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Essential dimension of the spin groups in characteristic 2

Burt Totaro*

Abstract. We determine the essential dimension of the spin group $\mathrm{Spin}(n)$ as an algebraic group over a field of characteristic 2, for n at least 15. In this range, the essential dimension is the same as in characteristic not 2. In particular, it is exponential in n . In contrast, the essential dimension of the orthogonal groups is smaller in characteristic 2. We also find the essential dimension of $\mathrm{Spin}(n)$ in characteristic 2 for n at most 10.

Mathematics Subject Classification (2010). 11E04, 11E72, 20G15.

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Introduction

The essential dimension of an algebraic group G is a measure of the number of parameters needed to describe all G -torsors over all fields. A major achievement of the subject was the calculation of the essential dimension of the spin groups over a field of characteristic not 2, started by Brosnan, Reichstein, and Vistoli, and completed by Chernousov, Merkurjev, Garibaldi, and Guralnick [3, 4, 7], [18, Theorem 9.1].

In this paper, we determine the essential dimension of the spin group $\mathrm{Spin}(n)$ for $n \geq 15$ over an arbitrary field (Theorem 2.1). We find that the answer is the same in all characteristics. In contrast, for the groups $O(n)$ and $SO(n)$, the essential dimension is smaller in characteristic 2, by Babic and Chernousov [1].

In characteristic not 2, the computation of essential dimension can be phrased to use a natural finite subgroup of $\mathrm{Spin}(2r + 1)$, namely an extraspecial 2-group, a central extension of $(\mathbf{Z}/2)^{2r}$ by $\mathbf{Z}/2$. A distinctive feature of the argument in characteristic 2 is that the analogous subgroup is a finite group scheme, a central extension of $(\mathbf{Z}/2)^r \times (\mu_2)^r$ by μ_2 , where μ_2 is the group scheme of square roots of unity.

In characteristic not 2, Rost and Garibaldi computed the essential dimension of $\mathrm{Spin}(n)$ for $n \leq 14$ [6, Table 23B], where case-by-case arguments seem to be needed. We show in Theorem 4.1 that for $n \leq 10$, the essential dimension of $\mathrm{Spin}(n)$

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is the same in characteristic 2 as in characteristic not 2. It would be interesting to compute the essential dimension of $\mathrm{Spin}(n)$ in the remaining cases, $11 \leq n \leq 14$ in characteristic 2.

1. Essential dimension

Let G be an affine group scheme of finite type over a field k . Write $H^1(k, G)$ for the set of isomorphism classes of G -torsors over k in the fppf topology. For G smooth over k , this is also the set of isomorphism classes of G -torsors over k in the étale topology.

Following Reichstein, the *essential dimension* $\mathrm{ed}(G)$ is the smallest natural number r such that for every G -torsor ξ over an extension field E of k , there is a subfield $k \subset F \subset E$ such that ξ is isomorphic to some G -torsor over F extended to E , and F has transcendence degree at most r over k . (It is essential that E is allowed to be any extension field of k , not just an algebraic extension field.) There are several survey articles on essential dimension, including [17, 19].

For example, let q_0 be a quadratic form of dimension n over a field k of characteristic not 2. Then $O(q_0)$ -torsors can be identified with quadratic forms of dimension n , up to isomorphism. (For convenience, we sometimes write $O(n)$ for $O(q_0)$.) Thus the essential dimension of $O(n)$ measures the number of parameters needed to describe all quadratic forms of dimension n . Indeed, every quadratic form of dimension n over a field of characteristic not 2 is isomorphic to a diagonal form $\langle a_1, \dots, a_n \rangle$. It follows that the orthogonal group $O(n)$ in characteristic not 2 has essential dimension at most n ; in fact, $O(n)$ has essential dimension equal to n , by one of the first computations of essential dimension [19, Example 2.5]. Reichstein also showed that the connected group $SO(n)$ in characteristic not 2 has essential dimension $n - 1$ for $n \geq 3$ [19, Corollary 3.6].

For another example, for an integer $n \geq 2$ and any field k , the group scheme μ_n of n th roots of unity is smooth over k if and only if n is invertible in k . Independent of that, $H^1(k, \mu_n)$ is always isomorphic to $k^*/(k^*)^n$. From that description, it is immediate that μ_n has essential dimension at most 1 over k . It is not hard to check that the essential dimension is in fact equal to 1.

One simple bound is that for any generically free representation V of a group scheme G over k (meaning that G acts freely on a nonempty open subset of V), the essential dimension of G is at most $\dim(V) - \dim(G)$ [18, Proposition 5.1]. It follows, for example, that the essential dimension of any affine group scheme of finite type over k is finite.

For a prime number p , the p -essential dimension $\mathrm{ed}_p(G)$ is a simplified invariant, defined by “ignoring field extensions of degree prime to p ”. In more detail, for a G -torsor ξ over an extension field E of k , define the p -essential dimension $\mathrm{ed}_p(\xi)$ to be the smallest number r such that there is a finite extension E'/E of degree

prime to p such that ξ over E' comes from a G -torsor over a subfield $k \subset F \subset E'$ of transcendence degree at most r over k . Then the p -essential dimension $\text{ed}_p(G)$ is defined to be the supremum of the p -essential dimensions of all G -torsors over all extension fields of k .

The *spin group* $\text{Spin}(n)$ is the simply connected double cover of $SO(n)$. It was a surprise when Brosnan, Reichstein, and Vistoli showed that the essential dimension of $\text{Spin}(n)$ over a field k of characteristic not 2 is exponentially large, asymptotic to $2^{n/2}$ as n goes to infinity [3]. As an application, they showed that the number of “parameters” needed to describe all quadratic forms of dimension $2r$ in I^3 over all fields is asymptotic to 2^r .

We now turn to quadratic forms over a field which may have characteristic 2. Define a quadratic form (q, V) over a field k to be *nondegenerate* if the radical V^\perp of the associated bilinear form is 0, and *nonsingular* if V^\perp has dimension at most 1 and q is nonzero on any nonzero element of V^\perp . (In characteristic not 2, nonsingular and nondegenerate are the same.) The orthogonal group is defined as the automorphism group scheme of a nonsingular quadratic form [13, Section VI.23]. For example, over a field k of characteristic 2, the quadratic form

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r}$$

is nonsingular of even dimension $2r$, while the form

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r} + x_{2r+1}^2$$

is nonsingular of odd dimension $2r+1$, with V^\perp of dimension 1. The *split* orthogonal group over k is the automorphism group of one of these particular quadratic forms.

