phi_{g_f}\big(f^j(z), c\big),$$

which with the definition (2.1) of the g_f-kernel function Φ_{g_f} completes the proof. □

3.2. Reduction to the critical dynamics. For any $n \in \mathbb{N}^*$ and any $0 < r \leq 1$, we set

$$L_n^r(f) := \frac{1}{d_n}\int_0^{2\pi} \log\big|p_n\big(f, re^{i\theta}\big)\big|\frac{d\theta}{2\pi}.$$

If f has no super-attracting cycles, for any $m, n \in \mathbb{N}^*$ with $m|n$ and any $r \in]0,1]$, we set

$$\begin{aligned} u_{m,n}(f,r) &:= \frac{1}{d^m+1}\Bigg(\sum_{c\in \mathrm{Crit}(f)} \log\big[f^m(c), c\big] - \sum_{\substack{z\in \mathrm{Fix}(f^m)\\ |(f^n)'(z)|<r}} \frac{1}{m}\log\big|(f^m)'(z)\big| \Bigg) \\ &= u_{m,m}\big(f, r^{m/n}\big). \end{aligned}$$

Lemma 3.3. *If f has no super-attracting cycles, then for any $r \in]0,1]$ and any $n \in \mathbb{N}^*$,*

$$\epsilon_n(f,r) := \frac{1}{n(d^n+1)} \sum_{\substack{z\in \mathrm{Fix}(f^n)\\ |(f^n)'(z)|\geq r}} \log\big|(f^n)'(z)\big| - L(f) = u_{n,n}(f,r) + \epsilon_n'(f),$$

where $(d^n+1)|\epsilon_n'(f)| \leq 2(2d-2)\sup_{\mathbb{P}^1}|g_f|$.

Proof. Pick $r \in]0,1]$ and $n \in \mathbb{N}^*$, and set $\mu_n := \sum_{z\in \mathrm{Fix}(f^n)} \delta_z$, taking into account the multiplicity of each $z \in \mathrm{Fix}(f^n)$. Since $(f^n)^\#(z) = |(f^n)'(z)|$ for any $z \in \mathrm{Fix}(f^n)$, integrating the equation (3.1) against μ_n gives

$$\begin{aligned} \frac{1}{n}\int_{\mathbb{P}^1} \log\big|(f^n)'\big|\,\mu_n - (d^n+1)L(f) &= \int_{\mathbb{P}^1} \log(f^\#)\,\mu_n - (d^n+1)L(f) \\ &= \sum_{c\in \mathrm{Crit}(f)} \int_{\mathbb{P}^1} \Phi_{g_f}(c,\cdot)\,\mu_n\,. \end{aligned}$$

This may be rewritten

$$\begin{aligned} (d^n+1)\epsilon_n(f,r) &= \frac{1}{n}\int_{\{|(f^n)'|\geq r\}} \log\big|(f^n)'\big|\,\mu_n - (d^n+1)L(f) \\ &= \sum_{c\in \mathrm{Crit}(f)} \int_{\mathbb{P}^1} \Phi_{g_f}(c,\cdot)\,\mu_n - \int_{\{|(f^n)'|<r\}} \frac{1}{n}\log\big|(f^n)'\big|\,\mu_n. \end{aligned}$$

Using again that $(f^n)^\#(z) = |(f^n)'(z)|$ for any $z \in \mathrm{Fix}(f^n)$ and that, by [40, Lemma 3.5],

$$\int_{\mathbb{P}^1} \Phi_{g_f}(a,\cdot)\,\mu_n = \Phi_{g_f}\big(f^n(a), a\big)$$

for every $a \in \mathbb{P}^1$, the definition (2.1) of the g_f-kernel function Φ_{g_f} completes the proof. □

Lemma 3.4. *If f has no super-attracting cycles, then for any $n \in \mathbb{N}^*$ and any $r \in]0,1]$,*

$$\left| L_n^r(f) - L(f) - \frac{1}{d_n}\sum_{m|n} \mu\Big(\frac{n}{m}\Big)(d^m+1)u_{m,n}(f,r)\right| \leq B(f,r)\frac{\sigma_0(n)}{d_n},$$

where $B(f,r) := (2d-2)(2\sup_{\mathbb{P}^1}|g_f| + |\log r|)$.

Proof. Pick $r \in]0,1]$ and $n \in \mathbb{N}^*$. By the definition of d_n, we have

$$1 = \frac{1}{d_n}\sum_{m|n}\mu\Big(\frac{n}{m}\Big)(d^m+1).$$

For any $m \in \mathbb{N}^*$ dividing n and any $z \in \mathrm{Fix}(f^m)$, we have $(f^n)'(z) = (f^m)'(z)^{n/m}$ by the chain rule, and have

$$\Bigg|\sum_{\substack{z\in\mathrm{Fix}(f^m)\\ |(f^n)'(z)|\ge r}}\log\big|(f^m)'(z)\big| - \sum_{z\in\mathrm{Fix}(f^m)}\log\max\big\{|(f^m)'(z)|, r^{m/n}\big\}\Bigg| \le m(2d-2)|\log r|$$

since the number of attracting periodic points of f of period dividing m is at most $(2d-2)m$. Recalling the definition (2.2) of p_n, we have

$$\begin{aligned}
L_n^r(f) &= \frac{1}{nd_n}\sum_{j=1}^{d_n}\int_0^{2\pi}\log\big|(f^n)'\big(z_j^{(n)}\big) - re^{i\theta}\big|\frac{\mathrm{d}\theta}{2\pi}\\
&= \frac{1}{nd_n}\sum_{z\in\mathrm{Fix}^*(f^n)}\log\max\big\{|(f^n)'(z)|, r\big\}\\
&= \frac{1}{nd_n}\sum_{m|n}\mu\Big(\frac{n}{m}\Big)\sum_{z\in\mathrm{Fix}(f^m)}\frac{n}{m}\log\max\big\{|(f^m)'(z)|, r^{m/n}\big\}\\
&= \frac{1}{d_n}\sum_{m|n}\mu\Big(\frac{n}{m}\Big)\frac{1}{m}\sum_{z\in\mathrm{Fix}(f^m)}\log\max\big\{|(f^m)'(z)|, r^{m/n}\big\},
\end{aligned}$$

where the third equality is by the Möbius inversion. Hence recalling the definition of $\sigma_0(n)$, by Lemma 3.3, we have

$$\begin{aligned}
&\Big|L_n^r(f) - L(f) - \frac{1}{d_n}\sum_{m|n}\mu\Big(\frac{n}{m}\Big)(d^m+1)u_{m,n}(f,r)\Big|\\
&\le \frac{1}{d_n}\sum_{m|n}(d^m+1)\big|\epsilon_m(f,r^{m/n}) - u_{m,m}(f,r^{m/n})\big| + \frac{(2d-2)|\log r|\cdot\sigma_0(n)}{d_n}\\
&\le B(f,r)\frac{\sigma_0(n)}{d_n},
\end{aligned}$$

which completes the proof. □

3.3. Proof of Theorem 3.1 using the parametric version of Przytycki's lemma.

Lemma 3.5. *For any $n \in \mathbb{N}^*$, any $z \in \mathrm{Fix}(f^n)$, and any $c \in \mathrm{Crit}(f)$,*

$$\big[f^n(c), c\big] \le 2\cdot M(f)^n\cdot[c,z].$$

Proof. Let $M_1 := \sup_{\mathbb{P}^1} f^\# > 1$. It is clear that the map f is M_1-Lipschitz in the chordal metric $[\cdot,\cdot]$. If $f^n(z) = z$, we have

$$\left[f^n(c), c\right] \le \left[f^n(c), z\right] + [c, z] \le (M_1^n + 1) \cdot [c, z]$$

and the conclusion follows since $M_1 > 1$. □

Proof of Theorem 3.1. As there is no persistent parabolic and super-attracting cycle in Rat_d, the set X of all elements in Rat_d having neither super-attracting nor parabolic cycles and no multiple critical points is the complement of a pluripolar subset in Rat_d, so X is dense in Rat_d. Pick $f \in X$, $n \in \mathbb{N}^*$, and $r \in]0, 1]$.

(i) For any $m \in \mathbb{N}^*$ dividing n, recalling the definition of $u_{m,n}(f, r)$, we have

$$u_{m,n}(f, r) = u_{m,n}(f, 1) + \frac{1}{d^m + 1} \sum_{\substack{z \in \mathrm{Fix}(f^m) \\ r \le |(f^n)'(z)| < 1}} \frac{1}{m} \log |(f^m)'(z)|,$$

and recalling that f has at most $(2d - 2)m$ attracting periodic points of period dividing m and that $|(f^m)'(z)| = |(f^n)'(z)|^{m/n}$ for any $z \in \mathrm{Fix}(f^m)$ by the chain rule, we have

$$\left| \sum_{\substack{z \in \mathrm{Fix}(f^m) \\ r \le |(f^n)'(z)| < 1}} \frac{1}{m} \log |(f^m)'(z)| \right| \le \frac{(2d - 2)m}{n} |\log r| \le (2d - 2)|\log r|.$$

Hence, recalling the definition of $\sigma_0(n)$, we have

$$\left| \frac{1}{d_n} \sum_{m|n} \mu\left(\frac{n}{m}\right)(d^m + 1)\big(u_{m,n}(f, r) - u_{m,n}(f, 1)\big) \right| \le (2d - 2)|\log r| \cdot \frac{\sigma_0(n)}{d_n}.$$

(ii) For any $m \in \mathbb{N}^*$ dividing n, we have

$$\sum_{\substack{z \in \mathrm{Fix}(f^m) \\ |(f^n)'(z)| < 1}} \sum_{j=0}^{m-1} \log \left[f^j(z), c\right] = m \cdot \sum_{\substack{z \in \mathrm{Fix}(f^m) \\ |(f^m)'(z)| < 1}} \log[z, c].$$

Recalling the definition of $u_{m,n}(f, 1)$ and applying Lemma 3.2 to each $z \in \mathrm{Fix}(f^m)$ such that $|(f^n)'(z)| < 1$, we have

$$\begin{aligned}
&\left| (d^m + 1)u_{m,n}(f, 1) - \sum_{c \in \mathrm{Crit}(f)} \left(\log \left[f^m(c), c\right] - \sum_{\substack{z \in \mathrm{Fix}(f^m) \\ |(f^m)'(z)| < 1}} \log[z, c] \right) \right| \\
&\quad = \left| \sum_{\substack{z \in \mathrm{Fix}(f^m) \\ |(f^n)'(z)| < 1}} \frac{1}{m} \left(\sum_{c \in \mathrm{Crit}(f)} \sum_{j=0}^{m-1} \log \left[f^j(z), c\right] - \log |(f^m)'(z)| \right) \right| \\
&\quad \le (2d - 2)m \cdot B_1(f),
\end{aligned}$$

where the last inequality holds since f has at most $(2d-2)m$ attracting periodic points of period dividing m. Hence recalling the definition of $\sigma_1(n)$, we have

$$\begin{aligned}\Bigg|\frac{1}{d_n}\sum_{m|n}\mu\Big(\frac{n}{m}\Big)(d^m+1)u_{m,n}(f,1)\\ -\frac{1}{d_n}\sum_{m|n}\mu\Big(\frac{n}{m}\Big)\sum_{c\in\mathrm{Crit}(f)}\Bigg(\log\big[f^m(c),c\big]-\sum_{\substack{z\in\mathrm{Fix}(f^m)\\ |(f^m)'(z)|<1}}\log[z,c]\Bigg)\Bigg|\\ \leq (2d-2)B_1(f)\frac{\sigma_1(n)}{d_n}.\end{aligned}$$

We finally reduced the proof of Theorem 3.1 to estimating

$$\delta_n(f)=\delta_n(f,1):=\frac{1}{d_n}\sum_{m|n}\mu\Big(\frac{n}{m}\Big)\sum_{c\in\mathrm{Crit}(f)}\Bigg(\log[f^m(c),c]-\sum_{\substack{z\in\mathrm{Fix}(f^m)\\ |(f^m)'(z)|<1}}\log[z,c]\Bigg).$$

(iii) We claim that for any $c\in\mathrm{Crit}(f)$ and any $m\in\mathbb{N}^*$ dividing n,

$$\Bigg|\log\big[f^m(c),c\big]-\sum_{\substack{z\in\mathrm{Fix}(f^m)\\ |(f^m)'(z)|<1}}\log[z,c]\Bigg|\leq 2(2d-2)m^2\Big(\log M(f)-\log\Big(\frac{\kappa}{2}\Big)\Big),$$

where $\kappa\in(0,1)$ is the absolute constant appearing in Lemma 2.3; recall that $\sup_{z,w\in\mathbb{P}^1}[z,w]\leq 1$. Assume first that $\kappa\cdot M(f)^{-m}\leq[f^m(c),c]$. Then by Lemma 3.5, we deduce that for any $z\in\mathrm{Fix}(f^m)$,

$$\kappa\cdot 2^{-1}M(f)^{-2m}\leq 2^{-1}M(f)^{-m}\big[f^m(c),c\big]\leq[z,c],$$

so that since f has at most $(2d-2)m$ attracting periodic points of period dividing m, we have

$$-m\log M(f)+\log\kappa\leq\log\big[f^m(c),c\big]-\sum_{\substack{z\in\mathrm{Fix}(f^m)\\ |(f^m)'(z)|<1}}\log[z,c],$$

and

$$\begin{aligned}\log\big[f^m(c),c\big]-\sum_{\substack{z\in\mathrm{Fix}(f^m)\\ |(f^m)'(z)|<1}}\log[z,c]&\leq(2d-2)m\Big(2m\log M(f)-\log\Big(\frac{\kappa}{2}\Big)\Big)\\ &\leq 2(2d-2)m^2\Big(\log M(f)-\log\Big(\frac{\kappa}{2}\Big)\Big).\end{aligned}$$

Assume next that $\kappa\cdot M(f)^{-m}>[f^m(c),c]$. By Lemma 2.3 applied to the trivial family (f) and its (constant) marked critical point c (recall that the constant κ given by Lemma 2.3 depends only on d), c belongs to the immediate basin of an attracting periodic point z_0 of f of period k dividing m, and we have

$$2\big[f^m(c), c\big] \geq [z_0, c] \quad \text{and} \quad [c, \mathcal{J}_f] \geq \kappa M(f)^{-m}.$$

Hence we have $-\log 2 \leq \log[f^m(c), c] - \log[z_0, c]$, so that

$$-\log 2 \leq \log\big[f^m(c), c\big] - \sum_{\substack{z \in \mathrm{Fix}(f^m) \\ |(f^m)'(z)|<1}} \log[z, c].$$

Noting that any attracting $z \in \mathrm{Fix}(f^m) \setminus \{z_0\}$ lies in a Fatou component of f which does not contain z_0, we also have $1 \geq [z, c] \geq [c, \mathcal{J}_f] \geq \kappa M(f)^{-m}$ for every such $z \in \mathrm{Fix}(f^m) \setminus \{z_0\}$ that $|(f^m)(z)| < 1$. Moreover, by Lemma 3.5, we have $[f^m(c), c] \leq M(f)^m [z_0, c]$. Hence, since f has at most $(2d-2)m$ attracting periodic points of period dividing m, we have

$$\begin{aligned}\log\big[f^m(c), c\big] - \sum_{\substack{z \in \mathrm{Fix}(f^m) \\ |(f^m)'(z)|<1}} \log[z, c] &\leq (2d-2)m \cdot \big(m \log M(f) - \log \kappa\big) \\ &\leq (2d-2)m^2 \big(\log M(f) - \log \kappa\big).\end{aligned}$$

Hence the claim holds.

Since f has exactly $2d-2$ critical points taking into account their multiplicities, letting $C_2 := 2(2d-2)^2 \max\{1, |\log(\kappa/2)|\}$, we have

$$|\delta_n(f)| \leq C_2 \cdot \big(\log M(f) + 1\big) \frac{\sigma_2(n)}{d_n},$$

by the definition of $\sigma_2(n)$.

(iv) Recall that $\frac{1}{2} \log d \leq L(f) = \int_{\mathbb{P}^1} \log(f^\#)\, \mu_f \leq \log(\sup_{\mathbb{P}^1} f^\#)$ and that by definition of d_n,

$$d_n = \sum_{m|n} \mu\Big(\frac{n}{m}\Big)(d^m + 1) \geq \sum_{m|n} \mu\Big(\frac{n}{m}\Big) d^m \geq \big(1 - d^{-1}\big)\, d^n.$$

Hence, all the above intermediate estimates yield

$$\left|\frac{1}{d_n} \int_0^{2\pi} \log\big|p_n\big(f, re^{i\theta}\big)\big| \frac{\mathrm{d}\theta}{2\pi} - L(f)\right| \leq B_3(f, r) \frac{\sigma_2(n)}{d^n} \tag{3.2}$$

for any $f \in X$, where $B_3(f, r) = C_3(\sup_{\mathbb{P}^1} |g_f| + \log(\sup_{\mathbb{P}^1} f^\#) + |\log r|)$ for some constant $C_3 > 0$ depending only on d. Since both sides of (3.2) depend continuously on $f \in \mathrm{Rat}_d$ and X is dense in Rat_d, the above estimate (3.2) still holds for any $f \in \mathrm{Rat}_d$. $\square$

3.4. Application: degeneration of the Lyapunov exponent. Consider a holomorphic family $(f_t)_{t\in\mathbb{D}^*}$ of degree $d>1$ rational maps parametrized by the punctured unit disk, and assume it extends to a meromorphic family over $\mathbb{D}$, i.e. $f_t\in\mathcal{O}(\mathbb{D})[t^{-1}](z)$.

Theorem 3.6. *There exists a non-negative $\alpha\in\mathbb{R}$ such that, as $t\to 0$,*

$$L(f_t)=\alpha\cdot\log|t|^{-1}+o\big(\log|t|^{-1}\big).$$

This is a special case of [21, Theorem C] and can also be obtained as the combination of [12, Proposition 3.1] and [11, Theorem 1.4]. We provide here a simple proof as an application of Theorem 3.1.

Proof. We can write $p_n(f_t,w)=t^{-N_n}h_n(t,w)$, where $h_n:\mathbb{D}\times\mathbb{C}\to\mathbb{C}$ is holomorphic and $N_n\in\mathbb{N}$. We rely on the following key lemma.

