Positively ratioed representations

Giuseppe Martone* and Tengren Zhang*

Abstract. Let S be a closed orientable surface of genus at least 2 and let G be a semisimple real algebraic group of non-compact type. We consider a class of representations from the fundamental group of S to G called positively ratioed representations. These are Anosov representations with the additional condition that certain associated cross ratios satisfy a positivity property. Examples of such representations include Hitchin representations and maximal representations. Using geodesic currents, we show that the corresponding length functions for these positively ratioed representations are well-behaved. In particular, we prove a systolic inequality that holds for all such positively ratioed representations.

Contents

1. Introduction

Let S be a closed, oriented, connected surface of genus at least 2 with fundamental group Γ. The Teichmüller space of S, denoted $\mathcal{T}(S)$, is the deformation space of hyperbolic structures on S. Via the holonomy, one can also think of $\mathcal{T}(S)$ as a connected component of the space

$$\mathcal{X}(\Gamma, \mathrm{PSL}(2, \mathbb{R})) := \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PGL}(2, \mathbb{R}).$$

The representations in $\mathcal{T}(S)$ can be characterized as the ones that are $[P]$-Anosov, where P is the unique (up to conjugation) parabolic subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Let $\mathcal{CG}(S)$ denote the set of free homotopy classes of closed curves (see also Definition 2.1). Every hyperbolic structure $\rho \in \mathcal{T}(S)$ induces a length function $\ell^\rho \colon \mathcal{CG}(S) \to \mathbb{R}$ which associates to $c \in \mathcal{CG}(S)$ the hyperbolic length, with respect to ρ, of the geodesic representative of c.

A *geodesic current* on S is a locally finite, Γ-invariant, Borel measure on the set of geodesics in the universal cover of S. Observe that the space of geodesic currents on S, denoted $\mathcal{C}(S)$, is an open convex cone in an infinite dimensional vector space. Furthermore, $\mathcal{CG}(S)$ can be identified with a subset of $\mathcal{C}(S)$ (see Section 2.2). Bonahon [1] showed that $\mathcal{C}(S)$ is naturally equipped with a continuous, bilinear *intersection pairing*

$$i \colon \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}^+ \cup \{0\}$$

which generalizes the geometric intersection number between free homotopy classes of closed curves in $\mathcal{CG}(S)$. Also, he proved that for every hyperbolic structure

$\rho \in \mathcal{T}(S)$, there is a unique geodesic current $\mu^\rho \in \mathcal{C}(S)$ with the property that for any $c \in \mathcal{CG}(S)$,

$$i(\mu^\rho, c) = \ell^\rho(c).$$

The geodesic current μ^ρ is known as the *intersection current* associated to ρ.

In this paper, we investigate the extent to which we can generalize this intersection current to the setting of $[P]$-Anosov representations $\rho \colon \Gamma \to G$, where G is a non-compact semisimple, real algebraic group and $[P]$ is the conjugacy class of a parabolic subgroup $P \subset G$. Every conjugacy class of parabolic subgroups of G determines a subset θ of the set of restricted simple roots Δ of G. We will assume, without loss of generality, that $\theta = \iota(\theta)$, where ι is the *opposition involution* on Δ (see Sections 2.3 and 2.4).

For each $\alpha \in \theta$, the corresponding restricted fundamental weight ω_α allows us to define a length function

$$\ell_\alpha^\rho \colon \mathcal{CG}(S) \to \mathbb{R}^+ \cup \{0\}$$

for ρ, which generalizes the length function associated to a hyperbolic structure in $\mathcal{T}(S)$. However, it is not true in general that there is a geodesic current ν so that $i(\nu, c) = \ell_\alpha^\rho(c)$ for every $c \in \mathcal{CG}(S)$.

As such, we introduce the notion of a $[P]$-*positively ratioed representation*. These are $[P]$-Anosov representations with the additional property that certain cross ratios associated to ω_α for all $\alpha \in \theta$ are always positive (see Section 2.2 for more details). Examples include $\mathrm{PSL}(n, \mathbb{R})$-Hitchin representations and $\mathrm{PSp}(2n, \mathbb{R})$-maximal representations. Combining the work of Hamenstädt [14, 15], Otal [28], and Tits [34], we have the following theorem.

Theorem 1.1. *If $\rho \colon \Gamma \to G$ is a $[P]$-positively ratioed representation, then for any $\alpha \in \theta$, there is a unique geodesic current μ_α^ρ so that $i(\mu_\alpha^\rho, c) = \ell_\alpha^\rho(c)$ for all $c \in \mathcal{CG}(S)$.*

By Theorem 1.1, to prove statements about ℓ_α^ρ, one needs only to prove the analogous statements in the setting of geodesic currents. Using this strategy, we prove the remaining results in this paper. In fact, all the results in this introduction can be stated in the more technical language of *period minimizing* geodesic currents with full support. These are geodesic currents with full support that satisfy the property that the number of closed geodesics $c \in \mathcal{CG}(S)$ so that $\ell_\nu(c) := i(c, \nu) < T$ is finite for all $T \in \mathbb{R}^+$. However, to emphasize the application we are interested in, we will state most of our results for positively ratioed representations in the introduction, and indicate the numbering of the analogous statement about geodesic currents in parenthesis.

We will need the following notation. For an essential subsurface $S' \subset S$, i.e. an incompressible subsurface with negative Euler characteristic, denote by $\mathcal{CG}(S')$ the set of free homotopy classes of unoriented closed curves in S'. Notice that S' is an orientable surface of genus g' with n' boundary components so that $2g' - 2 + n' > 0$.

The main point of this paper is that Theorem 1.1 can be exploited to study the length functions of positively ratioed representations. As a first example, we have

the following corollary about the asymptotic behavior of length functions along a sequence of positively ratioed representations. This was motivated by the work of Burger–Pozzetti [5].

Corollary 1.2 (Proposition 4.7). *Let $\{\rho_j \colon \Gamma \to G_j\}_{j=1}^{\infty}$ be a sequence of $[P_j]$-positively ratioed representations, let θ_j be a subset of the restricted simple roots of G_j determined by P_j, and let $\alpha_j \in \theta_j$. Fix an auxiliary hyperbolic structure on S. Then there is:*

- *a subsequence of $\{\rho_j\}_{j=1}^{\infty}$, also denoted $\{\rho_j\}_{j=1}^{\infty}$,*
- *a (possibly disconnected, possibly empty) essential subsurface $S' \subset S$,*
- *a (possibly empty) collection of pairwise non-intersecting, non-peripheral simple closed curves $\{c_1, \ldots, c_k\}$ in $\mathcal{CG}(S \setminus S')$*

so that $A := S' \cup \bigcup_{i=1}^{k} c_i$ is non-empty, and the following holds. Let $c \in \mathcal{CG}(S)$ be a closed curve so that $c \notin \mathcal{CG}(S \setminus A)$ and c is not a multiple of c_i for $i = 1, \ldots, k$.

(1) *If $d \in \mathcal{CG}(S \setminus A)$ or d is a multiple of c_i for some $i = 1, \ldots, k$, then*

$$\lim_{j \to \infty} \frac{\ell_{\alpha_j}^{\rho_j}(d)}{\ell_{\alpha_j}^{\rho_j}(c)} = 0.$$

(2) *If $d \in \mathcal{CG}(S)$ is a closed curve so that $d \notin \mathcal{CG}(S \setminus A)$ and d is not a multiple of c_i for $i = 1, \ldots, k$, then*

$$\lim_{j \to \infty} \frac{\ell_{\alpha_j}^{\rho_j}(d)}{\ell_{\alpha_j}^{\rho_j}(c)} \in \mathbb{R}^{+}.$$

In the case when $G_j = \mathrm{PSp}(2n, \mathbb{R})$ and ρ_j is maximal for all j, Corollary 1.2 is a result of Burger–Pozzetti [5, Theorem 1.1]. More informally, this corollary states that the closed curves in S whose lengths are growing at the fastest rate along a sequence of positively ratioed representations are exactly those that intersect a particular union of a subsurface of S with a collection of pairwise non-intersecting simple closed curves in S.

A second important consequence of Theorem 1.1 is that the length functions coming from positively ratioed representations behave as if they were the length functions of a negatively curved metric on S when we perform surgery (see Section 4.3).

Corollary 1.3 (Proposition 4.5). *Let $\rho \colon \Gamma \to G$ be P_θ-positively ratioed for some $\theta \subset \Delta$. For $c \in \mathcal{CG}(S)$ with $i(c, c) > 0$, let $c_1, c_2, c_3 \in \mathcal{CG}(S)$ be obtained via surgery at a point of self-intersection of c as in Proposition 4.5. Then for any $\alpha \in \theta$, we have*

$$\ell_\alpha^\rho(c_1) < \ell_\alpha^\rho(c) \quad and \quad \ell_\alpha^\rho(c_2) + \ell_\alpha^\rho(c_3) < \ell_\alpha^\rho(c).$$

For any period minimizing geodesic current $v \in \mathcal{C}(S)$, let $\ell_v \colon \mathcal{CG}(S) \to \mathbb{R}^+ \cup \{0\}$ be the function defined by $\ell_v(c) := i(v, c)$. Using this, we can define the following three quantities associated to connected essential subsurfaces $S' \subset S$. The first is the *entropy* of S', which is defined to be

$$h_v(S') = h(\ell_v, S') := \limsup_{T \to \infty} \frac{1}{T} \log \#\{c \in \mathcal{CG}(S') : \ell_v(c) \le T\},$$

and the second is the *systole length*, which is defined as

$$L_v(S') = L(\ell_v, S') := \min \{\ell_v(c) : c \in \mathcal{CG}(S')\}.$$

To define the third, one chooses a *minimal pants decomposition* $\mathcal{P}_{v,S'}$ of S', i.e. a maximal collection in $\mathcal{CG}(S')$ of pairwise non-intersecting simple closed geodesics $\{c_1, \ldots, c_{3g'-3+2n'}\}$ so that $c_{3g'-2+n'}, \ldots, c_{3g-3+2n'}$ are the boundary components and for all $j = 0, \ldots, 3g' - 4 + n'$, c_{j+1} is a non-peripheral systole in $\mathcal{CG}(S' \setminus \bigcup_{i=1}^{j} c_i)$. These exists because of Corollary 1.3 (see Section 4.5). The *panted systole length* is then the quantity

$$
\begin{aligned}
K_v(S') &= K(\ell_v, S') \\
&:= \min \{\ell_v(c) : c \in \mathcal{CG}(S') \text{ is not a multiple of a curve in } \mathcal{P}_{v,S'}\}.
\end{aligned}
$$

It turns out that the panted systole length does not depend on the choice of a minimal pants decomposition (see Lemma 6.2), and hence is an invariant of the geodesic current v.

In this setting, our main theorem is the following.

Theorem 1.4. *There is a constant $C \in \mathbb{R}^+$ depending only on the topology of S', so that for any period minimizing $v \in \mathcal{C}(S)$, we have the inequalities*

$$\frac{1}{4} \log(2) \le h_v(S') K_v(S') \le C \cdot \left(\log(4) + 1 + \log\left(1 + \frac{1}{x_0}\right) \right),$$

where x_0 is the unique positive solution to the equation $(1 + x)^{\left\lceil \frac{K_v(S')}{L_v(S')} - 1 \right\rceil} x = 1$.

Together, Theorem 1.1 and Theorem 1.4 give a systolic inequality that holds for all positively ratioed representations.

Let $h_\alpha^\rho := h(\ell_\alpha^\rho, S)$ and $L_\alpha^\rho := L(\ell_\alpha^\rho, S)$. As a first corollary to Theorem 1.4, we have the following.

Corollary 1.5 (Corollary 7.6). *There is a constant C depending only on the topology of S, so that for any $[P_\theta]$-positively ratioed representation $\rho \colon \Gamma \to G$, and any $\alpha \in \theta$, we have the inequality*

$$h_\alpha^\rho L_\alpha^\rho \le C.$$

We emphasize that C does not depend on G. In particular, if $\{\rho_j\colon \Gamma \to G_j\}_{j \in J}$ is a collection of $[P_j]$-positively ratioed representations on Γ so that $\{h_{\alpha_j}^{\rho_j}\}_{j \in J}$ is uniformly bounded below by a positive number, then Corollary 1.5 implies that $\{L_{\alpha_j}^{\rho_j}\}_{j \in J}$ is uniformly bounded from above.

Similarly, given a negatively curved Riemannian metric m on S' with totally geodesic (possibly empty) boundary, one can also define a length function $\ell_m\colon \mathcal{CG}(S') \to \mathbb{R}^+$ which assigns to each free homotopy class of closed curves the length of the geodesic representative in that free homotopy class. Theorem 1.4 also implies the following.

Corollary 1.6. *There is a constant C depending only on the topology of S', so that for any negatively curved Riemannian metric m on S' with totally geodesic boundary,*

$$h(\ell_m, S')L(\ell_m, S') \leq C.$$

Corollary 1.6 is a consequence of the work of Sabourau [33] in the case when S' is a closed surface. The constants in the statements of Theorem 1.4, Corollary 1.5 and Corollary 1.6 are explicit but not sharp, and depend exponentially on the Euler characteristic of the surface.

Another corollary of Theorem 1.4 is the following criterion for when the entropy along a sequence of "thick" positively ratioed representations converges to 0.

Corollary 1.7 (Corollary 7.8). *Let $\{\rho_j\colon \Gamma \to G_j\}_{j=1}^\infty$ be a sequence of $[P_j]$-positively ratioed representations, let θ_j be a subset of the positive roots of G_j determined by $[P_j]$. Also, let $\alpha_j \in \theta_j$ so that $\inf_j L_{\alpha_j}^{\rho_j} > 0$. Then $\lim_{j \to \infty} h_{\alpha_j}^{\rho_j} = 0$ if and only if for any subsequence of $\{\rho_j\}_{j=1}^\infty$, there is*

- *a further subsequence, which we also denote by $\{\rho_j\}_{j=1}^\infty$,*
- *a sequence $\{f_j\}_{j=1}^\infty$ of elements in the mapping class group of S,*
- *a (possibly empty) collection $\mathcal{D} \subset \mathcal{CG}(S)$ of pairwise non-intersecting simple closed curves,*

so that

$$\lim_{j \to \infty} \min\{\ell_{\alpha_j}^{f_j \cdot \rho_j}(c) : c \in \mathcal{CG}(S \setminus \mathcal{D}) \text{ is non-peripheral}\} = \infty$$

and

$$\sup_j \max\{\ell_{\alpha_j}^{f_j \cdot \rho_j}(c) : c \in \mathcal{D}\} < \infty.$$

Here, $f_j \cdot \rho_j := \rho_j \circ (f_j)_*$, where $(f_j)_*\colon \Gamma \to \Gamma$ is the group homomorphism induced by the mapping class $f_j\colon S \to S$.

Nie [26, 27] and the second author [36, 37] previously studied sequences of Hitchin representations whose entropy goes to zero. Corollary 1.7 includes all such sequences.

The rest of this article is organized as follows. In Section 2, we define positively ratioed representations and prove Theorem 1.1. Then, we show that Hitchin and maximal representations are examples of positively ratioed representations in Section 3. In Section 4, we prove Corollary 1.2 and some facts regarding geodesic currents and the intersection pairing, and Sections 5 and 6 are devoted to the proof of Theorem 1.4. Finally, in Section 7, we prove Corollary 1.5, Corollary 1.6, and Corollary 1.7.

2. Positively ratioed representations

The goal of this section is to describe a class of surface group representations, which we call positively ratioed representations. The main property these representations have is that certain length functions associated to them "arise from geodesic currents". This forces the length functions associated to these representations to satisfy some strong properties that are explained in Section 4.

2.1. Topological geodesics.
We begin by carefully specifying what we mean by a geodesic and a closed geodesic on a topological surface. The notation developed in this section will be used in the rest of the paper.

First, we will define closed geodesics. Let $[\Gamma]$ denote the set of conjugacy classes in Γ, and let $\sim$ be an equivalence relation on $[\Gamma]$ given by $[\gamma] \sim [\gamma^{-1}]$.

Definition 2.1. A *closed geodesic* in S is a non-identity equivalence class in $[\Gamma] / \sim$. Also, we say that a closed geodesic is *primitive* if it has a primitive representative in Γ (equivalently, all of its representatives in Γ are primitive).

We will denote the set of all closed geodesics in S by $\mathcal{CG}(S)$, and denote the equivalence class in $\mathcal{CG}(S)$ containing $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ by $[\![\gamma]\!]$. Observe that $\mathcal{CG}(S)$ is naturally in bijection with the free homotopy classes of closed curves on S. Hence, if we choose a hyperbolic structure Σ on S, then the closed geodesics in S are identified

with the closed hyperbolic geodesics in Σ since every free homotopy class of closed curves in Σ contains a unique closed hyperbolic geodesic.

Next, we will define geodesics. It is well known that Γ is a hyperbolic group, so it admits a Gromov boundary $\partial\Gamma$, which is topologically a circle.

Definition 2.2. A (unoriented) *geodesic in* $\widetilde{S}$ is an element of the topological space

$$\mathcal{G}(\widetilde{S}) := \{(x, y) \in \partial\Gamma \times \partial\Gamma : x \neq y\} / \sim,$$

where $\sim$ is the equivalence relation defined by $(x, y) \sim (y, x)$. Also, a *geodesic in* S is an element in $\mathcal{G}(S) := \mathcal{G}(\widetilde{S}) / \Gamma$.

Denote the equivalence classes in $\mathcal{G}(\widetilde{S})$ and $\mathcal{G}(S)$ containing (x, y) by $\{x, y\}$ and $[x, y]$ respectively. Observe that if we choose a hyperbolic structure Σ on S, then this induces a hyperbolic structure $\widetilde{\Sigma}$ on $\widetilde{S}$. The natural identification of $\partial\Gamma$ with the visual boundary $\partial\widetilde{\Sigma}$ of $\widetilde{\Sigma}$ then realizes geodesics in $\widetilde{S}$ (or S) as hyperbolic geodesics in $\widetilde{\Sigma}$ (or Σ).

Of course, closed geodesics in S and geodesics in S can be explicitly related in the following way. Any $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ has an attracting and a repelling fixed point in $\partial\Gamma$, which we denote by γ^+ and γ^- respectively. This allows us to define the map $F : \mathcal{CG}(S) \to \mathcal{G}(S)$ by $F : [\![\gamma]\!] \mapsto [\gamma^-, \gamma^+]$. More informally, this sends every closed geodesic to the bi-infinite geodesic that "wraps around" it. Note that the map F is not injective; if $\gamma \in \Gamma$ is primitive, then $F^{-1}(F[\![\gamma]\!]) = \{[\![\gamma^n]\!] : n \in \mathbb{Z} \setminus \{0\}\}$.

Finally, we have a notion for when two geodesics intersect transversely.

Definition 2.3. We say that $\{a, b\}, \{c, d\} \in \mathcal{G}(\widetilde{S})$ *intersect transversely* if a, c, b, d lie in $\partial\Gamma$ in that (strict) cyclic order. Similarly, two geodesics in $\mathcal{G}(S)$ *intersect transversely* if they have representatives in $\mathcal{G}(\widetilde{S})$ that intersect transversely, and two closed geodesics in $\mathcal{CG}(S)$ *intersect transversely* if their images under the map F described above intersect transversely in $\mathcal{G}(S)$.

If we equip S with a hyperbolic structure Σ, then a pair of geodesics or closed geodesics in S intersect transversely if and only if they intersect transversely as geodesics or closed geodesics for the metric on Σ.

2.2. Cross ratios and geodesic currents. The reader should be cautioned that there are many non-equivalent definitions of cross ratios in the literature, even in the restricted setting of Anosov representations. The definition we use here is one given by Ledrappier [24, Definition 1.f]. Consider the set

$$\partial\Gamma^{[4]} := \{(a, b, c, d) \in \partial\Gamma^4 : \{a, b\} \cap \{c, d\} = \emptyset\}.$$

Definition 2.4.

- A *cross ratio* is a continuous function $B : \partial\Gamma^{[4]} \to \mathbb{R}$ that is invariant under the diagonal action of Γ and satisfies the following:

  (1) (*Symmetry*) $B(x, y, z, w) = B(z, w, x, y)$;

(2) *(Additivity)* $B(x, y, z, w) + B(x, y, w, u) = B(x, y, z, u)$

for all $x, y, z, w, u \in \partial\Gamma$ such that $(x, y, z, w), (x, y, w, u) \in \partial\Gamma^{[4]}$.

- For any $[\![\gamma]\!] = c \in \mathcal{CG}(S)$, the *B-period* of c is $\ell_B(c) := B(\gamma^-, \gamma^+, \gamma \cdot a, a)$ for some $a \in \partial\Gamma - \{\gamma^-, \gamma^+\}$.

One easily shows that the B-period of c does not depend on the choice of a or γ. The following theorem of Otal [28, Theorem 2.2], (see also Ledrappier [24, Theorem 1.f]) states that any cross ratio B is determined by the B-periods.

Theorem 2.5 (Otal). *If $B_1, B_2 \colon \partial\Gamma^{[4]} \to \mathbb{R}$ are cross ratios so that $\ell_{B_1}(c) = \ell_{B_2}(c)$ for all $c \in \mathcal{CG}(S)$, then $B_1 = B_2$.*

Cross ratios are intimately related with geodesic currents, which we will now define. The notion of a geodesic current was first introduced by Bonahon [1], who used them to study Teichmüller space.

Definition 2.6. A *geodesic current on S* is a Γ-invariant, locally finite (non-signed) Borel measure on $\mathcal{G}(\widetilde{S})$.

Denote the space of geodesic currents on S by $\mathcal{C}(S)$. It can be naturally realized as an open cone in an infinite dimensional vector space equipped with the weak* topology (see Bonahon [2, Section 1]). The Γ-invariance in the above definition ensures that every geodesic current $\nu \in \mathcal{C}(S)$ descends to a finite measure $\hat{\nu}$ on the compact space $\mathcal{G}(S)$. However, the Γ-action on $\mathcal{G}(\widetilde{S})$ is not proper, so $\mathcal{G}(S)$ is not Hausdorff. As such, it is often more convenient to work with ν instead of $\hat{\nu}$.

An important example of geodesic currents are the ones associated to closed geodesics. Given any closed geodesic $c = [\![\gamma]\!] \in \mathcal{CG}(S)$, let $\mu_c \in \mathcal{C}(S)$ be the geodesic current defined by

$$\mu_c = \sum_{\eta \cdot \langle\gamma\rangle \in \Gamma/\langle\gamma\rangle} \delta_{\{\eta \cdot \gamma^+, \eta \cdot \gamma^-\}},$$

where $\delta_{\{x,y\}}$ is the Dirac measure supported at the point $\{x, y\} \in \mathcal{G}(\widetilde{S})$. When c is primitive, $\hat{\mu}_c$ is the Dirac measure on $F(c)$. This realizes $\mathcal{CG}(S)$ as a subset of $\mathcal{C}(S)$. *Henceforth, we will blur the distinction between $\mathcal{CG}(S)$ and the subset of $\mathcal{C}(S)$ it is identified with, by using c to denote μ_c.*

On the space of geodesic currents, Bonahon [1, Section 4.2] defined an important function that we will now describe.

Let $\mathcal{DG}(\widetilde{S}) \subset \mathcal{G}(\widetilde{S}) \times \mathcal{G}(\widetilde{S})$ be the open subset defined by

$$\mathcal{DG}(\widetilde{S}) := \left\{ (l_1, l_2) \in \mathcal{G}(\widetilde{S}) \times \mathcal{G}(\widetilde{S}) : l_1, l_2 \text{ intersect transversely} \right\}.$$

Note that $\mathcal{DG}(\widetilde{S})$ is stabilized by the diagonal Γ action on $\mathcal{G}(\widetilde{S}) \times \mathcal{G}(\widetilde{S})$, so we can define $\mathcal{DG}(S) := \mathcal{DG}(\widetilde{S})/\Gamma$. In this case, the Γ action on $\mathcal{DG}(\widetilde{S})$ is proper, so $\mathcal{DG}(S)$ is a Hausdorff space. For any $\mu, \nu \in \mathcal{C}(S)$, the Γ-invariant measure $\mu \times \nu$ on $\mathcal{DG}(\widetilde{S})$ descends to a measure $\widehat{\mu \times \nu}$ on $\mathcal{DG}(S)$.

Definition 2.7. The *intersection form* on $\mathcal{C}(S)$ is the map $i \colon \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}$ given by $i(\mu, \nu) = \widehat{\mu \times \nu}(\mathcal{DG}(S))$.

Bonahon proved that the intersection form is well-defined and continuous, and it is easy to verify that it is symmetric and bilinear. He also proves that if $c, c' \in \mathcal{CG}(S)$ then $i(c, c')$ gives the geometric intersection number between c and c'. More properties of the intersection form are later explained in Section 4.2.

There are several ways one can relate geodesic currents to cross ratios. One way to do so is to associate to every Hölder continuous cross ratio a Gibbs current (Hamenstädt [14], see also Ledrappier [24]). However, this is not useful for the purposes of this paper because it is not easy to read off the periods of the cross ratio from the Gibbs current. Instead, given a cross ratio B, we would like to find an intersection current, which we now define.

Definition 2.8. Let B be a cross ratio. A geodesic current μ is an *intersection current* for B if $\ell_B(c) = i(\mu, c)$ for all $c \in \mathcal{CG}(S)$.

Unfortunately, it turns out that there are cross ratios for which the intersection current does not exist. On the other hand, Hamenstädt observed [15, pp. 103–104] that one can always find intersection currents for cross ratios that satisfy the following positivity condition.

Definition 2.9. A cross ratio B is *positive* if for all $x, y, z, w \in \partial\Gamma$ in this cyclic order, one has $B(x, y, z, w) \geq 0$.

Theorem 2.10 (Hamenstädt). *If $B \colon \partial\Gamma^{[4]} \to \mathbb{R}$ is a positive cross ratio, then it has a unique intersection current.*

The proof of Theorem 2.10 is a standard argument from analysis. However, for lack of a good reference for the proof, we give the full proof in Appendix A.

2.3. Background on semisimple Lie groups. We would like to exploit the existence of intersection currents for positive cross ratios to study the length functions of a certain class of Anosov representations. To define Anosov representations, we need to recall some basic facts regarding non-compact, semisimple, real algebraic groups and their real representations. See Eberlein [8, Chapter 2], Helgason [16, Chapter VI.3], Humphreys [18, Chapters I–III], and Guéritaud–Guichard–Kassel–Wienhard [10, Section 4] for more details.

Let G be a non-compact, semisimple, Lie group with Lie algebra $\mathfrak{g}$. We will also assume that G is a finite union of connected components (for the real topology) of the real points $\mathbf{G}(\mathbb{R})$ of some algebraic group $\mathbf{G}$, and that the adjoint action of G on its Lie algebra is by inner automorphisms, i.e. $\mathrm{Ad}(G) \subset \mathrm{Aut}(\mathfrak{g})_0$. It is well known that there is a unique non-positively curved Riemannian symmetric space X on which G acts transitively by isometries, so that for any point $p \in X$, the stabilizer in G of p is a maximal compact Lie subgroup $K \subset G$. The transitivity of the G-action on X implies that $X \simeq G/K$ as G-spaces.

Let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie subalgebra of $K \subset G$. One can prove that $\mathfrak{k} \subset \mathfrak{g}$ is a maximal subspace on which the Killing form on $\mathfrak{g}$ is negative definite. Since $\mathfrak{g}$ is semisimple, the Killing form on $\mathfrak{g}$ is non-degenerate. This gives an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.

Definition 2.11. The *Cartan involution* $\tau_K : \mathfrak{g} \to \mathfrak{g}$ is the involution so that $\tau_K|_{\mathfrak{k}} = \mathrm{id}$ and $\tau_K|_{\mathfrak{p}} = -\mathrm{id}$.

Geometrically, via the canonical identification $T_p X \simeq \mathfrak{g}/\mathfrak{k} \simeq \mathfrak{p}$, the involution $\tau_K|_{\mathfrak{p}} : T_p X \to T_p X$ is the derivative of the geodesic involution of X at p.

A useful way to study G is to consider its linear representations. To do so, we will consider its restricted weights, which we now define. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$, then $\mathfrak{a} \subset \mathfrak{p} \simeq T_p X$ is the tangent space to a maximal flat F in X containing p, i.e. $\exp_p(\mathfrak{a}) = F$. Given an irreducible, real, finite dimensional, linear representation $r : G \to \mathrm{GL}(V)$ and $\alpha \in \mathfrak{a}^*$, define

$$V_\alpha := \{ w \in V : r(\exp(v))w = e^{\alpha(v)} w \text{ for all } v \in \mathfrak{a} \}.$$

Definition 2.12. We call α a *restricted weight* of the representation (r, V) if $\alpha \neq 0$ and V_α is non-empty. If α is a restricted weight, then V_α is a *restricted weight space*.

Let $\Phi(r, V)$ denote the set of restricted weights of (r, V). Since $r(\exp(\mathfrak{a}))$ is simultaneously diagonalizable over $\mathbb{R}$, we can decompose

$$V = V_0 + \sum_{\alpha \in \Phi(r,V)} V_\alpha$$

into its restricted weight spaces. If we specialize to the adjoint representation $(r, V) = (\mathrm{Ad}, \mathfrak{g})$, then the restricted weights of this representation are called the *restricted roots*, and the restricted weight spaces are called the *restricted root spaces*. In this case, we use the notation $\Sigma := \Phi(\mathrm{Ad}, \mathfrak{g})$ and $\mathfrak{g}_\alpha := V_\alpha$.

Let $\Phi := \bigcup_{(r,V)} \Phi(r, V)$, where the union is taken over all irreducible, real, finite dimensional, linear representations of G. It turns out that there is an easy description of Φ in terms of Σ:

$$\{0\} \cup \Phi = \left\{ \alpha \in \mathfrak{a}^* : 2 \frac{(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z} \text{ for all } \beta \in \Sigma \right\}.$$

where $(\cdot, \cdot)$ is the Killing form on $\mathfrak{a}^*$. In particular, $\{0\} \cup \Phi \subset \mathfrak{a}^*$ is a lattice. One would then like to find a base for the lattice $\{0\} \cup \Phi$.

To do so, choose any $v_0 \in \mathfrak{a}$ so that $\alpha(v_0) \neq 0$ for all $\alpha \in \Sigma$, and let

$$\Sigma^+ := \{ \alpha \in \Sigma : \alpha(v_0) > 0 \}.$$

It is a standard fact that $\alpha \in \Sigma$ if and only if $-\alpha \in \Sigma$, so $\Sigma = \Sigma^+ \cup \{-\alpha : \alpha \in \Sigma^+\}$.

Definition 2.13. A restricted root in Σ^+ is *simple* if it cannot be written as a non-trivial linear combination of the roots in Σ^+ with positive coefficients.

It turns out that the set of simple restricted roots, denoted Δ, is a basis for $\mathfrak{a}^*$. However, Δ is not a base for the lattice $\{0\} \cup \Phi$. To convert Δ into a base, we perform an additional "orthogonalization procedure" to every simple restricted root α. This gives the following definition.

Definition 2.14. For any $\alpha \in \Delta$, the *restricted fundamental weight* associated to α is the linear functional $\omega_\alpha \in \mathfrak{a}^*$ defined by

$$2\frac{(\omega_\alpha, \beta)}{(\beta, \beta)} = \delta_{\alpha,\beta} \quad \text{for all } \beta \in \Delta,$$

where $\delta_{\cdot,\cdot}$ is the Kronecker symbol.

It is well known that $\{\omega_\alpha : \alpha \in \Delta\}$ is a base for the lattice $\{0\} \cup \Phi$. The choice of v_0 induces a natural partial ordering $\leq$ on $\{0\} \cup \Phi$ defined as follows. For any $\omega_1, \omega_2 \in \{0\} \cup \Phi$, $\omega_1 \leq \omega_2$ if $\omega_2 - \omega_1$ is a non-negative linear combination of the simple roots in Δ. For any irreducible representation (r, V) of G, the set of weights $\{0\} \cup \Phi(r, V)$ has a unique maximal element in the partial ordering $\leq$. This is called the *highest restricted weight* of (r, V), and is a non-negative linear combination of the restricted fundamental weights.

With this, we can state the following theorem of Tits [34] that will play an important role later. Also, see Quint [30, Proposition 3.2] or Guéritaud–Guichard–Kassel–Wienhard [10, Lemma 4.5].