Babic and Chernousov computed the essential dimension of $O(n)$ and the smooth connected subgroup $O^+(n)$ over an infinite field k of characteristic 2 [1]. (We also write $SO(n)$ for $O^+(n)$ by analogy with the case of characteristic not 2, even though the whole group $O(2r)$ is contained in $SL(2r)$ in characteristic 2.) The answer is smaller than in characteristic not 2. Namely, $O(2r)$ has essential dimension $r+1$ (not $2r$) over k . Also, $O^+(2r)$ has essential dimension $r+1$ for r even, and either r or $r+1$ for r odd, not $2r-1$. Finally, the group scheme $O(2r+1)$ has essential dimension $r+2$ over k , and $O^+(2r+1)$ has essential dimension $r+1$. The lower bounds here are difficult, while the upper bounds are straightforward. For example, to show that $O(2r)$ has essential dimension at most $r+1$ in characteristic 2, write any quadratic form of dimension $2r$ as a direct sum of 2-dimensional forms, thus reducing the structure group to $(\mathbb{Z}/2)^r \times (\mu_2)^r$, and then use that the group $(\mathbb{Z}/2)^r$ has essential dimension only 1 over an infinite field of characteristic 2 [1, proof of Proposition 13.1].

In this paper, we determine the essential dimension of $\text{Spin}(n)$ in characteristic 2 for $n \leq 10$ or $n \geq 15$. Surprisingly, in view of what happens for $O(n)$ and $O^+(n)$, the results for spin groups are the same in characteristic 2 as in characteristic not 2.

For $n \leq 10$, the lower bound for the essential dimension is proved by constructing suitable cohomological invariants. It is not known whether a similar approach is possible for $n \geq 15$, either in characteristic 2 or in characteristic not 2.

2. Main result

Theorem 2.1. *Let k be a field. For every integer n at least 15, the essential dimension of the split group $\mathrm{Spin}(n)$ over k is given by:*

$$\mathrm{ed}(\mathrm{Spin}(n)) = \begin{cases} 2^{n-1} - n(n-1)/2 & \text{if } n \text{ is odd;} \\ 2^{(n-2)/2} - n(n-1)/2 & \text{if } n \equiv 2 \pmod{4}; \\ 2^{(n-2)/2} + 2^m - n(n-1)/2 & \text{if } n \equiv 0 \pmod{4}, \end{cases}$$

where 2^m is the largest power of 2 dividing n . The 2-essential dimension is the same number.

Proof. For k of characteristic 0, this was proved by Chernousov and Merkurjev, sharpening the results of Brosnan, Reichstein, and Vistoli [4, Theorem 2.2]. Their argument works in any characteristic not 2, using the results of Garibaldi and Guralnick for the upper bounds [7]. Namely, Garibaldi and Guralnick showed that for any field k and any n at least 15, $\mathrm{Spin}(n)$ acts generically freely on the spin representation for n odd, on each of the two half-spin representations if $n \equiv 2 \pmod{4}$, and on the direct sum of a half-spin representation and the standard representation if $n \equiv 0 \pmod{4}$. Moreover, for n at least 20 with $n \equiv 0 \pmod{4}$, $\mathrm{HSpin}(n) = \mathrm{Spin}(n)/\mu_2$ (the quotient different from $O^+(n)$) acts generically freely on a half-spin representation [7, Theorem 1.1].

It remains to consider a field k of characteristic 2. Garibaldi and Guralnick's result gives the desired upper bound in most cases. Namely, for n odd and at least 15, the spin representation has dimension $2^{(n-1)/2}$, and so

$$\mathrm{ed}(\mathrm{Spin}(n)) \leq 2^{(n-1)/2} - \dim(\mathrm{Spin}(n)) = 2^{(n-1)/2} - n(n-1)/2.$$

For $n \equiv 2 \pmod{4}$, the half-spin representations have dimension $2^{(n-2)/2}$, and so

$$\mathrm{ed}(\mathrm{Spin}(n)) \leq 2^{(n-2)/2} - n(n-1)/2.$$

For $n = 16$, since the spin group acts generically freely on the direct sum of a half-spin representation and the standard representation,

$$\mathrm{ed}(\mathrm{Spin}(n)) \leq 2^{(n-2)/2} + n - n(n-1)/2 (= 24).$$

For n at least 20 and divisible by 4, the optimal upper bound requires more effort. The following argument is modeled on Chernousov and Merkurjev's characteristic

zero argument [4, Theorem 2.2]. Namely, consider the map of exact sequences of k -group schemes:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Spin}(n) & \longrightarrow & \mathrm{HSpin}(n) \longrightarrow 1 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & O^+(n) & \longrightarrow & \mathrm{PGO}^+(n) \longrightarrow 1. \end{array}$$

Since $\mathrm{HSpin}(n)$ acts generically freely on a half-spin representation, which has dimension $2^{(n-2)/2}$, we have

$$\mathrm{ed}(\mathrm{HSpin}(n)) \leq 2^{(n-2)/2} - n(n-1)/2.$$

By Chernousov–Merkurjev or independently Lötscher, for any normal subgroup scheme C of an affine group scheme G over a field k ,

$$\mathrm{ed}(G) \leq \mathrm{ed}(G/C) + \max \mathrm{ed}[E/G],$$

where the maximum runs over all field extensions F of k and all G/C -torsors E over F [4, Proposition 2.1], [15, Example 3.4]. Thus $[E/G]$ is a gerbe over F banded by C .

Identifying $H^2(K, \mu_p)$ with the p -torsion in the Brauer group of K , we can talk about the index of an element of $H^2(K, \mu_p)$, meaning the degree of the corresponding division algebra over K . For a prime number p and a nonzero element E of $H^2(K, \mu_p)$ over a field K , the essential dimension (or also the p -essential dimension) of the corresponding μ_p -gerbe over K is equal to the index of E , by Karpenko and Merkurjev [11, Theorems 2.1 and 3.1].

By the diagram above, for any field F over k , the image of the connecting map

$$H^1(F, \mathrm{HSpin}(n)) \rightarrow H^2(F, \mu_2) \subset \mathrm{Br}(F)$$

is contained in the image of the other connecting map

$$H^1(F, \mathrm{PGO}^+(n)) \rightarrow H^2(F, \mu_2) \subset \mathrm{Br}(F).$$

In the terminology of the Book of Involutions, the image of the latter map consists of the classes $[A]$ of all central simple F -algebras A of degree n with a quadratic pair (σ, f) of trivial discriminant [13, Section 29.F]. Any torsor for $\mathrm{PGO}^+(n)$ is split by a field extension of degree a power of 2, by reducing to the corresponding fact about quadratic forms. So $\mathrm{ind}(A)$ must be a power of 2, but it also divides n , and so $\mathrm{ind}(A) \leq 2^m$, where 2^m is the largest power of 2 dividing n . We conclude that

$$\begin{aligned} \mathrm{ed}(\mathrm{Spin}(n)) &\leq \mathrm{ed}(\mathrm{HSpin}(n)) + 2^m \\ &\leq 2^{(n-2)/2} - n(n-1)/2 + 2^m. \end{aligned}$$

This completes the proof of the upper bound in Theorem 2.1.

