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## Essential dimension of the spin groups in characteristic 2

Burt Totaro\*

**Abstract.** We determine the essential dimension of the spin group  $\text{Spin}(n)$  as an algebraic group over a field of characteristic 2, for  $n$  at least 15. In this range, the essential dimension is the same as in characteristic not 2. In particular, it is exponential in  $n$ . In contrast, the essential dimension of the orthogonal groups is smaller in characteristic 2. We also find the essential dimension of  $\text{Spin}(n)$  in characteristic 2 for  $n$  at most 10.

**Mathematics Subject Classification (2010).** 11E04, 11E72, 20G15.

**Keywords.** Essential dimension, spin group, cohomological invariant, etale motivic cohomology.

### Introduction

The essential dimension of an algebraic group  $G$  is a measure of the number of parameters needed to describe all  $G$ -torsors over all fields. A major achievement of the subject was the calculation of the essential dimension of the spin groups over a field of characteristic not 2, started by Brosnan, Reichstein, and Vistoli, and completed by Chernousov, Merkurjev, Garibaldi, and Guralnick [3, 4, 7], [18, Theorem 9.1].

In this paper, we determine the essential dimension of the spin group  $\text{Spin}(n)$  for  $n \geq 15$  over an arbitrary field (Theorem 2.1). We find that the answer is the same in all characteristics. In contrast, for the groups  $O(n)$  and  $SO(n)$ , the essential dimension is smaller in characteristic 2, by Babic and Chernousov [1].

In characteristic not 2, the computation of essential dimension can be phrased to use a natural finite subgroup of  $\text{Spin}(2r + 1)$ , namely an extraspecial 2-group, a central extension of  $(\mathbb{Z}/2)^{2r}$  by  $\mathbb{Z}/2$ . A distinctive feature of the argument in characteristic 2 is that the analogous subgroup is a finite group scheme, a central extension of  $(\mathbb{Z}/2)^r \times (\mu_2)^r$  by  $\mu_2$ , where  $\mu_2$  is the group scheme of square roots of unity.

In characteristic not 2, Rost and Garibaldi computed the essential dimension of  $\text{Spin}(n)$  for  $n \leq 14$  [6, Table 23B], where case-by-case arguments seem to be needed. We show in Theorem 4.1 that for  $n \leq 10$ , the essential dimension of  $\text{Spin}(n)$

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is the same in characteristic 2 as in characteristic not 2. It would be interesting to compute the essential dimension of  $\text{Spin}(n)$  in the remaining cases,  $11 \leq n \leq 14$  in characteristic 2.

## 1. Essential dimension

Let  $G$  be an affine group scheme of finite type over a field  $k$ . Write  $H^1(k, G)$  for the set of isomorphism classes of  $G$ -torsors over  $k$  in the fppf topology. For  $G$  smooth over  $k$ , this is also the set of isomorphism classes of  $G$ -torsors over  $k$  in the etale topology.

Following Reichstein, the *essential dimension*  $\text{ed}(G)$  is the smallest natural number  $r$  such that for every  $G$ -torsor  $\xi$  over an extension field  $E$  of  $k$ , there is a subfield  $k \subset F \subset E$  such that  $\xi$  is isomorphic to some  $G$ -torsor over  $F$  extended to  $E$ , and  $F$  has transcendence degree at most  $r$  over  $k$ . (It is essential that  $E$  is allowed to be any extension field of  $k$ , not just an algebraic extension field.) There are several survey articles on essential dimension, including [17, 19].

For example, let  $q_0$  be a quadratic form of dimension  $n$  over a field  $k$  of characteristic not 2. Then  $O(q_0)$ -torsors can be identified with quadratic forms of dimension  $n$ , up to isomorphism. (For convenience, we sometimes write  $O(n)$  for  $O(q_0)$ .) Thus the essential dimension of  $O(n)$  measures the number of parameters needed to describe all quadratic forms of dimension  $n$ . Indeed, every quadratic form of dimension  $n$  over a field of characteristic not 2 is isomorphic to a diagonal form  $\langle a_1, \dots, a_n \rangle$ . It follows that the orthogonal group  $O(n)$  in characteristic not 2 has essential dimension at most  $n$ ; in fact,  $O(n)$  has essential dimension equal to  $n$ , by one of the first computations of essential dimension [19, Example 2.5]. Reichstein also showed that the connected group  $SO(n)$  in characteristic not 2 has essential dimension  $n - 1$  for  $n \geq 3$  [19, Corollary 3.6].

For another example, for an integer  $n \geq 2$  and any field  $k$ , the group scheme  $\mu_n$  of  $n$ th roots of unity is smooth over  $k$  if and only if  $n$  is invertible in  $k$ . Independent of that,  $H^1(k, \mu_n)$  is always isomorphic to  $k^*/(k^*)^n$ . From that description, it is immediate that  $\mu_n$  has essential dimension at most 1 over  $k$ . It is not hard to check that the essential dimension is in fact equal to 1.

One simple bound is that for any generically free representation  $V$  of a group scheme  $G$  over  $k$  (meaning that  $G$  acts freely on a nonempty open subset of  $V$ ), the essential dimension of  $G$  is at most  $\dim(V) - \dim(G)$  [18, Proposition 5.1]. It follows, for example, that the essential dimension of any affine group scheme of finite type over  $k$  is finite.

For a prime number  $p$ , the  $p$ -essential dimension  $\text{ed}_p(G)$  is a simplified invariant, defined by “ignoring field extensions of degree prime to  $p$ ”. In more detail, for a  $G$ -torsor  $\xi$  over an extension field  $E$  of  $k$ , define the  $p$ -essential dimension  $\text{ed}_p(\xi)$  to be the smallest number  $r$  such that there is a finite extension  $E'/E$  of degree

prime to  $p$  such that  $\xi$  over  $E'$  comes from a  $G$ -torsor over a subfield  $k \subset F \subset E'$  of transcendence degree at most  $r$  over  $k$ . Then the  $p$ -essential dimension  $\text{ed}_p(G)$  is defined to be the supremum of the  $p$ -essential dimensions of all  $G$ -torsors over all extension fields of  $k$ .

The *spin group*  $\text{Spin}(n)$  is the simply connected double cover of  $SO(n)$ . It was a surprise when Brosnan, Reichstein, and Vistoli showed that the essential dimension of  $\text{Spin}(n)$  over a field  $k$  of characteristic not 2 is exponentially large, asymptotic to  $2^{n/2}$  as  $n$  goes to infinity [3]. As an application, they showed that the number of “parameters” needed to describe all quadratic forms of dimension  $2r$  in  $I^3$  over all fields is asymptotic to  $2^r$ .

We now turn to quadratic forms over a field which may have characteristic 2. Define a quadratic form  $(q, V)$  over a field  $k$  to be *nondegenerate* if the radical  $V^\perp$  of the associated bilinear form is 0, and *nonsingular* if  $V^\perp$  has dimension at most 1 and  $q$  is nonzero on any nonzero element of  $V^\perp$ . (In characteristic not 2, nonsingular and nondegenerate are the same.) The orthogonal group is defined as the automorphism group scheme of a nonsingular quadratic form [13, Section VI.23]. For example, over a field  $k$  of characteristic 2, the quadratic form

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r}$$

is nonsingular of even dimension  $2r$ , while the form

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r} + x_{2r+1}^2$$

is nonsingular of odd dimension  $2r+1$ , with  $V^\perp$  of dimension 1. The *split* orthogonal group over  $k$  is the automorphism group of one of these particular quadratic forms.