Theorem 2.15 (Tits). *For any $\alpha \in \Delta$, there is an irreducible linear representation $r_\alpha : G \to \mathrm{SL}(n, \mathbb{R})$ so that highest restricted weight χ of $(r_\alpha, \mathbb{R}^n)$ is a positive integer multiple of the restricted fundamental weight ω_α, and the weight space $(\mathbb{R}^n)_\chi$ is one-dimensional.*

We will refer to the representation r_α guaranteed by Theorem 2.15 as an *α-Tits representation*.

The choice of v_0 also picks out a (closed) *positive Weyl chamber*

$$\overline{\mathfrak{a}^+} := \{v \in \mathfrak{a} : \alpha(v) \geq 0 \text{ for all } \alpha \in \Sigma^+\}.$$

One can show that $\overline{\mathfrak{a}^+} \subset T_p X \subset TX$ is a fundamental domain of the G-action on TX, i.e. for any $(p', v') \in TX$, there is a unique $v \in \overline{\mathfrak{a}^+}$ so that $g \cdot (p', v') = (p, v)$ for some $g \in G$. Since X is complete, this means that for any pair of points $x, y \in X$, there is a unique vector $v_{x,y} \in \overline{\mathfrak{a}^+}$ so that $g \cdot x = p$ and $g \cdot y = \exp_p(v_{x,y})$ for some $g \in G$. This allows us to define the Weyl chamber valued distance as follows.

Definition 2.16. The *Weyl chamber valued distance* is the function $d_{\overline{\mathfrak{a}^+}} : X \times X \to \overline{\mathfrak{a}^+}$ given by $d_{\overline{\mathfrak{a}^+}}(p, q) := v_{x,y}$.

It is easy to see that the Weyl chamber valued distance descends to an injective map $G \backslash (X \times X) \to \overline{\mathfrak{a}^+}$, and is thus a complete invariant of the G-orbits of pairs of points in X. Furthermore, $\|d_{\overline{\mathfrak{a}^+}}(p, q)\| = d(p, q)$, where $\|\cdot\|$ is the norm on $\overline{\mathfrak{a}^+}$

induced by the Riemannian metric on X and d is the distance on X. The classification of isometries on G also implies that for any $g \in G$, either

$$\inf \{d(q, g \cdot q) : q \in X\} = 0$$

or there is some $p \in X$ so that

$$\inf \{d(q, g \cdot q) : q \in X\} = d(p, g \cdot p).$$

Furthermore, if

$$\inf \{d(q, g \cdot q) : q \in X\} = d(p, g \cdot p) = d(p', g \cdot p'),$$

then $d_{\overline{\mathfrak{a}^+}}(p, g \cdot p) = d_{\overline{\mathfrak{a}^+}}(p', g \cdot p')$. Using this, one can define the Jordan projection geometrically.

Definition 2.17. The *Jordan projection* (sometimes also known as the *Lyapunov projection*) is the map $\lambda_G = \lambda \colon G \to \overline{\mathfrak{a}^+}$ defined by

- $\lambda \colon g \mapsto 0$ if $\inf\{d(q, g \cdot q) : q \in X\} = 0$,

- $\lambda \colon g \mapsto d_{\overline{\mathfrak{a}^+}}(p, g \cdot p)$ if there is some $p \in X$ so that $\inf\{d(q, g \cdot q) : q \in X\} = d(p, g \cdot p)$.

More algebraically, the Jordan projection can also be described as follows. The Jordan decomposition theorem (see Eberlein [8, Theorem 2.19.24]) ensures that any $g \in G$ can be written uniquely as a commuting product $g = g_h g_e g_u$, with g_h hyperbolic, g_e elliptic, and g_u unipotent. Furthermore, the fact that $\overline{\mathfrak{a}^+}$ is a fundamental domain of the G action on TX implies that there is a unique vector $v_g \in \overline{\mathfrak{a}^+}$ so that $\exp(v_g)$ is conjugate to g_h. Then the Jordan projection is the map $\lambda \colon G \to \overline{\mathfrak{a}^+}$ that sends $g \in G$ to v_g.

Since X is non-positively curved, it has a visual boundary ∂X that is topologically a sphere, and the G-action on X extends to a G-action on ∂X. One can then consider the stabilizers in G of points in ∂X.

Definition 2.18. A *parabolic subgroup* of G is the stabilizer of a point in ∂X. We say that two parabolic subgroups $P_1, P_2 \subset G$ are *opposite* if there is a geodesic in X with endpoints $x_1, x_2 \in \partial X$ so that $P_i = \mathrm{Stab}_G(x_i)$ for $i = 1, 2$.

If P_1 and P_1' are both parabolic subgroups that are opposite to P_2, then P_1 and P_1' must be conjugate in G. As such, we can say that two conjugacy classes $[P_1]$ and $[P_2]$ are *opposite* if for any representative P_1 of $[P_1]$ every opposite of P_1 lies in the conjugacy class $[P_2]$.

Using parabolic subgroups, we can define flag spaces.

Definition 2.19. Let $x \in \partial X$ and let $P := \mathrm{Stab}_G(x)$. The $[P]$-flag space, denoted $\mathcal{F}_{[P]}$, is the G-orbit of x.

Observe that the $[P]$-flag space is a G-homogeneous space, so $\mathcal{F}_{[P]} \simeq G/P$ as G-spaces. In particular, as an abstract space, $\mathcal{F}_{[P]}$ does not depend on x. Furthermore, if $P' = gPg^{-1}$ for some $g \in G$, then there is canonical isomorphism between the G-spaces G/P and G/P'. As such, $\mathcal{F}_{[P]}$ is well-defined, and depends only on the conjugacy class of $[P]$.

The decomposition of $\mathfrak{g}$ into its restricted root spaces can also be used to understand the parabolic subgroups of G. For any non-empty subset $\theta \subset \Delta$, the *standard θ-parabolic subgroup* is the parabolic subgroup $P_\theta \subset G$ with Lie algebra

$$\mathfrak{p}_\theta := \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Sigma^+ \cap \mathrm{Span}_{\mathbb{R}}(\Delta - \theta)} \mathfrak{g}_{-\alpha}.$$

Using this, we can define a map from non-empty subsets of Δ to conjugacy classes of parabolic subgroups in G by $\theta \mapsto [P_\theta]$. It turns out that this map is in fact a bijection.

It is also well known that there is some $g \in G$ so that $g \cdot \overline{\mathfrak{a}^+} = -\overline{\mathfrak{a}^+}$. In fact, for any $g' \in G$ so that $g' \cdot \overline{\mathfrak{a}^+} = -\overline{\mathfrak{a}^+}$, we know that $g' \cdot \mathfrak{a} = \mathfrak{a}$ and $g'|_{\mathfrak{a}} = g|_{\mathfrak{a}}$. Using this, we can define the *opposition involution* $i := -g \colon \overline{\mathfrak{a}^+} \to \overline{\mathfrak{a}^+}$. This gives an involution $\iota \colon \Delta \to \Delta$ defined by $\iota(\alpha) = \alpha \circ i$, which in turn induces an involution, also denoted by ι, on conjugacy classes of parabolic subgroups defined by $\iota[P_\theta] \mapsto [P_{\iota(\theta)}]$. Geometrically, this involution sends the conjugacy class of any parabolic subgroup to the conjugacy class of its opposite.

From the algebraic description of the Jordan projection λ, one can verify that for all $g \in G$ and for all $\alpha \in \Delta$, we have $\alpha \circ \lambda(g) = \iota(\alpha) \circ \lambda(g^{-1})$, which in turn implies that $\omega_\alpha \circ \lambda(g) = \omega_{\iota(\alpha)} \circ \lambda(g^{-1})$.

We finish this section by describing some of the objects defined above explicitly in the case when $G = \mathrm{SL}(n, \mathbb{R})$, as this will be of particular importance to us. We can choose K to be the maximal compact subgroup

$$\mathrm{SO}(n) := \{k \in \mathrm{SL}(n, \mathbb{R}) : k^T k = \mathrm{id}\} \subset \mathrm{SL}(n, \mathbb{R}).$$

In that case, $\mathfrak{k} = \mathfrak{so}(n) = \{A \in \mathfrak{sl}(n, \mathbb{R}) : A = -A^T\}$ and $\mathfrak{p} = \{A \in \mathfrak{sl}(n, \mathbb{R}) : A = A^T\}$, so the Cartan involution $\tau_K \colon \mathfrak{sl}(n, \mathbb{R}) \to \mathfrak{sl}(n, \mathbb{R})$ is given by $\tau \colon A \mapsto -A^T$. We can choose the maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ to be the vector space of traceless diagonal matrices in $\mathfrak{sl}(n, \mathbb{R})$. This allows us to naturally identify

$$\mathfrak{a} = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0\right\}.$$

With this identification, $\Sigma = \{\alpha_{i,j} \in \mathfrak{a}^* : i \neq j\}$, where $\alpha_{i,j} \colon \mathfrak{a} \to \mathbb{R}$ is given by $\alpha_{i,j} \colon (x_1, \ldots, x_n) \mapsto x_i - x_j$. By making an appropriate choice of $v_0 \in \mathfrak{a}$, we can

also ensure that

$$\Sigma^+ = \{\alpha_{i,j} \in \mathfrak{a}^* : i < j\},$$
$$\Delta = \{\alpha_{i,i+1} \in \mathfrak{a}^* : i = 1 \ldots, n-1\},$$
$$\overline{\mathfrak{a}^+} = \{(x_1, \ldots, x_n) \in \mathfrak{a} : x_1 \geq \cdots \geq x_n\}.$$

With these choices, the restricted fundamental weight $\omega_{\alpha_{i,i+1}}$ corresponding to $\alpha_{i,i+1}$ is given by the formula $\omega_{\alpha_{i,i+1}}(x_1, \ldots, x_n) = x_1 + \cdots + x_i$, and the involution ι can also explicitly be given by $\iota(\alpha_{i,i+1}) = \alpha_{n-i,n-i+1}$. Also, the Jordan projection evaluated on $g \in \mathrm{SL}(n, \mathbb{R})$ is

$$\lambda : g \mapsto \log \overline{g}$$

where $\overline{g} \in \mathrm{SL}(n, \mathbb{R})$ is the diagonal matrix whose diagonal entries are the absolute values of the generalized eigenvalues of g, listed in descreasing order down the diagonal. The group element $\overline{g}$ is a conjugate of g_h.

Finally, if $\theta = \{\alpha_{1,2}, \alpha_{n-1,n}\}$, then

$$\mathfrak{p}_\theta = \left\{ \begin{pmatrix} * & * & \ldots & * & * \\ 0 & * & \ldots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & \ldots & * & * \\ 0 & 0 & \ldots & 0 & * \end{pmatrix} \in \mathfrak{sl}(n, \mathbb{R}) \right\}$$

and $\mathcal{F}_{[P_\theta]} = \{(L, H) \in \mathbb{RP}^{n-1} \times (\mathbb{RP}^{n-1})^* : L \in H\}$. As such, we refer to the conjugacy class $[P_\theta]$ as the *line-hyperplane stabilizer* of $\mathrm{SL}(n, \mathbb{R})$.

2.4. Anosov representations and positively ratioed representations.
The notion of Anosov representations was first introduced by Labourie [21], and later refined by Guichard–Wienhard [12]. Several other characterizations have been provided by Kapovich–Leeb–Porti [19, 20] and Guéritaud–Guichard–Kassel–Wienhard [10]. In this article, we will only consider Anosov representations from the surface group Γ to a non-compact, semisimple, real algebraic group, G.

Given a representation $\rho : \Gamma \to G$, a ρ-equivariant map $\xi : \partial\Gamma \to \mathcal{F}_{[P]}$ is *dynamics-preserving* if for any $\gamma \in \Gamma \setminus \{\mathrm{id}\}$, $\xi(\gamma^+)$ is the attracting fixed point for the action of $\rho(\gamma)$ on $\mathcal{F}_{[P]}$. (In particular, $\rho(\gamma)$ has to have an attracting fixed point in $\mathcal{F}_{[P]}$.) A pair of maps $\xi : \partial\Gamma \to \mathcal{F}_{[P]}$ and $\eta : \partial\Gamma \to \mathcal{F}_{\iota[P]}$ is *transverse* if for all $x \neq y$, $(\xi(x), \eta(y))$ lies in the unique open G-orbit of $\mathcal{F}_{[P]} \times \mathcal{F}_{\iota[P]}$. With this, we can define Anosov representations using a characterization by Guéritaud–Guichard–Kassel–Wienhard (see [10, Theorem 1.7 and Proposition 2.2]).

Definition 2.20. A representation $\rho : \Gamma \to G$ is $[P]$-*Anosov* if

- there exist continuous, ρ-equivariant, dynamics-preserving and transverse maps $\xi : \partial\Gamma \to \mathcal{F}_{[P]}$ and $\eta : \partial\Gamma \to \mathcal{F}_{\iota[P]}$,

- there exist $C, c > 0$ such that $\alpha \circ \lambda \circ \rho(\gamma) > C \ell_\Gamma(\gamma) - c$ for all $\alpha \in \theta$ and $\gamma \in \Gamma$,

where $\ell_\Gamma(\gamma)$ is the translation distance of $\gamma \in \Gamma$ in the Cayley graph of Γ with respect to some finite generating set. (Recall that λ denotes the Jordan projection.) The maps ξ and η are called the *limit curves* of ρ.

Since the set of fixed points of group elements in Γ is dense in $\partial\Gamma$, the maps ξ and η are unique. In particular, $\xi = \eta$ necessarily when $\theta = \iota(\theta)$. Also, it is a result of Guichard–Wienhard [12, Lemma 3.18] that for any non-empty $\theta \subset \Delta$, $\rho\colon \Gamma \to G$ is $[P_\theta]$-Anosov if and only if it is $[P_{\theta \cap \iota(\theta)}]$-Anosov. Hence, we do not lose any generality by only considering parabolic subgroups of $P \subset G$ so that $[P] = \iota[P]$, i.e. non-empty subsets $\theta \subset \Delta$ so that $\theta = \iota(\theta)$. *We will do so for the rest of this article.* Under this assumption, we can associate to any $[P]$-Anosov representation some natural length functions.

Definition 2.21. Let $\rho\colon \Gamma \to G$ be a $[P_\theta]$-Anosov representation.

- For any $\alpha \in \theta$, the *α-length function of ρ* is the function

$$\ell_\alpha^\rho \colon \mathcal{CG}(S) \to \mathbb{R} \quad \text{given by} \quad \ell_\alpha^\rho(c) := \big(\omega_\alpha + \omega_{\iota(\alpha)}\big) \circ \lambda \circ \rho(\gamma),$$

where $[\![\gamma]\!] = c \in \mathcal{CG}(S)$.

- The *entropy of ℓ_α^ρ* is the quantity

$$h(\ell_\alpha^\rho) := \limsup_{T \to \infty} \frac{1}{T} \log \#\big\{c \in \mathcal{CG}(S) : \ell_\alpha^\rho(c) \le T\big\}.$$

One can verify that ℓ_α^ρ is well-defined and $\ell_\alpha^\rho = \ell_{\iota(\alpha)}^\rho$. It is also a well known consequence of the Anosovness of ρ that $h(\ell_\alpha^\rho) \in \mathbb{R}^+$ (for example, see Sambarino [31, Theorem B]). When $G = \mathrm{PSL}(2, \mathbb{R})$, one can choose $\mathfrak{a}$ to be the diagonal matrices in $\mathfrak{sl}(2, \mathbb{R})$. If $\rho\colon \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ is a Fuchsian representation, then it is an easy exercise to verify that $\Delta = \{\alpha\}$, where $\alpha\colon \mathfrak{a} \to \mathbb{R}$ is defined by

$$\alpha \colon \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} \mapsto 2t,$$

and ρ is $[P_\Delta]$-Anosov. In this case, for any $c \in \mathcal{CG}(S)$, $\ell_\alpha^\rho(c)$ is the hyperbolic length of the geodesic c measured in the hyperbolic structure Σ on S corresponding to ρ, and it is well known that $h(\ell_\alpha^\rho) = 1$.

For a general Anosov representation however, the length functions are so named purely by analogy as there is no natural metric on the surface that gives rise to these length functions.

As another example, we will consider projective Anosov representations.

Definition 2.22. Let $[P]$ be the line-hyperplane stabilizer of $\mathrm{SL}(n, \mathbb{R})$. A $[P]$-Anosov representation $\rho\colon \Gamma \to \mathrm{SL}(n, \mathbb{R})$ is a *projective Anosov representation*.

For any $g \in SL(n, \mathbb{R})$, let $\lambda(g) = (\lambda_1(g), \ldots, \lambda_n(g)) \in \overline{\mathfrak{a}^+}$ denote the Jordan projection of g. Recall that if $[P_\theta]$ is the line-hyperplane stabilizer of $SL(n, \mathbb{R})$, then $\theta = \{\alpha_{1,2}, \alpha_{n-1,n}\}$. Hence, there is only one length function of ρ, which we will abbreviate by $\ell^\rho \colon \mathcal{CG}(S) \to \mathbb{R}$. By the definition of ℓ^ρ, we see that

$$
\begin{aligned}
\ell^\rho(c) &= \left(\omega_{\alpha_{1,2}} + \omega_{\alpha_{n-1,n}}\right) \circ \lambda \circ \rho(\gamma) \\
&= \lambda_1(\rho(\gamma)) + \sum_{i=1}^{n-1} \lambda_i(\rho(\gamma)) \\
&= \lambda_1(\rho(\gamma)) - \lambda_n(\rho(\gamma)).
\end{aligned}
\tag{2.1}
$$

If $\rho \colon \Gamma \to SL(n, \mathbb{R})$ is projective Anosov, then the limit curve

$$
\xi \colon \partial\Gamma \to \mathcal{F}_{[P_\theta]} = \mathcal{F}_{\iota[P_\theta]}
$$

corresponding to ρ can be thought of as a pair of continuous, ρ-equivariant, maps

$$
\xi^{(1)} \colon \partial\Gamma \to \mathbb{RP}^{n-1} \quad \text{and} \quad \xi^{(n-1)} \colon \partial\Gamma \to (\mathbb{RP}^{n-1})^* \simeq \mathrm{Gr}(n-1, n)
$$

so that $\xi^{(1)}(x) \subset \xi^{(n-1)}(y)$ if and only if $x = y$. Since ξ is dynamics preserving, we see that for any $\gamma \in \Gamma \setminus \{\mathrm{id}\}$, if $\gamma^+, \gamma^- \in \partial\Gamma$ denote the attractor and repeller of γ, then $\xi^{(1)}(\gamma^+)$ and $\xi^{(n-1)}(\gamma^-)$ are the attracting line and repelling hyperplane of $\rho(\gamma)$ respectively.

In this case, Labourie [22] defined a function $L^\rho \colon \partial\Gamma^{[4]} \to \mathbb{R}$ given by

$$
L^\rho(x, y, z, w) = \frac{\langle \xi^{(n-1)}(x), \xi^{(1)}(y)\rangle \langle \xi^{(n-1)}(z), \xi^{(1)}(w)\rangle}{\langle \xi^{(n-1)}(x), \xi^{(1)}(w)\rangle \langle \xi^{(n-1)}(z), \xi^{(1)}(y)\rangle}.
$$

Here, for any $p, q = x, y, z, w$, we choose a covector representative in $(\mathbb{R}^n)^*$ for $\xi^{(n-1)}(p)$ and a vector representative in $\mathbb{R}^n$ for $\xi^{(1)}(q)$ to evaluate $\langle \xi^{(n-1)}(p), \xi^{(1)}(q)\rangle$. One can verify that $L^\rho(x, y, z, w)$ does not depend on any of the choices made.

We will refer to the function L^ρ as the *Labourie cross ratio*, even though it is not a cross ratio in the sense of Definition 2.4. It is easy to see that

$$
L^\rho(x, y, z, w) \cdot L^\rho(x, w, z, t) = L^\rho(x, y, z, t).
\tag{2.2}
$$

Furthermore, an easy computation proves that for all $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ and for all $a \in \partial\Gamma \setminus \{\gamma^+, \gamma^-\}$, we have

$$
L^\rho(\gamma^-, \gamma \cdot a, \gamma^+, a) = \frac{e^{\lambda_1(\rho(\gamma))}}{e^{\lambda_n(\rho(\gamma))}} = L^\rho(\gamma \cdot a, \gamma^-, a, \gamma^+).
\tag{2.3}
$$

Using (2.2), one can verify that the function

$$
B^\rho(x, y, z, w) := \frac{1}{2} \log\left(L^\rho(x, z, y, w) \cdot L^\rho(z, x, w, y)\right)
$$

is indeed a cross ratio. Furthermore, for any $c = [\![\gamma]\!] \in \mathcal{CG}(S)$, (2.1) and (2.3) imply that

$$\ell^\rho(c) = \frac{1}{2} \log \left(L^\rho(\gamma^-, \gamma \cdot a, \gamma^+, a) \cdot L^\rho(\gamma \cdot a, \gamma^-, a, \gamma^+) \right)$$
$$= \ell_{B^\rho}(c).$$

In particular, when $[P]$ is the line-hyperplane stabilizer of $\mathrm{SL}(n, \mathbb{R})$, the length function of $[P]$-Anosov representations are the periods of a unique cross ratio (the uniqueness is a consequence of Theorem 2.5). This is in fact true for any restricted simple root $\alpha \in \theta$ for any $[P_\theta]$-Anosov representation $\rho \colon \Gamma \to G$. To prove this, we need the following proposition, which is a special case of Proposition 4.3 of Guichard–Wienhard [12] (see also Guéritaud–Guichard–Kassel–Wienhard [10, Proposition 4.6]).

Proposition 2.23 (Guichard–Wienhard). *Let $\theta \subset \Delta$, let $\rho \colon \Gamma \to G$ be a $[P_\theta]$-Anosov representation, and let $\alpha \in \theta$. Also, let $r_\alpha \colon G \to \mathrm{SL}(n, \mathbb{R})$ be an α-Tits representation. Then $r_\alpha \circ \rho \colon \Gamma \to \mathrm{SL}(n, \mathbb{R})$ is $[P]$-Anosov, where $[P]$ is the line-hyperplane stabilizer of $\mathrm{SL}(n, \mathbb{R})$. Furthermore, the limit curve corresponding to ρ is $f_\alpha \circ \xi \colon \partial \Gamma \to \mathcal{F}_{[P]}$, where $f_\alpha \colon \mathcal{F}_{[P_\theta]} \to \mathcal{F}_{[P]}$ is the unique r_α-equivariant map.*

More explicitly, if $P \subset \mathrm{SL}(n, \mathbb{R})$ is the representative in the conjugacy class $[P]$ so that $r_\alpha^{-1}(P) = P_\theta$, then $f_\alpha \colon \mathcal{F}_{[P_\theta]} = G/P_\theta \to \mathrm{SL}(n, \mathbb{R})/P = \mathcal{F}_{[P]}$ is given by $f_\alpha \colon g \cdot P_\theta \mapsto r_\alpha(g) \cdot P$. Proposition 2.23 allows us to reduce the study of length functions of a general Anosov representation to the length function of a projective Anosov representations. In particular, we can prove the following.

Proposition 2.24. *Let $\theta \subset \Delta$ and let $\rho \colon \Gamma \to G$ be $[P_\theta]$-Anosov. For all $\alpha \in \theta$, there is a unique cross ratio B_α^ρ so that for all $c \in \mathcal{CG}(S)$,*

$$\ell_\alpha^\rho(c) = \ell_{B_\alpha^\rho}(c).$$

Proof. Let $r_\alpha \colon G \to \mathrm{SL}(n, \mathbb{R})$ be an α-Tits representation of G. Since $r_\alpha \circ \rho$ is Anosov with respect to the line-hyperplane stabilizer in $\mathrm{SL}(n, \mathbb{R})$, $r_\alpha \circ \rho(\gamma)$ has a largest eigenvalue of multiplicity 1 for all $\gamma \in \Gamma \setminus \{\mathrm{id}\}$. By Theorem 2.15, we have

$$\omega_{\alpha_{1,2}} \circ \lambda_{\mathrm{SL}(n,\mathbb{R})} \circ r_\alpha \circ \rho(\gamma) = k \cdot \omega_\alpha \circ \lambda_G \circ \rho(\gamma)$$

for some $k \in \mathbb{Z}^+$. Similarly, we have that

$$\omega_{\alpha_{n-1,n}} \circ \lambda_{\mathrm{SL}(n,\mathbb{R})} \circ r_\alpha \circ \rho(\gamma) = \omega_{\alpha_{1,2}} \circ \lambda_{\mathrm{SL}(n,\mathbb{R})} \circ r_\alpha \circ \rho(\gamma^{-1})$$
$$= k \cdot \omega_\alpha \circ \lambda_G \circ \rho(\gamma^{-1})$$
$$= k \cdot \omega_{\iota(\alpha)} \circ \lambda_G \circ \rho(\gamma).$$

Together, these imply that for any $c = [\![\gamma]\!] \in \mathcal{CG}(S)$,

$$\ell^{r_\alpha \circ \rho}(c) = \left(\omega_{\alpha_{1,2}} + \omega_{\alpha_{n-1,n}}\right) \circ \lambda_{\mathrm{SL}(n,\mathbb{R})} \circ r_\alpha \circ \rho(\gamma)$$
$$= \left(k \cdot \omega_\alpha + k \cdot \omega_{\iota(\alpha)}\right) \circ \lambda_G \circ \rho(\gamma)$$
$$= k \cdot \ell^\rho_\alpha(c).$$

Define $B^\rho_\alpha := \frac{1}{k} B^{r_\alpha \circ \rho}$. Since $\ell^{r_\alpha \circ \rho} = \ell_{B^{r_\alpha \circ \rho}}$, it immediately follows that $\ell^\rho_\alpha = \ell_{B^\rho_\alpha}$. The uniqueness of B^ρ_α is a consequence of Theorem 2.5. $\square$

Using Proposition 2.24, we can define positively ratioed representations.

Definition 2.25. A $[P_\theta]$-Anosov representation $\rho \colon \Gamma \to G$ is $[P_\theta]$-*positively ratioed* if the cross ratio B^ρ_α is positive for all $\alpha \in \theta$ (see Proposition 2.24 and Definition 2.9).

As a consequence of Theorem 2.10 and Proposition 2.24, we see that for any $\theta \subset \Delta$, any $[P_\theta]$-positively ratioed representation $\rho \colon \Gamma \to G$, and any $\alpha \in \theta$, there is a unique geodesic current $\mu^\rho_\alpha \in \mathcal{C}(S)$ so that $i(\mu^\rho_\alpha, c) = \ell^\rho_\alpha(c)$ for every $c \in \mathcal{CG}(S)$. This is stated as Theorem 1.1 in the introduction.

Let $\theta' \subset \theta$ be subsets of Δ. Guichard–Wienhard [12, Lemma 3.18] proved that if $\rho \colon \Gamma \to G$ is $[P_\theta]$-Anosov, then it is also $[P_{\theta'}]$-Anosov. It then follows from this definition that if ρ is $[P_\theta]$-positively ratioed, then it is also $[P_{\theta'}]$-positively ratioed.

The intersection currents arising from positively ratioed representations satisfy some basic properties that we will now explain.

Note that in the definition of a positive cross ratio, we used the weak inequality instead of the strict inequality. However, in the definition of positively ratioed representations, we can replace the weak inequality with a strict inequality without changing the definition. This is the content of the next proposition.

Notation 2.26. For any $x, y, z \in \partial\Gamma$, let $(x, y]_z$ denote the half-open subsegment of $\partial\Gamma$ that does not contain z and has x and y as its open and closed endpoints respectively. We will also use $(x, y)_z$, $[x, y)_z$ and $[x, y]_z$ to denote the interval $(x, y]_z$, but with the appropriate closed and open endpoints.

Proposition 2.27. *Let $\theta \subset \Delta$, let $\rho \colon \Gamma \to G$ be a $[P_\theta]$-positively ratioed representation, and let $\alpha \in \theta$. Then $B^\rho_\alpha(x, y, z, w) > 0$ for all $x, y, z, w \in \partial\Gamma$ in this cyclic order.*

Proof. Recall that $r_\alpha \circ \rho \colon \Gamma \to \mathrm{SL}(n, \mathbb{R})$ is projective Anosov, and that $B^\rho_\alpha = \frac{1}{k} B^{r_\alpha \circ \rho}$ for some $k \in \mathbb{Z}^+$. Therefore, we can assume that ρ is projective Anosov. Let

$$\xi = \left(\xi^{(1)}, \xi^{(n-1)}\right) \colon \partial\Gamma \to \mathbb{RP}^{n-1} \times (\mathbb{RP}^{n-1})^*$$

denote the limit curve of ρ. By assumption, $B^\rho(x, y, z, w) \geq 0$ for all $x, y, z, w \in \partial\Gamma$ in this cyclic order. Suppose for contradiction that there is some $x, y, z, w \in \partial\Gamma$ in this cyclic order so that $B^\rho(x, y, z, w) = 0$. This means that for all $t \in [z, w]_x$,

$B^\rho(x, y, t, w) = 0$. It then follows from the definition of B^ρ that $\xi^{(1)}([z, w]_x)$ lies in the proper subspace $(\xi^{(n-1)}(x) \cap \xi^{(n-1)}(y)) + \xi^{(1)}(w) \subset \mathbb{R}^n$.

Let $V \subset \mathbb{R}^n$ be the minimal subspace containing $\xi^{(1)}([z, w]_x)$, and let $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ so that its repellor γ^- lies in $(z, w)_x$. Since $\xi^{(1)}([z, w]_x) \subset \rho(\gamma) \cdot \xi^{(1)}([z, w]_x)$, we see that $V \subset \rho(\gamma) \cdot V$, so V is $\rho(\gamma)$-invariant. At the same time, observe that

$$\bigcup_{n=0}^{\infty} \gamma^n \cdot [z, w]_x = \partial\Gamma \setminus \{\gamma^+\},$$

so the continuity and ρ-equivariance of $\xi^{(1)}$ implies that $\xi^{(1)}(\partial\Gamma) \subset V$. However, $V \subset (\xi^{(n-1)}(x) \cap \xi^{(n-1)}(y)) + \xi^{(1)}(w)$, which means that

$$\xi^{(1)}(\partial\Gamma) \cap \xi^{(n-1)}(x) \subset \xi^{(n-1)}(x) \cap \xi^{(n-1)}(y).$$

In particular, $\xi^{(1)}(x) \in \xi^{(n-1)}(y)$, but this violates the transversality of ξ. $\qquad\square$

Definition 2.28. Let $\nu \in \mathcal{C}(S)$ be a geodesic current. We say that ν is *period minimizing* if

$$\left|\{c \in \mathcal{CG}(S) : i(\nu, c) < T\}\right| < \infty$$

for all $T \in \mathbb{R}^+$. Also, ν has *full support* if $\nu(U) > 0$ for any open set $U \subset \mathcal{C}(S)$.

It is well known that $|\{[\gamma] \in [\Gamma] : \ell_\Gamma(\gamma) < T\}| < \infty$ for all $T \in \mathbb{R}^+$. As such, an immediate consequence of the Anosovness of ρ that μ_α^ρ is period minimizing. In particular, measured laminations are not intersection currents coming from Anosov representations, because they are not period minimizing.

Let $x, y, z, w \in \partial\Gamma$ in this cyclic order, and let $\mathcal{G}_{[x,y]_z, [w,z]_y} \subset \mathcal{G}(\widetilde{S})$ be the set of geodesics in $\widetilde{S}$ with one endpoint in $[x, y]_z$ and one endpoint in $[w, z]_y$. By the construction of μ_α^ρ from B_α^ρ (see Appendix A), we see that

$$\mu_\alpha^\rho\left(\mathcal{G}_{[x,y]_z, [w,z]_y}\right) = B_\alpha^\rho(x, y, z, w). \tag{2.4}$$

In the degenerate case when $x = y \neq z = w$, this implies that $\mu_\alpha^\rho(\{x, z\}) = 0$, so the μ_α^ρ-measure of every point in $\mathcal{G}(\widetilde{S})$ is zero. As a consequence, the intersection current arising from an Anosov representation has no atoms.