Lemma 3.7. *There exist $C_1,C_2>0$ such that for any $t\in\mathbb{D}^*_{1/2}$,*

$$\sup_{z\in\mathbb{P}^1}\max\big\{|g_{f_t}(z)|,\log(f_t^{\#}(z))\big\}\le C_1\log|t|^{-1}+C_2.$$

Once Lemma 3.7 is at our disposal, by Theorem 3.1, there is $C>0$ such that for any $n\in\mathbb{N}^*$ and any $t\in\mathbb{D}^*_{1/2}$,

$$\left|L(f_t)-\frac{1}{d_n}\int_0^{2\pi}\log|p_n\big(f_t,e^{i\theta}\big)|\frac{d\theta}{2\pi}\right|\le C\big(C_1\log|t|^{-1}+C_2\big)\cdot\frac{\sigma_2(n)}{d^n},$$

so that dividing both sides by $\log|t|^{-1}$ and making $t\to 0$, there is $C'>0$ such that for all $n\in\mathbb{N}^*$

$$\frac{N_n}{d_n}-C'\frac{\sigma_2(n)}{d^n}\le\liminf_{t\to 0}\frac{L(f_t)}{\log|t|^{-1}}\le\limsup_{t\to 0}\frac{L(f_t)}{\log|t|^{-1}}\le\frac{N_n}{d_n}+C'\frac{\sigma_2(n)}{d^n}.$$

Indeed, as $p_n(f_t,w)=t^{-N_n}h_n(t,w)$, where h_n is analytic, we get

$$\frac{1}{\log|t|^{-1}}\int_0^{2\pi}\log\big|p_n\big(f_t,e^{i\theta}\big)\big|\frac{d\theta}{2\pi}=N_n+\int_0^{2\pi}\frac{\log\big|h_n\big(t,e^{i\theta}\big)\big|}{\log|t|^{-1}}\frac{d\theta}{2\pi}=N_n+o(1)$$

as $t\to 0$. Making $n\to\infty$, we get

$$\lim_{t\to 0}\frac{L(f_t)}{\log|t|^{-1}}=\lim_{n\to\infty}\frac{N_n}{d_n}=:\alpha\ge 0.$$

This concludes the proof. □

Proof of Lemma 3.7. There is a meromorphic family $(F_t)_{t\in\mathbb{D}}$ of homogeneous polynomial endomorphisms of $\mathbb{C}^2$ such that for every $t\in\mathbb{D}^*$, F_t is a lift of f_t and that the holomorphic function $t\mapsto\mathrm{Res}(F_t)$ on $\mathbb{D}$ may vanish only at $t=0$.

According to [12, Lemma 3.3] (or [21, Proposition 4.4]), there exist constants $C \geq 1$ and $\beta > 0$ such that for any $p \in \mathbb{C}^2 \setminus \{0\}$ and any $t \in \mathbb{D}^*$,

$$\frac{1}{C}|t|^\beta \leq \frac{\|F_t(p)\|}{\|p\|^d} \leq C. \tag{3.3}$$

For any $t \in \mathbb{D}^*$, set $u_t(z) := \log(\|F_t(p)\|/\|p\|^d)$ on $\mathbb{P}^1$, where $p \in \pi^{-1}(z)$. The function u_t on $\mathbb{P}^1$ is well-defined by the homogeneity of F_t. Recalling the definition of g_{F_t}, we have $g_{F_t}(z) = \sum_{n=0}^\infty (u_t \circ f_t^n(z))/d^{n+1}$ uniformly on $\mathbb{P}^1$, so that by (3.3),

$$\sup_{z\in\mathbb{P}^1} |g_{F_t}(z)| \leq \frac{1}{d-1} \sup_{z\in\mathbb{P}^1} |u_t(z)| \leq \frac{1}{d-1}\big(\beta \log|t|^{-1} + \log C\big).$$

Recalling the definition of $I_{g_{F_t}}$ and the formula $I_{g_{F_t}} = -(\log|\mathrm{Res}(F_t)|)/(d(d-1))$, we also have $g_{f_t} = g_{F_t} + (\log|\mathrm{Res}(F_t)|)/(2d(d-1))$ on $\mathbb{P}^1$ for every $t \in \mathbb{D}^*$. Hence we obtain the desired upper bound of $\sup_{z\in\mathbb{P}^1} |g_{f_t}(z)|$ since $t \mapsto \mathrm{Res}(F_t)$ is a holomorphic function on $\mathbb{D}$ vanishing only at $t = 0$.

To conclude the proof, we use the same strategy for giving an upper bound for $\log\sup_{z\in\mathbb{P}^1} f_t^\#(z)$. Recall the following formula

$$f_t^\#(z) = \frac{1}{d}|\det DF_t(p)| \frac{\|p\|^2}{\|F_t(p)\|^2}$$

on $\mathbb{P}^1$, where $p \in \pi^{-1}(z)$ (see, e.g., [29, Theorem 4.3]) and in particular, by (3.3), we have

$$\log(f_t^\#(z)) \leq \log\frac{|\det DF_t(p)|}{d\cdot\|p\|^{2d-2}} + 2\big(\beta\log|t|^{-1} + \log C\big).$$

Now write as

$$F_t = (P_t, Q_t), \quad P_t(z,w) = \sum_{j=0}^d a_j(t) z^j w^{d-j}, \quad Q_t(z,w) = \sum_{j=0}^d b_j(t) z^j w^{d-j}$$

with $a_j(t) = t^{-\gamma}\tilde{a}_j(t)$ and $b_j(t) = t^{-\gamma}\tilde{b}_j(t)$ for some $\gamma \in \mathbb{N}$ and some $\tilde{a}, \tilde{b} \in \mathcal{O}(\mathbb{D})$. In particular, there exists a constant $C' \geq 1$ such that for any $t \in \mathbb{D}(0, 1/2)$ and any $0 \leq j \leq d$, we have $\max\{|\tilde{a}_j(t)|, |\tilde{b}_j(t)|\} \leq C'$, so that

$$\begin{aligned}
|\det DF_t(p)| &= \left|\frac{\partial P_t}{\partial z}(p)\frac{\partial Q_t}{\partial w}(p) - \frac{\partial P_t}{\partial w}(p)\frac{\partial Q_t}{\partial z}(p)\right| \\
&\leq 2d^2|t|^{-2\gamma}\sum_{j,\ell=0}^d |\tilde{a}_j(t)\tilde{b}_\ell(t)|\cdot\|p\|^{2(d-1)} \\
&\leq 2d^4\cdot C'^2\|p\|^{2(d-1)}\cdot|t|^{-2\gamma}
\end{aligned}$$

for any $p \in \mathbb{C}^2$. This gives a constant $C'' \geq 1$ so that

$$\log(f_t^{\#}(z)) \leq 2(\beta+\gamma)\log|t|^{-1} + \log C'' \quad \text{on } \mathbb{P}^1,$$

which completes the proof. □

4. Equidistribution towards the bifurcation currents

Fix an integer $d > 1$. Let Λ be a quasi-projective variety either such that $\Lambda \subset \mathcal{M}_d$, or parametrizing an algebraic family $(f_\lambda)_{\lambda\in\Lambda}$ of degree d rational maps on $\mathbb{P}^1$.

4.1. The proof of Theorem 1.1. Pick any compact subset K in Λ, and set $C_1(K) := \sup_{\lambda\in K} C([f_\lambda]) \geq \frac{1}{2}\log d$, where $C([f_\lambda])$ is given by Theorem 3.1. We remark that for every $n \in \mathbb{N}^*$ and every $\rho \in]0,1]$,

$$T_n^1(\rho) := \frac{1}{d_n}\int_0^{2\pi} \big[\mathrm{Per}_n\,(\rho e^{i\theta})\big]\frac{\mathrm{d}\theta}{2\pi} = dd^c\Big(\frac{1}{d_n}\int_0^{2\pi} \log\big|p_n(\lambda,\rho e^{i\theta})\big|\frac{\mathrm{d}\theta}{2\pi}\Big).$$

Pick any $1 \leq p \leq \min\{m, 2d-2\}$, any $\underline{n} = (n_1,\dots,n_p) \in (\mathbb{N}^*)^p$, and any $\underline{\rho} = (\rho_1,\dots,\rho_p) \in]0,1]^p$.

Assume first that $p = 1$, i.e. $\underline{n} = n \in \mathbb{N}^*$ and $\underline{\rho} = \rho \in]0,1]$, and pick any continuous DSH $(m-1,m-1)$-form Ψ on Λ supported in K. By definition, we can write $dd^c\Psi = T^+ - T^-$, where $T^\pm$ are positive measures of finite masses on Λ. By Stokes's formula and Theorem 3.1, we have

$$\begin{aligned}
\big|\langle T_n^1(\rho) - T_{\mathrm{bif}}, \Psi\rangle\big| &= \left|\int_K \Big(\frac{1}{d_n}\int_0^{2\pi} \log\big|p_n(\lambda,\rho e^{i\theta})\big|\frac{\mathrm{d}\theta}{2\pi} - L(\lambda)\Big) dd^c\Psi\right| \\
&\leq \int_K \left|\frac{1}{d_n}\int_0^{2\pi} \log\big|p_n(\lambda,\rho e^{i\theta})\big|\frac{\mathrm{d}\theta}{2\pi} - L(\lambda)\right| (T^+ + T^-) \\
&\leq \Big(AC_1(K)\big(1+|\log\rho|\big)\frac{\sigma_2(n)}{d^n}\Big)\big(\|T^+\| + \|T^-\|\big),
\end{aligned}$$

which completes the proof of Theorem 1.1 in this case by the definition of $\|\Psi\|^*_{\mathrm{DSH}}$.

We now assume that $2 \leq p \leq \min\{m, 2d-2\}$. Setting $S_j = S_j(\underline{n},\underline{\rho}) := (\bigwedge_{1\leq \ell<j} T_{\mathrm{bif}}) \wedge (\bigwedge_{j<k\leq p} T_{n_k}(\rho_k))$ for any $1 \leq j \leq p$, which is a positive closed current of bidegree $(p-1,p-1)$ on Λ, we have

$$T_{\underline{n}}^p(\underline{\rho}) - T_{\mathrm{bif}}^p = \sum_{j=1}^p S_j \wedge \big(T_{n_j}(\rho_j) - T_{\mathrm{bif}}\big). \tag{4.1}$$

Pick any continuous DSH $(m-p,m-p)$-form Ψ on Λ supported in K, and write $dd^c\Psi = T^+ - T^-$ where $T^\pm$ are positive closed $(m-p+1,m-p+1)$ currents of finite masses on Λ. Then by Stokes's formula, we have

$$\begin{aligned}\left\langle T^p_{\underline{n}}(\underline{\rho}) - T^p_{\mathrm{bif}}, \Psi\right\rangle &= \sum_{j=1}^{p}\left\langle S_j \wedge \left(T^1_{n_j}(\rho_j) - T_{\mathrm{bif}}\right), \Psi\right\rangle \\ &= \sum_{j=1}^{p}\int_K\left(\frac{1}{d_n}\int_0^{2\pi}\log\left|p_n\left(\lambda,\rho e^{i\theta}\right)\right|\frac{d\theta}{2\pi} - L(\lambda)\right)S_j \wedge dd^c\Psi.\end{aligned}$$

Since the masses can be computed in cohomology, by (2.3), there is $C_2 > 0$ independent of K, Ψ and $T^{\pm}$, such that for every $1 \leq j \leq p$,

$$\int_\Lambda S_j \wedge (T^+ + T^-) \leq C_2\left(\|T^+\| + \|T^-\|\right).$$

Then by Theorem 3.1, we have

$$\begin{aligned}&\left|\int_K\left(\frac{1}{d_n}\int_0^{2\pi}\log\left|p_n\left(\lambda,\rho e^{i\theta}\right)\right|\frac{d\theta}{2\pi} - L(\lambda)\right)S_j \wedge dd^c\Psi\right| \\ &\qquad\leq \left(AC_1(K)\left(1 + |\log\rho_j|\right)\frac{\sigma_2(n_j)}{d^{n_j}}\right)\int_\Lambda S_j \wedge (T^+ + T^-) \\ &\qquad\leq AC_1(K)C_2\left(1 + |\log\rho_j|\right)\frac{\sigma_2(n_j)}{d^{n_j}}\left(\|T^+\| + \|T^-\|\right)\end{aligned}$$

for any $1 \leq j \leq p$, which completes the proof of Theorem 1.1. □

Remark 4.1. As in [2], we deduce from Theorem 1.1 the density in the support of T^p_{bif} of parameters having p distinct neutral cycles. We can actually give a more precise statement: taking any sequence of p-tuple of integers $(\underline{n}_k)$ in $(\mathbb{N}^*)^p$ such that $\min_j n_{j,k} \to \infty$, we have that the set of parameters λ such that f_λ has p distinct neutral cycles of exact periods $n_{1,k}, \ldots, n_{p,k}$ for some $k \in \mathbb{N}^*$ is dense in the support of T^p_{bif}.

4.2. The proof of Corollary 1.2. Pick $1 \leq p \leq \min\{m, 2d-2\}$. We recall some basics on PB measures. For each $\rho > 0$, let $\lambda_{\mathbb{S}_\rho}$ the Lebesgue probability measure on the circle $\mathbb{S}_\rho$. Let $\theta\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function with compact support in $]0,1[$ such that $\int_0^1 \theta(x)dx = 1$. We consider the smooth measure $\widetilde{\nu}$ defined as

$$\widetilde{\nu} := \bigotimes_{j=1}^{p}\int_0^1 \lambda_{\mathbb{S}_{\rho_j}}\theta(\rho_j)d\rho_j.$$

We say that a probability measure ν on $(\mathbb{P}^1)^p$ is PB (or has bounded potential) if there exists a constant $C \geq 0$ such that

$$|\langle \nu - \widetilde{\nu}, \varphi\rangle| \leq C\|\varphi\|^*_{\mathrm{DSH}}$$

for all φ which is DSH on $(\mathbb{P}^1)^p$, and then let $C_\nu \geq 0$ be the minimal $C \geq 0$ satisfying the above inequality for every φ. For example, $\widetilde{\nu}$ is PB on $(\mathbb{P}^1)^p$, and $\lambda_{\mathbb{S}_\rho}$ is PB on $\mathbb{P}^1$. We claim that the positive closed (p, p)-current

$$T_{\underline{n}}^p(\nu) := \frac{1}{d_{|\underline{n}|}} \int_{(\mathbb{P}^1)^p} \bigwedge_{j=1}^{p} \left[\mathrm{Per}_{n_j}(w_j)\right] \nu(w_1, \dots, w_p)$$

on Λ is well-defined for any PB measure ν on $(\mathbb{P}^1)^p$ and any $\underline{n} = (n_1, \dots, n_p) \in (\mathbb{N}^*)^p$. Indeed, the set of all $\underline{w} = (w_1, \dots, w_p) \in \mathbb{C}^p$ such that $\bigcap_{i=1}^p \mathrm{Per}_{n_i}(w_i)$ is not of pure codimension p in Λ is analytic. Hence for any $\underline{w} = (w_1, \dots, w_p) \in \mathbb{C}^p$ except for a pluripolar subset and any $\underline{n} = (n_1, \dots, n_p) \in (\mathbb{N}^*)^p$, the current $\bigwedge_{i=1}^p [\mathrm{Per}_{n_i}(w_i)]$ on Λ is well defined. In particular, since PB measures on $(\mathbb{P}^1)^p$ give no mass to pluripolar sets, the current $T_{\underline{n}}^p(\nu)$ is also well defined.

Observe that $T_{\underline{n}}^p(\widetilde{\nu})$ give no mass to pluripolar sets (hence to analytic sets) since it has bounded potentials. So for any PB measures on $(\mathbb{P}^1)^p$, $T_{\underline{n}}^p(\nu)$ gives no mass to analytic sets.

Here is another description of $(d_{|\underline{n}|})^{-1} \bigwedge_{i=1}^p [\mathrm{Per}_{n_i}(w_i)]$ and $T_{\underline{n}}^p(\nu)$; let $\Gamma_{\underline{n}}$ be the analytic set of dimension m in $\Lambda \times (\mathbb{P}^1)^p$ defined as

$$\Gamma_{\underline{n}} := \left\{(\lambda, (z_1, \dots, z_p)) \in \Lambda \times (\mathbb{P}^1)^p : z_j \in \mathrm{Fix}^*(f_\lambda^{n_j}) \text{ for every } 1 \leq j \leq p\right\}.$$

Let $F_{\underline{n}}\colon \Gamma_{\underline{n}} \to (\mathbb{P}^1)^p$ be a holomorphic map defined by

$$F_{\underline{n}}(\lambda, z_1, \dots, z_p) = \left((f_\lambda^{n_1})'(z_1), \dots, (f_\lambda^{n_p})'(z_p)\right),$$

and $\mathscr{P}\colon \Gamma_{\underline{n}} \to \Lambda$ be the restriction to $\Gamma_{\underline{n}}$ of the projection $\Lambda \times (\mathbb{P}^1)^p \to \Lambda$. Consider $\pi\colon \widetilde{\Gamma}_{\underline{n}} \to \Gamma_{\underline{n}}$ a desingularization of $\Gamma_{\underline{n}}$. The map $\widetilde{F}_{\underline{n}} := F_{\underline{n}} \circ \pi$ is holomorphic and the map $\widetilde{\mathscr{P}} := \mathscr{P} \circ \pi$ is an analytic map. If ν is a smooth PB measure in $(\mathbb{P}^1)^p$, then:

$$T_{\underline{n}}^p(\nu) = \frac{1}{\prod_j n_j d_{n_j}} \widetilde{\mathscr{P}}_*\left(\left(\widetilde{F}_{\underline{n}}|_{\widetilde{\Gamma}_{\underline{n}}}\right)^*(\nu)\right) \quad \text{on } \Lambda. \tag{4.2}$$

Indeed, observe that, when testing against a smooth form, there is always one term that is smooth when computing the pull-back and push-forward.