Babic and Chernousov computed the essential dimension of  $O(n)$  and the smooth connected subgroup  $O^+(n)$  over an infinite field  $k$  of characteristic 2 [1]. (We also write  $SO(n)$  for  $O^+(n)$  by analogy with the case of characteristic not 2, even though the whole group  $O(2r)$  is contained in  $SL(2r)$  in characteristic 2.) The answer is smaller than in characteristic not 2. Namely,  $O(2r)$  has essential dimension  $r+1$  (not  $2r$ ) over  $k$ . Also,  $O^+(2r)$  has essential dimension  $r+1$  for  $r$  even, and either  $r$  or  $r+1$  for  $r$  odd, not  $2r-1$ . Finally, the group scheme  $O(2r+1)$  has essential dimension  $r+2$  over  $k$ , and  $O^+(2r+1)$  has essential dimension  $r+1$ . The lower bounds here are difficult, while the upper bounds are straightforward. For example, to show that  $O(2r)$  has essential dimension at most  $r+1$  in characteristic 2, write any quadratic form of dimension  $2r$  as a direct sum of 2-dimensional forms, thus reducing the structure group to  $(\mathbf{Z}/2)^r \times (\mu_2)^r$ , and then use that the group  $(\mathbf{Z}/2)^r$  has essential dimension only 1 over an infinite field of characteristic 2 [1, proof of Proposition 13.1].

In this paper, we determine the essential dimension of  $\text{Spin}(n)$  in characteristic 2 for  $n \leq 10$  or  $n \geq 15$ . Surprisingly, in view of what happens for  $O(n)$  and  $O^+(n)$ , the results for spin groups are the same in characteristic 2 as in characteristic not 2.

For  $n \leq 10$ , the lower bound for the essential dimension is proved by constructing suitable cohomological invariants. It is not known whether a similar approach is possible for  $n \geq 15$ , either in characteristic 2 or in characteristic not 2.

## 2. Main result

**Theorem 2.1.** *Let  $k$  be a field. For every integer  $n$  at least 15, the essential dimension of the split group  $\text{Spin}(n)$  over  $k$  is given by:*

$$\text{ed}(\text{Spin}(n)) = \begin{cases} 2^{n-1} - n(n-1)/2 & \text{if } n \text{ is odd;} \\ 2^{(n-2)/2} - n(n-1)/2 & \text{if } n \equiv 2 \pmod{4}; \\ 2^{(n-2)/2} + 2^m - n(n-1)/2 & \text{if } n \equiv 0 \pmod{4}, \end{cases}$$

where  $2^m$  is the largest power of 2 dividing  $n$ . The 2-essential dimension is the same number.

*Proof.* For  $k$  of characteristic 0, this was proved by Chernousov and Merkurjev, sharpening the results of Brosnan, Reichstein, and Vistoli [4, Theorem 2.2]. Their argument works in any characteristic not 2, using the results of Garibaldi and Guralnick for the upper bounds [7]. Namely, Garibaldi and Guralnick showed that for any field  $k$  and any  $n$  at least 15,  $\text{Spin}(n)$  acts generically freely on the spin representation for  $n$  odd, on each of the two half-spin representations if  $n \equiv 2 \pmod{4}$ , and on the direct sum of a half-spin representation and the standard representation if  $n \equiv 0 \pmod{4}$ . Moreover, for  $n$  at least 20 with  $n \equiv 0 \pmod{4}$ ,  $\text{HSpin}(n) = \text{Spin}(n)/\mu_2$  (the quotient different from  $O^+(n)$ ) acts generically freely on a half-spin representation [7, Theorem 1.1].

It remains to consider a field  $k$  of characteristic 2. Garibaldi and Guralnick's result gives the desired upper bound in most cases. Namely, for  $n$  odd and at least 15, the spin representation has dimension  $2^{(n-1)/2}$ , and so

$$\text{ed}(\text{Spin}(n)) \leq 2^{(n-1)/2} - \dim(\text{Spin}(n)) = 2^{(n-1)/2} - n(n-1)/2.$$

For  $n \equiv 2 \pmod{4}$ , the half-spin representations have dimension  $2^{(n-2)/2}$ , and so

$$\text{ed}(\text{Spin}(n)) \leq 2^{(n-2)/2} - n(n-1)/2.$$

For  $n = 16$ , since the spin group acts generically freely on the direct sum of a half-spin representation and the standard representation,

$$\text{ed}(\text{Spin}(n)) \leq 2^{(n-2)/2} + n - n(n-1)/2 (= 24).$$

For  $n$  at least 20 and divisible by 4, the optimal upper bound requires more effort. The following argument is modeled on Chernousov and Merkurjev's characteristic

zero argument [4, Theorem 2.2]. Namely, consider the map of exact sequences of  $k$ -group schemes:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \text{Spin}(n) & \longrightarrow & \text{HSpin}(n) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & O^+(n) & \longrightarrow & \text{PGO}^+(n) \longrightarrow 1. \end{array}$$

Since  $\text{HSpin}(n)$  acts generically freely on a half-spin representation, which has dimension  $2^{(n-2)/2}$ , we have

$$\text{ed}(\text{HSpin}(n)) \leq 2^{(n-2)/2} - n(n-1)/2.$$

By Chernousov–Merkurjev or independently Löttscher, for any normal subgroup scheme  $C$  of an affine group scheme  $G$  over a field  $k$ ,

$$\text{ed}(G) \leq \text{ed}(G/C) + \max \text{ed}[E/G],$$

where the maximum runs over all field extensions  $F$  of  $k$  and all  $G/C$ -torsors  $E$  over  $F$  [4, Proposition 2.1], [15, Example 3.4]. Thus  $[E/G]$  is a gerbe over  $F$  banded by  $C$ .

Identifying  $H^2(K, \mu_p)$  with the  $p$ -torsion in the Brauer group of  $K$ , we can talk about the index of an element of  $H^2(K, \mu_p)$ , meaning the degree of the corresponding division algebra over  $K$ . For a prime number  $p$  and a nonzero element  $E$  of  $H^2(K, \mu_p)$  over a field  $K$ , the essential dimension (or also the  $p$ -essential dimension) of the corresponding  $\mu_p$ -gerbe over  $K$  is equal to the index of  $E$ , by Karpenko and Merkurjev [11, Theorems 2.1 and 3.1].

By the diagram above, for any field  $F$  over  $k$ , the image of the connecting map

$$H^1(F, \text{HSpin}(n)) \rightarrow H^2(F, \mu_2) \subset \text{Br}(F)$$

is contained in the image of the other connecting map

$$H^1(F, \text{PGO}^+(n)) \rightarrow H^2(F, \mu_2) \subset \text{Br}(F).$$

In the terminology of the Book of Involutions, the image of the latter map consists of the classes  $[A]$  of all central simple  $F$ -algebras  $A$  of degree  $n$  with a quadratic pair  $(\sigma, f)$  of trivial discriminant [13, Section 29.F]. Any torsor for  $\text{PGO}^+(n)$  is split by a field extension of degree a power of 2, by reducing to the corresponding fact about quadratic forms. So  $\text{ind}(A)$  must be a power of 2, but it also divides  $n$ , and so  $\text{ind}(A) \leq 2^m$ , where  $2^m$  is the largest power of 2 dividing  $n$ . We conclude that

$$\begin{aligned} \text{ed}(\text{Spin}(n)) &\leq \text{ed}(\text{HSpin}(n)) + 2^m \\ &\leq 2^{(n-2)/2} - n(n-1)/2 + 2^m. \end{aligned}$$

This completes the proof of the upper bound in Theorem 2.1.

We now prove the corresponding lower bound for the 2-essential dimension of the spin group over a field  $k$  of characteristic 2. Since  $\text{ed}_2(\text{Spin}(n)) \leq \text{ed}(\text{Spin}(n))$ , this will imply that the 2-essential dimension and the essential dimension are both equal to the number given in Theorem 2.1.