We now prove the corresponding lower bound for the 2-essential dimension of the spin group over a field k of characteristic 2. Since $\text{ed}_2(\text{Spin}(n)) \leq \text{ed}(\text{Spin}(n))$, this will imply that the 2-essential dimension and the essential dimension are both equal to the number given in Theorem 2.1.

Write $O(2r)$ for the orthogonal group of the quadratic form

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r}$$

over k , and $O(2r + 1)$ for the orthogonal group of

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r} + x_{2r+1}^2.$$

Then we have an inclusion $O(2r) \subset O(2r + 1)$. Note that $O(2r)$ is smooth over k , with

$$O(2r)/O^+(2r) \cong \mathbf{Z}/2.$$

The group scheme $O(2r + 1)$ is not smooth over k , but it contains a smooth connected subgroup $O^+(2r + 1)$ with

$$O(2r + 1) \cong O^+(2r + 1) \times \mu_2.$$

It follows that $O(2r)$ is contained in $O^+(2r + 1)$. Using the subgroup $\mathbf{Z}/2 \times \mu_2$ of $O(2)$, we have a k -subgroup scheme

$$K := (\mathbf{Z}/2 \times \mu_2)^r \subset O(2r) \subset O^+(2r + 1).$$

Let G be the inverse image of K in the double cover $\text{Spin}(2r + 1)$ of $O^+(2r + 1)$. Thus G is a central extension

$$1 \rightarrow \mu_2 \rightarrow G \rightarrow (\mathbf{Z}/2)^r \times (\mu_2)^r \rightarrow 1.$$

(Essentially the same “finite Heisenberg group scheme” appeared in the work of Mumford and Sekiguchi on abelian varieties [20, Appendix A].)

To describe the structure of G in more detail, think of $K = (\mu_2)^r$ as the 2-torsion subgroup scheme of a fixed maximal torus $T_{SO} \cong (G_m)^r$ in $O^+(2r + 1)$, where G_m is the multiplicative group. The character group of T_{SO} is the free abelian group $\mathbf{Z}\{x_1, \dots, x_r\}$, and the Weyl group $W = N(T_{SO})/T_{SO}$ of $O^+(2r + 1)$ is the semi-direct product $S_r \ltimes (\mathbf{Z}/2)^r$. Here S_r permutes the characters x_1, \dots, x_r of T_{SO} , and the subgroup $E_r = (\mathbf{Z}/2)^r$ of W , with generators $\epsilon_1, \dots, \epsilon_r$, acts by: ϵ_i changes the sign of x_i and fixes x_j for $j \neq i$. The character group of $K = T_{SO}[2]$ is $\mathbf{Z}/2\{x_1, \dots, x_r\}$. The group E_r centralizes K , and the group

$$(\mathbf{Z}/2)^r \times (\mu_2)^r \subset O^+(2r + 1)$$

above is $E_r \times K$.

Let L be the inverse image of K in $\mathrm{Spin}(2r + 1)$, which is contained in a maximal torus T of $\mathrm{Spin}(2r + 1)$, the inverse image of T_{SO} . The character group $X^*(T)$ is

$$\mathbb{Z}\{x_1, \dots, x_r, A\}/(2A = x_1 + \dots + x_r).$$

Therefore, the character group $X^*(L)$ is

$$\mathbb{Z}\{x_1, \dots, x_r, A\}/(2x_i = 0, 2A = x_1 + \dots + x_r).$$

(Thus $X^*(L)$ is isomorphic to $(\mathbb{Z}/4) \times (\mathbb{Z}/2)^{r-1}$, and so L is isomorphic to $\mu_4 \times (\mu_2)^{r-1}$.) The Weyl group W of $\mathrm{Spin}(2r + 1)$ is the same as that of $O^+(2r + 1)$, namely $S_r \ltimes E_r$. In particular, the element ϵ_i of E_r acts on $X^*(T)$ by changing the sign of x_i and fixing x_j for $j \neq i$, and hence it sends A to $A - x_i$.

The subset S of $X^*(L)$ corresponding to characters of L which are faithful on the center μ_2 of L is the complement of the subgroup $X^*(K) = \mathbb{Z}/2\{x_1, \dots, x_r\}$. Therefore, S has order 2^r . The group $E_r = (\mathbb{Z}/2)^r$ acts freely and transitively on S , since

$$\left(\prod_{i \in I} \epsilon_i\right)(A) = A - \sum_{i \in I} x_i$$

for any subset I of $\{1, \dots, r\}$.

The group $G = E_r \cdot L$ is the central extension considered above. Now, let V be a representation of G over k on which the center $\mu_2 \subset L$ acts faithfully by scalars. Then the restriction of V to L is fixed (up to isomorphism) by the action of E_r on $X^*(L)$. By the previous paragraph, the 2^r 1-dimensional representations of L that are nontrivial on the center μ_2 all occur with the same multiplicity in V . Therefore, V has dimension a multiple of 2^r . This bound is optimal, since the spin representation W of $\mathrm{Spin}(2r + 1)$ has dimension 2^r over k , and the center μ_2 acts faithfully by scalars on W .

We use the following result of Merkurjev's [16, Theorem 5.2], [11, Remark 4.5]. (The first reference covers the case of the group scheme μ_p in characteristic p , as needed here.)

Theorem 2.2. *Let k be a field and p be a prime number. Let $1 \rightarrow \mu_p \rightarrow G \rightarrow Q \rightarrow 1$ be a central extension of affine group schemes over k . For a field extension K of k , let $\partial_K: H^1(K, Q) \rightarrow H^2(K, \mu_p)$ be the boundary homomorphism in fppf cohomology. Then the maximal value of the index of $\partial_K(E)$, as K ranges over all field extensions of k and E ranges over all Q -torsors over K , is equal to the greatest common divisor of the dimensions of all representations of G on which μ_p acts by its standard representation.*

As mentioned above, for a prime number p and a nonzero element E of $H^2(K, \mu_p)$ over a field K , the essential dimension (or also the p -essential dimension) of the corresponding μ_p -gerbe over K is equal to the index of E .

Finally, consider a central extension $1 \rightarrow \mu_p \rightarrow G \rightarrow Q \rightarrow 1$ of finite group schemes over a field k . Generalizing an argument of Brosnan–Reichstein–Vistoli, Karpenko and Merkurjev showed that the p -essential dimension of G (and hence the essential dimension of G) is at least the p -essential dimension of the μ_p -gerbe over K associated to any Q -torsor over any field K/k [11, Theorem 4.2]. By the analysis above of representations of the finite subgroup scheme G of $\mathrm{Spin}(2r+1)$ over a field k of characteristic 2, we find that $\mathrm{ed}_2(G) \geq 2^r$. For a closed subgroup scheme G of a group scheme L over a field k and any prime number p , we have

$$\mathrm{ed}_p(L) + \dim(L) \geq \mathrm{ed}_p(G) + \dim(G)$$

[17, Corollary 4.3] (which covers the case of fppf torsors for non-smooth group schemes, as needed here). Applying this to the subgroup scheme G of $\mathrm{Spin}(2r)$, we conclude that

$$\mathrm{ed}_2(\mathrm{Spin}(2r+1)) \geq 2^r - \dim(\mathrm{Spin}(2r+1)) = 2^r - r(2r+1).$$

Combining this with the upper bound discussed above, we have

$$\mathrm{ed}(\mathrm{Spin}(2r+1)) = \mathrm{ed}_2(\mathrm{Spin}(2r+1)) = 2^r - r(2r+1)$$

for $r \geq 7$.

