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The L^2-torsion function and the Thurston norm of 3-manifolds

Stefan Friedl and Wolfgang Lück

Abstract. Let M be an oriented irreducible 3-manifold with infinite fundamental group and empty or toroidal boundary which is not $S^1 \times D^2$. Consider any element ϕ in the first cohomology of M with integer coefficients. Then one can define the ϕ-twisted L^2-torsion function of the universal covering which is a function from the set of positive real numbers to the set of real numbers. By earlier work of the second author and Schick the evaluation at $t = 1$ determines the volume.

In this paper we show that the degree of the L^2-torsion function, which is a number extracted from its asymptotic behavior at 0 and at ∞, agrees with the Thurston norm of ϕ.

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0. Introduction

Reidemeister torsion is one of the first invariants in algebraic topology which are able to distinguish the homeomorphism type of closed manifolds which are homotopy equivalent. A prominent example is the complete classification of lens spaces, see for instance [6]. The Alexander polynomial, which is one of the basic invariants for knots and 3-manifolds, can be interpreted as Reidemeister torsion, see for instance [31]. The Reidemeister torsion of a 3-manifold can be generalized in two ways. Either one can twist it with an element in the first cohomology which leads for example to the twisted Alexander polynomial, see for instance [14], or one can consider the L^2-version of appropriate coverings resulting in L^2-torsion invariants, see for instance [27, Chapter 3]. Recently there have been attempts to combine these two generalizations and consider twisted L^2-versions. Such generalizations have been considered under the name of *L^2-Alexander torsion* or *L^2-Alexander Conway* invariants for knots or 3-manifolds, for instance in [7–11, 19–21].

In all of these papers one has to make certain assumptions to ensure that the twisted L^2-versions are well-defined. They concern L^2-acyclicity and determinant class. Either these conditions were just assumed to hold, or verified in special cases by a direct computation. A systematic study of these twisted L^2-invariants under the name *L^2-torsion function* has been carried out in [28]. We summarize some of the results of [28] for 3-manifolds. Let M be a 3-manifold. (Here and throughout the paper we assume that all 3-manifolds are compact, connected and oriented with empty or toroidal boundary, unless we say explicitly otherwise.) If M is irreducible and if it has infinite fundamental group, then it was shown in [28] that all these necessary conditions are satisfied for the universal covering $\widetilde{M}$ and an element $\phi \in H^1(M;\mathbb{Z})$. The upshot is that we obtain an invariant that is an equivalence class of functions

$$\overline{\rho}^{(2)}(M;\phi)\colon (0,\infty) \to \mathbb{R}$$

where we call two functions $f, g\colon (0,\infty) \to \mathbb{R}$ equivalent if for some integer m we have $f(t) - g(t) = m \cdot \ln(t)$. We recall the definition in Section 1.3. Note though that this invariant is *minus the logarithm* of the function defined and studied in the aforementioned papers. In those papers the corresponding function was usually referred to as the L^2-Alexander torsion. The convention of this paper brings us in line with [27]. We refer to Section 1.4 and to (1.6) for a short discussion which relates the L^2-torsion function $\overline{\rho}^{(2)}(M;\phi)$ to the L^2-Alexander torsion $\tau^{(2)}(M,\phi)$ of the aforementioned papers.

The evaluation of $\overline{\rho}^{(2)}(M;\phi)$ at $t = 1$ is well-defined and by definition it equals the "usual" L^2-torsion $\rho^{(2)}(M)$ of M. It was shown by the second author and Schick [29, Theorem 0.7] that for any irreducible 3-manifold we have

$$\overline{\rho}^{(2)}(M;\phi)(t = 1) = \rho^{(2)}(M) = -\tfrac{1}{6\pi}\operatorname{vol}(M),$$

where $\operatorname{vol}(M)$ equals the sum of the volumes of the hyperbolic pieces in the JSJ-decomposition of M.

In the sequence of papers [9], [7], [8] and [28] the behavior of $\overline{\rho}^{(2)}(M;\phi)$ as t "goes to the extremes," i.e. as $t \to 0$ and $t \to \infty$, was studied. In particular in [28] it was shown that for any representative ρ there exist constants $C \geq 0$ and $D \geq 0$ such that we get for $0 < t \leq 1$

$$C \cdot \ln(t) - D \leq \rho(t) \leq -C \cdot \ln(t) + D,$$

and for $t \geq 1$

$$-C \cdot \ln(t) - D \leq \rho(t) \leq C \cdot \ln(t) + D.$$

Hence $\limsup_{t\to 0} \frac{\rho(t)}{\ln(t)}$ and $\liminf_{t\to\infty} \frac{\rho(t)}{\ln(t)}$ exist and we can define the *degree* of $\overline{\rho}^{(2)}(M;\phi)$ to be

$$\deg\big(\overline{\rho}^{(2)}(M;\phi)\big) := \limsup_{t\to\infty} \frac{\rho(t)}{\ln(t)} - \liminf_{t\to 0} \frac{\rho(t)}{\ln(t)}.$$

It is obviously independent of the choice of the representative ρ.

Thurston [37] assigned to $\phi \in H^1(M;\mathbb{Z}) = H_2(M,\partial M;\mathbb{Z})$ another invariant, its *Thurston norm* $x_M(\phi)$, which measures the minimal complexity of a surface dual to ϕ. We we will review the precise definition of the Thurston norm in Subsection 1.7.

The main result of our paper says that the functions $\overline{\rho}^{(2)}(M;\phi)$ not only determine the volume of a 3-manifold but that they also determine the Thurston norm. More precisely, we have the following theorem.

Theorem 0.1. *Let M be an irreducible 3-manifold with infinite fundamental group and empty or toroidal boundary which is not homeomorphic to $S^1 \times D^2$. Then we get for any element $\phi \in H^1(M;\mathbb{Q})$ that*

$$\deg\big(\overline{\rho}^{(2)}(M;\phi)\big) = -x_M(\phi).$$

Actually we get a much more general result, where we can consider not only the universal covering but appropriate G-coverings $G \to \overline{M} \to M$ and get estimates for the L^2-function for all times $t \in (0,\infty)$ which imply the equality of the degree and the Thurston norm, see Theorem 5.1.

The equality in Theorem 0.1 had been proved initially by Dubois–Wegner [10, 11] for the exteriors of torus knots. This result of Dubois–Wegner was later generalized by Ben Aribi [3] and Herrmann [18] to Seifert fibered spaces and graph manifolds. Furthermore the equality in the theorem was shown by Dubois and the authors to hold for fibered 3-manifolds [9, Theorem 1.3]. It is a well-established feature of "Alexander type" invariants, e.g. (twisted) Alexander polynomials or non-commutative Alexander polynomials, that their degrees give lower bounds on the knot genus or the Thurston norm. We refer to [5, 13, 16, 30, 35, 40], and [8] for details. Even though it is technically non-trivial to show the "$\leq$"-inequality in the theorem, it is not entirely surprising. What is much more striking, in our opinion, is that we can in fact prove that the equality holds.

In the following we quickly summarize the key ideas behind the proofs of Theorem 0.1. First note, that if the underlying group G is $\mathbb{Z}$, the Fulgede–Kadison determinant agrees with the Mahler measure and the behaviour of the Mahler measure is well understood if one varies the coefficients of an element in the group ring, which is in this case essentially a polynomial in one variable $p(z)$. In particular the asymptotic of the Mahler measure of $p(tz)$ for $t \to 0+$ and $t \to \infty$ is completely known. Using limit formulas for Mahler measures one can extend this to the case $G = \mathbb{Z}^n$, or equivalently, to polynomials in several variables. Now approximation techniques allow to estimate the Fuglede–Kadison determinant and its asymptotic behaviour for $t \to 0+$ and $t \to \infty$ of an element in the group ring by looking at a sequence of epimorphisms from G to finitely generated free abelian groups, whose kernels become smaller and smaller. The Virtual Fibering Theorem of Agol [1, 2], Wise [43] and Przytycki–Wise [32] is used to reduce the general case by a delicate continuity arguments to the case of a mapping torus, where the L^2-torsion function can be computed explicitly. The main technical difficulty is that the Fuglede–Kadison determinant is in general *not* continuous in the coefficients of an element in the group ring and is a much more sophisticated invariant than the Murray-von Neumann dimension.

Added in proof. Liu [23] has given, almost simultaneously, a completely independent proof of Theorem 0.1. The techniques used in both papers are at times somewhat similar. Liu [23] goes on to prove several other very interesting results that are not covered in this paper. In particular he proves Theorem 0.1 also for real classes and shows the continuity of the L^2-torsion function.

Conventions and notations. Given a group G we view elements in $(\mathbb{Z}G)^k$ always as row vectors. Given a group G and an $m \times n$-matrix over $\mathbb{Z}G$ we denote by r_A the homomorphism $(\mathbb{Z}G)^m \to (\mathbb{Z}G)^n$ given by right multiplication by A. Furthermore, given a group homomorphism $\gamma\colon G \to H$ we denote by $\gamma(A)$ the matrix over $\mathbb{Z}H$ given by applying γ to all entries. Throughout the paper we assume that all 3-manifolds are compact, connected and oriented, unless we say explicitly otherwise.

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1. Review of the ϕ-twisted L^2-torsion function and the Thurston norm

In this section we recall some basic definitions, notions and results from [7, 9, 28].

1.1. L^2-Betti numbers, Fuglede–Kadison determinants and L^2-torsion. Unfortunately it would take up too much space to recall the definitions of all the L^2-invariants that we use in this paper. Therefore we refer to [27] for the precise definitions. In this section we only quickly recall the notation and the basic setup for L^2-invariants from [27].

(1) Given a group G we denote by $\mathcal{N}(G)$ the group von Neumann algebra as defined in [27, Definition 1.1 on p. 15].

(2) We refer to [27, Definition 1.5 on p. 16] for the definition of a Hilbert $\mathcal{N}(G)$-module and for the definition of a morphism between Hilbert $\mathcal{N}(G)$-modules. As an example, if G is a group and if V is a based free left $\mathbb{C}G$-module, then $L^2(G) \otimes_{\mathbb{C}G} V$

is naturally a Hilbert $\mathcal{N}(G)$-module. Furthermore, if $f: V \to W$ is a homomorphism between based free left $\mathbb{C}G$-module, then $\mathrm{id} \otimes f: L^2(G) \otimes_{\mathbb{C}G} V \to L^2(G) \otimes_{\mathbb{C}G} W$ is a morphism of Hilbert $\mathcal{N}(G)$-modules.

(3) We refer to [27, Definition 1.10 on p. 21] for the definition of the *von Neumann dimension* $\dim_{\mathcal{N}(G)}(V)$ of a Hilbert $\mathcal{N}(G)$-module V.

(4) Given a Hilbert $\mathcal{N}(G)$-chain complex (C_*, c_*) we refer to

$$b_p(C_*) := \dim_{\mathcal{N}(G)} \big(\ker(c_p)/\overline{\mathrm{im}(c_{p+1})}\big)$$

as the *p-th L^2-Betti number* of the chain complex. We refer to [27, Definition 1.15 on p. 24] for details.

(5) Given a finite CW-complex X together with a G-covering $\overline{X}$ we view $C_*(\overline{X}, \mathbb{C})$ as chain complex of based free left $\mathbb{C}G$-modules, where a basis is given by lifts of the cells from X to $\overline{X}$. We denote by

$$b_n^{(2)}\big(\overline{X}; \mathcal{N}(G)\big) = b_n^{(2)}\big(L^2(G) \otimes_{\mathbb{C}G} C_*(\overline{X}, \mathbb{C})\big)$$

the corresponding L^2-Betti number. This definition can also be generalized to any topological space, see [27, Definition 6.50 on p. 263] for details.

(6) Let $f: U \to V$ be a morphism of finite dimensional Hilbert $\mathcal{N}(G)$-modules. The definition of f being of *determinant class* is given in [27, Definition 3.11 on p. 140]. If f is of determinant class, then op. cit. gives the definition of the corresponding Fuglede–Kadison determinant $\det_{\mathcal{N}(G)}(f) \in (0, \infty)$. The Fuglede–Kadison determinant shares many properties of the usual determinant, but at times it behaves strikingly different. For example the Fuglede–Kadison determinant for every zero morphism, regardless of U and V, equals 1 (see [27, p. 127]).