Write  $O(2r)$  for the orthogonal group of the quadratic form

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r}$$

over  $k$ , and  $O(2r+1)$  for the orthogonal group of

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r} + x_{2r+1}^2.$$

Then we have an inclusion  $O(2r) \subset O(2r+1)$ . Note that  $O(2r)$  is smooth over  $k$ , with

$$O(2r)/O^+(2r) \cong \mathbf{Z}/2.$$

The group scheme  $O(2r+1)$  is not smooth over  $k$ , but it contains a smooth connected subgroup  $O^+(2r+1)$  with

$$O(2r+1) \cong O^+(2r+1) \times \mu_2.$$

It follows that  $O(2r)$  is contained in  $O^+(2r+1)$ . Using the subgroup  $\mathbf{Z}/2 \times \mu_2$  of  $O(2)$ , we have a  $k$ -subgroup scheme

$$K := (\mathbf{Z}/2 \times \mu_2)^r \subset O(2r) \subset O^+(2r+1).$$

Let  $G$  be the inverse image of  $K$  in the double cover  $\text{Spin}(2r+1)$  of  $O^+(2r+1)$ . Thus  $G$  is a central extension

$$1 \rightarrow \mu_2 \rightarrow G \rightarrow (\mathbf{Z}/2)^r \times (\mu_2)^r \rightarrow 1.$$

(Essentially the same ‘‘finite Heisenberg group scheme’’ appeared in the work of Mumford and Sekiguchi on abelian varieties [20, Appendix A].)

To describe the structure of  $G$  in more detail, think of  $K = (\mu_2)^r$  as the 2-torsion subgroup scheme of a fixed maximal torus  $T_{SO} \cong (G_m)^r$  in  $O^+(2r+1)$ , where  $G_m$  is the multiplicative group. The character group of  $T_{SO}$  is the free abelian group  $\mathbf{Z}\{x_1, \dots, x_r\}$ , and the Weyl group  $W = N(T_{SO})/T_{SO}$  of  $O^+(2r+1)$  is the semi-direct product  $S_r \ltimes (\mathbf{Z}/2)^r$ . Here  $S_r$  permutes the characters  $x_1, \dots, x_r$  of  $T_{SO}$ , and the subgroup  $E_r = (\mathbf{Z}/2)^r$  of  $W$ , with generators  $\epsilon_1, \dots, \epsilon_r$ , acts by:  $\epsilon_i$  changes the sign of  $x_i$  and fixes  $x_j$  for  $j \neq i$ . The character group of  $K = T_{SO}[2]$  is  $\mathbf{Z}/2\{x_1, \dots, x_r\}$ . The group  $E_r$  centralizes  $K$ , and the group

$$(\mathbf{Z}/2)^r \times (\mu_2)^r \subset O^+(2r+1)$$

above is  $E_r \times K$ .

Let  $L$  be the inverse image of  $K$  in  $\text{Spin}(2r+1)$ , which is contained in a maximal torus  $T$  of  $\text{Spin}(2r+1)$ , the inverse image of  $T_{SO}$ . The character group  $X^*(T)$  is

$$\mathbf{Z}\{x_1, \dots, x_r, A\}/(2A = x_1 + \dots + x_r).$$

Therefore, the character group  $X^*(L)$  is

$$\mathbf{Z}\{x_1, \dots, x_r, A\}/(2x_i = 0, 2A = x_1 + \dots + x_r).$$

(Thus  $X^*(L)$  is isomorphic to  $(\mathbf{Z}/4) \times (\mathbf{Z}/2)^{r-1}$ , and so  $L$  is isomorphic to  $\mu_4 \times (\mu_2)^{r-1}$ .) The Weyl group  $W$  of  $\text{Spin}(2r+1)$  is the same as that of  $O^+(2r+1)$ , namely  $S_r \ltimes E_r$ . In particular, the element  $\epsilon_i$  of  $E_r$  acts on  $X^*(T)$  by changing the sign of  $x_i$  and fixing  $x_j$  for  $j \neq i$ , and hence it sends  $A$  to  $A - x_i$ .

The subset  $S$  of  $X^*(L)$  corresponding to characters of  $L$  which are faithful on the center  $\mu_2$  of  $L$  is the complement of the subgroup  $X^*(K) = \mathbf{Z}/2\{x_1, \dots, x_r\}$ . Therefore,  $S$  has order  $2^r$ . The group  $E_r = (\mathbf{Z}/2)^r$  acts freely and transitively on  $S$ , since

$$\left( \prod_{i \in I} \epsilon_i \right)(A) = A - \sum_{i \in I} x_i$$

for any subset  $I$  of  $\{1, \dots, r\}$ .

The group  $G = E_r \cdot L$  is the central extension considered above. Now, let  $V$  be a representation of  $G$  over  $k$  on which the center  $\mu_2 \subset L$  acts faithfully by scalars. Then the restriction of  $V$  to  $L$  is fixed (up to isomorphism) by the action of  $E_r$  on  $X^*(L)$ . By the previous paragraph, the  $2^r$  1-dimensional representations of  $L$  that are nontrivial on the center  $\mu_2$  all occur with the same multiplicity in  $V$ . Therefore,  $V$  has dimension a multiple of  $2^r$ . This bound is optimal, since the spin representation  $W$  of  $\text{Spin}(2r+1)$  has dimension  $2^r$  over  $k$ , and the center  $\mu_2$  acts faithfully by scalars on  $W$ .

We use the following result of Merkurjev's [16, Theorem 5.2], [11, Remark 4.5]. (The first reference covers the case of the group scheme  $\mu_p$  in characteristic  $p$ , as needed here.)

**Theorem 2.2.** *Let  $k$  be a field and  $p$  be a prime number. Let  $1 \rightarrow \mu_p \rightarrow G \rightarrow Q \rightarrow 1$  be a central extension of affine group schemes over  $k$ . For a field extension  $K$  of  $k$ , let  $\partial_K: H^1(K, Q) \rightarrow H^2(K, \mu_p)$  be the boundary homomorphism in fppf cohomology. Then the maximal value of the index of  $\partial_K(E)$ , as  $K$  ranges over all field extensions of  $k$  and  $E$  ranges over all  $Q$ -torsors over  $K$ , is equal to the greatest common divisor of the dimensions of all representations of  $G$  on which  $\mu_p$  acts by its standard representation.*

As mentioned above, for a prime number  $p$  and a nonzero element  $E$  of  $H^2(K, \mu_p)$  over a field  $K$ , the essential dimension (or also the  $p$ -essential dimension) of the corresponding  $\mu_p$ -gerbe over  $K$  is equal to the index of  $E$ .

Finally, consider a central extension  $1 \rightarrow \mu_p \rightarrow G \rightarrow Q \rightarrow 1$  of finite group schemes over a field  $k$ . Generalizing an argument of Brosnan–Reichstein–Vistoli, Karpenko and Merkurjev showed that the  $p$ -essential dimension of  $G$  (and hence the essential dimension of  $G$ ) is at least the  $p$ -essential dimension of the  $\mu_p$ -gerbe over  $K$  associated to any  $Q$ -torsor over any field  $K/k$  [11, Theorem 4.2]. By the analysis above of representations of the finite subgroup scheme  $G$  of  $\mathrm{Spin}(2r+1)$  over a field  $k$  of characteristic 2, we find that  $\mathrm{ed}_2(G) \geq 2^r$ . For a closed subgroup scheme  $G$  of a group scheme  $L$  over a field  $k$  and any prime number  $p$ , we have

$$\mathrm{ed}_p(L) + \dim(L) \geq \mathrm{ed}_p(G) + \dim(G)$$

[17, Corollary 4.3] (which covers the case of fppf torsors for non-smooth group schemes, as needed here). Applying this to the subgroup scheme  $G$  of  $\mathrm{Spin}(2r)$ , we conclude that

$$\mathrm{ed}_2(\mathrm{Spin}(2r+1)) \geq 2^r - \dim(\mathrm{Spin}(2r+1)) = 2^r - r(2r+1).$$

Combining this with the upper bound discussed above, we have

$$\mathrm{ed}(\mathrm{Spin}(2r+1)) = \mathrm{ed}_2(\mathrm{Spin}(2r+1)) = 2^r - r(2r+1)$$

for  $r \geq 7$ .