Finally, the intersection currents arising from Anosov representations have full support. To see this, observe that

$$\left\{\mathcal{G}_{(x,y)_z, (w,z)_y} \subset \mathcal{G}(\widetilde{S}) : x, y, z, w \in \partial\Gamma \text{ in this cyclic order}\right\}$$

generates the topology on $\mathcal{G}(\widetilde{S})$. Since $\mu_\alpha^\rho(\mathcal{G}_{[x,y]_z, [w,z]_y}) = \mu_\alpha^\rho(\mathcal{G}_{(x,y)_z, (w,z)_y})$, Equation (2.4) and Proposition 2.27 immediately imply that μ_α^ρ has full support.

3. Examples of positively ratioed representations

In this section, we provide several important examples of positively ratioed representations to motivate the definition.

3.1. Hitchin representations. The *Teichmüller space of S* can be defined to be

$$\mathcal{T}(S) := \{\text{discrete, faithful } \rho\colon \Gamma \to \mathrm{PSL}(2,\mathbb{R})\}/\mathrm{PGL}(2,\mathbb{R}).$$

This is the space of holonomy representations of hyperbolic structures on S. If we equip $\mathcal{T}(S)$ with the compact-open topology, it is well known that $\mathcal{T}(S)$ is topologically a cell of dimension $6g - 6$. Let

$$\iota_n\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$$

be the projectivization of the unique (up to post-composing by an automorphism of $\mathrm{SL}(n,\mathbb{R})$) n-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(n,\mathbb{R})$. If we equip

$$\mathcal{X}(S,\mathrm{PSL}(n,\mathbb{R})) := \mathrm{Hom}(\Gamma,\mathrm{PSL}(n,\mathbb{R}))/\mathrm{PGL}(n,\mathbb{R})$$

with the compact-open topology, this gives us an embedding

$$i_n\colon \mathcal{T}(S) \to \mathcal{X}(S,\mathrm{PSL}(n,\mathbb{R}))$$

defined by $i_n[\rho] = [\iota_n \circ \rho]$. In particular, $i_n(\mathcal{T}(S)) \subset \mathcal{X}(S,\mathrm{PSL}(n,\mathbb{R}))$ is connected.

Definition 3.1. The $\mathrm{PSL}(n,\mathbb{R})$-*Hitchin component* $\mathrm{Hit}_n(S)$ is the connected component of $\mathcal{X}(S,\mathrm{PSL}(n,\mathbb{R}))$ that contains $i_n(\mathcal{T}(S))$. The representations in $\mathrm{Hit}_n(S)$ are known as $\mathrm{PSL}(n,\mathbb{R})$-*Hitchin representations*.

Often, we will simply use a representative ρ in the conjugacy class $[\rho]$ to denote an element in $\mathrm{Hit}_n(S)$. It is classically known that $\mathcal{T}(S)$ is a connected component of $\mathcal{X}(S,\mathrm{PSL}(2,\mathbb{R}))$, so $\mathrm{Hit}_2(S) = \mathcal{T}(S)$. As such, one can think of $\mathrm{Hit}_n(S)$ as a generalization of $\mathcal{T}(S)$.

The Hitchin component was first studied by Hitchin [17], who used Higgs bundle techniques to parameterize $\mathrm{Hit}_n(S)$ using certain holomorphic differentials on a Riemann surface homeomorphic to S. In particular, he showed that $\mathrm{Hit}_n(S)$ is topologically a cell of dimension $(n^2 - 1)(2g - 2)$, where g is the genus of S. With this, the global topology of $\mathrm{Hit}_n(S)$ is completely understood. However, the geometric properties of the representations in $\mathrm{Hit}_n(S)$ remained a mystery until a seminal theorem of Labourie.

To explain this theorem, we first need the notion of a Frenet curve. Let $\mathcal{F}(\mathbb{R}^n)$ denote the space of complete flags in $\mathbb{R}^n$, i.e. $F \in \mathcal{F}(\mathbb{R}^n)$ is a properly nested sequence $F^{(1)} \subset \cdots \subset F^{(n-1)}$ of linear subspaces in $\mathbb{R}^n$, where each $F^{(i)}$ has dimension i. When $G = \mathrm{PSL}(n,\mathbb{R})$, it is easy to verify that $\mathcal{F}(\mathbb{R}^n) = \mathcal{F}_{[P_\Delta]}$.

Definition 3.2. A continuous map $\xi\colon S^1 \to \mathcal{F}(\mathbb{R}^n)$ is *Frenet* if the following hold:

- For all $x_1,\ldots,x_k \in S^1$ pairwise distinct and $m_1,\ldots,m_k \in \mathbb{Z}^+$ such that $k \leq n$ and $m_1 + \cdots + m_k = n$, we have that

$$\bigoplus_{i=1}^{k} \xi(x_i)^{(m_i)} = \mathbb{R}^n.$$

- Let $m_1, \ldots, m_k \in \mathbb{Z}^+$ such that $k \leq n$ and $m_1 + \cdots + m_k = m \leq n$, and let $\{(x_1^j, \ldots, x_k^j)\}_{j=1}^{\infty}$ be a sequence a k-tuples of pairwise distinct points. If there is some $x \in S^1$ so that $\lim_{j \to \infty} x_i^j = x$ for all $i = 1, \ldots, k$, then

$$\lim_{j \to \infty} \bigoplus_{i=1}^k \xi(x_i^j)^{(m_i)} = \xi(x)^{(m)}.$$

Labourie [21, Theorem 4.1] proved that $\mathrm{PSL}(n, \mathbb{R})$-Hitchin representations preserve a ρ-equivariant Frenet curve. Later, Guichard [11, Theorem 1] proved the converse to this, thus giving us the following theorem.

Theorem 3.3 (Labourie, Guichard). *Let $\rho \in \mathcal{X}(S, \mathrm{PSL}(n, \mathbb{R}))$. Then $\rho \in \mathrm{Hit}_n(S)$ if and only if there is a ρ-equivariant Frenet curve $\xi : \partial\Gamma \to \mathcal{F}_{[P_\Delta]}$.*

As a consequence of this, we know that every $\rho \in \mathrm{Hit}_n(S)$ is $[P_\Delta]$-Anosov. In particular, for all $\rho \in \mathrm{Hit}_n(S)$ and $\alpha \in \Delta$, we can define ℓ_α^ρ and the corresponding cross ratios B_α^ρ as per Section 2.4 and Section 2.2 respectively. In fact, we have the following theorem.

Theorem 3.4. *If $\rho \in \mathrm{Hit}_n(S)$, then ρ is $[P_\Delta]$-positively ratioed.*

To prove Theorem 3.4, we will use Theorem 3.3 to construct positive cross ratios $B_i^\rho : \partial\Gamma^{[4]} \to \mathbb{R}$ for $i = 1, \ldots, n-1$ in the following way.

Notation 3.5. Given flags $(F_1, F_2, \ldots, F_s)$ for every $l = 1, \ldots, s$ choose vectors $f_{1,l}, \ldots, f_{n-1,l} \in \mathbb{R}^n$ so that

$$\mathrm{Span}_{\mathbb{R}}\{f_{1,l}, \ldots, f_{i,l}\} = F_l^{(i)}$$

for all $i = 1, \ldots, n-1$. Fix once and for all an identification $\bigwedge^n \mathbb{R}^n \cong \mathbb{R}$. For any integers $i_l \geq 0$ with $\sum_l i_l = n$, denote by $F_1^{(i_1)} \wedge \cdots \wedge F_s^{(i_s)}$ the real number

$$f_{1,1} \wedge \cdots \wedge f_{i_1,1} \wedge \cdots \wedge f_{1,s} \wedge \cdots \wedge f_{i_s,s}.$$

This notation involves some choices, but none of the quantities we define using this notation will depend on them.

Let $\widetilde{B}_i^\rho : \partial\Gamma^{[4]} \to \mathbb{R}$ be the function

$$\widetilde{B}_i^\rho(x, y, z, w) := \log \left| \frac{\xi(x)^{(n-i)} \wedge \xi(z)^{(i)}}{\xi(x)^{(n-i)} \wedge \xi(w)^{(i)}} \frac{\xi(y)^{(n-i)} \wedge \xi(w)^{(i)}}{\xi(y)^{(n-i)} \wedge \xi(z)^{(i)}} \right|$$

and set $B_i^\rho := \frac{1}{2}\left(\widetilde{B}_i^\rho + \widetilde{B}_{n-i}^\rho\right)$.

Lemma 3.6. *For $i = 1, \ldots, n-1$, B_i^ρ is a positive cross ratio.*

Proof. Additivity and symmetry of B_i^ρ are easy to check thanks to the explicit formula above. Continuity of B_i^ρ follows from the continuity of ξ. Hence B_i^ρ is a cross ratio for every $i = 1, \ldots, n-1$. To show positivity of B_i^ρ, we will write it as a sum of functions on $\partial\Gamma^{[4]}$ that are positive when evaluated on points x, y, z, w lying in this cyclic order along $\partial\Gamma$.

Fix $i \in \{1, \ldots, n-1\}$. For $(x, y, z, w) \in \partial\Gamma^{[4]}$, $k = 1, \ldots, n-i$ and $j = 1, \ldots, i$, define

$$(x, y, z, w)_{i,j,k} := \log \left| \frac{\xi(x)^{(n-i-k+1)} \wedge \xi(y)^{(k-1)} \wedge \xi(z)^{(i-j+1)} \wedge \xi(w)^{(j-1)}}{\xi(x)^{(n-i-k+1)} \wedge \xi(y)^{(k-1)} \wedge \xi(z)^{(i-j)} \wedge \xi(w)^{(j)}} \right|$$
$$+ \log \left| \frac{\xi(x)^{(n-i-k)} \wedge \xi(y)^{(k)} \wedge \xi(z)^{(i-j)} \wedge \xi(w)^{(j)}}{\xi(x)^{(n-i-k)} \wedge \xi(y)^{(k)} \wedge \xi(z)^{(i-j+1)} \wedge \xi(w)^{(j-1)}} \right|,$$

and observe that for all $(x, y, z, w) \in \partial\Gamma^{[4]}$

$$\widetilde{B}_i^\rho(x, y, z, w) = \sum_{k=1}^{n-i} \sum_{j=1}^{i} (x, y, z, w)_{i,j,k}.$$

The functions $(x, y, z, w)_{i,j,k}$ were studied by the second author, who proved [36, Proposition 2.12], that each $(x, y, z, w)_{i,j,k}$ is positive on quadruples of points x, y, z, w in this cyclic order along $\partial\Gamma$. This shows the positivity of B_i^ρ. $\square$

Proof of Theorem 3.4. By Lemma 3.6, it is sufficient to show that $B_{\alpha_{i,i+1}}^\rho = B_i^\rho$. For any element $g \in \mathrm{PSL}(n, \mathbb{R})$, let $\lambda(g) = (\lambda_1(g), \ldots, \lambda_n(g)) \in \overline{\mathfrak{a}^+}$ be its Jordan projection. An easy computation, using the explicit formula for the restricted fundamental weights, shows that for all $\gamma \in \Gamma \setminus \{\mathrm{id}\}$,

$$\ell_{\alpha_{i,i+1}}^\rho [\![\gamma]\!] = \left(\omega_{\alpha_{i,i+1}} + \omega_{\alpha_{n-i,n-i+1}} \right) \circ \lambda \circ \rho(\gamma) = \sum_{k=1}^{i} \lambda_k(\rho(\gamma)) - \sum_{k=n-i+1}^{n} \lambda_k(\rho(\gamma)).$$

By Theorem 2.5, it is thus sufficient to show that $\ell_{B_i^\rho} = \ell_{\alpha_{i,i+1}}^\rho$.

Choose a basis $e_1, \ldots, e_n \subset \mathbb{R}^n$ such that e_i spans the line

$$\xi(\gamma^+)^{(i)} \cap \xi(\gamma^-)^{(n-i+1)}.$$

Then for this basis we have

$$\left| \xi(\gamma^-)^{(n-i)} \wedge \xi(\gamma \cdot x)^{(i)} \right| = \left| e^{\lambda_1(\rho(\gamma)) + \cdots + \lambda_i(\rho(\gamma))} \xi(\gamma^-)^{(n-i)} \wedge \xi(x)^{(i)} \right|;$$
$$\left| \xi(\gamma^+)^{(n-i)} \wedge \xi(\gamma \cdot x)^{(i)} \right| = \left| e^{\lambda_{n-i+1}(\rho(\gamma)) + \cdots + \lambda_n(\rho(\gamma))} \xi(\gamma^+)^{(n-i)} \wedge \xi(x)^{(i)} \right|.$$

Hence,

$$
\begin{aligned}
2B_i^\rho(\gamma^-, \gamma^+, \gamma x, x) &= \log \left| \frac{\xi(\gamma^-)^{(n-i)} \wedge \xi(\gamma \cdot x)^{(i)}}{\xi(\gamma^-)^{(n-i)} \wedge \xi(x)^{(i)}} \frac{\xi(\gamma^+)^{(n-i)} \wedge \xi(x)^{(i)}}{\xi(\gamma^+)^{(n-i)} \wedge \xi(\gamma \cdot x)^{(i)}} \right| \\
&\quad + \log \left| \frac{\xi(\gamma^-)^{(i)} \wedge \xi(\gamma \cdot x)^{(n-i)}}{\xi(\gamma^-)^{(i)} \wedge \xi(x)^{(n-i)}} \frac{\xi(\gamma^+)^{(i)} \wedge \xi(x)^{(n-i)}}{\xi(\gamma^+)^{(i)} \wedge \xi(\gamma \cdot x)^{(n-i)}} \right| \\
&= \sum_{k=1}^{i} \lambda_k(\rho(\gamma)) - \sum_{k=n-i+1}^{n} \lambda_k(\rho(\gamma)) \\
&\qquad\qquad + \sum_{k=1}^{n-i} \lambda_k(\rho(\gamma)) - \sum_{k=i+1}^{n} \lambda_k(\rho(\gamma)) \\
&= 2 \left(\sum_{k=1}^{i} \lambda_k(\rho(\gamma)) - \sum_{k=n-i+1}^{n} \lambda_k(\rho(\gamma)) \right) = 2\ell_{\alpha_{i,i+1}}^\rho [\![\gamma]\!]. \qquad \square
\end{aligned}
$$

3.2. Maximal representations.

Another important feature of $\mathrm{PSL}(2, \mathbb{R})$ is that it is a Lie group of Hermitian type.

Definition 3.7. A connected semisimple Lie group G is of *Hermitian type* if it has finite center, has no compact factors and the associated Riemannian symmetric space X admits a G-invariant complex structure.

For our purposes, the main example of Lie group of Hermitian type will be $G = \mathrm{PSp}(2n, \mathbb{R})$. Let g be the Riemannian metric on the symmetric space X and J the G-invariant complex structure. This allows us to define a non-degenerate two-form ω_X by

$$
\omega_X(V, W) := g(JV, W)
$$

for any two vector fields V, W on X. One can show (Burger–Iozzi–Labourie–Wienhard [3, Lemma 2.1]) that (X, ω_X) is a Kähler manifold. For any representation $\rho \colon \Gamma \to G$, the symplectic form ω_X defines an important invariant for ρ as follows. Consider the bundle $S \times_\Gamma X := (\widetilde{S} \times X)/\Gamma$ over S, where Γ acts on $\widetilde{S}$ via deck transformations and on X via ρ. The fiber of this bundle is X, which is contractible, so $S \times_\Gamma X$ admits a smooth section. Equivalently, there exists a smooth Γ-equivariant map $f \colon \widetilde{S} \to X$. The pull back $f^*(\omega_X)$ is a Γ-invariant two-form on $\widetilde{S}$, which descends to the two-form $\widehat{f^*(\omega_X)}$ on the compact surface S. We can define the *Toledo invariant* of ρ as

$$
T(\rho) := \frac{1}{2\pi} \int_S \widehat{f^*(\omega_X)}.
$$

Since any two ρ-equivariant maps $f, f' \colon \widetilde{S} \to X$ are homotopic, $T(\rho)$ is well-defined.

If $\mathrm{rank}_{\mathbb{R}} X$ is the real rank of the symmetric space X, the Toledo invariant satisfies the inequality

$$
|T(\rho)| \leq -\chi(S)\mathrm{rank}_{\mathbb{R}} X
$$

(see Turaev [35], Dominic–Toledo [7], Clerc–Ørsted [6]). In the case $G = \mathrm{PSL}(2, \mathbb{R})$, this is the classical Milnor–Wood inequality [25]. Goldman [13] showed that $\mathcal{T}(S)$ is the unique connected component of $\mathcal{X}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))$ with Toledo invariant $2g - 2$ (the real rank in this case is 1). This motivated Burger–Iozzi–Wienhard [4] to define the following class of representations.

Definition 3.8. A representation $\rho\colon \Gamma \to G$, with G a Lie group of Hermitian type is *maximal* if $|T(\rho)| = -\chi(S)\mathrm{rank}_{\mathbb{R}} X$.

For the rest of this section, fix the target group to be $G = \mathrm{PSp}(2n, \mathbb{R})$. We will show that in this case, maximal representations are also positively ratioed with respect to a particular parabolic subgroup. Recall that the maximal compact subgroup of $\mathrm{PSp}(2n, \mathbb{R})$ is isomorphic to $U(n)$. At the level of Lie algebras, we can write $\mathfrak{sp}(2n, \mathbb{R}) = \mathfrak{k} + \mathfrak{p}$ with

$$\mathfrak{u}(n) \cong \mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix}\colon A, B \in M_n(\mathbb{R}),\ A^t = -A,\ B^t = B \right\}$$
$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix}\colon A, B \in M_n(\mathbb{R}),\ A^t = A,\ B^t = B \right\}$$

where $M_n(\mathbb{R})$ is the set of $n \times n$ matrices. The maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ can therefore be identified with diagonal, traceless matrices in $\mathfrak{sp}(2n, \mathbb{R})$. With this identification, the restricted simple roots can be chosen to be

$$\alpha_{i,i+1}(x_1, \ldots, x_n, -x_n, \ldots, -x_1) = x_i - x_{i+1} \text{ for } i = 1, \ldots, n.$$

Moreover, the opposition involution is the identity.

Burger–Iozzi–Labourie–Wienhard [3, Theorem 6.1] proved that maximal representations are Anosov.

Theorem 3.9 (Burger–Iozzi–Labourie–Wienhard). *If $\rho\colon \Gamma \to G$ is a maximal representation, then ρ is $[P_{\alpha_{n,n+1}}]$-Anosov.*

The quotient $\mathcal{F}_{[P_{\alpha_{n,n+1}}]}$ is the Grassmannian of Lagrangian subspaces in $\mathbb{R}^{2n}$. Consider four Lagrangian subspaces $L_1, L_2, L_3, L_4 \in \mathcal{F}_{[P_{\alpha_{n,n+1}}]}$ so that L_1, L_3 and L_2, L_4 are transverse pairs of Lagrangians, and let $(e_j^1, \ldots, e_j^n)$ be a basis for L_j. For any $i, j = 1, \ldots, 4$, let $A_{i,j}$ be the matrix whose (k, m)-th entry is

$$(A_{i,j})_{k,m} = \Omega(e_i^k, e_j^m),$$

where Ω is the symplectic form on $\mathbb{R}^{2n}$ preserved by the $\mathrm{Sp}(2n, \mathbb{R})$ action. Using this, define

$$\mathbb{B}(L_1, L_2, L_3, L_4) := \frac{\det(A_{1,2}) \cdot \det(A_{3,4})}{\det(A_{1,4}) \cdot \det(A_{3,2})}.$$

Labourie [23, Section 4.2] proved the following.

Theorem 3.10 (Labourie). *If $\rho\colon \Gamma \to G$ is a maximal representation with flag curve $\xi\colon \partial\Gamma \to \mathcal{F}_{[P_{\alpha_{n,n+1}}]}$, then*

$$B(x, y, z, w) := \log \left| \mathbb{B}\big(\xi(x), \xi(z), \xi(y), \xi(w)\big) \right|$$

is a cross ratio. Also, $\ell_B(c) = 2(\lambda_1(\rho(\gamma)) + \lambda_2(\rho(\gamma)) + \cdots + \lambda_n(\rho(\gamma)))$ for all $c = [\![\gamma]\!] \in \mathcal{CG}(S)$. Moreover, for any four distinct points x, y, z, w in this cyclic order along $\partial\Gamma$, we have that $B(x, y, z, w) > 0$.

Combining Theorem 3.9 and Theorem 3.10, we obtain the following corollary.

Corollary 3.11. *If $\rho\colon \Gamma \to G$ is a maximal representation, then ρ is $[P_{\alpha_{n,n+1}}]$-positively ratioed.*

Proof. The restricted fundamental weight $\omega_{\alpha_{n,n+1}}$ corresponding to $\alpha_{n,n+1}$ is given by $\omega_{\alpha_{n,n+1}}(x_1, \ldots, x_n, -x_n, \ldots, -x_1) = x_1 + \cdots + x_n$ and therefore

$$\ell^\rho_{\alpha_{n,n+1}}[\![\gamma]\!] = 2(\lambda_1(\rho(\gamma)) + \lambda_2(\rho(\gamma)) + \cdots + \lambda_n(\rho(\gamma))).$$

Theorem 2.5 then implies that $B^\rho_{\alpha_{n,n+1}}$ is the cross ratio B defined in Theorem 3.10, which is a positive cross ratio. $\qquad\square$

4. Background on geodesic currents

In this section, we will introduce some notation, terminology and basic lemmas to study length functions on subsurfaces of S induced by geodesic currents on S.

4.1. Extension to subsurfaces. We begin by a definition for the kind of subsurfaces of S that we consider.

Definition 4.1. Let $\mathcal{D}$ be a (possibly empty) collection of pairwise non-intersecting, pairwise non-homotopic, non-contractible, simple, closed curves on S. An *essential subsurface* S' of S is a union of connected components of $S \setminus \mathcal{D}$.

If S' is connected, let Γ' be the fundamental group of S' and let $\widetilde{S}'$ denote the universal cover of S'. By choosing appropriate base points in $\widetilde{S}$ and $\widetilde{S}'$, the inclusion $S' \subset S$ induces inclusions $\Gamma' \subset \Gamma$ and $\widetilde{S}' \subset \widetilde{S}$. Also, the inclusion $\Gamma' \subset \Gamma$ realizes the Gromov boundary $\partial\Gamma'$ of Γ' as a subset of $\partial\Gamma$.

If we choose a hyperbolic structure Σ on S, then any connected essential subsurface S' of S is homotopic to a connected subsurface $\Sigma' \subset \Sigma$ with totally geodesic boundary. Also, denote the universal cover of Σ' by $\widetilde{\Sigma}'$, then the inclusion $\widetilde{S}' \subset \widetilde{S}$ gives an inclusion $\widetilde{\Sigma}' \subset \widetilde{\Sigma}$ as the convex hull in $\widetilde{\Sigma}$ of $\partial\Gamma' \subset \partial\Gamma \simeq \partial\widetilde{\Sigma}$.

Previously (see Definitions 2.1 and 2.2), we defined a topological notion of geodesics in $\widetilde{S}$ and S, as well as closed geodesics in S using only Γ. Note that we can define oriented geodesics and geodesics in $\widetilde{S}'$ and S', as well as closed geodesics in S' in the same way, using Γ' in place of Γ. We will denote the set of geodesics in $\widetilde{S}'$, the set of geodesics in S', and the set of closed geodesics in S' by $\mathcal{G}(\widetilde{S}')$, $\mathcal{G}(S')$ and $\mathcal{CG}(S')$ respectively.

Since the closed geodesics in S' are in a natural bijection with the free homotopy classes of closed curves on S', we say that a closed geodesic in S' is *simple* if its

corresponding free homotopy class contains a simple curve, and we say that it is *peripheral* if its corresponding free homotopy class is peripheral.

If $S' = \bigcup_{i=1}^{k} S_i$ is a disconnected union, then we define $\mathcal{CG}(S') := \bigcup_{i=1}^{k} \mathcal{CG}(S_i)$ and $\mathcal{G}(S') := \bigcup_{i=1}^{k} \mathcal{G}(S_i)$.

For the rest of this paper, we will use the notations $S' \subset S$, $\widetilde{S}' \subset \widetilde{S}$, $\Gamma' \subset \Gamma$, $\partial \Gamma' \subset \partial \Gamma$, $\mathcal{CG}(S') \subset \mathcal{CG}(S)$, $\mathcal{G}(\widetilde{S}') \subset \mathcal{G}(\widetilde{S})$ and $\mathcal{G}(S') \subset \mathcal{G}(S)$ as above. Also, whenever we choose a hyperbolic structure on S, we will identify Σ', Σ, $\widetilde{\Sigma}'$ and $\widetilde{\Sigma}$ with S', S, $\widetilde{S}'$ and $\widetilde{S}$ respectively without any further comment.

4.2. Properties of the intersection form.

Although the intersection form (see Definition 2.7) was defined purely topologically as the measure of the set $\mathcal{DG}(S)$, it is often convenient to choose a hyperbolic structure Σ on S. This choice allows us to use the following description of $\mathcal{DG}(S)$, which will be useful for computing the intersection form.

The tangent bundle of the Poincaré disc $T\mathbb{D}$ is a vector bundle over $\mathbb{D}$, so we can projectivize its fibers to obtain a fiber bundle over $\mathbb{D}$, which we denote by $\mathbb{P}(T\mathbb{D})$. Let $\mathbb{P}(T\mathbb{D}) \oplus \mathbb{P}(T\mathbb{D})$ be the fiber bundle over $\mathbb{D}$ obtained by taking the fiber-wise product of $\mathbb{P}(T\mathbb{D})$ with itself. An element of $\mathbb{P}(T\mathbb{D}) \oplus \mathbb{P}(T\mathbb{D})$ is thus a triple (p, l_1, l_2), where $p \in \mathbb{D}$ and l_1, l_2 are lines through the origin in $T_p\mathbb{D}$. Clearly, the $\mathrm{PGL}(2, \mathbb{R}) = \mathrm{Isom}(\mathbb{D})$ action on $\mathbb{P}(T\mathbb{D}) \oplus \mathbb{P}(T\mathbb{D})$ leaves invariant the subset

$$\mathrm{Trans}\big(\mathbb{P}(T\mathbb{D}) \oplus \mathbb{P}(T\mathbb{D})\big) := \big\{(p, l_1, l_2) \in \mathbb{P}(T\mathbb{D}) \oplus \mathbb{P}(T\mathbb{D}) : l_1 \neq l_2\big\}.$$

A choice of a hyperbolic metric Σ on S induces a unique (up to post composition by $\mathrm{PGL}(2, \mathbb{R})$) isometry between $\widetilde{S}$ and $\mathbb{D}$. The action of Γ on $\widetilde{S}$ by deck transformations then conjugates to a free and proper Γ action on $\mathbb{D}$, which in turn induces a free and proper action of Γ on $\mathbb{P}(T\mathbb{D}) \oplus \mathbb{P}(T\mathbb{D})$ that stabilizes $\mathrm{Trans}\big(\mathbb{P}(T\mathbb{D}) \oplus \mathbb{P}(T\mathbb{D})\big)$. This allows us to define the Hausdorff space

$$\mathcal{Q}(\Sigma) := \mathrm{Trans}\big(\mathbb{P}(T\mathbb{D}) \oplus \mathbb{P}(T\mathbb{D})\big) / \Gamma.$$

The isometry between $\widetilde{S}$ and $\mathbb{D}$ also induces an obvious Γ-equivariant homeomorphism between $\mathcal{DG}(\widetilde{S})$ and $\mathrm{Trans}\big(\mathbb{P}(T\mathbb{D}) \oplus \mathbb{P}(T\mathbb{D})\big)$, which descends to a homeomorphism between $\mathcal{DG}(S)$ and $\mathcal{Q}(\Sigma)$. This identification allows us to prove Lemma 4.4 below. However, to state Lemma 4.4, we first need to develop some notation.

Notation 4.2. For any $q, p \in \mathbb{D}$, let $(q, p]$ denote the half-open geodesic in $\mathbb{D}$ with open endpoint q and closed endpoint p. Similarly, (q, p), $[q, p)$ and $[q, p]$ will denote the interval $(q, p]$, but with the appropriate open and closed endpoints.

Notation 4.3. For any $q, p \in \mathbb{D}$, let I be one of the four geodesic segments in $\mathbb{D}$ described in Notation 4.2 with endpoints p and q. Then let $G(I)$ denote the set of geodesics in $\mathbb{D}$ that intersect I transversely. Similarly, for any $x, y, z, w \in \partial \mathbb{D}$ in that

cyclic order, let L be the geodesic in $\mathbb{D}$ with endpoints x, w and let J be either of the following four subsegments of $\partial\Gamma$:

$$(y, z)_x = (y, z)_w, \quad [y, z)_x = [y, z)_w, \quad (y, z]_x = (y, z]_w \quad \text{or} \quad [y, z]_x = [y, z]_w.$$

Then let $G_{\{x,w\}}(J)$ denote the set of geodesics in $\mathbb{D}$ that intersect L and have one endpoint in J.

Lemma 4.4. *Let* $v \in \mathcal{C}(S)$ *and let* $c = [\![\gamma]\!] \in \mathcal{CG}(S)$. *Let* $\{x, y\}$ *be the set of fixed points of* γ. *Also, choose any hyperbolic structure on* S *and let* L *be the axis of* γ *in* $\widetilde{S} = \mathbb{D}$. *Finally, let* $q \in \mathbb{D}$ *and let* $z \in \partial\mathbb{D} \setminus \{x, y\}$. *Then the following hold:*

(1) *If* $q \in L$, *then*

$$i(c, v) = v\big(G(q, \gamma \cdot q]\big) = v\big(G_{\{x,y\}}(z, \gamma \cdot z]\big).$$

(2) *If* $q \notin L$, *then*

$$i(c, v) \leq v\big(G(q, \gamma \cdot q]\big)$$

and the inequality holds strictly if v *has full support.*

Proof. Note that $\{x, y\}$ is the set of endpoints for L in $\partial\mathbb{D}$.

(1) Let $\eta \in \Gamma$ be the primitive element so that $\gamma = \eta^k$ for some positive integer k, and let $c' := [\![\eta]\!] \in \mathcal{CG}(S)$. By definition, $i(c, v)$ is the mass of a fundamental domain of the Γ-action on $\mathcal{DG}(\widetilde{S}) = \text{Trans}(\mathbb{P}(T\mathbb{D}) \oplus \mathbb{P}(T\mathbb{D}))$ in the measure $c \times v$. Since the support of $c \times v$ lies in the set

$$\big\{(p, l_1, l_2) \in \text{Trans}\big(\mathbb{P}(T\mathbb{D}) \oplus \mathbb{P}(T\mathbb{D})\big) : \exp_p(l_1) \text{ is a lift of } c \text{ to } \mathbb{D}\big\},$$

this means that

$$\begin{aligned}
i(c, v) &= (c \times v)\big(\{(p, l_1, l_2) : p \in (q, \eta \cdot q] \text{ and } \exp_p(l_1) = L\}\big) \\
&= k(c' \times v)\big(\{(p, l_1, l_2) : p \in (q, \eta \cdot q] \text{ and } \exp_p(l_1) = L\}\big) \\
&= (c' \times v)\big(\{(p, l_1, l_2) : p \in (q, \gamma \cdot q] \text{ and } \exp_p(l_1) = L\}\big) \\
&= v\big(G(q, \gamma \cdot q]\big).
\end{aligned}$$

Next, we will prove that $v(G(q, \gamma \cdot q]) = v(G_{\{x,y\}}(z, \gamma \cdot z])$. For any $k \in \mathbb{Z}$, let $G^k(q, \gamma \cdot q] \subset G(q, \gamma \cdot q]$ be the subset of geodesics with one endpoint in $\gamma^k \cdot (z, \gamma \cdot z]_x$. It is clear that $G(q, \gamma \cdot q]$ can be written as the disjoint union

$$G(q, \gamma \cdot q] = \bigcup_{k \in \mathbb{Z}} G^k(q, \gamma \cdot q].$$

Also, for any $k \in \mathbb{Z}$, let $G^k_{\{x,y\}}(z, \gamma \cdot z] \subset G_{\{x,y\}}(z, \gamma \cdot z]$ be the subset of geodesics that intersect $\gamma^k \cdot (q, \gamma \cdot q]$. As before, $G_{\{x,y\}}(z, \gamma \cdot z]$ can be written as the disjoint union

$$G_{\{x,y\}}(z, \gamma \cdot z] = \bigcup_{k \in \mathbb{Z}} G^k_{\{x,y\}}(z, \gamma \cdot z].$$

Finally, observe that $G^k(q, \gamma \cdot q] = \gamma^k \cdot G^{-k}_{\{x,y\}}(z, \gamma \cdot z]$. Hence,

$$
\begin{aligned}
\nu\big(G(q, \gamma \cdot q]\big) &= \sum_{k \in \mathbb{Z}} \nu\big(G^k(q, \gamma \cdot q]\big) \\
&= \sum_{k \in \mathbb{Z}} \nu\big(\gamma^k \cdot G^{-k}_{\{x,y\}}(z, \gamma \cdot z]\big) \\
&= \sum_{k \in \mathbb{Z}} \nu\big(G^{-k}_{\{x,y\}}(z, \gamma \cdot z]\big) \\
&= \nu\big(G_{\{x,y\}}(z, \gamma \cdot z]\big).
\end{aligned}
$$

(2) Let p be the foot of the perpendicular from q to L and let L' be the bi-infinite geodesic through q and $\gamma \cdot q$. Observe that $\gamma \cdot p$ is also the foot of the perpendicular from $\gamma \cdot q$ to L, and $L \cap L'$ is empty. Let $z, w \in \partial \mathbb{D}$ be the endpoints of L' so that $z, x, y, w \in \partial \mathbb{D}$ in that order (see Figure 1). By (1), we know that $i(c, \nu) = \nu(G(p, \gamma \cdot p])$, so it is sufficient to show that $\nu(G(q, \gamma \cdot q]) \geq \nu(G(p, \gamma \cdot p])$, and that this inequality is strict when $\nu(U) > 0$ for all open sets $U \subset \mathcal{G}(\widetilde{S})$.
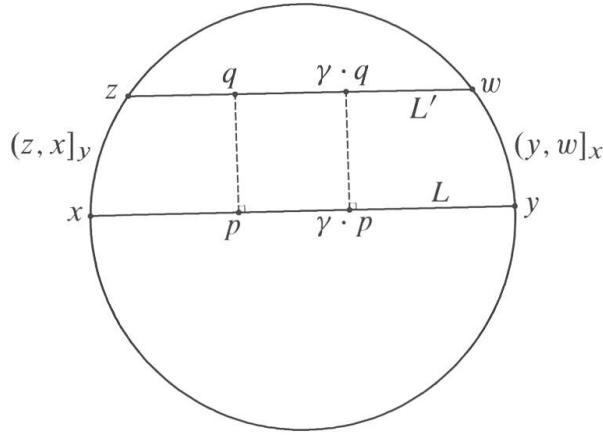
Figure 1. Proof of (2) of Lemma 4.4.