Theorem 4.2. *Let Λ be a quasi-projective variety either such that $\Lambda \subset \mathcal{M}_d$, or parametrizing an algebraic family $(f_\lambda)_{\lambda\in\Lambda}$ of degree d rational maps on $\mathbb{P}^1$. Then for any compact subset K in Λ, there exists $C(K) > 0$ such that for any $1 \leq p \leq \min\{m, 2d-2\}$, any $\underline{n} = (n_1, \dots, n_p) \in (\mathbb{N}^*)^p$, any PB measure ν in $(\mathbb{P}^1)^p$, and any continuous* DSH*-form Ψ of bidegree $(m-p, m-p)$ on Λ supported in K, we have*

$$\left|\left\langle T_{\underline{n}}^p(\nu) - T_{\mathrm{bif}}^p, \Psi\right\rangle\right| \leq C(K) \cdot (1 + C_\nu) \max_{1\leq j\leq p} \left(\frac{1}{n_j}\right) \|\Psi\|_{\mathrm{DSH}}.$$

Proof. Pick $1 \leq p \leq \min\{m, 2d-2\}$ and a PB measure ν on $(\mathbb{P}^1)^p$. Consider first the case where ν is smooth. Pick $\underline{n} = (n_1, \ldots, n_p) \in (\mathbb{N}^*)^p$ and a smooth DSH form Ψ of bidegree $(m-p, m-p)$ on Λ with compact support in K. By Theorem 1.1 and our choice of $\widetilde{\nu}$, there is $C(K) > 0$ depending only on K such that

$$\left|\left\langle T^p_{\underline{n}}(\widetilde{\nu}) - T^p_{\mathrm{bif}}, \Psi\right\rangle\right| \leq C(K) \max_{1\leq j\leq p} \left(\frac{\sigma_2(n_j)}{d^{n_j}}\right) \|\Psi\|_{\mathrm{DSH}},$$

and we will show that

$$\left|\left\langle T^p_{\underline{n}}(\widetilde{\nu}) - T^p_{\underline{n}}(\nu), \Psi\right\rangle\right| \leq C(K) C_\nu \max_{1\leq j\leq p} \left(\frac{1}{n_j}\right) \|\Psi\|_{\mathrm{DSH}}.$$

By the above description of $T^p_{\underline{n}}(\nu)$ and the definition of PB measures, we have

$$\left|\left\langle T^p_{\underline{n}}(\widetilde{\nu}) - T^p_{\underline{n}}(\nu), \Psi\right\rangle\right| \leq C_\nu \left\| \frac{1}{\prod_j n_j d_{n_j}} \left(\widetilde{F}_{\underline{n}}\right)_* \left(\widetilde{\mathcal{P}}^*(\Psi)\right) \right\|^*_{\mathrm{DSH}}.$$

As taking dd^c commutes with taking pull-pack or push-forward, writing as $dd^c\Psi = T^+ - T^-$, where $T^\pm$ are smooth (because Ψ is smooth) positive closed currents of bidegree $(m-p+1, m-p+1)$ of finite masses on Λ, one simply has to estimate the mass $\|(\widetilde{F}_{\underline{n}})_*(\widetilde{\mathcal{P}}^*(T^\pm))\|$. Computing those masses can be done in cohomology testing against $\bigwedge_{i\neq j} \omega_i$ for all $1 \leq j \leq p$, where ω_i is the Fubini Study form on the i-th factor of $(\mathbb{P}^1)^p$. Set $S_{\underline{n},j} := (\prod_{\ell=1}^p n_\ell d_{n_\ell})^{-1} \widetilde{\mathcal{P}}_*((\widetilde{F}_{\underline{n}})^*(\bigwedge_{i\neq j} \omega_i))$.

By duality, this computation is the same as controlling $\langle S_{\underline{n},j},\ T^\pm\rangle$ for any j. Finally, for any j, one has to control the mass $\|S_{\underline{n},j}\|$. By symmetry, consider the case where $j = p$. Let $\underline{n}' = (n_1, \ldots, n_{p-1})$ and consider the associated map $\widetilde{F}_{\underline{n}'}$. Now take a generic point $(z_1^0, \ldots z_{p-1}^0) \in (\mathbb{P}^1)^{p-1}$ and consider the line

$$L := \left\{z = (z_1, \ldots, z_p) \in (\mathbb{P}^1)^p,\ \forall i \leq p-1,\ z_i = z_i^0\right\}.$$

Then the degree of $\widetilde{F}_{\underline{n}}^{-1}(L)$ equals d_{n_p} times the degree of $\widetilde{F}_{\underline{n}'}^{-1}(z_1^0, \ldots z_{p-1}^0))$. So pushing-forward, we see that

$$\|S_{\underline{n},p}\| \leq C \frac{1}{n_p},$$

for some constant $C \geq 0$ that does not depend on $\underline{n}$.

In particular, we deduce that

$$\prod_{j=1}^p (n_j d_{n_j})^{-1} \left\|\left(\widetilde{F}_{\underline{n}}\right)_* (\widetilde{\mathcal{P}}^*(\Psi))\right\|^*_{\mathrm{DSH}} \leq C \cdot \max_{1\leq j\leq p} \left(\frac{1}{n_j}\right) \|\Psi\|_{\mathrm{DSH}},$$

where $C \geq 0$ is (another) constant that does not depend on $\underline{n}$, which implies the wanted result for Ψ and ν smooth. By a regularization argument [17], the result follows for Ψ continuous, replacing $C(K)$ by a constant given by a (small) larger neighborhood of K. Finally, we extend the result to any PB measure ν using again an approximation of ν. $\square$

Corollary 1.2 follows from Theorem 4.2 using classical pluripotential techniques as in [16] or [28].

Remark 4.3. The order $O(\max_j(n_j^{-1}))$ as $\min_j(n_j) \to \infty$ in the right-hand side is sharp. Indeed, for the quadratic polynomials family $(z^2+\lambda)_{\lambda\in\mathbb{C}}$, it has been shown in [7] that the sequence $(2^{-n+1}[\mathrm{Per}(n,e^{2n})])_n$ (recall $2_n \sim 2^n$ as $n\to\infty$) of measures on $\mathbb{C}$ converges to $dd^c \max\{g, 4-2\log 2\}$, where g is the Green function of the Mandelbrot set. Since $4-2\log 2 > 0$, this measure is not proportional to μ_{bif}. On the other hand, if $\nu_n = \lambda_{\mathbb{S}_{e^{2n}}}$ we have $C_{\nu_n} = O(n)$ as $n\to\infty$, where $\lambda_{\mathbb{S}_{e^{2n}}}$ is the probability Lebesgue measure on the circle of center 0 and radius e^{2n} in $\mathbb{C}$, which is PB. So one cannot improve the order $O(n^{-1})$ as $n\to\infty$ in the right-hand side for this family; otherwise, $2^{-n}[\mathrm{Per}(n,e^{2n})]$ would tend to μ_{bif} as $n\to\infty$.

5. Transversality of periodic critical orbit relations

5.1. Infinitesimal deformations of rational maps. Pick $f \in \mathrm{Rat}_d$. The orbit

$$\mathcal{O}(f) := \left\{\phi^{-1}\circ f\circ\phi \in \mathrm{Rat}_d : \phi \in \mathrm{PSL}_2(\mathbb{C})\right\}$$

of f under the conjugacy action of $\mathrm{PSL}_2(\mathbb{C})$ on Rat_d is a 3 dimensional complex analytic submanifold in Rat_d.

A tangent vector to Rat_d at f is an equivalence class of holomorphic maps $\phi\colon \mathbb{D}\to\mathrm{Rat}_d$ such that $\phi(0)=f$ under the relation $\phi\sim\psi$ iff $\phi'(0)=\psi'(0)$. The vector space of all tangent vectors at f is denoted by $T_f\,\mathrm{Rat}_d$. A tangent vector $\zeta\in T_f\,\mathrm{Rat}_d$ can be identified to a section of the line bundle $f^*(T\mathbb{P}^1)$, where $T\mathbb{P}^1$ denotes the holomorphic tangent bundle on $\mathbb{P}^1$. Moreover, to any tangent vector $\zeta\in T_f\,\mathrm{Rat}_d$, we attach a rational vector field η_ζ on $\mathbb{P}^1$ whose poles are in $\mathrm{Crit}(f)$ by letting

$$\eta_\zeta(z) := -\,D_z f^{-1}\cdot\zeta(z) \in T_z\mathbb{P}^1, \quad z\in\mathbb{P}^1.$$

If f has only simple critical points, then η_ζ also has only simple poles (see [6] for more details).

If f is postcritically finite, i.e., the postcritical set

$$\mathcal{P}(f) := \bigcup_{n\in\mathbb{N}^*} f^n(\mathrm{Crit}(f))$$

of f is a finite subset in $\mathbb{P}^1$, then we denote by $\mathcal{T}(\mathcal{P}(f))$ the vector field on $\mathcal{P}(f)$, and a vector field $\tau\in\mathcal{T}(\mathcal{P}(f))$ is said to be *guided* by $\zeta\in T_f\,\mathrm{Rat}_d$ if

$$\tau = f^*\tau + \eta_\zeta \text{ on } \mathcal{P}(f) \quad\text{and}\quad \tau\circ f = \zeta \text{ on } \mathrm{Crit}(f).$$

For the sequel, we will rely on the following crucial result (see [6, 22]).

Proposition 5.1 (Buff–Epstein). *If $f \in \mathrm{Rat}_d$ is postcritically finite and neither is conjugate to $z^{\pm 2}$ nor is a Lattès map, then a tangent vector $\zeta \in T_f \mathrm{Rat}_d$ is tangent to $\mathcal{O}(f)$ if and only if there is a vector field $\tau \in \mathcal{T}(\mathcal{P}(f))$ guided by ζ.*

5.2. A transversality of periodic critical orbit relations. Let $f \in \mathrm{Rat}_d$ be postcritically finite and hyperbolic of disjoint type, and let $c_1, \dots, c_{2d-2}$ be $2d-2$ distinct critical points of f. For any $1 \le i \le 2d-2$, there is $p_i \in \mathbb{N}^*$ such that $c_i \in \mathrm{Fix}^*(f^{p_i})$, and there is an open neighborhood U of f in Rat_d small enough so that $c_1, \dots, c_{2d-2}$ can be followed holomorphically on U, that is for any $1 \le i \le 2d-2$, there is a holomorphic map $c_i \colon U \to \mathbb{P}^1$ such that $c_i(f) = c_i$ and that $c_i(g) \in \mathrm{Crit}(g)$ for every $g \in U$.

We can choose an atlas of $\mathbb{P}^1$ such that there is an affine chart of $\mathbb{P}^1$ containing $c_1(g), \dots, c_{2d-2}(g)$ for every $g \in U$, and define a map $\mathcal{V} \colon U \longrightarrow \mathbb{C}^{2d-2}$ by

$$\mathcal{V}(g) := \big(g^{p_1}(c_1(g)) - c_1(g), \dots, g^{p_{2d-2}}(c_{2d-2}(g)) - c_{2d-2}(g)\big), \quad g \in U.$$

We will need the following.

Theorem 5.2. *Let $f \in \mathrm{Rat}_d$ be postcritically finite and hyperbolic of disjoint type. If f is not conjugate to $z^{\pm 2}$, then the linear map $D_f\mathcal{V} \colon T_f \mathrm{Rat}_d \to T_0\mathbb{C}^{2d-2}$ is surjective and* $\ker(D_f\mathcal{V}) = T_f\mathcal{O}(f)$.

Though this result seems folklore, we could not find it in the above form in the literature. We provide here a proof for the sake of completeness, which is very much inspired by [6, 20] (see also [22]).

5.3. The proof of Theorem 5.2. From now on, we write

$$\dot{u} := \left.\frac{du_t}{dt}\right|_{t=0}$$

for any holomorphic map $t \mapsto u_t$ defined on a disk $\mathbb{D}$.

Proof of Theorem 5.2. Under our assumption, the postcritically finite map f is neither a Lattès map, nor conjugate to $z^{\pm 2}$. Let us pick $\zeta \in \ker(D_f\mathcal{V})$, and choose a holomorphic disc $t \mapsto f_t \in \mathrm{Rat}_d$ with $f_0 = f$ and such that $\dot{f} = \zeta$. We shall use Proposition 5.1 and build a vector field $\tau \in \mathcal{T}(\mathcal{P}(f))$ which is guided by ζ. Then counting dimensions will complete the proof.

For any $n \in \mathbb{N}$ and any $1 \le i \le 2d-2$, we set $c_i(t) = c_i(f_t)$,

$$v_{n,i}(t) := f_t^n(c_i(t)),$$

$c_i := c_i(0)$, and $v_{n,i} := v_{n,i}(0)$. It is clear that for any $n \ge 0$, we have

$$\dot{v}_{n+1,i} = \zeta(v_{n,i}) + D_{v_{n,i}} f \cdot \dot{v}_{n,i}\,. \tag{5.1}$$

We shall deduce the following from this equation.

Lemma 5.3. *Fix $1 \leq i \leq 2d-2$. For all $n, m \in \mathbb{N}^*$, if $v_{n,i} = v_{m,i}$, then $\dot{v}_{n,i} = \dot{v}_{m,i}$.*

Taking this result for granted, we continue to define a vector field τ on $\mathcal{P}(f)$ that is guided by ζ. For any $x \in \mathcal{P}(f)$, we set $\tau(x) := \dot{v}_{n,i}$ for some $1 \leq i \leq 2d-2$ and some $n \in \mathbb{N}$ such that $x = v_{n,i}$. Since f is of disjoint type, the previous lemma shows that τ is well-defined at x. It remains to check that τ is guided by ζ. The equality $\tau(f(c_i)) = \zeta(c_i)$ follows from the definition of τ and (5.1). When $x = v_{n,i}$ is not a critical point, then multiplying (5.1) by $D_x f^{-1}$ gives $\tau = f^*\tau + \eta_\zeta$ at x.

When $x = c_i$ is a critical point, since x is a simple critical point, we may choose coordinates z at c_i and w at $f(c_i)$ such that

$$w = f_t(z) = z^2 + t(a + O(z)) + O(t^2).$$

Since we may follow the critical point for $|t|$ small, we may also suppose that $c_i(t) = 0$ for all t so that

$$f_t(z) = z^2 + t(a + O(z^2)) + O(t^2).$$

We thus obtain $\zeta(z) = (a + O(z))\frac{\partial}{\partial w}$, and $\eta_\zeta(z) = (-\frac{a}{2z} + O(z))\frac{\partial}{\partial z}$. Observe that in our coordinates we have $\tau(c_i) = \dot{c}_i = 0$, and $\tau(P(c_i)) = \frac{d}{dt}|_{t=0} f_t(c_i(t)) = a\,\frac{\partial}{\partial z}$. We may thus extend τ locally at c_i and $P(c_i)$ holomorphically by setting $\tau(z) \equiv 0$ and $\tau(w) \equiv a$. It follows that

$$f^*\tau(z) + \eta_\zeta(z) - \tau(z) = \frac{a}{f'(z)}\frac{\partial}{\partial z} + \Big(-\frac{a}{2z} + O(z)\Big)\frac{\partial}{\partial z} - 0 = O(z)\frac{\partial}{\partial z}\,.$$

It follows that $f^*\tau + \eta_\zeta = \tau$ at any critical point, which concludes the proof. □

Proof of Lemma 5.3. To simplify notation we write v_k, c, p instead of $v_{k,i}, c_i, p_i$ respectively. For any $l \geq 1$, p is really the exact period iterating the assertion (5.1) and using the fact that $D(f^p)$ is vanishing at all points of the cycle containing c, so in particular $D_{v_{(l-1)p}} f^p = 0$, and that $v_{k+p} = v_k$ for all $k \geq 0$, give

$$\begin{aligned}
\dot{v}_{lp} &= \zeta(v_{lp-1}) + D_{v_{lp-1}} f \cdot \zeta(v_{lp-2}) + \cdots + D_{v_{(l-1)p+1}} f^{p-1} \cdot \zeta(v_{(l-1)p}) \\
&\qquad\qquad + D_{v_{(l-1)p}} f^p \cdot \dot{v}_{(l-1)p} \\
&= \zeta(v_{lp-1}) + D_{v_{lp-1}} f \cdot \zeta(v_{lp-2}) + \cdots + D_{v_{(l-1)p+1}} f^{p-1} \cdot \zeta(v_{(l-1)p}) \\
&= \zeta(v_{p-1}) + D_{v_{p-1}} f \cdot \zeta(v_{p-2}) + \cdots + D_{v_1} f^{p-1} \cdot \zeta(v_0) = \dot{v}_p
\end{aligned}$$

Since $\zeta \in \ker(D_f\mathcal{V})$, we also have $\dot{v}_0 - \dot{v}_p = \dot{c} - \dot{v}_p = D_f\mathcal{V} \cdot \zeta = 0$, whence $\dot{v}_{lp} = \dot{v}_0$ for all $l \geq 1$. Again by (5.1) we get

$$\dot{v}_{lp+1} = \zeta(v_{lq}) + D_{v_{lp}} f \cdot v_{lp} = \zeta(v_0) + D_{v_0} f \cdot v_0 = \dot{v}_1\,.$$

An immediate induction on $k \geq 0$ then proves $\dot{v}_{lp+k} = \dot{v}_k$ for all $l \geq 0$. This proves the lemma. □

5.4. Application to the space Rat_d^0. Denote by Rat_d^0 the space of degree d rational maps on $\mathbb{P}^1$ fixing 0, 1 and ∞. To be more precise, let us parametrize Rat_d by

$$f\big([z:t]\big) = \left[\sum_{i=0}^{d} a_i z^i t^{d-i} : \sum_{i=0}^{d} b_i z^i t^{d-i}\right], \ [z:t] \in \mathbb{P}^1,$$

with $[a_d : \cdots : a_0 : b_d : \cdots : b_0] \in \mathbb{P}^{2d+1} \setminus \{\mathrm{Res} = 0\}$. The space Rat_d^0 is then determined by the equations $b_d = 0, a_0 = 0, \sum_i a_i = \sum_j b_j$ and is thus clearly a smooth subvariety of Rat_d of pure dimension $2d-2$.

Lemma 5.4. *The complex submanifolds* Rat_d^0 *and* $\mathcal{O}(f)$ *in* Rat_d *intersect transversely at any* $f \in \mathrm{Rat}_d^0$.

Proof. Let $\zeta \in T_f\mathcal{O}(f) \cap T_f \mathrm{Rat}_d^0$. Then there exists a holomorphic germ $m_t \in \mathrm{Aut}(\mathbb{P}^1)$ centered at $m_0 = \mathrm{id}$ such that $f_t = m_t^{-1} \circ f \circ m_t$ and $\zeta = \dot{f}$. Moreover, since $(f_t)_t$ is tangent to Rat_d^0, we can assume there are fixed points of f_t satisfying $x_t \equiv 0$, $y_t \equiv 1$ and $z_t \equiv \infty$.

