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Commentarii Mathematici Helvetici

Lagrangian isotopies and symplectic function theory

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Abstract. We study two related invariants of Lagrangian submanifolds in symplectic manifolds. For a Lagrangian torus these invariants are functions on the first cohomology of the torus.

The first invariant is of topological nature and is related to the study of Lagrangian isotopies with a given Lagrangian flux. More specifically, it measures the length of straight paths in the first cohomology that can be realized as the Lagrangian flux of a Lagrangian isotopy.

The second invariant is of analytical nature and comes from symplectic function theory. It is defined for Lagrangian submanifolds admitting fibrations over a circle and has a dynamical interpretation.

We partially compute these invariants for certain Lagrangian tori.

Mathematics Subject Classification (2010). 53D12.

Keywords. Symplectic manifold, Lagrangian submanifold, Lagrangian isotopy, Poisson bracket, symplectic function theory.

Contents

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1. Introduction

In this paper we study two related invariants of Lagrangian submanifolds which are invariant under symplectomorphisms of the ambient symplectic manifolds.

Let (M, ω) be a symplectic manifold (possibly with boundary) and $L \subset (M, \omega)$ a closed Lagrangian submanifold. Let $\iota: L \to M$ denote here and further on the natural inclusion of L in M.

The first invariant of L comes from the consideration of Lagrangian isotopies of L with a given Lagrangian flux path in $H^1(L; \mathbb{R})$. Namely, recall that if $\psi := \{\psi_t: L \to M\}_{0 \le t \le T}$, $\psi_0 = \iota$, is a Lagrangian isotopy, one can associate to it a *Lagrangian flux path*

$$\{Flux(\psi)_t\}_{0 \le t \le T} \subset H^1(L; \mathbb{R}), \quad Flux(\psi)_0 = 0,$$

as follows: given a closed curve $C \subset L$ and $t \in [0, T]$, consider the trace of C under the Lagrangian isotopy $\{\psi_\tau: L \to M\}_{0 \le \tau \le t}$ and integrate ω over the resulting surface. The resulting (real) number depends only on the homology class of C in $H_1(L)$ (see [36]). The numbers obtained for all C in this way are the periods of a uniquely defined *Lagrangian flux* class $Flux(\psi)_t \in H^1(L; \mathbb{R})$.

The notion of Lagrangian flux immediately raises the following question.

Question 1.1. *Which paths in $H^1(L; \mathbb{R})$ based at 0 have the form $\{Flux(\psi)_t\}$ for some Lagrangian isotopy ψ of L?*

In general, this seems to be a very difficult question. In this paper we investigate its weaker version:

Question 1.2. *Which* straight *paths in $H^1(L; \mathbb{R})$ based at 0 have the form $\{Flux(\psi)_t\}$ for some Lagrangian isotopy ψ of L?*

The *deformation invariant* of L is a function $def_L\colon H^1(L;\mathbb{R}) \to (0,+\infty]$ defined as follows. Given $\alpha \in H^1(L;\mathbb{R})$, set

$$def_L(\alpha) := \sup T,$$

where we take the supremum over all $T \in \mathbb{R}_{\geq 0}$ for which there exists a Lagrangian isotopy $\psi = \{\psi_t\colon L \to M\}_{0\leq t\leq T}$, $\psi_0 = \iota$, of L such that

$$Flux(\psi)_t = -t\alpha \quad \text{for all } 0 \leq t \leq T, \tag{1.1}$$

or, in other words, the path $\{-t\alpha\}_{0\leq t\leq T}$ in $H^1(L;\mathbb{R})$ is the path $\{Flux(\psi)_t\}$ for some Lagrangian isotopy ψ of L. It is easy to show that $def_L(\alpha)$ is always non-zero — thus, $def_L(\alpha)$ takes values in $(0,+\infty]$.

In case $(M,\omega = d\lambda)$ is an exact symplectic manifold the invariant def_L can be related to the notion of a symplectic shape studied in [11,35]. The symplectic shape of $(M,d\lambda)$ associated to L and a homomorphism $h\colon H^1(M;\mathbb{R}) \to H^1(L;\mathbb{R})$ is the subset of $H^1(L;\mathbb{R})$ formed by the Liouville classes $[e^*\lambda] \in H^1(L;\mathbb{R})$ for all possible Lagrangian embeddings $e\colon L \to (M,d\lambda)$ such that

$$h = e^*\colon H^1(M;\mathbb{R}) \to H^1(L;\mathbb{R}).$$

Assume $M = \mathbb{T}^n \times U \subset \mathbb{T}^n \times \mathbb{R}^n = T^*\mathbb{T}^n$, where $U \subset \mathbb{R}^n$ is a connected open set, and λ is the standard Liouville form on $\mathbb{T}^n \times \mathbb{R}^n$. Let $\mathbf{x} \in U$ and let $L := \mathbb{T}^n \times \{\mathbf{x}\} \subset \mathbb{T}^n \times U$ be the corresponding Lagrangian submanifold. Let

$$h\colon H^1(\mathbb{T}^n \times U;\mathbb{R}) \to H^1(\mathbb{T}^n \times \{\mathbf{x}\};\mathbb{R})$$

be induced by the embedding $\mathbb{T}^n \times \{\mathbf{x}\} \hookrightarrow \mathbb{T}^n \times U$. Then the Benci–Sikorav theorem [35] (cf. [11]) says that the shape of $(\mathbb{T}^n \times U, d\lambda)$ associated to L and h is $U \subset \mathbb{R}^n \cong H^1(\mathbb{T}^n \times \{\mathbf{x}\};\mathbb{R})$. Rephrasing this result in terms of the deformation invariant gives us the following theorem.

Theorem 1.3. *With U and $(M = \mathbb{T}^n \times U, d\lambda)$, and $L = \mathbb{T}^n \times \{\mathbf{x}\}$ as above, for all $\alpha \in H^1(L;\mathbb{R})$*

$$def_L(\alpha) = \sup\{\, t > 0 \mid \mathbf{x} - t\alpha \in U \,\}.$$

To prove Theorem 1.3 note that in this setting $Flux(\psi)_t$ can be represented as the difference of the Liouville classes of $\psi_t(L)$ and L:

$$Flux(\psi)_t = \psi_t^*\lambda - \psi_0^*\lambda.$$

Then the inequality $def_L(\alpha) \geq \sup\{\, t > 0 \mid \mathbf{x} - t\alpha \in U \,\}$ follows from the existence of an obvious Lagrangian isotopy $\psi_t(\mathbb{T}^n \times \{\mathbf{x}\}) = \mathbb{T}^n \times \{\mathbf{x} - t\alpha\}$, while the opposite inequality follows directly from the Benci–Sikorav theorem.

The function $1/def_L$ is obviously non-negatively homogeneous. In the case when U is star-shaped with respect to $\mathbf{x}$, it is, by Theorem 1.3, the Minkowski functional associated to U — which in turn completely determines U.

The study of def_L can also be viewed as a relative analogue of a deformation problem for symplectic forms on closed symplectic manifolds considered in [32]: how far can one deform a symplectic form ω within a family of symplectic forms such that the cohomology class of the deformed form changes along a straight ray originating at $[\omega]$ and such that its restriction to a given ω-symplectic submanifold remains symplectic? To see how a relative version of this question is related to def_L we denote by $[\omega]_L \in H^2(M, L; \mathbb{R})$ and $[\omega]_t^{\psi} \in H^2(M, \psi_t(L); \mathbb{R})$ the relative symplectic area cohomology classes of, respectively, L and $\psi_t(L)$. The isotopy ψ defines a canonical isomorphism

$$H^2(M, L; \mathbb{R}) \cong H^2(M, \psi_t(L); \mathbb{R})$$

and thus $[\omega]_t^{\psi}$ can be viewed as an element of $H^2(M, L; \mathbb{R})$. Let $\partial: H^1(L; \mathbb{R}) \to H^2(M, L; \mathbb{R})$ be the connecting homomorphism. Then

$$[\omega]_t^{\psi} = [\omega]_L + \partial\, Flux(\psi)_t$$

and condition (1.1) becomes

$$[\omega]_t^{\psi} = [\omega]_L - t\,\partial\alpha \in H^2(M, L; \mathbb{R}), \quad 0 \le t \le T. \tag{1.2}$$

Therefore, as long as $\partial\alpha \neq 0$, the number $def_L(\alpha)$ measures how far one can deform $[\omega]_L$ in a Lagrangian isotopy ψ satisfying (1.2).

This viewpoint enables us to study def_L using methods of "hard" symplectic topology. Namely, the existence of pseudo-holomorphic curves with boundaries on $\psi_t(L)$ for $0 \le t \le T$ may yield constraints on the time-length T of the deformation, since $[\omega]_t^{\psi}$ evaluates positively on such curves. All the upper bounds on $def_L(\alpha)$ known to us and appearing further in this paper are obtained in this way.

On the other hand, as we explain below, lower bounds on def_L can be obtained by "soft" constructions. These bounds come from the study of the second invariant of L, called the *Poisson-bracket invariant of* L. It is defined only for L admitting a fibration over $\mathbb{S}^1$: it is a function on the set of isotopy classes of smooth fibrations of L over $\mathbb{S}^1$. In the case when L is a Lagrangian torus with a choice of an isotopy class of its smooth parametrizations $\mathbb{T}^n \to L$, we reduce this invariant to a function

$$bp_L: H^1(L) \to (0, +\infty].$$

Postponing the precise definition of $bp_L: H^1(L) \to (0, +\infty]$ until Section 3 we give a short and informal definition here.

Namely, assume L is a Lagrangian torus equipped with a parametrization $\mathbb{T}^n \to L$ and $a \in H^1(L)$, $a \neq 0$ (for $a = 0$ we set $bp_L(0) := +\infty$). Consider a fibration $f: L \to \mathbb{S}^1$ such that a is the pull-back under f of the standard generator of $H^1(\mathbb{S}^1)$ (the parametrization of L is used in the construction of f; see Section 3.3). Cut $\mathbb{S}^1$ into four consecutive arcs, denote their preimages under f by X_0, Y_1, X_1, Y_0 (so that $X_0 \cap X_1 = Y_0 \cap Y_1 = \emptyset$, $X_0 \cup Y_1 \cup X_1 \cup X_0 = L$) and set

$$bp_L(a) := 1/pb_4^+(X_0, X_1, Y_0, Y_1).$$

Here pb_4^+ is the Poisson-bracket invariant of a quadruple of sets defined in [14] — it is a refined version of the pb_4-invariant introduced in [4] and it admits a dynamical interpretation in terms of the existence of connecting trajectories of sufficiently small time-length between X_0 and X_1 for certain Hamiltonian flows; see Section 3 for details. The relation between bp_L and def_L is given by the following inequality (which will be proved in a stronger form in Theorem 3.5).