For any $k \in \mathbb{Z}$, let $G_k(p, \gamma \cdot p] \subset G(p, \gamma \cdot p]$ be the subset of geodesics that intersect $\gamma^k \cdot (q, \gamma \cdot q]$. Then $G(p, \gamma \cdot p]$ can be written as the disjoint union

$$
G(p, \gamma \cdot p] = \bigcup_{k \in \mathbb{Z}} G_k(p, \gamma \cdot p].
$$

Similarly, for any $k \in \mathbb{Z}$, let $G_k(q, \gamma \cdot q] \subset G(q, \gamma \cdot q]$ be the subset of geodesics that intersect $\gamma^k \cdot (p, \gamma \cdot p]$. Also, let $A \subset G(q, \gamma \cdot q]$ be the subset of geodesics with one endpoint in $(z, x]_y$ and let $B \subset G(q, \gamma \cdot q]$ be the subset of geodesics with one endpoint in $[y, w)_x$. Observe that $G(q, \gamma \cdot q]$ can again be written as the disjoint union

$$
G(q, \gamma \cdot q] = A \cup B \cup \bigcup_{k \in \mathbb{Z}} G_k(q, \gamma \cdot q].
$$

Since $G_k(p, \gamma \cdot p] = \gamma^k \cdot G_{-k}(q, \gamma \cdot q]$ for all $k \in \mathbb{Z}$, we have

$$
\begin{aligned}
\nu\big(G(q, \gamma \cdot q]\big) &= \nu(A) + \nu(B) + \sum_{k \in \mathbb{Z}} \nu\big(G_k(q, \gamma \cdot q]\big) \\
&= \nu(A) + \nu(B) + \sum_{k \in \mathbb{Z}} \nu\big(\gamma^k \cdot G_{-k}(q, \gamma \cdot q]\big) \\
&= \nu(A) + \nu(B) + \sum_{k \in \mathbb{Z}} \nu\big(G_k(p, \gamma \cdot p]\big) \\
&= \nu(A) + \nu(B) + \nu\big(G(p, \gamma \cdot p]\big) \\
&\geq \nu\big(G(p, \gamma \cdot p]\big).
\end{aligned}
$$

It is clear that A and B contain open subsets of geodesics in $\mathcal{G}(S)$, so the strictness statement holds. $\square$

4.3. Surgery and lengths. Let $c \in \mathcal{CG}(S')$ be a primitive closed geodesic of S' with positive geometric self-intersection number. Choose a representative $\bar{c}$ in the free homotopy class of closed curves corresponding to c, so that $\bar{c}$ has only transverse self-intersections and minimal self-intersection number. We can also assume that $\bar{c}$ only has simple self-intersection points, i.e. if we choose a parameterization of c by $\mathbb{S}^1$, then $c(t_1) = c(t_2) = c(t_3)$ implies that $t_1 \in \{t_2, t_3\}$. Let p be a point of self-intersection for $\bar{c}$.

There is a well known procedure one can apply to $\bar{c}$ known as *surgery at p* to obtain new closed curves in S'. To do so, choose a small topological disc in $U \subset S'$ so that $\bar{c} \cap \partial U$ is four points x, y, z, w that lie along ∂U in that order, and $\bar{c} \cap U$ is the union of two simple paths that intersect at p, one with endpoints x and z, and the other with endpoints y and w. We can then modify the curve $\bar{c}$ by replacing the two simple paths $\bar{c} \cap U$ that intersect at p with two simple paths in U that do not intersect. There are two ways to do so; we can either replace $\bar{c} \cap U$ with two simple, non-intersecting paths in U with endpoints x, y and z, w, or we can replace $\bar{c} \cap U$ with two simple, non-intersecting paths in U with endpoints y, z and x, w.

These two different ways of performing surgery to $\bar{c}$ at p yield either one closed curve $\bar{c}_1$ in S' or two closed curves $\bar{c}_2$ and $\bar{c}_3$ in S'. For $i = 1, 2, 3$, let $c_i \in \mathcal{CG}(S')$ correspond to the free homotopy class of closed curves in S' that contains $\bar{c}_i$ (see Figure 2). It is easy to see that c_1, c_2 and c_3 do not depend on the choice of U. Moreover, an easy homotopy argument shows that c_1, c_2 and c_3 also do not depend on $\bar{c}$ in the following sense. If $\bar{c}'$ is another closed curve in the free homotopy class corresponding to c with minimal geometric self-intersection number and only simple, transverse self-intersection points, then the homotopy between $\bar{c}$ and $\bar{c}'$ gives a bijection

$$
h_{\bar{c}, \bar{c}'} \colon \{\text{self-intersection points of } \bar{c}\} \to \{\text{self-intersection points of } \bar{c}'\}.
$$

If we perform surgery to $\bar{c}'$ at the self-intersection point $h_{\bar{c},\bar{c}'}(p)$ in both ways, then the closed geodesics corresponding to the free homotopy classes of closed curves we obtain are exactly c_1, c_2 and c_3.
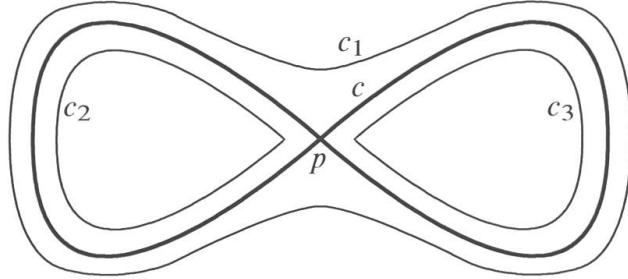
Figure 2. Surgery.

The following proposition gives useful inequalities involving $i(c_1, \nu)$, $i(c_2, \nu)$, $i(c_3, \nu)$ and $i(c, \nu)$.

Proposition 4.5. *Let $\nu \in \mathcal{C}(S)$ and let $c \in \mathcal{CG}(S)$ be a primitive closed geodesic so that $i(c,c) > 0$. By performing surgery to c at a point of self-intersection in two different ways (see discussion above), we obtain either a single geodesic c_1 or a pair of geodesics c_2, c_3. Then*

$$i(c_1, \nu) \leq i(c, \nu) \quad and \quad i(c_2, \nu) + i(c_3, \nu) \leq i(c, \nu).$$

Furthermore, these inequalities are strict when ν has full support.

Proof. Let $\gamma \in \Gamma$ be a group element so that $[\![\gamma]\!] = c \in \mathcal{CG}(S)$. Choose a hyperbolic structure on S, and let $\bar{c}$ be a closed curve homotopic to c with transverse, minimal self-intersection and only simple self-intersection points. The homotopy between $\bar{c}$ and c gives a surjection

$$h_{\bar{c},c} \colon \{\text{self-intersection points of } \bar{c}\} \to \{\text{self-intersection points of } c\}.$$

Let q be the self-intersection point of $\bar{c}$ where the surgeries to obtain c_1, c_2 and c_3 are performed, and let $p = h_{\bar{c},c}(q)$.

Let $L \subset \widetilde{S} = \mathbb{D}$ be the axis of γ, and observe that L is a lift of the geodesic c. Let $\tilde{p}$ be a point in L whose image under the covering map $\Pi \colon \widetilde{S} \to S$ is p. Then $\gamma \cdot \tilde{p}$ also lies in L and $\Pi(\gamma \cdot \tilde{p}) = p$ as well. Let $\gamma_2, \gamma_3 \in \Gamma$ be the group elements so that $[\![\gamma_2]\!] = c_2$, $[\![\gamma_3]\!] = c_3$, $\gamma = \gamma_3 \gamma_2$ and $\gamma_2 \cdot \tilde{p} \in (\tilde{p}, \gamma \cdot \tilde{p}]$ (see Notation 4.2). It is clear that $\Pi(\gamma_2 \cdot \tilde{p}) = p$ and $[\![\gamma_1 := \gamma_3^{-1}\gamma_2]\!] = c_1$.

We will first prove the inequality $i(c_2, \nu) + i(c_3, \nu) \leq i(c, \nu)$. By (2) of Lemma 4.4, we have

$$\begin{aligned}
i(c_2, \nu) + i(c_3, \nu) &\leq \nu\big(G(\tilde{p}, \gamma_2 \cdot \tilde{p}]\big) + \nu\big(G(\gamma_2 \cdot \tilde{p}, \gamma_3\gamma_2 \cdot \tilde{p}]\big) \\
&= \nu\big(G(\tilde{p}, \gamma \cdot \tilde{p}]\big) \\
&= i(c, \nu).
\end{aligned}$$

To prove the inequality $i(c_1, \nu) \le i(c, \nu)$, first make the following observation. For any $\widetilde{q} \in \mathbb{D}$, let L' be the geodesic in $\widetilde{S}$ containing $[\widetilde{q}, \gamma_3 \cdot \widetilde{q}]$. If $\nu(\{L'\}) = 0$, then $\nu(\gamma_3 \cdot \{L'\}) = 0$ as well, so the ν-measure of the set of geodesics through $\gamma_3 \cdot \widetilde{q}$ that are transverse to L', is equal to the ν-measure of the set of geodesics through $\gamma_3 \cdot \widetilde{q}$, which is again equal to the ν-measure of the set of geodesics through $\gamma_3 \cdot \widetilde{q}$ transverse to $\gamma_3 \cdot L'$. This is in turn equal to the ν-measure of the set of geodesics through $\widetilde{q}$ transverse to L'. Hence, we may conclude that if $\nu(\{L'\}) = 0$, then

$$\nu\big(G(\widetilde{q}, \gamma_3^{-1} \cdot \widetilde{q}]\big) = \nu\big(G[\widetilde{q}, \gamma_3 \cdot \widetilde{q})\big) = \nu\big(G(\widetilde{q}, \gamma_3 \cdot \widetilde{q}]\big).$$

Now, observe that the geodesic containing $[\gamma_2 \cdot \widetilde{p}, \gamma_3 \gamma_2 \cdot \widetilde{p}]$ is L. Hence, the previous observation, together with (2) Lemma 4.4, allows us to conclude that if $\nu(\{L\}) = 0$, then

$$\begin{aligned}
i(c_1, \nu) &\le \nu\big(G(\widetilde{p}, \gamma_1 \cdot \widetilde{p}]\big) \\
&\le \nu\big(G(\widetilde{p}, \gamma_2 \cdot \widetilde{p}]\big) + \nu\big(G(\gamma_2 \cdot \widetilde{p}, \gamma_3^{-1} \gamma_2 \cdot \widetilde{p}]\big) \\
&= \nu\big(G(\widetilde{p}, \gamma_2 \cdot \widetilde{p}]\big) + \nu\big(G(\gamma_2 \cdot \widetilde{p}, \gamma_3 \gamma_2 \cdot \widetilde{p}]\big) \\
&= \nu\big(G(\widetilde{p}, \gamma \cdot \widetilde{p}]\big) \\
&= i(c, \nu).
\end{aligned}$$

With this, we can prove $i(c_1, \nu) \le i(c, \nu)$ for general ν. Since c_1 is obtained from c by performing surgery, it is clear that $i(c_1, c) \le i(c, c)$. Also, since L is a lift of c, we can write $\nu = \mu + kc$ for some $\mu \in \mathcal{C}(S)$ so that $\mu(\{L\}) = 0$. This means that

$$i(c_1, \nu) = i(c_1, \mu) + ki(c_1, c) \le i(c, \mu) + ki(c, c) = i(c, \nu).$$

Finally, we argue that these inequalities are strict when ν has full support. By the strictness statement in (2) of Lemma 4.4, it is sufficient to show that $\widetilde{p}$ does not lie along the axes of γ_1, γ_2 and γ_3. This is obvious, since the axes of γ, γ_1, γ_2 and γ_3 are pairwise disjoint. $\qquad\square$

As a consequence of Theorem 1.1, Proposition 2.27 and Proposition 4.5, we have Corollary 1.3.

4.4. Asymptotics of lengths. In [5], Burger and Pozzetti consider a metric compactification of the space of $\mathrm{Sp}(2n, \mathbb{R})$-maximal representations. The limit points correspond to actions via isometries of Γ on certain asymptotic cones. They prove [5, Theorem 1.1] that a boundary point determines a decomposition of S into essential subsurfaces. This decomposition is defined in terms of the asymptotic behavior of the length function.

In this section, we obtain an analogous result for sequences of positively ratioed representations as a consequence of Theorem 1.1. Here, we use the compactness of the space of *projectivized geodesic currents* $\mathcal{PC}(S) := \mathcal{C}(S)/\mathbb{R}^+$ [2, Corollary 5] to describe the limit points. First, we need the following lemma.

Lemma 4.6. *Let $v \in \mathcal{C}(S)$ and $\overline{e} \in \mathcal{CG}(S')$ be a primitive non-simple curve. Then there is some geodesic pair of pants $P \subset S'$ and $e \in \mathcal{CG}(P)$ so that:*

- *$i(e, v) \leq i(\overline{e}, v)$;*

- *e is primitive and has a unique self-intersection point p;*

- *the three closed geodesics e_1, e_2 and e_3 obtained by performing surgery to e at p in the two different ways specified in Section 4.3 are the three boundary components of P.*

- *if a curve $c \in \mathcal{CG}(S')$ intersects $\overline{e}$ transversely, then c intersects e transversely.*

Proof. Let q be any self intersection point of $\overline{e}$ and let $\overline{e}_1$, $\overline{e}_2$ and $\overline{e}_3$ be the three closed geodesics obtained by performing surgery to $\overline{e}$ at q. It is clear that the self-intersection numbers of $\overline{e}_1$, $\overline{e}_2$ and $\overline{e}_3$ are less than that of $\overline{e}$. Also, Proposition 4.5 implies that $i(\overline{e}_j, v) \leq i(\overline{e}, v)$ for all $j = 1, 2, 3$. Suppose that there is some $j_0 = 1, 2, 3$ so that $\overline{e}_{j_0}$ is not a multiple of a simple curve. Then let $\overline{\overline{e}}$ be the closed geodesic so that $\overline{\overline{e}} = [\![\gamma]\!]$ for some primitive $\gamma \in \Gamma$ with the property that $[\![\gamma^k]\!] = \overline{e}_{j_0}$ for some $k \in \mathbb{Z}$. Then $\overline{\overline{e}}$ is primitive, non-simple, has fewer self-intersection points than $\overline{e}$, and $i(\overline{\overline{e}}, v) \leq i(\overline{e}_{j_0}, v) \leq i(\overline{e}, v)$. Replace $\overline{e}$ with $\overline{\overline{e}}$.

Iterate the replacement procedure above. This iteration will terminate to give a non-simple $e \in \mathcal{CG}(S')$ so that $i(e, v) \leq i(\overline{e}, v)$, and for any self-intersection point p of e, the three closed geodesics e_1, e_2 and e_3 obtained by performing surgery to e at p are multiples of simple curves in S'. This then implies that e must have a unique self-intersection point. In particular, e_1, e_2 and e_3 are simple and are pairwise non-intersecting. The homotopy from e_1, e_2 and e_3 to e is a pair of pants P that contains e, and has e_1, e_2 and e_3 as its boundary components. $\square$

Proposition 4.7. *Fix an auxiliary hyperbolic structure on S. Let $\{v_j\}_{j=1}^{\infty} \subset \mathcal{C}(S)$ be a sequence of non-zero geodesic currents. There is*

- *a subsequence of $\{v_j\}_{j=1}^{\infty}$, also denoted $\{v_j\}_{j=1}^{\infty}$,*

- *a (possibly disconnected, possibly empty) essential subsurface $S' \subset S$,*

- *a (possibly empty) collection of pairwise non-intersecting, simple closed geodesics $\{c_1, \ldots, c_k\} \subset \mathcal{CG}(S \setminus S')$*

so that $A := S' \cup \bigcup_{i=1}^{k} c_i \subset S$ is non-empty, and the following property holds. Let $c \in \mathcal{CG}(S)$ be a closed curve so that $c \notin \mathcal{CG}(S \setminus A)$ and c is not a multiple of c_i for $i = 1, \ldots, k$.

(1) *If d is a multiple of c_i for some $i = 1, \ldots, k$ or $d \in \mathcal{CG}(S \setminus A)$, then*

$$\lim_{j \to \infty} \frac{i(v_j, d)}{i(v_j, c)} = 0.$$

(2) *If $d \in \mathcal{CG}(S)$ is a closed curve so that $d \notin \mathcal{CG}(S \setminus A)$ and d is not a multiple of c_i for $i = 1, \ldots, k$, then*

$$\lim_{j \to \infty} \frac{i(v_j, d)}{i(v_j, c)} \in \mathbb{R}^+.$$

Proof. Since the weak* topology on $\mathcal{C}(S)$ is metrizable and $\mathcal{PC}(S)$ is compact, there exist

- a subsequence of $\{\nu_j\}_{j=1}^{\infty}$, also denoted $\{\nu_j\}_{j=1}^{\infty}$,
- a sequence of positive real numbers $\{\lambda_j\}_{j=1}^{\infty}$,
- a non-zero geodesic current ν,

such that $\lim_{j\to\infty} \lambda_j \nu_j = \nu$. Define

$$\operatorname{supp} \nu := \left\{ g \in \mathcal{G}(\widetilde{S}): \nu(U_g) > 0 \text{ for all neighborhoods } U_g \text{ of } g \right\},$$

and consider $B := (\operatorname{supp} \nu)/\Gamma \subset \mathcal{G}(S)$.

For our choice of an auxiliary hyperbolic metric on S, let $\{c_1', c_2', \ldots, c_l'\}$ be a maximal (possibly empty) collection of pairwise non-intersecting simple closed geodesics that do not have transverse intersections with any geodesic in B. Then let $\{S_1, \ldots, S_m\}$ be the list of connected components of $S \setminus \{c_1', c_2', \ldots, c_l'\}$ and define

$$S' := \bigcup_{\{t\in\{1,\ldots,m\}: S_t \cap B \neq \emptyset\}} S_t.$$

If $B \cap \{c_1', c_2', \ldots, c_l'\} = \emptyset$, set k to be 0. Otherwise, let $c_1, \ldots, c_k$ be the closed geodesics in $B \cap \{c_1', \ldots, c_l'\}$.

Notice that A is non-empty because $\operatorname{supp} \nu$ is non-empty. Since the intersection pairing is continuous, for any $a \in \mathcal{CG}(S)$ and for any b intersecting transversely a geodesic in B, up to passing to a further subsequence, we have

$$\lim_{j\to\infty} \frac{i(\nu_j, a)}{i(\nu_j, b)} = \lim_{j\to\infty} \frac{i(\lambda_j \nu_j, a)}{i(\lambda_j \nu_j, b)} = \frac{i(\nu, a)}{i(\nu, b)}.$$

Also, $i(\nu, a) > 0$ if and only if a intersects some geodesic in B transversely. It is thus sufficient to show that for any $a \in \mathcal{CG}(S)$, a intersects a geodesic in B transversely if and only if a is not a multiple of c_i for $i = 1, \ldots, k$ and $a \notin \mathcal{CG}(S \setminus A)$.

Clearly, if a is a multiple of c_i for some $i = 1, \ldots, k$ or $a \in \mathcal{CG}(S \setminus A)$, then a does not intersect any geodesic in $\operatorname{supp} \nu$ transversely. To prove the converse, suppose that a is not a multiple of c_i for $i = 1, \ldots, k$, and $a \notin \mathcal{CG}(S \setminus A)$. If a intersects c_i transversely for some $i = 1, \ldots, k$, we are done. Hence, for the rest of the proof, we will assume that a intersects the interior of S'. The proof proceeds in two cases.

Case 1: Suppose $a \in \mathcal{CG}(S')$. By the way S' is constructed, if a is simple, then it must intersect a geodesic in B (otherwise the maximality of $\{c_1', \ldots, c_l'\}$ is contradicted). Hence, we may assume that a is non-simple. By Lemma 4.6, we may also assume that a is contained in a geodesic pair of pants $P \subset S'$. Since a is a non-peripheral geodesic in P, it has transverse intersections with every non-peripheral geodesic in $\mathcal{G}(P)$ and every geodesic segment in P with endpoints in ∂P. Also, because $P \subset S'$, there is some geodesic in B that intersects the interior of P. Hence, a intersects some geodesic in B transversely.

Case 2: Suppose a is not entirely contained in S'. Let S'_0 be a connected component of S' so that a intersects the interior of S'_0, and let B'_0 be the set of geodesics in B that lie in S'_0. Let x_1 and x_2 be a pair of points where a intersects the boundary of S'_0 in S, so that there is a subsegment e of a with endpoints x_1 and x_2 that is entirely contained in S'_0. Let b_1 and b_2 be the boundary components of $\overline{S'_0}$ containing x_1 and x_2, respectively.

For $s = 1, 2$, choose a parameterization $f_s \colon [0, 1] \to b_s$ so that $f_s(0) = f_s(1) = x_s$ and choose a parameterization $g \colon [0, 1] \to e$ so that $g(0) = x_1$ and $g(1) = x_2$. Consider, the closed curve $\gamma = g^{-1} * f_2^2 * g * f_1^2$, where $*$ is the symbol for concatenation. Observe that γ is freely homotopic to a non-peripheral geodesic d in S'_0. By the previous case, we know that d intersects a geodesic in B'_0, so γ also intersects a geodesic in B'_0. Since b_1 and b_2 are boundary geodesics, they do not intersect any geodesics in B'_0. This means that e intersects a geodesic in B'_0. Moreover, since e is a geodesic segment, this intersection is transverse. Hence, a intersects a geodesic in B transversely in this case as well. $\qquad\square$

As a consequence of Theorem 1.1 and Proposition 4.7, we have Corollary 1.2.

4.5. Systoles and minimal pants decompositions. We will now explore the consequences of Proposition 4.5 on systole lengths of any essential subsurface $S' \subset S$. If $v \in \mathcal{C}(S)$ is period minimizing, then the function $\mathcal{CG}(S) \to \mathbb{R}$ given by $c \mapsto i(c, v)$ is minimized at some $c \in \mathcal{CG}(S)$. The same idea gives us a notion of systoles for essential subsurfaces, which we will now define.

Definition 4.8. Let $S' \subset S$ be an essential subsurface, and let $v \in \mathcal{C}(S)$ be period minimizing. The v-*systole length* of S' is

$$L_v(S') := \min \{ i(c, v) : c \in \mathcal{CG}(S') \},$$

and a v-*systole* of S' is a closed geodesic $c \in \mathcal{CG}(S')$ so that $i(c, v) = L_v(S')$. Also, define the v-*interior systole length* of S' to be

$$L_v^{\mathrm{int}}(S') := \min \{ i(c, v) : c \in \mathcal{CG}(S') \text{ is non-peripheral} \},$$

and a v-*interior systole* of S' is a non-peripheral closed geodesic $c \in \mathcal{CG}(S')$ so that $i(c, v) = L_v^{\mathrm{int}}(S')$. In the case when $S = S'$, we will denote $L_v := L_v(S) = L_v^{\mathrm{int}}(S)$.

Using Proposition 4.5, we can prove the following corollary.

Corollary 4.9. *Let $S' \subset S$ be a connected essential subsurface and let $v \in \mathcal{C}(S)$ be period minimizing. Suppose that S' is not a pair of pants. Then the following hold.*

(1) *There is a v-interior systole of S' that is simple.*

(2) *If v has full support, then every v-interior systole of S' is simple.*

Proof. Let c be a ν-interior systole of S'. We may assume without loss of generality that c is primitive. Suppose that $c = [\![\gamma]\!]$ has $k \geq 1$ self-intersections. Then we can perform surgery to c at some point of self-intersection to obtain $c_1 = [\![\gamma_1]\!]$, $c_2 = [\![\gamma_2]\!]$ and $c_3 = [\![\gamma_3]\!]$ with $\gamma = \gamma_3\gamma_2$ and $\gamma_1 = \gamma_3^{-1}\gamma_2$. If c_1, c_2 and c_3 are all peripheral, then the relation $\gamma_1 = \gamma_3^{-1}\gamma_2$ implies that S' is a pair of pants, which contradicts the hypothesis of the corollary. Hence, for some $j_0 = 1, 2, 3$, c_{j_0} is a non-peripheral closed geodesic whose self-intersection number is strictly less than the self-intersection number of c.

(1) By Proposition 4.5, we know that c_{j_0} is also a ν-interior systole of S', so we can iterate the above procedure with c_{j_0} in place of c. This will eventually terminate after at most k steps to give a ν-interior systole that is simple.

(2) In the case when ν has full support, Proposition 4.5 tells us that $i(c_{j_0}, \nu) < i(c, \nu)$. This contradicts the fact that c is a ν-interior systole. $\qquad\square$

In particular, if we have a period minimizing $\nu \in \mathcal{C}(S)$, we can build a ν-*minimal pants decomposition*, denoted $\mathcal{P}_\nu(S')$, on any essential subsurface $S' \subset S$. Let $c_1, \ldots, c_n$ be the n boundary components of S'. If S' is a disjoint union of pairs of pants, then n is three times the number of components of S' and $\mathcal{P}_\nu(S') = \{c_1, \ldots, c_n\}$. Otherwise, Corollary 4.9 implies that there is a ν-interior systole of S' that is simple. Let c_{n+1} be such a ν-interior systole of S', then $S' \setminus c_{n+1}$ is again an essential subsurface of S. Hence, we can iterate this procedure until we have a pants decomposition $\mathcal{P}_\nu(S')$. Denote $\mathcal{P}_\nu(S)$ simply by $\mathcal{P}_\nu$.

5. Combinatorial description of $\mathcal{CG}(S')$

In this section, fix some $\nu \in \mathcal{C}(S)$ that is period minimizing. An important ingredient in the proof of Theorem 1.4 is a finite combinatorial description, defined below, for each conjugacy class in Γ' that is adapted to ν. The methods in this section and the following one are inspired by work of the second author [37].

5.1. Minimal pants decompositions and related structures. First, we need to equip S with an ideal triangulation which depends on S' and ν.

Definition 5.1. An *ideal triangulation* of $\widetilde{S}$ is a maximal Γ-invariant subset $\widetilde{\mathcal{T}} \subset \mathcal{G}(\widetilde{S})$ such that the following hold:

(1) Any two pairs of geodesics $\{x, y\}, \{z, w\} \in \widetilde{\mathcal{T}}$ do not intersect transversely.

(2) For any geodesic $\{x, y\} \in \widetilde{\mathcal{T}}$, one of the following must hold:

- There is some z in $\partial\Gamma$ such that $\{x, z\}, \{y, z\} \in \widetilde{\mathcal{T}}$.
- There is some $\gamma \in \Gamma$ such that $\{x, y\}$ is the set of fixed points of γ.

An *ideal triangulation* of S is then the quotient $\mathcal{T} := \widetilde{\mathcal{T}}/\Gamma$ for some ideal triangulation of $\widetilde{\mathcal{T}}$ of $\widetilde{S}$. A *triangle* is an unordered triple of geodesics in $\widetilde{\mathcal{T}}$ of the form $\{\{x, y\}, \{y, z\}, \{z, x\}\}$.

If we choose a hyperbolic structure on S, then every ideal triangulation $\widetilde{\mathcal{T}}$ of $\widetilde{S}$ can be realized as an ideal triangulation of $\widetilde{S} = \mathbb{D}$ (in the classical sense) by assigning to each pair $\{x, y\} \in \widetilde{\mathcal{T}}$ the unique hyperbolic geodesic in $\mathbb{D}$ with endpoints $x, y \in \partial \mathbb{D}$. Moreover, this ideal triangulation is Γ-invariant, so $\mathcal{T}$ can be thought of as an ideal triangulation (in the classical sense) of S.

For our purposes, we will consider a particular ideal triangulation $\mathcal{T}_{\nu,S'}$ of S, defined as follows. *Choose an orientation on S*. Recall that we previously constructed a ν-minimal pants decomposition $\mathcal{P}_\nu(S')$ of S' as a consequence of Corollary 4.9. Extend this to a pants decomposition $\mathcal{P}_{\nu,S'}$ of S, and let $P_1, \ldots, P_{2g-2}$ be the pairs of pants given by $\mathcal{P}_{\nu,S'}$ where g is the genus of S. For each $j = 1, \ldots, 2g-2$, orient each component of ∂P_j so that P_j lies on the left of the boundary component. Let $\gamma_{1,j}, \gamma_{2,j}, \gamma_{3,j} \in \Gamma$ be primitive group elements corresponding to the three boundary components of P_j equipped with their orientations, so that $\gamma_{3,j}\gamma_{2,j}\gamma_{1,j} = \mathrm{id}$. For each $i = 1, 2, 3$ and $j = 1, \ldots, 2g-2$, let $\gamma_{i,j}^+, \gamma_{i,j}^- \in \partial \Gamma$ denote the attracting and repelling fixed points of $\gamma_{i,j}$ respectively.

Let $\widetilde{\mathcal{Q}}_j$ and $\widetilde{\mathcal{P}}_j$ be the subsets of $\mathcal{G}(\widetilde{S})$ defined by

$$\widetilde{\mathcal{Q}}_j := \bigcup_{\gamma \in \Gamma} \{\gamma \cdot \{\gamma_{1,j}^-, \gamma_{2,j}^-\}, \gamma \cdot \{\gamma_{2,j}^-, \gamma_{3,j}^-\}, \gamma \cdot \{\gamma_{3,j}^-, \gamma_{1,j}^-\}\},$$

$$\widetilde{\mathcal{P}}_j := \bigcup_{\gamma \in \Gamma} \{\gamma \cdot \{\gamma_{1,j}^-, \gamma_{1,j}^+\}, \gamma \cdot \{\gamma_{2,j}^-, \gamma_{2,j}^+\}, \gamma \cdot \{\gamma_{3,j}^-, \gamma_{3,j}^+\}\},$$

and note that both $\widetilde{\mathcal{Q}}_j$ and $\widetilde{\mathcal{P}}_j$ do not depend on the choice of $\gamma_{1,j}, \gamma_{2,j}$ and $\gamma_{3,j}$. They are also Γ-invariant, so we can define $\mathcal{Q}_j := \widetilde{\mathcal{Q}}_j/\Gamma$ and $\mathcal{P}_j := \widetilde{\mathcal{P}}_j/\Gamma$. With this, define

$$\widetilde{\mathcal{Q}} := \bigcup_{l=1}^{2g-2} \widetilde{\mathcal{Q}}_j, \quad \widetilde{\mathcal{P}} := \bigcup_{l=1}^{2g-2} \widetilde{\mathcal{P}}_j, \quad \mathcal{P} := \bigcup_{l=1}^{2g-2} \mathcal{P}_j, \quad \mathcal{Q} := \bigcup_{l=1}^{2g-2} \mathcal{Q}_j,$$

and observe that $\widetilde{\mathcal{T}}_{\nu,S'} := \widetilde{\mathcal{Q}} \cup \widetilde{\mathcal{P}}$ and $\mathcal{T}_{\nu,S'} := \mathcal{Q} \cup \mathcal{P}$ are ideal triangulations of $\widetilde{S}$ and S respectively.