Writing $m_t(z) = (a_t z + b_t)/(c_t z + d_t)$ with $a_t d_t - c_t b_t = 1$, we get

$$x_t = -b_t/a_t, \quad y_t = (b_t - d_t)/(c_t - a_t) \quad \text{and} \quad z_t = -d_t/c_t.$$

As $m_0 = \mathrm{id}$, we have

$$\begin{aligned} a_t &= 1 + \alpha t + O(t^2), \quad b_t = \beta t + O(t^2), \\ c_t &= \gamma t + O(t^2) \quad \text{and} \quad d_t = 1 + \delta t + O(t^2). \end{aligned}$$

We thus get

$$\begin{gathered} -\beta t + O(t^2) \equiv 0, \quad -\gamma t + O(t^2) \equiv 0, \\ 1 + (\alpha + \delta)t + O(t^2) = 1 \quad \text{and} \quad 1 + (\delta - \beta + \gamma - \alpha)t + O(t^2) \equiv 1, \end{gathered}$$

whence $\alpha = \beta = \gamma = \delta = 0$. As a consequence, $m_t = \mathrm{id} + O(t^2)$ and $m_t^{-1} = \mathrm{id} + O(t^2)$. Finally, differentiating $f_t = m_t^{-1} \circ f \circ m_t$ with respect to t and evaluating at $t = 0$ gives

$$\zeta = Df \cdot \dot{m} = Df \cdot 0 = 0\,.$$

This proves $T_f\mathcal{O}(f) \cap T_f \mathrm{Rat}_d^0 = \{0\}$. □

As above, we pick $f \in \mathrm{Rat}_d^0$ which is hyperbolic and postcritically finite with simple critical points $c_1, \ldots, c_{2d-2}$. We also assume that for $1 \le i \le 2d-2$, there exists $p_i \ge 1$ such that $c_i \in \mathrm{Fix}^*(f^{p_i})$. Let $U \subset \mathrm{Rat}_d^0$ be a neighborhood of f in which c_i can be followed holomorphically as a critical point $c_i(g)$ of g for all i. We can choose an atlas of $\mathbb{P}^1$ such that there is an affine chart containing $c_1(g), \ldots, c_{2d-2}(g)$ for every $g \in U$. We let

$$\mathcal{V}(g) := \big(g^{p_1}(c_1(g)) - c_1(g), \ldots, g^{p_{2d-2}}(c_{2d-2}(g)) - c_{2d-2}(g)\big), \quad g \in U.$$

From Theorem 5.2 and Lemma 5.4, we directly get the following.

Corollary 5.5. *Pick any postcritically finite and hyperbolic $f \in \mathrm{Rat}_d^0$ of disjoint type. The map $\mathcal{V}_f\colon U \to \mathbb{C}^{2d-2}$ is a local biholomorphism at f.*

6. Counting the centers of hyperbolic components of disjoint type in $\mathcal{M}_d$

6.1. The marked spaces $\mathrm{Rat}_d^{0,\mathrm{fm}}$ and $\mathrm{Rat}_d^{0,\mathrm{tm}}$ and $\mathcal{M}_d^{\mathrm{fm}}$. We follow closely [35, Section 9].

A *fixed marked* degree d rational map on $\mathbb{P}^1$ is a $(d+2)$-tuple $(f, x_1, \ldots, x_{d+1})$ where $f \in \mathrm{Rat}_d$, and $(x_1, \ldots, x_{d+1}) \in (\mathbb{P}^1)^{d+1}$ is a $(d+1)$-tuple of all the fixed points of f, taking into account their multiplicities. A *totally marked* degree d rational map on $\mathbb{P}^1$ is a $3d$-tuple $(f, x_1, \ldots, x_{d+1}, c_1, \ldots, c_{2d-2})$ where $(f, x_1, \ldots, x_{d+1})$ is a fixed marked rational map and $(c_1, \ldots, c_{2d-2}) \in (\mathbb{P}^1)^{2d-2}$ is a $(2d-2)$-tuples of all critical points of f, taking into account their multiplicities.

Let $\mathrm{Rat}_d^{0,\mathrm{fm}}$ be the space of all fixed marked degree d rational maps $(f, x_1, \ldots, x_{d+1})$ such that $x_1 = 0$, $x_2 = 1$ and $x_3 = \infty$. Let also $\mathrm{Rat}_d^{0,\mathrm{tm}}$ be the space of all totally marked degree d rational maps $(f, x_1, \ldots, x_{d+1}, c_1, \ldots, c_{2d-2})$ such that $x_1 = 0$, $x_2 = 1$ and $x_3 = \infty$. It is clear that both $\mathrm{Rat}_d^{0,\mathrm{fm}}$ and $\mathrm{Rat}_d^{0,\mathrm{tm}}$ are smooth and quasi-projective of dimension $2d-2$.

In both fixed marked spaces, the action by conjugation of $\mathrm{PSL}_2(\mathbb{C})$ extends naturally, by respecting the marking. Note that, in both cases where $x_1 = 0$, $x_2 = 1$ and $x_3 = \infty$, two tuples which are distinct cannot be conjugated, since an element in $\mathrm{PSL}_2(\mathbb{C})$ fixes three distinct points if and only if it is the identity. Moreover, the conjugacy class of any marked tuple $(f, \star)$ where f has no multiple fixed point admits a representative in with $x_1 = 0$, $x_2 = 1$ and $x_3 = \infty$.

We finally let $\mathcal{M}_d^{\mathrm{fm}}$ be the quotient of this action on the space of fixed marked rational maps. The space $\mathcal{M}_d^{\mathrm{fm}}$ is an irreducible quasi-projective variety of dimension $2d-2$ and its singular points are contained in the subvariety of $\mathcal{M}_d^{\mathrm{fm}}$ consisting of all classes $[(f, x_1, \ldots, x_{d+1})]$ such that $\#\{x_1, \ldots, x_{d+1}\} \leq 2$. In particular, $\mathcal{M}_d^{\mathrm{fm}}$ is smooth at any class $[(f, x_1, \ldots, x_{d+1})]$ such that f is hyperbolic. By the above, it is clear that the natural projection $\mathrm{Rat}_d^{0,\mathrm{fm}} \setminus \mathrm{Per}_1(1) \to \mathcal{M}_d^{\mathrm{fm}} \setminus \mathrm{Per}_1(1)$ is actually a bijection.

We note that the same construction of $\mathcal{L}$, μ_{bif}, $\mathrm{Per}_n(w)$, and $T_{\underline{n}}^{\underline{p}}(\rho)$ works on all the spaces introduced above (and even more generally) exactly as in the case of $\mathcal{M}_d$.

6.2. Parameterizing hyperbolic components of $\mathrm{Rat}_d^{0,\mathrm{fm}}$ of disjoint type. Let Ω be a hyperbolic component in $\mathrm{Rat}_d^{0,\mathrm{fm}}$. If $\mathcal{J}_f$ is connected for any $(f, x_1, \ldots, x_{d+1}) \in \Omega$, then Ω is simply connected and contains a *center*, which is by definition the unique point $(f, x_1, \ldots, x_{d+1}) \in \Omega$ such that $\#\mathcal{P}(f) < \infty$, by [35, Theorem 9.3].

For any $\underline{n} = (n_1, \ldots, n_{2d-2}) \in (\mathbb{N}^*)^{2d-2}$ and any hyperbolic component Ω in $\mathrm{Rat}_d^{0,\mathrm{fm}}$ of type $\underline{n}$, $\mathcal{J}_f$ is connected for any $(f, x_1, \ldots, x_{d+1}) \in \Omega$, since all Fatou components of f are then topological disks by [41, Proposition, p. 231]. In particular, Ω has a center. We will also use the following in the sequel.

Lemma 6.1. *For any $\underline{n}=(n_1,\dots,n_{2d-2})\in(\mathbb{N}^*)^{2d-2}$, any hyperbolic component Ω in $\mathrm{Rat}_d^{0,\mathrm{fm}}$ of type $\underline{n}$ is simply connected and the fixed points and critical points are marked throughout Ω. More precisely, there are a holomorphic maps $x_1,\dots,x_{d+1},c_1,\dots,c_{2d-2}\colon\Omega\to\mathbb{P}^1$ such that $\mathrm{Fix}(f_\lambda)=\{x_1(\lambda),\dots,x_{d+1}(\lambda)\}$ and $\mathrm{Crit}(f_\lambda)=(c_1(\lambda),\dots,c_{2d-2}(\lambda))$ for any $\lambda\in\Omega$.*

Proof. We have already seen that Ω is simply connected. Let $\tau\colon\mathrm{Rat}_d^{0,\mathrm{tm}}\to\mathrm{Rat}_d^{0,\mathrm{fm}}$ be the natural finite branched cover. For any component $\widetilde{\Omega}$ of $\tau^{-1}(\Omega)$, $\widetilde{\Omega}$ is a hyperbolic component of disjoint type in $\mathrm{Rat}_d^{0,\mathrm{tm}}$ and $\tau|_{\widetilde{\Omega}}\colon\widetilde{\Omega}\to\Omega$ is an unramified cover, so is a biholomorphism, since Ω is simply connected.

In particular, we have a holomorphic map

$$(\tau|_{\widetilde{\Omega}})^{-1}\colon\Omega\ni\lambda\mapsto\big(f_\lambda,x_1(\lambda),\dots,x_{d+1}(\lambda),c_1(\lambda),\dots,c_{2d-2}(\lambda)\big)\in\widetilde{\Omega}$$

and the holomorphic maps $x_1,\dots,x_{d+1}$, $c_1,\dots,c_{2d-2}$ follow all the fixed points and critical points of f_λ, respectively. $\square$

Pick any hyperbolic component Ω in $\mathrm{Rat}_d^{0,\mathrm{fm}}$ of type $\underline{n}=(n_1,\dots,n_{2d-2})\in(\mathbb{N}^*)^{2d-2}$, and let $x_1,\dots,x_{d+1},c_1,\dots,c_{2d-2}\colon\Omega\to\mathbb{P}^1$ be the marking of all the fixed points and critical points of $(f_\lambda)_{\lambda\in\Omega}$ given by Lemma 6.1. For any $i\in\{1,\dots 2d-2\}$ and any $\lambda\in\Omega$, let $w_i(\lambda)\in\mathbb{D}$ be the multiplier of the attracting cycle $\mathscr{C}_i(\lambda)$ of f_λ of exact period n_i and whose immediate attractive basin contains $c_i(\lambda)$. The *multiplier map* $\mathcal{W}_\Omega\colon\Omega\to\mathbb{D}^{2d-2}$ on Ω is defined by

$$\mathcal{W}_\Omega(\lambda):=(w_1(\lambda),\dots,w_{2d-2}(\lambda)),\quad\lambda\in\Omega.$$

Let λ_Ω be the center in Ω. Noting also that $\#\mathcal{P}(f_\lambda)<\infty$ for any $\lambda\in\mathcal{W}_\Omega^{-1}\{0\}$, we have $\mathcal{W}_\Omega^{-1}\{0\}=\{\lambda_\Omega\}$.

Theorem 6.2. *The map $\mathcal{W}_\Omega\colon\Omega\to\mathbb{D}^{2d-2}$ is a biholomorphism.*

Proof. Write $\mathcal{W}$ for $\mathcal{W}_\Omega$. First, we prove that $\mathcal{W}$ is surjective and finite. According to [26, §3, p. 179], this implies that $\mathcal{W}$ is a finite and possibly ramified covering. Next, we show that $\mathcal{W}$ is locally invertible at λ_Ω. Since $\mathcal{W}^{-1}\{0\}=\{\lambda_\Omega\}$, this implies $\mathcal{W}$ has degree 1, i.e. is a biholomorphism.

Let us first prove that $\mathcal{W}$ is surjective. We proceed using the classical surgery argument: for any $0<\varepsilon<1$, we construct a continuous map $\sigma\colon\mathbb{D}(0,1-\varepsilon)^{2d-2}\to\Omega$ such that $\mathcal{W}\circ\sigma=\mathrm{id}$. We sketch the construction referring to [8, Theorem VIII.2.1] or [13] for detail.

Choose $\lambda=(f,x_1,\dots,x_{d+1})\in\Omega$ and for any $1\le i\le 2d-2$, let $U_{1,i},\dots,U_{n_i,i}$ be all the components of the immediate basin of the attracting cycle $\mathcal{C}_i(\lambda)$ such that $c_i(\lambda) \in U_{1,i}$. Since $c_i(\lambda)$ is a simple critical point of f, $U_{1,i}$ is simply connected and there exists a conformal map $\varphi_i\colon U_{1,i} \to \mathbb{D}$ such that

$$\varphi_i \circ f^{n_i} \circ \varphi_i^{-1}(\xi) = \xi \cdot \frac{\xi + w_i}{1 + \bar{w}_i \xi}, \quad |\xi| < 1,$$

where $w_i := w_i(\lambda)$. Fix $\epsilon \in (0,1)$. For any $\rho = (\rho_1, \dots, \rho_{2d-2}) \in \mathbb{D}(0, 1-\varepsilon)^{2d-2}$, we can define a continuous map $\widetilde{f}_\rho$ by setting $\widetilde{f}_\rho = f$ outside the union of all $U_{j,i}$, and such that

$$\varphi_i \circ \widetilde{f}_\rho^{n_i} \circ \varphi_i^{-1}(\xi) = \xi \cdot \frac{\xi + \rho_i}{1 + \bar{\rho}_i \xi}$$

on the open disk $|\xi| < 1 - r$ containing the critical point of the Blashke product in the right hand side. Notice that $(\widetilde{f}_\rho)_\rho$ is a continuous family of quasiregular maps of $\mathbb{P}^1$. We now solve the Beltrami equation for the unique Beltrami form which is 0 on the complement the $U_{j,i}$'s and invariant under $\widetilde{f}_\rho$: there is a continuous family of quasiconformal homeomorphism $\psi_\rho\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $f_\rho := \psi_\rho \circ \widetilde{f}_\rho \circ \psi_\rho^{-1}$ is a rational map and depends again continuously on ρ and that $\psi_\rho(x_1(\lambda)) = 0$, $\psi_\rho(x_2(\lambda)) = 1$, and $\psi_\rho(x_3(\lambda)) = \infty$. Then the $d+2$ tuple $(\psi_\rho \circ \widetilde{f}_\rho \circ \psi_\rho^{-1}, \psi_\rho(x_1(\lambda)), \dots, \psi_\rho(x_{d+1}(\lambda)))$ lies in Ω by the above continuous dependence and is mapped to ρ by $\mathcal{W}$ by the chain rule.

Let us show that $\mathcal{W}$ is finite, i.e., $\#\mathcal{W}^{-1}(\underline{w}) < \infty$ for any $\underline{w} = (w_1, \dots, w_{2d-2}) \in \mathbb{D}^{2d-2}$. Suppose to the contrary that for some $\underline{w} \in \mathbb{D}^{2d-2}$, $\#\mathcal{W}^{-1}(\underline{w}) = \infty$. Then there is an infinite set contained in $\bigcap_i \mathrm{Per}_{n_i}(w_i) \cap \Omega$. In particular, the quasi-projective subvariety $\Lambda := \bigcap_i \mathrm{Per}_{n_i}(w_i)$ has dimension > 0 and any $\lambda \in \Lambda$ has $2d-2$ distinct attracting cycles of respecting periods $n_1, \dots, n_{2d-2}$. The holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ thus has no bifurcations. Since $\mathcal{J}_{f_\lambda} \neq \mathbb{P}^1$ for some $\lambda \in \Lambda$, $(f_\lambda)_{\lambda\in\Lambda}$ is not a family of Lattès maps. Hence by [32, Theorem 2.2], $(f_\lambda)_{\lambda\in\Lambda}$ is trivial. Since the natural projection $\mathrm{Rat}_d^{0,\mathrm{fm}} \to \mathcal{M}_d$ has finite fibers, this implies that the quasi-projective variety Λ is a finite set. This is a contradiction.

Let us finally see the local invertibility of $\mathcal{W}$ at λ_Ω. Since p is a biholomorphism on Ω, the map $\mathcal{W}$ is locally invertible at λ_Ω if and only if W is locally invertible at $a := (f(\lambda_\Omega), x_1(\lambda_\Omega), \dots, x_{d+1}(\lambda_\Omega)) \in \mathrm{Rat}_d^{0,\mathrm{fm}}$. The conclusion follows from Lemma 6.3 below by the inverse function theorem. □

Lemma 6.3. *The linear map $D_a W$ is invertible.*

Proof. Let $\pi\colon \mathrm{Rat}_d^{0,\mathrm{fm}} \to \mathrm{Rat}_d^0 \subset \mathrm{Rat}_d$ be the natural branched cover, and set $\widehat{\Omega} := \pi(\Omega)$, which is the hyperbolic component in Rat_d^0 containing f with $\lambda_\Omega = [(f, x_1, \dots, x_{d+1})]$. Let us remark that, since f has only simple fixed points, the restriction $\pi_\Omega\colon \Omega \to \widehat{\Omega}$ of π to Ω is an (unramified) cover. We can choose an atlas of $\mathbb{P}^1$ such that there is an affine chart of $\mathbb{P}^1$ containing $\{c_1(g), \dots, c_{2d-2}(g)\}$ for every $g \in \widehat{\Omega}$, and define $V\colon \widehat{\Omega} \to \mathbb{C}^{2d-2}$ by

$$V(g) := \big(g^{n_1}(c_1(g)) - c_1(g), \dots, g^{n_{2d-2}}(c_{2d-2}(g)) - c_{2d-2}(g)\big), \quad g \in \widehat{\Omega}.$$

According to Corollary 5.5, we have $\ker(D_f V) = \{0\}$. Beware that $\widehat{W} := \mathcal{W} \circ \pi^{-1}$ is a holomorphic map from an open neighborhood of f in $\widehat{\Omega}$ to $\mathbb{C}^{2d-2}$, so it is sufficient to prove that $\ker(D_f \widehat{W}) \subset \ker(D_f V)$.