The proof of the lower bound for $\mathrm{ed}_2(\mathrm{Spin}(2r))$ when r is odd is similar. The intersection of the subgroup $K = (\mu_2 \times \mathbb{Z}/2)^r \subset O(2r)$ with $O^+(2r)$ is

$$K_1 \cong (\mu_2)^r \times (\mathbb{Z}/2)^{r-1},$$

where $(\mathbb{Z}/2)^{r-1}$ denotes the kernel of the sum $(\mathbb{Z}/2)^r \rightarrow \mathbb{Z}/2$. As a result, the double cover $\mathrm{Spin}(2r)$ contains a subgroup G_1 which is a central extension

$$1 \rightarrow \mu_2 \rightarrow G_1 \rightarrow (\mathbb{Z}/2)^{r-1} \times (\mu_2)^r \rightarrow 1.$$

In this case, an argument analogous to the one for G shows that every representation of G_1 on which the center μ_2 acts by its standard representation has dimension a multiple of 2^{r-1} (rather than 2^r). The argument is otherwise identical to the argument for $\mathrm{Spin}(2r+1)$, and we find that

$$\mathrm{ed}_2(\mathrm{Spin}(2r)) \geq 2^{r-1} - r(2r-1).$$

For r odd at least 9, this agrees with the lower bound found earlier, which proves the theorem on $\mathrm{Spin}(n)$ for $n \equiv 0 \pmod{4}$.

It remains to show that for n a multiple of 4, with 2^m the largest power of 2 dividing n , we have

$$\mathrm{ed}_2(\mathrm{Spin}(n)) \geq 2^{(n-2)/2} + 2^m - n(n-1)/2.$$

The argument follows that of Merkurjev in characteristic not 2 [17, Theorem 4.9].

Namely, for n a multiple of 4, the center C of $G := \text{Spin}(n)$ is isomorphic to $\mu_2 \times \mu_2$, and $H := G/C$ is the group $\text{PGO}^+(n)$. An H -torsor over a field L over k is equivalent to a central simple algebra A of degree n over L with a quadratic pair (σ, f) and with trivialized discriminant, meaning an isomorphism from the center of the Clifford algebra $C(A, \sigma, f)$ to $L \times L$ [13, Section 29.F]. The image of the homomorphism from $C^* \cong (\mathbb{Z}/2)^2$ to the Brauer group of L is equal to $\{0, [A], [C^+], [C^-]\}$, where C^+ and C^- are the simple components of the Clifford algebra; each is a central simple algebra of degree $2^{(n-2)/2}$ over L . By Merkurjev, there is a field L over k and an H -torsor E over L such that

$$\text{ind}(C^+) = \text{ind}(C^-) = 2^{(n-2)/2} \quad \text{and} \quad \text{ind}(A) = 2^m$$

[16, Section 4.4 and Theorem 5.2]. We use the following result [17, Example 3.7]:

Lemma 2.3. *Let L be a field, p a prime number, and r a natural number. Let C be the group scheme $(\mu_p)^r$, and let Y be a C -gerbe over L . Then the p -essential dimension of Y , and also the essential dimension of Y , is the minimum, over all bases u_1, \dots, u_r for C^* , of $\sum_{i=1}^r \text{ind}(u_i(Y))$.*

It follows that the 2-essential dimension of the $(\mu_2)^2$ -gerbe E/G over L associated to the H -torsor E above is

$$\text{ed}_2(E/G) = \text{ind}(A) + \text{ind}(C^+) = 2^{(n-2)/2} + 2^m.$$

It follows that

$$\begin{aligned} \text{ed}(\text{Spin}(n)) &\geq \text{ed}_2(\text{Spin}(n)) \\ &\geq \text{ed}_2(E/G) - \dim(G/C) \\ &= 2^{(n-2)/2} + 2^m - n(n-1)/2. \end{aligned} \quad \square$$

3. Etale motivic cohomology

In this section, we summarize the properties of etale motivic cohomology of fields, the natural home of mod p cohomological invariants for group schemes over a field of characteristic p .

For a field k of characteristic $p > 0$, let $H^{i,j}(k)$ be the etale motivic cohomology group $H_{\text{et}}^i(k, \mathbb{Z}/p(j))$, or equivalently

$$H_{\text{et}}^i(k, \mathbb{Z}/p(j)) \cong H_{\text{et}}^{i-j}(k, \Omega_{\log}^j),$$

where Ω_{\log}^j is the subgroup of the group Ω^j of differential forms on the separable closure k_s over \mathbb{F}_p spanned by products $(da_1/a_1) \wedge \dots \wedge (da_j/a_j)$ with $a_1, \dots, a_j \in k_s^*$ [9]. The group $H^{i,j}(k)$ is zero except when i equals j or $j+1$,

because k has p -cohomological dimension at most 1 [21, Section II.2.2]. The symbol $\{a_1, \dots, a_{n-1}, b\}$ denotes the element of $H^{n,n-1}(k)$ which is the product of the elements

$$a_i \in k^*/(k^*)^p \cong H^{1,1}(k) \quad \text{and} \quad b \in k/\{a^p - a : a \in k\} \cong H^{1,0}(k).$$

Also, for a field k of characteristic 2, let $W(k)$ denote the Witt ring of symmetric bilinear forms over k , and let $I_q(k)$ be the Witt group of nondegenerate quadratic forms over k . (By the conventions in Section 1, $I_q(k)$ consists only of even-dimensional forms.) Then $I_q(k)$ is a module over $W(k)$ via tensor product [5, Lemma 8.16]. Let I be the kernel of the homomorphism $\text{rank}: W(k) \rightarrow \mathbb{Z}/2$, and let

$$I_q^m(k) = I^{m-1} \cdot I_q(k),$$

following [5, p. 53]. To motivate the notation, observe that the class of an m -fold quadratic Pfister form $\langle\langle a_1, \dots, a_{m-1}, b \rangle\rangle$ lies in $I_q^m(k)$. By definition, for a_1, \dots, a_{m-1} in k^* and b in k , $\langle\langle a_1, \dots, a_{m-1}, b \rangle\rangle$ is the quadratic form

$$\langle\langle a_1 \rangle\rangle_b \otimes \cdots \otimes \langle\langle a_{m-1} \rangle\rangle_b \otimes \langle\langle b \rangle\rangle$$

of dimension 2^m , where $\langle\langle a \rangle\rangle_b$ is the bilinear form $\langle 1, a \rangle$ and $\langle\langle b \rangle\rangle$ is the quadratic form

$$[1, b] = x^2 + xy + by^2.$$

In analogy with the Milnor conjecture, Kato proved the isomorphism

$$I_q^m(F)/I_q^{m+1} \cong H^{m,m-1}(F)$$

for every field F of characteristic 2 [5, Fact 16.2]. The isomorphism takes the quadratic Pfister form $\langle\langle a_1, \dots, a_{m-1}, b \rangle\rangle$ to the symbol $\{a_1, \dots, a_{m-1}, b\}$. (For this paper, it would suffice to have Kato's homomorphism, without knowing that it is an isomorphism.)