(7) A Hilbert $\mathcal{N}(G)$-chain complex C_* is called *L^2-acyclic* if its L^2-Betti numbers are zero. It is called det-*L^2-acyclic* if C_* is L^2-acyclic and if all the boundary maps $c_p: C_p \to C_{p-1}$ are of determinant class. If C_* is det-L^2-acyclic we define its *L^2-torsion* by

$$\rho^{(2)}(C_*) := -\sum_{p \in \mathbb{Z}} (-1)^p \cdot \ln\big(\det_{\mathcal{N}(G)}(c_p)\big).$$

The following proposition gives a convenient criterion for a matrix to be of determinant class. It is a consequence of [34, Theorem 1.21], see also [27, Chapter 13] or [12, Theorem 5].

Proposition 1.1. *Let G be a group and let A be an $m \times n$-matrix over $\mathbb{Z}G$. If G is residually finite, then the morphism $L^2(G)^m \to L^2(G)^n$ of $\mathcal{N}(G)$-modules defined by right multiplication by A is of determinant class.*

1.2. Euler structures and Spin^c-structures. Let X be a finite CW complex and let $p\colon \widetilde{X} \to X$ be the universal covering of X. Following Turaev [38–40], we define a *fundamental family of cells* to be a choice for each open cell in X of precisely one open cell in $\widetilde{X}$ which projects to the given cell in X.

We write $\pi = \pi_1(X)$ and we denote by $\psi\colon \pi \to H_1(\pi;\mathbb{Z}) = H_1(X;\mathbb{Z})$ the abelianization map. Now let $\{e_i\}_{i\in I}$ and $\{\widehat{e}_i\}_{i\in I}$ be two fundamental families of cells. After reordering them we can arrange that for each $i \in I$ we have $e_i = g_i\widehat{e}_i$ for some $g_i \in \pi$. We say that two fundamental families of cells are *equivalent* if

$$\sum_{i\in I}(-1)^{\dim(e_i)}\psi(g_i) = 0.$$

The set of equivalence classes of fundamental families of cells on X is called the set $\operatorname{Eul}(X)$ of *Euler structures on* X. Note that the set of Euler structures on X admits a free and transitive action by $H_1(X;\mathbb{Z})$.

We recall some basic facts regarding Spin^c-structures on 3-manifolds, with empty or toroidal boundary. We refer to [40, Chapter XI] for a detailed discussion. Given a 3-manifold M we denote by $\operatorname{Spin}^c(M)$ the set of Spin^c-structures on M. The set $\operatorname{Spin}^c(M)$ comes with a canonical free and transitive action by $H_1(M;\mathbb{Z})$. Given $\mathfrak{s} \in \operatorname{Spin}^c(M)$ we denote by $c_1(\mathfrak{s}) \in H^2(M,\partial M;\mathbb{Z}) = H_1(M;\mathbb{Z})$ its Chern class. The Chern class has the property that for each $\mathfrak{s} \in \operatorname{Spin}^c(M)$ and $h \in H_1(M;\mathbb{Z})$ the following equality holds

$$c_1(h\mathfrak{s}) = 2h + c_1(\mathfrak{s}). \tag{1.1}$$

In [40, 41] Turaev shows that given any CW-structure X for M there exists a canonical $H_1(M;\mathbb{Z}) = H_1(X;\mathbb{Z})$-equivariant bijection $\operatorname{Spin}^c(M) \to \operatorname{Eul}(X)$.

1.3. (L^2-acyclic) admissible pairs and the ϕ-twisted L^2-torsion function. In [7, 9] the authors and Dubois introduced the ϕ-twisted L^2-torsion function of a 3-manifold. This definition was later generalized and analyzed in [28, Section 7] for G-coverings of compact connected manifolds in all dimensions.

We start out with the following definitions.

Definition 1.2. (1) In the following, given any abelian group A we write

$$A_f := A/\operatorname{tors}(A).$$

(2) We say that a group homomorphism $\mu\colon \pi \to G$ is *$(H_1)_f$-factorizing*, if the projection map $\pi \to H_1(\pi;\mathbb{Z})_f$ factors through μ.

(3) An *admissible pair* (M,μ) consists of an irreducible 3-manifold $M \neq S^1 \times D^2$ with infinite fundamental group and a $(H_1)_f$-factorizing group homomorphism $\mu\colon \pi_1(M) \to G$ to a residually finite countable group G. Denote by $\overline{M} \to M$ the G-covering associated to μ. We say that (M,μ) is *L^2-acyclic* if the n-th L^2-Betti number $b_n^{(2)}(\overline{M};\mathcal{N}(G))$ vanishes for every $n \geq 0$.

Many of the subsequent results will hold in more general situations, e.g., it is not always necessary to assume that G is residually finite or that μ is $(H_1)_f$-factorizing. Nonetheless, in an attempt to keep the paper readable we will not state all the results in the maximal generality.

Convention 1.3. If $\mu\colon \pi \to G$ is a $(H_1)_f$-factorizing epimorphism, then we can and will identify $\operatorname{Hom}(\pi, \mathbb{R})$ with $\operatorname{Hom}(G, \mathbb{R})$. Furthermore, given any space X we make the usual identifications $H^1(X;\mathbb{R}) = \operatorname{Hom}(H_1(X;\mathbb{Z}),\mathbb{R}) = \operatorname{Hom}(\pi_1(X),\mathbb{R})$. In particular, if $(M, \mu\colon \pi_1(M) \to G)$ is an admissible pair, such that the cokernel of μ is finite, then any $\phi \in H^1(M;\mathbb{R})$ induces a unique homomorphism $G \to \mathbb{R}$ that, by a slight abuse of notation, we also denote by ϕ.

Lemma 1.4. *If $M \neq S^1 \times D^2$ is an irreducible 3-manifold with infinite fundamental group, then $(M, \mathrm{id}_{\pi_1(M)})$ is an L^2-acyclic admissible pair.*

Proof. Let $M \neq S^1 \times D^2$ be an irreducible 3-manifold with infinite fundamental group. It is a well-known consequence of the Geometrization Theorem proved by Perelman, that $\pi_1(M)$ is residually finite. We refer to [36, Theorem 3.3] and [17, Theorem 1.1] for details. (Note that Hempel writes "Haken manifold", but in the proof Hempel only uses the fact that Haken manifolds were known to satisfy the Geometrization Conjecture.) Furthermore, it is a consequence of the Geometrization Theorem and [24, Theorem 0.1] that M is L^2-acyclic. □

Now consider an L^2-acyclic admissible pair $(M; \mu\colon \pi_1(M) \to G)$ with Spin^c-structure $\mathfrak{s} \in \operatorname{Spin}^c(M)$. Let $\phi \in H^1(M;\mathbb{Q})$. We pick a CW-structure for M, which by abuse of notation we denote again by M. We denote by $\widetilde{M}$ the universal cover of M and we write $\pi = \pi_1(M)$. We pick a fundamental family of cells in $\widetilde{M}$ that corresponds to $\mathfrak{s}$.

This fundamental family of cells turns $C_*(\widetilde{M})$ into a chain complex of based free $\mathbb{Z}\pi$-left modules. (The basis is now unique up to permutation and multiplying each element with ± 1 which will not affect the Hilbert space structure and hence the ϕ-twisted L^2-torsion function below.) We view $\mathbb{Z}G$ as a right $\mathbb{Z}\pi$-module via the homomorphism μ. We obtain the chain complex $\mathbb{Z}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M})$ of based free $\mathbb{Z}G$-left modules.

Now let $t \in (0,\infty)$. We denote by $\phi^*\mathbb{C}_t$ the based 1-dimensional complex G-representation whose underlying complex vector space is $\mathbb{C}$ and on which $g \in G$ acts by multiplication with $t^{\phi(g)}$. Let $f\colon \mathbb{C}G \to \mathbb{C}G$ be a $\mathbb{C}G$-left linear map that is given by right multiplication with an element $\sum_{g\in G} \lambda_g \cdot g$. We define the $\phi^*\mathbb{C}_t$-twist of f as the $\mathbb{C}G$-left linear map $\mathbb{C}G \to \mathbb{C}G$ that is given by right multiplication with the element $\sum_{g\in G} \lambda_G \cdot t^{\phi(g)} \cdot g$. Similarly we can twist left linear maps $\mathbb{C}G^m \to \mathbb{C}G^n$ and the maps in the chain complex $\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M})$. Thus twisting $\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M})$ with $\phi^*\mathbb{C}_t$ yields a finite free $\mathbb{C}G$-chain complex $\eta_{\phi^*\mathbb{C}_t}(\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}))$ with a $\mathbb{C}G$-basis. (A more formal definition is to say, that given a $\mathbb{C}G$-left module V we define $\eta_{\phi^*\mathbb{C}_t}(V) := \phi^*\mathbb{C}_t \otimes_{\mathbb{C}} V$ where G acts diagonally on both terms. This point of view extends to chain complexes in an obvious way.)

Given a $\mathbb{C}G$-left linear map $A: \mathbb{C}G^m \to \mathbb{C}G^n$, we obtain by applying $L^2(G) \otimes_{\mathbb{C}G} -$, a morphism $\Lambda^G(A)$ of finitely generated Hilbert $\mathcal{N}(G)$-modules $L^2(G)^m \to L^2(G)^n$. Thus we obtain from $\eta_{\phi^*\mathbb{C}_t}(\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}))$ by applying $L^2(G) \otimes_{\mathbb{C}G} -$, a finite Hilbert $\mathcal{N}(G)$-chain complex denoted by $\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}))$. Note that for $t = 1$ this chain complex is just the chain complex $L^2(G) \otimes_{\mathbb{Z}G} C_*(\widetilde{M})$ which is L^2-acyclic by our hypothesis. Furthermore all boundary matrices are given by matrices defined over $\mathbb{Z}G$. Since G is residually finite it follows from Proposition 1.1 that the chain complex is of determinant class. It follows from [28, Theorem 6.7] that the chain complex $\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}))$ is in fact det-L^2-acyclic for any $t \in (0, \infty)$. In particular the $\mathcal{N}(G)$-chain complex $\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}))$ has well-defined L^2-torsion for any $t \in (0, \infty)$. We pick a homomorphism $\nu: G \to H_1(\pi; \mathbb{Z})$ such that the homomorphism $\nu \circ \mu: \pi \to H_1(\pi; \mathbb{Z})$ is just the usual projection map. Now define the *ϕ-twisted L^2-torsion function*

$$\begin{aligned} \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)\colon (0, \infty) &\to \mathbb{R} \\ t &\mapsto \rho^{(2)}\big(\Lambda^G \circ \eta_{(\phi\circ\nu)^*\mathbb{C}_t}\big(\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M})\big)\big). \end{aligned} \tag{1.2}$$

The right hand side is indeed independent of the choice of ν. Namely, if G' is the image of μ and $\mu': \pi \to G'$ is the epimorphism induced by μ, then there is precisely one homomorphism $\nu': G' \to H_1(\pi; \mathbb{Z})$ such that $\nu' \circ \mu'$ agrees with the projection $\pi \to H_1(\pi; \mathbb{Z})$ and we get from [28, Theorem 7.7 (7)] that

$$\rho^{(2)}\big(\Lambda^G \circ \eta_{(\phi\circ\nu)^*\mathbb{C}_t}(\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}))\big) = \rho^{(2)}\big(\Lambda^{G'} \circ \eta_{(\phi\circ\nu')^*\mathbb{C}_t}(\mathbb{C}G' \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}))\big).$$

More details of this construction and the proof that it is well-defined can be found in [28, Section 7] and, with slightly different conventions, in [9].

If μ is the identity homomorphism, then we drop it from the notation. Put differently, we write $\rho^{(2)}(M, \mathfrak{s}; \phi) := \rho^{(2)}(M, \mathfrak{s}; \mathrm{id}_{\pi_1(M)}, \phi)$.

1.4. Comparing the ϕ-twisted L^2-torsion function and the L^2-Alexander torsion. The ϕ-twisted L^2-torsion function $\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)\colon (0, \infty) \to \mathbb{R}$, as considered in this paper and in [28], is designed in an additive setup, as it is the main convention when dealing with related invariants such as topological L^2-torsion, analytic L^2-torsion, analytic Ray–Singer torsion and so on. When dealing with torsion invariants in dimension 3, the multiplicative setting is standard, which is the reason why we defined for instance in [7,9] the L^2-Alexander torsion multiplicatively as a function $\tau^{(2)}(M, \mathfrak{s}; \phi, \mu)\colon (0, \infty) \to [0, \infty)$.