Let $v \in T_f \operatorname{Rat}_d^0$, and pick a holomorphic disk $(f_t)_{t\in\mathbb{D}}$ in Rat_d^0 such that $f_0 = f$ and $\dot{f} = v$. For any $t \in \mathbb{D}$ and any $1 \le i \le 2d-2$, set

$$w_i(t) := w_i(f_t), \quad c_i(t) := c_i(f_t), \quad \widehat{W}(t) = \widehat{W}(f_t) \quad \text{and} \quad V(t) = V(f_t).$$

For any $t \in \mathbb{D}$ and any $1 \le i \le 2d-2$, let $\mathcal{C}_i(t)$ be the attracting cycle of f_t whose immediate attractive basin contains $c_i(t)$, so that there is a holomorphic function z_i on $\mathbb{D}$ such that $z_i(t) \in \mathcal{C}_i(t)$ for any $t \in \mathbb{D}$ (so $w_i(t) = (f_t^{n_i})'(z_i(t)))$ and that $z_i(0) = c_i(0)$. Then for any $1 \le i \le 2d-2$, we find

$$\begin{aligned}\dot{w}_i &= \frac{d(f_t^{n_i})'}{dt}\Big|_{t=0}(z_i(0)) + (f^{n_i})''(z_i(0)) \cdot \dot{z}_i \\ &= \frac{\partial(f_t^{n_i})'}{\partial t}\Big|_{t=0}(c_i(0)) + (f^{n_i})''(c_i(0)) \cdot \dot{z}_i,\end{aligned}$$

and since $(f_t^{n_i})'(c_i(t)) = 0$ for any $t \in \mathbb{D}$, we also have

$$0 = \frac{\partial(f_t^{n_i})'}{\partial t}\Big|_{t=0}(c_i(0)) + (f^{n_i})''(c_i(0)) \cdot \dot{c}_i(0).$$

Hence for any $1 \le i \le 2d-2$,

$$\dot{w}_i = (f^{n_i})''(c_i(0)) \cdot (\dot{z}_i - \dot{c}_i),$$

and we also note that $(f^{n_i})''(c_i(0)) \neq 0$ since f is hyperbolic of disjoint type. If in addition $v \in \ker(D_f \widehat{W})$, then for any $1 \le i \le 2d-2$, $\dot{w}_i = 0$ (and by definition $z_i(0) = c_i(0)$) hence we have $z_i(t) - c_i(t) = O(t^2)$. For any $1 \le i \le 2d-2$, the i-th component of $V(t)$ is

$$\begin{aligned}f_t^{n_i}(c_i(t)) - c_i(t) &= f_t^{n_i}(z_i(t)) + (f_t^{n_i})'(z_i(t))\big(c_i(t) - z_i(t)\big) \\ &\qquad + O\big((c_i(t) - z_i(t))^2\big) - c_i(t) \\ &= \big(1 - w_i(t)\big)\big(z_i(t) - c_i(t)\big) + O(t^4) = O(t^2)\end{aligned}$$

as $t \to 0$, so that $v \in \ker(D_f V)$. □

6.3. Counting hyperbolic components: the mass of μ_{bif} in $\mathcal{M}_d$. We now prove Theorem 1.6 and Corollary 1.7. To avoid confusions, for any $\underline{\rho}$ and any $\underline{n}$, denote $T_{\underline{n}}^{2d-2}(\underline{\rho})$ and μ_{bif} on $\operatorname{Rat}_d^{0,\mathrm{fm}}$ by $T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho})$ and $\mu_{\mathrm{bif}}^{\mathrm{fm}}$, respectively.

Observe first the following.

Lemma 6.4. *Fix any* $\underline{\rho} \in]0,1[^{2d-2}$ *and any* $\underline{n} \in (\mathbb{N}^*)^{2d-2}$. *Then*

(1) *the measure* $T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho})$ *has full mass on the union of all hyperbolic components in* $\mathrm{Rat}_d^{0,\mathrm{fm}}$ *of type* $\underline{n}$, *and for any such component* Ω^{fm},

$$\big(T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho})\big)(\Omega^{\mathrm{fm}}) = \frac{\#\mathrm{Stab}(\underline{n},\underline{\rho})}{d_{|\underline{n}|}}.$$

(2) *the measure* $T_{\underline{n}}^{2d-2}(\underline{\rho})$ *has full mass on the union of all hyperbolic components in* $\mathcal{M}_d$ *of type* $\underline{n}$, *and for any such component* Ω,

$$\big(T_{\underline{n}}^{2d-2}(\underline{\rho})\big)(\Omega) = \frac{\#\mathrm{Stab}(\underline{n},\underline{\rho})}{d_{|\underline{n}|}}.$$

Proof. Consider the case $\mathrm{Rat}_d^{0,\mathrm{fm}}$ first. Pick $\underline{\rho} \in]0,1[^{2d-2}$, and observe that

$$T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho}) = \frac{1}{(2\pi)^{2d-2}d_{|\underline{n}|}} \int_{[0,2\pi]^{2d-2},\forall i\neq j,\ \theta_i\neq\theta_j} \bigwedge_{j=1}^{2d-2} \big[\mathrm{Per}_{n_j}\big(\rho_j e^{i\theta_j}\big)\big]\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{2d-2},$$

as finite measures on $\mathrm{Rat}_d^{0,\mathrm{fm}}$, since we only remove a set of Lebesgue measure zero in $[0,2\pi]^{2d-2}$. Hence $T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho})$-almost every point has $2d-2$ distinct attracting cycles. For the second part, let Ω be a hyperbolic component in $\mathrm{Rat}_d^{0,\mathrm{fm}}$ of type $\underline{n} = (n_1,\ldots,n_{2d-2})$. By Theorem 6.2, we know that the multiplier map $\mathcal{W} = (\mathcal{W}_1,\ldots,\mathcal{W}_{2d-2})\colon \Omega \to \mathbb{D}^{2d-2}$ is a biholomorphism. In particular, the intersection $\bigcap_{j=0}^{2d-2}\mathrm{Per}_{n_j}(w_i)$ is smooth and transverse in Ω for all $\underline{w} \in \mathbb{D}^{2d-2}$. This implies

$$d_{|\underline{n}|}T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho}) = \sum_{\sigma\in\mathrm{Stab}(\underline{n},\underline{\rho})} \bigwedge_{i=1}^{2d-2} dd^c \log\max\big\{|\mathcal{W}_{\sigma(i)}|,\rho_{\sigma(i)}\big\} \tag{6.1}$$

on Ω, which has mass $\#\mathrm{Stab}(\underline{n},\underline{\rho})$ on Ω. This concludes the proof for the case of $\mathrm{Rat}_d^{0,\mathrm{fm}}$.

Let $p\colon \mathrm{Rat}_d^{0,\mathrm{fm}} \to \mathcal{M}_d$ be the natural finite branched cover of degree $(d+1)!$. Observe that the restriction

$$\tilde{p} := p|_{\mathrm{Rat}_d^{0,\mathrm{fm}} \setminus \mathrm{Per}_1(1)}$$

of p to $\mathrm{Rat}_d^{0,\mathrm{fm}} \setminus \mathrm{Per}_1(1)$ is a finite unbranched cover of $\mathcal{M}_d \setminus \mathrm{Per}_1(1)$. Indeed, p can only branch at parameters $(f, x_1, \ldots, x_{d+1})$ where at least two of $x_1, \ldots, x_{d+1}$ coincide, and those parameters are contained in $\mathrm{Per}_1(1)$.

Let $\Omega \subset \mathcal{M}_d$ be a hyperbolic component of type $\underline{n}$ and let $\widetilde{\Omega}$ be a connected component of $p^{-1}(\Omega)$. Then the restriction $p|_{\widetilde{\Omega}}: \widetilde{\Omega} \to \Omega$ is an unbranched cover. Since multipliers do not depend on the marking of critical points, the multiplier map $\mathcal{W}_{\widetilde{\Omega}}: \widetilde{\Omega} \to \mathbb{D}^{2d-2}$ descends to a biholomorphism $\Omega \to \mathbb{D}^{2d-2}$. We now observe that $T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho}) = (p)^*(T_{\underline{n}}^{2d-2}(\underline{\rho}))$ and the conclusion follows as above. □

Fix any $\underline{\rho} \in]0,1[^{2d-2}$ and any $\underline{n} \in (\mathbb{N}^*)^{2d-2}$. Note that by construction,

$$T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho}) = (\widetilde{p})^*(T_{\underline{n}}^{2d-2}(\underline{\rho})) \quad \text{and} \quad \mu_{\mathrm{bif}}^{\mathrm{fm}} = (\widetilde{p})^*(\mu_{\mathrm{bif}}).$$

Since $\mathrm{Rat}_d^{0,\mathrm{fm}}$ is an affine variety, we can assume $\mathrm{Rat}_d^{0,\mathrm{fm}} \subset \mathbb{C}^N$ for some N. Consider the function $\log^+|Z|$, defined on $\mathbb{C}^N$, and let $\varphi: \mathrm{Rat}_d^{0,\mathrm{fm}} \to \mathbb{R}$ be its restriction to $\mathrm{Rat}_d^{0,\mathrm{fm}}$. The function φ is psh, continuous, non-negative and $dd^c\varphi$ has finite mass in $\mathrm{Rat}_d^{0,\mathrm{fm}}$. We have the lemma:

Lemma 6.5. *There exist constants $C_1, C_2 > 0$ that depend only on d such that, for any compact subset K of $\mathrm{Rat}_d^{0,\mathrm{fm}}$, if $C(K)$ is the constant in Theorem 1.1, then we have the following inequality:*

$$C(K) \leq C_1 \cdot \|\varphi\|_{\infty,K} + C_2.$$

Proof. We follow closely the proof of [12, Proposition 3.1] (see also [21, Proposition 4.4]) and adapt it to the present situation. Since $H_1(\mathrm{Rat}_d, \mathbb{R}) = 0$, by [1, Lemma 4.9], there exists a family of non-degenerate homogeneous polynomial lifts to $\mathbb{C}^2$ of the family Rat_d. We thus may choose a family F of non-degenerate homogeneous polynomial lifts to $\mathbb{C}^2$ of the family $\mathrm{Rat}_d^{0,\mathrm{fm}}$. Set $V := \mathrm{Rat}_d^{0,\mathrm{fm}}$. We may regard this family F as a homogeneous non-degenerate polynomial maps with coefficients in the ring $\mathbb{C}[V]$. Note that $\mathrm{Res}(F) \in \mathbb{C}[V]$ and, in particular, $\big|\log|\mathrm{Res}(F)|\big| \leq \alpha\varphi(f) + \beta$ for some constants $\alpha, \beta \geq 0$ independent of $f \in V$.

We now want to prove that there exists $m \geq 1$ and $C > 0$ such that for any $f \in \mathrm{Rat}_d^{0,\mathrm{fm}}$ and any $(x, y) \in \mathbb{C}^2 \setminus \{0\}$,

$$\frac{1}{C} e^{-m\varphi(f)} \leq \frac{\|F(x,y)\|^2}{\|(x,y)\|^{2d}} \leq Ce^{m\varphi(f)}.$$

We work with the maximum norm $\|(x,y)\| = \max\{|x|, |y|\}$ on $\mathbb{C}^2$. The upper bound follows easily from the fact that $F_1, F_2 \in \mathbb{C}[V][x,y]$ and the triangle inequality. By the homogeneity of F_1 and F_2, it is sufficient to verify the lower bound whenever $\|(x,y)\| = 1$.

By the item (c) of [45, Proposition 2.13], there exists homogeneous polynomials $g_1, g_2, h_1, h_2 \in \mathbb{C}[V][x, y]_{d-1}$ such that

$$g_1(x, y)F_1(x, y) + g_2(x, y)F_2(x, y) = \mathrm{Res}(F)x^{2d-1} \tag{6.2}$$

and
$$h_1(x, y)F_1(x, y) + h_2(x, y)F_2(x, y) = \mathrm{Res}(F)y^{2d-1}. \tag{6.3}$$

Again, since $g_1, g_2, h_1, h_2 \in \mathbb{C}[V][x, y]$, there are constants $A, B \geq 0$ independent of $f \in V$ such that

$$\max\left\{|g_1(x, y)|, |g_2(x, y)|, |h_1(x, y)|, |h_2(x, y)|\right\} \leq Ae^{B\varphi(f)} \quad \text{if } \|(x, y)\| \leq 1.$$

When $x = 1$, equation (6.2) gives

$$|\mathrm{Res}(F)| \leq 4 \max\left\{|g_1(x, y)|, |g_2(x, y)|\right\} \cdot \|F(x, y)\| \leq 4A^{B\varphi(f)}\|F(x, y)\|.$$

We proceed similarly with equation (6.3) when $y = 1$ and the conclusion follows.

Following exactly the proof of Lemma 3.7 gives $C_1, C_2 \geq 0$ such that

$$\max\Big\{\sup_{z\in\mathbb{P}^1} \log f^{\#}(z), \sup_{z\in\mathbb{P}^1} |g_f(z)|\Big\} \leq C_1\varphi(f) + C_2$$

for any $f \in \mathrm{Rat}_d^{0,\mathrm{fm}}$. □

Recall that we picked $\underline{n} \in (\mathbb{N}^*)^{2d-2}$. Let $\varepsilon > 0$, and set $\underline{\rho} = (1/2, \dots, 1/2)$, so in particular that $\mathrm{Stab}(\underline{n}, \underline{\rho}) = \mathrm{Stab}(\underline{n})$. Take $R > 0$ large enough so that $\mathrm{supp}(T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho}))$ is contained in the intersection $B(0, R)$ between V and the open ball in $\mathbb{C}^N$ of radius R and centered at 0. Observe that this is possible since there are at most finitely many of type $\underline{n}$ and for a hyperbolic component Ω of type $\underline{n}$, $\mathcal{W}_\Omega^{-1}(\mathbb{D}_{1/2}^{2d-2}) \subset \Omega$ is relatively compact in V (for $d = 2$, this is known to be true for the whole component Ω [19]).

For any $A > 0$, we pick the following test function

$$\Psi_A := \frac{1}{A}\min\left\{\max(\varphi, A) - 2A, 0\right\} \quad \text{on } V.$$

Then, Ψ_A is continuous and DSH on V and $dd^c\Psi_A = T_A^+ - T_A^-$ for some positive closed currents of finite masses, where $\|T_A^\pm\| \leq C'/A$ and for some $C' > 0$ depending neither on A nor on $T_A^\pm$ (e.g. [17, Lemma 2.2.6]). Then observe that Ψ_A is equal to -1 in $B(0, e^A)$, and 0 outside $B(0, e^{2A})$. Applying Theorem 1.1 with the control of Lemma 6.5 implies:

$$\begin{aligned}&\left|\left\langle T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho}), \Psi_A\right\rangle - \left\langle \mu_{\mathrm{bif}}^{\mathrm{fm}}, \Psi_A\right\rangle\right| \\ &\qquad\leq (1 + \log 2)\big(C_1\varphi(e^{2A}) + C_2\big) \max_{1\leq j\leq 2d-2}\left(\frac{\sigma_2(n_j)}{d^{n_j}}\right)\frac{C'}{A}.\end{aligned}$$

Taking $A = \log R$ so that $\varphi(e^{2A}) = 2A$, there is a constant $C_d > 0$ depending only on d, such that $|\|T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho})\| - \langle \mu_{\mathrm{bif}}^{\mathrm{fm}}, -\Psi_A\rangle| \leq C_d \max_j(\sigma_2(n_j)/d^{n_j})$. As $R \to \infty$, we have $\langle \mu_{\mathrm{bif}}^{\mathrm{fm}}, -\Psi_A\rangle \to \|\mu_{\mathrm{bif}}^{\mathrm{fm}}\|$ and in turn

$$\left|\|T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho})\| - \|\mu_{\mathrm{bif}}^{\mathrm{fm}}\|\right| \leq C_d \max_{1\leq j\leq 2d-2}\left(\frac{\sigma_2(n_j)}{d^{n_j}}\right).$$

Let us go back to $\mathcal{M}_d$. Since the measures $T_{\underline{n}}^{2d-2}(\underline{\rho})$ and μ_{bif} (resp. $T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho})$ and $\mu_{\mathrm{bif}}^{\mathrm{fm}}$) give no mass to algebraic subvarieties of $\mathcal{M}_d$ (resp. of $\mathrm{Rat}_d^{0,\mathrm{fm}}$), we have

$$\left|\|T_{\underline{n}}^{2d-2}(\underline{\rho})\| - \|\mu_{\mathrm{bif}}\|\right| = \frac{1}{\deg(\widetilde{p})}\left|\|T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho})\| - \|\mu_{\mathrm{bif}}^{\mathrm{fm}}\|\right|,$$

together with $\|T_{\underline{n}}^{2d-2}(\underline{\rho})\| = \#\mathrm{Stab}(\underline{n})N(\underline{n})/d_{|\underline{n}|}$ (by Lemma 6.4) completes the proof of Theorem 1.6. □

Proof of Corollary 1.7. Let us begin with describing [30, Theorem 1.1] by Kiwi and Rees. Taking $m > n \geq 2$, they computed, in the critically marked moduli space $\mathcal{M}_2^{\mathrm{cm}}$, the number $n_{IV}(n,m)$ of all hyperbolic components Ω in $\mathcal{M}_2^{\mathrm{cm}}$ of type (n,m) such that any $[(f,c_1,c_2)] \in \Omega$ has two distinct attracting cycles of exact periods j,k with $j|n$ and $k|m$, respectively and their immediate attractive basins contain c_1,c_2, respectively. They prove

$$n_{IV}(n,m) = \left(\frac{5}{3}2^{n-3} + \frac{1}{12} - \frac{1}{4}\sum_{q=2}^{n}\frac{\phi(q)\nu_q(n)}{2^q-1}\right)2^m + \varepsilon_1(n,m),$$

where $|\varepsilon_1(n,m)| \leq 2^n + 2^{2\gcd(n,m)}$ and $|\nu_q(n) - 2^n/(2(2^q-1))| \leq 1/2$. Their computation in particular yields that for any $m > n \geq 2$,

$$n_{IV}(n,m) = \left(\frac{1}{3} - \frac{1}{8}\sum_{q=1}^{n}\frac{\phi(q)}{(2^q-1)^2}\right)2^{n+m} + \varepsilon_2(n,m),$$

where $|\varepsilon_2(n,m)| \leq C\cdot 2^m$ for some $C \geq 1$ independent of n,m. We now note that the natural projection $\pi\colon \mathcal{M}_2^{\mathrm{cm}} \to \mathcal{M}_2$ is of degree 2 and is unramified over any hyperbolic $[f] \in \mathcal{M}_2$ of disjoint type, and for any $[(f,c_1,c_2)] \in \mathcal{M}_2^{\mathrm{cm}}$, $\pi^{-1}\{[f]\} = \{[f,c,c'],[f,c',c]\}$. In particular,

$$n_{IV}(n,n+1) = \sum_{j|n,\,k|(n+1)} N(j,k).$$

Since we have $N(j,k) \leq C\cdot 2^{j+k}$ by Bézout's theorem and $d_n\cdot d_{n+1} = 2^{2n+1} + O(2^n)$ as $n \to \infty$, the above gives

$$\frac{N(n,n+1)}{d_n\cdot d_{n+1}} = \frac{n_{IV}(n,n+1)}{2^{2n+1}} + o(1) = \frac{1}{3} - \frac{1}{8}\sum_{q=1}^{+\infty}\frac{\phi(q)}{(2^q-1)^2} + o(1), \quad \text{as } n \to \infty.$$

The conclusion follows from Theorem 1.6, since $\#\mathrm{Stab}(n,n+1) = 1$. □

6.4. Weak genericity of postcritically finite hyperbolic rational maps. The moduli space $\mathcal{M}_d$ of degree d rational maps is known to be an irreducible affine variety of dimension $2d-2$ which is defined over $\mathbb{Q}$ (see [34, 44]); and all non-flexible Lattès postcritically finite degree d rational maps are known to be defined over $\bar{\mathbb{Q}}$ (see e.g. [45]). These properties are, for the moduli space of critically marked degree d polynomials, the starting point of the work [22]. The idea developed there is to apply Yuan's equidistribution theorem [46] to get the equidistribution of pcf maps towards the bifurcation measure.