It is clear that $\mathcal{P} = F(\mathcal{P}_{\nu,S'})$ (recall that $F: \mathcal{CG}(S) \to \mathcal{G}(S)$ sends $[\![\gamma]\!]$ to $[\gamma^-, \gamma^+]$). Also, if we choose a hyperbolic structure on S, then for all $j = 1, \ldots, 2g-2$, the three geodesics in $\mathcal{Q}_j$ correspond to three simple, pairwise non-intersecting geodesics in the hyperbolic pair of pants P_j that each "spiral" towards two different boundary components of P_j (see Figure 3).
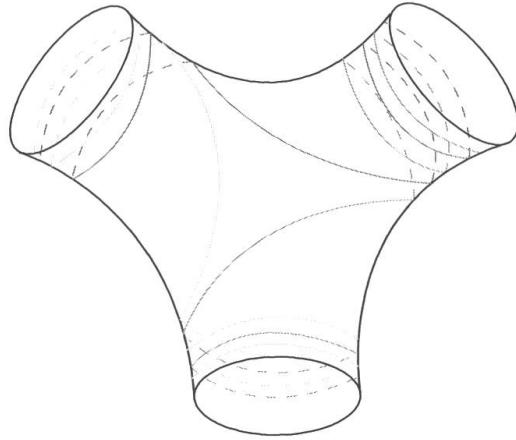
Figure 3. Curves in $\mathcal{Q}_j$.

The ideal triangulation by itself is insufficient to give a finite combinatorial description for the geodesics in $\mathcal{CG}(S')$. We need to make some additional choices, which we will now specify. *Choose:*

- *an orientation on each simple closed geodesic in $\mathcal{P}_{\nu,S'}$.*

- *a hyperbolic structure Σ on S.*

Since we have chosen orientations on every $c \in \mathcal{P}_{\nu,S'}$, c can be viewed as a conjugacy class in $[\Gamma]$. For any such c, let $\gamma_c \in \Gamma$ be a primitive group element so that $[\gamma_c] = c \in [\Gamma]$. Then let

$$V(\gamma_c^{\pm}) := \left\{ x \in \partial\Gamma \setminus \{\gamma_c^-, \gamma_c^+\} : \{x, \gamma_c^{\pm}\} \in \widetilde{\mathcal{T}}_{\nu,S'} \right\}$$

and define

$$\widetilde{\mathcal{N}}(\gamma_c^{\pm}) := \left\{ \{x, y\} \in \widetilde{\mathcal{T}}_{\nu,S'} : x, y \in V(\gamma_c^{\pm}) \right\}.$$

Observe that $V(\gamma_c^{\pm})$ and $\widetilde{\mathcal{N}}(\gamma_c^{\pm})$ are both invariant under the cyclic subgroup $\langle \gamma_c \rangle \subset \Gamma$. Also, the geodesics in $\widetilde{\mathcal{N}}(\gamma_c^-) \cup \widetilde{\mathcal{N}}(\gamma_c^+)$ are realized as hyperbolic geodesics in $\widetilde{S} \simeq \mathbb{D}$, and their union bounds a simply connected, convex domain $\Omega_{\gamma_c} \subset \widetilde{S}$ that contains the axis of γ_c. Let P_1 and P_2 be the two pairs of pants given by $\mathcal{P}_{\nu,S'}$ that have c as a common boundary component, so that P_1 and P_2 lie on the left and right of c respectively. (It is possible that $P_1 = P_2$).

Choose a point $r^{\pm}$ on a hyperbolic geodesic in $\widetilde{\mathcal{N}}(\gamma_c^{\pm})$, and let $p_{\gamma_c}^{\pm} \in \langle \gamma_c \rangle \cdot r^{\pm}$ be a point so that

$$\nu\left(G[p_{\gamma_c}^+, p_{\gamma_c}^-]\right) = \min\left\{ \nu\left(G[p^+, p^-]\right) : p^{\pm} \in \langle \gamma_c \rangle \cdot r^{\pm} \right\}.$$

Observe that this minimum exists because

$$\lim_{n-m \to \pm\infty} \nu\left(G[\gamma_c^n \cdot r^-, \gamma_c^m \cdot r^+]\right) = \infty.$$

Also, let $x_{\gamma_c}^{\pm}, y_{\gamma_c}^{\pm} \in \partial\Gamma$ be the points so that $\{x_{\gamma_c}^{\pm}, y_{\gamma_c}^{\pm}\} \in \mathcal{G}(\widetilde{S})$ correspond to the hyperbolic geodesics in $\widetilde{\mathcal{N}}(\gamma_c^{\pm})$ that contain $p_{\gamma_c}^{\pm}$ (see Figure 4).
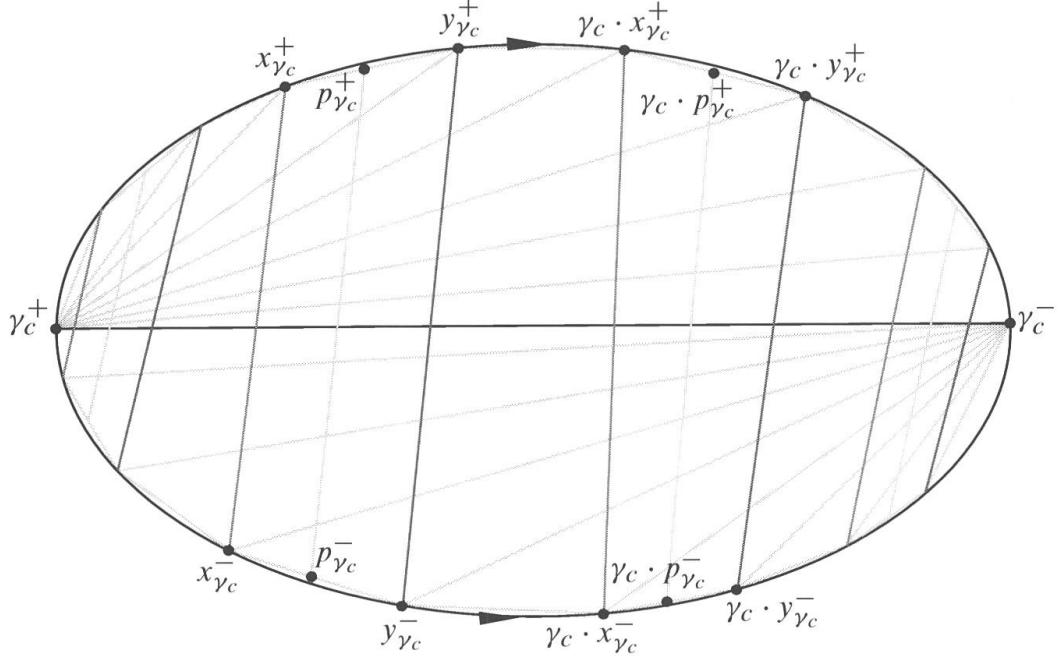
Figure 4. $\{x_{\gamma_c}^-, x_{\gamma_c}^+\}$, $\{y_{\gamma_c}^-, y_{\gamma_c}^+\}$ and $\{p_{\gamma_c}^-, p_{\gamma_c}^+\}$.

By reversing the labeling of $x_{\gamma_c}^+$ and $y_{\gamma_c}^+$ if necessary, we can assume without loss of generality that the hyperbolic geodesics corresponding to $\{x_{\gamma_c}^+, x_{\gamma_c}^-\}$ and $\{y_{\gamma_c}^+, y_{\gamma_c}^-\}$ do not intersect. Then define

$$\widetilde{\mathcal{R}}_1(\gamma_c) := \bigcup_{k \in \mathbb{Z}} \{\gamma_c{}^k \cdot \{x_{\gamma_c}^+, x_{\gamma_c}^-\}, \gamma_c^k \cdot \{y_{\gamma_c}^+, y_{\gamma_c}^-\}\} \subset \mathcal{G}(S), \quad \widetilde{\mathcal{R}}_1(c) := \bigcup_{\eta \in \Gamma} \eta \cdot \widetilde{\mathcal{R}}_1(\gamma_c),$$

and

$$\widetilde{\mathcal{R}}_2(\gamma_c) := \{\gamma_c^k \cdot [p_{\gamma_c}^+, p_{\gamma_c}^-] : k \in \mathbb{Z}\}, \quad \widetilde{\mathcal{R}}_2(c) := \bigcup_{\eta \in \Gamma} \eta \cdot \widetilde{\mathcal{R}}_2(\gamma_c).$$

Note that γ_c induces orderings on $\widetilde{\mathcal{R}}_1(\gamma_c)$ and $\widetilde{\mathcal{R}}_2(\gamma_c)$. Also, for $i = 1, 2$, $\widetilde{\mathcal{R}}_i(\gamma_c)/\langle\gamma_c\rangle = \widetilde{\mathcal{R}}_i(c)/\Gamma$, which consists of two geodesics in $\mathcal{G}(S)$ when $i = 1$ and one geodesic in $\mathcal{G}(S)$ when $i = 2$.

5.2. Binodal edges and winding. Let $[\gamma] \in [\Gamma']$ be the conjugacy class of any non-identity element. We can now define (given all the choices made above) a finite combinatorial description for each conjugacy class $[\gamma] \in [\Gamma']$, which is adapted to ν.

Recall that we have already chosen a hyperbolic structure on S.

Definition 5.2. Let $I \subset \widetilde{S}$ be either a geodesic or geodesic subsegment. Also, for any $\gamma \in \Gamma \setminus \{\mathrm{id}\}$, let $L_\gamma \subset \mathbb{D}$ be the axis of γ.

- Let $\widetilde{\mathcal{A}}(I)$ be the set of geodesics in $\widetilde{\mathcal{Q}}$ that intersect I transversely. A point in $\partial\Gamma$ is a *node* of I if it is the common endpoint of two distinct geodesics in $\widetilde{\mathcal{A}}(I)$. We call a geodesic in $\widetilde{\mathcal{A}}(I)$ *binodal* if both of its endpoints in $\partial\mathbb{D}$ are nodes. Denote the set of binodal edges in $\widetilde{\mathcal{A}}(I)$ by $\widetilde{\mathcal{B}}(I)$.

- In the case when $I = L_\gamma$, observe that $\widetilde{\mathcal{A}}(\gamma) := \widetilde{\mathcal{A}}(L_\gamma)$ and $\widetilde{\mathcal{B}}(\gamma) := \widetilde{\mathcal{B}}(L_\gamma)$ are both $\langle\gamma\rangle$-invariant, so we can define $\mathcal{A}[\gamma] := \widetilde{\mathcal{A}}(\gamma)/\langle\gamma\rangle$ and $\mathcal{B}[\gamma] := \widetilde{\mathcal{B}}(\gamma)/\langle\gamma\rangle$.

Observe that we can think of $\mathcal{A}[\gamma]$ and $\mathcal{B}[\gamma]$ as cyclic sequences of geodesics in S. In that case, they depend only on the conjugacy class of γ, and not on γ itself. Also, $\mathcal{B}[\gamma]$ is finite, and is empty if and only if $L_\gamma \in \widetilde{\mathcal{P}}$. *For the rest of this section, we will assume that $\mathcal{B}[\gamma]$ is non-empty unless stated otherwise.*

The orientation on L_γ induces a natural ordering $\prec$ on $\widetilde{\mathcal{A}}(\gamma)$. Also, since $\widetilde{\mathcal{A}}(\gamma)$ does not contain any of its accumulation points, we can define a bijective successor map suc: $\widetilde{\mathcal{A}}(\gamma) \to \widetilde{\mathcal{A}}(\gamma)$. Moreover, the ordering $\prec$ induces a cyclic order on $\mathcal{A}[\gamma]$, and the successor map suc: $\widetilde{\mathcal{A}}(\gamma) \to \widetilde{\mathcal{A}}(\gamma)$ descends to a successor map suc: $\mathcal{A}[\gamma] \to \mathcal{A}[\gamma]$.

The orientation on S induces an orientation on $\partial\mathbb{D} = \partial\Gamma$. Let $s_0(\gamma)$ and $s_1(\gamma)$ be the two connected components of $\partial\Gamma \setminus \{\gamma^-, \gamma^+\}$, oriented from γ^- to γ^+, so that the orientation on $s_0(\gamma)$ agrees with the orientation on $\partial\Gamma$.
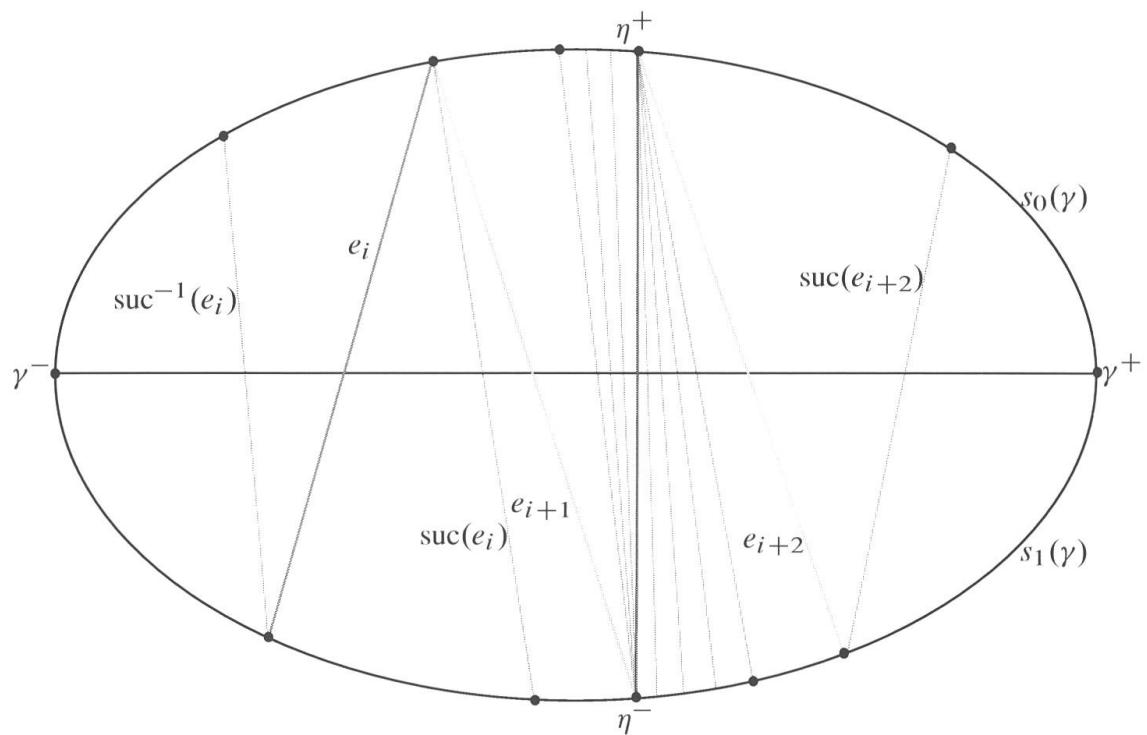

Figure 5. e_i is of S-type. e_{i+1} and e_{i+2} are of Z-type. Notice $\mathrm{suc}(e_i) = \mathrm{suc}^{-1}(e_{i+1})$.

Definition 5.3 (See Figure 5). Let $\{x, y\}$ be an edge in $\widetilde{\mathcal{B}}(\gamma)$ and assume without loss of generality that x lies in $s_0(\gamma)$ and y lies in $s_1(\gamma)$. We say $\{x, y\}$ is

- *Z-type* if $\big(\mathrm{suc}\{x, y\}\big) \cap \{x, y\} = \{y\}$ and $\big(\mathrm{suc}^{-1}\{x, y\}\big) \cap \{x, y\} = \{x\}$,
- *S-type* if $\big(\mathrm{suc}\{x, y\}\big) \cap \{x, y\} = \{x\}$ and $\big(\mathrm{suc}^{-1}\{x, y\}\big) \cap \{x, y\} = \{y\}$.

Let $\widetilde{\mathcal{Z}}(\gamma)$ be the edges in $\widetilde{\mathcal{B}}(\gamma)$ that are Z-type and $\widetilde{\mathcal{S}}(\gamma)$ be the edges in $\widetilde{\mathcal{B}}(\gamma)$ that are S-type. Since $\widetilde{\mathcal{Z}}(\gamma)$ and $\widetilde{\mathcal{S}}(\gamma)$ are $\langle\gamma\rangle$-invariant, we can define $\mathcal{Z}[\gamma] := \widetilde{\mathcal{Z}}(\gamma)/\langle\gamma\rangle$ and $\mathcal{S}[\gamma] := \widetilde{\mathcal{S}}(\gamma)/\langle\gamma\rangle$.

Again, $\mathcal{S}[\gamma]$ and $\mathcal{Z}[\gamma]$, when viewed as a sequence of geodesics in S, depend only on the conjugacy class of γ. Also, note that $\mathcal{Z}[\gamma] \cup \mathcal{S}[\gamma] = \mathcal{B}[\gamma]$, and the cyclic order on $\mathcal{A}[\gamma]$ induces cyclic orders on $\mathcal{Z}[\gamma]$, $\mathcal{S}[\gamma]$ and $\mathcal{B}[\gamma]$. Let e and e' be consecutive geodesics in $\mathcal{B}[\gamma]$ so that e precedes e'. Then the following must hold:

(1) If e and e' are not of the same type, then there are representatives $\widetilde{e}, \widetilde{e}' \in \widetilde{\mathcal{B}}(\gamma)$ of e, e' respectively so that $\widetilde{e}$ and $\widetilde{e}'$ share a common endpoint in $\partial\mathbb{D}$, and $\widetilde{e} \prec \widetilde{e}'$.

(2) If e and e' are of the same type, then there are representatives $\widetilde{e}, \widetilde{e}' \in \widetilde{\mathcal{B}}(\gamma)$ of e, e' respectively so that there is a geodesic in $\widetilde{\mathcal{P}}$ that has a common endpoint with each of $\widetilde{e}$ and $\widetilde{e}'$, and $\widetilde{e} \prec \widetilde{e}'$.

If (1) holds, let $\gamma(\widetilde{e}, \widetilde{e}') \in \Gamma$ be the primitive group element that has the common vertex of $\widetilde{e}$ and $\widetilde{e}'$ as a fixed point, and so that the conjugacy class $[\gamma(\widetilde{e}, \widetilde{e}')]$ corresponds to an oriented closed geodesic in $\mathcal{P}_{\nu,S'}$. On the other hand, if (2) holds, let $\gamma(\widetilde{e}, \widetilde{e}')$ be the element in Γ whose axis is the geodesic in $\widetilde{\mathcal{P}}$ that has common endpoints with $\widetilde{e}$ and $\widetilde{e}'$, and so that the conjugacy class $[\gamma(\widetilde{e}, \widetilde{e}')]$ corresponds to an oriented closed geodesic in $\mathcal{P}_{\nu,S'}$. If $\gamma \in \Gamma'$, the closed geodesic in $\mathcal{P}_{\nu,S'}$ corresponding to $\gamma(\widetilde{e}, \widetilde{e}')$ is in $\mathcal{P}_\nu(S')$.

Notation 5.4. For $i = 1, 2$, let $t_i(e, e') = t_{i,\gamma}(e, e')$ be the signed number of edges in $\widetilde{\mathcal{R}}_i\big(\gamma(\widetilde{e}, \widetilde{e}')\big)$ that intersect L_γ. Here, the sign is positive if the orderings on these edges induced by $\gamma(\widetilde{e}, \widetilde{e}')$ and by γ agree, and is negative otherwise.

The quantities $t_i(e, e')$ for $i = 1, 2$ do not depend on the choice of $\widetilde{e}$ and $\widetilde{e}'$. Also, they do not depend on the choice of $\gamma \in [\gamma]$ in the following sense: if $\overline{\gamma} = \eta\gamma\eta^{-1}$ for some $\eta \in \Gamma'$, then $\eta \cdot e$ and $\eta \cdot e'$ are consecutive elements in $\mathcal{B}[\overline{\gamma}]$, and $t_{i,\gamma}(e, e') = t_{i,\overline{\gamma}}(\eta \cdot e, \eta \cdot e')$.

Notation 5.5. Let $[p, q] \subset \overline{\Omega}_{\gamma_c}$ be a geodesic segment that intersects the geodesics in $\widetilde{\mathcal{R}}_1(\gamma_c) \cup \widetilde{\mathcal{R}}_2(\gamma_c)$ transversely. For $i = 1, 2$, let $w_i[p, q]$ denote the number of edges in $\widetilde{\mathcal{R}}_i(\gamma_c)$ that intersect $[p, q]$ respectively.

It is clear that $[\mathrm{suc}^{-1}(\widetilde{e}) \cap L_\gamma, \mathrm{suc}(\widetilde{e}') \cap L_\gamma] \subset \overline{\Omega}_{\gamma(\widetilde{e},\widetilde{e}')}$, and that $|t_i(e, e')| = w_i[\mathrm{suc}^{-1}(\widetilde{e}) \cap L_\gamma, \mathrm{suc}(\widetilde{e}') \cap L_\gamma]$.

Cyclically enumerate $\mathcal{B}[\gamma] = \{e_{m+1} = e_1, e_2, \ldots, e_m\}$, and for each $i = 1, \ldots, m$, let T_i be the type (Z or S) of e_i. Then define the cyclic sequence of tuples

$$\psi_{\mathcal{P}_{\nu,S'}}[\gamma] = \psi[\gamma] := \big\{(\mathrm{suc}^{-1}(e_i), e_i, \mathrm{suc}(e_i), T_i, t_1(e_i, e_{i+1}))\big\}_{i=1}^m.$$

This is the combinatorial description of $[\gamma] \in [\Gamma]$ mentioned at the start of the section.
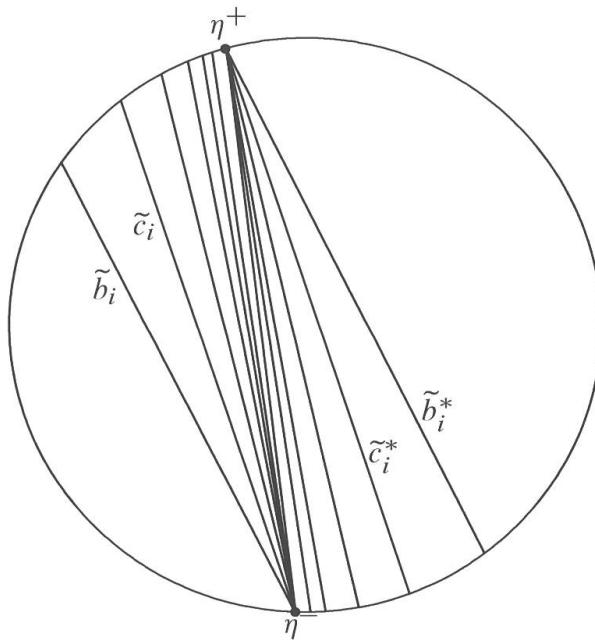
Figure 6. $\widetilde{b}_i, \widetilde{c}_i, \widetilde{b}_i^*$ and $\widetilde{c}_i^*$.

Let Ψ be the collection of cyclic sequences of the form $\{(a_i, b_i, c_i, T_i, t_i)\}_{i=1}^m$, where a_i, b_i, c_i are the three distinct edges in $\mathcal{Q}_j$ for some j, T_i is the symbol S or Z, and $t_i \in \mathbb{Z}$. For any term $\{a_i, b_i, c_i, T_i, t_i\}$ of the sequence $\{(a_i, b_i, c_i, T_i, t_i)\}_{i=1}^m \in \Psi$, let

$$T_i^* := \begin{cases} \text{S} & \text{if } T_i = \text{Z}, \\ \text{Z} & \text{if } T_i = \text{S}. \end{cases}$$

Also, let $\widetilde{b}_i$ and $\widetilde{c}_i$ be lifts of b_i and c_i respectively that share a common endpoint in $\partial\Gamma$, and let $\eta \in \Gamma$ be the group element whose repelling fixed point is this common endpoint. Then there are exactly two geodesics b_i^* and c_i^* in $\mathcal{Q}$ with lifts $\widetilde{b}_i^*$ and $\widetilde{c}_i^*$ in $\mathbb{D}$ respectively that have η^+ as a common endpoint. Let a_i^* be the edge in $\mathcal{Q}$ so that $\{a_i^*, b_i^*, c_i^*\} = \mathcal{Q}_j$ for some j (see Figure 6).

Definition 5.6. We say a sequence $\{(a_i, b_i, c_i, T_i, t_i)\}_{i=1}^m$ in Ψ is *admissible* if for all $i = 1, \ldots, m$, $(a_{i+1}, b_{i+1}, c_{i+1}, T_{i+1})$ is one of the following:

$$(b_i, c_i, a_i, T_i^*), (c_i, b_i, a_i, T_i^*), (b_i^*, c_i^*, a_i^*, T_i), (c_i^*, b_i^*, a_i^*, T_i).$$

(Notice that the last two cases correspond to γ crossing the pants curve $[\![\eta]\!]$.) Let Ψ' denote the set of admissible sequences in Ψ.

Observe that ψ can be viewed as a map from $[\Gamma]$ to Ψ'. The most important property of ψ is its injectivity, which was previously proven by the second author [37, Proposition 4.5].

Proposition 5.7. *Let γ_0, γ_1 be elements in Γ'. Then $\psi[\gamma_0] = \psi[\gamma_1]$ if and only if $[\gamma_0] = [\gamma_1]$.*

Notation 5.8.

- For any cyclic sequence $\sigma = \{(a_i, b_i, c_i, T_i, t_i)\}_{i=1}^m \in \Psi$, let $B(\sigma) := m$ and let $W_1(\sigma) := \sum_{i=1}^m |t_i|$.

- If $c = [\![\gamma]\!] \in \mathcal{CG}(S')$, let

$$p(c) := \sum_{d \in \mathcal{P}_\nu(S')} i(c, d), \quad b(c) := |\mathcal{B}[\gamma]|,$$

and for $i = 1, 2$, let

$$w_i(c) := \sum_{j=1}^m |t_i(e_j, e_{j+1})| = \sum_{j=1}^m w_i\big[\mathrm{suc}^{-1}(e_j) \cap L_\gamma, \mathrm{suc}(e_{j+1}) \cap L_\gamma\big].$$

Note that $p(c)$, $b(c)$, $w_1(c)$ and $w_2(c)$ are well-defined as they do not depend on the orientation on c induced by $[\gamma]$. Also, note that $b(c) = B(\psi[\gamma])$ and $w_1(c) = W_1(\psi[\gamma])$. Informally, $p(c)$ is the number of times c cuts across pants curves, $b(c)$ is the number of times c crosses a binodal edge in $\mathcal{Q}$, and $w_1(c)$ and $w_2(c)$ are two different ways of measuring how many times c "winds around" collar neighborhoods of the curves in $\mathcal{P}_\nu(S')$.

The advantage of $w_1(c)$ over $w_2(c)$ is that $w_1(c)$ can be read off the combinatorial description $\psi(\gamma)$. On the other hand, we will later obtain a lower bound for $i(c, \nu)$ in terms of $w_2(c)$. In the following lemma, we make the relationship between $w_1(c)$ and $w_2(c)$ explicit.

Lemma 5.9. *Let $\gamma \in \Gamma'$ and let $c = [\![\gamma]\!] \in \mathcal{CG}(S')$. Then*

$$\frac{1}{2}w_1(c) - b(c) \leq w_2(c) \leq \frac{1}{2}w_1(c) + b(c).$$

Proof. First, observe that for any consecutive pair $e, e' \in \mathcal{B}[\gamma]$ with e preceding e', we have

$$\frac{1}{2}|t_1(e, e')| - 1 \leq |t_2(e, e')| \leq \frac{1}{2}|t_1(e, e')| + 1.$$

Summing the above inequality over all consecutive pairs in $\mathcal{B}[\gamma]$ yields the required inequality. $\qquad \square$

6. Lengths and geodesic currents

In this section, we will prove some inequalities about lengths of closed geodesics which depend on their intersections with a ν-minimal pants decomposition $\mathcal{P}_\nu(S')$ and the corresponding ideal triangulation $\mathcal{T}_{\nu,S'}$ as defined in Section 5. *For the rest of this section, fix a period minimizing geodesic current $\nu \in \mathcal{CG}(S)$, a hyperbolic*

structure Σ on S, and an essential subsurface S' of S. The goal of this section is to prove Theorem 6.8. For any $c \in \mathcal{CG}(S')$, this theorem gives a lower bound of $i(v, c)$ in terms of the v-panted systole length, the v-systole length, and the combinatorial description $b(c)$ and $w_2(c)$ defined in Section 5.

6.1. Length lower bounds: intersection with pants curves.

We begin first by finding a lower bound for $i(c, v)$ in terms of the number of times $c \in \mathcal{CG}(S')$ intersects $\mathcal{P}_v(S')$. To do so, we define the following quantity.

Definition 6.1. Let $\mathcal{P}_v(S')$ be a v-minimal pants decomposition of S'. Define the *v-panted systole length* to be

$$K_v(S') := \min \left\{ i(c, v) : c \in \mathcal{CG}(S') \text{ is not a multiple of a geodesic in } \mathcal{P}_v(S') \right\}.$$

Lemma 6.2. *The v-panted systole length does not depend on the choice of a minimal pants decomposition $\mathcal{P}_v(S')$. Namely,*

$$K_v(S') = \min \left\{ i(c, v) : c \in \mathcal{CG}(S') \text{ is not a multiple of a geodesic in } \overline{\mathcal{P}_v(S')} \right\}.$$

for any minimal pants decomposition $\overline{\mathcal{P}_v(S')}$.

Proof. Assume there exists a minimal pants decomposition $\overline{\mathcal{P}_v(S')}$ such that

$$\overline{K_v(S')} := \min \left\{ i(c, v) : c \in \mathcal{CG}(S') \text{ is not a multiple of a geodesic in } \overline{\mathcal{P}_v(S')} \right\}$$

is greater or equal to $K_v(S')$. We claim that this implies $K_v(S') = \overline{K_v(S')}$. Let $c_0 = [\![\gamma_0]\!] \in \mathcal{CG}(S')$ so that c_0 is not a multiple of a geodesic in $\mathcal{P}_v(S')$, and $i(c_0, v) = K_v(S')$. The minimality of $K_v(S')$ implies that c_0 is primitive.

If c_0 is not a simple closed geodesic in $\overline{\mathcal{P}_v(S')}$, then we are done because

$$\overline{K_v(S')} \leq i(c_0, v) = K_v(S') \leq \overline{K_v(S')}.$$

On the other hand, if $c_0 \in \overline{\mathcal{P}_v(S')}$, then the fact that c_0 is simple implies that there are closed geodesics $c \in \mathcal{P}_v(S')$ so that $i(c, c_0) \neq 0$. Let $c_1 \in \mathcal{P}_v(S')$ be such a closed geodesic so that $i(c_1, v)$ is minimal, and observe that $i(c_1, v) \leq i(c_0, v)$ by the definition of a v-minimal pants decomposition. Also, since $i(c_0, c_1) \neq 0$, we have $c_1 \notin \overline{\mathcal{P}_v(S')}$, so $\overline{K_v(S')} \leq i(c_1, v)$. Therefore,

$$\overline{K_v(S')} \leq i(c_1, v) \leq i(c_0, v) = K_v(S') \leq \overline{K_v(S')}.$$

One finishes the proof by reversing the roles of $\mathcal{P}_v(S')$ and $\overline{\mathcal{P}_v(S')}$ if $\overline{K_v(S')} \leq K_v(S')$. $\square$

With the notion of a panted systole length, we have the following lemma.