A cohomological invariant gives a lower bound for the essential dimension, as follows. This is standard for mod l invariants with $l \neq p = \text{char}(k)$ [17, Theorem 5.3], and we now give the analogous statement for mod p invariants of a k -group scheme G . Define a cohomological invariant f of G with values in $H^{n,n-1}$ to be *nontrivial* if there is a field F containing an algebraic closure of k and a G -torsor u over F such that $f(u)$ is not zero.

Lemma 3.1. *Let G be an affine group scheme of finite type over a field k of characteristic $p > 0$. If there is a nontrivial cohomological invariant for G with values in $H^{n,n-1}$, then $\text{ed}(G) \geq \text{ed}_p(G) \geq n$.*

Proof. Let f be the given cohomological invariant for G . It suffices to prove a lower bound on the essential dimension after enlarging k . So we can replace k by its algebraic closure. Then every field F of transcendence degree less than n

over k has $H^{n,n-1}(F) = 0$, by Kato and Kuzumaki [12, Section 3, Corollary 2]. By assumption, there is a G -torsor u over a field E over k such that $f(u)$ is not zero in $H^{n,n-1}(E)$. Thanks to the transfer maps on Galois cohomology (viewing $H^{n,n-1}(E)$ as $H^1(E, \Omega_{\log}^{n-1}(E_s))$), this element remains nonzero in $H^{n,n-1}(E')$ for any finite extension E'/E of degree prime to p . Therefore, the G -torsor u extended up to E' cannot be defined over a subfield F of E' with transcendence degree less than n over k . So $\text{ed}(G) \geq \text{ed}_p(G) \geq n$. \square

Corollary 3.2. *Let G be an affine group scheme of finite type over a field k of characteristic $p > 0$. Let f be a cohomological invariant for G with values in $H^{n,n-1}$. Suppose that for any field F over k and any a_1, \dots, a_{n-1} in F^* and a_n in F , there is a G -torsor u over F with*

$$f(u) = \{a_1, \dots, a_{n-1}, a_n\}$$

in $H^{n,n-1}(F)$. Then $\text{ed}(G) \geq \text{ed}_p(G) \geq n$.

Proof. Let \bar{k} be an algebraic closure of k , and let E be the rational function field $\bar{k}(a_1, \dots, a_n)$. By assumption, there is a G -torsor u over E such that

$$f(u) = \{a_1, \dots, a_{n-1}, a_n\}.$$

This symbol in $H^{n,n-1}(E)$ is not zero, by Izhboldin's calculation of $H^{n,n-1}$ of a rational function field [10, Theorem 4.5]. Thus f is nontrivial, in the sense above. By Lemma 3.1, $\text{ed}(G) \geq \text{ed}_p(G) \geq n$. \square

4. Low-dimensional spin groups

Rost and Garibaldi determined the essential dimension of the spin groups $\text{Spin}(n)$ with $n \leq 14$ in characteristic not 2 [6, Table 23B]. It should be possible to compute the essential dimension of low-dimensional spin groups in characteristic 2 as well. The following section carries this out for $\text{Spin}(n)$ with $n \leq 10$. We find that in this range (as for $n \geq 15$), the essential dimension of the spin group is the same in characteristic 2 as in characteristic not 2, unlike what happens for $O(n)$ and $SO(n)$.

For $n \leq 10$, we give group-theoretic proofs which work almost the same way in any characteristic, despite the distinctive features of quadratic forms in characteristic 2.

Theorem 4.1. *For $n \leq 10$, the essential dimension, as well as the 2-essential dimension, of the split group $\mathrm{Spin}(n)$ over a field k of any characteristic is given by:*

n	$\mathrm{ed}(\mathrm{Spin}(n))$
≤ 6	0
7	4
8	5
9	5
10	4

Proof. As discussed above, it suffices to consider the case of a field k of characteristic 2. For $2 \leq n \leq 6$, every $\mathrm{Spin}(n)$ -torsor over a field is trivial, for example by the exceptional isomorphisms

$$\begin{aligned} \mathrm{Spin}(2) &\cong G_m, & \mathrm{Spin}(3) &\cong SL(2), & \mathrm{Spin}(4) &\cong SL(2) \times SL(2), \\ \mathrm{Spin}(5) &\cong Sp(4), & \text{and} & & \mathrm{Spin}(6) &\cong SL(4). \end{aligned}$$

It follows that $\mathrm{ed}(\mathrm{Spin}(n)) = 0$ for $2 \leq n \leq 6$.

We will use the following standard approach to bounding the essential dimension of a group.

Lemma 4.2. *Let G be an affine group scheme of finite type over a field k . Suppose that G acts on a k -scheme Y with a nonempty open orbit U . Suppose that for every G -torsor E over an infinite field F over k , the twisted form $(E \times Y)/G$ of Y over F has a Zariski-dense set of F -points. Finally, suppose that U has a k -point x , and let N be the stabilizer k -group scheme of x in G . Then*

$$H^1(F, N) \rightarrow H^1(F, G)$$

is surjective for every infinite field F over k (or for every field F over k , if G is smooth and connected). As a result, $\mathrm{ed}_k(G) \leq \mathrm{ed}_k(N)$.

The proof is short, the same as that of [6, Theorem 9.3]. (Note that even if k is finite, we get the stated upper bound for the essential dimension of G : a G -torsor over a finite field F that contains k causes no problem, because F has transcendence degree 0 over k .) If G is smooth and connected, then $H^1(F, G)$ is in fact trivial for every finite field F that contains k , by Lang [14]; that implies the statement in the theorem that $H^1(F, N) \rightarrow H^1(F, G)$ is surjective for every field F over k .

The assumption about a Zariski-dense set of rational points holds, for example, if Y is a linear representation V of G , or if Y is the associated projective space $P(V)$ to a representation, or (as we use later) a product $P(V) \times P(W)$.