Theorem 1.4. *$bp_L \leq def_L$ on $H^1(L)$.*

This relation, albeit in a different language, was already exploited in [14], where upper bounds on def_L were obtained by symplectic rigidity methods in a setting where the Lagrangian isotopy class of a Lagrangian torus L does not contain (weakly) exact Lagrangian tori. This was then used to prove the existence of connecting trajectories of Hamiltonian flows. In this paper we get upper bounds on def_L in new cases by using several strong symplectic rigidity results, including some recent ones. Namely, Theorem 2.6 (the original idea of whose proof belongs to E. Opshtein) and Theorem 2.11 rely on Gromov's famous work [19], while Theorem 2.16 and Theorem 2.19 rely on the recent powerful rigidity results of, respectively, K. Cieliebak–K. Mohnke [7] and G. Dimitroglou Rizell–E. Goodman–A. Ivrii [10]. At the same time we use new soft dynamical constructions to get lower bounds on bp_L, and hence on def_L, in many new settings.

In Section 2 we partially compute the functions bp_L and def_L for several classes of Lagrangian tori. Section 2.7 is then devoted to a discussion of the results and further directions. In Section 3 we discuss in detail the definition of bp_L. The sections following Section 3 contain the proofs of the results from Section 2.

2. The main results

We now present results about def_L and bp_L for several examples of Lagrangian tori in symplectic manifolds. (For general properties of def_L and bp_L see Section 3.4).

2.1. Symplectic manifolds without weakly exact Lagrangian submanifolds. Recall that a Lagrangian submanifold $L \subset (M, \omega)$ is called *weakly exact*, if $[\omega]_L \equiv 0$.

Theorem 2.1. *Let L be a closed Lagrangian submanifold of a symplectic manifold (M, ω) (possibly with boundary). Assume that (M, ω) does not admit weakly exact Lagrangian submanifolds in the Lagrangian isotopy class of L. Assume that $\alpha \in H^1(L; \mathbb{R})$ and*

$$\partial\alpha = \frac{1}{C}[\omega]_L$$

for some $C > 0$. Then $def_L(\alpha) \leq C$.

Proof. Let $\psi = \{\psi_t : L \to M\}$, $0 \leq t \leq T$, be a Lagrangian isotopy of L such that

$$[\omega]_t^{\psi} = [\omega]_L - t\,\partial\alpha. \tag{2.1}$$

Since, by the hypothesis of the theorem, $\partial\alpha = \frac{1}{C}[\omega]_L$, we get

$$[\omega]_t^{\psi} = [\omega]_L(1 - t/C).$$

Then $t = C$ cannot lie in the interval $[0, T]$, because if it did, we would have $[\omega]_C^{\psi} = 0$, implying that the Lagrangian submanifold $\psi_C(L)$, which is Lagrangian isotopic to L, is weakly exact, in contradiction with the hypothesis of the theorem. Hence $T < C$. Since this is true for any Lagrangian isotopy of L satisfying (2.1), we get $def_L(\alpha) \leq C$. □

A similar result for a particular class of Lagrangian submanifolds and a particular α was proved in the same way in [14].

Let us note that symplectic manifolds that do not admit weakly exact Lagrangian submanifolds at all or in a particular Lagrangian isotopy class are plentiful and include, in particular, symplectic vector spaces [19] and complex projective spaces.

2.2. Lagrangian tori in symplectic surfaces. Suppose (M^2, ω) is a connected symplectic surface (possibly with boundary) and $L \subset (M, \omega)$ is a simple closed oriented curve (that is, a 1-dimensional Lagrangian torus) lying in the interior of M. The orientation of L defines an isomorphism $H^1(L) \cong \mathbb{Z}$. This allows to define bp_L on $H^1(L)$ (see Section 3.3). Denote the positive generator of $H^1(L)$ by e.

We distinguish between two possibilities: when $M \setminus L$ is disconnected and when $M \setminus L$ is connected. We present precise statements in both cases.

In the first case L divides M into two connected components: M_+ and M_- of areas A_+, A_-, where $0 < A_\pm \leq +\infty$. The signs $+$ and $-$ here are determined by the usual orientation convention.

Theorem 2.2. *For $k \in \mathbb{N}$ we have*

$$bp_L(ke) = def_L(ke) = \frac{A_+}{k},$$
$$bp_L(-ke) = def_L(-ke) = \frac{A_-}{k},$$

In the second case let $L \subset (M, \omega)$ be a simple closed curve such that $M \setminus L$ is connected.

Theorem 2.3. *For $k \in \mathbb{Z}$ we have*

$$bp_L(ke) = def_L(ke) = +\infty.$$

The case $k = 1$ was proved in [33]. For the proofs of Theorems 2.2 and 2.3 see Section 4.

2.3. Toric orbits in symplectic toric manifolds. Let $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. Denote by

$$Lie\,\mathbb{T}^n = \mathbb{R}^n$$

the Lie algebra of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and by

$$Lie^*\,\mathbb{T}^n = (\mathbb{R}^n)^*$$

its dual space. Denote by $(\mathbb{Z}^n)^*$ the integral lattice in $(\mathbb{R}^n)^*$.

Let (M^{2n}, ω) be a connected (not necessarily closed) symplectic manifold equipped with an effective Hamiltonian action of $\mathbb{T}^n$. Denote the moment map of the action by $\Phi\colon M \to (\mathbb{R}^n)^*$. Assume that Φ is proper, the fibers of Φ are exactly the orbits of the action and the image of Φ is a convex set $\Delta \subset (\mathbb{R}^n)^*$ with non-empty interior so that its interior points are exactly the regular values of Φ (by the Atiyah–Guillemin–Sternberg theorem [1, 20], these conditions are automatically satisfied if M is closed).

Given $\mathbf{x} \in \Delta$, denote by $L_\mathbf{x} := \Phi^{-1}(\mathbf{x})$ the corresponding fiber of Φ. If $\mathbf{x} \in \operatorname{Int}\Delta$, then $L_\mathbf{x}$ is a Lagrangian torus and the Hamiltonian $\mathbb{T}^n$-action on $L_\mathbf{x}$ gives us a preferred isotopy class of diffeomorphisms $\mathbb{T}^n \to L_\mathbf{x}$. Thus the pair $H^1(L_\mathbf{x}) \subset H^1(L_\mathbf{x}; \mathbb{R})$ is naturally identified with the pair $(\mathbb{Z}^n)^* \subset (\mathbb{R}^n)^*$. We denote by

$$bp_\mathbf{x} := bp_{L_\mathbf{x}}\colon (\mathbb{Z}^n)^* \longrightarrow \mathbb{R}$$

and by

$$def_\mathbf{x} := def_{L_\mathbf{x}}\colon (\mathbb{R}^n)^* \longrightarrow \mathbb{R}$$

the Poisson bracket and the deformation invariants of $L_\mathbf{x}$.

For $\mathbf{x} \in \operatorname{Int}\Delta$ and $\alpha \in H^1(L; \mathbb{R})$ define $l_\mathbf{x}(\alpha)$ as the largest $t > 0$ for which $\mathbf{x} - t\alpha \in \Delta$ and let $\mathcal{I}(\mathbf{x}, \alpha)$ be the open segment of the open ray $\mathbf{x} - t\alpha$, $t \in (0, +\infty)$, connecting $\mathbf{x}$ and $\mathbf{x} - l_\mathbf{x}(\alpha)\alpha$:

$$\mathcal{I}(\mathbf{x}, \alpha) := \{\mathbf{x} - t\alpha,\ 0 < t < l_\mathbf{x}(\alpha)\}.$$

If no such t exists, set $l_\mathbf{x}(\alpha) := +\infty$ and let $\mathcal{I}(\mathbf{x}, \alpha)$ be the whole open ray. In other words, $\mathcal{I}(\mathbf{x}, \alpha)$ is the interior of the closed segment obtained by the intersection of the ray with Δ and $l_\mathbf{x}(\alpha)$ is the ratio of the rational length of α and the rational length of this segment[1].

Theorem 2.4. *Let $\alpha \in (\mathbb{Z}^n)^*$. Then*

$$l_\mathbf{x}(\alpha) \le bp_\mathbf{x}(\alpha).$$

For the proof see Section 5.

[1]Recall that the rational length of a vector cv, $v \in \mathbb{Z}^n$, is defined as $|c|$.

Remark 2.5. Note that there exists a symplectomorphism $\Phi^{-1}(\operatorname{Int}\Delta) \to \mathbb{T}^n \times \operatorname{Int}\Delta$ that identifies $\Phi^{-1}(\mathbf{x})$ with $\mathbb{T}^n \times \{\mathbf{x}\}$ for each $\mathbf{x} \in \operatorname{Int}\Delta$. Then Theorem 1.3, applied to the symplectic toric manifold $\Phi^{-1}(\operatorname{Int}\Delta)$, implies that *for $L_{\mathbf{x}}$ viewed as a Lagrangian submanifold of* $\Phi^{-1}(\operatorname{Int}\Delta)$ for each α the deformation invariant $def_{\mathbf{x}}(\alpha)$ equals $l_{\mathbf{x}}(\alpha)$ and thus, by Theorem 2.4,

$$bp_{\mathbf{x}}(\alpha) = def_{\mathbf{x}}(\alpha) = l_{\mathbf{x}}(\alpha)$$

for all $\alpha \in H^1(L)$.

For $L_{\mathbf{x}}$ viewed as a Lagrangian submanifold of the whole M the problem of finding $bp_{\mathbf{x}}(\alpha)$ and $def_{\mathbf{x}}(\alpha)$ is more difficult and the results below that we have been able to obtain are weaker.

2.4. Lagrangian tori in symplectic vector spaces. Let $M = \mathbb{C}^n$ be equipped with the standard symplectic structure ω and let $z_1, \dots, z_n$ be the complex coordinates on $\mathbb{C}^n$. Given $x_1, \dots, x_n > 0$, set $\mathbf{x} := (x_1, \dots, x_n)$. Define a *split Lagrangian torus* $T^n(\mathbf{x}) \subset \mathbb{C}^n$ by

$$T^n(\mathbf{x}) := \{\pi|z_i|^2 = x_i,\ i = 1, \dots, n\}.$$

The standard Hamiltonian $\mathbb{T}^n$-action gives us a preferred isotopy class of diffeomorphisms $\mathbb{T}^n \to T^n(\mathbf{x})$ and we naturally identify $H^1(L_{\mathbf{x}}) \subset H^1(L_{\mathbf{x}}; \mathbb{R})$ with $(\mathbb{Z}^n)^* \subset (\mathbb{R}^n)^*$.