The proof of the lower bound for  $\mathrm{ed}_2(\mathrm{Spin}(2r))$  when  $r$  is odd is similar. The intersection of the subgroup  $K = (\mu_2 \times \mathbf{Z}/2)^r \subset O(2r)$  with  $O^+(2r)$  is

$$K_1 \cong (\mu_2)^r \times (\mathbf{Z}/2)^{r-1},$$

where  $(\mathbf{Z}/2)^{r-1}$  denotes the kernel of the sum  $(\mathbf{Z}/2)^r \rightarrow \mathbf{Z}/2$ . As a result, the double cover  $\mathrm{Spin}(2r)$  contains a subgroup  $G_1$  which is a central extension

$$1 \rightarrow \mu_2 \rightarrow G_1 \rightarrow (\mathbf{Z}/2)^{r-1} \times (\mu_2)^r \rightarrow 1.$$

In this case, an argument analogous to the one for  $G$  shows that every representation of  $G_1$  on which the center  $\mu_2$  acts by its standard representation has dimension a multiple of  $2^{r-1}$  (rather than  $2^r$ ). The argument is otherwise identical to the argument for  $\mathrm{Spin}(2r+1)$ , and we find that

$$\mathrm{ed}_2(\mathrm{Spin}(2r)) \geq 2^{r-1} - r(2r-1).$$

For  $r$  odd at least 9, this agrees with the lower bound found earlier, which proves the theorem on  $\mathrm{Spin}(n)$  for  $n \equiv 0 \pmod{4}$ .

It remains to show that for  $n$  a multiple of 4, with  $2^m$  the largest power of 2 dividing  $n$ , we have

$$\mathrm{ed}_2(\mathrm{Spin}(n)) \geq 2^{(n-2)/2} + 2^m - n(n-1)/2.$$

The argument follows that of Merkurjev in characteristic not 2 [17, Theorem 4.9].

Namely, for  $n$  a multiple of 4, the center  $C$  of  $G := \text{Spin}(n)$  is isomorphic to  $\mu_2 \times \mu_2$ , and  $H := G/C$  is the group  $PGO^+(n)$ . An  $H$ -torsor over a field  $L$  over  $k$  is equivalent to a central simple algebra  $A$  of degree  $n$  over  $L$  with a quadratic pair  $(\sigma, f)$  and with trivialized discriminant, meaning an isomorphism from the center of the Clifford algebra  $C(A, \sigma, f)$  to  $L \times L$  [13, Section 29.F]. The image of the homomorphism from  $C^* \cong (\mathbb{Z}/2)^2$  to the Brauer group of  $L$  is equal to  $\{0, [A], [C^+], [C^-]\}$ , where  $C^+$  and  $C^-$  are the simple components of the Clifford algebra; each is a central simple algebra of degree  $2^{(n-2)/2}$  over  $L$ . By Merkurjev, there is a field  $L$  over  $k$  and an  $H$ -torsor  $E$  over  $L$  such that

$$\text{ind}(C^+) = \text{ind}(C^-) = 2^{(n-2)/2} \quad \text{and} \quad \text{ind}(A) = 2^m$$

[16, Section 4.4 and Theorem 5.2]. We use the following result [17, Example 3.7]:

**Lemma 2.3.** *Let  $L$  be a field,  $p$  a prime number, and  $r$  a natural number. Let  $C$  be the group scheme  $(\mu_p)^r$ , and let  $Y$  be a  $C$ -gerbe over  $L$ . Then the  $p$ -essential dimension of  $Y$ , and also the essential dimension of  $Y$ , is the minimum, over all bases  $u_1, \dots, u_r$  for  $C^*$ , of  $\sum_{i=1}^r \text{ind}(u_i(Y))$ .*

It follows that the 2-essential dimension of the  $(\mu_2)^2$ -gerbe  $E/G$  over  $L$  associated to the  $H$ -torsor  $E$  above is

$$\text{ed}_2(E/G) = \text{ind}(A) + \text{ind}(C^+) = 2^{(n-2)/2} + 2^m.$$

It follows that

$$\begin{aligned} \text{ed}(\text{Spin}(n)) &\geq \text{ed}_2(\text{Spin}(n)) \\ &\geq \text{ed}_2(E/G) - \dim(G/C) \\ &= 2^{(n-2)/2} + 2^m - n(n-1)/2. \end{aligned} \quad \square$$

### 3. Etale motivic cohomology

In this section, we summarize the properties of etale motivic cohomology of fields, the natural home of mod  $p$  cohomological invariants for group schemes over a field of characteristic  $p$ .

For a field  $k$  of characteristic  $p > 0$ , let  $H^{i,j}(k)$  be the etale motivic cohomology group  $H_{\text{et}}^i(k, \mathbb{Z}/p(j))$ , or equivalently

$$H_{\text{et}}^i(k, \mathbb{Z}/p(j)) \cong H_{\text{et}}^{i-j}(k, \Omega_{\log}^j),$$

where  $\Omega_{\log}^j$  is the subgroup of the group  $\Omega^j$  of differential forms on the separable closure  $k_s$  over  $\mathbf{F}_p$  spanned by products  $(da_1/a_1) \wedge \dots \wedge (da_j/a_j)$  with  $a_1, \dots, a_j \in k_s^*$  [9]. The group  $H^{i,j}(k)$  is zero except when  $i$  equals  $j$  or  $j+1$ ,

because  $k$  has  $p$ -cohomological dimension at most 1 [21, Section II.2.2]. The symbol  $\{a_1, \dots, a_{n-1}, b\}$  denotes the element of  $H^{n, n-1}(k)$  which is the product of the elements

$$a_i \in k^*/(k^*)^p \cong H^{1,1}(k) \quad \text{and} \quad b \in k/\{a^p - a : a \in k\} \cong H^{1,0}(k).$$

Also, for a field  $k$  of characteristic 2, let  $W(k)$  denote the Witt ring of symmetric bilinear forms over  $k$ , and let  $I_q(k)$  be the Witt group of nondegenerate quadratic forms over  $k$ . (By the conventions in Section 1,  $I_q(k)$  consists only of even-dimensional forms.) Then  $I_q(k)$  is a module over  $W(k)$  via tensor product [5, Lemma 8.16]. Let  $I$  be the kernel of the homomorphism  $W(k) \rightarrow \mathbf{Z}/2$ , and let

$$I_q^m(k) = I^{m-1} \cdot I_q(k),$$

following [5, p. 53]. To motivate the notation, observe that the class of an  $m$ -fold quadratic Pfister form  $\langle\!\langle a_1, \dots, a_{m-1}, b \rangle\!\rangle$  lies in  $I_q^m(k)$ . By definition, for  $a_1, \dots, a_{m-1}$  in  $k^*$  and  $b$  in  $k$ ,  $\langle\!\langle a_1, \dots, a_{m-1}, b \rangle\!\rangle$  is the quadratic form

$$\langle\!\langle a_1 \rangle\!\rangle_b \otimes \cdots \otimes \langle\!\langle a_{m-1} \rangle\!\rangle_b \otimes \langle\!\langle b \rangle\!\rangle$$

of dimension  $2^m$ , where  $\langle\!\langle a \rangle\!\rangle_b$  is the bilinear form  $\langle 1, a \rangle$  and  $\langle\!\langle b \rangle\!\rangle$  is the quadratic form

$$[1, b] = x^2 + xy + by^2.$$

In analogy with the Milnor conjecture, Kato proved the isomorphism

$$I_q^m(F)/I_q^{m+1} \cong H^{m, m-1}(F)$$

for every field  $F$  of characteristic 2 [5, Fact 16.2]. The isomorphism takes the quadratic Pfister form  $\langle\!\langle a_1, \dots, a_{m-1}, b \rangle\!\rangle$  to the symbol  $\{a_1, \dots, a_{m-1}, b\}$ . (For this paper, it would suffice to have Kato's homomorphism, without knowing that it is an isomorphism.)