Now suppose that (M, μ) is L^2-acyclic. As we had already pointed out in the previous section, it follows from [28, Theorem 6.7] that the function $\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)$ is defined on all of $(0, \infty)$. It follows immediately from comparing the definition of $\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)$ in Section 1.1 and the previous section of the present paper with the definition of $\tau^{(2)}(M, \mathfrak{s}; \phi, \mu)$ in [9, Section 3.1] that these two invariants are related by the formula

$$\tau^{(2)}(M, \mathfrak{s}; \phi, \mu) = \exp\big(-\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)\big). \tag{1.3}$$

Notice that this discussion shows that $\tau^{(2)}(M, \mathfrak{s}; \phi, \mu)$ never takes the value zero. This is a consequence of [28, Theorem 6.7] which was not available when [9] was finished. In the following we will cite results from [7, 9] about $\tau^{(2)}(M, \mathfrak{s}; \phi, \mu)$, which via (1.3) we reinterpret as results on $\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)$.

1.5. Properties of the ϕ-twisted L^2-torsion function. The following theorem summarizes some of the key properties of the ϕ-twisted L^2-torsion function.

Theorem 1.5 (Properties of the twisted L^2-torsion function). *Let (M, μ) be an L^2-acyclic admissible pair, let $\phi \in H^1(M; \mathbb{R})$ and let $\mathfrak{s} \in \mathrm{Spin}^c(M)$.*

(1) Pinching estimate. *There exist constants C and D such that we get for $0 < t \leq 1$*

$$C \cdot \ln(t) - D \leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \leq -C \cdot \ln(t) + D,$$

and for $t \geq 1$

$$-C \cdot \ln(t) - D \leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \leq C \cdot \ln(t) + D;$$

(2) Dependence on the Spin^c-structure. *For any $h \in H_1(M; \mathbb{Z})$ we have*

$$\rho^{(2)}(M, h\mathfrak{s}; \mu, \phi) = \rho^{(2)}(M, \mathfrak{s}; \mu, \phi) + \ln(t) \cdot \phi(h).$$

(3) Covering formula. *Let $p\colon \widehat{M} \to M$ be a finite regular covering such that $\ker(\mu) \subset \widehat{\pi} := \pi_1(\widehat{M})$. We write $\widehat{\phi} := p^*\phi$ and we denote by $\widehat{\mu}$ the restriction of μ to $\widehat{\pi}$. Finally we write $\widehat{\mathfrak{s}} := p^*(\mathfrak{s})$. Then for all t we have*

$$\rho^{(2)}(\widehat{M}, \widehat{\mathfrak{s}}; \widehat{\phi}, \widehat{\mu})(t) = [\widehat{M} : M] \cdot \rho^{(2)}(M, \mathfrak{s}, \phi, \mu)(t).$$

(4) Scaling ϕ. *Let $r \in \mathbb{R}$. Then we get for all $t \in (0, \infty)$*

$$\rho^{(2)}(M, \mathfrak{s}; \mu, r\phi)(t) = \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t^r).$$

(5) Symmetry. *For any $t \in (0, \infty)$ we have*

$$\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t^{-1}) = \phi(c_1(\mathfrak{s})) \cdot \ln(t) + \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t).$$

Statement (1) is proved in [28, Theorem 7.4 (i)], it is one of the main results of that paper. Statement (2) is proved in [9] and [7]. Statement (3) is proved in [28, Theorem 5.7 (6)] and [9, Lemma 5.3] without explicitly mentioning Spin^c-structures. Nonetheless, it is straightforward to see that the proofs provided in the literature also imply the statement about Spin^c-structures. Statement (4) is basically a tautology, see [28, Theorem 7.4 (5)] and [9, Lemma 5.2]. Finally Statement (5) is obtained in the proof of Theorem 1.1 of [7].

Define two functions $f_0, f_1\colon (0,\infty) \to \mathbb{R}$ to be *equivalent* if there is an $m \in \mathbb{R}$ such that $f_1(t) - f_0(t) = m \cdot \ln(t)$ holds. Because of Theorem 1.5 (2) the equivalence class of the function $\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)$ defined in (1.2) is independent of the choice of the Spin^c-structure, and will be denoted by

$$\overline{\rho}^{(2)}(M; \mu, \phi). \tag{1.4}$$

Theorem 1.5 (1) allows us to define the degree of $\overline{\rho}^{(2)}(M; \mu, \phi)$ by

$$\deg\bigl(\overline{\rho}^{(2)}(M; \mu, \phi)\bigr) = \limsup_{t\to\infty} \frac{\rho(t)}{\ln(t)} - \liminf_{t\to 0} \frac{\rho(t)}{\ln(t)} \tag{1.5}$$

for any representative $\rho\colon (0,\infty) \to \mathbb{R}$ of $\overline{\rho}^{(2)}(M; \mu, \phi)$.

Remark 1.6. Notice the minus sign appearing in the formula (1.3). This has the consequence that the degree $\deg(\tau^{(2)}(M, \phi, \mu))$ defined in [9] and the degree $\deg(\overline{\rho}^{(2)}(M, \mu, \phi))$ defined in the introduction and later again in (1.5) are related by

$$\deg\bigl(\tau^{(2)}(M, \phi, \mu)\bigr) = -\deg\bigl(\overline{\rho}^{(2)}(M, \mu, \phi)\bigr). \tag{1.6}$$

1.6. Approximation. The following is a consequence of one of the main technical results of [28].

Theorem 1.7 (Twisted Approximation inequality). *Let $\phi\colon G \to \mathbb{R}$ be a group homomorphism whose image is finitely generated.*

Consider a nested sequence of normal subgroups of G

$$G \supseteq G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

such that G_i is contained in $\ker(\phi)$ and the intersection $\bigcap_{i\ge 0} G_i$ is trivial. Suppose that the index $[\ker(\phi) : G_i]$ is finite for all $i \ge 0$. Put $Q_i := G/G_i$. Let $\phi_i\colon Q_i \to \mathbb{R}$ be the homomorphism uniquely determined by $\phi_i \circ \mathrm{pr}_i = \phi$, where $\mathrm{pr}_i\colon G \to Q_i$ is the canonical projection.

Fix an (r,s)-matrix $A \in M_{r,s}(\mathbb{Z}G)$. Denote by $A[i]$ the image of A under the map $M_{r,s}(\mathbb{Z}G) \to M_{r,s}(\mathbb{Z}Q_i)$ induced by the projection pr_i.

Then we get

$$\dim_{\mathcal{N}(G)}\bigl(\ker\bigl(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_A)\bigr)\bigr) = \lim_{i\to\infty} \dim_{\mathcal{N}(Q_i)}\bigl(\ker\bigl(\Lambda^{Q_i} \circ \eta_{\phi_i^*\mathbb{C}_t}(r_{A[i]})\bigr)\bigr)$$

and

$$\det_{\mathcal{N}(G)}\bigl(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_A)\bigr) \ge \limsup_{i\to\infty} \det_{\mathcal{N}(Q_i)}\bigl(\Lambda^{Q_i} \circ \eta_{\phi_i^*\mathbb{C}_t}(r_{A[i]})\bigr).$$

Proof. Since the image of ϕ is finitely generated, we can choose a monomorphism $j\colon \mathbb{Z}^d \to \mathbb{R}$ and an epimorphism $\phi'\colon G \to \mathbb{Z}^d$ with $\phi = j \circ \phi$. Now we apply [28, Theorem 6.52] to ϕ' in the special case $V = j^*\mathbb{C}_t$. □

1.7. The Thurston norm. Recall the definition in [37] of the *Thurston norm* $x_M(\phi)$ of a 3-manifold M and an element $\phi \in H^1(M;\mathbb{Z}) = \operatorname{Hom}(\pi_1(M),\mathbb{Z})$:

$$x(\phi) := \min\bigl\{\chi_-(F) \mid F \subset M \text{ properly embedded surface dual to } \phi\bigr\},$$

where, given a surface F with connected components $F_1, F_2, \dots, F_k$, we define

$$\chi_-(F) := \sum_{i=1}^{k} \max\bigl\{-\chi(F_i), 0\bigr\}.$$

Thurston [37] showed that this defines a seminorm on $H^1(M;\mathbb{Z})$ which can be extended to a seminorm on $H^1(M;\mathbb{R})$ which we also denote by x_M. In particular we get for $r \in \mathbb{R}$ and $\phi \in H^1(M;\mathbb{R})$

$$x_M(r\cdot\phi) = |r|\cdot x_M(\phi). \tag{1.7}$$

If $p\colon \widetilde{M} \to M$ is a finite covering with n sheets, then Gabai [15, Corollary 6.13] showed that

$$x_{\widetilde{M}}(p^*\phi) = n\cdot x_M(\phi). \tag{1.8}$$

If $F \to M \xrightarrow{p} S^1$ is a fiber bundle for a 3-manifold M and compact surface F, and $\phi \in H^1(M;\mathbb{Z})$ is given by $H_1(p)\colon H_1(M) \to H_1(S^1) = \mathbb{Z}$, then by [37, Section 3] we have

$$x_M(\phi) = \begin{cases} -\chi(F), & \text{if } \chi(F) \le 0;\\ 0, & \text{if } \chi(F) \ge 0.\end{cases} \tag{1.9}$$

2. Calculating the ϕ-twisted L^2-torsion function

The following theorem says that given M and $\psi \in H^1(M;\mathbb{Q})$ the corresponding L^2-torsion functions can be computed using one fixed square matrix over $\mathbb{Z}\pi_1(M)$ together with a well-understood error term.

Theorem 2.1. *Let M be a 3-manifold with $b_1(M) > 0$ and let $\mathfrak{s} \in \operatorname{Spin}^c(M)$. We write $\pi = \pi_1(M)$.*

(1) *Suppose ∂M is non-empty and toroidal. Then there exists an $s \in \pi_1(M)$ and a square matrix A over $\mathbb{Z}\pi$ such that the following conditions are satisfied for any $(H_1)_f$-factorizing homomorphism $\mu\colon \pi \to G$ and any homomorphism $\phi\colon G \to \mathbb{R}$:*

   (a) *$b_n^{(2)}(\overline{M};\mathcal{N}(G)) = 0$ holds for all $n \ge 0$ if and only if*

$$\dim_{\mathcal{N}(G)}\bigl(\ker\bigl(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_{\mu(A)})\bigr)\bigr)$$

   vanishes for all $t > 0$.

(b) *If* (a) *is the case, then* (M,μ) *is* ϕ*-twisted* det-L^2*-acyclic (in the sense of* [28, Definition 7.1]) *and we get*

$$\rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) = -\ln\bigl(\det_{\mathcal{N}(G)}\bigl(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_{\mu(A)})\bigr)\bigr) + \eta(t)$$

where $\eta(t)$ *is given by*

$$\eta(t) = \max\bigl\{0, |\phi(s)|\cdot\ln(t)\bigr\}.$$

(2) *Suppose* M *is closed. Then there exist* $s,s'\in\pi_1(M)$ *and a square matrix* A *over* $\mathbb{Z}\pi$ *such that the following conditions are satisfied for any* $(H_1)_f$*-factorizing homomorphism* $\mu\colon\pi\to G$ *and any homomorphism* $\phi\colon G\to\mathbb{R}$*:*

(a) $b_n^{(2)}(\overline{M};\mathcal{N}(G)) = 0$ *holds for all* $n\geq 0$ *if and only if*

$$\dim_{\mathcal{N}(G)}\bigl(\ker\bigl(\Lambda^G\circ\eta_{\phi^*\mathbb{C}_t}(r_{\mu(A)})\bigr)\bigr)$$

vanishes for all $t>0$.