We use Garibaldi and Guralnick's calculation of the stabilizer group scheme of a general k -point in the spin (for n odd) or a half-spin (for n even) representation W

of the split group $\mathrm{Spin}(n)$, listed in Table 1 here [7, Table 1]. Here $\mathrm{Spin}(n)$ has an open orbit on the projective space $P(W)$ of lines in W if $n \leq 12$ or $n = 14$, and an open orbit on W if $2 \leq n \leq 6$ or $n = 10$. (To be precise, we will use that even if k is finite, there is a k -point in the open orbit for which the stabilizer k -group scheme is the *split* group listed in the table.)

n	$\mathrm{char} k \neq 2$	$\mathrm{char} k = 2$
6	$SL(3) \cdot (G_a)^3$	same
7	G_2	same
8	$\mathrm{Spin}(7)$	same
9	$\mathrm{Spin}(7)$	same
10	$\mathrm{Spin}(7) \cdot (G_a)^8$	same
11	$SL(5)$	$\mathbf{Z}/2 \ltimes SL(5)$
12	$SL(6)$	$\mathbf{Z}/2 \ltimes SL(6)$
13	$SL(3) \times SL(3)$	$\mathbf{Z}/2 \ltimes (SL(3) \times SL(3))$
14	$G_2 \times G_2$	$\mathbf{Z}/2 \ltimes (G_2 \times G_2)$

Table 1. Generic stabilizer of spin (or half-spin) representation of $\mathrm{Spin}(n)$.

We now begin to compute the essential dimension of the split group $G = \mathrm{Spin}(7)$ over a field k of characteristic 2. Let W be the 8-dimensional spin representation of G . Then G has an open orbit on the projective space $P(W)$ of lines in W . By Table 1, there is a k -point x in W whose image in $P(W)$ is in the open orbit such that the stabilizer of x in G is the split exceptional group G_2 . Since G preserves a quadratic form on W , the stabilizer H of the corresponding k -point in $P(W)$ is at most $G_2 \times \mu_2$. In fact, H is equal to $G_2 \times \mu_2$, because the center μ_2 of G acts trivially on $P(W)$.

By Lemma 4.2, the inclusion $G_2 \times \mu_2 \hookrightarrow G$ induces a surjection

$$H^1(F, G_2 \times \mu_2) \rightarrow H^1(F, G)$$

for every field F over k . Over any field F , G_2 -torsors up to isomorphism can be identified with 3-fold quadratic Pfister forms $\langle\langle a_1, a_2, b \rangle\rangle$ (with $a_1, a_2 \in F^*$ and $b \in F$), and so G_2 has essential dimension 3 [21, Théorème 11]. Since μ_2 has essential dimension 1, the surjectivity above implies that $G = \mathrm{Spin}(7)$ has essential dimension at most 4.

Next, a G -torsor determines two quadratic forms of dimension 8. Besides the obvious homomorphism $\chi_1: G \hookrightarrow \mathrm{Spin}(8) \rightarrow SO(8)$ (which is trivial on the center μ_2 of G), we have the spin representation $\chi_2: G \rightarrow SO(8)$, on which μ_2 acts faithfully by scalars. Thus a G -torsor u over a field F over k determines two quadratic forms of dimension 8 over F , which we call q_1 and q_2 .

To describe these quadratic forms in more detail, use that every G -torsor comes from a torsor for $G_2 \times \mu_2$. The two homomorphisms $G_2 \hookrightarrow G \rightarrow SO(8)$ (via χ_1 and χ_2) are both conjugate to the standard inclusion. Also, χ_1 is trivial on the μ_2 factor, while χ_2 acts faithfully by scalars on the μ_2 factor. It follows that q_1 is a quadratic Pfister form, $\langle\langle a, b, c \rangle\rangle$ (the form associated to a G_2 -torsor), while q_2 is a scalar multiple of that form, $d\langle\langle a, b, c \rangle\rangle$.

Therefore, a G -torsor u canonically determines a 4-fold quadratic Pfister form,

$$q_1 + q_2 = \langle\langle d, a, b, c \rangle\rangle.$$

Define $f_4(u)$ to be the associated element of $H^{4,3}(F)$,

$$f_4(u) = \{d, a, b, c\}.$$

By construction, this is well-defined and an invariant of u . By considering the subgroup $G_2 \times \mu_2 \subset \text{Spin}(7)$, where there is a $G_2 \times \mu_2$ -torsor associated to any elements a, b, d in F^* and c in F , we see that a, b, c, d can be chosen arbitrarily. By Corollary 3.2, $G = \text{Spin}(7)$ has 2-essential dimension at least 4, and hence essential dimension at least 4.

The opposite inequality was proved above, and so $\text{Spin}(7)$ has essential dimension equal to 4. Since the lower bound is proved by constructing a mod 2 cohomological invariant, this argument also shows that $\text{Spin}(7)$ has 2-essential dimension equal to 4. For the same reason, the computations of essential dimension below (for $\text{Spin}(n)$ with $8 \leq n \leq 10$) also give the 2-essential dimension.

Next, we turn to $\text{Spin}(8)$. At first, let $G = \text{Spin}(2r)$ for a positive integer r over a field k of characteristic 2. Let V be the standard $2r$ -dimensional representation of G . Then G has an open orbit in the projective space $P(V)$ of lines in V . The stabilizer k -group scheme H of a general k -point in $P(V)$ is conjugate to $\text{Spin}(2r-1) \cdot Z$, where Z is the center of $\text{Spin}(2r)$, with $\text{Spin}(2r-1) \cap Z = \mu_2$. (In more detail, a general line in V is spanned by a vector x with $q(x) \neq 0$, where q is the quadratic form on V . Then the stabilizer of x in $SO(V)$ is isomorphic to $SO(S)$, where $S := x^\perp$ is a hyperplane in V on which q restricts to a nonsingular quadratic form of dimension $2r-1$, with S^\perp equal to the line $k \cdot x \subset S$.) Here

$$Z \cong \begin{cases} \mu_2 \times \mu_2 & \text{if } r \text{ is even,} \\ \mu_4 & \text{if } r \text{ is odd.} \end{cases}$$

In particular, if r is even, then $H \cong \text{Spin}(2r-1) \times \mu_2$. Thus, for r even, the inclusion $\text{Spin}(2r-1) \times \mu_2 \hookrightarrow G$ induces a surjection

$$H^1(F, \text{Spin}(2r-1) \times \mu_2) \rightarrow H^1(F, G)$$

for every field F over k , by Lemma 4.2.

It follows that, for r even, the essential dimension of $\mathrm{Spin}(2r)$ is at most 1 plus the essential dimension of $\mathrm{Spin}(2r - 1)$. Since $\mathrm{Spin}(7)$ has essential dimension 4, $G = \mathrm{Spin}(8)$ has essential dimension at most 5.