Lemma 6.3. *Let c be a simple v-interior systole of S', and let $p, q \in \widetilde{S'} \subset \mathbb{D}$ be points such that the interval $(p, q]$ intersects $\Pi^{-1}(c)$ transversely. Then*

$$v\big(G(p, q]\big) \geq \left(\big|[p, q] \cap \Pi^{-1}(c)\big| - 2\right) \cdot \frac{K_v(S')}{10},$$

where $G(p, q] \subset \mathcal{G}(S)$ is the set of geodesics defined in Notation 4.3.

Proof. First, observe that since $[p, q] \subset \widetilde{S'} \subset \mathbb{D}$ is compact, $[p, q] \cap \Pi^{-1}(c)$ is finite. Also, if $[p, q] \cap \Pi^{-1}(c) = 0, 1$ or 2, then the desired inequality clearly holds. Thus, we will assume for the rest of this proof that $|[p, q] \cap \Pi^{-1}(c)| \geq 3$. Let $p_1, p_2, \ldots, p_k$ be the points in $[p, q] \cap \Pi^{-1}(c)$ in that order along $[p, q]$, where $k = |[p, q] \cap \Pi^{-1}(c)|$. For any $j = 1, \ldots, k$, let $\gamma_j \in \Gamma'$ denote a group element so that

- $[\![\gamma_j]\!] = c \in \mathcal{CG}(S')$,

- the axis L_j of γ_j contains p_j.
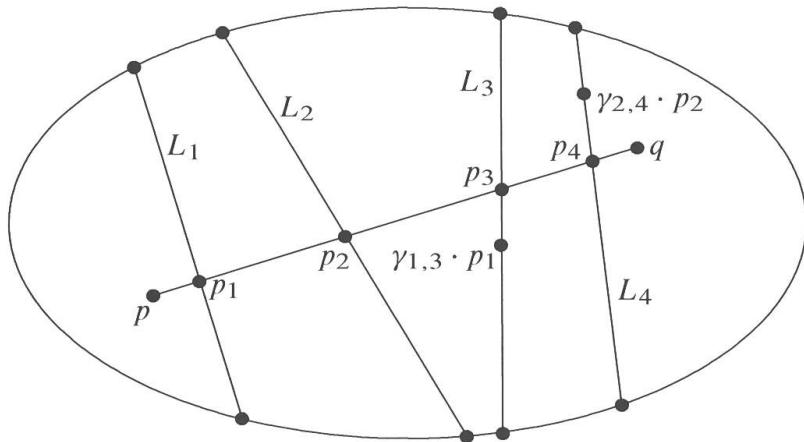
Figure 7. Case 1 of proof of Lemma 6.3.

The proof will proceed in two cases from here.

Case 1: $i(c, v) \geq 2K_v(S')/5$. Then for $j = 1, \ldots, k - 2$, let $\gamma_{j,j+2} \in \Gamma'$ be a group element so that

- $\gamma_{j,j+2} \cdot L_j = L_{j+2}$,

- $v\big(G(\gamma_{j,j+2} \cdot p_j, p_{j+2}]\big) = \min\big\{v\big(G(\gamma \cdot p_j, p_{j+2}]\big) : \gamma \in \Gamma, \gamma \cdot L_j = L_{j+2}\big\}$,

and let $c_{j,j+2} \in \mathcal{CG}(S)$ be the closed geodesic such that $[\![\gamma_{j,j+2}]\!] = c_{j,j+2}$ (see Figure 7). Note that $c_{j,j+2}$ is not a multiple of a curve in $\mathcal{P}_v(S')$ because it has positive geometric intersection number with c, so $i(c_{j,j+2}, v) \geq i(c, v)$. By (1) of

Lemma 4.4, we have $v(G(\gamma_{j+2} \cdot r, r]) = i(c, v)$ for all $r \in L_{j+2}$. The definition of $\gamma_{j,j+2}$ implies that

$$v\big(G[\gamma_{j,j+2} \cdot p_j, p_{j+2})\big) \leq \frac{1}{2} i(c, v)$$

for all $j = 1, \ldots, k - 2$. Then by (2) of Lemma 4.4, we have that

$$
\begin{aligned}
v\big(G(p, q]\big) &\geq \frac{1}{2} \cdot \sum_{j=1}^{k-2} v\big(G(p_j, p_{j+2}]\big) \\
&\geq \frac{1}{2} \cdot \sum_{j=1}^{k-2} \big(v\big(G(p_j, \gamma_{j,j+2} \cdot p_j]\big) - v\big(G[\gamma_{j,j+2} \cdot p_j, p_{j+2})\big)\big) \\
&\geq \frac{1}{2} \cdot \sum_{j=1}^{k-2} \big(i(c_{j,j+2}, v) - \frac{1}{2} i(c, v)\big) \\
&\geq (k-2) \cdot \frac{i(c, v)}{4} \geq (k-2) \cdot \frac{K_v(S')}{10}.
\end{aligned}
$$

Case 2: $i(c, v) < 2K_v(S')/5$. Let $\mathcal{P}_v(S')$ be a v-minimal pants decomposition of S' that contains c. For any $j = 1, \ldots, k-1$, let $\gamma_{j,j+1} := \gamma_j \cdot \gamma_{j+1} \in \Gamma'$ and let $c_{j,j+1} \in \mathcal{CG}(S')$ be the closed geodesic such that $[\![\gamma_{j,j+1}]\!] = c_{j,j+1}$. Note that $c_{j,j+1}$ is not a multiple of a curve in $\mathcal{P}_v(S')$ because $\gamma_j \neq \gamma_{j+1}$ and $c_{j,j+1}$ has positive geometric self-intersection number, so $i(c_{j,j+1}, v) \geq K_v(S')$. Thus, by Lemma 4.4,

$$
\begin{aligned}
v\big(G(p, q]\big) & \\
&\geq \frac{1}{2} \cdot \big(v\big(G(p_1, p_k]\big) + v\big(G(p_k, p_1]\big)\big) \\
&= \frac{1}{2} \cdot \sum_{j=1}^{k-1} \big(v\big(G(\gamma_j^{-1} \cdot p_j, p_j]\big) + v\big(G(p_j, p_{j+1}]\big) - v\big(G(\gamma_j^{-1} \cdot p_j, p_j]\big)\big) \\
&\quad + \frac{1}{2} \cdot \sum_{j=1}^{k-1} \Big(v\big(G(p_{j+1}, \gamma_{j+1} \cdot p_{j+1}]\big) + v\big(G(\gamma_{j+1} \cdot p_{j+1}, \gamma_{j+1} \cdot p_j]\big) \\
&\hspace{6cm} - v\big(G(p_{j+1}, \gamma_{j+1} \cdot p_{j+1}]\big)\Big) \\
&\geq \frac{1}{2} \cdot \sum_{j=1}^{k-1} \big(v\big(G(\gamma_j^{-1} \cdot p_j, \gamma_{j+1} \cdot p_j]\big) - 2i(c, v)\big) \\
&= \frac{1}{2} \cdot \sum_{j=1}^{k-1} \big(v\big(G(p_j, \gamma_j \gamma_{j+1} \cdot p_j]\big) - 2i(c, v)\big)
\end{aligned}
$$

$$\geq \frac{1}{2} \cdot \sum_{j=1}^{k-1} \big(i(c_{j,j+1}, v) - 2i(c, v) \big)$$

$$\geq (k-2) \cdot \frac{K_v(S')}{10}. \qquad \square$$

As a consequence of the above lemma, we obtain the following corollary.

Corollary 6.4. *Let c be a simple v-interior systole of S'. For any $d \in \mathcal{CG}(S')$,*

$$i(d, v) \geq \big(i(d, c) - 1 \big) \cdot \frac{K_v(S')}{10}.$$

Proof. If $i(d, c) = 0$ or 1, the corollary clearly holds. For the rest of this proof, we will assume that $i(d, c) \geq 2$. Choose a hyperbolic structure Σ on S. Then c and d are realized as closed geodesics in S'. Choose a point $p \in c \cap d$ and a point $\widetilde{p} \in \widetilde{S}$ so that $\Pi(\widetilde{p}) = p$. Let $\gamma_d \in \Gamma'$ be a group element so that $[\![\gamma_d]\!] = d$ and $\widetilde{p}$ lies in the axis L_d of γ_d. Then $|[\widetilde{p}, \gamma_d \cdot \widetilde{p}] \cap \Pi^{-1}(c)| = i(d, c) + 1$. Hence, by (1) of Lemma 4.4 and Lemma 6.3, we have

$$i(d, v) = v \big(G(\widetilde{p}, \gamma_d \cdot \widetilde{p}] \big) \geq \big(i(d, c) - 1 \big) \cdot \frac{K_v(S')}{10}. \qquad \square$$

By applying Lemma 6.3 to all the curves in a v-minimal pants decomposition on S', we can also obtain the following lower bound on $i(c, v)$ in terms of the number of times c intersects the curves in a v-minimal pants decomposition $\mathcal{P}_v(S')$.

Lemma 6.5. *Suppose that $S' \subset S$ is a connected essential subsurface of genus g with n boundary components. Let $\mathcal{P}_v(S') = \{c_1, \dots, c_{3g-3+2n}\}$ so that the boundary components of S' are $c_{3g-3+n+1}, \dots, c_{3g-3+2n}$, and let $c \in \mathcal{CG}(S')$. Then*

$$i(c, v) \geq \left(\sum_{j=1}^{3g-3+n} i(c, c_j) \right) \frac{K_v(S')}{10 \cdot 3^{3g-3+n}}.$$

Proof. Assume without loss of generality that $i(v, c_j) \leq i(v, c_{j+1})$ for all $j = 1, \dots, 3g - 3 + n - 1$. If

$$\sum_{j=1}^{3g-3+n} i(c, c_j) = 0$$

(this has to happen when S' is a pair of pants), the desired inequality holds, so we assume that $\sum_{j=1}^{3g-3+n} i(c, c_j) > 0$ in the rest of this proof.

Let $\gamma \in \Gamma'$ so that $[\![\gamma]\!] = c \in \mathcal{CG}(S')$ and let $\widetilde{p} \in \widetilde{S'} \subset \mathbb{D}$ so that

$$\widetilde{p} \in \Pi^{-1}\left(c \cap \left(\bigcup_{j=1}^{3g-3+n} c_j \right) \right).$$

Then let $m = 3^{3g-3+n}$ and let $\tilde{p} = \tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_k = \gamma^m \cdot \tilde{p}$ be the points in

$$[\tilde{p}, \gamma^m \cdot \tilde{p}] \cap \left(\bigcup_{j=1}^{3g-3+n} \Pi^{-1}(c_j) \right),$$

enumerated so that $\tilde{p}_j \in (\tilde{p}_{j-1}, \tilde{p}_{j+1}]$ for all $j = 1, \ldots, k-1$. Observe that $k = m \cdot \sum_{j=1}^{3g-3+n} i(c, c_j)$.

Choose any $j \in \{0, \ldots, k-m\}$. If we can show that $\nu(G(\tilde{p}_j, \tilde{p}_{j+m}]) \geq \frac{K_\nu(S')}{10}$, then by (1) of Lemma 4.4,

$$\begin{aligned}
i(c, \nu) &= \nu\big(G(\tilde{p}, \gamma \cdot \tilde{p}]\big) \\
&= \frac{1}{m} \cdot \nu\big(G(\tilde{p}, \gamma^m \cdot \tilde{p}]\big) \\
&= \frac{1}{m} \cdot \sum_{j=0}^{\frac{k}{m}-1} \nu\big(G(\tilde{p}_{j\cdot m}, \tilde{p}_{(j+1)\cdot m}]\big) \\
&\geq \frac{1}{m} \frac{k}{m} \frac{K_\nu(S')}{10},
\end{aligned}$$

which proves the lemma.

We will now show that $\nu\big(G(\tilde{p}_j, \tilde{p}_{j+m}]\big) \geq \frac{K_\nu(S')}{10}$ for all $j \in \{0, \ldots, k-m\}$. If the interval $(\tilde{p}_j, \tilde{p}_{j+m}]$ intersects $\Pi^{-1}(c_1)$ at least thrice, then Lemma 6.3 implies that

$$\nu\big(G(\tilde{p}_j, \tilde{p}_{j+m}]\big) \geq \frac{K_\nu(S')}{10},$$

and we are done. (This is necessarily the case if $3g-3+n = 1$.) On the other hand, if $(\tilde{p}_j, \tilde{p}_{j+m}]$ intersects $\Pi^{-1}(c_1)$ at most twice, then by the pigeon hole principle, there is some $j_1 \in \{j, \ldots, j + \frac{2m}{3}\}$ so that $(\tilde{p}_{j_1}, \tilde{p}_{j_1+\frac{m}{3}}]$ does not intersect $\Pi^{-1}(c_1)$. In other words, there is a component S_1 of $S' \setminus c_1$ so that the interval $(\tilde{p}_{j_1}, \tilde{p}_{j_1+\frac{m}{3}}]$ lies in some lift $\widetilde{S}_1 \subset \widetilde{S}$ of the subsurface $S_1 \subset S$. Since $\frac{m}{3} = 3^{3g-3+n-1} \geq 1$, it follows that S_1 cannot be a pair of pants.

If $(\tilde{p}_{j_1}, \tilde{p}_{j_1+\frac{m}{3}}]$ intersects $\Pi^{-1}(c_2)$ at least thrice, then Lemma 6.3 again implies that

$$\nu\big(G(\tilde{p}_j, \tilde{p}_{j+m}]\big) \geq \nu\big(G(\tilde{p}_{j_1}, \tilde{p}_{j_1+\frac{m}{3}}]\big) \geq \frac{K_\nu(S_1)}{10} \geq \frac{K_\nu(S')}{10}.$$

(This is necessarily the case if $3g-3+n = 2$.) Otherwise, $(\tilde{p}_{j_1}, \tilde{p}_{j_1+\frac{m}{3}}]$ intersects $\Pi^{-1}(c_2)$ at most twice, so there must be some $j_2 \in \{j_1, \ldots, j_1 + \frac{2m}{9}\} \subset \{j, \ldots, j + \frac{8m}{9}\}$ with the property that $(\tilde{p}_{j_2}, \tilde{p}_{j_2+\frac{m}{9}}]$ does not intersect $\Pi^{-1}(c_1 \cup c_2)$. Hence, there is a component S_2 of $S' \setminus (c_1 \cup c_2)$ so that $(\tilde{p}_{j_2}, \tilde{p}_{j_2+\frac{m}{9}}]$ lies in some lift $\widetilde{S}_2 \subset \mathbb{D}$ of the subsurface $S_2 \subset S$. As before, S_2 cannot be a pair of pants because $\frac{m}{9} = 3^{3g-3+n-2} \geq 1$.

By iterating this procedure, $3g-3+n-1$ times, we will have either already proven that $v(G(\widetilde{p}_j, \widetilde{p}_{j+m})) \geq \frac{K_v(S')}{10}$, or have some $j_{3g-3+n-1} \in \{j, \ldots, j+m-3\}$ and some component $S_{3g-3+n-1}$ of $S' \setminus (c_1 \cup \cdots \cup c_{3g-3+n-1})$ so that:

- $S_{3g-3+n-1}$ is not a pair of pants;

- $(\widetilde{p}_{j_{3g-3+n-1}}, \widetilde{p}_{j_{3g-3+n-1}+3}]$ lies in some lift $\widetilde{S}_{3g-3+n-1} \subset \mathbb{D}$ of the subsurface $S_{3g-3+n-1} \subset S$.

In this case, the unique simple closed geodesic in $S_{3g-3+n-1}$ is c_{3g-3+n}, and $(\widetilde{p}_{j_{3g-3+n-1}}, \widetilde{p}_{j_{3g-3+n-1}+3}]$ necessarily intersects $\Pi^{-1}(c_{3g-3+n})$ at

$$\widetilde{p}_{j_{3g-3+n-1}+1}, \quad \widetilde{p}_{j_{3g-3+n-1}+2}, \quad \text{and} \quad \widetilde{p}_{j_{3g-3+n-1}+3}.$$

Lemma 6.3 then implies that

$$v\big(G(\widetilde{p}_j, \widetilde{p}_{j+m})\big) \geq v\big(G(\widetilde{p}_{j_{3g-3+n-1}}, \widetilde{p}_{j_{3g-3+n-1}+3}]\big)$$
$$\geq \frac{K_v(S_{3g-3+n-1})}{10} \geq \frac{K_v(S')}{10}. \qquad \square$$

6.2. Length lower bounds: winding and intersection with binodal edges.

In this section, fix a v-minimal pants decomposition $\mathcal{P}_v(S')$. Next, we want a lower bound of $i(c, v)$ in terms of $b(c)$ and $w_2(c)$. To do so, we need the following two technical lemmas. Informally, Lemma 6.6 tells us how much length c has to pick up if it crosses sufficiently many binodal edges. On the other hand, Lemma 6.7 tells us how much length c has to pick up if it "winds around" a lot between binodal edges.

Lemma 6.6. *Let $P \subset S'$ be a pair of pants given by $\mathcal{P}_v(S')$ and let $\widetilde{P} \subset \widetilde{S}'$ be the universal cover of P. Also, let $p, q \in \widetilde{P}$ be points so that $[p, q]$ intersects the geodesics in $\widetilde{Q}$ transversely (if at all). Then*

$$v\big(G(p, q]\big) \geq \max\{|\widetilde{\mathcal{B}}[p, q]| - 8, 0\} \cdot \frac{K_v(P)}{16}.$$

(See Definition 5.2 for definition of $\widetilde{\mathcal{B}}[p, q]$.)

Proof. If $k := |\widetilde{\mathcal{B}}[p, q]| = 1, \ldots, 8$, the desired inequality holds, so we will assume for the rest of this proof that $k \geq 9$. Let $p_1, \ldots, p_k$ be the points along $[p, q]$ that also lie in the geodesics in $\widetilde{\mathcal{B}}[p, q]$, enumerated so that they lie along $[p, q]$ in that order. Suppose that for all $j = 1, \ldots, k-8$, we have $v(G(p_j, p_{j+8}]) \geq \frac{1}{2} K_v(P)$. Then

$$v\big(G(p, q]\big) \geq v\big(G(p_1, p_k]\big)$$
$$\geq \frac{1}{8} \sum_{j=1}^{k-8} v\big(G(p_j, p_{j+8}]\big)$$
$$\geq (k-8) \cdot \frac{K_v(P)}{16}.$$

It is thus sufficient to show that $\nu(G(p_j, p_{j+8})) \geq \frac{1}{2} K_\nu(P)$ for all $j = 1, \ldots, k - 8$. Fix any $j = 1, \ldots, k - 8$. For all $i = 0, \ldots, 8$, let $q_i := p_{j+i}$ and let L_i be the geodesic in $\widetilde{Q}$ that contains q_i. Observe that L_i and L_{i+1} share a common endpoint in $\partial \Gamma$, which is the repelling fixed point of some primitive $\gamma_i \in \Gamma$ so that $[\![\gamma_i]\!] \in \mathcal{CG}(S)$ is a boundary component of S'. Denote this common endpoint by γ_i^-. We will first prove the following claim: there exist $i_1, i_2 \in \{1, \ldots, 6\}$ so that $i_1 \neq i_2$ and

$$\left(\gamma_{i_1} \cdot (q_0, q_8]\right) \cap (q_0, q_8] \neq \emptyset \neq \left(\gamma_{i_2} \cdot (q_0, q_8]\right) \cap (q_0, q_8].$$

This will be done in the following cases.
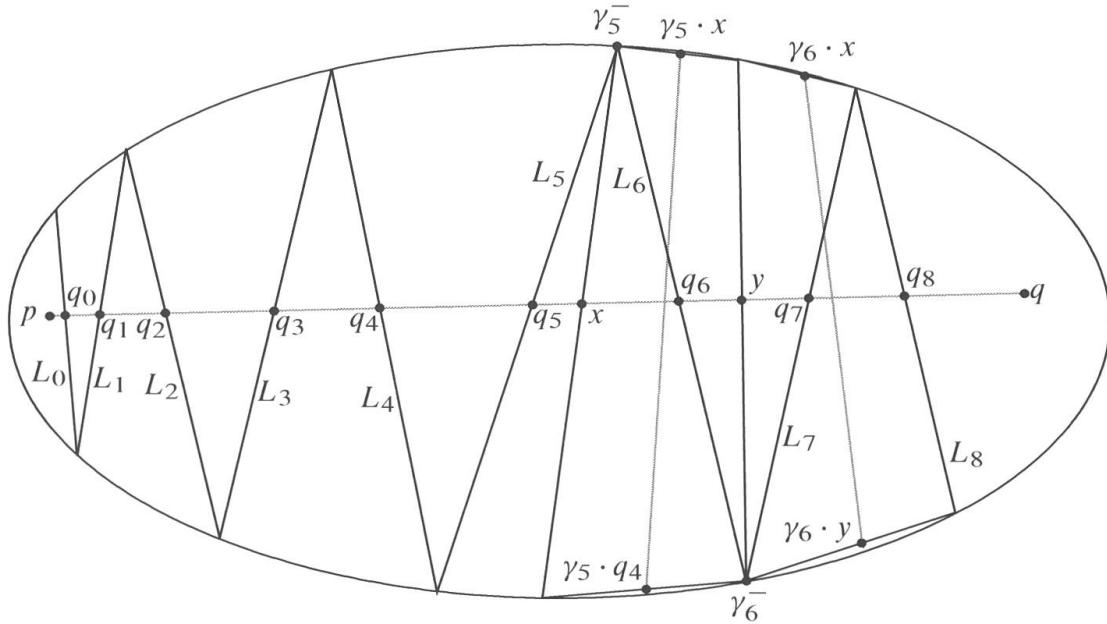
Figure 8. Case 1 of proof of Lemma 6.6, with $k_1 = 5$ and $k_2 = 6$.

Case 1: There is some $k_1, k_2 \in \{1, \ldots, 6\}$ so that $k_1 \neq k_2$ and $\mathrm{suc}(L_{k_t}) \neq L_{k_t+1}$ for $t = 1, 2$. In this case, let $i_t = k_t$. By replacing γ_{i_t} with $\gamma_{i_t}^{-1}$ if necessary, we can assume that $\mathrm{suc}^2(L_{i_t}) = \gamma_{i_t} \cdot L_{i_t}$. Observe that $\gamma_{i_t} \cdot \mathrm{suc}^{-1}(L_{i_t})$ is an edge in $\widetilde{Q}$ that forms a triangle with $\mathrm{suc}(L_{i_t})$ and $\mathrm{suc}^2(L_{i_t})$. On the other hand, $\gamma_{i_t} \cdot \mathrm{suc}(L_{i_t+1})$ is an edge in $\widetilde{Q}$ whose endpoints in $\partial \mathbb{D}$ both lie in $(\gamma_{i_t}^-, \gamma_{i_t+2}^-)_{\gamma_{i_t+1}^-}$ (see Notation 2.26). Thus, $(\gamma_{i_t} \cdot (q_0, q_8]) \cap (q_0, q_8]$ is non-empty (see Figure 8).

Case 2: There is a unique $k \in \{1, \ldots, 6\}$ so that $\mathrm{suc}(L_k) \neq L_{k+1}$. In this case, let $i_1 = k$, let $i_2 = 5$ if $i_1 \leq 3$ and let $i_2 = 2$ if $i_1 \geq 4$. The same argument as Case 1 will show that $(\gamma_{i_1} \cdot (q_0, q_8]) \cap (q_0, q_8]$ is non-empty. We will now prove that $(\gamma_{i_2} \cdot (q_0, q_8]) \cap (q_0, q_8]$ is non-empty when $i_1 \leq 3$; the case when $i_1 \geq 4$ is similar. By replacing γ_5 by γ_5^{-1} if necessary, we can assume that $\gamma_5 \cdot \gamma_4^- = \gamma_7^-$. Observe then that $\gamma_5 \cdot \gamma_3^- = \gamma_6^-$, $\gamma_5 \cdot L_8 \subset [\gamma_5^-, \gamma_7^-]_{\gamma_6^-}$, and $\gamma_5 \cdot \mathrm{suc}^{-1}(L_4) = \{\gamma_6^-, \gamma_8^-\}$ (see Figure 9). In particular, $(\gamma_{i_2} \cdot (q_0, q_8]) \cap (q_0, q_8]$ is non-empty.
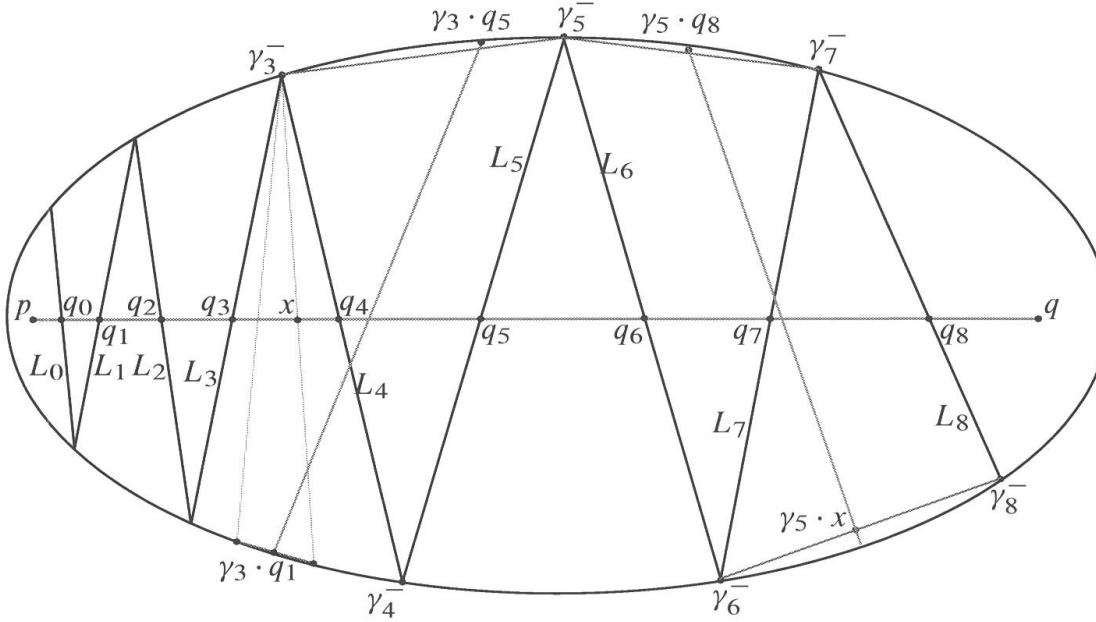
Figure 9. Case 2 of proof of Lemma 6.6, with $i_1 = 3$ and $i_2 = 5$.

Case 3: *For all* $k \in \{1, \ldots, 6\}$, $\mathrm{suc}(L_k) = L_{k+1}$. In this case, let $i_1 = 2$ and let $i_2 = 5$. The argument given in Case 2 proves that $(\gamma_{i_t} \cdot (q_0, q_8]) \cap (q_0, q_8]$ is non-empty for $t = 1, 2$. This concludes the proof of the claim.

Next, we will use the claim to prove the lemma. Assume without loss of generality that $i_1 < i_2$. Let $x_1, x_2 \in (q_0, q_8]$ be points so that $\gamma_{i_t} \cdot x_t \in (q_0, q_8]$. (They exist because of the claim.) By replacing each γ_{i_t} with $\gamma_{i_t}^{-1}$ if necessary, we can assume that $x_t, \gamma_{i_t} \cdot x_t$ lie along $(q_0, q_8]$ in that order. Observe then that x_1 has to lie in (q_{i_1-1}, q_{i_1+1}), $\gamma_{i_1} \cdot x_1$ has to lie in (q_{i_1}, q_{i_1+2}), x_2 has to lie in (q_{i_2-1}, q_{i_2+1}) and $\gamma_{i_2} \cdot x_2$ has to lie in (q_{i_2}, q_{i_2+2}). In particular, $x_1, \gamma_{i_2} \cdot x_2$ lie along $(q_0, q_8]$ in that order, and $\gamma_{i_1} \cdot x_1, x_2 \in (x_1, \gamma_{i_2} \cdot x_2]$. It is clear that $[\![\gamma_{i_2} \cdot \gamma_{i_1}]\!] \in \mathcal{CG}(P)$ is non-peripheral. Hence, Lemma 4.4 implies that

$$
\begin{aligned}
2\nu\big(G(q_0, q_8]\big) &\geq \nu\big(G(x_1, \gamma_{i_2} \cdot x_2]\big) + \nu\big(G(x_2, \gamma_{i_1} \cdot x_1]\big) \\
&= \nu\big(G(x_1, \gamma_{i_2} \cdot x_2]\big) + \nu\big(G(\gamma_{i_2} \cdot x_2, (\gamma_{i_2}\gamma_{i_1}) \cdot x_1]\big) \\
&\geq \nu\big(G(x_1, (\gamma_{i_2}\gamma_{i_1}) \cdot x_1]\big) \\
&\geq i\big([\![\gamma_{i_2} \cdot \gamma_{i_1}]\!], \nu\big) \\
&\geq K_\nu(P). \qquad\qquad\qquad\qquad \square
\end{aligned}
$$

Lemma 6.7. *Let $c \in \mathcal{P}_\nu(S')$, let $\gamma_c \in \Gamma'$ so that $[\![\gamma_c]\!] = c$, and let $p, q \in \overline{\Omega}_{\gamma_c} \subset \mathbb{D}$ so that $[p, q]$ intersects*

$$
\bigcup_{j \in \mathbb{Z}} \gamma_c^j \cdot [p_{\gamma_c}^+, p_{\gamma_c}^-]
$$

transversely. Then

$$v\big(G[p,q]\big) \geq \big(w_2[p,q]-1\big) \cdot \frac{i(c,v)}{2}.$$

(See Section 5.1 for the definition of $p_{\gamma_c}^{\pm}$ and Notation 5.5 for the definition of $w_2[p,q]$.)
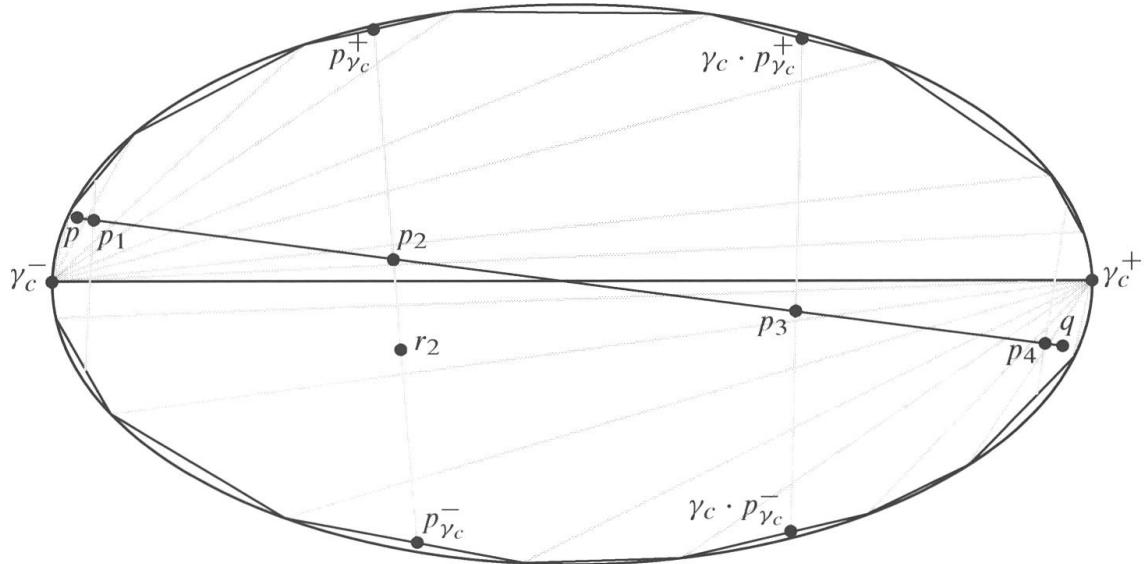
Figure 10. Proof of Lemma 6.7 when $j = 2$.