We first consider the case of Lagrangian tori in $\mathbb{C}^2$.

Lagrangian tori in $\mathbb{C}^2$. We first present computations of def_L for general Lagrangian tori in $\mathbb{C}^2$. We then restrict to the cases of split and Chekanov tori.

Let $L \subset (\mathbb{C}^2, \omega)$ be a Lagrangian torus.

We say that an almost complex structure J (on $\mathbb{C}^2$) compatible with ω is *regular (for L) with respect to a point $p \in L$*, if for any $C \in H_2(\mathbb{C}^2, L)$ the moduli space of (non-parameterized) somewhere injective J-holomorphic disks in $\mathbb{C}^2$ with boundary in L and with one marked boundary point that represent the class C and pass through p (that is, the marked point coincides with p) is a (transversally cut out) smooth manifold of the expected dimension. For any $p \in L$ a generic almost complex structure J compatible with ω has this property; see Section 6.

The original idea of the proof of the following theorem belongs to E. Opshtein.

Theorem 2.6. *Assume that $H_2(\mathbb{C}^2, L) \simeq \mathbb{Z}\langle A, B\rangle$, where $\omega(A) > 0$. Let $\alpha \in H^1(L; \mathbb{R})$ so that $\partial\alpha(A) =: \sigma > 0$ and $\partial\alpha(B) =: \rho$. Assume that for some $k \geq 0$*

$$\mu(A) = 2, \quad \mu(B) = 2k,$$

and

$$\rho/\sigma \leq k + 1 \leq \omega(B)/\omega(A).$$

Then:

(A) *For any $p \in L$ and any almost complex structure J compatible with ω and regular with respect to p the mod-2 number $n_A(p, J)$ of (non-parameterized) somewhere injective J-holomorphic disks with boundary in L in the class A passing through p is well-defined and independent of the choice of p and J.*

(B) *If $n_A(p, J) \neq 0$ for some p and J as in (A), then*

$$def_L(\alpha) \leq \frac{\omega(A)}{\sigma}.$$

For the proof see Section 6.

Remark 2.7. As it can be seen from the proof, Theorem 2.6 remains true if $\mathbb{C}^2$ is replaced by any 4-dimensional symplectic manifold (M, ω) which satisfies $\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0$, is geometrically bounded in the sense of [2], or convex at infinity in the sense of [12], and

$$H_2(M, L) \cong \mathbb{Z}\langle A, B\rangle \oplus Im(\pi_2(M) \to \pi_2(M, L)),$$

where A, B satisfy the same conditions as in Theorem 2.6.

Note that Theorem 2.6 applies to certain split Lagrangian tori $T^2(\mathbf{x}) \subset \mathbb{C}^2$. Indeed, by [8], the standard complex structure J on $\mathbb{C}^2$ (which is, of course, compatible with ω) is regular (for $T^2(\mathbf{x})$) with respect to any point $p \in T^2(\mathbf{x})$. It is also easy to see that for any point $p \in T^2(\mathbf{x})$ there is exactly one regular (non-parametrized) J-holomorphic disk in the class A (with one marked point) that passes through p, if A is any one of the two standard generators of $H_2(\mathbb{C}^2, T^2(\mathbf{x}))$ with positive symplectic area.

Theorem 2.6, together with Theorem 1.4 and Theorem 2.14 (this general statement for split Lagrangian tori in $\mathbb{C}^n$ will appear later), yields the following corollary for computations of def_L and bp_L for split tori in $\mathbb{C}^2$. We state the result in the case $x_1 \leq x_2$ — the corresponding result in the case $x_1 > x_2$ can be deduced from it using the obvious symmetry of $bp_{\mathbf{x}}$ and $def_{\mathbf{x}}$ with respect to permutations of coordinates in $\mathbf{x}$ (see (7.1) below).

Corollary 2.8. *Assume $m, n \in \mathbb{Z}$, $\mathbf{x} = (x_1, x_2)$, $0 < x_1 \leq x_2$. Under these assumptions the following claims are true:*

(A) *If $m, n \leq 0$, then*

$$bp_{\mathbf{x}}(m, n) = def_{\mathbf{x}}(m, n) = +\infty.$$

(B) *If $x_1 < x_2/l$ and $1 \leq n \leq l$, then*

$$bp_{\mathbf{x}}(0, n) = def_{\mathbf{x}}(0, n) = +\infty.$$

(C) *If* $nx_1 - mx_2 \leq 0, m > 0$, *then*

$$x_1/m \leq bp_{\mathbf{x}}(m,n) \leq def_{\mathbf{x}}(m,n).$$

(D) *If* $nx_1 - mx_2 \geq 0, n > 0$, *then*

$$x_2/n \leq bp_{\mathbf{x}}(m,n) \leq def_{\mathbf{x}}(m,n).$$

(E) *Assume for* $\mathbf{x} = (x_1, x_2)$ *that* $2x_1 \leq x_2$. *If* $m > 0$, $n - 2m \leq 0$, *then*

$$x_1/m = bp_{\mathbf{x}}(m,n) = def_{\mathbf{x}}(m,n). \qquad \square$$

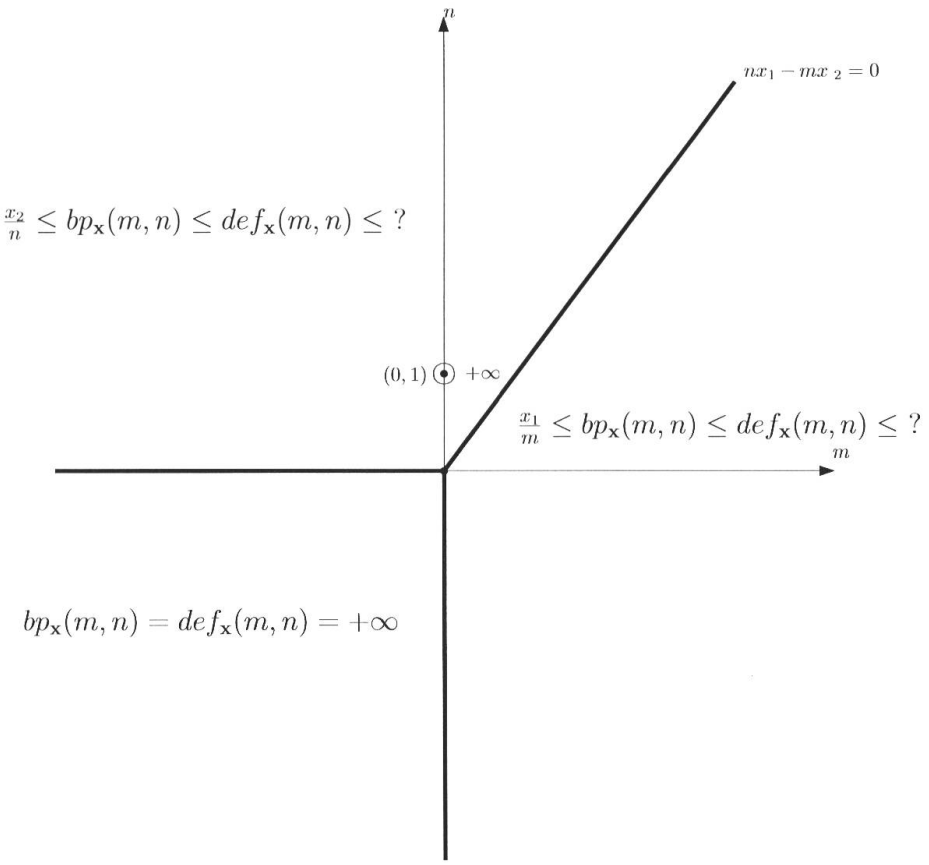

Figure 1. Regions of validity in Corollary 2.8 for $x_1 < x_2$.

Parts (A)–(D) of Corollary 2.8 follow directly from Theorems 1.4 and 2.14. Part (E) follows from the inequalities:

$$x_1/m \leq bp_{\mathbf{x}}(m,n) \leq def_{\mathbf{x}}(m,n) \leq x_1/m.$$

Here the first inequality follows from part (C) of Corollary 2.8, the second one from Theorem 1.4, and the third one from Theorem 2.6 with A, B being the standard basis

of $H_2(\mathbb{C}^2, T^2(\mathbf{x}))$ (so that $\omega(A) = x_1$, $\omega(B) = x_2$), $\alpha = me_1 + ne_2$, $a = m$, $b = n$, and $k = 1$.