(b) *If* (a) *is the case, then* (M,μ) *is* ϕ*-twisted* det-L^2*-acyclic and we get*

$$\rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) = -\ln\bigl(\det_{\mathcal{N}(G)}(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_{\mu(A)}))\bigr) + \eta(t)$$

where $\eta(t)$ *is given by*

$$\eta(t) = \max\bigl\{0, |\phi(s)|\cdot\ln(t)\bigr\} + \max\bigl\{0, |\phi(s')|\cdot\ln(t)\bigr\}.$$

Proof. We only treat the case, where ∂M is empty, and leave it to the reader to figure out the details for the case of a non-empty boundary using the proof of [26, Theorem 2.4]. From [30, Proof of Theorem 5.1] we obtain the following:

(1) a compact 3-dimensional CW-complex X together with a homeomorphism $f\colon X\to M$ (in the following we identify $\pi=\pi_1(M)=\pi_1(X)$ using $\pi_1(f)$),

(2) two sets of generators $\{s_1,\dots,s_a\}$ and $\{s'_1,\dots,s'_a\}$ of π,

(3) an $a\times a$-matrix F over $\mathbb{Z}\pi$,

such that the cellular $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{X})$ of the universal cover $\widetilde{X}$, for an appropriate fundamental family of cells, is isomorphic to

$$\mathbb{Z}\pi \xrightarrow{\prod_{i=1}^a r_{s'_i-1}} \bigoplus_{i=1}^a \mathbb{Z}\pi \xrightarrow{r_F} \bigoplus_{i=1}^a \mathbb{Z}\pi \xrightarrow{\bigoplus_{i=1}^a r_{s_i-1}} \mathbb{Z}\pi.$$

It follows that the based $\mathbb{Z}G$-chain complex $\mathbb{Z}G\otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})$ is isomorphic to

$$\mathbb{Z}G \xrightarrow{\prod_{i=1}^a r_{\mu(s'_i)-1}} \bigoplus_{i=1}^a \mathbb{Z}G \xrightarrow{r_{\mu(F)}} \bigoplus_{i=1}^a \mathbb{Z}G \xrightarrow{\bigoplus_{i=1}^a r_{\mu(s_i)-1}} \mathbb{Z}G.$$

Then the Hilbert $\mathcal{N}(G)$-chain complex $\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(C_*(\overline{X}))$ is isomorphic to

$$L^2(G) \xrightarrow{\prod_{i=1}^{a} \Lambda^G\left(r_{t^{\phi(s'_1)}\cdot\mu(s'_i)-1}\right)} \bigoplus_{i=1}^{a} L^2(G) \xrightarrow{\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_{\mu(F)})} \bigoplus_{i=1}^{a} L^2(G) \xrightarrow{\bigoplus_{i=1}^{a} \Lambda^G\left(r_{t^{\phi(s_i)}\cdot\mu(s_i)-1}\right)} L^2(G).$$

Since $b_1(M) > 0$ is non-trivial there exist $i, j \in \{1, \ldots, a\}$ such that s_i and s'_j represent non-zero elements in $H_1(M;\mathbb{Z})_f$. We write $s = s_i$ and $s' = s'_j$. For later we record that, given any $(H_1)_f$-factorizing homomorphism $\mu\colon \pi \to G$, the images $\mu(s)$ and $\mu(s')$ have infinite order. We denote by A the matrix that is obtained from F by removing the i-th column and the j-th row.

For $g \in G$ and $t \in (0, \infty)$ let $D(g,t)_*$ be the Hilbert $\mathcal{N}(G)$-chain complex that is given by

$$0 \to L^2(G) \xrightarrow{\Lambda^G\left(r_{t^{\phi(g)}\cdot g-1}\right)} L^2(G) \to 0$$

where the non-zero terms are in the degrees 1 and 0. Provided that $|g| = \infty$ holds, $D(g,t)_*$ is det-L^2-acyclic and a direct computation using [27, Theorem 3.14 (6) on p. 129 and (3.23) on p. 136] shows

$$\rho^{(2)}\big(D(g,t)_*\big) = \ln\big(\det_{\mathcal{N}(G)}\big(\Lambda^G\big(r_{t^{-\phi(g)}\cdot g-1}\big)\big)\big) = \max\big\{|\phi(g)| \cdot \ln(t), 0\big\}. \quad (2.1)$$

Now let $\mathfrak{s} \in \mathrm{Spin}^c(M)$ be the Spin^c-structure that corresponds to the above fundamental family of cells. It follows from [9, Lemma 3.2] that the above group elements s, s' and the matrix A have all the desired properties regarding the L^2-Betti numbers and the L^2-torsion. This concludes the proof of the theorem in the closed case for the Spin^c-structure $\mathfrak{s}$.

If $\mathfrak{t} \in \mathrm{Spin}^c(M)$ is a different Spin^c-structure, then we can write $\mathfrak{t} = h\mathfrak{s}$ for some $h \in H_1(M;\mathbb{Z})$. We pick a representative $g \in \pi$ of h and we multiply one column of A by h to obtain the matrix with the desired properties. □

3. Lower bounds

The elementary proof of the next lemma can be found in [28, Lemma 6.9].

Lemma 3.1. *Let $f\colon L^2(G)^m \to L^2(G)^n$ be a bounded G-equivariant operator. Then*

$$\det_{\mathcal{N}(G)}(f) \le \|f\|^{\dim_{\mathcal{N}(G)}(\overline{\mathrm{im}(f)})}.$$

Here $\|f\|$ denotes the operator norm of f, i.e.

$$\|f\| = \sup\big\{\|f(v)\|_{L^2(G)^n} \,\big|\, v \in L^2(G)^m \text{ with } \|v\|_{L^2(G)^m} = 1\big\}.$$

Before we state the next lemma we introduce the following definition. We say that a bounded G-equivariant operator $f\colon L^2(G)^m \to L^2(G)^m$ is det-L^2-*acyclic* if the chain complex

$$0 \to L^2(G)^m \xrightarrow{f} L^2(G)^m \to 0$$

is det-L^2-acyclic. Now we can formulate the next result which is an improvement of [9, Proposition 9.5].

Lemma 3.2. *Consider bounded G-equivariant operators $f_0, f_1\colon L^2(G)^m \to L^2(G)^m$. For $t > 0$ we define*

$$f[t] := f_0 + t \cdot f_1.$$

Suppose that for every $t > 0$ the operator $f[t]\colon L^2(G)^m \to L^2(G)^m$ is det-L^2-*acyclic. Put*

$$\rho\colon (0,\infty) \to (0,\infty), \quad t \mapsto \ln\big(\det_{\mathcal{N}(G)}\big(f[t]\big)\big).$$

Then we get

$$\begin{aligned} &\rho(t) \le m \cdot \max\big\{0, \ln\big(\|f_0\| + \|f_1\|\big)\big\} && \textit{for } t \le 1;\\ &\rho(t) \le \dim_{\mathcal{N}(G)}\big(\overline{\operatorname{im}(f_1)}\big) \cdot \ln(t) + m \cdot \max\big\{0, \ln\big(2 \cdot \|f_0\| + \|f_1\|\big)\big\} && \textit{for } t \ge 1. \end{aligned}$$

In particular we get

$$\begin{aligned} \limsup_{t\to\infty} \frac{\rho(t)}{\ln(t)} &\le \dim_{\mathcal{N}(G)}\big(\overline{\operatorname{im}(f_1)}\big);\\ \liminf_{t\to 0} \frac{\rho(t)}{\ln(t)} &\ge 0;\\ \limsup_{t\to\infty} \frac{\rho(t)}{\ln(t)} - \liminf_{t\to 0} \frac{\rho(t)}{\ln(t)} &\le \dim_{\mathcal{N}(G)}\big(\overline{\operatorname{im}(f_1)}\big). \end{aligned}$$

Proof. It suffices to prove the two inequalities for $\rho(t)$, then the other claims follow.

We begin with the case $t \le 1$. We get from Lemma 3.1

$$\det_{\mathcal{N}(G)}\big(f[t]\big) \le \|f[t]\|^{\dim_{\mathcal{N}(G)}(\overline{\operatorname{im}(f[t])})}.$$

If $\|f[t]\| \le 1$, this implies $\det_{\mathcal{N}(G)}(f[t]) \le 1$ and the claim follows. Hence it remains to treat the case $\|f[t]\| > 1$. Because of $\dim_{\mathcal{N}(G)}(\overline{\operatorname{im}(f)}) \le m$ we get that

$$\begin{aligned} \det_{\mathcal{N}(G)}\big(f[t]\big) &\le \|f[t]\|^m\\ &= \|f_0 + t \cdot f_1\|^m\\ &\le \big(\|f_0\| + t \cdot \|f_1\|\big)^m\\ &\overset{t\le 1}{\le} \big(\|f_0\| + \|f_1\|\big)^m. \end{aligned}$$

Next we consider the case $t \geq 1$. We have the orthogonal decomposition

$$L^2(G)^m = \overline{\operatorname{im}(f_1)} \oplus \overline{\operatorname{im}(f_1)}^{\perp}.$$

With respect to this decomposition we get for any bounded G-equivariant operator $g\colon L^2(G)^m \to L^2(G)^m$ the decomposition

$$g = \begin{pmatrix} g^{(1,1)} & g^{(1,2)} \\ g^{(2,1)} & g^{(2,2)} \end{pmatrix}.$$

We estimate for $t \geq 1$ using [27, Theorem 3.14 (1) and (2) on p. 128]

$$\begin{aligned}
\frac{\det_{\mathcal{N}(G)}(f[t])}{t^{\dim_{\mathcal{N}(G)}(\overline{\operatorname{im}(f_1)})}} &= \det_{\mathcal{N}(G)}\left(\begin{pmatrix} t^{-1}\cdot \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}\right) \cdot \det_{\mathcal{N}(G)}\big(f[t]\big) \\
&= \det_{\mathcal{N}(G)}\left(\begin{pmatrix} t^{-1}\cdot \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix} \circ f[t]\right) \\
&\overset{\text{Lem. 3.1}}{\leq} \left\| \begin{pmatrix} t^{-1}\cdot \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix} \circ f[t] \right\|^m.
\end{aligned}$$

If $\|\left(\begin{smallmatrix} t^{-1}\cdot \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right) \circ f[t]\| \leq 1$ the claim is obviously true. Hence it remains to treat the case $\|\left(\begin{smallmatrix} t^{-1}\cdot \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right) \circ f[t]\| \geq 1$. Then we get

$$\begin{aligned}
&\frac{\det_{\mathcal{N}(G)}(f[t])}{t^{\dim_{\mathcal{N}(G)}(\overline{\operatorname{im}(f_1)})}} \\
&\leq \left\| \begin{pmatrix} t^{-1}\cdot \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix} (f_0 + t\cdot f_1) \right\|^m \\
&= \left\| \begin{pmatrix} t^{-1} f_0^{(1,1)} & t^{-1} f_0^{(1,2)} \\ f_0^{(2,1)} & f_0^{(2,2)} \end{pmatrix} + \begin{pmatrix} f_1^{(1,1)} & f_1^{(1,2)} \\ 0 & 0 \end{pmatrix} \right\|^m \\
&\leq \left(t^{-1} \cdot \left\| \begin{pmatrix} f_0^{(1,1)} & f_0^{(1,2)} \\ 0 & 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 & 0 \\ f_0^{(2,1)} & f_0^{(2,2)} \end{pmatrix} \right\| + \left\| \begin{pmatrix} f_1^{(1,1)} & f_1^{(1,2)} \\ 0 & 0 \end{pmatrix} \right\| \right)^m \\
&\overset{t^{-1}\leq 1}{\leq} \left(\left\| \begin{pmatrix} f_0^{(1,1)} & f_0^{(1,2)} \\ 0 & 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 & 0 \\ f_0^{(2,1)} & f_0^{(2,2)} \end{pmatrix} \right\| + \left\| \begin{pmatrix} f_1^{(1,1)} & f_1^{(1,2)} \\ 0 & 0 \end{pmatrix} \right\| \right)^m \\
&\leq \left(\left\| \begin{pmatrix} f_0^{(1,1)} & f_0^{(1,2)} \\ f_0^{(2,1)} & f_0^{(2,2)} \end{pmatrix} \right\| + \left\| \begin{pmatrix} f_0^{(1,1)} & f_0^{(1,2)} \\ f_0^{(2,1)} & f_0^{(2,2)} \end{pmatrix} \right\| + \left\| \begin{pmatrix} f_1^{(1,1)} & f_1^{(1,2)} \\ 0 & 0 \end{pmatrix} \right\| \right)^m \\
&= \big(2\cdot \|f_0\| + \|f_1\|\big)^m.
\end{aligned}$$

This finishes the proof of Lemma 3.2. □

For an element $x = \sum_{g\in G} x_g \cdot g$ in $\mathbb{C}G$ define $|x|_1 := \sum_{g\in G} |x_g|$. Given a matrix $A \in M_{r,s}(\mathbb{C}G)$ define

$$\|A\|_1 = r\cdot s\cdot \max\big\{|a_{j,k}|_1 \;\big|\; 1 \leq j \leq r, 1 \leq k \leq s\big\}. \tag{3.1}$$

The next theorem can be viewed as saying that, in the acyclic case, the degree of the ϕ-twisted L^2-torsions gives lower bounds on the Thurston norm. This result is thus an analogue of the classical fact, mentioned already in the introduction, that the degree of the Alexander polynomial gives a lower bound on the knot genus [35].