The use of this equidistribution result requires:

(1) defining an adelic semi-positive metric on an ample line bundle $L \to \bar{\mathcal{M}}_d$, where the associated height function h satisfies $h([f]) = 0$ for all non-Lattès pcf map f, and where the induced Monge–Ampère measure is proportional to μ_{bif}.

(2) showing that *any* sequence (X_k) of Galois invariant finite sets of postcritically finite parameters is *weakly generic* in $\mathcal{M}_d$ in that $\mathrm{Card}(X_k \cap C) = o\,(\mathrm{Card}(X_k))$ as $\mathrm{Card}(X_k) \to \infty$ for any proper affine subvariety C in $\mathcal{M}_d$ defined over $\mathbb{Q}$. This is stronger than the Zariski density of $\bigcup_k X_k$ in $\mathcal{M}_d$.

Contrary to the case of polynomials, item (1) seems very difficult to establish and could even be wrong as stated. Here we focus on item (2).

For any $\underline{n} = (n_1, \dots, n_{2d-2}) \in (\mathbb{N}^*)^{2d-2}$, we set

$$X_{\underline{n}} := \big\{[f] \in \mathcal{M}_d : f \text{ has } 2d-2 \text{ periodic critical points } c_1, \dots, c_{2d-2} \\ \text{of respective exact periods } n_1, \dots, n_{2d-2}\big\},$$

so that $\mathrm{C}_{\underline{n}} \subset X_{\underline{n}}$. A consequence of our counting of hyperbolic components is that any sequences of sets of centers of hyperbolic components of disjoint type is weakly generic.

Theorem 6.6. *For any sequence $(\underline{n}(k))_k$ of $(2d-2)$-tuples $\underline{n}(k) = (n_{1,k}, \dots, n_{2d-2,k})$ in $(\mathbb{N}^*)^{2d-2}$ satisfying $\min_j(n_j(k)) \to \infty$ as $k \to \infty$, the sequence $(X_{\underline{n}(k)})_k$ is Galois-invariant and weakly generic in $\mathcal{M}_d$.*

Remark 6.7. This result in particular implies that $\bigcup_k X_{\underline{n}(k)}$ is Zariski dense in $\mathcal{M}_d$, which refines [12, Theorem A].

For proving this weak genericity property, we prove a stronger result in the moduli space $\mathcal{M}_d^{\mathrm{cm}}$ of *critically marked* degree d rational maps on $\mathbb{P}^1$, i.e., the orbit space of $\mathrm{PSL}_2(\mathbb{C})$ in the space $\mathrm{Rat}_d^{\mathrm{cm}}$ of *critically marked* degree d rational maps $(f, c_1, \dots, c_{2d-2})$, where $f \in \mathrm{Rat}_d$ and $(c_1, \dots, c_{2d-2})$ is a $(2d-2)$-tuple of all critical points of f, counted with multiplicity. This is also an irreducible affine variety of dimension $2d-2$ which is defined over $\mathbb{Q}$ and the natural finite branched cover $p\colon \mathcal{M}_d^{\mathrm{cm}} \to \mathcal{M}_d$ is of degree $(2d-2)!$ and also defined over $\mathbb{Q}$.

For any $n \in \mathbb{N}^*$ and any $1 \le j \le 2d-2$, let $\mathrm{Per}_j(n)$ be the analytic hypersurface

$$\mathrm{Per}_j(n) := \big\{[(f, c_1, \dots, c_{2d-2})] \in \mathcal{M}_d^{\mathrm{cm}} : \Phi_n^*(F, C_j) = 0\big\}$$

in $\mathcal{M}_d^{\mathrm{cm}}$, where F and C_j are lifts of f and c_j, respectively (see Section 2.4 for the definition of Φ_n^*); the degree of the hypersurface $\mathrm{Per}_j(n)$ is bounded from above by Cd^n for some constant $C \geq 1$ depending only on d (see e.g. [45] for more details), and since $\mathcal{M}_d^{\mathrm{cm}}$ is quasi-projective, $\mathrm{Per}_j(n)$ is actually an algebraic hypersurface of $\mathcal{M}_d^{\mathrm{cm}}$.

For any $\underline{n} = (n_1, \dots, n_{2d-2}) \in (\mathbb{N}^*)^{2d-2}$, set $Y_{\underline{n}} := \bigcap_{j=1}^{2d-2} \mathrm{Per}_j(n_j) \subset \mathcal{M}_d^{\mathrm{cm}}$. We prove here the following as an application of our counting result.

Theorem 6.8. *For any proper algebraic subvariety V in $\mathcal{M}_d^{\mathrm{cm}}$, there exists a constant $C > 0$ such that for all $\underline{n} \in (\mathbb{N}^*)^{2d-2}$, we have*

$$\mathrm{Card}(Y_{\underline{n}} \cap V)/\mathrm{Card}(Y_{\underline{n}}) \leq C \cdot d^{-(\min_j n_j)/2}.$$

For the proof, we follow the strategy of [22, Theorem 5.3] and we rely on the following, which is just an adaptation of [22, Lemma 5.4].

Lemma 6.9. *Let V be any irreducible algebraic subvariety of dimension q in $\mathcal{M}_d^{\mathrm{cm}}$ and let p be a smooth point in $\mathcal{M}_d^{\mathrm{cm}}$. Assume that V is also smooth at p. Pick hypersurfaces $H_1, \dots, H_{2d-2}$ intersecting transversely at p. Then there is $I \subset \{1, \dots, 2d-2\}$ of cardinality $2d-2-q$ such that p is an isolated point of $V \cap \bigcap_{j \in I} H_j$.*

Proof of Theorem 6.8. The case $\dim V = 0$ is an immediate consequence of Theorem 1.6, since V is a finite set in that case. We thus assume $q := \dim V \in \{1, \dots, 2d-3\}$.

Pick $\underline{n} = (n_1, \dots, n_{2d-2}) \in (\mathbb{N}^*)^{2d-2}$. Let $Z_{\underline{n}}$ be the subset in $Y_{\underline{n}}$ consisting of all $[(f, c_1, \dots, c_{2d-2})] \in \mathcal{M}_d$ such that the orbits of $c_1, \dots, c_{2d-2}$ of f are also disjoint. We claim that there is a constant $C' > 0$ depending only on d such that

$$\mathrm{Card}\left(Y_{\underline{n}} \setminus Z_{\underline{n}}\right) \leq C' d^{|\underline{n}|-(\min_i n_i)/2};$$

for, since $Y_{\underline{n}} \setminus Z_{\underline{n}}$ consists of all $[f] \in \mathcal{M}_d^{\mathrm{cm}}$ such that f has a super-attracting cycles of exact period n_i and containing at least two distinct critical points for some i, we have

$$Y_{\underline{n}} \setminus Z_{\underline{n}} \subset \bigcup_{i=1}^{2d-2} \Bigg(\bigcup_{\substack{j \neq i \\ n_i = n_j}} \bigcup_{k=0}^{[n_i/2]} \{f^k(c_i) = c_j\} \cap \bigcap_{\ell : \ell \neq i} \mathrm{Per}_\ell(n_\ell) \Bigg).$$

Since also $\deg(\{f^k(c_i) = c_j\}) \leq Cd^k$ for some $C > 0$ independent of i, j and k, by Bézout's theorem, there exists constant C_0, C' depending only on d such that

$$\begin{aligned}
\mathrm{Card}(Y_{\underline{n}} \setminus Z_{\underline{n}}) &\leq \sum_{i=1}^{2d-2} \sum_{\substack{j \neq i \\ n_i = n_j}} \sum_{k=0}^{[n_i/2]} Cd^k \prod_{\ell : \ell \neq i} Cd^{n_\ell} \\
&\leq \sum_{i=1}^{2d-2} \sum_{\substack{j \neq i \\ n_i = n_j}} C_0 d^{|\underline{n}| - n_i/2} \leq C' d^{|\underline{n}|-(\min_i n_i)/2}.
\end{aligned}$$

Set $N = N_q := 2d - 2 - q \in \mathbb{N}^*$ and recall that $N = \dim V$. Let V_{reg} be the regular locus of V. We also claim that there is a constant $C'' > 0$ depending only on d such that

$$\mathrm{Card}(V_{\mathrm{reg}} \cap Z_{\underline{n}}) \leq C'' \deg(V) \sum_I d^{\sum_{j=1}^N n_{i_j}},$$

where here and below the sums $\sum_I$ range over all N-tuples $I = (i_1, \dots, i_N)$ of distinct indices in $\{1, \dots, 2d-2\}$; indeed, for any such choice I, we set $Y_I := \bigcap_{j=1}^N \mathrm{Per}_{i_j}(n_{i_j})$, and let F_I be the set of all isolated points of $V \cap Y_I$. By Bézout's theorem, we have

$$\mathrm{Card}(F_I) \leq \deg(V) \prod_{j=1}^N \deg\left(\mathrm{Per}_{i_j}(n_{i_j})\right) = C_2 \deg(V) d^{\sum_{j=1}^N n_{i_j}} \tag{6.4}$$

for some constant $C_2 > 0$ depending only on d. Since $\mathrm{Card}\left(V_{\mathrm{reg}} \cap Z_{\underline{n}}\right) \leq \sum_I \mathrm{Card}(F_I)$ by Lemma 6.9, the claim holds. According to Theorem 1.6, there is a constant $C_4 > 0$ depending only on d such that $\mathrm{Card}(Y_{\underline{n}}) \geq \mathrm{Card}(Z_{\underline{n}}) \geq C_4 d^{|\underline{n}|}$ provided $\min_j n_j$ is large enough.

Hence, the above two claims imply

$$\mathrm{Card}(V_{\mathrm{reg}} \cap Z_{\underline{n}}) \leq C_5 \sum_I d^{-\sum_{j \notin I} n_{i_j}} \mathrm{Card}(Y_{\underline{n}})$$

and

$$\mathrm{Card}(Y_{\underline{n}} \setminus Z_{\underline{n}}) \leq C_5 d^{-(\min_i n_i)/2} \mathrm{Card}(Y_{\underline{n}}),$$

where $C_5 > 0$ depends only on V, d and q. Since $V_{\mathrm{sing}} := V \setminus V_{\mathrm{reg}}$ is an algebraic subset in $\mathcal{M}_d^{\mathrm{cm}}$ of codimension $2d - 2 - q + 1$, the proof is complete by a finite induction. $\square$

Proof of Theorem 6.6. Pick any $\underline{n} = (n_1, \dots, n_{2d-2}) \in (\mathbb{N}^*)^{2d-2}$, and for any permutation $\sigma \in \mathfrak{S}_{2d-2}$, set $\sigma(\underline{n}) := (n_{\sigma(1)}, \dots, n_{\sigma(2d-2)})$. Let us first remark that $p^{-1}(X_{\underline{n}}) = \bigcup_{\sigma \in \mathfrak{S}_{2d-2}} Y_{\sigma(\underline{n})}$. The Galois-invariance of $X_{\underline{n}}$ follows from the Galois-invariance of $Y_{\sigma(\underline{n})}$ for any $\sigma \in \mathfrak{S}_{2d-2}$. Similarly, for any irreducible subvariety $Z \subset \mathcal{M}_d$ defined over $\mathbb{Q}$, we can apply Theorem 6.8 to any irreducible component of the algebraic subset $V = p^{-1}(Z)$ in $\mathcal{M}_d^{\mathrm{cm}}$. The fact that p is a finite branched cover together with the assumption $\min_j(n_j(k)) \to +\infty$ as $k \to \infty$ completes the proof. $\square$

7. Distribution of hyperbolic maps with given multipliers in $\mathcal{M}_d$

This section is devoted to the proof of Theorem 1.8. Pick any $\underline{n} = (n_1, \dots, n_{2d-2}) \in (\mathbb{N}^*)^{2d-2}$ and any $\underline{w} = (w_1, \dots, w_{2d-2}) \in \mathbb{D}^{2d-2}$.

7.1. A reduction to work on $\mathcal{M}_d^{\mathrm{fm}}$. Let $\pi_{\mathrm{fm}}\colon \mathcal{M}_d^{\mathrm{fm}} \to \mathcal{M}_d$ be the natural quotient map. It is a finite branched cover of degree $\deg(\pi_{\mathrm{fm}}) = (d+1)!$ and recall the definition of $C_{\underline{n},\underline{w}}$ from the introduction. Again, for any $\underline{\rho}$ and any $\underline{n}$, denote $T_{\underline{n}}^{2d-2}(\underline{\rho})$ and μ_{bif} on $\mathcal{M}_d^{\mathrm{fm}}$ by $T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho})$ and $\mu_{\mathrm{bif}}^{\mathrm{fm}}$, respectively (though we already used those notations on $\mathrm{Rat}_d^{0,\mathrm{fm}}$, this is not an issue since the projection $\mathrm{Rat}_d^{0,\mathrm{fm}} \setminus \mathrm{Per}_1(1) \to \mathcal{M}_d^{\mathrm{fm}} \setminus \mathrm{Per}_1(1)$ is a bijection and none of the considered objects give mass to $\mathrm{Per}_1(1)$). Set $C_{\underline{n},\underline{w}}^{\mathrm{fm}} := \pi_{\mathrm{fm}}^{-1}(C_{\underline{n},\underline{w}})$,

$$\mu_{\underline{n},\underline{w}}^{\mathrm{fm}} := \frac{\#\mathrm{Stab}(\underline{n},\underline{w})}{d_{|\underline{n}|}} \sum_{C_{\underline{n},\underline{w}}^{\mathrm{fm}}} \delta_{[(f,x_1,\dots,x_{d+1})]},$$

and $\mu_{\underline{n}}^{\mathrm{fm}} := \mu_{\underline{n},(0,\dots,0)}^{\mathrm{fm}}$. Then $\mu_{\underline{n},\underline{w}}^{\mathrm{fm}} = (\pi_{\mathrm{fm}})^*(\mu_{\underline{n},\underline{w}})$ and $\mu_{\underline{n}}^{\mathrm{fm}} = (\pi_{\mathrm{fm}})^*(\mu_{\underline{n}})$. In particular, for any DSH and continuous function $\widetilde{\Psi}$ on $\mathcal{M}_d$ with compact support,

$$\begin{aligned}\langle \mu_{\underline{n},\underline{w}} - \mu_{\mathrm{bif}}, \widetilde{\Psi}\rangle &= \frac{1}{(d+1)!}\langle \mu_{\underline{n},\underline{w}} - \mu_{\mathrm{bif}}, (\pi_{\mathrm{fm}})_*(\pi_{\mathrm{fm}})^*\widetilde{\Psi}\rangle \\ &= \frac{1}{(d+1)!}\langle \mu_{\underline{n},\underline{w}}^{\mathrm{fm}} - \mu_{\mathrm{bif}}^{\mathrm{fm}}, (\pi_{\mathrm{fm}})^*\widetilde{\Psi}\rangle.\end{aligned}$$

Hence, it is sufficient to prove the desired estimates in the fixed marked moduli space $\mathcal{M}_d^{\mathrm{fm}}$. So, pick any compact set K *in* $\mathcal{M}_d^{\mathrm{fm}}$, and any either $\mathscr{C}^1$ or $\mathscr{C}^2$ function Ψ on $\mathcal{M}_d^{\mathrm{fm}}$ with support in K. Set

$$\underline{\rho} = (\rho_1,\dots,\rho_{2d-2}) := \big(\max\big(|w_1|, 1/2\big),\dots,\max\big(|w_{2d-2}|, 1/2\big)\big) \in \big([1/2, 1[\big)^{2d-2},$$

so that $\rho_j \in [|w_j|, 1[$ for any $1 \le j \le 2d-2$. By Theorem 1.1 and the upper bound of the DSH-norm by the $\mathscr{C}^2$-norm, there is $C(K) > 0$ depending only on K such that

$$\big|\langle T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho}) - \mu_{\mathrm{bif}}^{\mathrm{fm}}, \Psi\rangle\big| \le C(K) \max_{1\le j\le 2d-2}\Big(\frac{\sigma_2(n_j)}{d^{n_j}}\Big)\|\Psi\|_{\mathscr{C}^2} \quad \text{if } \Psi \text{ is } \mathscr{C}^2$$

and then, by interpolation between Banach spaces, that

$$\big|\langle T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho}) - \mu_{\mathrm{bif}}^{\mathrm{fm}}, \Psi\rangle\big| \le C(K) \max_{1\le j\le 2d-2}\Big(\frac{\sigma_2(n_j)}{d^{n_j}}\Big)^{1/2}\|\Psi\|_{\mathscr{C}^1} \quad \text{if } \Psi \text{ is } \mathscr{C}^1.$$

Whence the proof of Theorem 1.8 reduces to showing

$$\big|\langle \mu_{\underline{n}}^{\mathrm{fm}} - T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho}), \Psi\rangle\big| \le C \max_{1\le j\le 2d-2}\Big(\frac{1}{d^{n_j}}\Big)\cdot\|\Psi\|_{\mathscr{C}^2} \quad \text{if } \Psi \text{ is } \mathscr{C}^2, \tag{7.1}$$

and

$$\big|\langle \mu_{\underline{n},\underline{w}}^{\mathrm{fm}} - T_{\underline{n}}^{2d-2,\mathrm{fm}}(\underline{\rho}), \Psi\rangle\big| \le C \max_{1\le j\le 2d-2}\Big(\frac{-1}{d^{n_j}\log\rho_j}\Big)^{1/2}\|\Psi\|_{\mathscr{C}^1} \quad \text{if } \Psi \text{ is } \mathscr{C}^1, \tag{7.2}$$

where $C > 0$ depends only on d.