A cohomological invariant gives a lower bound for the essential dimension, as follows. This is standard for mod  $l$  invariants with  $l \neq p = \text{char}(k)$  [17, Theorem 5.3], and we now give the analogous statement for mod  $p$  invariants of a  $k$ -group scheme  $G$ . Define a cohomological invariant  $f$  of  $G$  with values in  $H^{n, n-1}$  to be *nontrivial* if there is a field  $F$  containing an algebraic closure of  $k$  and a  $G$ -torsor  $u$  over  $F$  such that  $f(u)$  is not zero.

**Lemma 3.1.** *Let  $G$  be an affine group scheme of finite type over a field  $k$  of characteristic  $p > 0$ . If there is a nontrivial cohomological invariant for  $G$  with values in  $H^{n, n-1}$ , then  $\text{ed}(G) \geq \text{ed}_p(G) \geq n$ .*

*Proof.* Let  $f$  be the given cohomological invariant for  $G$ . It suffices to prove a lower bound on the essential dimension after enlarging  $k$ . So we can replace  $k$  by its algebraic closure. Then every field  $F$  of transcendence degree less than  $n$

over  $k$  has  $H^{n,n-1}(F) = 0$ , by Kato and Kuzumaki [12, Section 3, Corollary 2]. By assumption, there is a  $G$ -torsor  $u$  over a field  $E$  over  $k$  such that  $f(u)$  is not zero in  $H^{n,n-1}(E)$ . Thanks to the transfer maps on Galois cohomology (viewing  $H^{n,n-1}(E)$  as  $H^1(E, \Omega_{\log}^{n-1}(E_s))$ ), this element remains nonzero in  $H^{n,n-1}(E')$  for any finite extension  $E'/E$  of degree prime to  $p$ . Therefore, the  $G$ -torsor  $u$  extended up to  $E'$  cannot be defined over a subfield  $F$  of  $E'$  with transcendence degree less than  $n$  over  $k$ . So  $\text{ed}(G) \geq \text{ed}_p(G) \geq n$ .  $\square$

**Corollary 3.2.** *Let  $G$  be an affine group scheme of finite type over a field  $k$  of characteristic  $p > 0$ . Let  $f$  be a cohomological invariant for  $G$  with values in  $H^{n,n-1}$ . Suppose that for any field  $F$  over  $k$  and any  $a_1, \dots, a_{n-1}$  in  $F^*$  and  $a_n$  in  $F$ , there is a  $G$ -torsor  $u$  over  $F$  with*

$$f(u) = \{a_1, \dots, a_{n-1}, a_n\}$$

*in  $H^{n,n-1}(F)$ . Then  $\text{ed}(G) \geq \text{ed}_p(G) \geq n$ .*

*Proof.* Let  $\bar{k}$  be an algebraic closure of  $k$ , and let  $E$  be the rational function field  $\bar{k}(a_1, \dots, a_n)$ . By assumption, there is a  $G$ -torsor  $u$  over  $E$  such that

$$f(u) = \{a_1, \dots, a_{n-1}, a_n\}.$$

This symbol in  $H^{n,n-1}(E)$  is not zero, by Izhboldin's calculation of  $H^{n,n-1}$  of a rational function field [10, Theorem 4.5]. Thus  $f$  is nontrivial, in the sense above. By Lemma 3.1,  $\text{ed}(G) \geq \text{ed}_p(G) \geq n$ .  $\square$

#### 4. Low-dimensional spin groups

Rost and Garibaldi determined the essential dimension of the spin groups  $\text{Spin}(n)$  with  $n \leq 14$  in characteristic not 2 [6, Table 23B]. It should be possible to compute the essential dimension of low-dimensional spin groups in characteristic 2 as well. The following section carries this out for  $\text{Spin}(n)$  with  $n \leq 10$ . We find that in this range (as for  $n \geq 15$ ), the essential dimension of the spin group is the same in characteristic 2 as in characteristic not 2, unlike what happens for  $O(n)$  and  $SO(n)$ .

For  $n \leq 10$ , we give group-theoretic proofs which work almost the same way in any characteristic, despite the distinctive features of quadratic forms in characteristic 2.

**Theorem 4.1.** *For  $n \leq 10$ , the essential dimension, as well as the 2-essential dimension, of the split group  $\text{Spin}(n)$  over a field  $k$  of any characteristic is given by:*

$n$	$\text{ed}(\text{Spin}(n))$
$\leq 6$	0
7	4
8	5
9	5
10	4

*Proof.* As discussed above, it suffices to consider the case of a field  $k$  of characteristic 2. For  $2 \leq n \leq 6$ , every  $\text{Spin}(n)$ -torsor over a field is trivial, for example by the exceptional isomorphisms

$$\begin{aligned} \text{Spin}(2) &\cong G_m, & \text{Spin}(3) &\cong SL(2), & \text{Spin}(4) &\cong SL(2) \times SL(2), \\ \text{Spin}(5) &\cong Sp(4), & \text{and} & & \text{Spin}(6) &\cong SL(4). \end{aligned}$$

It follows that  $\text{ed}(\text{Spin}(n)) = 0$  for  $2 \leq n \leq 6$ .

We will use the following standard approach to bounding the essential dimension of a group.

**Lemma 4.2.** *Let  $G$  be an affine group scheme of finite type over a field  $k$ . Suppose that  $G$  acts on a  $k$ -scheme  $Y$  with a nonempty open orbit  $U$ . Suppose that for every  $G$ -torsor  $E$  over an infinite field  $F$  over  $k$ , the twisted form  $(E \times Y)/G$  of  $Y$  over  $F$  has a Zariski-dense set of  $F$ -points. Finally, suppose that  $U$  has a  $k$ -point  $x$ , and let  $N$  be the stabilizer  $k$ -group scheme of  $x$  in  $G$ . Then*

$$H^1(F, N) \rightarrow H^1(F, G)$$

*is surjective for every infinite field  $F$  over  $k$  (or for every field  $F$  over  $k$ , if  $G$  is smooth and connected). As a result,  $\text{ed}_k(G) \leq \text{ed}_k(N)$ .*

The proof is short, the same as that of [6, Theorem 9.3]. (Note that even if  $k$  is finite, we get the stated upper bound for the essential dimension of  $G$ : a  $G$ -torsor over a finite field  $F$  that contains  $k$  causes no problem, because  $F$  has transcendence degree 0 over  $k$ .) If  $G$  is smooth and connected, then  $H^1(F, G)$  is in fact trivial for every finite field  $F$  that contains  $k$ , by Lang [14]; that implies the statement in the theorem that  $H^1(F, N) \rightarrow H^1(F, G)$  is surjective for *every* field  $F$  over  $k$ .

The assumption about a Zariski-dense set of rational points holds, for example, if  $Y$  is a linear representation  $V$  of  $G$ , or if  $Y$  is the associated projective space  $P(V)$  to a representation, or (as we use later) a product  $P(V) \times P(W)$ .