Theorem 3.3 (Lower bound). *Let M be an irreducible 3-manifold with infinite fundamental group π. Let $\mathfrak{s} \in \mathrm{Spin}^c(M)$. Then for any $\phi \in H^1(M;\mathbb{Q})$ there exists a constant $D \geq 0$ such that for any $(H_1)_f$-factorizing homomorphism $\mu\colon \pi_1(M) \to G$, for which (M,μ) is L^2-acyclic, we have*

$$\tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) + x_M(\phi)\big) \cdot \ln(t) - D \leq \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \quad \text{for } t \leq 1;$$
$$\tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) - x_M(\phi)\big) \cdot \ln(t) - D \leq \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \quad \text{for } t \geq 1.$$

In [9, Theorem 1.5] we proved the analogous statement under the extra assumption that $\mu\colon \pi_1(M) \to G$ is a homomorphism to a virtually abelian group.

In the proof of Theorem 3.3 we will make use of the following elementary lemma. Before we state the lemma, recall that a cohomology class in $H^1(X;\mathbb{Z}) = \mathrm{Hom}(\pi_1(X),\mathbb{Z})$ is *primitive* if the corresponding homomorphism $\pi_1(X) \to \mathbb{Z}$ is surjective.

Lemma 3.4. *Let M be an irreducible 3-manifold with infinite fundamental group and let $\mathfrak{s} \in \mathrm{Spin}^c(M)$. If the conclusion of Theorem 3.3 holds for all primitive $\phi \in H^1(M;\mathbb{Z})$, then it holds for all $\phi \in H^1(M;\mathbb{Q})$.*

Proof. If ϕ is trivial, then clearly there is nothing to prove. So let $\phi \in H^1(M;\mathbb{Q})$ be non-zero. We pick an $r \in \mathbb{Q}_{>0}$ such that $r\phi \in H^1(M;\mathbb{Z})$ is primitive. We denote by D the constant of Theorem 3.3 corresponding to the primitive class $r\phi$.

From Theorem 1.5 (4) and from (1.7) we get for any $(H_1)_f$-factorizing homomorphism $\mu\colon \pi_1(M) \to G$, for which (M,μ) is L^2-acyclic, that

$$\rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) = \rho^{(2)}(M,\mathfrak{s};\mu,r\phi)\big(t^{\frac{1}{r}}\big);$$
$$x_M(r\phi) = r \cdot x_M(\phi).$$

Combining these equalities with the elementary equalities

$$\ln\big(t^{\frac{1}{r}}\big) = \frac{1}{r}\ln(t);$$
$$(r\phi)(c_1(\mathfrak{s})) = r \cdot \phi(c_1(\mathfrak{s})),$$

it is straightforward to see that the desired inequalities also hold for μ and ϕ. □

Proof of Theorem 3.3. By Lemma 3.4 it suffices to prove the statement for every primitive $\phi \in H^1(M;\mathbb{Z})$. We start out with the following claim.

Claim. Given a primitive $\phi \in H^1(M;\mathbb{Z})$ there exists an $\mathfrak{s} \in \mathrm{Spin}^c(M)$ such that for any $(H_1)_f$-factorizing homomorphism $\mu\colon \pi_1(M) \to G$, for which (M,μ) is L^2-acyclic, the following inequalities hold

$$\begin{aligned} -D &\le \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \quad \text{for } t \le 1;\\ -x_M(\phi)\cdot \ln(t) - D &\le \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \quad \text{for } t \ge 1. \end{aligned}$$

In the following we abbreviate

$$\rho(\mu,\phi) = \rho^{(2)}(M,\mathfrak{s};\mu,\phi).$$

We conclude by inspecting the proof of [9, Proposition 9.1 in Section 9.1] that there exists:

(1) a Spin^c-structure $\mathfrak{s}$,

(2) integers k,l,m with $k,l \ge 0$ and $x_M(\phi) = k - l$,

(3) an element $\gamma \in \pi$ with $\phi(\gamma) = 1$, and

(4) a matrix $A \in M_{k+m,k+m}(\mathbb{Z}K)$, where $K = \ker(\phi)$,

such that for any $(H_1)_f$-factorizing homomorphism $\mu\colon \pi_1(M) \to G$, for which (M,μ) is L^2-acyclic, the following equality holds

$$\rho(\mu,\phi)(t) = -\ln\big(\max\{1,t\}^{-l}\cdot \det_{\mathcal{N}(G)}\big(\Lambda^G(r_{\mu(A)}) + t\cdot\mu(\gamma)\cdot \mathrm{id}_{L^2(G)^k} \oplus 0_{L^2(G)^m}\big)\big).$$

This implies

$$\rho(\mu,\phi)(t) = \begin{cases} -\ln\big(\det_{\mathcal{N}(G)}\big(\Lambda^G(r_{\mu(A)}) + t\cdot\mu(\gamma)\cdot \mathrm{id}_{L^2(G)^k} \oplus 0_{L^2(G)^m}\big)\big) \\ \hfill \text{if } t \le 1;\\ l\cdot\ln(t) - \ln\big(\det_{\mathcal{N}(G)}\big(\Lambda^G(r_{\mu(A)}) + t\cdot\mu(\gamma)\cdot \mathrm{id}_{L^2(G)^k} \oplus 0_{L^2(G)^m}\big)\big) \\ \hfill \text{if } t \ge 1. \end{cases}$$

Define

$$D = (k+m)\cdot \ln\big(2\cdot(\|A\|_1 + 1)\big).$$

Note that D depends on ϕ but not on μ. We conclude from [28, Lemma 6.3] and the monotonicity of ln that

$$\begin{aligned} D &\ge (k+m)\cdot\ln\big(2\cdot\|\Lambda^G(r_{\mu(A)})\| + \|\mathrm{id}_{L^2(G)^k}\oplus 0_{L^2(G)^m}\|\big)\\ &\ge (k+m)\cdot\ln\big(\|\Lambda^G(r_{\mu(A)})\| + \|\mathrm{id}_{L^2(G)^k}\oplus 0_{L^2(G)^m}\|\big). \end{aligned}$$

Therefore we conclude from Lemma 3.2, applied to the case $f_0 = \Lambda^G(r_{\mu(A)})$ and $f_1 = \mu(\gamma)\cdot \mathrm{id}_{L^2(G)^k}\oplus 0_{L^2(G)^m}$, that

$$\ln\big(\det_{\mathcal{N}(G)}\big(\Lambda^G(r_{\mu(A)}) + t\cdot\mu(\gamma)\cdot\mathrm{id}_{L^2(G)^k}\oplus 0_{L^2(G)^m}\big)\big) \le \begin{cases} D, & t \le 1;\\ k\cdot\ln(t) + D, & t \ge t. \end{cases}$$

This implies

$$\begin{aligned} -D &\le \rho(\mu,\phi)(t) \quad \text{for } t \le 1; \\ -(k-l)\cdot \ln(t) - D &\le \rho(\mu,\phi)(t) \quad \text{for } t \ge 1. \end{aligned}$$

Since $x_M(\phi) = k - l$, this implies the claim.

We now turn to the proof of the desired inequalities in the theorem. Using Theorem 1.5 (2) and equality (1.1) one can easily see that if the desired inequalities hold for one Spin^c-structure of M, then they also hold for all other Spin^c-structures of M. Now let $\mathfrak{s} \in \operatorname{Spin}^c(M)$ be the Euler structure from the claim. Then:

$$\begin{aligned} -D &\le \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \quad \text{for } t \le 1; \\ -x_M(\phi)\cdot \ln(t) - D &\le \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \quad \text{for } t \ge 1. \end{aligned}$$

By Theorem 1.5 (5) we also know that

$$\rho(M,\mathfrak{s};\mu,\phi)(t) = -\phi(c_1(\mathfrak{s}))\ln(t) + \rho(M,\mathfrak{s};\mu,\phi)(t^{-1})$$

for all $t \in (0,\infty)$. Combining this equality with the above inequalities we obtain that

$$\begin{aligned} (\phi(c_1(\mathfrak{s})) + x_M(\phi))\cdot \ln(t) - D &\le \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \quad \text{for } t \le 1; \\ \phi(c_1(\mathfrak{s}))\cdot \ln(t) - D &\le \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \quad \text{for } t \ge 1. \end{aligned}$$

Adding the two inequalities for $t \le 1$ and dividing by two, and doing the same for the inequalities for $t \ge 1$ gives us the desired inequalities

$$\begin{aligned} \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) + x_M(\phi)\big)\cdot \ln(t) - D &\le \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \quad \text{for } t \le 1; \\ \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) - x_M(\phi)\big)\cdot \ln(t) - D &\le \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \quad \text{for } t \ge 1. \qquad \square \end{aligned}$$

4. Upper bounds

Before we can provide upper bounds on the Thurston norm we will need to prove one preliminary result. This lemma will ensure that some information which is only available at 0 and ∞ leads to uniform estimates for all $t > 0$. This will be a key ingredient when we want to apply approximation techniques.

Lemma 4.1. *Let $\phi\colon G \to \mathbb{Z}$ be a non-trivial group homomorphism with finite kernel. Let $A \in M_{m,m}(\mathbb{Z}G)$ be a matrix such that $\Lambda^G(r_A)\colon L^2(G)^m \to L^2(G)^m$ is a weak isomorphism. Then $\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_A)\colon L^2(G)^m \to L^2(G)^m$ is* det-L^2*-acyclic for any $t > 0$. Put*

$$\rho\colon (0,\infty) \to \mathbb{R}, \quad t \mapsto \ln\big(\det_{\mathcal{N}(G)}\big(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_A)\big)\big).$$

Suppose that there are real numbers C and D and integers k and l such that

$$\lim_{t\to 0} \rho(t) - k \cdot \ln(t) = C;$$
$$\lim_{t\to \infty} \rho(t) - l \cdot \ln(t) = D.$$

Then we get for all $t > 0$

$$k \cdot \ln(t) + C \le \rho(t);$$
$$l \cdot \ln(t) + D \le \rho(t).$$

Proof. Choose an integer $n \ge 1$ and an epimorphism $\phi' \colon G \to \mathbb{Z}$ such that $\phi = n \cdot \mathrm{id}_{\mathbb{Z}} \circ \phi'$. Then we get for the two functions ρ and ρ' associated to ϕ and ϕ' from Theorem 1.5 (4)

$$\rho'(t) = \rho(t^n).$$

Hence we can assume without loss of generality that ρ is surjective, otherwise replace ϕ by ϕ'.

Choose a group homomorphism $s \colon \mathbb{Z} \to G$ with $\phi \circ s = \mathrm{id}$. Choose a map of sets $\sigma \colon \mathrm{im}(s)\backslash G \to G$ whose composition with the projection $\mathrm{pr} \colon G \to \mathrm{im}(s)\backslash G$ is the identity and whose composition with $\phi \colon G \to \mathbb{Z}$ is the constant map with value $0 \in \mathbb{Z}$. Let $B \in M_{m\cdot|\ker(\phi)|, m\cdot|\ker(\phi)|}(\mathbb{Z}[\mathbb{Z}])$ be the matrix describing the restriction of $r_A \colon \mathbb{Z}G^m \to \mathbb{Z}G^m$ with s, see [28, (6.40)]. Then a direct computation shows for all $t \in (0, \infty)$

$$s^*\big(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_A)\big) = \Lambda^{\mathbb{Z}} \circ \eta_{(\phi\circ s)^*\mathbb{C}_t}(r_B) \colon L^2(\mathbb{Z})^{m\cdot|\ker(\phi)|} \to L^2(\mathbb{Z})^{m\cdot|\ker(\phi)|},$$

where s^* denotes restriction with s. We get from [27, Theorem 3.14 (5) on p. 128]

$$\ln\big(\det_{\mathcal{N}(G)}\big(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_A)\big)\big) = \frac{\ln\big(\det_{\mathcal{N}(\mathbb{Z})}\big(s^*\big(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_A)\big)\big)\big)}{|\ker(\phi)|}.$$

Hence we can assume without loss of generality that $G = \mathbb{Z}$ and $\phi = \mathrm{id}_{\mathbb{Z}}$, otherwise replace $\phi \colon G \to \mathbb{Z}$ by $\phi \circ s = \mathrm{id} \colon \mathbb{Z} \to \mathbb{Z}$ and A by B.