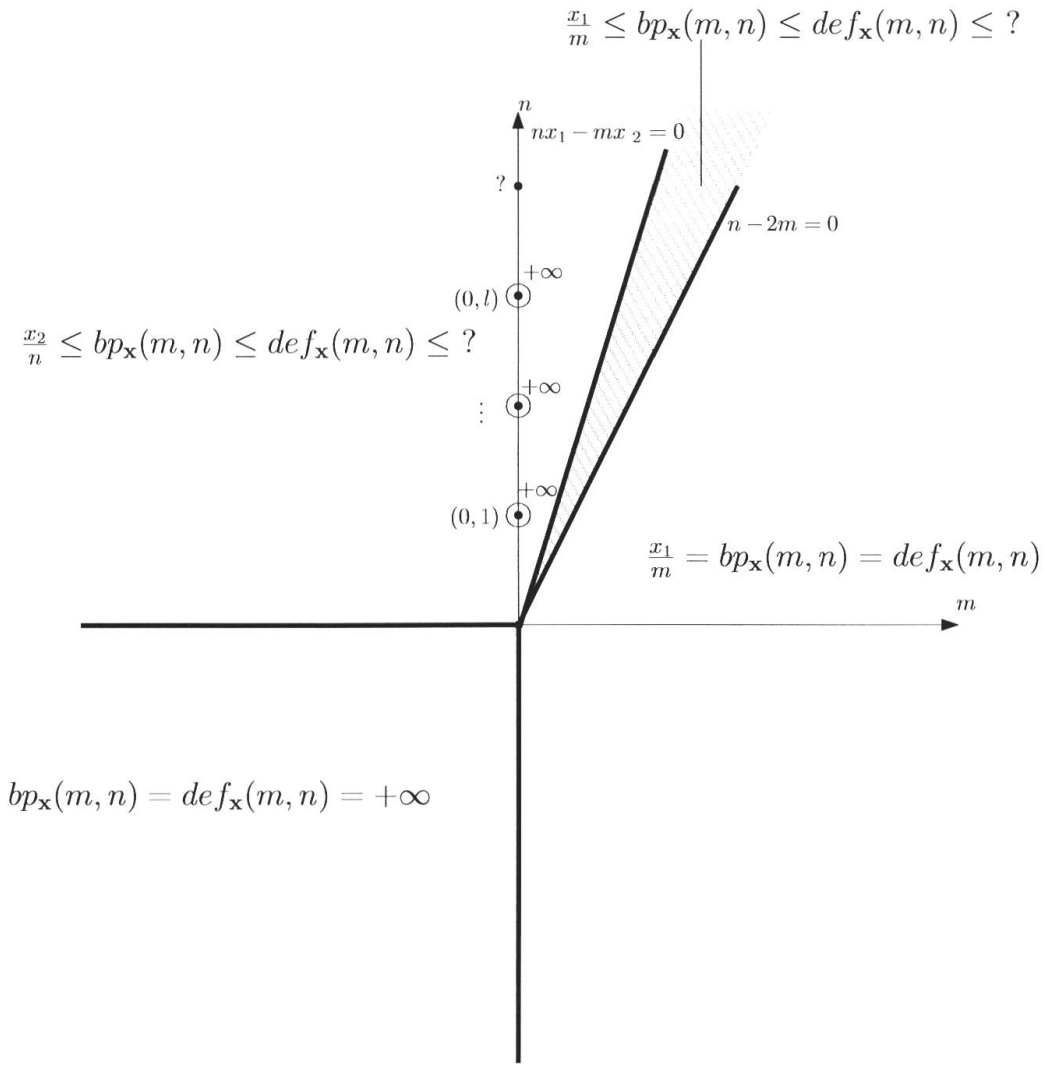

Figure 2. Regions of validity in Corollary 2.8 for $x_1 < x_2/l$ for $l \geq 2$.

In the case of the split monotone Lagrangian torus in $\mathbb{C}^2$ Corollary 2.8, together with the obvious homogeneity property of $def_{\mathbf{x}}$, $bp_{\mathbf{x}}$ with respect to $\mathbf{x}$ (see part (B) of Proposition 7.1 below) and Theorem 2.1 (since $\mathbb{C}^2$ does not admit weakly exact Lagrangian submanifolds by [19]), yields the following result.

Corollary 2.9. *Assume $m, n \in \mathbb{Z}$, $\mathbf{x} = (x, x)$, $x > 0$. Then:*

$$\begin{aligned} +\infty = bp_{\mathbf{x}}(m,n) &= def_{\mathbf{x}}(m,n), && \text{if } m,n \leq 0, \\ \frac{x}{m} = bp_{\mathbf{x}}(m,m) &= def_{\mathbf{x}}(m,m), && \text{if } m = n > 0, \\ \frac{x}{\max\{m,n\}} \leq bp_{\mathbf{x}}(m,n) &\leq def_{\mathbf{x}}(m,n), && \text{if } m > 0 \text{ or } n > 0. \end{aligned}$$

□

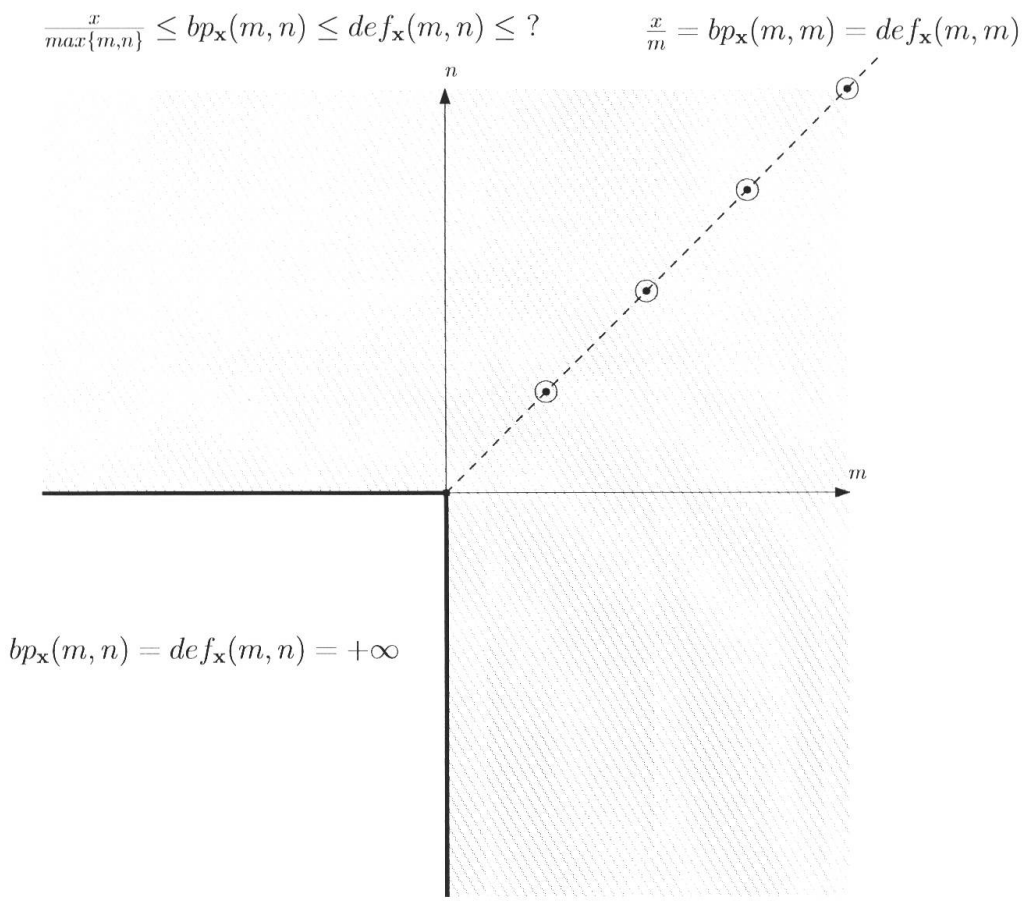

Figure 3. Regions of validity in Corollary 2.9 for monotone Lagrangian tori.

Chekanov tori in $\mathbb{C}^2$. Chekanov tori Θ_a, $a > 0$, in $\mathbb{C}^2$ were originally introduced in [5] (cf. [13]). The torus $\Theta_a \subset \mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ is defined as follows: consider the first open quadrant Q of $\mathbb{C}$ and a point $q \in Q$. Fix a foliation of $Q \setminus q$ by simple closed curves (each winding once around q) so that for each $a > 0$ there is exactly one curve in the foliation that bounds a disk of area a in Q.

Pick $a > 0$ and let $\eta(t)$ be the counterclockwise regular parameterization of the corresponding curve in the foliation. Then

$$\Theta_a := \left\{ \frac{1}{\sqrt{2}} \left(e^{2\pi i s} \eta(t), e^{-2\pi i s} \eta(t) \right) \right\}.$$

Consider the basis Γ, γ of $H_1(\Theta_a)$, where Γ, γ are, respectively, the homology classes of the curves

$$t \mapsto \frac{1}{\sqrt{2}} \left(\eta(t), \eta(t) \right) \quad \text{and} \quad s \mapsto \frac{1}{\sqrt{2}} \left(e^{2\pi i s} \eta(0), e^{-2\pi i s} \eta(0) \right)$$

in Θ_a. Let $\widehat{\Gamma}, \widehat{\gamma}$ be the integral basis of $H^1(\Theta_a)$ dual to Γ, γ. Define the functions

$$def_a\colon (\mathbb{Z}^2)^* \to (0, +\infty], \quad bp_a\colon (\mathbb{Z}^2)^* \to (0, +\infty]$$

by

$$def_a(m,n) := def_{\Theta_a}(m\widehat{\Gamma} + n\widehat{\gamma}), \quad bp_a(m,n) := bp_{\Theta_a}(m\widehat{\Gamma} + n\widehat{\gamma}).$$

Theorem 2.10. *The following claims hold for the Chekanov tori in $\mathbb{C}^2$:*

(A) $bp_a(m,n) \geq a/m$, *if* $m > 0$.

(B) $bp_a(m,n) = def_a(m,n) = +\infty$ *for* $m < 0$.

For the proof see Section 6.1.

Lagrangian tori in $\mathbb{C}^n$ for general $n \in \mathbb{N}$. We first generalize the statement of Theorem 2.6 to Lagrangian tori in $\mathbb{C}^n$ for a general $n \in \mathbb{N}$.

Let $L \subset (\mathbb{C}^n, \omega)$ be a Lagrangian torus.

As above, we say that an almost complex structure J (on $\mathbb{C}^n$) compatible with ω is *regular (for L) with respect to a point $p \in L$*, if for any $C \in H_2(\mathbb{C}^n, L)$ the moduli space of non-parameterized somewhere injective J-holomorphic disks in $\mathbb{C}^n$ with boundary on L and with one marked point that represent the class C and pass through p (that is, the marked point coincides with p) is a (transversally cut out) smooth manifold of the expected dimension. A generic J (on $\mathbb{C}^n$) compatible with ω has this property; see Section 7.2.

Theorem 2.11. *Assume $H_2(\mathbb{C}^n, L) \simeq \mathbb{Z}\langle A_1, \dots, A_n\rangle$, where*

$$\mu(A_i) = 2, \quad i = 1, \dots, n,$$

and

$$\omega(A_1) =: a > 0, \quad \omega(A_2) = \cdots = \omega(A_n) =: b.$$

Let $\alpha \in H^1(L; \mathbb{R})$ so that

$$\partial\alpha(A_1) =: \sigma > 0, \quad \partial\alpha(A_i) =: \rho, \quad i = 2, \dots, n.$$

Assume

$$\rho/\sigma \leq \frac{n+2}{2} \leq b/a, \quad \text{if } n \text{ is even},$$
$$\rho/\sigma \leq \frac{n+3}{2} \leq b/a, \quad \text{if } n \text{ is odd}.$$

Then:

(A) *For an almost complex structure J compatible with ω and regular (for L) with respect to a point $p \in L$ the mod-2 number $n_{A_1}(p, J)$ of (non-parameterized somewhere injective) J-holomorphic disks with boundary in L in the class A_1 with one marked point passing through p is well-defined and independent of the choice of p and J.*

(B) *If* $n_{A_1}(p, J) \neq 0$ *for some* p *and* J *as in (A), then*

$$def_L(\alpha) \leq \frac{\omega(A_1)}{\sigma} = \frac{a}{\sigma}.$$

For the proof see Section 7.2.