We use Garibaldi and Guralnick's calculation of the stabilizer group scheme of a general  $k$ -point in the spin (for  $n$  odd) or a half-spin (for  $n$  even) representation  $W$

of the split group  $\text{Spin}(n)$ , listed in Table 1 here [7, Table 1]. Here  $\text{Spin}(n)$  has an open orbit on the projective space  $P(W)$  of lines in  $W$  if  $n \leq 12$  or  $n = 14$ , and an open orbit on  $W$  if  $2 \leq n \leq 6$  or  $n = 10$ . (To be precise, we will use that even if  $k$  is finite, there is a  $k$ -point in the open orbit for which the stabilizer  $k$ -group scheme is the *split* group listed in the table.)

$n$	$\text{chark} \neq 2$	$\text{chark} = 2$
6	$SL(3) \cdot (G_a)^3$	same
7	$G_2$	same
8	$\text{Spin}(7)$	same
9	$\text{Spin}(7)$	same
10	$\text{Spin}(7) \cdot (G_a)^8$	same
11	$SL(5)$	$\mathbb{Z}/2 \ltimes SL(5)$
12	$SL(6)$	$\mathbb{Z}/2 \ltimes SL(6)$
13	$SL(3) \times SL(3)$	$\mathbb{Z}/2 \ltimes (SL(3) \times SL(3))$
14	$G_2 \times G_2$	$\mathbb{Z}/2 \ltimes (G_2 \times G_2)$

Table 1. Generic stabilizer of spin (or half-spin) representation of  $\text{Spin}(n)$ .

We now begin to compute the essential dimension of the split group  $G = \text{Spin}(7)$  over a field  $k$  of characteristic 2. Let  $W$  be the 8-dimensional spin representation of  $G$ . Then  $G$  has an open orbit on the projective space  $P(W)$  of lines in  $W$ . By Table 1, there is a  $k$ -point  $x$  in  $W$  whose image in  $P(W)$  is in the open orbit such that the stabilizer of  $x$  in  $G$  is the split exceptional group  $G_2$ . Since  $G$  preserves a quadratic form on  $W$ , the stabilizer  $H$  of the corresponding  $k$ -point in  $P(W)$  is at most  $G_2 \times \mu_2$ . In fact,  $H$  is equal to  $G_2 \times \mu_2$ , because the center  $\mu_2$  of  $G$  acts trivially on  $P(W)$ .

By Lemma 4.2, the inclusion  $G_2 \times \mu_2 \hookrightarrow G$  induces a surjection

$$H^1(F, G_2 \times \mu_2) \rightarrow H^1(F, G)$$

for every field  $F$  over  $k$ . Over any field  $F$ ,  $G_2$ -torsors up to isomorphism can be identified with 3-fold quadratic Pfister forms  $\langle\!\langle a_1, a_2, b \rangle\!\rangle$  (with  $a_1, a_2 \in F^*$  and  $b \in F$ ), and so  $G_2$  has essential dimension 3 [21, Théorème 11]. Since  $\mu_2$  has essential dimension 1, the surjectivity above implies that  $G = \text{Spin}(7)$  has essential dimension at most 4.

Next, a  $G$ -torsor determines two quadratic forms of dimension 8. Besides the obvious homomorphism  $\chi_1: G \hookrightarrow \text{Spin}(8) \rightarrow SO(8)$  (which is trivial on the center  $\mu_2$  of  $G$ ), we have the spin representation  $\chi_2: G \rightarrow SO(8)$ , on which  $\mu_2$  acts faithfully by scalars. Thus a  $G$ -torsor  $u$  over a field  $F$  over  $k$  determines two quadratic forms of dimension 8 over  $F$ , which we call  $q_1$  and  $q_2$ .

To describe these quadratic forms in more detail, use that every  $G$ -torsor comes from a torsor for  $G_2 \times \mu_2$ . The two homomorphisms  $G_2 \hookrightarrow G \rightarrow SO(8)$  (via  $\chi_1$  and  $\chi_2$ ) are both conjugate to the standard inclusion. Also,  $\chi_1$  is trivial on the  $\mu_2$  factor, while  $\chi_2$  acts faithfully by scalars on the  $\mu_2$  factor. It follows that  $q_1$  is a quadratic Pfister form,  $\langle\!\langle a, b, c \rangle\!\rangle$  (the form associated to a  $G_2$ -torsor), while  $q_2$  is a scalar multiple of that form,  $d \langle\!\langle a, b, c \rangle\!\rangle$ .

Therefore, a  $G$ -torsor  $u$  canonically determines a 4-fold quadratic Pfister form,

$$q_1 + q_2 = \langle\!\langle d, a, b, c \rangle\!\rangle.$$

Define  $f_4(u)$  to be the associated element of  $H^{4,3}(F)$ ,

$$f_4(u) = \{d, a, b, c\}.$$

By construction, this is well-defined and an invariant of  $u$ . By considering the subgroup  $G_2 \times \mu_2 \subset \text{Spin}(7)$ , where there is a  $G_2 \times \mu_2$ -torsor associated to any elements  $a, b, d$  in  $F^*$  and  $c$  in  $F$ , we see that  $a, b, c, d$  can be chosen arbitrarily. By Corollary 3.2,  $G = \text{Spin}(7)$  has 2-essential dimension at least 4, and hence essential dimension at least 4.

The opposite inequality was proved above, and so  $\text{Spin}(7)$  has essential dimension equal to 4. Since the lower bound is proved by constructing a mod 2 cohomological invariant, this argument also shows that  $\text{Spin}(7)$  has 2-essential dimension equal to 4. For the same reason, the computations of essential dimension below (for  $\text{Spin}(n)$  with  $8 \leq n \leq 10$ ) also give the 2-essential dimension.

Next, we turn to  $\text{Spin}(8)$ . At first, let  $G = \text{Spin}(2r)$  for a positive integer  $r$  over a field  $k$  of characteristic 2. Let  $V$  be the standard  $2r$ -dimensional representation of  $G$ . Then  $G$  has an open orbit in the projective space  $P(V)$  of lines in  $V$ . The stabilizer  $k$ -group scheme  $H$  of a general  $k$ -point in  $P(V)$  is conjugate to  $\text{Spin}(2r-1) \cdot Z$ , where  $Z$  is the center of  $\text{Spin}(2r)$ , with  $\text{Spin}(2r-1) \cap Z = \mu_2$ . (In more detail, a general line in  $V$  is spanned by a vector  $x$  with  $q(x) \neq 0$ , where  $q$  is the quadratic form on  $V$ . Then the stabilizer of  $x$  in  $SO(V)$  is isomorphic to  $SO(S)$ , where  $S := x^\perp$  is a hyperplane in  $V$  on which  $q$  restricts to a nonsingular quadratic form of dimension  $2r-1$ , with  $S^\perp$  equal to the line  $k \cdot x \subset S$ .) Here

$$Z \cong \begin{cases} \mu_2 \times \mu_2 & \text{if } r \text{ is even,} \\ \mu_4 & \text{if } r \text{ is odd.} \end{cases}$$

In particular, if  $r$  is even, then  $H \cong \text{Spin}(2r-1) \times \mu_2$ . Thus, for  $r$  even, the inclusion  $\text{Spin}(2r-1) \times \mu_2 \hookrightarrow G$  induces a surjection

$$H^1(F, \text{Spin}(2r-1) \times \mu_2) \rightarrow H^1(F, G)$$

for every field  $F$  over  $k$ , by Lemma 4.2.

It follows that, for  $r$  even, the essential dimension of  $\text{Spin}(2r)$  is at most 1 plus the essential dimension of  $\text{Spin}(2r - 1)$ . Since  $\text{Spin}(7)$  has essential dimension 4,  $G = \text{Spin}(8)$  has essential dimension at most 5.