One easily checks that

$$r_{\det_{\mathbb{C}[\mathbb{Z}]}(\eta_{\mathbb{C}_t}(r_A))} = \eta_{\mathbb{C}_t}(r_{\det_{\mathbb{C}[\mathbb{Z}]}(A)}) \colon L^2(\mathbb{Z}) \to L^2(\mathbb{Z}).$$

Because of [28, Lemma 6.25] we can without loss of generality assume that $m = 1$, otherwise replace A by the $(1,1)$-matrix given by $\det_{\mathbb{C}[\mathbb{Z}]}(A)$.

Let $p(z) \in \mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$ be the only entry in the $(1,1)$-matrix A. Since $\Lambda^{\mathbb{Z}}(r_A)$ is a weak isomorphism by assumption, p is non-trivial. We can write

$$p(z) = \sum_{n=n_0}^{n_1} c_n \cdot z^n$$

for integers n_0 and n_1 with $n_0 \leq n_1$, complex numbers $c_{n_0}, c_{n_0+1}, \dots, c_{n_1}$ with $c_{n_0} \neq 0$ and $c_{n_1} \neq 0$. We can also write

$$p(z) = c_{n_1} \cdot z^r \cdot \prod_{i=1}^{s} (z - a_i)$$

for an integer $s \geq 0$, non-zero complex numbers $a_1, \dots, a_r$ and an integer r. We get from [27, (3.23) on p. 136]

$$\det_{\mathcal{N}(\mathbb{Z})}\big(\Lambda^{\mathbb{Z}}(r_p)\big) = |c_{n_1}| \cdot \prod_{\substack{i=1,\dots,s \\ |a_i| \geq 1}} |a_i|.$$

For $t \in (0, \infty)$ we get

$$p(t \cdot z) = c_{n_1} \cdot (tz)^r \cdot \prod_{i=1}^{s} (tz - a_i) = t^{r+s} \cdot c_{n_1} \cdot z^r \cdot \prod_{i=1}^{s} \big(z - \tfrac{a_i}{t}\big),$$

and hence

$$\det_{\mathcal{N}(\mathbb{Z})}\big(\Lambda^{\mathbb{Z}}(r_{p(tz)})\big) = t^{r+s} \cdot |c_{n_1}| \cdot \prod_{\substack{i=1,\dots,s \\ |a_i/t| \geq 1}} \big|\tfrac{a_i}{t}\big| = t^{r+s} \cdot |c_{n_1}| \cdot \prod_{\substack{i=1,\dots,s \\ |a_i| \geq t}} \tfrac{|a_i|}{t}.$$

This implies for $t \in (0, \infty)$

$$\rho(t) = (r+s) \cdot \ln(t) + \ln\big(|c_{n_1}|\big) + \sum_{\substack{i=1,\dots,s \\ |a_i| \geq t}} \big(\ln\big(|a_i|\big) - \ln(t)\big). \tag{4.1}$$

Define positive real numbers

$$T_0 = \min\big\{|a_i| \mid i = 1, 2, \dots, s\big\};$$
$$T_\infty = \max\big\{|a_i| \mid i = 1, 2, \dots, s\big\}.$$

Then we get

$$\rho(t) = \begin{cases} r \cdot \ln(t) + \ln\big(|c_{n_1}|\big) + \sum_{i=1}^{s} \ln\big(|a_i|\big) & \text{for } t \leq T_0; \\ (r+s) \cdot \ln(t) + \ln\big(|c_{n_1}|\big) & \text{for } t \geq T_\infty. \end{cases}$$

Since by assumption there are real numbers C and D and integers k and l such that

$$\lim_{t \to 0} \rho(t) - k \cdot \ln(t) = C;$$
$$\lim_{t \to \infty} \rho(t) - l \cdot \ln(t) = D,$$

we must have $r = k$, $r + s = l$, $C = \ln(|c_{n_1}|) + \sum_{i=1}^{s} \ln(|a_i|)$, and $D = \ln(|c_{n_1}|)$. Equation (4.1) becomes

$$\rho(t) = l \cdot \ln(t) + D + \sum_{\substack{i=1,\dots,s \\ |a_i| \geq t}} \bigl(\ln\bigl(|a_i|\bigr) - \ln(t)\bigr).$$

Since $(\ln(|a_i|) - \ln(t)) \geq 0$ for $|a_i| \geq t$, we get $l \cdot \ln(t) + \ln(D) \leq \rho(t)$ for all $t > 0$. We estimate for $t > 0$

$$\begin{aligned}
&k \cdot \ln(t) + C \\
&= k \cdot \ln(t) + D + \sum_{i=1}^{s} \ln\bigl(|a_i|\bigr) \\
&= k \cdot \ln(t) + D + \sum_{\substack{i=1,\dots,s \\ |a_i| \geq t}} \ln\bigl(|a_i|\bigr) + \sum_{\substack{i=1,\dots,s \\ |a_i| < t}} \ln\bigl(|a_i|\bigr) \\
&= r \cdot \ln(t) + D + s \cdot \ln(t) + \sum_{\substack{i=1,\dots,s \\ |a_i| \geq t}} \bigl(\ln\bigl(|a_i|\bigr) - \ln(t)\bigr) + \sum_{\substack{i=1,\dots,s \\ |a_i| < t}} \bigl(\ln\bigl(|a_i|\bigr) - \ln(t)\bigr) \\
&= l \cdot \ln(t) + D + \sum_{\substack{i=1,\dots,s \\ |a_i| \geq t}} \bigl(\ln\bigl(|a_i|\bigr) - \ln(t)\bigr) + \sum_{\substack{i=1,\dots,s \\ |a_i| < t}} \bigl(\ln\bigl(|a_i|\bigr) - \ln(t)\bigr) \\
&\leq l \cdot \ln(t) + D + \sum_{\substack{i=1,\dots,s \\ |a_i| \geq t}} \bigl(\ln\bigl(|a_i|\bigr) - \ln(t)\bigr) = \rho(t).
\end{aligned}$$

This finishes the proof of Lemma 4.1. □

Definition 4.2 (Fibered classes). Let M be a 3-manifold and consider an element $\phi \in H^1(M;\mathbb{Q}) = \mathrm{Hom}(\pi_1(M),\mathbb{Q})$. We say that ϕ is *fibered* if there exists a locally trivial fiber bundle $p\colon M \to S^1$ and a $k \in \mathbb{Q}$, $k > 0$ such that the induced map $p_*\colon \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ coincides with $k \cdot \phi$.

Theorem 4.3. *Let $M \neq S^1 \times D^2$ be an irreducible 3-manifold. Then the following two statements hold:*

(1) *If M is fibered, then for any $(H_1)_f$-factorizing homomorphism $\mu\colon \pi_1(M) \to G$ to a residually finite group the pair (M,μ) is L^2-acyclic.*

(2) *If $\phi \in H^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ is a primitive fibered class, then there exists a $T \geq 1$ such that for any $\mathfrak{s} \in \mathrm{Spin}^c(M)$ and for any $(H_1)_f$-factorizing homomorphism $\mu\colon \pi_1(M) \to G$ to a residually finite group the following equalities hold*

$$\rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) = \tfrac{1}{2}\bigl(\phi(c_1(\mathfrak{s})) + x_M(\phi)\bigr) \cdot \ln(t) \quad \textit{for } t < \tfrac{1}{T};$$
$$\rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) = \tfrac{1}{2}\bigl(\phi(c_1(\mathfrak{s})) - x_M(\phi)\bigr) \cdot \ln(t) \quad \textit{for } t > T.$$

In fact one can choose T to be the entropy of the monodromy.

Proof. The first statement follows from [25, Theorem 2.1]. Now we denote by T the entropy of the monodromy of the primitive fibered class ϕ. By Theorem 8.5 of [9] there exists an $\mathfrak{s} \in \mathrm{Spin}^c(M)$ such that

$$\begin{aligned} 0 &= \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t < \tfrac{1}{T}; \\ -x_M(\phi) \cdot \ln(t) &= \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t > T. \end{aligned}$$

The statement of the theorem follows from these inequalities in precisely the same way as we concluded the proof of Theorem 3.3. □

The next lemma improves on Theorem 4.3 in so far as it gives us some control over $\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)$ for all t. In particular the set of t's for which we have control does not depend on the choice of fibered ϕ.

Lemma 4.4. *Let $(M, \mu\colon \pi_1(M) \to G)$ be an admissible pair and let $\mathfrak{s} \in \mathrm{Spin}^c(M)$. Then for any fibered $\phi \in H^1(M; \mathbb{Q})$ we have*

$$\begin{aligned} \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) &\leq \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) + x_M(\phi)\big) \cdot \ln(t) \quad \textit{for } t \leq 1; \\ \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) &\leq \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) - x_M(\phi)\big) \cdot \ln(t) \quad \textit{for } t \geq 1. \end{aligned}$$

Proof. Let $(M, \mu\colon \pi_1(M) \to G)$ be an admissible pair. By Theorem 4.3 the pair (M, μ) is L^2-acyclic. Let $\mathfrak{s} \in \mathrm{Spin}^c(M)$. The argument of the proof of Lemma 3.4 shows that it suffices to prove the lemma for primitive fibered classes. So let $\phi \in H^1(M; \mathbb{Z}) = \mathrm{Hom}(\pi_1(M), \mathbb{Z})$ be a primitive fibered class.

Consider a nested sequence of normal subgroups of G:

$$G \supseteq G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

such that G_i is contained in $\ker(G \to H_1(G; \mathbb{Z})_f)$, the index $[\ker(G \to H_1(G; \mathbb{Z})_f) : G_i]$ is finite for $i \geq 0$ and the intersection $\bigcap_{i \geq 0} G_i$ is trivial. Put $Q_i := G/G_i$. Denote by $\mathrm{pr}_i\colon G \to Q_i$ the obvious projection. Let $\mu_i\colon \pi_1(M) \to Q_i$ be the composition $\mathrm{pr}_i \circ \mu$. The homomorphisms μ_i are again $(H_1)_f$-factorizing.

In the following we consider only the case where M is closed, the case with boundary is analogous. We apply Theorem 2.1 (2) to M. We denote the resulting square matrix over $\mathbb{Z}\pi$ by A and the resulting elements in the group π by s, s'. We write $A_i = \mathrm{pr}_i(A)$. Define

$$\eta(t) = \max\big\{0, |\phi(s)| \cdot \ln(t)\big\} + \max\big\{0, |\phi(s')| \cdot \ln(t)\big\}.$$

As above, the pair (M, μ_i) is L^2-acyclic. Our choice of A and s, s' ensures that

$$\begin{aligned} \rho^{(2)}(M, \mathfrak{s}; \mu, \phi) &= \eta(t) - \ln\big(\det_{\mathcal{N}(G)}\big(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_A)\big)\big); \\ \rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi) &= \eta(t) - \ln\big(\det_{\mathcal{N}(Q_i)}\big(\Lambda^{Q_i} \circ \eta_{\phi^*\mathbb{C}_t}(r_{A_i})\big)\big). \end{aligned}$$

We conclude from Theorem 1.7

$$\ln\big(\det_{\mathcal{N}(G)}\big(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_A)\big)\big) \geq \limsup_{i\to\infty} \ln\big(\det_{\mathcal{N}(Q_i)}\big(\Lambda^{Q_i} \circ \eta_{\phi^*\mathbb{C}_t}(r_{A_i})\big)\big). \quad (4.2)$$

By Theorem 4.3 there exists a $T \geq 1$ such that for any natural number i we have

$$\begin{aligned} \rho^{(2)}(M,\mathfrak{s};\mu_i,\phi)(t) &= \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) + x_M(\phi)\big)\cdot\ln(t) \quad \text{for } t < \tfrac{1}{T};\\ \rho^{(2)}(M,\mathfrak{s};\mu_i,\phi)(t) &= \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) - x_M(\phi)\big)\cdot\ln(t) \quad \text{for } t > T. \end{aligned}$$

This implies

$$\begin{aligned} \ln\big(\det_{\mathcal{N}(Q_i)}\big(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_{A_i})\big)\big) &= \eta(t) - \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) + x_M(\phi)\big)\cdot\ln(t) \quad \text{for } t < \tfrac{1}{T};\\ \ln\big(\det_{\mathcal{N}(Q_i)}\big(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_{A_i})\big)\big) &= \eta(t) - \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) - x_M(\phi)\big)\cdot\ln(t) \quad \text{for } t > T. \end{aligned}$$