Remark 2.12. For Lagrangian tori $L \subset \mathbb{C}^2$ satisfying $H_2(\mathbb{C}^2, L) \simeq \mathbb{Z}\langle A_1, A_2\rangle$, where $\mu(A_1) = \mu(A_2) = 2$, Theorem 2.11 gives the same result as Theorem 2.6.

Remark 2.13. As it can be seen from the proof, and similarly to Remark 2.7, Theorem 2.11 remains true if $\mathbb{C}^n$ is replaced by any $2n$-dimensional symplectic manifold (M, ω) which satisfies $\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0$, is geometrically bounded in the sense of [2], or convex at infinity in the sense of [12], and the following holds:

$$H_2(M, L) \simeq \mathbb{Z}\langle A_1, \ldots, A_m\rangle \oplus Im(\pi_2(M) \to \pi_2(M, L)), \tag{2.2}$$

for some $m \in \mathbb{N}$ (not necessarily equal to n!), where

$$\mu(A_i) = 2, \quad i = 1, \ldots, m,$$
$$\omega(A_1) =: a > 0, \quad \omega(A_2) = \cdots = \omega(A_m) =: b,$$

and $\alpha \in H^1(L; \mathbb{R})$ satisfies

$$\partial\alpha(A_1) =: \sigma > 0, \quad \partial\alpha(A_i) =: \rho, \quad i = 2, \ldots, m,$$
$$\rho/\sigma \leq \frac{n+2}{2} \leq b/a, \quad \text{if } n \text{ is even},$$
$$\rho/\sigma \leq \frac{n+3}{2} \leq b/a, \quad \text{if } n \text{ is odd}.$$

In such a case Theorem 2.11 can be applied not only to M, L and α but also to $(\widehat{M} := M \times T^*\mathbb{S}^1, \widehat{\omega} := \omega \oplus d\theta \wedge dr)$ (where $\theta \in \mathbb{S}^1, r \in \mathbb{R}$ are the standard coordinates on $T^*\mathbb{S}^1$), $\widehat{L} := L \times \mathbb{S}^1 \subset \widehat{M}$ (where $\mathbb{S}^1 = \{r = 0\}$ is the zero-section of $T^*\mathbb{S}^1$) and $\widehat{\alpha}$, which is the image of α under the inclusion $H^1(L; \mathbb{R}) \to H^1(\widehat{L}; \mathbb{R}) = H^1(L; \mathbb{R}) \oplus \mathbb{R}$. (Note that $H_2(M, L) \simeq H_2(\widehat{M}, \widehat{L})$).

Moreover, if an almost complex structure J on M is compatible with ω and regular (for L) with respect to a point $p \in L$, then $\widehat{J} := J \oplus j$ (where j is the standard complex structure on $T^*\mathbb{S}^1$) is an almost complex structure on $\widehat{M}$ compatible with $\widehat{\omega}$ and regular (for $\widehat{L}$) with respect to a point $\widehat{p} := p \times q \in \widehat{L} = L \times \mathbb{S}^1$ (where q is a point in $\mathbb{S}^1$), and $n_{A_1}(p, J) = n_{A_1}(\widehat{p}, \widehat{J})$.

Then, if $\alpha \in H^1(L)$, we can apply Theorem 2.11 to $\widehat{M}, \widehat{L}, \widehat{\alpha}$ and, together with Theorem 1.4, it yields

$$bp_{\widehat{L}}(\widehat{\alpha}) \leq def_{\widehat{L}}(\widehat{\alpha}) \leq \frac{a}{\sigma},$$

or, equivalently,

$$pb_4^+(X_0 \times \mathbb{S}^1, Y_0 \times \mathbb{S}^1, X_1 \times \mathbb{S}^1, Y_1 \times \mathbb{S}^1) \geq \frac{\sigma}{a},$$

where X_0, X_1, Y_0, Y_1 are the closed subsets of $L = X_0 \cup X_1 \cup Y_0 \cup Y_1$ used to define $bp_L(\alpha)$.

By [14] (cf. Theorem 3.1 below), this yields the existence of connecting trajectories from X_0 to X_1 for Hamiltonian flows (defined for all times) generated by time-periodic Hamiltonians $H: M \times \mathbb{S}^1 \to \mathbb{R}$ such that

$$\Delta_H := \min_{Y_1 \times \mathbb{S}^1} H - \max_{Y_0 \times \mathbb{S}^1} H > 0.$$

Such a connecting trajectory will have time-length $\leq a/\sigma\Delta_H$.

Note that Theorem 2.11 applies to certain split Lagrangian tori in $(\mathbb{C}^n, \omega)$ and the standard basis $A_1, \ldots, A_n$ of $H_2(\mathbb{C}^n, T^n(\mathbf{x}))$ – indeed, for the standard complex structure J on $\mathbb{C}^n$ (which is, of course, compatible with ω) there is exactly one (non-parametrized) J-holomorphic disk in the class A_1 passing through any point of $T^n(\mathbf{x})$. The regularity of J (for $T^n(\mathbf{x})$) with respect to any $p \in T^n(\mathbf{x})$ follows again from [8].

The following results give partial information about the function $bp_{\mathbf{x}}$ for split Lagrangian tori. Set $x_{\min} := \min\{x_1, \ldots, x_n\}$. Denote by $e_1, \ldots, e_n$ the standard generators of $H^1(T^n(\mathbf{x})) \cong (\mathbb{Z}^n)^*$.

Theorem 2.14.

(A) *If $m_i \leq 0$ for all $i = 1, \ldots, n$, then*

$$bp_{\mathbf{x}}(m_1, \ldots, m_n) = def_{\mathbf{x}}(m_1, \ldots, m_n) = +\infty.$$

Otherwise

$$\min_{i, m_i > 0} x_i/m_i \leq bp_{\mathbf{x}}(m_1, \ldots, m_n).$$

(B) *If $x_1 = \cdots = x_n =: x$ and $k \in \mathbb{N}$, then*

$$bp_{\mathbf{x}}(k, \ldots, k) = def_{\mathbf{x}}(k, \ldots, k) = x/k.$$

(C) *Let $l \in \mathbb{N}$ and assume $x_{\min} < x_i/l$. Then for all $k \in \mathbb{N}$ with $k \leq l$,*

$$bp_{\mathbf{x}}(ke_i) = def_{\mathbf{x}}(ke_i) = +\infty.$$

For the proof see Section 7.3.

Combining Theorem 2.11 and Theorem 2.14 we obtain for certain split tori $T^n(\mathbf{x})$ the following result.

Corollary 2.15. *Assume* $x_2 = \cdots = x_n =: y$ *and let* $0 < x_1 \leq 2y/(n+2)$ *if* n *is even and* $0 < x_1 \leq 2y/(n+1)$ *if* n *is odd. For* $m_1 \in \mathbb{N}$ *let the integer* $m_2 = \cdots = m_n \leq y\, m_1/x_1$. *Then we have*

$$bp_{\mathbf{x}}(m_1, \ldots, m_n) = def_{\mathbf{x}}(m_1, \ldots, m_n) = x_1/m_1.$$

In particular, for any $m_1 \in \mathbb{N}$

$$bp_{\mathbf{x}}(m_1 e_1) = def_{\mathbf{x}}(m_1 e_1) = x_1/m_1.$$

Furthermore, assume for $l \in \mathbb{N}$ *that* $x_1 < y/l$. *Then for all* $1 \leq k \leq l$ *and all* $2 \leq i \leq n$ *we have*

$$bp_{\mathbf{x}}(k e_i) = def_{\mathbf{x}}(k e_i) = +\infty.$$

2.5. Lagrangian tori in complex projective spaces. Let $M = \mathbb{C}P^n$ and let ω be the standard Fubini–Study symplectic form on $\mathbb{C}P^n$ normalized so that $\int_{\mathbb{C}P^1} \omega = 1$.

Theorem 2.16. *Let* $L \subset (\mathbb{C}P^n, \omega)$ *be a Lagrangian torus and* $\alpha \in H^1(L)$. *Consider*

$$[\omega]_t := [\omega]_L - t\,\partial\alpha \in H^2(\mathbb{C}P^n, L; \mathbb{R})$$

for $t \geq 0$. *If there exists a* $C > 0$ *such that* $[\omega]_C \in H^2(\mathbb{C}P^n, L; \frac{1}{n}\mathbb{Z})$, *then*

$$def_L(\alpha) \leq C.$$

For the proof see Section 8.1.

We provide a precise statement in the case of Lagrangian torus fibers. Consider the standard Hamiltonian $\mathbb{T}^n$-action on $\mathbb{C}P^n$ and denote its moment map by $\Phi\colon \mathbb{C}P^n \to (\mathbb{R}^n)^*$. Its image is the simplex

$$\Delta := \left\{(x_1, \ldots, x_n) \in (\mathbb{R}^n)^* \mid x_1, \ldots, x_n \geq 0,\ 0 \leq x_1 + \cdots + x_n \leq 1\right\}.$$

As in Section 2.3, for $\mathbf{x} \in \operatorname{Int}\Delta$ denote $L_{\mathbf{x}} := \Phi^{-1}(\mathbf{x})$ the corresponding Lagrangian torus fiber of Φ and set

$$def_{\mathbf{x}}(\alpha) := def_{L_{\mathbf{x}}}(\alpha), \quad bp_{\mathbf{x}}(\alpha) := bp_{L_{\mathbf{x}}}(\alpha)$$

for each $\alpha \in (\mathbb{Z}^n)^* \cong H^1(L_{\mathbf{x}})$.

By Theorem 2.4, for all $\alpha \in (\mathbb{Z}^n)^*$

$$l_{\mathbf{x}}(\alpha) \leq bp_{\mathbf{x}}(\alpha). \tag{2.3}$$

For certain $\alpha \in (\mathbb{Z}^n)^*$ and $\mathbf{x} \in \operatorname{Int}\Delta$ we obtain an upper bound on $def_{\mathbf{x}}(\alpha)$ from Theorem 2.16.