7.2. A reduction to work on algebraic curves. Observe that the measure

$$\mu^{\mathrm{fm}}_{\underline{n},\underline{w}} - T^{2d-2,\mathrm{fm}}_{\underline{n}}(\underline{\rho}) \quad \text{on } \mathcal{M}^{\mathrm{fm}}_d$$

has its support contained in the union of all hyperbolic components of type $\underline{n}$. Let Ω be such a component, and

$$\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_{2d-2})\colon \Omega \to \mathbb{D}^{2d-2}$$

the multiplier map on Ω. Letting λ_r be the normalized Lebesgue measure on $\partial\mathbb{D}_r$, by (6.1) we have on Ω,

$$\begin{aligned} &d_{|\underline{n}|}\big(T^{2d-2,\mathrm{fm}}_{\underline{n}}(\underline{\rho}) - \mu^{\mathrm{fm}}_{\underline{n},\underline{w}}\big) \\ &\quad = \#\mathrm{Stab}(\underline{n},\underline{w})\mathcal{W}^*\big(\lambda_{\rho_1}\otimes\cdots\otimes\lambda_{\rho_{2d-2}}\big) - \#\mathrm{Stab}(\underline{n},\underline{w}) \bigwedge_{i=1}^{2d-2}\big[\mathrm{Per}_{n_i}(w_i)\big] \\ &\quad = \sum_{\sigma\in\mathrm{Stab}(\underline{n},\underline{w})} \mathcal{W}^*_{\sigma(1)}(\lambda_{\rho_1})\wedge\cdots\wedge\mathcal{W}^*_{\sigma(2d-2)}(\lambda_{\rho_{2d-2}}) - \mathcal{W}^*_{\sigma(1)}(\delta_{w_1})\wedge\dots \\ &\qquad\qquad\qquad\qquad \cdots\wedge\mathcal{W}^*_{\sigma(2d-2)}(\delta_{w_{2d-2}}) \\ &\quad = \sum_{\sigma\in\mathrm{Stab}(\underline{n},\underline{w})}\sum_{j=1}^{2d-2} S_{\sigma,j}, \end{aligned}$$

where for any $\sigma\in\mathrm{Stab}(\underline{n})$ and any $1\le j\le 2d-2$, the measure $S_{\sigma,j}$ is defined as

$$\begin{aligned} S_{\sigma,j} &:= \Big(\bigwedge_{1\le i<j}\mathcal{W}^*_{\sigma(i)}(\lambda_{\rho_i})\Big)\wedge\big(\mathcal{W}^*_{\sigma(j)}(\lambda_{\rho_j}) - \mathcal{W}^*_{\sigma(j)}(\delta_{w_j})\big)\wedge\Big(\bigwedge_{k>j}\mathcal{W}^*_{\sigma(k)}(\delta_{w_k})\Big) \\ &= \int_{\widetilde{\mathbb{S}}_{j-1}}\bigwedge_{1\le i<j}\mathcal{W}^*_{\sigma(i)}(\delta_{u_i})\wedge\big(\mathcal{W}^*_{\sigma(j)}(\lambda_{\rho_j}) - \mathcal{W}^*_{\sigma(j)}(\delta_{w_j})\big) \\ &\qquad\qquad \wedge\bigwedge_{k>j}\mathcal{W}^*_{\sigma(k)}(\delta_{w_k})\,\mathrm{d}\lambda_{\widetilde{\mathbb{S}}_{j-1}}(u_1,\dots,u_{j-1}) \end{aligned}$$

on Ω, setting $\lambda_{\widetilde{\mathbb{S}}_{j-1}} := \lambda_{\rho_1}\otimes\cdots\otimes\lambda_{\rho_{j-1}}$ on $\widetilde{\mathbb{S}}_{j-1} := \partial\mathbb{D}_{\rho_1}\times\cdots\times\partial\mathbb{D}_{\rho_{j-1}}$ if $j>1$ and $S_{\sigma,1} := (\mathcal{W}^*_{\sigma(1)}(\lambda_{\rho_1}) - \mathcal{W}^*_{\sigma(1)}(\delta_{w_1}))\wedge\bigwedge_{k>1}\mathcal{W}^*_{\sigma(k)}(\delta_{w_k})$. Recall that a wedge-product over the empty set is equal to 1 and that an intersection over the empty set is the whole space.

For any $\sigma\in\mathrm{Stab}(\underline{n},\underline{w})$, any $1\le j\le 2d-2$, and any $u\in\widetilde{\mathbb{S}}_{j-1}$ (if $j>1$), let $\Lambda_{\sigma,j}(u)$ or $\Lambda_{\sigma,1}$ be the set of all $[(f,x_1,\dots,x_{d+1})]\in\mathcal{M}^{\mathrm{fm}}_d$ having a cycle of exact period $n_{\sigma(i)}$ and multiplier $u_i\in\partial\mathbb{D}_{\rho_i}$ for any $1\le i<j$ and a cycle of exact period $n_{\sigma(k)}$ and multiplier w_k for any $k>j$. Hence

$$\Lambda_{\sigma,j}(u)\subset\bigcap_{1\le i<j}\mathrm{Per}_{n_i}\big(u_{\sigma(i)}\big)\cap\bigcap_{k>j}\mathrm{Per}_{n_k}\big(w_{\sigma(k)}\big)$$

is an algebraic curve and, by Bézout's theorem, its area is $\leq C \cdot d^{|\underline{n}|-n_j}$ for some constant C depending only on d. Set $\mathcal{W}_{\Lambda_{\sigma,j}(u)} := p_j \circ (\mathcal{W}|_{\Omega\cap\Lambda_{\sigma,j}(u)})$ or $\mathcal{W}_{\Lambda_{\sigma,1}} := p_1 \circ (\mathcal{W}|_{\Omega\cap\Lambda_{\sigma,1}})$, where $p_j\colon \mathbb{D}^{2d-2} \to \mathbb{D}$ is the projection on the j-th coordinate. Then the measure

$$\bigwedge_{1\leq i<j} \mathcal{W}^*_{\sigma(i)}(\delta_{u_i}) \wedge \big(\mathcal{W}^*_{\sigma(j)}(\lambda_{\rho_j}) - \mathcal{W}^*_{\sigma(j)}(\delta_{w_j})\big) \wedge \bigwedge_{k>j} \mathcal{W}^*_{\sigma(k)}(\delta_{w_k})$$

is equal to

$$\mathcal{W}^*_{\Lambda_{\sigma,j}(u)}(\lambda_{\rho_j} - \delta_{w_j}). \tag{7.3}$$

on $\Omega \cap \Lambda_{\sigma,j}(u)$ if $j > 1$, and the measure $S_{\sigma,1}$ is itself equal to $\mathcal{W}_{\Lambda_{\sigma,1}}(\lambda_{\rho_1} - \delta_{w_1})$ on $\Omega \cap \Lambda_{\sigma,1}$.

7.3. Proof of (7.2): the case of arbitrary multipliers in $\mathbb{D}^{2d-2}$. Assume that Ψ is $\mathscr{C}^1$ and test (7.3) against Ψ. Pick any $\sigma \in \mathrm{Stab}(\underline{n}, \underline{w})$, any $1 \leq j \leq 2d-2$, and any $u := (u_i)_{i<j} \in \widetilde{\mathbb{S}}_{j-1}$ (if $j > 1$). We continue to fix Ω as in Subsection (7.2) and let $O := \mathcal{W}^{-1}_{\Lambda_{\sigma,j}(u)}(w_j)$. Then

$$\int_{\Lambda_{\sigma,j}(u)\cap\Omega} \Psi \cdot \mathcal{W}^*_{\Lambda_{\sigma,j}(u)}(\lambda_{\rho_j} - \delta_{w_j}) = \int_{\Lambda_{\sigma,j}(u)\cap\Omega} (\Psi - \Psi(O)) \cdot \mathcal{W}^*_{\Lambda_{\sigma,j}(u)}(\lambda_{\rho_j}),$$

so that by the mean value inequality:

$$\begin{aligned}\left|\int_{\Lambda_{\sigma,j}(u)\cap\Omega} \Psi \cdot \mathcal{W}^*_{\Lambda_{\sigma,j}(u)}(\lambda_{\rho_j} - \delta_{w_j})\right| &\leq \int_{\Lambda_{\sigma,j}(u)\cap\Omega} |\Psi - \Psi(O)| \cdot \mathcal{W}^*_{\Lambda_{\sigma,j}(u)}(\lambda_{\rho_j}) \\ &\leq C \cdot \|\Psi\|_{\mathscr{C}^1} \mathrm{diam}\big(\mathcal{W}^{-1}_{\Lambda_{\sigma,j}(u)}\big(\mathbb{D}(0,\rho_j)\big)\big),\end{aligned}$$

where the diameter is computed with respect to the distance induced by β and the constant $C > 0$ only depends on the choice of the $\mathscr{C}^1$-norm. By the length-area estimate (Lemma 2.4), we have

$$\mathrm{diam}\big(\mathcal{W}^{-1}_{\Lambda_{\sigma,j}(u)}\big(\mathbb{D}(0,\rho_j)\big)\big)^2 \leq \tau \cdot \frac{\mathrm{Area}(\Omega \cap \Lambda_{\sigma,j}(u))}{\min\{1, \frac{1}{2\pi}\log(1/\rho_j)\}} = \tau \cdot \frac{\mathrm{Area}(\Omega \cap \Lambda_{\sigma,j}(u))}{|\log \rho_j|/(2\pi)}$$

since $\mathcal{W}^{-1}_{\Lambda_{\sigma,j}(u)}(\mathbb{D}(0,\rho_j)) \Subset \mathcal{W}^{-1}_{\Lambda_{\sigma,j}(u)}(\mathbb{D}(0,1)) = \Omega \cap \Lambda_{\sigma,j}(u)$ in $\Lambda_{\sigma,j}(u)$ are holomorphic disks and $\rho_j \geq 1/2$. Using the Cauchy–Schwarz inequality gives

$$\begin{aligned}&\left|\sum_{\Omega}\int_{\Lambda_{\sigma,j}(u)\cap\Omega} \Psi \cdot \mathcal{W}^*_{\Lambda_{\sigma,j}(u)}(\lambda_{\rho_j} - \delta_{\rho_j})\right| \\ &\qquad\leq C \cdot \|\Psi\|_{\mathscr{C}^1} \cdot \sum_{\Omega}\left(\tau \cdot \frac{\mathrm{Area}(\Omega \cap \Lambda_{\sigma,j}(u))}{|\log \rho_j|/(2\pi)}\right)^{1/2}\end{aligned}$$

$$\le C\Big(\sum_{\Omega}\frac{\tau}{|\log\rho_j|/(2\pi)}\Big)^{1/2}\Big(\sum_{\Omega}\mathrm{Area}(\Omega\cap\Lambda_{\sigma,j}(u))\Big)^{1/2}\|\Psi\|_{\mathcal{C}^1}$$
$$\le C\Big(\frac{\tau}{|\log\rho_j|/(2\pi)}\Big)^{1/2}N_{\mathrm{fm}}(\underline{n})^{1/2}\big(\mathrm{Area}(\Lambda_{\sigma,j}(u))\big)^{1/2}\|\Psi\|_{\mathcal{C}^1},$$

so that recalling that $N_{\mathrm{fm}}(\underline{n}) = N_{\mathcal{M}_d^{\mathrm{fm}}}(\underline{n}) \le C_1 d^{|\underline{n}|}$ by Bézout's theorem and $\mathrm{Area}(\Lambda_{\sigma,j}(u)) \le C_2 d^{|\underline{n}|-n_j}$, where $C_1, C_2 > 0$ depend only on d, we have

$$\Big|\sum_{\Omega}\int_{\Lambda_{\sigma,j}(u)\cap\Omega}\Psi\cdot\mathcal{W}^*_{\Lambda_{\sigma,j}(u)}\big(\lambda_{\rho_j}-\delta_{\rho_j}\big)\Big| \le C_3\Big(\frac{d^{-n_j}\tau}{|\log\rho_j|/(2\pi)}\Big)^{1/2}d^{|\underline{n}|}\|\Psi\|_{\mathcal{C}^1},$$

where $C_3 > 0$ depends only on d. Similarly,

$$\Big|\sum_{\Omega}\int_{\Lambda_{\sigma,1}\cap\Omega}\Psi\cdot\mathcal{W}^*_{\Lambda_{\sigma,1}}\big(\lambda_{\rho_1}-\delta_{\rho_1}\big)\Big| \le C_3\Big(\frac{d^{-n_1}\tau}{|\log\rho_1|/(2\pi)}\Big)^{1/2}d^{|\underline{n}|}\|\Psi\|_{\mathcal{C}^1}.$$

Since the right-hand sides are independent of u and σ, recalling (7.3), we have

$$\big|\langle\mu^{\mathrm{fm}}_{\underline{n},\underline{\rho}} - T^{2d-2,\mathrm{fm}}_{\underline{n}}(\underline{\rho}),\Psi\rangle\big| \le C_4\max_{1\le j\le 2d-2}\Big(\frac{1}{d^{n_j}|\log\rho_j|}\Big)^{1/2}\|\Psi\|_{\mathcal{C}^1},$$

where $C_4 > 0$ depends only on d (and actually not on K). Hence (7.2) holds.

7.4. Proof of (7.1): the center of components. Assume that Ψ is $\mathcal{C}^2$ and test (7.3) against Ψ. Pick any $\sigma \in \mathrm{Stab}(\underline{n})$, any $1 \le j \le 2d-2$, and any $(u_i)_{i<j} \in \widetilde{\mathbb{S}}_{j-1}$ (if $j > 1$). We continue to fix Ω as in Subsection (7.2) for a while and let $O := \mathcal{W}^{-1}_{\Lambda_{\sigma,j}(u)}(0,\dots,0)$ be the center of the disk $\Omega\cap\Lambda_{\sigma,j}(u)$. Then we have

$$\int_{\Lambda_{\sigma,j}(u)\cap\Omega}\Psi\cdot\mathcal{W}^*_{\Lambda_{\sigma,j}(u)}\big(\lambda_{\rho_j}-\delta_0\big)$$
$$= \int_{\Lambda_{\sigma,j}(u)\cap\Omega}\big(\Psi-\Psi(O)\big)\cdot\mathcal{W}^*_{\Lambda_{\sigma,j}(u)}(\lambda_{\rho_j})$$
$$= \int_{\Lambda_{\sigma,j}(u)\cap\Omega}\big(\Psi(z)-(\mathrm{D}_O\Psi)(z-O)-\Psi(O)\big)\big((\mathcal{W}_{\Lambda_{\sigma,j}(u)})^*(\lambda_{\rho_j})\big)(z),$$

the latter equality holding by the mean value theorem for harmonic functions, that is:

$$\int_{\Lambda_{\sigma,j}(u)\cap\Omega}(\mathrm{D}_O\Psi)\,(z-O)\cdot(\mathcal{W}_{\Lambda_{\sigma,j}(u)})^*(\lambda_{\rho_j})(z) = (\mathrm{D}_O\Psi)\,(O-O) = 0.$$

By the mean value inequality, we have

$$\begin{aligned}
&\left|\int_{\Lambda_{\sigma,j}(u)\cap\Omega}\Psi\cdot\mathcal{W}^*_{\Lambda_{\sigma,j}(u)}\big(\lambda_{\rho_j}-\delta_0\big)\right| \\
&\qquad\le\int_{\Lambda_{\sigma,j}(u)\cap\Omega}\big|\Psi(z)-(\mathrm{D}_O\Psi)(z-O)-\Psi(O)\big|\cdot(\mathcal{W}_{\Lambda_{\sigma,j}(u)})^*(\lambda_{\rho_j})(z) \\
&\qquad\le C\cdot\|\Psi\|_{\mathcal{C}^2}\cdot\operatorname{diam}\big(\mathcal{W}^{-1}_{\Lambda_{\sigma,j}(u)}(\mathbb{D}(0,\rho_j))\big)^2,
\end{aligned}$$

where the diameter is again computed with respect to the distance induced by β and the constant $C>0$ depends only on the choice of the $\mathcal{C}^2$-norm. Again by the length-area estimate (Lemma 2.4), we have

$$\left|\int_{\Lambda_{\sigma,j}(u)\cap\Omega}\Psi\cdot\mathcal{W}^*_{\Lambda_{\sigma,j}(u)}\big(\lambda_{\rho_j}-\delta_0\big)\right|\le C\cdot\|\Psi\|_{\mathcal{C}^2}\cdot\left(\tau\cdot\frac{\operatorname{Area}(\Omega\cap\Lambda_{\sigma,j}(u))}{|\log(1/2)|/(2\pi)}\right)$$

since $\mathcal{W}^{-1}_{\Lambda_{\sigma,j}(u)}(\mathbb{D}(0,\rho_j))\Subset\mathcal{W}^{-1}_{\Lambda_{\sigma,j}(u)}(\mathbb{D}(0,1))=\Omega\cap\Lambda_{\sigma,j}(u)$ in $\Lambda_{\sigma,j}(u)$ are holomorphic disks (here, $\rho_j\equiv 1/2$ by definition). Hence

$$\left|\int_{\Lambda_{\sigma,j}(u)\cap\Omega}\Psi\cdot\mathcal{W}^*_{\Lambda_{\sigma,j}(u)}\big(\lambda_{\rho_j}-\delta_0\big)\right|\le C'\cdot\|\Psi\|_{\mathcal{C}^2}\cdot\operatorname{Area}(\Omega\cap\Lambda_{\sigma,j}(u)),$$

where $C'>0$ depends only on d, so recalling that

$$\sum_{\Omega}\operatorname{Area}(\Omega\cap\Lambda_{\sigma,j}(u))\le\operatorname{Area}(\Lambda_{\sigma,j}(u))\le C_1 d^{|\underline{n}|-n_j},$$

where $C_1>0$ depends only on d,

$$\left|\sum_{\Omega}\int_{\Lambda_{\sigma,j}(u)\cap\Omega}\Psi\cdot\mathcal{W}^*_{\Lambda_{\sigma,j}(u)}\big(\lambda_{\rho_j}-\delta_0\big)\right|\le C'C_1\cdot\|\Psi\|_{\mathcal{C}^2}d^{|\underline{n}|-n_j},$$

and similarly,

$$\left|\sum_{\Omega}\int_{\Lambda_{\sigma,1}\cap\Omega}\Psi\cdot\mathcal{W}^*_{\Lambda_{\sigma,1}}\big(\lambda_{\rho_1}-\delta_0\big)\right|\le C'C_1\cdot\|\Psi\|_{\mathcal{C}^2}d^{|\underline{n}|-n_1}.$$

Since the right-hand sides are independent of u and σ, recalling (7.3), we have

$$\big|\big\langle\mu^{\mathrm{fm}}_{\underline{n}}-T^{2d-2,\mathrm{fm}}_{\underline{n}}(\underline{\rho}),\Psi\big\rangle\big|\le C''\|\Psi\|_{\mathcal{C}^2}\max_{1\le j\le 2d-2}d^{-n_j},$$

where $C''>0$ depends only on d. Hence (7.1) holds. □

Remark 7.1. One cannot hope the (even qualitative) convergence $\mu_{\underline{n}}\to\mu_{\mathrm{bif}}$ for all bounded DSH observables; indeed, consider the DSH function

$$\phi_A:=\min\big\{0,\max\{\log|Z|,-A\}/A\big\}$$

on $\mathbb{C}^{2d-2}$ for some $A > 0$, which is identically equal to 0 outside the ball $B(0, \exp(-A))$ and equal to -1 at 0. Furthermore, it is DSH and its DSH norm can be taken arbitrarily small for $A \gg 1$. By a change of coordinates, one can then construct a DSH function in $\mathcal{M}_d$ which is equal to -1 at the center of a given hyperbolic component and 0 outside that component, with arbitrarily small DSH norm. Summing this constructions over sufficiently many hyperbolic components, we can construct an observable ψ_A which is bounded and DSH with $\langle \mu_{\underline{n}}, \psi_A \rangle \not\to \langle \mu_{\mathrm{bif}}, \psi_A \rangle$. Nevertheless, it would be interesting to find a space of test functions independent of the choice of coordinates for which a similar statement as Theorem 1.8 holds.