Before proving that equality holds, let us analyze  $G$ -torsors in more detail. We know that  $H^1(F, \text{Spin}(7) \times \mu_2) \rightarrow H^1(F, G)$  is onto, for all fields  $F$  over  $k$ . Also, we showed earlier that  $H^1(F, G_2 \times \mu_2) \rightarrow H^1(F, \text{Spin}(7))$  is surjective. Therefore,

$$H^1(F, G_2 \times \mu_2 \times \mu_2) \rightarrow H^1(F, G)$$

is surjective for all fields  $F$  over  $k$ , where  $Z = \mu_2 \times \mu_2$  is the center of  $G$ . As discussed earlier,  $G_2$ -torsors up to isomorphism can be identified with 3-fold quadratic Pfister forms. It follows that every  $G$ -torsor is associated to some 3-fold quadratic Pfister form  $\langle\!\langle a, b, c \rangle\!\rangle$  and some elements  $d, e$  in  $F^*$ , which yield elements of  $H^1(F, \mu_2) = F^*/(F^*)^2$ .

Next, observe that a  $G$ -torsor determines several quadratic forms. Besides the obvious double covering  $\chi_1: G \rightarrow SO(8)$ , the two half-spin representations of  $G$  give two other homomorphisms  $\chi_2, \chi_3: G \rightarrow SO(8)$ . (These three homomorphisms can be viewed as the quotients of  $G$  by the three  $k$ -subgroup schemes of order 2 in  $Z$ . They are permuted by the group  $S_3$  of “triality” automorphisms of  $G$ .) Thus a  $G$ -torsor  $u$  over a field  $F$  over  $k$  determines three quadratic forms of dimension 8, which we call  $q_1, q_2, q_3$ .

To describe how these three quadratic forms are related, use that every  $G$ -torsor comes from a torsor for  $G_2 \times \mu_2 \times \mu_2$ . The three homomorphisms  $G_2 \rightarrow G \rightarrow SO(8)$  (via  $\chi_1, \chi_2$ , and  $\chi_3$ ) are all conjugate to the standard inclusion, whereas the three homomorphisms send  $\mu_2 \times \mu_2$  to the center  $\mu_2 \subset SO(8)$  by the three possible surjections. It follows that the three quadratic forms can be written as  $q_1 = d \langle\!\langle a, b, c \rangle\!\rangle, q_2 = e \langle\!\langle a, b, c \rangle\!\rangle$ , and  $q_3 = de \langle\!\langle a, b, c \rangle\!\rangle$ .

Note that a scalar multiple  $q = d \langle\!\langle a_1, \dots, a_{m-1}, b \rangle\!\rangle$  of a quadratic Pfister form (as a quadratic form up to isomorphism) uniquely determines the associated quadratic Pfister form  $q_0 = \langle\!\langle a_1, \dots, a_{m-1}, b \rangle\!\rangle$  up to isomorphism. (Proof: it suffices to show that if  $q$  and  $r$  are  $m$ -fold quadratic Pfister forms over  $F$  with  $aq \cong r$  for some  $a$  in  $F^*$ , then  $q \cong r$ . Since  $r$  takes value 1, so does  $aq$ , and so  $q$  takes value  $a^{-1}$ . But then  $a^{-1}q \cong q$  by the multiplicativity of quadratic Pfister forms [5, Corollary 9.9]. Therefore,  $r \cong aq \cong q$ .)

We now define an invariant for  $G = \text{Spin}(8)$  over  $k$  with values in  $H^{5,4}$ . Given a  $G$ -torsor  $u$  over a field  $F$  over  $k$ , consider the three associated quadratic forms  $q_1, q_2, q_3$  as above. By the previous paragraph,  $q_1 = d \langle\!\langle a, b, c \rangle\!\rangle$  determines the quadratic Pfister form  $q_0 = \langle\!\langle a, b, c \rangle\!\rangle$ . So  $u$  determines the 5-fold quadratic Pfister form

$$q_0 + q_1 + q_2 + q_3 = \langle\!\langle d, e, a, b, c \rangle\!\rangle.$$

The associated class

$$f_5(u) = \{d, e, a, b, c\} \in H^{5,4}(F)$$

is therefore an invariant of  $u$ .

By considering the subgroup  $G_2 \times Z \subset G = \text{Spin}(8)$ , where  $Z = \mu_2 \times \mu_2$ , there is a  $G_2 \times Z$ -torsor associated to any elements  $a, b, d, e$  in  $F^*$  and  $c$  in  $F$ , and  $f_5$  of the associated  $G$ -torsor is  $\{d, e, a, b, c\}$  in  $H^{5,4}(F)$ . By Corollary 3.2,  $G$  has essential dimension at least 5. Since the opposite inequality was proved above,  $G = \text{Spin}(8)$  has essential dimension over  $k$  equal to 5.

Next, let  $G = \text{Spin}(9)$  over a field  $k$  of characteristic 2. Let  $W$  be the spin representation of  $G$ , of dimension 16, corresponding to a homomorphism  $G \rightarrow SO(16)$ . (A reference for the fact that this self-dual representation is orthogonal in characteristic 2, as in other characteristics, is [8, Theorem 9.2.2].) By Table 1,  $G$  has an open orbit on the space  $P(W)$  of lines in  $W$ , and the stabilizer in  $G$  of a general  $k$ -point in  $W$  is conjugate to  $\text{Spin}(7)$ . (This is not the standard inclusion of  $\text{Spin}(7)$  in  $\text{Spin}(9)$ , but rather a lift of the spin representation  $\chi_2: \text{Spin}(7) \rightarrow SO(8)$  to  $\text{Spin}(8)$  followed by the standard inclusion  $\text{Spin}(8) \hookrightarrow \text{Spin}(9)$ . In particular, the image of  $\text{Spin}(7)$  does not contain the center  $\mu_2$  of  $G = \text{Spin}(9)$ .) Since  $G$  preserves a quadratic form on  $W$ , it follows that the stabilizer in  $G$  of a general  $k$ -point in  $P(W)$  is conjugate to  $\text{Spin}(7) \times \mu_2$ , where  $\mu_2$  is the center of  $\text{Spin}(9)$  (which acts faithfully by scalars on  $W$ ). Therefore, by Lemma 4.2, the inclusion of  $\text{Spin}(7) \times \mu_2$  in  $G = \text{Spin}(9)$  induces a surjection

$$H^1(F, \text{Spin}(7) \times \mu_2) \rightarrow H^1(F, G)$$

for every field  $F$  over  $k$ .

Since  $\text{Spin}(7)$  has essential dimension 4 over  $k$  as shown above,  $G = \text{Spin}(9)$  has essential dimension at most  $4 + 1 = 5$ .

Next, a  $G$ -torsor determines several quadratic forms. Besides the obvious homomorphism  $R: G \hookrightarrow \text{Spin}(10) \rightarrow SO(10)$ , we have the spin representation  $S: G \rightarrow SO(16)$ . Thus a  $G$ -torsor over a field  $F$  over  $k$  determines a quadratic form  $r$  of dimension 10 and a quadratic form  $s$  of dimension 16.

To describe how these forms are related, use that every  $G$ -torsor comes from a torsor for the subgroup  $\text{Spin}(7) \times \mu_2$  described above. The restriction of  $R$  to the given subgroup  $\text{Spin}(7)$  is the composition of the spin representation  $\chi_2: \text{Spin}(7) \rightarrow SO(8)$  with the obvious inclusion  $SO(8) \hookrightarrow SO(10)$ . The restriction of  $S$  to the given subgroup  $\text{Spin}(7)$  is the direct sum of the standard representation  $\chi_1: \text{Spin}(7) \rightarrow SO(8)$  and the spin representation  $\chi_2: \text{Spin}(7) \rightarrow SO(8)$ . Finally,  $R$  is trivial on the second factor  $\mu_2$  (the center of  $G$ ), whereas  $S$  acts faithfully by scalars on  $S$ .