Namely, define $d_{\mathbf{x}}(\alpha)$ as the smallest $t > 0$ for which $\mathbf{x} - t\alpha \in \frac{1}{n} \cdot (\mathbb{Z}^n)^*$ and let $\mathcal{J}(\mathbf{x}, \alpha)$ be the open segment of the same open ray $\mathbf{x} - t\alpha$, $t \in (0, +\infty)$, as above connecting $\mathbf{x}$ and $\mathbf{x} - d_{\mathbf{x}}(\alpha)\alpha$:

$$\mathcal{J}(\mathbf{x}, \alpha) := \{\mathbf{x} - t\alpha,\ 0 < t < d_{\mathbf{x}}(\alpha)\}.$$

If no such t exists, set $d_{\mathbf{x}}(\alpha) := +\infty$ and let $\mathcal{J}(\mathbf{x}, \alpha)$ be the whole open ray. In other words, $\mathcal{J}(\mathbf{x}, \alpha)$ is the open part of the segment of the ray connecting the origin $\mathbf{x}$ of the ray to the closest point of the lattice $\frac{1}{n} \cdot (\mathbb{Z}^n)^*$ on the ray.

Corollary 2.17. *With the setup as above,*

$$def_{\mathbf{x}}(\alpha) \leq d_{\mathbf{x}}(\alpha). \tag{2.4}$$

□

In case $l_{\mathbf{x}}(\alpha) = d_{\mathbf{x}}(\alpha)$ the lower and the upper bounds (2.3), (2.4) yield $def_{\mathbf{x}}(\alpha) = d_{\mathbf{x}}(\alpha) = l_{\mathbf{x}}(\alpha) = bp_{\mathbf{x}}(\alpha)$. For instance, this happens when the intervals $\mathcal{I}(\mathbf{x}, \alpha)$ and $\mathcal{J}(\mathbf{x}, \alpha)$ coincide. Thus, we get the following corollary.

Corollary 2.18. *Assume that $\mathbf{x} = \kappa\alpha$ for some $\kappa > 0$, $\alpha \in H^1(L_{\mathbf{x}})$, and that the open interval $\{t\mathbf{x},\ 0 < t < \kappa\}$, does not contain points of the lattice $\frac{1}{n} \cdot (\mathbb{Z}^n)^*$.*

Then this interval coincides with $\mathcal{I}(\mathbf{x}, \alpha)$ and $\mathcal{J}(\mathbf{x}, \alpha)$ and therefore

$$def_{\mathbf{x}}(\alpha) = bp_{\mathbf{x}}(\alpha) = l_{\mathbf{x}}(\alpha) = d_{\mathbf{x}}(\alpha) = \kappa.$$ □

2.6. Lagrangian tori in $\mathbb{S}^2 \times \mathbb{S}^2$. Let $(\mathbb{S}^2, \sigma)$ be the standard symplectic sphere with normalized symplectic area $\int_{\mathbb{S}^2} \sigma = 1$.

Theorem 2.19. *Let $L \subset (\mathbb{S}^2 \times \mathbb{S}^2, \sigma \oplus \sigma)$ be a Lagrangian torus and $\alpha \in H^1(L)$. Consider*

$$[\omega]_t := [\omega]_L - t\,\partial\alpha \in H^2(\mathbb{S}^2 \times \mathbb{S}^2, L; \mathbb{R})$$

for $t \geq 0$. If there exists a $C > 0$ such that $[\omega]_C \in H^2(\mathbb{S}^2 \times \mathbb{S}^2, L; \mathbb{Z})$, then

$$def_L(\alpha) \leq C.$$

For the proof see Section 8.2.

We provide a precise statement in the case of Lagrangian torus fibers. Consider the standard Hamiltonian $\mathbb{T}^2$-action on $\mathbb{S}^2 \times \mathbb{S}^2$ and denote its moment map by $\Phi\colon \mathbb{S}^2 \times \mathbb{S}^2 \to (\mathbb{R}^2)^*$. Its image is given by $\Delta := [0, 1] \times [0, 1]$. As in Section 2.3, for $\mathbf{x} \in \operatorname{Int}\Delta$ denote $L_{\mathbf{x}} := \Phi^{-1}(\mathbf{x})$ the corresponding Lagrangian torus fiber of Φ and set

$$def_{\mathbf{x}}(\alpha) := def_{L_{\mathbf{x}}}(\alpha), \quad bp_{\mathbf{x}}(\alpha) := bp_{L_{\mathbf{x}}}(\alpha)$$

for each $\alpha \in (\mathbb{Z}^2)^* \cong H^1(L_{\mathbf{x}})$.

By Theorem 2.4, for all $\alpha \in (\mathbb{Z}^2)^*$

$$l_{\mathbf{x}}(\alpha) \leq bp_{\mathbf{x}}(\alpha).$$

For certain $\alpha \in (\mathbb{Z}^2)^*$ and $\mathbf{x} \in \operatorname{Int}\Delta$ we obtain an upper bound on $def_{\mathbf{x}}(\alpha)$ from Theorem 2.19.

Namely, define $\rho_{\mathbf{x}}(\alpha)$ as the smallest $t > 0$ for which $\mathbf{x} - t\alpha \in (\mathbb{Z}^2)^*$ and let $\mathcal{K}(\mathbf{x}, \alpha)$ be the open segment of the same open ray $\mathbf{x} - t\alpha$, $t \in (0, +\infty)$, as above connecting $\mathbf{x}$ and $\mathbf{x} - \rho_{\mathbf{x}}(\alpha)\alpha$:

$$\mathcal{K}(\mathbf{x}, \alpha) := \{\mathbf{x} - t\alpha,\ 0 < t < \rho_{\mathbf{x}}(\alpha)\}.$$

If no such t exists, set $\rho_{\mathbf{x}}(\alpha) := +\infty$ and let $\mathcal{K}(\mathbf{x}, \alpha)$ be the whole open ray. In other words, $\mathcal{K}(\mathbf{x}, \alpha)$ is the open part of the segment of the ray connecting the origin $\mathbf{x}$ of the ray to the closest point of the lattice $(\mathbb{Z}^2)^*$ on the ray.

Corollary 2.20. *With the setup as above,*

$$l_{\mathbf{x}}(\alpha) \leq bp_{\mathbf{x}}(\alpha) \leq def_{\mathbf{x}}(\alpha) \leq \rho_{\mathbf{x}}(\alpha).$$

In particular, if $\mathcal{K}(\mathbf{x}, \alpha)$ connects $\mathbf{x}$ to either of the four vertices of Δ, then $\mathcal{I}(\mathbf{x}, \alpha) = \mathcal{K}(\mathbf{x}, \alpha)$ then

$$l_{\mathbf{x}}(\alpha) = bp_{\mathbf{x}}(\alpha) = def_{\mathbf{x}}(\alpha) = \rho_{\mathbf{x}}(\alpha). \qquad \square$$

2.7. Discussion and open questions. The results above reflect first steps in the study of the invariants bp_L and def_L. In this section we discuss the main difficulty in the current approach and a possible direction of further investigation of these invariants.

As we have already mentioned in the introduction, the lower bounds on bp_L come from "soft" constructions, while the upper bounds on def_L are based on "rigid" symplectic methods – foremost, on strong results yielding the existence of pseudo-holomorphic disks with boundary on Lagrangian submanifolds appearing in appropriate Lagrangian isotopies of L. Unfortunately, it seems that these strong rigidity results are not strong enough to get upper bounds on $def_L(\alpha)$ for many α even for the basic examples of Lagrangian tori considered above. The (well-known) difficulty comes from the fact that the pseudo-holomorphic disks in a given relative homology class of L in M may not persist in a Lagrangian isotopy $\psi = \{\psi_t : L \to M\}$ of L (since bubbling-off of pseudo-holomorphic disks is a codimension-1 phenomenon), which makes it very difficult to track, as t changes, the relative homology classes of $\psi_t(L)$ carrying the disks and, accordingly, the symplectic areas of these disks. (As above, we use the Lagrangian isotopy to identify the relative homology groups of all $\psi_t(L)$). In our case the difficulty is compounded by the need to track the pseudo-holomorphic disks and their areas for an *arbitrary* Lagrangian isotopy ψ of L satisfying the cohomological condition (1.1), with no *a priori* geometric information about it. Such a Lagrangian isotopy ψ typically involves non-monotone Lagrangians which limits even more the control over the disks and their areas.

Thus, the progress on upper bounds for def_L depends on getting more precise information on pseudo-holomorphic/symplectic disks with boundary on (possibly non-monotone) Lagrangian submanifolds Lagrangian isotopic to L.

Here is an example of possible additional helpful information on the disks. We will present it in the case of Lagrangian tori in the standard symplectic $\mathbb{C}^2$.

Assume there is a way to associate to any non-monotone Lagrangian torus $L \subset \mathbb{C}^2$ an ordered integral basis B_L of $H_2(\mathbb{C}^2, L)$ with the following properties:

(a) For any Lagrangian isotopy L_t of L *among non-monotone Lagrangian tori* the following conditions hold:

- the bases B_{L_t} for different t are all identified with each other under the isomorphisms between the groups $H_2(\mathbb{C}^2, L_t)$ defined by the Lagrangian isotopy;
- the symplectic areas of the elements of B_{L_t} are positive and change continuously with t.

(b) For $L = T^2(x_1, x_2)$, $x_1 < x_2$, the basis B_L is the standard basis of

$$H_2(\mathbb{C}^2, T^2(x_1, x_2)).$$

If such bases exist, a rather straightforward argument would allow to strengthen Corollary 2.8 and show that if $0 < x_1 \leq x_2$ and $nx_1 - mx_2 \leq 0$, $m > 0$, then

$$x_1/m = bp_{\mathbf{x}}(m, n) = def_{\mathbf{x}}(m, n).$$

This, in turn, would allow to strengthen Corollary 2.9 and show that for a monotone split Lagrangian torus $T^2(x, x)$ one has

$$bp_{\mathbf{x}}(m, n) = def_{\mathbf{x}}(m, n) = \frac{x}{\max\{0, m, n\}}$$

for all $m, n \in \mathbb{Z}$.

The question about the existence of a basis B_L is motivated by the folklore conjecture that any non-monotone torus L in $\mathbb{C}^2$ is Hamiltonian isotopic to a split torus $T^2(x_1, x_2)$.

Indeed, assume the conjecture is true. Then, by a theorem of Y.Chekanov [5], the ordered pair (x_1, x_2) is uniquely determined by L, as long as we require $x_1 < x_2$. A Hamiltonian isotopy between L and $T^2(x_1, x_2)$ identifies the standard basis of $H_2(\mathbb{C}^2, T(x_1, x_2))$ with an integral basis B_L of $H_2(\mathbb{C}^2, L)$. The basis B_L does not depend on the choice of the Hamiltonian isotopy (since any Hamiltonian isotopy of $\mathbb{C}^2$ preserving $T^2(x_1, x_2)$ as a set acts trivially on the homology of $T^2(x_1, x_2)$, by a theorem of M.-L. Yau [38]). It is not hard to check that B_L satisfies (a) and (b).