Proof. Let $k = w_2[p,q]$ and note that if $k = 0, 1$ there is nothing to prove. Therefore, assume $k \geq 2$ and let $p_1, \ldots, p_k$ be the points in

$$[p,q] \cap \left(\bigcup_{j \in \mathbb{Z}} \gamma_c^j \cdot [p_{\gamma_c}^+, p_{\gamma_c}^-] \right)$$

in that order along $[p,q]$. Fix any $j = 1, \ldots, k-1$, let $r_j := \gamma_c^{-1} \cdot p_{j+1}$ and assume without loss of generality that $p_j \in [p_{\gamma_c}^+, p_{\gamma_c}^-]$. Also, assume that $p_{\gamma_c}^+, p_j, r_j, p_{\gamma_c}^-$ lie along $[p_{\gamma_c}^+, p_{\gamma_c}^-]$ in that order; the other case is similar. Then

$$
\begin{aligned}
v\big(G[p_j, r_j]\big) &= v\big(G[p_{\gamma_c}^+, p_{\gamma_c}^-]\big) - v\big(G[p_{\gamma_c}^+, p_j]\big) - v\big(G[r_j, p_{\gamma_c}^-]\big) \\
&\leq v\big(G[p_{\gamma_c}^+, \gamma_c \cdot p_{\gamma_c}^-]\big) - v\big(G[p_{\gamma_c}^+, p_j]\big) - v\big(G[p_{j+1}, \gamma_c \cdot p_{\gamma_c}^-]\big) \\
&\leq v\big(G[p_j, p_{j+1}]\big),
\end{aligned}
$$

where the first inequality above is a consequence of the way $p_{\gamma_c}^+$ and $p_{\gamma_c}^-$ are defined.

By (2) of Lemma 4.4, we have

$$
\begin{aligned}
i(c, v) &\leq v\big(G[p_{j+1}, r_j]\big) \\
&\leq v\big(G[p_{j+1}, p_j]\big) + v\big(G[p_j, r_j]\big) \\
&\leq v\big(G[p_{j+1}, p_j]\big) + v\big(G[p_j, p_{j+1}]\big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
v\big(G[p, q]\big) &\geq \frac{1}{2}\big(v\big(G[p_1, p_k]\big) + v\big(G(p_1, p_k]\big)\big) \\
&= \frac{1}{2}\sum_{j=1}^{k-1}\big(v\big(G[p_j, p_{j+1}]\big) + v\big(G(p_j, p_{j+1}]\big)\big) \\
&\geq (k-1) \cdot \frac{i(c, v)}{2}. \qquad\qquad\square
\end{aligned}
$$

6.3. Length lower bounds: the combinatorial description. Combining the previous lemmas in this section, we can obtain the following lower bound for $i(c, v)$ in terms of the v-panted systole length and the v-systole length.

Theorem 6.8. *Let $S' \subset S$ be a connected essential subsurface of genus g with n boundary components, let*

$$
\bar{K}_v(S') := \frac{K_v(S')}{400 \cdot 3^{3g-3+n} + 96}
$$

and let

$$
\bar{L}_v(S') := \frac{L_v(S')}{400 \cdot 3^{3g-3+n} + 96}.
$$

Then

$$
i(c, v) \geq b(c) \cdot \bar{K}_v(S') + w_1(c) \cdot \bar{L}_v(S').
$$

Proof. By Lemma 6.5, we know that

$$
5 \cdot 3^{3g-3+n} \cdot i(c, v) \geq p(c) \cdot \frac{K_v(S')}{2}. \tag{6.1}
$$

Let $\gamma \in \Gamma'$ so that $[\![\gamma]\!] = c \in \mathcal{CG}(S')$, let $\tilde{q} \in \tilde{S}' \subset \mathbb{D}$ so that

$$
\tilde{q} \in \Pi^{-1}\left(c \cap \left(\bigcup_{j=1}^{3g-3+n} c_j\right)\right),
$$

(recall $\mathcal{P}_v(S') = \{c_1, \ldots, c_{3g-3+2n}\}$, where $c_{3g-3+n+1}, \ldots, c_{3g-3+2n}$ are the boundary components of S') and let $\tilde{q} = \tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_{p(c)} = \gamma \cdot \tilde{q}$ be the points in

$$
[\tilde{q}, \gamma \cdot \tilde{q}] \cap \left(\bigcup_{j=1}^{3g-3+n} \Pi^{-1}(c_j)\right),
$$

enumerated so that $\tilde{q}_j \in (\tilde{q}_{j-1}, \tilde{q}_{j+1})$ for all $j = 1, \ldots, k-1$.

Note that for any pair of pants P given by $\mathcal{P}_\nu(S')$, $K_\nu(P) \geq K_\nu(S')$. Hence, by Lemma 6.6, we have

$$
\begin{aligned}
i(c, \nu) &= \sum_{j=0}^{p(c)-1} \nu\big(G(\tilde{q}_j, \tilde{q}_{j+1}]\big) \\
&\geq \sum_{j=0}^{p(c)-1} \max\big\{\big|\tilde{\mathcal{B}}[\tilde{q}_j, \tilde{q}_{j+1}]\big| - 8, 0\big\} \frac{K_\nu(P)}{16} \\
&\geq \sum_{j=0}^{p(c)-1} \big(\big|\tilde{\mathcal{B}}[\tilde{q}_j, \tilde{q}_{j+1}]\big| - 8\big) \cdot \frac{K_\nu(S')}{16} \\
&\geq b(c) \cdot \frac{K_\nu(S')}{16} - p(c) \cdot \frac{K_\nu(S')}{2}.
\end{aligned}
\tag{6.2}
$$

Adding the inequalities (6.1) and (6.2) then gives

$$
(5 \cdot 3^{3g-3+n} + 1)i(c, \nu) \geq b(c) \cdot \frac{K_\nu(S')}{16}.
\tag{6.3}
$$

Let $k_j := |\tilde{\mathcal{B}}[\tilde{q}_j, \tilde{q}_{j+1}]|$. For each interval $[\tilde{q}_j, \tilde{q}_{j+1}]$, let $\tilde{e}_{j,1}, \ldots, \tilde{e}_{j,k_j}$ be the edges in $\tilde{\mathcal{B}}[\tilde{q}_j, \tilde{q}_{j+1}]$, enumerated so that $\tilde{p}_{j,i} \in (\tilde{p}_{j,i-1}, \tilde{p}_{j,i+1})$ for all $i = 2, \ldots, k_j - 1$, where $\tilde{p}_{j,i} := \tilde{e}_{j,i} \cap [\tilde{q}_j, \tilde{q}_{j+1}]$. We previously observed (see discussion after Notation 5.4) that the interval

$$
\big(\mathrm{suc}^{-1}(\tilde{e}_{j,i}) \cap [\tilde{q}_j, \tilde{q}_{j+1}], \mathrm{suc}(\tilde{e}_{j,i+1}) \cap [\tilde{q}_j, \tilde{q}_{j+1}]\big]
$$

lies in $\Omega_{\gamma(\tilde{e}_{j,i}, \tilde{e}_{j,i+1})}$. Also, it is clear that for every point $r \in [\tilde{q}_j, \tilde{q}_{j+1}]$, there are at most four different values of i so that

$$
r \in \big[\mathrm{suc}^{-1}(\tilde{e}_{j,i}) \cap [\tilde{q}_j, \tilde{q}_{j+1}], \mathrm{suc}(\tilde{e}_{j,i+1}) \cap [\tilde{q}_j, \tilde{q}_{j+1}]\big].
$$

Thus, by Lemma 6.7,

$$
\begin{aligned}
4i(c, \nu) &= \sum_{j=0}^{p(c)-1} 4\nu\big(G(\tilde{q}_j, \tilde{q}_{j+1}]\big) \\
&\geq \sum_{j=0}^{p(c)-1} \sum_{i=1}^{k_j} \nu\big(G\big[\mathrm{suc}^{-1}(\tilde{e}_{j,i}) \cap [\tilde{q}_j, \tilde{q}_{j+1}], \mathrm{suc}(\tilde{e}_{j,i+1}) \cap [\tilde{q}_j, \tilde{q}_{j+1}]\big]\big) \\
&\geq \sum_{j=0}^{p(c)-1} \sum_{i=1}^{k_j} \big(w_2\big[\mathrm{suc}^{-1}(\tilde{e}_{j,i}) \cap [\tilde{q}_j, \tilde{q}_{j+1}], \mathrm{suc}(\tilde{e}_{j,i+1}) \cap [\tilde{q}_j, \tilde{q}_{j+1}]\big] - 1\big) \cdot \frac{L_\nu(S')}{2} \\
&= (w_2(c) - b(c)) \cdot \frac{L_\nu(S')}{2}.
\end{aligned}
$$

We finish the proof by combining the above inequalities to obtain a positive lower bound for $i(c, v)$. To ensure positivity of this lower bound, we need to add the above inequality to a multiple (strictly greater than 16) of inequality (6.3). We choose this multiple to be 20 to improve readability.

$$
\begin{aligned}
(100 \cdot 3^{3g-3+n} + 24)i(c, v) &\geq (w_2(c) - b(c)) \cdot \frac{L_v(S')}{2} + b(c) \cdot \frac{5 \cdot K_v(S')}{4} \\
&\geq w_2(c) \cdot \frac{L_v(S')}{2} + b(c) \cdot \frac{3 \cdot K_v(S')}{4} \\
&\geq \left(\frac{1}{2}w_1(c) - b(c)\right) \cdot \frac{L_v(S')}{2} + b(c) \cdot \frac{3 \cdot K_v(S')}{4} \\
&\geq w_1(c) \cdot \frac{L_v(S')}{4} + b(c) \cdot \frac{K_v(S')}{4},
\end{aligned}
$$

where the third inequality follows from Lemma 5.9. Dividing both sides by

$$
100 \cdot 3^{3g-3+n} + 24
$$

yields the required inequality. $\square$

7. Vanishing of entropy and a systolic inequality

For the rest of this section, let $S' \subset S$ be a connected essential subsurface of genus g with n boundary components. If we choose a period minimizing $v \in \mathcal{C}(S)$, we can associate to S' a quantity which we call the topological entropy.

Definition 7.1. Let $v \in \mathcal{C}(S)$ be a period minimizing geodesic current. The *v-topological entropy* of S' is

$$
h_v(S') := \limsup_{T \to \infty} \frac{1}{T} \log \left|\{c \in \mathcal{CG}(S') : i(c, v) \leq T\}\right|.
$$

When $S' = S$, we will use the notation $h_v := h_v(S)$.

7.1. A systolic inequality.

The goal of this section is to prove Theorem 1.4. The constant C that arises from our proof is $400 \cdot 3^{3g-3+n} + 96$. We will divide the proof of Theorem 1.4 into three lemmas. The first two lemmas give us the first inequality.

Lemma 7.2. *Let $v \in \mathcal{C}(S)$ be a period minimizing geodesic current and $\mathcal{P}_v(S')$ be a v-minimal pants decompositions of S'. Then there exists a pair of pants $P \subset S'$ and a closed geodesic $e \in \mathcal{CG}(P)$ so that*

- *e has a unique self-intersection point p,*
- *$i(e, v) \leq 4K_v(S')$,*

- *the three closed geodesics obtained by performing surgery to e at p are the three boundary components of P.*

Proof. By Lemma 4.6, it is sufficient to construct a primitive, non-simple $\overline{e} \in \mathcal{CG}(S')$ so that $i(\overline{e}, v) \leq 4K_v(S')$.

Let $d \in \mathcal{CG}(S')$ be a closed geodesic that is not a multiple of a curve in $\mathcal{P}_v(S')$, and so that $i(d, v) = K_v(S')$. Note that d is primitive. If d is non-simple, set $\overline{e}$ to be d and we are done. If d is simple, then there is some $c \in \mathcal{P}_v(S')$ that intersects d transversely, so that $i(c, v) \leq i(d, v)$. Choose a hyperbolic structure Σ on S. There exists $\gamma, \eta \in \Gamma'$ so that $[\![\gamma]\!] = c$, $[\![\tau]\!] = d$, and the axes L_τ and L_γ of τ and γ respectively intersect transversely. Let $p \in \widetilde{S}$ be the intersection point of L_τ and L_γ.

By Lemma 4.4, we have that

$$
\begin{aligned}
i\left([\![\tau\gamma\tau\gamma^{-1}]\!], v\right) &\leq v\left(G(\gamma \cdot p, \tau\gamma\tau\gamma^{-1} \cdot (\gamma \cdot p)]\right) \\
&\leq v\left(G(\gamma \cdot p, p]\right) + v\left(G(p, \tau \cdot p]\right) + v\left(G(\tau \cdot p, \tau\gamma \cdot p]\right) \\
&\qquad\qquad\qquad\qquad + v\left(G(\tau\gamma \cdot p, \tau\gamma\tau \cdot p]\right) \\
&= v\left(G(\gamma \cdot p, p]\right) + v\left(G(p, \tau \cdot p]\right) + v\left(G(p, \gamma \cdot p]\right) \\
&\qquad\qquad\qquad\qquad + v\left(G(p, \tau \cdot p]\right) \\
&= 2i(c, v) + 2i(d, v) \\
&\leq 4i(d, v) \\
&= 4K_v(S').
\end{aligned}
$$

It is easy to see that $[\![\tau\gamma\tau\gamma^{-1}]\!]$ is non-simple, so we can set $\overline{e}$ to be $[\![\tau\gamma\tau\gamma^{-1}]\!]$. $\square$

Lemma 7.3. *Let $v \in \mathcal{C}(S)$ be a period minimizing geodesic current. Then*

$$
h_v(S') \geq \frac{\log(2)}{4K_v(S')}.
$$

Proof. Let $\mathcal{P}_v(S')$, P, e and p be as in Lemma 7.2. Let $\gamma_1, \gamma_2, \gamma_3 \in \pi_1(P)$ be primitive elements so that $\gamma_3 \cdot \gamma_2 \cdot \gamma_1 = \mathrm{id}$, and so that the closed geodesics $[\![\gamma_1]\!]$, $[\![\gamma_2]\!]$, $[\![\gamma_3]\!]$ are the boundary components of P. Then e has to be either

$$
[\![\gamma_3^{-1} \cdot \gamma_2]\!], \quad [\![\gamma_2^{-1} \cdot \gamma_1]\!] \quad \text{or} \quad [\![\gamma_1^{-1} \cdot \gamma_3]\!].
$$

Assume without loss of generality that $e = [\![\gamma_2^{-1} \cdot \gamma_1]\!]$. Then

$$
\begin{aligned}
i(e, v) &= v\left(G(p, \gamma_2^{-1}\gamma_1 \cdot p]\right) \\
&= v\left(G(\gamma_2 \cdot p, \gamma_1 \cdot p]\right) \\
&= v\left(G(\gamma_2 \cdot p, p]\right) + v\left(G(p, \gamma_1 \cdot p]\right) \\
&= v\left(G(p, \gamma_2^{-1} \cdot p]\right) + v\left(G(p, \gamma_1^{-1} \cdot p]\right),
\end{aligned}
$$

so $i(e, v) \geq v(G(\gamma_2 \cdot p, p)), v(G(p, \gamma_1 \cdot p)), v(G(p, \gamma_2^{-1} \cdot p))$ and $v(G(p, \gamma_1^{-1} \cdot p))$. Since $\pi_1(P) \subset \Gamma'$ is a free group of rank 2 generated by γ_1 and γ_2, no two distinct elements of the form

$$\gamma_1^{\epsilon_1} \gamma_2^{\delta_1} \cdots \gamma_1^{\epsilon_t} \gamma_2^{\delta_t} \gamma_1^2$$

are conjugate, where $\epsilon_i, \delta_i = \pm 1$.

By Lemma 4.4, we have that for any $f = [\![\gamma_1^{\epsilon_1} \gamma_2^{\delta_1} \cdots \gamma_1^{\epsilon_t} \gamma_2^{\delta_t} \gamma_1^2]\!] \in \mathcal{CG}(S')$,

$$
\begin{aligned}
i(f, v) &\leq v\big(G(p, \gamma_1^{\epsilon_1} \gamma_2^{\delta_1} \cdots \gamma_1^{\epsilon_t} \gamma_2^{\delta_t} \gamma_1^2 \cdot p)\big) \\
&\leq v\big(G(p, \gamma_1^{\epsilon_1} \cdot p)\big) + v\big(G(\gamma_1^{\epsilon_1} \cdot p, \gamma_1^{\epsilon_1} \gamma_2^{\delta_1} \cdot p)\big) + \cdots \\
&\quad \cdots + v\big(G(\gamma_1^{\epsilon_1} \gamma_2^{\delta_1} \cdots \gamma_1^{\epsilon_t} \gamma_2^{\delta_t} \cdot p, \gamma_1^{\epsilon_1} \gamma_2^{\delta_1} \cdots \gamma_1^{\epsilon_t} \gamma_2^{\delta_t} \gamma_1^2 \cdot p)\big) \\
&= v\big(G(p, \gamma_1^{\epsilon_1} \cdot p)\big) + v\big(G(p, \gamma_2^{\delta_1} \cdot p)\big) + \cdots + v\big(G(p, \gamma_2^{\delta_t} \cdot p)\big) \\
&\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad + 2v\big(G(p, \gamma_1 \cdot p)\big) \\
&\leq (2t + 2)i(e, v) \\
&\leq 8(t + 1)K_v(S').
\end{aligned}
$$

This means that

$$
\begin{aligned}
&\Big|\{[\gamma] \in [\Gamma' \setminus \{\mathrm{id}\}] : i([\![\gamma]\!], v) \leq T\}\Big| \\
&\geq \Big|\Big\{[\gamma_1^{\epsilon_1} \gamma_2^{\delta_1} \cdots \gamma_1^{\epsilon_t} \gamma_2^{\delta_t} \gamma_1^2] \in [\pi_1(P) \setminus \{\mathrm{id}\}] : \epsilon_i, \delta_i \in \{-1, 1\} \text{ and } t \leq \frac{T}{8K_v(S')} - 1\Big\}\Big| \\
&\geq 4^{\lfloor \frac{T}{8K_v(S')} \rfloor - 1},
\end{aligned}
$$

so $h_v(S') \geq \frac{\log(2)}{4K_v(S')}$. $\square$

Now, we finish the proof of Theorem 1.4 by proving the second inequality.

Lemma 7.4. *There is a constant $C \in \mathbb{R}^+$ which depends only on the topology of S', so that for any period minimizing $v \in \mathcal{C}(S)$, we have*

$$h_v(S')K_v(S') \leq C \cdot \left(\log(4) + 1 + \log\left(1 + \frac{1}{x_0}\right)\right),$$

where x_0 is the unique positive solution to the equation $(1 + x)^{\lceil \frac{K_v(S')}{L_v(S')} - 1 \rceil} x = 1$.

Proof. Simplify notation by denoting $\bar{K}_v(S')$ and $\bar{L}_v(S')$ defined in the statement of Theorem 6.8 simply by K and L respectively. Choose a v-minimal pants

decomposition $\mathcal{P}_\nu(S')$. By Theorem 6.8 and Proposition 5.7, we have

$$
\left|\left\{c \in \mathcal{CG}(S') : i(c, \nu) \leq T\right\}\right| \leq \left|\left\{c \in \mathcal{CG}(S') : b(c) \cdot K + w_1(c) \cdot L \leq T\right\}\right|
$$

$$
\leq \frac{1}{2}\left|\left\{\sigma \in \Psi' : B(\sigma) \cdot K + W_1(\sigma) \cdot L \leq T\right\}\right|
$$

$$
= \frac{1}{2}\sum_{i=1}^{\left\lfloor\frac{T}{K}\right\rfloor}\left|\left\{\sigma \in \Psi' : B(\sigma) = i, W_1(\sigma) \leq \left\lfloor\frac{T - Ki}{L}\right\rfloor\right\}\right|.
$$

(See Definition 5.6 for the definition of Ψ'.)

If $\sigma = \{(u_i, v_i, w_i, T_i, t_i)\}_{i=1}^m$, let σ' be the cyclic sequence

$$
\sigma' := \{(u_i, v_i, w_i, T_i)\}_{i=1}^m.
$$

For any $e \in \mathcal{Q}$, let $e', e'' \in \mathcal{Q}$ be the geodesics so that $\{e, e', e''\} = \mathcal{Q}_j$ for some j. If $v_i = e$, then there are four possibilities for (u_i, v_i, w_i, T_i), namely

$$
(e', e, e'', S), (e', e, e'', Z), (e'', e, e', S), (e'', e, e', Z).
$$

Since $|\{e \in \mathcal{Q} : e \subset S'\}| = 6g - 6 + 3n$, we see from the definition of Ψ' that

$$
\left|\left\{\sigma' : \sigma \in \Psi', B(\sigma) = i\right\}\right| \leq \frac{(24g - 24 + 12n) \cdot 4^{i-1}}{i}.
$$

Hence,

$$
\left|\left\{\sigma : B(\sigma) = i, W_1(\sigma) \leq \left\lfloor\frac{T - Ki}{L}\right\rfloor\right\}\right| \leq \frac{(24g - 24 + 12n) \cdot 4^{i-1}}{i} \cdot \binom{\left\lfloor\frac{T-Ki}{L}\right\rfloor + i}{i},
$$

which implies that for $T \gg K$,

$$
\left|\left\{c \in \mathcal{CG}(S') : i(c, \nu) \leq T\right\}\right| \leq \frac{1}{2}\sum_{i=1}^{\left\lfloor\frac{T}{K}\right\rfloor} \frac{(24g - 24 + 12n) \cdot 4^{i-1}}{i} \cdot \binom{\left\lfloor\frac{T-Ki}{L}\right\rfloor + i}{i}
$$

$$
\leq \frac{1}{2}(24g - 24 + 12n) \cdot 4^{\left\lfloor\frac{T}{K}\right\rfloor - 1} \cdot \binom{\left\lfloor\frac{T-KQ}{L}\right\rfloor + Q}{Q},
$$

where $Q = Q(T, K, L) \in \{0, \ldots, \left\lfloor\frac{T}{K}\right\rfloor\}$ is the integer so that for all $i \in \{0, \ldots, \left\lfloor\frac{T}{K}\right\rfloor\}$,

$$
\binom{\left\lfloor\frac{T-KQ}{L}\right\rfloor + Q}{Q} \geq \binom{\left\lfloor\frac{T-Ki}{L}\right\rfloor + i}{i}.
$$

As a consequence, we have

$$h_v(S') \cdot K \leq \log(4) + \limsup_{T \to \infty} \frac{K}{T} \log \left(\frac{\left\lfloor \frac{T - KQ}{L} \right\rfloor + Q}{Q} \right)$$

$$\leq \log(4) + 1 + \log \left(1 + \frac{1}{x_0} \right),$$

where the last inequality is a computation that we do in Appendix B (see Proposition B.1). Since $K = \frac{K_v(S')}{400 \cdot 3^{3g-3+n} + 96}$, we have proven the lemma. $\qquad \square$

7.2. Corollaries of the systolic inequality. Theorem 1.4 has several interesting corollaries, which we will now explain. The first is a slight simplification of the inequality in Theorem 1.4 from which we can deduce all our other corollaries.

Corollary 7.5. *There is a constant $C \in \mathbb{R}^+$ which depends only on the topology of S', so that for any period minimizing $v \in \mathcal{C}(S)$, we have*

$$\frac{1}{4} \log(2) \leq h_v(S') K_v(S') \leq C \cdot \left(\log(4) + 1 + \log \left(1 + \frac{\sqrt{5} + 1}{2} \cdot \frac{K_v(S')}{L_v(S')} \right) \right).$$

Proof. Let $a := \left\lceil \frac{K_v(S')}{L_v(S')} - 1 \right\rceil$, and consider the function $f_a \colon [0, \infty) \to \mathbb{R}$ defined by $f_a(x) = (1+x)^a \cdot x$. Observe that f_a is increasing, $f_a(0) = 0$, and $\lim_{x \to \infty} f_a(x) = \infty$. Also, let $x_0 = x_0(a)$ be the unique point in $[0, 1)$ so that $f_a(x_0) = 1$. It is sufficient to show that for all $a \geq 0$,

$$\frac{1}{x_0(a)} \leq \frac{\sqrt{5} + 1}{2} \cdot \frac{K_v(S')}{L_v(S')}.$$

First, consider the case when $a = 0$. Then $\frac{K_v(S')}{L_v(S')} = 1$ and $x_0(0) = 1$. We see immediately that in this case, the required inequality holds.

Next, consider the case when $a \geq 1$. The equation $f_a(x_0) = 1$ can be rearranged as

$$a \cdot x_0 = -\frac{x_0 \log(x_0)}{\log(1 + x_0)}.$$

Since the function $g \colon (0, 1) \to \mathbb{R}$ given by $g(x) = -\frac{x \log(x)}{\log(1+x)}$ is positive and strictly decreasing, we see that $a \cdot x_0(a)$ is minimized over all $a \geq 1$ when $x_0(a)$ is maximized. From the definition of f_a, it is clear that $x_0(a)$ is strictly decreasing with a, so $a \cdot x_0(a)$ is minimized over all $a \geq 1$ when $a = 1$. It is easy to compute that $x_0(1) = \frac{\sqrt{5}-1}{2}$, so $a \cdot x_0(a) \geq \frac{\sqrt{5}-1}{2}$ for all $a \geq 1$. Hence,

$$\frac{1}{x_0(a)} \leq \frac{\sqrt{5} + 1}{2} \cdot \left\lceil \frac{K_v(S')}{L_v(S')} - 1 \right\rceil \leq \frac{\sqrt{5} + 1}{2} \cdot \frac{K_v(S')}{L_v(S')}. \qquad \square$$

Using Corollary 7.5, we have the following universal upper bound on the systole length renormalized by the entropy.

Corollary 7.6. *There is a constant* $C \in \mathbb{R}^+$ *which depends only on the topology of* S', *so that for any period minimizing* $v \in \mathcal{C}(S)$, *we have*

$$h_v(S')L_v(S') \leq C.$$

Proof. By Corollary 7.5, we see that there is a constant C' depending only on the topology of S', so that for any period minimizing $v \in \mathcal{C}(S)$,

$$h_v(S')L_v(S')$$
$$\leq C' \cdot \left((\log(4) + 1) \cdot \frac{L_v(S')}{K_v(S')} + \frac{L_v(S')}{K_v(S')} \cdot \log \left(1 + \frac{\sqrt{5}+1}{2} \cdot \frac{K_v(S')}{L_v(S')} \right) \right)$$
$$\leq C' \cdot \left(\log(4) + 1 + \frac{\sqrt{5}+1}{2} \right) =: C,$$

where the last inequality holds because $x \log(1 + \frac{k}{x}) \leq k$ for all $x > 0$ and $k > 0$. $\square$

Corollary 7.6 together with Theorem 1.1 proves Corollary 1.5. Also, given any negatively curved Riemannian metric m' on S' with geodesic boundary, one can always find a negatively curved Riemannian metric m on S whose restriction to S' is m'. It is then known (see Otal [29, Proposition 3]) that the Lebesgue–Liouville current v_m of m has the property that $i(v_m, c) = \ell_m(c)$, where $\ell_m : \mathcal{CG}(S) \to \mathbb{R}$ is the length function induced by m. This fact combined with Corollary 7.6 then gives us Corollary 1.6.

Corollary 7.5 also gives us a criterion that determines when the topological entropy of a sequence of geodesic currents in the "ϵ-thick" part of $\mathcal{C}(S)$ converges to 0. Before we state the corollary, we first define what the "ϵ-thick" part of $\mathcal{C}(S)$ is.

Definition 7.7. Let $\mathcal{C}(S)^{\min} \subset \mathcal{C}(S)$ be the set of period minimizing geodesic currents and let $\epsilon > 0$. Define

$$\mathcal{C}(S')_\epsilon^{\min} := \left\{ v \in \mathcal{C}(S)^{\min} : L_v(S') \geq \epsilon \right\}$$

and $\mathcal{M}(S')_\epsilon := \mathcal{C}(S')_\epsilon^{\min} / MCG(S')$.

Observe that if $\mu, v \in \mathcal{C}(S')_\epsilon^{\min}$ lie in the same equivalence class in $\mathcal{M}(S')_\epsilon$, then $h_\mu(S') = h_v(S')$. Thus, we can think of $h_-(S')$ as a function from $\mathcal{M}(S')_\epsilon$ to $\mathbb{R}$.

Corollary 7.8. *Let ϵ be any positive number and let $\{[v_k]\}_{k=1}^\infty$ be a sequence in $\mathcal{M}(S')_\epsilon$. Then $\lim_{k \to \infty} h_{v_k}(S') = 0$ if and only if the following condition holds: for each k, there is a (possibly empty) collection $\mathcal{D}_k$ of pairwise non-intersecting simple closed geodesics in S' so that*

- $\sup_k \max\{i(c, v_k) : c \in \mathcal{D}_k\} < \infty$, *and*

- $\lim_{k \to \infty} \min\{i(c, v_k) : c \in \mathcal{CG}(S' \setminus \mathcal{D}_k) \text{ is non-peripheral}\} = \infty$.

Proof. Let us choose an appropriate collection $\mathcal{D}_k$ of closed geodesics. For each v_k, let $\mathcal{P}_{v_k}(S') = \{c_{1,k}, \ldots, c_{3g-3+2n,k}\}$ be a minimal pants decomposition, where $c_{1,k}, \ldots, c_{3g-3+n,k}$ are non-peripheral and enumerated so that $i(c_{j,k}, v_k) \leq i(c_{j+1,k}, v_k)$. Let $j_0 \in \{0, \ldots, 3g - 3 + n\}$ be the number so that

- $\limsup_{k \to \infty} i(c_{j,k}, v_k) < \infty$ for all $j \leq j_0$,

- $\limsup_{k \to \infty} i(c_{j,k}, v_k) = \infty$ for all $j \in \{j_0 + 1, \ldots, 3g - 3 + n\}$.

(We use the convention $j_0 = 0$ if $\limsup_{k \to \infty} i(c_{1,k}, v_k) = \infty$ and $j_0 = 3g - 3 + n$ if $\limsup_{k \to \infty} i(c_{3g-3+n,k}, v_k) < \infty$.) Let

$$\mathcal{D}_k := \begin{cases} \{c_{1,k} \ldots, c_{j_0,k}\} & \text{if } j_0 > 0, \\ \emptyset & \text{if } j_0 = 0. \end{cases}$$

First, we show that if the condition does not hold, then $\limsup_{k \to \infty} h_{v_k}(S') > 0$. Since the condition does not hold, there is a constant C so that for each k, there is a component S_k'' of $S' \setminus \mathcal{D}_k$ and a non-peripheral primitive closed geodesic $d_k \in \mathcal{CG}(S_k'')$ satisfying $i(d_k, v_k) \leq C$. Notice that when $j_0 = 3g - 3 + n$, S_k'' is a pair of pants which implies that d_k is primitive and non-simple. Likewise, if $j_0 < 3g - 3 + n$, for sufficiently large k, d_k is primitive and non-simple because $\lim_{k \to \infty} i(c_{j,k}, v_k) = \infty$ for all $j > j_0$. In either case, $K_{v_k}(S_k'') \leq i(d_k, v_k) \leq C$ for all k. Thus, $h_{v_k}(S_k'') \geq \frac{\log(2)}{4C}$ by Corollary 7.5. Since $h_{v_k}(S') \geq h_{v_k}(S_k'')$, we see that $\limsup_{k \to \infty} h_{v_k}(S') > 0$.

Next, we suppose that the condition holds and we prove that $\lim_{k \to \infty} h_{v_k}(S') = 0$. Observe that for k big enough, the curves in $\mathcal{D}_k$ are part of a minimal pants decomposition $\mathcal{P}_{v_k}(S')$. Let $d_k \in \mathcal{CG}(S')$ be a closed geodesic that is not a multiple of an element in $\mathcal{P}_{v_k}(S')$ so that $i(d_k, v_k) = K_{v_k}(S')$. By Corollary 7.5, it is sufficient to show that $\lim_{k \to \infty} i(d_k, v_k) = \infty$. Choose a hyperbolic structure Σ on S, set

$$A := \sup_k \max \{i(c, v_k) : c \in \mathcal{D}_k\},$$

and set

$$B_k := \min \{i(c, v_k) : c \in \mathcal{CG}(S' \setminus \mathcal{D}_k) \text{ is non-peripheral}\}.$$

Since the condition holds, $0 < A < \infty$ and $\lim_{k \to \infty} B_k = \infty$.