8. Distribution of hyperbolic maps in $\mathcal{P}_d^{\mathrm{cm}}$

8.1. A good parametrization of $\mathcal{P}_d^{\mathrm{cm}}$. We refer to [15, §5] and [28, §2] for the material of this section. Recall that the *critically marked moduli space* $\mathcal{P}_d^{\mathrm{cm}}$ of degree d polynomials is the space of affine conjugacy classes of degree d polynomials with $d-1$ marked critical points in $\mathbb{C}$. We define a finite branched cover of $\mathbb{C}^{d-1} \to \mathcal{P}_d^{\mathrm{cm}}$ as follows. For $c = (c_1, \ldots, c_{d-2}) \in \mathbb{C}^{d-2}$ and $a \in \mathbb{C}$, let

$$P_{c,a}(z) := \frac{1}{d} z^d + \sum_{j=2}^{d-1} (-1)^{d-j} \frac{\sigma_{d-j}(c)}{j} z^j + a^d \,, \quad z \in \mathbb{C} \,,$$

where $\sigma_k(c)$ is the monic elementary degree k symmetric polynomial in the c_i's. This family is known to be a finite branched cover of $\mathcal{P}_d^{\mathrm{cm}}$. Remark also that the (finite) critical points of $P_{c,a}$ are exactly $c_0, c_1, \ldots, c_{d-2}$, taking into account their multiplicity, where we set $c_0 := 0$, and that they depend algebraically on $(c, a) \in \mathbb{C}^{d-1}$. From now on, we work on the parameter space $\mathbb{C}^{d-1}$ of the family $(P_{c,a})_{(c,a) \in \mathbb{C}^{d-1}}$ rather than $\mathcal{P}_d^{\mathrm{cm}}$ itself, without loss of generality.

The dynamical Green function of $P_{c,a}$ is the continuous psh function $g_{c,a} \colon \mathbb{C} \to \mathbb{R}_+$ defined by $g_{c,a}(z) := \lim_{n\to\infty} d^{-n} \log^+ |P_{c,a}^n(z)|$, $z \in \mathbb{C}$, where the convergence is locally uniform in $(c, a, z) \in \mathbb{C}^d$. For any $0 \le j \le d-2$, the function $g_j(c, a) := g_{c,a}(c_j)$ is psh and continuous on $\mathbb{C}^{d-1}$ and, setting $T_j := dd^c g_j$, we have $dd^c L = \sum_j T_j$ and $T_j \wedge T_j = 0$.

In this family it is now classical to define the bifurcation measure on $\mathbb{C}^{d-1}$ as a *probability* measure $\mu_{\mathrm{bif}} := \bigwedge_{j=0}^{d-2} T_j = \frac{1}{(d-1)!} (dd^c L)^{d-1}$ on $\mathbb{C}^{d-1}$. Then $\operatorname{supp} \mu_{\mathrm{bif}}$ is *compact* and coincides with the Shilov boundary of the connectedness locus

$$\mathcal{C}_d := \big\{(c, a) \in \mathbb{C}^{d-1} : \mathcal{J}_{P_{c,a}} \text{ is connected}\big\} = \big\{(c, a) \in \mathbb{C}^{d-1} : \max_j g_j(c, a) = 0\big\}.$$

For any $n \in \mathbb{N}^*$, we set

$$D_n := \sum_{k|n} \mu\Big(\frac{n}{k}\Big) d^k;$$

$d^n = \sum_{k|n} D_k$ by Möbius inversion, and $D_n = d_n - 1$ if $n = 1$ and $D_n = d_n$ if $n \geq 2$.

For any $n \in \mathbb{N}^*$, the n-th *dynatomic polynomial* of $P_{c,a}$ is defined as

$$\Phi_n^*(P_{c,a}, z) := \prod_{k|n} \big(P_{c,a}^k(z) - z\big)^{\mu(n/k)},$$

and for any $0 \leq j \leq d-1$ and any $n \in \mathbb{N}^*$, we set

$$\mathrm{Per}_j(n) := \big\{(c,a) \in \mathbb{C}^{d-1} \ ; \ \Phi_n^*(P_{c,a}, c_j) = 0\big\}$$

(cf. Subsection 6.4 for $\mathcal{M}_d^{\mathrm{cm}}$). The variety $\mathrm{Per}_j(n)$ is an algebraic hypersurface of $\mathbb{C}^{d-1}$ of degree D_n (*and* of degree d_n for $n \geq 2$) and is contained in $\{(c,a) \in \mathbb{C}^{d-1} \ ; \ g_{c,a}(c_j) = 0\}$. Moreover, the following holds (see [22, Theorem 6.1]).

Theorem 8.1. *For any $\underline{n} = (n_0, \ldots, n_{d-2}) \in (\mathbb{N}^*)^{d-1}$ satisfying $\min_j n_j > 1$ and any $(c,a) \in \bigcap_{j=0}^{d-1} \mathrm{Per}_j(n_j)$ such that $P_{c,a}$ has only simple critical points in $\mathbb{C}$, the $(d-1)$ hypersurfaces $\mathrm{Per}_j(n_j)$ are smooth and intersect transversely at (c,a).*

Pick any $\underline{n} = (n_0, \ldots, n_{d-2}) \in (\mathbb{N}^*)^{d-1}$. We say a hyperbolic component $\mathcal{H}$ in $\mathbb{C}^{d-1}$ (or the family $(P_{c,a})_{(c,a)\in\mathbb{C}^{d-1}}$) to be of (disjoint) type $\underline{n}$ if for every $(c,a) \in \mathcal{H}$, $P_{c,a}$ admits $d-1$ distinct attracting cycles of respective exact periods $n_0, \ldots, n_{d-2}$ in $\mathbb{C}$. Then all critical points of $P_{c,a}$ in $\mathbb{C}$ for $(c,a) \in \mathcal{H}$ are simple. For each $0 \leq i \leq d-2$, we let $w_i(c,a) \in \mathbb{D}$ be the multiplier of the attracting cycle that has exact period n_i. In this way we get a holomorphic map $\mathcal{W} = \mathcal{W}_{\mathcal{H}} \colon \mathcal{H} \to \mathbb{D}^{d-1}$ defined by

$$\mathcal{W}(c,a) := \big(w_0(c,a), \ldots, w_{d-2}(c,a)\big), \quad (c,a) \in \mathcal{H}.$$

The following (see [22, Theorem 6.8]) will also be useful in the sequel.

Theorem 8.2. *The map $\mathcal{W} \colon \mathcal{H} \to \mathbb{D}^{d-1}$ is a biholomorphism.*

8.2. Counting hyperbolic components of disjoint type. As in the case of rational maps, we denote by $N_{\mathcal{P}}(\underline{n})$ the number of hyperbolic components of type $\underline{n} = (n_0, \ldots, n_{d-2})$ in the family $(P_{c,a})_{(c,a)\in\mathbb{C}^{d-1}}$. When $n_j \neq n_\ell$ for all $j \neq \ell$ and $n_j \geq 1$ for all j, we have $N_{\mathcal{P}}(\underline{n}) = (d-1)! \cdot d_{|\underline{n}|}$. This result is an immediate consequence of Theorem 8.2. Indeed, all such components contain one postcritically finite parameter, counted with multiplicity, and all of them are contained in $\mathcal{C}_d$. The result follows from Bézout's theorem and the fact that $\deg(\mathrm{Per}_j(n_j)) = d_{n_j}$.

Our aim here is to give a good generalization of the above statement, including the case when $n_j = n_\ell$ for all j, ℓ. The first observation is that any hyperbolic component $\mathcal{H}$ in $\mathbb{C}^{d-1}$ of type $\underline{n}$ is contained in the compact set $\mathcal{C}_d$. We rely on the following lemma, which is an immediate adaptation of Lemma 6.4 (hence we omit the proof).

Lemma 8.3. *For any $\underline{\rho} \in]0,1[^{d-1}$ and any $\underline{n} = (n_0, \dots, n_{d-2}) \in (\mathbb{N}^*)^{d-1}$ with $\min_j n_j \geq 2$, the measure $T_{\underline{n}}^{d-1}(\underline{\rho})$ has full mass on the union of all hyperbolic components $\Omega \subset \mathcal{C}_d$ such that for all $(c, a) \in \Omega$, $P_{c,a}$ has $d-1$ distinct attracting cycles in $\mathbb{C}$ of respective exact periods $n_0, \dots, n_{d-2}$. Furthermore, it gives mass $\#\mathrm{Stab}(\underline{n})/d_{|\underline{n}|}$ to each of those components.*

Here is the precise statement.

Theorem 8.4. *There exists a constant $C \geq 1$ depending only on d, such that for any $\underline{n} = (n_0, \dots, n_{d-2}) \in (\mathbb{N}^*)^{d-1}$ with $\min_j n_j \geq 2$, we have*

$$0 \leq 1 - \frac{\#\mathrm{Stab}(\underline{n}) \cdot N_{\mathcal{P}}(\underline{n})}{(d-1)! \cdot d_{|\underline{n}|}} \leq C \max_{0 \leq j \leq d-2} \frac{\sigma_2(n_j)}{d^{n_j}}.$$

Proof. Pick any $\underline{n} = (n_0, \dots, n_{d-2}) \in (\mathbb{N}^*)^{d-1}$ with $\min_j n_j \geq 2$. Set $\underline{\rho} := (1/2, \dots, 1/2)$, and pick a smooth cut-off function Ψ on $\mathbb{C}^{d-1}$ such that $\Psi = 1$ on $\mathcal{C}_d$. Applying Theorem 1.1 yields

$$\left| \left\langle T_{\underline{n}}^{d-1}(\underline{\rho}), \Psi \right\rangle - \left\langle (dd^c L)^{d-1}, \Psi \right\rangle \right| \leq C \|\Psi\|_{\mathrm{DSH}}^* \max_{0 \leq j \leq d-2} \frac{\sigma_2(n_j)}{d^{n_j}},$$

where $C > 0$ only depends on $\mathrm{supp}(\Psi)$ and d. As seen in the previous Subsection, we have $\langle (dd^c L)^{d-1}, \Psi \rangle = (d-1)!$, and by Lemma 8.3 and $\mathrm{supp}(T_{\underline{n}}^{d-1}(\underline{\rho})) \subset \mathcal{C}_d$,

$$\left\langle T_{\underline{n}}^{d-1}(\underline{\rho}), \Psi \right\rangle = \frac{\#\mathrm{Stab}(\underline{n})}{d_{|\underline{n}|}} \cdot N_{\mathcal{P}}(\underline{n}).$$

Now the proof is complete also by $N_{\mathcal{P}}(\underline{n}) \leq d_{|\underline{n}|}/\#\mathrm{Stab}(\underline{n})$. □

Remark 8.5. This result is coherent with the above remark concerning the case $n_j \neq n_\ell$ for all $j \neq \ell$, since in that case, $\mathrm{Stab}(\underline{n}) = \{\mathrm{id}\}$. The above statement can also be interpreted as follows; *the number of postcritically finite parameters for which all critical points are periodic with prescribed exact periods $n_0, \dots, n_{d-2} \geq 2$ and at least 2 critical points lie in the same super-attracting cycle, counted with multiplicity of intersection of the $\mathrm{Per}_{\tau(j)}(n_j)$ for all $\tau \in \mathfrak{S}_{d-1}$, is bounded from above by $C \max_{j \leq d-2}(\sigma_2(n_j)/d^{n_j}) \cdot d^{|\underline{n}|}$.* This is a much better estimate than the one we can obtain by naive arguments. Indeed, without taking the multiplicity into account we can naively get a bound from above of the form $Cd^{|\underline{n}| - \min_j n_j/2} \max_{j \leq d-2} n_j$, see e.g. the proof of the upper bound on $\mathrm{Card}(Y_{\underline{n}} \setminus Z_{\underline{n}})$ in the proof of Theorem 6.8.

An immediate application of this theorem is the following:

Corollary 8.6. *For any integer $n \geq 2$, we have*

$$0 \leq 1 - \frac{N_{\mathcal{P}}(n,\dots,n)}{(d_n)^{d-1}} \leq C\Big(\frac{\sigma_2(n)}{d^n}\Big),$$

where $C \geq 1$ is given by Theorem 8.4. In particular,

$$\frac{N_{\mathcal{P}}(n,\dots,n)}{(d_n)^{d-1}} = 1 + O\Big(\frac{\sigma_2(n)}{d^n}\Big), \quad \text{as } n \to \infty.$$

Proof. In the present case, we have $\#\mathrm{Stab}(n,\dots,n) = (d-1)!$ and $d_{|(n,\dots,n)|} = (d_n)^{d-1}$. Since μ_{bif} is a probability measure, the result follows from Theorem 8.4 above. □

Remark 8.7. In fact, we have proved that, counted with multiplicity, the number of intersection points of the $\mathrm{Per}_j(n)$ for which at least two critical points lie in the same periodic orbit is bounded from above by a constant times $\sigma_2(n)d^{(d-2)n}$.

8.3. Distribution of polynomials with $(d-1)$ attracting cycles. Pick $\underline{n} = (n_0,\dots,n_{d-2}) \in (\mathbb{N}^*)^{d-1}$ with $\min_j n_j \geq 2$ and $\underline{w} := (w_0,\dots,w_{d-2}) \in \mathbb{D}^{d-1}$. As in the case of rational maps, we let $C_{\underline{n},\underline{w}}$ be the (finite) set of parameters $(c,a) \in \mathbb{C}^{d-1}$ such that $P_{c,a}$ has $d-1$ distinct attracting cycles in $\mathbb{C}$ of respective exact periods $n_0,\dots,n_{d-2}$ and multipliers $w_0,\dots,w_{d-2}$. We also let

$$\nu_{\underline{n},\underline{w}} := \frac{\#\mathrm{Stab}(\underline{n},\underline{w})}{(d-1)!\cdot d_{|\underline{n}|}} \sum_{(c,a)\in C_{\underline{n},\underline{w}}} \delta_{(c,a)}.$$

The only modification from the case of rational maps is the multiplication by $1/(d-1)!$. From the normalization $\mu_{\mathrm{bif}} = (dd^cL)^{d-1}/(d-1)!$, we see easily that this factor should also appear in the definition of $\nu_{\underline{n},\underline{w}}$. An argument similar to that in the proof of Theorem 1.8 gives the following.

Theorem 8.8. *There exists a constant $C > 0$ depending only on d such that*

(1) *for any $\Psi \in \mathcal{C}_c^2(\mathbb{C}^{d-1})$ and any $\underline{n} = (n_0,\dots,n_{d-2}) \in (\mathbb{N}^*)^{d-1}$ with $\min_j n_j \geq 2$,*

$$\Big|\int_{\mathbb{C}^{d-1}} \Psi\,\nu_{\underline{n},\underline{0}} - \int_{\mathbb{C}^{d-1}} \Psi\,\mu_{\mathrm{bif}}\Big| \leq C \max_{0\leq j\leq d-2}\Big(\frac{\sigma_2(n_j)}{d^{n_j}}\Big)\|\Psi\|_{\mathcal{C}^2}.$$

(2) *for any $\Psi \in \mathcal{C}_c^1(\mathbb{C}^{d-1})$, any $\underline{w} = (w_0,\dots,w_{d-2}) \in \mathbb{D}^{d-1}$ and any $\underline{n} = (n_0,\dots,n_{d-2}) \in (\mathbb{N}^*)^{d-1}$ with $\min_j n_j \geq 2$,*

$$\Big|\int_{\mathbb{C}^{d-1}} \Psi\,\nu_{\underline{n},\underline{w}} - \int_{\mathbb{C}^{d-1}} \Psi\,\mu_{\mathrm{bif}}\Big| \leq C \max_{0\leq j\leq d-2}\Big\{\Big(\frac{-1}{d^{n_j}\log|w_j|}\Big)^{1/2}, \Big(\frac{\sigma_2(n_j)}{d^{n_j}}\Big)^{1/2}\Big\}\|\Psi\|_{\mathcal{C}^1}.$$

Remark 8.9. The key difference with the case of the moduli space $\mathcal{M}_d$ of degree d rational maps is the existence of a *universal* constant $C > 0$. This is a consequence of the fact that $C_{\underline{n},\underline{w}} \cup \mathrm{supp}(\mu_{\mathrm{bif}}) \subset \mathcal{C}_d$, which is compact in $\mathbb{C}^{d-1}$, for all $\underline{n}$ and all $\underline{w}$. This compactness property implies the existence of a universal constant $C_1 > 0$ in the conclusion of Theorem 1.1 in the family $(P_{c,a})_{(c,a)\in\mathbb{C}^{d-1}}$.

We now come to our last result in the spirit of Theorem B of [28]: for any $n \in \mathbb{N}^*$, we want to prove the measure equidistributed on parameters $(c,a) \in \mathbb{C}^{d-1}$ satisfying $c_j \in \mathrm{Fix}^*(P^n_{c,a})$ for any $0 \le j \le d-2$ converges towards the bifurcation measure, with an exponential speed of convergence.

Corollary 8.10. *There exists a constant $C > 0$ depending only on d such that for any integer $n \geq 2$ and any $\Psi \in \mathcal{C}^2_c(\mathbb{C}^{d-1})$, we have*

$$\left| \frac{1}{(d_n)^{d-1}} \int_{\mathbb{C}^{d-1}} \Psi \bigwedge_{j=0}^{d-2} \left[\mathrm{Per}_j(n) \right] - \int_{\mathbb{C}^{d-1}} \Psi \, \mu_{\mathrm{bif}} \right| \le C \cdot \frac{\sigma_2(n)}{d^n} \cdot \|\Psi\|_{\mathcal{C}^2}.$$

Proof. For any integer $n \geq 2$, we have $\bigwedge_{j=0}^{d-2}[\mathrm{Per}_j(n)] = (d_n)^{d-1} \nu_{n,(0,\dots,0)}$, so that we can directly apply Theorem 8.8. □

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