The existence of bases B_L satisfying (a) and (b) is, of course, much weaker than the conjecture and, accordingly, might be easier to prove and to generalize to higher dimensions.

3. The invariant bp_L and its properties

The definition of bp_L is based on the following construction related to the Poisson bracket.

3.1. Poisson bracket invariants. Let (M^{2n}, ω) be a connected symplectic manifold, possibly with boundary. Let $C^\infty(M)$ and $C_c^\infty(M)$ denote, respectively, the spaces of all and of compactly supported smooth functions on M (in the latter case the support is allowed to intersect the boundary of M).

Our sign convention for the Poisson bracket on (M, ω) will be as follows. For $G \in C^\infty(M)$ define a vector field sgrad G by $i_{\mathrm{sgrad}\, G}\omega = -dG$. Given $F, G \in C^\infty(M)$, define the Poisson bracket $\{F, G\}$ by

$$\begin{aligned}\{F, G\} := \omega(\mathrm{sgrad}\, G, \mathrm{sgrad}\, F) &= dF(\mathrm{sgrad}\, G) \\ &= -dG(\mathrm{sgrad}\, F) = L_{\mathrm{sgrad}\, G} F = -L_{\mathrm{sgrad}\, F} G.\end{aligned}$$

We say that sets $X_0, X_1, Y_0, Y_1 \subset M$ form an *admissible quadruple*, if they are compact and $X_0 \cap X_1 = Y_0 \cap Y_1 = \emptyset$.

Assume $X_0, X_1, Y_0, Y_1 \subset M$ is an admissible quadruple. Recall from [14] (cf. [4]) the following definition:

$$pb_4^+(X_0, X_1, Y_0, Y_1) := \inf_{(F,G)\in\mathcal{F}} \max_M \{F, G\},$$

where $\mathcal{F} = \mathcal{F}(X_0, X_1, Y_0, Y_1)$ is the set of all pairs (F, G), $F, G \in C_c^\infty(M)$, such that

$$F|_{X_0} \le 0, \quad F|_{X_1} \ge 1, \quad G|_{Y_0} \le 0, \quad G|_{Y_1} \ge 1. \tag{3.1}$$

One can show (see [4, 14]) that $\mathcal{F}$ can be replaced in the definition of pb_4^+ by a smaller set $\mathcal{F}' = \mathcal{F}'(X_0, X_1, Y_0, Y_1)$ of pairs (F, G), $F, G \in C_c^\infty(M)$, for which the inequalities in (3.1) are replaced by equalities on some open neighborhoods of the sets X_0, X_1, Y_0, Y_1.

If it is clear from the context what X_0, X_1, Y_0, Y_1 are meant, we will omit the corresponding indices and sets in the notation for pb_4^+, $\mathcal{F}$, $\mathcal{F}'$.

The number $1/pb_4^+(X_0, X_1, Y_0, Y_1)$ has the following dynamical interpretation [14] (cf. [4]):

Consider the set $\mathfrak{S}$ of complete Hamiltonians $G\colon M \to \mathbb{R}$ such that

$$\Delta_G := \min_{Y_1} G - \max_{Y_0} G > 0.$$

For each such G define $T_G \in (0, +\infty]$ as the supremum of all $t > 0$ such that there is no trajectory of the Hamiltonian flow of G of time-length $\le t/\Delta_G$ from X_0 to X_1 (such a trajectory is called a *chord of* G). We recall:

Theorem 3.1 ([14, Theorem 1.11]).

$$\sup_{G\in\mathfrak{S}} T_G = 1/pb_4^+(X_0, X_1, Y_0, Y_1).$$

Thus, if there exists a complete Hamiltonian $G\colon M \to \mathbb{R}$ with

$$\min_{Y_1} G - \max_{Y_0} G > 0$$

that has no chords from X_0 to X_1 then

$$1/pb_4^+(X_0, X_1, Y_0, Y_1) = +\infty.$$

3.2. The invariant bp_L for general Lagrangian submanifolds. The pb_4^+-invariant of admissible quadruples can be used to define a symplectic invariant of Lagrangian submanifolds fibered over the circle in the following way.

Let $L \subset (M, \omega)$ be a closed, connected Lagrangian submanifold admitting a fibration over $\mathbb{S}^1$. Let $\mathcal{J}'(L)$ denote the set of smooth fibrations $L \to \mathbb{S}^1$. The right action of $\mathrm{Diff}_0(L)$ defines an equivalence relation on $\mathcal{J}'(L)$ and we denote the resulting quotient set by $\mathcal{J}(L)$. Now slice $\mathbb{S}^1$ into 4 consecutive closed arcs γ_1, γ_2, γ_3, γ_4 *in the counterclockwise order.* For a chosen smooth fibration $f\colon L \to \mathbb{S}^1$ representing a class $[f] \in \mathcal{J}(L)$ we define

$$X_0 := f^{-1}(\gamma_1), \quad X_1 := f^{-1}(\gamma_3), \quad Y_0 := f^{-1}(\gamma_4), \quad Y_1 := f^{-1}(\gamma_2).$$

Roughly speaking, we slice L into four parts along cuts parallel to a fiber of f; see Figure 4.

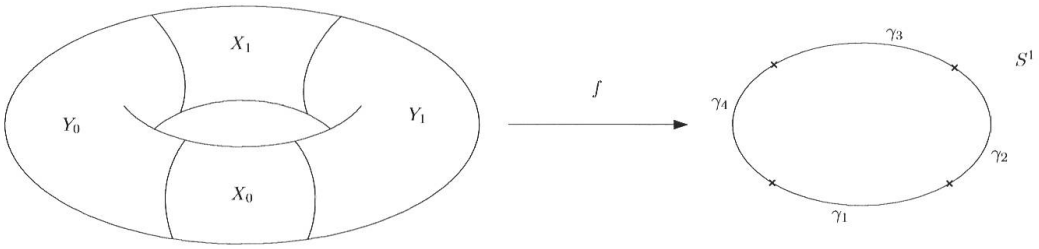

Figure 4. The four sets X_0, X_1, Y_0, Y_1 arising from a fibration f.

One easily sees that $X_0, X_1, Y_0, Y_1 \subset M$ is an admissible quadruple and $X_0 \cup X_1 \cup Y_0 \cup Y_1 = L$. We call such a quadruple *an admissible quadruple associated to* f. We set

$$bp_L([f]) := 1/pb_4^+(X_0, X_1, Y_0, Y_1).$$

If $pb_4^+(X_0, X_1, Y_0, Y_1) = 0$, we set

$$bp_L([f]) := +\infty.$$

Thus, $bp_L([f])$ takes values in $(0, +\infty]$.

Equivalently, $bp_L([f])$ can be described as the infimum of all $T > 0$ such that for *any* complete Hamiltonian $G: M \to \mathbb{R}$ satisfying $G|_{Y_0} \leq 0$, $G|_{Y_1} \geq 1$, there exists a chord of G from X_0 to X_1 of time-length $\leq T$. If no such T exists, we set

$$bp_L([f]) := +\infty.$$

In applications we will often prove a lower bound $T \leq bp_L([f])$ by constructing for any $\varepsilon > 0$ a complete Hamiltonian G that satifies $G|_{Y_0} \leq 0$, $G|_{Y_1} \geq 1$ and has no chords from X_0 to X_1 of time-length $< T - \varepsilon$.

Remark 3.2. The letters in the notation *bp* stand for the "Poisson bracket" and their inverse order ("*b*" before "*p*") reflects the fact that in the definition of *bp* we take the inverse of the maximum of the Poisson bracket.

Proposition 3.3. *$bp_L([f])$ is well-defined i.e. it does not depend on the choice of representative of $[f] \in \mathcal{I}(L)$ and the choice of division of $\mathbb{S}^1$ into 4 arcs.*

Proof. Let $f: L \to \mathbb{S}^1$ be a smooth fibration. We first show that for $\phi \in \mathrm{Diff}_0(L)$ the bp_L-invariants of f and $f \circ \phi$ are equal.

Note that any $\phi \in \mathrm{Diff}_0(L)$ is a time-1 flow of a time-dependent vector field on L. Denote this vector field on L by X_t and the corresponding flow by ϕ_t. Let $W \cong U \subset T^*L$ be a Weinstein neighborhood of L in M. Identify L with the 0-section in U. We can extend X_t in U to a Hamiltonian vector field as follows: in canonical coordinates (q, p) we define the Hamiltonian $H_t: U \to \mathbb{R}$ by setting $H_t(q, p) = p(X_t(q))$. A short calculation reveals that the Hamiltonian vector field $X_{H_t} \equiv X_t$ on the zero-section. By multiplying H_t with a suitable cut-off function one obtains a time-dependent Hamiltonian with compact support in W such that the induced flow on L coincides with the flow of ϕ_t. Therefore, by invariance of pb_4^+ under symplectomorphism, bp_L does not depend on the choice of representative of a class in $\mathcal{I}(L)$.

We now show independence of the choice of four arcs γ_i. Choose four other arcs γ_i' in the same fashion. Note that there exists a $\varphi \in \mathrm{Diff}_0(\mathbb{S}^1)$ such that $\varphi(\gamma_i') = \gamma_i$. Using the notation of Section 3.2, this implies that the image of $\varphi \circ f$ and f under ϱ' lie in the same path-connected component of $\mathcal{N}'(L)$. Using Proposition 3.4 we see that $\varphi \circ f$ and f represent the same equivalence class in $\mathcal{I}(L)$. Hence our definition is independent of choice of arcs. □

Clearly, the resulting function $bp_L: \mathcal{I}(L) \to (0, +\infty]$ is a symplectic invariant of Lagrangian submanifolds fibered over the circle. We call this function the *Poisson-bracket invariant of L*.

We now relate $\mathcal{I}(L)$ to the integral cohomology of L.

Consider the set $\mathcal{N}'(L)$ of all non-singular (that is, non-vanishing) closed 1-forms on L representing non-zero integral classes in $H^1(L)$. It follows easily from Moser's method [30] that the path-connected components of $\mathcal{N}'(L)$ are exactly the orbits of

the natural $\mathrm{Diff}_0(L)$-action on $\mathcal{N}'(L)$. Let $\mathcal{N}(L) = \mathcal{N}'(L)/\mathrm{Diff}_0(L)$ be the set of the path-connected components of $\mathcal{N}'(L)$.