Then Lemma 4.1 applied to $\phi\colon Q_i \to \mathbb{Z}$ yields

$$\begin{aligned} \ln\big(\det_{\mathcal{N}(Q_i)}\big(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_{A_i})\big)\big) &\geq \eta(t) - \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) + x_M(\phi)\big)\cdot\ln(t) \quad \text{for } t \leq 1;\\ \ln\big(\det_{\mathcal{N}(Q_i)}\big(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_{A_i})\big)\big) &\geq \eta(t) - \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) - x_M(\phi)\big)\cdot\ln(t) \quad \text{for } t \geq 1. \end{aligned}$$

Since this holds for all $i \geq 0$ and all $t > 0$, we conclude from (4.2)

$$\begin{aligned} \ln\big(\det_{\mathcal{N}(G)}\big(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_A)\big)\big) &\geq \eta(t) - \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) + x_M(\phi)\big)\cdot\ln(t) \quad \text{for } t \leq 1;\\ \ln\big(\det_{\mathcal{N}(G)}\big(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_A)\big)\big) &\geq \eta(t) - \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) - x_M(\phi)\big)\cdot\ln(t) \quad \text{for } t \geq 1. \end{aligned}$$

This implies

$$\begin{aligned} \rho^{(2)}(M,\mathfrak{s};\mu,\phi) &\leq \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) + x_M(\phi)\big)\cdot\ln(t) \quad \text{for } t \leq 1;\\ \rho^{(2)}(M,\mathfrak{s};\mu,\phi) &\leq \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) - x_M(\phi)\big)\cdot\ln(t) \quad \text{for } t \geq 1. \end{aligned} \qquad \square$$

Lemma 4.5. *Let Γ be a group that is virtually finitely generated free abelian. Consider a finite subset $S \subseteq \Gamma$. Then for any natural number n the function*

$$\{A \in M_{n,n}(\mathbb{C}\Gamma) \mid \operatorname{supp}_\Gamma(A) \subseteq S\} \to [0,\infty],$$
$$A \mapsto \begin{cases} \det_{\mathcal{N}(\Gamma)}\big(\Lambda^\Gamma(r_A)\big) & \text{if } \Lambda^\Gamma(r_A) \text{ is a weak isomorphism};\\ 0 & \text{otherwise}, \end{cases}$$

is continuous with respect to the standard topology on the source coming from the structure of a finite-dimensional complex vector space.

Proof. Let $i\colon \mathbb{Z}^d \to \Gamma$ be an inclusion whose image has finite index in Γ. Fix a map of sets $\sigma\colon \operatorname{im}(i)\backslash\Gamma \to \Gamma$ whose composition with the projection $\Gamma \to \operatorname{im}(i)\backslash\Gamma$ is the identity. Put $m = [\Gamma : \operatorname{im}(i)]$. With this choice the finitely generated free $\mathbb{C}[\mathbb{Z}^d]$-module $i^*\mathbb{C}\Gamma$ obtained from $\mathbb{C}\Gamma$ by restriction with i inherits a preferred

$\mathbb{C}[\mathbb{Z}^d]$-basis. Hence there is a finite subset $T \subseteq \mathbb{Z}^d$ and a $\mathbb{C}$-linear (and hence continuous) map

$$i^*\colon \{A \in M_{n,n}(\mathbb{C}\Gamma) \mid \operatorname{supp}_\Gamma(A) \subseteq S\} \\ \to \{B \in M_{mn,mn}(\mathbb{C}[\mathbb{Z}^d]) \mid \operatorname{supp}_{\mathbb{Z}^d}(B) \subseteq T\}$$

such that $i^*\Lambda^\Gamma(r_A) = \Lambda^{\mathbb{Z}^d}(r_{i^*A})$. Since

$$\det_{\mathcal{N}(\mathbb{Z}^d)}\big(i^*\Lambda^\Gamma(r_A)\big) = m \cdot \det_{\mathcal{N}(\Gamma)}(r_A)$$

holds for any $A \in M_{n,n}(\mathbb{C}\Gamma)$ by [27, Theorem 3.14 (5) on p. 128], it suffices to prove the claim in the special case $\Gamma = \mathbb{Z}^d$.

As $\det_{\mathbb{C}[\mathbb{Z}^d]}\colon M_{n,n}(\mathbb{C}[\mathbb{Z}^d]) \to M_{1,1}(\mathbb{C}[\mathbb{Z}^d])$ is continuous and since for every $A \in M_{n,n}(\mathbb{C}\mathbb{Z}^d)$ with $\operatorname{supp}_{\mathbb{Z}^d}(A) \subset S$ we have

$$\operatorname{supp}_{\mathbb{Z}^d}\left(\det_{\mathbb{C}[\mathbb{Z}^d]}\right) \subseteq S^n$$

for $S^n = \{g_1 \cdot g_2 \cdot \dots \cdot g_n \mid g_i \in S\}$, we conclude from [28, Lemma 6.25] that it suffices to treat the case $n = 1$. Since the Mahler measure of a non-trivial element $p \in \mathbb{C}[\mathbb{Z}^d]$ is equal to

$$\det_{\mathbb{C}[\mathbb{Z}^d]}\big(\Lambda^{\mathbb{Z}^d}(r_p)\colon L^2(\mathbb{Z}^d) \to L^2(\mathbb{Z}^d)\big)$$

and defined to be zero for $p = 0$, Lemma 4.5 follows from a continuity theorem for Mahler measures proved by Boyd [4, p. 127]. $\square$

Definition 4.6 (Quasi-fibered classes). Let N be a 3-manifold. We call an element $\phi \in H^1(N;\mathbb{R})$ *quasi-fibered*, if there exists a sequence of fibered elements $\phi_n \in H^1(N;\mathbb{Q})$ converging to ϕ in $H^1(N;\mathbb{R})$.

Notice that obviously any fibered ϕ is non-trivial. The next theorem generalizes the inequalities of Lemma 4.4 for fibered classes to quasi-fibered classes. This theorem can be viewed as the key technical result of this paper.

Theorem 4.7 (Upper bound in the quasi-fibered case). *Let (M,μ) be an admissible pair, $\mathfrak{s} \in \operatorname{Spin}^c(M)$ and let $\phi \in H^1(M;\mathbb{R})$ be a quasi-fibered class. Then*

$$\rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \le \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) + x_M(\phi)\big)\cdot \ln(t) \quad \textit{for } t \le 1;$$
$$\rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \le \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) - x_M(\phi)\big)\cdot \ln(t) \quad \textit{for } t \ge 1.$$

Proof. We only treat the case, where ∂M is empty, the other case is completely analogous: in the proof below one needs to replace Theorem 2.1 (2) by Theorem 2.1 (1). We write $\pi = \pi_1(M)$ and we pick $\mathfrak{s} \in \operatorname{Spin}^c(M)$.

First recall that our assumption that $\mu\colon \pi \to G$ is $(H_1)_f$-factorizing implies that the projection $\pi \to H_1(M)_f$ factors through μ and a map $\nu\colon G \to H_1(M)_f$. Since G is residually finite we can choose a sequence of normal subgroups of G

$$G \supseteq G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

such that G_i is contained in $\ker(\nu\colon G \to H_1(M)_f)$, the index $[\ker(\nu) : G_i]$ is finite for $i \geq 0$ and the intersection $\bigcap_{i \geq 0} G_i$ is trivial. Put $Q_i := G/G_i$. Denote by $\mu_i\colon \pi \to Q_i$ the composition of the projection $\mathrm{pr}_i\colon G \to Q_i$ with μ. Note that μ_i is again a $(H_1)_f$-factorizing homomorphism. Recall that this implies in particular that we can make the identifications

$$H^1(M;\mathbb{R}) = \mathrm{Hom}(H_1(\pi)_f, \mathbb{R}) = \mathrm{Hom}(\pi, \mathbb{R}) = \mathrm{Hom}(G, \mathbb{R}) = \mathrm{Hom}(G_i, \mathbb{R}).$$

We apply Theorem 2.1 (2) to M and $\mathfrak{s}$. We denote the resulting square matrix over $\mathbb{Z}\pi$ by A and the resulting elements in π by s, s'. For each $i \in \mathbb{N}$ we write $A_i = \mathrm{pr}_i(A)$. Define for any homomorphism $\psi\colon H_1(M)_f \to \mathbb{R}$

$$\xi(\psi)(t) = \max\big\{0, \big(|\psi \circ \nu \circ \mu(s)| + |\psi \circ \nu \circ \mu(s')|\big) \cdot \ln(t)\big\}.$$

We start out with the following claim.

Claim. For each $i \in \mathbb{N}$ we have the inequalities

$$\rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi_i)(t) \leq \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) + x_M(\phi)\big) \cdot \ln(t) \quad \text{for } t \leq 1;$$
$$\rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi_i)(t) \leq \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) - x_M(\phi)\big) \cdot \ln(t) \quad \text{for } t \geq 1.$$

Let $i \in \mathbb{N}$. Since $\phi \in H^1(M;\mathbb{R})$ is quasi-fibered there exists a sequence of fibered elements $\phi_n \in H^1(M;\mathbb{Q})$ converging to ϕ. By Lemma 4.4 we know that for each i and n we have

$$\rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi_n)(t) \leq \tfrac{1}{2}\big(\phi_n(c_1(\mathfrak{s})) + x_M(\phi_n)\big) \cdot \ln(t) \quad \text{for } t \leq 1; \tag{4.3}$$
$$\rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi_n)(t) \leq \tfrac{1}{2}\big(\phi_n(c_1(\mathfrak{s})) - x_M(\phi_n)\big) \cdot \ln(t) \quad \text{for } t \geq 1. \tag{4.4}$$

By Theorem 2.1 (2) we have

$$\rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi_n)(t) = \xi(\phi_n)(t) - \ln\big(\det_{\mathcal{N}(Q_i)}\big(\Lambda^{Q_i} \circ \eta_{\phi_n^*\mathbb{C}_t}(r_{A_i})\big)\big); \tag{4.5}$$
$$\rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi)(t) = \xi(\phi)(t) - \ln\big(\det_{\mathcal{N}(Q_i)}\big(\Lambda^{Q_i} \circ \eta_{\phi^*\mathbb{C}_t}(r_{A_i})\big)\big). \tag{4.6}$$

Since ϕ_n converges to ϕ and the kernel of the projection map $Q_i \to H_1(M)_f$ is finite, we get from Lemma 4.5 that

$$\lim_{n\to\infty} \ln\big(\det_{\mathcal{N}(Q_i)}(\Lambda^{Q_i} \circ \eta_{\phi_n^*\mathbb{C}_t}(r_{A_i}))\big) = \ln\big(\det_{\mathcal{N}(Q_i)}(\Lambda^{Q_i} \circ \eta_{\phi^*\mathbb{C}_t}(r_{A_i}))\big).$$

This equality, together with Equations (4.5) and (4.6) and the observation that for any $t \in (0,\infty)$ the equality $\lim_{n\to\infty} \xi(\phi_n)(t) = \xi(\phi)(t)$ holds, implies that

$$\rho^{(2)}(M,\mathfrak{s};\mu_i,\phi)(t) = \lim_{n\to\infty} \rho^{(2)}(M,\mathfrak{s};\mu_i,\phi_n)(t) \quad \text{for all } t \in (0,\infty).$$

The desired inequalities for $\rho^{(2)}(M,\mathfrak{s};\mu_i,\phi)(t)$ now follow from (4.3) and (4.4). This concludes the proof of the claim.

Now the theorem follows from the claim we just proved and the following claim.

Claim. For each $t \in (0,\infty)$ we have

$$\rho^{(2)}(M,\mathfrak{s},\mu,\phi)(t) \le \liminf_{i\to\infty} \rho^{(2)}(M,\mathfrak{s},\mu_i,\phi_i)(t).$$

This claim is proved as follows. By Theorem 4.3 we know that the pairs (M,μ) and (M,μ_i) are L^2-acyclic. By Theorem 2.1 (2) we have

$$\rho^{(2)}(M,\mathfrak{s},\mu,\phi)(t) = \xi(\phi)(t) - \ln\bigl(\det_{\mathcal{N}(G)}\bigl(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_A)\bigr)\bigr). \tag{4.7}$$

Recall that the kernel of $Q_i \to H_1(M)_f$ is finite und that $Q_i \to H_1(M)_f$ is surjective. Now we apply Theorem 1.7 to $\phi\colon G \to \mathbb{R}$. For all $t \in (0,\infty)$ we obtain

$$\ln\bigl(\det_{\mathcal{N}(G)}\bigl(\Lambda^G \circ \eta_{\phi^*\mathbb{C}_t}(r_A)\bigr)\bigr) \ge \limsup_{i\to\infty} \ln\bigl(\det_{\mathcal{N}(Q_i)}\bigl(\Lambda^{Q_i} \circ \eta_{\phi_i^*\mathbb{C}_t}(r_{A_i})\bigr)\bigr).$$

Now apply (4.6) and (4.7). This finishes the proof of Theorem 4.7. $\square$

For convenience we also state the result which follows from combining Theorem 3.3 with Theorem 4.7.