Before proving that equality holds, let us analyze G -torsors in more detail. We know that $H^1(F, \mathrm{Spin}(7) \times \mu_2) \rightarrow H^1(F, G)$ is onto, for all fields F over k . Also, we showed earlier that $H^1(F, G_2 \times \mu_2) \rightarrow H^1(F, \mathrm{Spin}(7))$ is surjective. Therefore,

$$H^1(F, G_2 \times \mu_2 \times \mu_2) \rightarrow H^1(F, G)$$

is surjective for all fields F over k , where $Z = \mu_2 \times \mu_2$ is the center of G . As discussed earlier, G_2 -torsors up to isomorphism can be identified with 3-fold quadratic Pfister forms. It follows that every G -torsor is associated to some 3-fold quadratic Pfister form $\langle\langle a, b, c \rangle\rangle$ and some elements d, e in F^* , which yield elements of $H^1(F, \mu_2) = F^*/(F^*)^2$.

Next, observe that a G -torsor determines several quadratic forms. Besides the obvious double covering $\chi_1: G \rightarrow \mathrm{SO}(8)$, the two half-spin representations of G give two other homomorphisms $\chi_2, \chi_3: G \rightarrow \mathrm{SO}(8)$. (These three homomorphisms can be viewed as the quotients of G by the three k -subgroup schemes of order 2 in Z . They are permuted by the group S_3 of “triality” automorphisms of G .) Thus a G -torsor u over a field F over k determines three quadratic forms of dimension 8, which we call q_1, q_2, q_3 .

To describe how these three quadratic forms are related, use that every G -torsor comes from a torsor for $G_2 \times \mu_2 \times \mu_2$. The three homomorphisms $G_2 \rightarrow G \rightarrow \mathrm{SO}(8)$ (via χ_1, χ_2 , and χ_3) are all conjugate to the standard inclusion, whereas the three homomorphisms send $\mu_2 \times \mu_2$ to the center $\mu_2 \subset \mathrm{SO}(8)$ by the three possible surjections. It follows that the three quadratic forms can be written as $q_1 = d \langle\langle a, b, c \rangle\rangle$, $q_2 = e \langle\langle a, b, c \rangle\rangle$, and $q_3 = de \langle\langle a, b, c \rangle\rangle$.

Note that a scalar multiple $q = d \langle\langle a_1, \dots, a_{m-1}, b \rangle\rangle$ of a quadratic Pfister form (as a quadratic form up to isomorphism) uniquely determines the associated quadratic Pfister form $q_0 = \langle\langle a_1, \dots, a_{m-1}, b \rangle\rangle$ up to isomorphism. (Proof: it suffices to show that if q and r are m -fold quadratic Pfister forms over F with $aq \cong r$ for some a in F^* , then $q \cong r$. Since r takes value 1, so does aq , and so q takes value a^{-1} . But then $a^{-1}q \cong q$ by the multiplicativity of quadratic Pfister forms [5, Corollary 9.9]. Therefore, $r \cong aq \cong q$.)

We now define an invariant for $G = \mathrm{Spin}(8)$ over k with values in $H^{5,4}$. Given a G -torsor u over a field F over k , consider the three associated quadratic forms q_1, q_2, q_3 as above. By the previous paragraph, $q_1 = d \langle\langle a, b, c \rangle\rangle$ determines the quadratic Pfister form $q_0 = \langle\langle a, b, c \rangle\rangle$. So u determines the 5-fold quadratic Pfister form

$$q_0 + q_1 + q_2 + q_3 = \langle\langle d, e, a, b, c \rangle\rangle.$$

The associated class

$$f_5(u) = \{d, e, a, b, c\} \in H^{5,4}(F)$$

is therefore an invariant of u .

By considering the subgroup $G_2 \times Z \subset G = \text{Spin}(8)$, where $Z = \mu_2 \times \mu_2$, there is a $G_2 \times Z$ -torsor associated to any elements a, b, d, e in F^* and c in F , and f_5 of the associated G -torsor is $\{d, e, a, b, c\}$ in $H^{5,4}(F)$. By Corollary 3.2, G has essential dimension at least 5. Since the opposite inequality was proved above, $G = \text{Spin}(8)$ has essential dimension over k equal to 5.

Next, let $G = \text{Spin}(9)$ over a field k of characteristic 2. Let W be the spin representation of G , of dimension 16, corresponding to a homomorphism $G \rightarrow \text{SO}(16)$. (A reference for the fact that this self-dual representation is orthogonal in characteristic 2, as in other characteristics, is [8, Theorem 9.2.2].) By Table 1, G has an open orbit on the space $P(W)$ of lines in W , and the stabilizer in G of a general k -point in W is conjugate to $\text{Spin}(7)$. (This is not the standard inclusion of $\text{Spin}(7)$ in $\text{Spin}(9)$, but rather a lift of the spin representation $\chi_2: \text{Spin}(7) \rightarrow \text{SO}(8)$ to $\text{Spin}(8)$ followed by the standard inclusion $\text{Spin}(8) \hookrightarrow \text{Spin}(9)$. In particular, the image of $\text{Spin}(7)$ does not contain the center μ_2 of $G = \text{Spin}(9)$.) Since G preserves a quadratic form on W , it follows that the stabilizer in G of a general k -point in $P(W)$ is conjugate to $\text{Spin}(7) \times \mu_2$, where μ_2 is the center of $\text{Spin}(9)$ (which acts faithfully by scalars on W). Therefore, by Lemma 4.2, the inclusion of $\text{Spin}(7) \times \mu_2$ in $G = \text{Spin}(9)$ induces a surjection

$$H^1(F, \text{Spin}(7) \times \mu_2) \rightarrow H^1(F, G)$$

for every field F over k .

Since $\text{Spin}(7)$ has essential dimension 4 over k as shown above, $G = \text{Spin}(9)$ has essential dimension at most $4 + 1 = 5$.

Next, a G -torsor determines several quadratic forms. Besides the obvious homomorphism $R: G \hookrightarrow \text{Spin}(10) \rightarrow \text{SO}(10)$, we have the spin representation $S: G \rightarrow \text{SO}(16)$. Thus a G -torsor over a field F over k determines a quadratic form r of dimension 10 and a quadratic form s of dimension 16.

To describe how these forms are related, use that every G -torsor comes from a torsor for the subgroup $\text{Spin}(7) \times \mu_2$ described above. The restriction of R to the given subgroup $\text{Spin}(7)$ is the composition of the spin representation $\chi_2: \text{Spin}(7) \rightarrow \text{SO}(8)$ with the obvious inclusion $\text{SO}(8) \hookrightarrow \text{SO}(10)$. The restriction of S to the given subgroup $\text{Spin}(7)$ is the direct sum of the standard representation $\chi_1: \text{Spin}(7) \rightarrow \text{SO}(8)$ and the spin representation $\chi_2: \text{Spin}(7) \rightarrow \text{SO}(8)$. Finally, R is trivial on the second factor μ_2 (the center of G), whereas S acts faithfully by scalars on S .