Now, let  $(u_1, e)$  be a  $\text{Spin}(7) \times \mu_2$ -torsor over  $k$ , where  $u_1$  is a  $\text{Spin}(7)$ -torsor and  $e$  is in  $H^1(F, \mu_2) = F^*/(F^*)^2$ , which we lift to an element  $e$  of  $F^*$ . By the earlier analysis of the quadratic forms associated to a  $\text{Spin}(7)$ -torsor, the quadratic form associated to  $u_1$  via the standard representation  $\chi_1: \text{Spin}(7) \rightarrow SO(8)$  is a 3-fold quadratic Pfister form  $\langle\langle a, b, c \rangle\rangle$ , while the quadratic form associated to  $u_1$  via the spin representation  $\chi_2: \text{Spin}(7) \rightarrow SO(8)$  is a multiple of the same form,  $d \langle\langle a, b, c \rangle\rangle$ .

By the analysis of representations two paragraphs back, it follows that the quadratic form associated to  $(u_1, e)$  via the representation  $R: G \rightarrow SO(10)$  is  $r = H + d\langle\langle a, b, c \rangle\rangle$ , where  $H$  is the hyperbolic plane. Also, the quadratic form associated to  $(u_1, e)$  via the representation  $S: G \rightarrow SO(16)$  is  $s = e\langle\langle a, b, c \rangle\rangle + de\langle\langle a, b, c \rangle\rangle$ .

Next,  $r$  determines the quadratic form  $r_0 = d\langle\langle a, b, c \rangle\rangle$  by Witt cancellation [5, Theorem 8.4], and that in turn determines the quadratic Pfister form  $q_0 = \langle\langle a, b, c \rangle\rangle$  as shown above. Therefore, a  $G$ -torsor  $u$  determines the 5-fold quadratic Pfister form

$$q_0 + r_0 + s = \langle\langle d, e, a, b, c \rangle\rangle$$

up to isomorphism.

Therefore, defining

$$f_5(u) = \{d, e, a, b, c\}$$

in  $H^{5,4}(F)$  yields an invariant of  $u$ . By our earlier description of  $\text{Spin}(7)$ -torsors, we can take  $a, b, d, e$  to be any elements of  $F^*$  and  $c$  any element of  $F$ . By Corollary 3.2,  $G$  has essential dimension at least 5. Since the opposite inequality was proved earlier,  $G = \text{Spin}(9)$  over  $k$  has essential dimension equal to 5.

Finally, let  $G = \text{Spin}(10)$  over a field  $k$  of characteristic 2. Let  $V$  be the 10-dimensional standard representation of  $G$ , corresponding to the double covering  $G \rightarrow SO(10)$ , and let  $W$  be one of the 16-dimensional half-spin representations of  $G$ , corresponding to a homomorphism  $G \rightarrow SL(16)$ . (The other half-spin representation of  $G$  is the dual  $W^*$ .)

As discussed above for any group  $\text{Spin}(2r)$ ,  $G = \text{Spin}(10)$  has an open orbit on  $P(V)$ , with generic stabilizer  $\text{Spin}(9) \cdot \mu_4$ . (Here  $\mu_4$  is the center of  $G$ , which contains the center  $\mu_2$  of  $\text{Spin}(9)$ .) Consider the action of  $G$  on  $P(V) \times P(W) \cong \mathbf{P}^9 \times \mathbf{P}^{15}$ . As discussed above,  $\text{Spin}(9)$  (and hence  $\text{Spin}(9) \cdot \mu_4$ ) has an open orbit on  $P(W)$ . As a result,  $G$  has an open orbit on  $P(V) \times P(W)$ . Moreover, the generic stabilizer of  $\text{Spin}(9)$  on  $P(W)$  is  $\text{Spin}(7) \times \mu_2$ , where the inclusion  $\text{Spin}(7) \hookrightarrow \text{Spin}(9)$  is the composition of the spin representation  $\text{Spin}(7) \hookrightarrow \text{Spin}(8)$  with the standard inclusion into  $\text{Spin}(9)$ ; in particular, the image does not contain the center  $\mu_2$  of  $\text{Spin}(9)$ . Therefore, the generic stabilizer of  $\text{Spin}(9) \cdot \mu_4 \subset \text{Spin}(10)$  on  $P(W)$  is  $\text{Spin}(7) \times \mu_4$ . We conclude that  $G$  has an open orbit on  $P(V) \times P(W)$ , with generic stabilizer  $\text{Spin}(7) \times \mu_4$ . It follows that

$$H^1(F, \text{Spin}(7) \times \mu_4) \rightarrow H^1(F, G)$$

is surjective for every field  $F$  over  $k$ , by Lemma 4.2.

The image  $H_2$  of the subgroup  $H = \text{Spin}(7) \times \mu_4 \subset G$  in  $SO(10)$  is  $\text{Spin}(7) \times \mu_2$ , where  $\text{Spin}(7)$  is contained in  $SO(8)$  (and contains the center  $\mu_2$  of  $SO(8)$ ) and  $\mu_2$  is the center of  $SO(10)$ . In terms of the subgroup  $SO(8) \times SO(2)$  of  $SO(10)$ , we can also describe  $H_2$  as  $\text{Spin}(7) \times \mu_2$ , where  $\text{Spin}(7)$  is contained in  $SO(8)$  and  $\mu_2$  is contained in  $SO(2)$ . Thus  $H_2$  is contained in  $\text{Spin}(7) \times SO(2)$ . Therefore,  $H$  is

contained in  $\mathrm{Spin}(7) \times G_m \subset G = \mathrm{Spin}(10)$ , where  $G_m$  is the inverse image in  $G$  of  $SO(2) \subset SO(10)$ . It follows that

$$H^1(F, \mathrm{Spin}(7) \times G_m) \rightarrow H^1(F, G)$$

is surjective for every field  $F$  over  $k$ . Since every  $G_m$ -torsor over a field is trivial,

$$H^1(F, \mathrm{Spin}(7)) \rightarrow H^1(F, G)$$

is surjective for every field  $F$  over  $k$ .

Here  $\mathrm{Spin}(7)$  maps into  $\mathrm{Spin}(8)$  by the spin representation, and then

$$\mathrm{Spin}(8) \hookrightarrow G = \mathrm{Spin}(10)$$

by the standard inclusion. By the description above of the 8-dimensional quadratic form associated to a  $\mathrm{Spin}(7)$ -torsor by the spin representation, it follows that the quadratic form associated to a  $G$ -torsor is of the form  $H + d \langle\!\langle a, b, c \rangle\!\rangle$ .

Every 10-dimensional quadratic form in  $I_q^3$  over a field is associated to some  $G$ -torsor. So we have given another proof that every 10-dimensional quadratic form in  $I_q^3$  is isotropic. This was proved in characteristic not 2 by Pfister, and it was extended to characteristic 2 by Baeza and Tits, independently [2, pp. 129–130], [22, Theorem 4.4.1(ii)].

Since  $\mathrm{Spin}(7)$  has essential dimension 4, the surjectivity above implies that  $G = \mathrm{Spin}(10)$  has essential dimension at most 4. To prove equality, we define an invariant for  $G$  with values in  $H^{4,3}$  by the same argument used for  $\mathrm{Spin}(7)$ . Namely, a  $G$ -torsor  $u$  over a field  $F$  over  $k$  determines a 4-fold quadratic Pfister form

$$\langle\!\langle d, a, b, c \rangle\!\rangle$$

up to isomorphism, and hence the element

$$f_4(u) = \{d, a, b, c\}$$

in  $H^{4,3}(F)$ . By Corollary 3.2, this completes the proof that  $G = \mathrm{Spin}(10)$  over  $k$  has essential dimension equal to 4. As in the previous cases, since the lower bound is proved using a mod 2 cohomological invariant,  $G$  also has 2-essential dimension equal to 4.  $\square$

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