Theorem 4.8 (Lower and upper bounds combined in the quasi-fibered case). *Let $M \neq S^1 \times D^2$ be an irreducible 3-manifold with infinite fundamental group π. Let $\phi \in H^1(M;\mathbb{Q})$ be a quasi-fibered class.*

Then there exists a $D \in \mathbb{R}$ such that for any $\mathfrak{s} \in \operatorname{Spin}^c(M)$ and any $(H_1)_f$-factorizing homomorphism $\mu\colon \pi_1(M) \to G$, where G is residually finite and countable, the pair (M,μ) is L^2-acyclic and such that for $t \le 1$

$$\tfrac{1}{2}\bigl(\phi(c_1(\mathfrak{s})) + x_M(\phi)\bigr)\ln t - D \le \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \le \tfrac{1}{2}\bigl(\phi(c_1(\mathfrak{s})) + x_M(\phi)\bigr)\ln t$$

and such that for $t \ge 1$

$$\tfrac{1}{2}\bigl(\phi(c_1(\mathfrak{s})) - x_M(\phi)\bigr)\ln t - D \le \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \le \tfrac{1}{2}\bigl(\phi(c_1(\mathfrak{s})) - x_M(\phi)\bigr)\ln t.$$

In particular we get

$$\deg\bigl(\rho(M,\mathfrak{s};\mu,\phi)\bigr) = -x_M(\phi).$$

5. Proof of the main theorem

The following is the main theorem of this paper.

Theorem 5.1 (Main theorem). *Let M be an irreducible 3-manifold with infinite fundamental group π which is not a closed graph manifold and not homeomorphic to $S^1 \times D^2$. Let $\mathfrak{s} \in \operatorname{Spin}^c(M)$ and write $\pi = \pi_1(M)$.*

Then there exists a $(H_1)_f$-factorizing epimorphism $\alpha\colon \pi \to \Gamma$ to a virtually finitely generated free abelian group such that the following holds: For any $\phi \in H^1(M;\mathbb{Q})$ and any factorization of $\alpha\colon \pi \to \Gamma$ into group homomorphisms $\pi \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$ for a residually finite countable group G, there exists a real number D depending only on ϕ but not on μ such that for $t \leq 1$

$$\tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) + x_M(\phi)\big)\ln t - D \leq \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \leq \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) + x_M(\phi)\big)\ln t$$

and such that for $t \geq 1$

$$\tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) - x_M(\phi)\big)\ln t - D \leq \rho^{(2)}(M,\mathfrak{s};\mu,\phi)(t) \leq \tfrac{1}{2}\big(\phi(c_1(\mathfrak{s})) - x_M(\phi)\big)\ln t.$$

In particular we get

$$\deg\big(\rho(M,\mathfrak{s};\mu,\phi)\big) = -x_M(\phi).$$

Proof. As explained in [9, Section 10], we conclude from combining [1,2,22,32,33,42,43] that there exists a finite regular covering $p\colon N \to M$ such that for any $\phi \in H^1(M;\mathbb{R})$ its pullback $p^*\phi \in H^1(N;\mathbb{R})$ is quasi-fibered. Let k be the number of sheets of p. Let $\mathrm{pr}_N\colon \pi_1(N) \to H_1(N)_f$ and $\mathrm{pr}_M\colon \pi_1(M) \to H_1(M)_f$ be the canonical projections. The kernel of pr_N is a characteristic subgroup of $\pi_1(N)$. The regular finite covering p induces an injection

$$\pi_1(p)\colon \pi_1(N) \to \pi_1(M)$$

such that the image of $\pi_1(p)$ is a normal subgroup of $\pi_1(M)$ of finite index. Let Γ be the quotient of $\pi_1(M)$ by the normal subgroup $\pi_1(p)(\ker(\mathrm{pr}_N))$. Let $\alpha\colon \pi_1(M) \to \Gamma$ be the projection. Since $H_1(p;\mathbb{Z})_f \circ \mathrm{pr}_N = \mathrm{pr}_M \circ \pi_1(p)$ we know that $\pi_1(p)(\ker(\mathrm{pr}_N))$ is contained in the kernel of the canonical projection $\mathrm{pr}_M\colon \pi_1(M) \to H_1(M)_f$. This implies that $\alpha\colon \pi_1(M) \to \Gamma$ is $(H_1)_f$-factorizing, which means in particular that there exists precisely one epimorphism

$$\beta\colon \Gamma \to H_1(M)_f$$

satisfying $\mathrm{pr}_M = \beta \circ \alpha$. Moreover, $\alpha \circ \pi_1(p)$ factorizes over pr_N into an injective homomorphism $j\colon H_1(N)_f \to \Gamma$ with finite cokernel. Hence Γ is virtually finitely generated free abelian.

Consider any factorization of the homomorphism $\alpha\colon \pi_1(M) \to \Gamma$ into group homomorphisms

$$\pi_1(M) \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$$

with residually finite countable G.

Let G' be the quotient of $\pi_1(N)$ by the normal subgroup $\pi_1(p)^{-1}(\ker(\mu))$ and $\mu' \colon \pi_1(N) \to G'$ be the projection. Since the kernels of μ' and of $\mu \circ \pi_1(p)$ agree, there is precisely one injective group homomorphism $i \colon G' \to G$ satisfying $\mu \circ \pi_1(p) = i \circ \mu'$. The kernel of μ' is contained in the kernel of $\mathrm{pr}_N \colon \pi_1(N) \to H_1(N)_f$ since j is injective and we have $j \circ \mathrm{pr}_N = \nu \circ i \circ \mu'$. Hence there is precisely one group homomorphism

$$\nu' \colon G' \to H_1(N)_f$$

satisfying $\nu' \circ \mu' = \mathrm{pr}_N$. In particular μ' is a $(H_1)_f$-factorizing homomorphism. One easily checks that the following diagram commutes, and all vertical arrows are injective, the indices $[\pi_1(N) : \mathrm{im}(\pi_1(p))]$ and $[\Gamma : H_1(N)_f]$ are finite, and μ', ν' and β are surjective:

$$\begin{tikzcd}
\pi_1(N) \arrow[r, "\mu'"] \arrow[d, "\pi_1(p)"] \arrow[rr, bend left, "\mathrm{pr}_N"] & G' \arrow[r, "\nu'"] \arrow[d, "i"] & H_1(N)_f \arrow[d, "j"] \arrow[dr, "H_1(p)_f"] & \\
\pi_1(M) \arrow[r, "\mu"] \arrow[rr, bend right, "\alpha"'] \arrow[rrr, bend right=40, "\mathrm{pr}_M"'] & G \arrow[r, "\nu"] & \Gamma \arrow[r, "\beta"] & H_1(M)_f
\end{tikzcd}$$

Since G is residually finite and countable, the group G' is residually finite and countable.

Now let $\mathfrak{s} \in \mathrm{Spin}^c(M)$ and let $\phi \in H^1(M;\mathbb{Q}) = \mathrm{Hom}(H_1(M)_f;\mathbb{Q})$. We write $\mathfrak{s}' = p^*(\mathfrak{s})$ and $\phi' = p^*(\phi)$. Furthermore we put $c = c_1(\mathfrak{s})$ and $c' = c_1(\mathfrak{s}')$. Since $\phi' \in H^1(N;\mathbb{Q}) = \mathrm{Hom}(H_1(N)_f;\mathbb{Q})$ is quasi-fibered we can appeal to Theorem 4.8. In our context it says that (N, μ') is L^2-acyclic and that there exists a real number D' depending only on ϕ' but not on μ' such that for $t \leq 1$

$$\tfrac{1}{2}\big(\phi'(c') + x_N(\phi')\big) \ln t - D' \leq \rho^{(2)}(N, \mathfrak{s}'; \mu', \phi')(t) \leq \tfrac{1}{2}\big(\phi'(c') + x_N(\phi')\big) \ln t$$

and such that for $t \geq 1$

$$\tfrac{1}{2}\big(\phi'(c') - x_N(\phi')\big) \ln t - D' \leq \rho^{(2)}(N, \mathfrak{s}'; \mu, \phi')(t) \leq \tfrac{1}{2}\big(\phi'(c') - x_N(\phi')\big) \ln t.$$

We now set $D := \frac{1}{k} D'$. The theorem now follows from these inequalities and the following equalities

$$\begin{aligned}
x_M(\phi) &= \tfrac{1}{k} x_N(\phi') \\
\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) &= \tfrac{1}{k} \rho^{(2)}(N, \mathfrak{s}'; \mu', \phi')(t) \quad \text{for all } t \\
\phi(c_1(\mathfrak{s})) &= \tfrac{1}{k} \phi'(c_1(\mathfrak{s}')).
\end{aligned}$$

Here the first equality is (1.8) and the second one Theorem 1.5 (3). The third one follows easily from the definitions. $\square$

Remark 5.2 (Graph manifolds). The proof of Theorem 4.8 does not cover closed graph manifolds. However, for a graph manifold M together with a $(H_1)_f$-factorizing homomorphism $\mu\colon \pi_1(M) \to G$, for which (M, μ) is L^2-acyclic, together with a class $\phi \in H^1(M;\mathbb{R})$ the L^2-torsion function $\overline{\rho}^{(2)}(M;\mu,\phi)(t)$ has been computed explicitly in [9, Theorem 8.2] and in [18] to be equivalent to $\min\{0, -x_M(\phi)\cdot \ln(t)\}$, provided that the image of the regular fiber under μ is an element of infinite order and M is neither $S^1 \times D^2$ nor $S^1 \times S^2$. This implies

$$\deg\bigl(\overline{\rho}^{(2)}(M;\mu,\phi)\bigr) = -x_M(\phi).$$

Remark 5.3 (The role of Γ). In Theorem 5.1 the group Γ is in some sense optimal. Namely, one cannot expect $\Gamma = H_1(M)_f$ and $\beta = \mathrm{id}_\Gamma$ in Theorem 5.1. For instance, let $K \subseteq S^3$ be a non-trivial knot. Take M to be the 3-manifold given by the complement of an open tubular neighborhood of the knot. Then $\deg(\overline{\rho}(M;\mu,\phi))$ for $\mu\colon \pi_1(M) \to H_1(M)_f$ the canonical projection and $\phi\colon H_1(M)_f \xrightarrow{\cong} \mathbb{Z}$ an isomorphism is just the degree of the Alexander polynomial of the knot K which is not the Thurston norm $x_M(\phi)$ in general, see [9, Section 7.3].

Example 5.4 ($S^1 \times D^2$ and $S^1 \times S^2$). Consider a homomorphism

$$\phi\colon H_1(S^1 \times D^2) \xrightarrow{\cong} \mathbb{Z}.$$

Let k be the index $[\mathbb{Z} : \mathrm{im}(\phi)]$ if ϕ is non-trivial, and let $k = 0$ if ϕ is trivial. Then we conclude from Theorem 1.5 (4), (1.9), and [28, Theorem 7.10]

$$\begin{aligned} x_{S^1\times D^2}(\phi) &= 0;\\ \deg\bigl(\overline{\rho}(\widetilde{S^1 \times D^2};\phi)\bigr) &= k. \end{aligned}$$

Hence we have to exclude $S^1 \times D^2$ in Theorem 5.1. Analogously we get

$$\begin{aligned} x_{S^1\times S^2}(\phi) &= 0;\\ \deg\bigl(\overline{\rho}(\widetilde{S^1 \times S^2};\phi)\bigr) &= 2\cdot k, \end{aligned}$$

so that we cannot replace "irreducible" by "prime" in Theorem 5.1.

We conclude the paper with the proof of Theorem 0.1.

Proof of Theorem 0.1. Let M be an irreducible 3-manifold with infinite fundamental group. If M is a graph manifold, then the statement is proved in Remark 5.2. Now suppose that M is not a graph manifold. In this case the theorem follows from Theorem 5.1, applied in the special case $G = \pi_1(M)$, $\mu = \mathrm{id}_{\pi_1(M)}$, and $\nu = \alpha$. Here we use the fact, mentioned in the proof of Lemma 1.4, that fundamental groups of 3-manifolds are residually finite. $\square$

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