If $d_k \in \mathcal{CG}(S' \setminus \mathcal{D}_k)$, then it is non-peripheral, so

$$i(d_k, v_k) \geq B_k.$$

On the other hand, if d_k does not lie in $\mathcal{CG}(S' \setminus \mathcal{D}_k)$, let S_k'' be a component of $S' \setminus \mathcal{D}_k$ that intersects d_k. Let $\gamma_k \in \Gamma'$ be the group element so that $[\![\gamma_k]\!] = d_k$

and let L_{γ_k} be the axis of γ_k in $\widetilde{S}'$. Choose distinct points $p_k, q_k \in L_{\gamma_k}$ so that $\Pi(p_k), \Pi(q_k) \in \partial S_k''$ and $\Pi(r) \notin \partial S_k''$ for all $r \in [p_k, q_k)$. It is clear that

$$v\big(G(p_k, q_k]\big), v\big(G(q_k, p_k]\big) \leq i(d_k, v_k).$$

Let $\eta_k, \tau_k \in \pi_1(S_k'')$ be the group elements so that $[\![\eta_k]\!], [\![\tau_k]\!] \in \mathcal{D}_k$, and p_k and q_k lie in the axes of η_k and τ_k respectively. Observe that $i([\![\eta_k]\!], v_k), i([\![\tau_k]\!], v_k) \leq A$. Hence, by Lemma 4.4,

$$
\begin{aligned}
i\big([\![\tau_k\eta_k]\!], v_k\big) &\leq v_k\big(G(\eta_k^{-1} \cdot p_k, \tau_k\eta_k \cdot (\eta_k^{-1} \cdot p_k)]\big) \\
&\leq v_k\big(G(\eta_k^{-1} \cdot p_k, p_k]\big) + v_k\big(G(p_k, q_k]\big) + v_k\big(G(q_k, \tau_k \cdot q_k]\big) \\
&\qquad\qquad\qquad\qquad + v_k\big(G(\tau_k \cdot q_k, \tau_k \cdot p_k]\big) \\
&= i\big([\![\eta_k]\!], v_k\big) + i\big([\![\tau_k]\!], v_k\big) + v_k\big(G(p_k, q_k]\big) + v_k\big(G[p_k, q_k)\big) \\
&\leq 2A + 2i(d_k, v_k).
\end{aligned}
$$

Similarly, $i([\![\tau_k^{-1}\eta_k]\!], v_k) \leq 2A + 2i(d_k, v_k)$. Since $[\![\tau_k\eta_k]\!], [\![\tau_k^{-1}\eta_k]\!] \in \mathcal{CG}(S_k'')$ cannot both be peripheral, either $i([\![\tau_k\eta_k]\!], v_k) \geq B_k$ or $i([\![\tau_k^{-1}\eta_k]\!], v_k) \geq B_k$. Thus,

$$i(d_k, v_k) \geq \frac{1}{2}B_k - A.$$

This implies that $\lim_{k\to\infty} i(d_k, v_k) \geq \frac{1}{2}\lim_{k\to\infty} B_k - A = \infty$. $\qquad\square$

Consider the case when $S' = S$. In the above theorem, each $\mathcal{D}_k$ can be completed to a pants decomposition of S. Since there are only finitely many mapping class group orbits of pants decompositions of S, we can apply Corollary 7.8 and Theorem 1.1 to deduce Corollary 1.7.

A. From positive cross ratios to geodesic currents

In this appendix, we give a proof Theorem 2.10 (which was previously observed by Hamenstädt [15]) for the convenience of the reader. Recall that an *algebra* $\mathcal{A}$ on a set X is a family of subsets of X so that

(1) for all A in $\mathcal{A}$, the complement A^c of A is in $\mathcal{A}$,

(2) for all $A_1, A_2 \in \mathcal{A}$, the union $A_1 \cup A_2 \in \mathcal{A}$.

Also, a *premeasure* v on $\mathcal{A}$ is a function

$$v \colon \mathcal{A} \to [0, \infty]$$

such that

(1) $v(\emptyset) = 0$;

(2) if $\{A_i\}_{i\in\mathbb{N}}$ is a countable family of pairwise disjoint sets in $\mathcal{A}$ whose union lies in $\mathcal{A}$, then

$$\nu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \nu(A_i).$$

Choose an orientation on $\partial\Gamma$. For the rest of this appendix, we will assume that all intervals in $\partial\Gamma$ are of the form $[x,y)_z$ (see Notation 2.26) unless otherwise stated, where x,y,z lie in $\partial\Gamma$ in the order specified by the orientation. We will start by defining a particular algebra in $\mathcal{G}(\widetilde{S})$.

Definition A.1. Let $\mathcal{L} = \{\{I_1, J_1\}, \{I_2, J_2\}, \ldots, \{I_n, J_n\}\}$ be a finite list of pairs of proper subintervals of $\partial\Gamma$.

- $\mathcal{L}$ is an *admissible list* if for all $k = 1, \ldots, n$, either $I_k = J_k$ or I_k and J_k are disjoint.

- For any admissible list $\mathcal{L}$, let $\mathcal{G}_{\mathcal{L}}$ denote the set of geodesics $\{a, b\} \in \mathcal{G}(\widetilde{S})$ such that there exists a $k \in \{1, \ldots, n\}$ with I_k and J_k each containing one endpoint of $\{a, b\}$.

For the rest of this appendix, let $\mathcal{A} := \{\mathcal{G}_{\mathcal{L}} : \mathcal{L}$ is an admissible list$\}$. Note that every $\mathcal{G}_{\mathcal{L}} \in \mathcal{A}$ can be written as a finite disjoint union

$$\mathcal{G}_{\mathcal{L}} = \bigcup_{k=1}^{n} \mathcal{G}_{\{I_k, J_k\}}.$$

Lemma A.2. *$\mathcal{A}$ is an algebra.*

Proof. For all $\mathcal{G}_{\mathcal{L}}, \mathcal{G}_{\mathcal{L}'} \in \mathcal{A}$, note that $\mathcal{G}_{\mathcal{L}} \cup \mathcal{G}_{\mathcal{L}'} = \mathcal{G}_{\mathcal{L}\cup\mathcal{L}'} \in \mathcal{A}$. To prove closure under complements, it is sufficient to show that $\mathcal{G}_{\{I,J\}}^c \in \mathcal{A}$ and $\mathcal{G}_{\{I_1, I_2\}} \cap \mathcal{G}_{\{J_1, J_2\}} \in \mathcal{A}$ because of De Morgan's laws.

First, we will show that $\mathcal{G}_{\{I,J\}}^c \in \mathcal{A}$. If I and J have disjoint interiors, let x, y be the endpoints of I and z, w be the endpoints of J. Also, let $K := [y, z)_x$ and $L := [w, x)_y$, and observe that $\partial\Gamma = K \cup L \cup I \cup J$ is a disjoint union. It is then easy to see that $(\mathcal{G}_{\{I,J\}})^c = \mathcal{G}_{\mathcal{L}}$, where

$$\mathcal{L} := \{\{I, I\}, \{I, K\}, \{I, L\}, \{K, K\}, \{K, J\}, \{K, L\}, \{J, J\}, \{J, L\}, \{L, L\}\}.$$

On the other hand, if $I = J$, let $K := \partial\Gamma - I$. Then $(\mathcal{G}_{\{I,J\}})^c = \mathcal{G}_{\mathcal{L}}$, where $\mathcal{L} := \{\{I, K\}, \{K, K\}\}$.

Next, we will show that $\mathcal{G}_{\{I_1, I_2\}} \cap \mathcal{G}_{\{J_1, J_2\}} \in \mathcal{A}$. Let $K_{i,j,1}, \ldots, K_{i,j,t_{i,j}}$ be intervals so that $I_i \cap J_j = \bigcup_{k=1}^{t_{i,j}} K_{i,j,k}$ is a disjoint union. (Note that $t_{i,j}$ is either 0, 1 or 2.) Then observe that $\mathcal{G}_{\{I_1, I_2\}} \cap \mathcal{G}_{\{J_1, J_2\}} = \mathcal{G}_{\mathcal{L}}$, where

$$\mathcal{L} := \big\{\{K_{1,1,1}, K_{2,2,1}\}, \{K_{1,1,t_{1,1}}, K_{2,2,1}\}, \{K_{1,1,1}, K_{2,2,t_{2,2}}\}, \{K_{1,1,t_{1,1}}, K_{2,2,t_{2,2}}\},$$
$$\{K_{1,2,1}, K_{2,1,1}\}, \{K_{1,2,t_{1,2}}, K_{2,1,1}\}, \{K_{1,2,1}, K_{2,1,t_{2,1}}\}, \{K_{1,2,t_{1,2}}, K_{2,1,t_{2,1}}\}\big\}.$$

$\square$

Next, we will use the positive cross ratio B to define a premeasure on $\mathcal{A}$. Let $v_B \colon \mathcal{A} \to [0, \infty]$ be a function defined as follows:

(1) $v_B(\emptyset) = 0$.

(2) $v_B(\mathcal{G}_{\{I,J\}}) = \infty$ when $I = J$ or I and J share a common endpoint.

(3) If the intervals I and J are non-empty and have disjoint closures, let x and y be the endpoints of I and z and w the endpoints of J, so that x, y, z, w lie in this cyclic order along $\partial \Gamma$. Then define

$$v_B(\mathcal{G}_{\{I,J\}}) := B(x, y, z, w).$$

(4) If $\mathcal{G}_{\mathcal{L}} = \bigcup_{k=1}^{n} \mathcal{G}_{\{I_k, J_k\}}$ is a disjoint union, define

$$v_B(\mathcal{G}_{\mathcal{L}}) := \sum_{1 \le k \le n} v_B(\mathcal{G}_{\{I_k, J_k\}}).$$

Since $B(x, y, z, w) = B(z, w, x, y)$, $v_B(\mathcal{G}_{\{I,J\}})$ is well-defined. Also, if

$$\mathcal{G}_{\mathcal{L}} = \bigcup_{k=1}^{n} \mathcal{G}_{\{I_k, J_k\}} = \bigcup_{k=1}^{m} \mathcal{G}_{\{I_k', J_k'\}}$$

are two ways to write $\mathcal{G}_{\mathcal{L}}$ as disjoint unions, then by taking intersections, we can write $\mathcal{G}_{\mathcal{L}}$ as the disjoint union

$$\mathcal{G}_{\mathcal{L}} = \bigcup_{k=1}^{l} \mathcal{G}_{\{I_k'', J_k''\}}$$

so that for all $k = 1, \dots, l$, $\mathcal{G}_{\{I_k'', J_k''\}}$ is a connected component of $\mathcal{G}_{\{I_s, J_s\}} \cap \mathcal{G}_{\{I_t', J_t'\}}$ for some $s \in \{1, \dots, n\}$ and $t \in \{1, \dots, m\}$. By the additive property of the cross ratio, if I_1, I_2, I, J are subintervals of $\partial \Gamma$ so that $I_1 \cup I_2 = I$, then

$$v_B(\mathcal{G}_{\{I_1, J\}}) + v_B(\mathcal{G}_{\{I_2, J\}}) = v_B(\mathcal{G}_{\{I, J\}}).$$

This implies that

$$\sum_{k=1}^{n} v_B(\mathcal{G}_{\{I_k, J_k\}}) = \sum_{k=1}^{l} v_B(\mathcal{G}_{\{I_k'', J_k''\}}) = \sum_{k=1}^{m} v_B(\mathcal{G}_{\{I_k', J_k'\}}),$$

so v_B is well-defined.

It is also clear from the definition that v_B is finitely additive, i.e. if $\mathcal{G}_{\mathcal{L}_1}, \dots, \mathcal{G}_{\mathcal{L}_n}$ are pairwise disjoint, then

$$v_B\left(\bigcup_{i=1}^{n} \mathcal{G}_{\mathcal{L}_i} \right) = \sum_{i=1}^{n} v_B(\mathcal{G}_{\mathcal{L}_i}).$$

Furthermore, the positivity of B, ensures that v_B takes values in $[0, \infty]$.

Proposition A.3. *For any positive cross ratio B, ν_B is a premeasure on $\mathcal{A}$.*

Proof. Set $\nu := \nu_B$ to simplify the notation. By definition, $\nu(\emptyset) = 0$, so we need only to prove countable additivity. Let $\{\mathcal{G}_{\mathcal{L}_k}\}_{k=1}^{\infty}$ be a family of disjoint sets in $\mathcal{A}$ with $\bigcup_{k=1}^{\infty} \mathcal{G}_{\mathcal{L}_k} = \mathcal{G}_{\mathcal{L}}$ for some admissible list $\mathcal{L}$. Up to repartitioning and renumbering, we can assume the following:

- For all k, $\mathcal{L}_k = \{I_k, J_k\}$ for some intervals I_k, J_k,
- There is some admissible list $\{\{I_1', J_1'\}, \ldots, \{I_t', J_t'\}\}$ so that $\mathcal{G}_{\mathcal{L}} = \bigcup_{s=1}^{t} \mathcal{G}_{\{I_s', J_s'\}}$ is a disjoint union.

By finite additivity, we have that for all $n \in \mathbb{Z}^+$,

$$
\nu(\mathcal{G}_{\mathcal{L}}) = \nu\left(\bigcup_{k=1}^{n} \mathcal{G}_{\mathcal{L}_k}\right) + \nu\left(\mathcal{G}_{\mathcal{L}} \setminus \bigcup_{k=1}^{n} \mathcal{G}_{\mathcal{L}_k}\right)
$$

$$
\geq \nu\left(\bigcup_{k=1}^{n} \mathcal{G}_{\mathcal{L}_k}\right) = \sum_{k=1}^{n} \nu(\mathcal{G}_{\mathcal{L}_k}).
$$

Thus, $\nu(\mathcal{G}_{\mathcal{L}}) \geq \sum_{k=1}^{\infty} \nu(\mathcal{G}_{\mathcal{L}_k})$.

To finish the proof, we need to show that $\nu(\mathcal{G}_{\mathcal{L}}) \leq \sum_{k=1}^{\infty} \nu(\mathcal{G}_{\mathcal{L}_k})$. First consider the case where $\nu(\mathcal{G}_{\mathcal{L}}) < \infty$, then $\nu(\mathcal{G}_{\{I_s', J_s'\}}), \nu(\mathcal{G}_{\mathcal{L}_k}) < \infty$ for all $s = 1, \ldots, t$, $k \in \mathbb{Z}^+$. Since B is continuous, for any $\varepsilon > 0$ and any $s = 1, \ldots, t$, we can find compact subintervals $I_s'' \subset I_s'$ and $J_s'' \subset J_s'$ such that

$$
\nu(\mathcal{G}_{\{I_s', J_s'\}}) - \nu(\mathcal{G}_{\{I_s'', J_s''\}}) < \frac{\varepsilon}{t}. \tag{A.1}
$$

Similarly, for any $k \in \mathbb{N}$ we can find open intervals $I_k^* \supset I_k$ and $J_k^* \supset J_k$ such that

$$
\nu(\mathcal{G}_{\{I_k^*, J_k^*\}}) - \nu(\mathcal{G}_{\mathcal{L}_k}) < \frac{\varepsilon}{2^k}. \tag{A.2}
$$

Observe that the open sets $\{\mathcal{G}_{\{I_k^*, J_k^*\}}\}_{k=1}^{\infty}$ is an open cover of the compact set $\bigcup_{s=1}^{t} \mathcal{G}_{\{I_s'', J_s''\}}$, so it has a finite subcover $\{\mathcal{G}_{\{I_k^*, J_k^*\}}\}_{k=1}^{N}$. By using inequalities (A.1) and (A.2), we have

$$
\nu(\mathcal{G}_{\mathcal{L}}) < \varepsilon + \nu\left(\bigcup_{s=1}^{t} \mathcal{G}_{\{I_s'', J_s''\}}\right) \leq \varepsilon + \sum_{k=1}^{N} \nu(\mathcal{G}_{\{I_k^*, J_k^*\}})
$$

$$
\leq \varepsilon + \sum_{k=1}^{N} \left(\nu(\mathcal{G}_{\mathcal{L}_k}) + \frac{\varepsilon}{2^k}\right) \leq 2\varepsilon + \sum_{k=1}^{\infty} \nu(\mathcal{G}_{\mathcal{L}_k}).
$$

Since ε was arbitrary, this proves that $\nu(\mathcal{G}_{\mathcal{L}}) \leq \sum_{k=1}^{\infty} \nu(\mathcal{G}_{\mathcal{L}_k})$ when $\nu(\mathcal{G}_{\mathcal{L}}) < \infty$.

Next, consider the case where $\nu(\mathcal{G}_{\mathcal{L}}) = \infty$. This means that $\nu(\mathcal{G}_{\{I_{s_0}', J_{s_0}'\}}) = \infty$ for some $s_0 = 1, \ldots, t$, so either I_{s_0}' and J_{s_0}' are disjoint and share an endpoint,

or $I'_{s_0} = J'_{s_0}$. In either case, we can find subintervals $\bar{I} \subset I'_{s_0}$ and $\bar{J} \subset J'_{s_0}$ that are disjoint and share an endpoint. Let p be the common endpoint of $\bar{I}$ and $\bar{J}$, let $p_{\bar{J}}$ be the endpoint of $\bar{J}$ that is not p, and for any $q \in \bar{J}$ let $\bar{J}_q$ be the subinterval of $\bar{J}$ with endpoints q and $p_{\bar{J}}$. Since

$$\lim_{q \to p_{\bar{J}}} \nu(\mathcal{G}_{\{\bar{I}, \bar{J}_q\}}) = 0, \quad \lim_{q \to p} \nu(\mathcal{G}_{\{\bar{I}, \bar{J}_q\}}) = \infty,$$

we know that for every $t \in \mathbb{R}$, there exists q_t in the interior of $\bar{J}$ such that $\nu(\mathcal{G}_{\{\bar{I}, \bar{J}_{q_t}\}}) = t$. Using the previous case, we know that for all $q \in \bar{J}$,

$$\nu(\mathcal{G}_{\{\bar{I}, \bar{J}_q\}}) \leq \sum_{k=1}^{\infty} \nu(\mathcal{G}_{\mathcal{L}_k}).$$

Thus, $\sum_{k=1}^{\infty} \nu(\mathcal{G}_{\mathcal{L}_k}) = \infty = \nu(\mathcal{G}_{\mathcal{L}})$. $\qquad\square$

We will now recall a standard procedure to obtain a unique measure extending a premeasure. See for example Chapter 1 of Folland [9] for more details. Given a premeasure ν on $\mathcal{A}$ and $E \subset \mathcal{G}(\tilde{S})$, define the *outer measure*

$$\nu^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \nu(A_k) \colon A_k \in \mathcal{A}, \ E \subset \bigcup_{k=1}^{\infty} A_k \right\}.$$

A premeasure $\nu \colon \mathcal{A} \to [0, \infty]$ is σ-*finite* if X can be written as a union of countably many sets with finite outer measure. The following theorem [9, Theorem 1.14] relates σ-finite premeasures and measures.

Theorem A.4. *Let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. Then, $\mu := \nu^*_{|\mathcal{M}}$ is a measure on $\mathcal{M}$ and the restriction of μ to $\mathcal{A}$ is ν. If ν is σ-finite, then μ is unique.*

Lemma A.5. *If B is a positive cross ratio, then the premeasure ν_B is σ-finite.*

Proof. Let Δ be a countable dense subset of $\partial\Gamma$. For any $m \in \mathbb{N}$ and $p \in \Delta$, let I_m^p be the open interval in $\partial\Gamma$ centered at p of width $1/m$. Define

$$\Delta_m^2 := \left\{ (p, q) \in \Delta^2 : I_m^p \text{ and } I_m^q \text{ have disjoint closures and } p \text{ precedes } q \right\}.$$

Clearly

$$\mathcal{G}(\tilde{S}) = \bigcup_{m=1}^{\infty} \bigcup_{(p,q) \in \Delta_m^2} \mathcal{G}_{\{I_m^p, I_m^q\}}. \tag{A.3}$$

Observe that the right hand side of Equation A.3 is a countable union of sets, each with finite measure. $\qquad\square$

Proof of Theorem 2.10. Since the premeasure ν_B is σ-finite, Theorem A.4 ensures that there is a unique measure μ_B on the σ-algebra $\mathcal{M}$ generated by $\mathcal{A}$. The Γ-invariance of B ensures that μ_B is also Γ-invariant. It is easy to see that the topology on $\mathcal{G}(\widetilde{S})$ lies in $\mathcal{M}$. Thus, μ_B is a geodesic current.

Next, we show that for all $c \in \mathcal{CG}(S)$, $i(c, \mu_B) = \ell_B(c)$. By (1) of Lemma 4.4, we know that

$$
\begin{aligned}
i(c, \mu_B) &= \mu_B(\mathcal{G}_{\{[\gamma^+, \gamma^-)_z, [z, \gamma \cdot z)_{\gamma^-}\}}) \\
&= \nu_B(\mathcal{G}_{\{[\gamma^+, \gamma^-)_z, [z, \gamma \cdot z)_{\gamma^-}\}}) \\
&= B(\gamma^-, \gamma^+, \gamma \cdot z, z) \\
&= \ell_B(c).
\end{aligned}
$$
$\square$

B. Computation for proof of Theorem 1.4

The goal of this appendix is to prove the following statement, which we need to finish the proof of Theorem 1.4.

Proposition B.1. *Let T, K, L be positive numbers so that $K \geq L$. Also, let $Q = Q(T, K, L) \in \{0, \ldots, \lfloor \frac{T}{K} \rfloor\}$ be an integer so that for all $i \in \{0, \ldots, \lfloor \frac{T}{K} \rfloor\}$, we have*

$$
\binom{\lfloor \frac{T - KQ}{L} \rfloor + Q}{Q} \geq \binom{\lfloor \frac{T - Ki}{L} \rfloor + i}{i}.
$$

Then

$$
\limsup_{T \to \infty} \frac{K}{T} \log \binom{\lfloor \frac{T - KQ}{L} \rfloor + Q}{Q} \leq 1 + \log\left(1 + \frac{1}{x_0}\right),
$$

where x_0 is the unique positive solution to the equation $(1 + x)^{\lceil \frac{K}{L} - 1 \rceil} x = 1$.

The proof of Proposition B.1 is a refinement of the argument given in Appendix B of [36]. However, we will make this appendix self-contained for the convenience of the reader. The main tool in this computation is an old result known as Stirling's formula, which we state here.

Theorem B.2 (Stirling's formula). $\displaystyle \lim_{n \to \infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} = 1.$

In order to use Stirling's formula, we need to prove the following lemma. For the rest of this appendix, let

$$
F = F(T, K, L) := \left\lfloor \frac{T - KQ}{L} \right\rfloor.
$$

Lemma B.3. *Let* $K \geq L > 0$ *be fixed numbers,* $a := \lceil \frac{K}{L} - 1 \rceil$ *and* $b := \lfloor \frac{K}{L} - 1 \rfloor$. *Then the following hold:*

(1) $\lim\limits_{T \to \infty} F = \infty$.

(2) $\lim\limits_{T \to \infty} Q = \infty$.

(3) $1 \leq \left(1 + \liminf\limits_{T \to \infty} \frac{Q}{F}\right)^a \cdot \liminf\limits_{T \to \infty} \frac{Q}{F}$.

(4) $1 \geq \left(1 + \limsup\limits_{T \to \infty} \frac{Q}{F}\right)^b \cdot \limsup\limits_{T \to \infty} \frac{Q}{F}$.

Proof. From the definition of Q, we see that

$$
1 \leq \left(\begin{array}{c} \lfloor \frac{T-QK}{L} \rfloor + Q \\ Q \end{array}\right) \bigg/ \left(\begin{array}{c} \lfloor \frac{T-(Q+1)K}{L} \rfloor + Q + 1 \\ Q + 1 \end{array}\right)
$$

$$
= \frac{(\lfloor \frac{T-QK}{L} \rfloor + Q)(\lfloor \frac{T-QK}{L} \rfloor + Q - 1)\cdots(\lfloor \frac{T-(Q+1)K}{L} \rfloor + Q + 2)(Q+1)}{\lfloor \frac{T-QK}{L} \rfloor(\lfloor \frac{T-QK}{L} \rfloor - 1)\cdots(\lfloor \frac{T-(Q+1)K}{L} \rfloor + 1)},
$$
(B.1)

which can be rearranged to be

$$
\frac{Q+1}{F} \geq \frac{(\lfloor \frac{T-QK}{L} \rfloor - 1)(\lfloor \frac{T-QK}{L} \rfloor - 2)\cdots(\lfloor \frac{T-(Q+1)K}{L} \rfloor + 1)}{(\lfloor \frac{T-QK}{L} \rfloor + Q)(\lfloor \frac{T-QK}{L} \rfloor + Q - 1)\cdots(\lfloor \frac{T-(Q+1)K}{L} \rfloor + Q + 2)}.
$$
(B.2)

Similarly, the definition of Q also tells us that

$$
1 \geq \left(\begin{array}{c} \lfloor \frac{T-(Q-1)K}{L} \rfloor + Q - 1 \\ Q - 1 \end{array}\right) \bigg/ \left(\begin{array}{c} \lfloor \frac{T-QK}{L} \rfloor + Q \\ Q \end{array}\right)
$$

$$
= \frac{(\lfloor \frac{T-(Q-1)K}{L} \rfloor + Q - 1)(\lfloor \frac{T-(Q-1)K}{L} \rfloor + Q - 2)\cdots(\lfloor \frac{T-QK}{L} \rfloor + Q + 1)Q}{\lfloor \frac{T-(Q-1)K}{L} \rfloor(\lfloor \frac{T-(Q-1)K}{L} \rfloor - 1)\cdots(\lfloor \frac{T-QK}{L} \rfloor + 1)},
$$
(B.3)

which implies

$$
\frac{Q}{F+1} \leq \frac{(\lfloor \frac{T-(Q-1)K}{L} \rfloor)\cdots(\lfloor \frac{T-QK}{L} \rfloor + 2)}{(\lfloor \frac{T-(Q-1)K}{L} \rfloor + Q - 1)\cdots(\lfloor \frac{T-QK}{L} \rfloor + Q + 1)} \leq 1.
$$
(B.4)

(1) Suppose for contradiction that $\liminf\limits_{T \to \infty} F < \infty$. By the definition of F, we see that $\limsup\limits_{T \to \infty} Q = \infty$. Thus,

$$
\limsup\limits_{T \to \infty} \frac{Q}{F+1} = \infty,
$$

which contradicts (B.4).

(2) Suppose again for contradiction that $\liminf_{T \to \infty} Q < \infty$. By taking an appropriate subsequence, we can assume that $\lim_{T \to \infty} Q < \infty$. Hence, $\lim_{T \to \infty} F = \infty$, so we have

$$\lim_{T \to \infty} \frac{F}{Q+1} = \infty.$$

On the other hand, if $\lim_{T \to \infty} Q < \infty$ and $\lim_{T \to \infty} F = \infty$, then the right hand side of the inequality (B.2) converges to 1 as $T \to \infty$, which implies that

$$\lim_{T \to \infty} \frac{Q+1}{F} \geq 1.$$

This is a contradiction.

(3) Since (1) and (2) hold, taking the limit infimum of (B.1) as $T \to \infty$ gives

$$1 \leq \liminf_{T \to \infty} \left(1 + \frac{Q}{\left\lfloor \frac{T-QK}{L} \right\rfloor}\right)\left(1 + \frac{Q}{\left\lfloor \frac{T-QK}{L} \right\rfloor - 1}\right) \cdots$$
$$\cdots \left(1 + \frac{Q}{\left\lfloor \frac{T-(Q+1)K}{L} \right\rfloor + 2}\right) \cdot \frac{Q+1}{\left\lfloor \frac{T-(Q+1)K}{L} \right\rfloor + 1}$$
$$\leq \left(1 + \liminf_{T \to \infty} \frac{Q}{F}\right)^a \cdot \liminf_{T \to \infty} \frac{Q}{F}.$$

(4) Similarly, by taking limit supremum of (B.3) as $T \to \infty$, we get

$$1 \geq \limsup_{T \to \infty} \left(1 + \frac{Q-1}{\left\lfloor \frac{T-(Q-1)K}{L} \right\rfloor}\right)\left(1 + \frac{Q-1}{\left\lfloor \frac{T-(Q-1)K}{L} \right\rfloor - 1}\right) \cdots$$
$$\cdots \left(1 + \frac{Q-1}{\left\lfloor \frac{T-QK}{L} \right\rfloor + 2}\right) \cdot \frac{Q}{\left\lfloor \frac{T-QK}{L} \right\rfloor + 1}$$
$$\geq \left(1 + \limsup_{T \to \infty} \frac{Q}{F}\right)^b \cdot \limsup_{T \to \infty} \frac{Q}{F}. \qquad \square$$

By (3) and (4) of Lemma B.3, we see that $\limsup_{T \to \infty} \frac{F}{Q}$ is a positive real number, which we will denote by D in the sequel. We now use (3) of Lemma B.3 to find an inequality relating D to $a := \left\lceil \frac{K}{L} - 1 \right\rceil$.

Lemma B.4. *For any positive numbers $K \geq L > 0$, let $a := \left\lceil \frac{K}{L} - 1 \right\rceil$. Then*

$$D \leq \frac{1}{x_0},$$

where x_0 is the unique positive solution to the equation $(1 + x)^a x = 1$.

Proof. Consider the function $f_a \colon [0, \infty) \to \mathbb{R}$ defined by $f_a(x) = (1 + x)^a \cdot x$, and observe that f_a is increasing, $f_a(0) = 0$, and $\lim_{x \to \infty} f(x) = \infty$. By (3) of Lemma B.3, we know that $f_a\left(\frac{1}{D}\right) \geq 1$, so $\frac{1}{D} \geq f_a^{-1}(1) = x_0$. $\square$

With Lemma B.3 and Lemma B.4, we are now ready to prove Proposition B.1.

Proof of Proposition B.1. Since (1) and (2) of Lemma B.3 hold, we can apply Stirling's formula to obtain

$$\lim_{T \to \infty} \left(\binom{F + Q}{Q} \cdot \sqrt{\frac{2\pi Q F}{F + Q}} \cdot \left(\frac{Q}{F + Q}\right)^Q \cdot \left(\frac{F}{F + Q}\right)^F \right) = 1.$$

Taking the logarithm and multiplying by $\frac{K}{T}$ then gives an expression that can be rearranged to yield

$$
\begin{aligned}
\limsup_{T \to \infty} \frac{K}{T} \log & \binom{F + Q}{Q} \\
&= \limsup_{T \to \infty} \frac{K}{2T} \log \left(\frac{1}{Q} + \frac{1}{F}\right) + \limsup_{T \to \infty} \frac{KQ}{T} \log \left(1 + \frac{F}{Q}\right) \\
&\quad + \limsup_{T \to \infty} \frac{KQ}{T} \cdot \frac{F}{Q} \log \left(1 + \frac{Q}{F}\right) + \limsup_{T \to \infty} \frac{K}{2T} \log \left(\frac{1}{2\pi}\right) \quad \text{(B.5)} \\
&= \limsup_{T \to \infty} \frac{KQ}{T} \log \left(1 + \frac{F}{Q}\right) + \limsup_{T \to \infty} \frac{KQ}{T} \cdot \frac{F}{Q} \log \left(1 + \frac{Q}{F}\right).
\end{aligned}
$$

By the definition of F, we have

$$\limsup_{T \to \infty} \frac{Q}{F} = \frac{L}{K} \limsup_{T \to \infty} \frac{1}{\frac{T}{KQ} - 1},$$

which implies that

$$\limsup_{T \to \infty} \frac{KQ}{T} = \frac{K \cdot \limsup_{T \to \infty} \frac{Q}{F}}{L + K \cdot \limsup_{T \to \infty} \frac{Q}{F}} \leq 1.$$

Applying this to the inequality (B.5) then gives

$$
\begin{aligned}
\limsup_{T \to \infty} \frac{K}{T} \log \binom{F + Q}{Q} &\leq \limsup_{T \to \infty} \log \left(1 + \frac{F}{Q}\right) + \limsup_{T \to \infty} \frac{F}{Q} \log \left(1 + \frac{Q}{F}\right) \\
&\leq \log(1 + D) + 1 \\
&\leq \log \left(1 + \frac{1}{x_0}\right) + 1,
\end{aligned}
$$

where the second inequality is a consequence of the fact that $x \log(1 + \frac{1}{x}) \leq 1$ for all $x > 0$, and the final inequality is Lemma B.4. $\square$

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G. Martone, Department of Mathematics, University of Michigan,
Ann Arbor, MI 48109-1043, USA
E-mail: martone@umich.edu

T. Zhang, Department of Mathematics, National University of Singapore,
10 Lower Kent Ridge Road, 119076, Singapore
E-mail: matzt@nus.edu.sg