Now, let (u_1, e) be a $\text{Spin}(7) \times \mu_2$ -torsor over k , where u_1 is a $\text{Spin}(7)$ -torsor and e is in $H^1(F, \mu_2) = F^*/(F^*)^2$, which we lift to an element e of F^* . By the earlier analysis of the quadratic forms associated to a $\text{Spin}(7)$ -torsor, the quadratic form associated to u_1 via the standard representation $\chi_1: \text{Spin}(7) \rightarrow \text{SO}(8)$ is a 3-fold quadratic Pfister form $\langle\langle a, b, c \rangle\rangle$, while the quadratic form associated to u_1 via the spin representation $\chi_2: \text{Spin}(7) \rightarrow \text{SO}(8)$ is a multiple of the same form, $d\langle\langle a, b, c \rangle\rangle$.

By the analysis of representations two paragraphs back, it follows that the quadratic form associated to (u_1, e) via the representation $R: G \rightarrow SO(10)$ is $r = H + d\langle\langle a, b, c \rangle\rangle$, where H is the hyperbolic plane. Also, the quadratic form associated to (u_1, e) via the representation $S: G \rightarrow SO(16)$ is $s = e\langle\langle a, b, c \rangle\rangle + de\langle\langle a, b, c \rangle\rangle$.

Next, r determines the quadratic form $r_0 = d\langle\langle a, b, c \rangle\rangle$ by Witt cancellation [5, Theorem 8.4], and that in turn determines the quadratic Pfister form $q_0 = \langle\langle a, b, c \rangle\rangle$ as shown above. Therefore, a G -torsor u determines the 5-fold quadratic Pfister form

$$q_0 + r_0 + s = \langle\langle d, e, a, b, c \rangle\rangle$$

up to isomorphism.

Therefore, defining

$$f_5(u) = \{d, e, a, b, c\}$$

in $H^{5,4}(F)$ yields an invariant of u . By our earlier description of $\text{Spin}(7)$ -torsors, we can take a, b, d, e to be any elements of F^* and c any element of F . By Corollary 3.2, G has essential dimension at least 5. Since the opposite inequality was proved earlier, $G = \text{Spin}(9)$ over k has essential dimension equal to 5.

Finally, let $G = \text{Spin}(10)$ over a field k of characteristic 2. Let V be the 10-dimensional standard representation of G , corresponding to the double covering $G \rightarrow SO(10)$, and let W be one of the 16-dimensional half-spin representations of G , corresponding to a homomorphism $G \rightarrow SL(16)$. (The other half-spin representation of G is the dual W^* .)

As discussed above for any group $\text{Spin}(2r)$, $G = \text{Spin}(10)$ has an open orbit on $P(V)$, with generic stabilizer $\text{Spin}(9) \cdot \mu_4$. (Here μ_4 is the center of G , which contains the center μ_2 of $\text{Spin}(9)$.) Consider the action of G on $P(V) \times P(W) \cong \mathbf{P}^9 \times \mathbf{P}^{15}$. As discussed above, $\text{Spin}(9)$ (and hence $\text{Spin}(9) \cdot \mu_4$) has an open orbit on $P(W)$. As a result, G has an open orbit on $P(V) \times P(W)$. Moreover, the generic stabilizer of $\text{Spin}(9)$ on $P(W)$ is $\text{Spin}(7) \times \mu_2$, where the inclusion $\text{Spin}(7) \hookrightarrow \text{Spin}(9)$ is the composition of the spin representation $\text{Spin}(7) \hookrightarrow \text{Spin}(8)$ with the standard inclusion into $\text{Spin}(9)$; in particular, the image does not contain the center μ_2 of $\text{Spin}(9)$. Therefore, the generic stabilizer of $\text{Spin}(9) \cdot \mu_4 \subset \text{Spin}(10)$ on $P(W)$ is $\text{Spin}(7) \times \mu_4$. We conclude that G has an open orbit on $P(V) \times P(W)$, with generic stabilizer $\text{Spin}(7) \times \mu_4$. It follows that

$$H^1(F, \text{Spin}(7) \times \mu_4) \rightarrow H^1(F, G)$$

is surjective for every field F over k , by Lemma 4.2.

The image H_2 of the subgroup $H = \text{Spin}(7) \times \mu_4 \subset G$ in $SO(10)$ is $\text{Spin}(7) \times \mu_2$, where $\text{Spin}(7)$ is contained in $SO(8)$ (and contains the center μ_2 of $SO(8)$) and μ_2 is the center of $SO(10)$. In terms of the subgroup $SO(8) \times SO(2)$ of $SO(10)$, we can also describe H_2 as $\text{Spin}(7) \times \mu_2$, where $\text{Spin}(7)$ is contained in $SO(8)$ and μ_2 is contained in $SO(2)$. Thus H_2 is contained in $\text{Spin}(7) \times SO(2)$. Therefore, H is

contained in $\mathrm{Spin}(7) \times G_m \subset G = \mathrm{Spin}(10)$, where G_m is the inverse image in G of $SO(2) \subset SO(10)$. It follows that

$$H^1(F, \mathrm{Spin}(7) \times G_m) \rightarrow H^1(F, G)$$

is surjective for every field F over k . Since every G_m -torsor over a field is trivial,

$$H^1(F, \mathrm{Spin}(7)) \rightarrow H^1(F, G)$$

is surjective for every field F over k .

Here $\mathrm{Spin}(7)$ maps into $\mathrm{Spin}(8)$ by the spin representation, and then

$$\mathrm{Spin}(8) \hookrightarrow G = \mathrm{Spin}(10)$$

by the standard inclusion. By the description above of the 8-dimensional quadratic form associated to a $\mathrm{Spin}(7)$ -torsor by the spin representation, it follows that the quadratic form associated to a G -torsor is of the form $H + d\langle\langle a, b, c \rangle\rangle$.

Every 10-dimensional quadratic form in I_q^3 over a field is associated to some G -torsor. So we have given another proof that every 10-dimensional quadratic form in I_q^3 is isotropic. This was proved in characteristic not 2 by Pfister, and it was extended to characteristic 2 by Baeza and Tits, independently [2, pp. 129–130], [22, Theorem 4.4.1(ii)].

Since $\mathrm{Spin}(7)$ has essential dimension 4, the surjectivity above implies that $G = \mathrm{Spin}(10)$ has essential dimension at most 4. To prove equality, we define an invariant for G with values in $H^{4,3}$ by the same argument used for $\mathrm{Spin}(7)$. Namely, a G -torsor u over a field F over k determines a 4-fold quadratic Pfister form

$$\langle\langle d, a, b, c \rangle\rangle$$

up to isomorphism, and hence the element

$$f_4(u) = \{d, a, b, c\}$$

in $H^{4,3}(F)$. By Corollary 3.2, this completes the proof that $G = \mathrm{Spin}(10)$ over k has essential dimension equal to 4. As in the previous cases, since the lower bound is proved using a mod 2 cohomological invariant, G also has 2-essential dimension equal to 4. \square

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