Define a map $\varrho'\colon \mathcal{I}'(L) \to \mathcal{N}'(L)$ as follows. Given an element $f \in \mathcal{I}'(L)$, i.e. a fibration $f\colon L \to \mathbb{S}^1$, let $\varrho'(f)$ be the non-singular 1-form $f^*d\theta$, where $d\theta$ is the standard angle 1-form on $\mathbb{S}^1$. Clearly, $\varrho'\colon \mathcal{I}'(L) \to \mathcal{N}'(L)$ induces a map $\varrho\colon \mathcal{I}(L) \to \mathcal{N}(L)$.

Proposition 3.4. *The map $\varrho\colon \mathcal{I}(L) \to \mathcal{N}(L)$ is invertible.*

Proof. Define a map $\rho'\colon \mathcal{N}'(L) \to \mathcal{I}'(L)$ as follows. Fix a point $x_0 \in L$. Given a non-singular form $\alpha \in \mathcal{N}'(L)$, let $\rho'(\alpha)$ be the map $L \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ that sends each $x \in L$ to $\int_{x_0}^{x} \alpha \bmod 1$. Here the integral is taken along any smooth path from x_0 to x in L (recall that L is assumed to be connected); a different choice of path changes the integral by an integral value, since the cohomology class of α is integral. One easily checks that $\rho'(\alpha)$ is a fibration of L over $\mathbb{S}^1$ and thus ρ' is well-defined. Clearly, ρ' induces a map $\rho\colon \mathcal{N}(L) \to \mathcal{I}(L)$.

Note that $\varrho' \circ \rho' = \mathrm{Id}$ and for any fibration $f \in \mathcal{I}'(L)$ the fibration $\rho' \circ \varrho'(f)$ lies in the $\mathrm{Diff}_0(L)$-orbit of f. This shows that $\rho = \varrho^{-1}$. □

Thus bp_L is also defined as a function $bp_L\colon \mathcal{N}(L) \to (0, +\infty]$.

Note that a path in $\mathcal{N}'(L)$ has to lie in the same cohomology class. This defines a map $\Upsilon\colon \mathcal{N}(L) \to H^1(L) \setminus 0$. For general L the map Υ does not have to be either surjective or injective — to check whether it is surjective for a particular L is, in general, a very non-trivial task, see [15, 21, 23, 37]. However, if $\dim L \leq 3$, then Υ is injective; see e.g. [25]. Using $\Upsilon\colon \mathcal{N}(L) \to H^1(L) \setminus 0$ we define a version $\mathfrak{bp}_L$ of the invariant on the image of Υ in $H^1(L) \setminus 0$: for $\alpha \in \mathrm{im}(\Upsilon) \subset H^1(L) \setminus 0$ we set

$$\mathfrak{bp}_L(\alpha) := \sup_{A \in \Upsilon^{-1}(\alpha)} bp_L(A),$$

and we extend $\mathfrak{bp}_L$ to 0 via $\mathfrak{bp}_L(0) := +\infty$.

Theorem 3.5. *Let $L \subset (M, \omega)$ be a closed Lagrangian submanifold admitting fibrations over $\mathbb{S}^1$. Then for all $\alpha \in \mathrm{im}(\Upsilon) \cup \{0\} \subset H^1(L)$ we have*

$$\mathfrak{bp}_L(\alpha) \leq def_L(\alpha).$$

Proof. For $\alpha = 0$ we have $\mathfrak{bp}_L(\alpha) = def_L(\alpha) = +\infty$ verifying the claim.

Let us assume $\alpha \neq 0$. We will prove the inequality $\mathfrak{bp}_L(\alpha) \leq def_L(\alpha)$ by a method developed in [4] (cf. [14]). Namely, given $\alpha \in H^1(L)$, consider any smooth fibration $f\colon L \to \mathbb{S}^1$ such that $\Upsilon \circ \varrho([f]) = \alpha$ (see Section 3.2 for the notation). Let X_0, X_1, Y_0, Y_1 be an admissible quadruple used in the definition of $bp_L([f]) = 1/pb_4^+(X_0, X_1, Y_0, Y_1)$. Let $(F, G) \in \mathcal{F}'(X_0, X_1, Y_0, Y_1)$. Note that, since $L = X_0 \cup X_1 \cup Y_0 \cup Y_1$ and since F is constant on some neighborhoods of X_0 and X_1 and G is constant on some neighborhoods of Y_0 and Y_1,

$$dF \wedge dG \equiv 0 \text{ on a neighborhood of } L, \tag{3.2}$$

and thus $FdG|_L$ is a closed 1-form on L. An easy direct computation shows that

$$[FdG|_L] = -\alpha \in H^1(L). \tag{3.3}$$

Consider the deformation

$$\omega_t := \omega + t\,dF \wedge dG, \quad t \in \mathbb{R}_{\geq 0}.$$

A direct calculation shows that

$$dF \wedge dG \wedge \omega^{n-1} = -\frac{1}{n}\{F, G\}\omega^n,$$

and thus

$$\omega_t^n = \big(1 - t\{F, G\}\big)\omega^n.$$

Thus, ω_t is symplectic for any $t \in I_{(F,G)}$, where

$$I_{(F,G)} := \Big[0, \frac{1}{\max_M\{F, G\}}\Big).$$

Fix an arbitrary $t \in I_{(F,G)}$. Since F, G are compactly supported, the form ω can be mapped (using Moser's method [30]) to ω_t by a compactly supported isotopy $\vartheta_t\colon (M, \omega_t) \to (M, \omega)$. Since, by (3.2), L is Lagrangian with respect to ω_t, we get that $L_t := \vartheta_t(L)$ is a Lagrangian submanifold of (M, ω) Lagrangian isotopic to L.

Using (3.3) we readily see that, under the identification

$$H^2(M, L; \mathbb{R}) \cong H^2(M, L_t; \mathbb{R})$$

induced by the isotopy, the class $[\omega]_{L_t}$ is identified with $\omega - t\,\partial\alpha$. Since this is true for all $t \in I_{(F,G)}$ we obtain

$$\frac{1}{\max_M\{F, G\}} \leq def_L(\alpha).$$

Now the latter inequality holds for all $(F, G) \in \mathcal{F}'(X_0, X_1, Y_0, Y_1)$, this gives us

$$bp_L\big([f]\big) := 1/pb_4^+(X_0, X_1, Y_0, Y_1) \leq def_L(\alpha).$$

This is true for any $[f]$ such that $\Upsilon \circ \varrho([f]) = \alpha$, hence

$$\mathfrak{bp}_L(\alpha) \leq def_L(\alpha). \qquad \square$$

Question 3.6. *Do there exist L and $\alpha \in \mathrm{im}(\Upsilon) \subset H^1(L) \setminus 0$ for which*

$$\mathfrak{bp}_L(\alpha) \neq def_L(\alpha)?$$

3.3. The invariant bp_L for Lagrangian tori. Let us now assume that L is diffeomorphic to a torus $\mathbb{T}^n$. In this case the map Υ is clearly surjective. If $n \leq 3$, then, as mentioned above, Υ is injective and hence bijective. Thus $bp_L = \mathfrak{bp}_L$ coincide and by abuse of notation we write the invariant

$$bp_L \colon H^1(L) \longrightarrow \mathbb{R}$$

in these cases (we extend bp_L to 0 via $bp_L(0) := +\infty$).

For a torus of dimension strictly greater than 3 the map Υ may not be injective. For instance, Υ is known to be not injective if $n > 5$; see [24,34]. In any case, each isotopy class of diffeomorphisms $s \colon \mathbb{T}^n \to L$ defines a right inverse $\Psi_s \colon H^1(L) \setminus 0 \to \mathcal{N}(L)$ of Υ as follows: given $a \in H^1(L) \setminus 0$, represent $s^*a \in H^1(\mathbb{T}^n) \setminus 0$ by a linear form and let $\Psi_s(a)$ be the path-connected component of $\mathcal{N}'(L)$ containing the pull-back of this linear form under s^{-1}. Thus for any isotopy class of diffeomorphisms $s \colon \mathbb{T}^n \to L$ we can define

$$bp_L^s := bp_L \circ \Psi_s \colon H^1(L) \longrightarrow \mathbb{R},$$

where we again extend bp_L to 0 via $bp_L^s(0) := +\infty$. Clearly,

$$bp_L^s \leq \mathfrak{bp}_L. \tag{3.4}$$

In case the class of parametrizations $s \colon \mathbb{T}^n \to L$ is clear we sometimes write bp_L by abuse of notation. For instance, if L is a regular fiber of a Hamiltonian $\mathbb{T}^n$-action there is an obvious preferred isotopy class of diffeomorphisms $\mathbb{T}^n \to L$.

Theorem 3.5 and (3.4) yield

$$bp_L \leq def_L\,. \tag{3.5}$$

Remark 3.7. The discussion above shows that for a Lagrangian torus L, a cohomology class $\alpha \in H^1(L) \setminus 0$ and an isotopy class s of diffeomorphisms $\mathbb{T}^n \to L$ we have

$$bp_L^s(\alpha) = 1/pb_4^+(X_0, X_1, Y_0, Y_1).$$

Here (X_0, X_1, Y_0, Y_1) is an admissible quadruple associated to a fibration $f_\alpha \colon L \to \mathbb{S}^1$ such that the $\mathrm{Diff}_0(L)$-orbit of the 1-form $f_\alpha^* d\theta$ on L (whose cohomology class is α) is $\Psi_s(\alpha) \in \mathcal{N}(L)$. By the same token,

$$bp_L^s(-\alpha) = 1/pb_4^+(X_0, X_1, Y_1, Y_0) = 1/pb_4^+(X_1, X_0, Y_0, Y_1).$$

Indeed, $f_{-\alpha}$ can be constructed by composing f_α with an orientation-reversing diffeomorphism of $\mathbb{S}^1$.

Now assume $k \in \mathbb{N}$ and $\alpha \in H^1(L)$ is a primitive class (that is, a class which is not a positive integral multiple of another class in $H^1(L)$). Then an admissible quadruple

(X_0, X_1, Y_0, Y_1) associated to $f_{k\alpha}$ can be described in terms of f_α. Namely, divide $\mathbb{S}^1$ in consecutive closed arcs $\gamma_1, \dots, \gamma_{4k}$ in the counterclockwise order. Set

$$X_0 := f_\alpha^{-1}\Big(\bigcup_{i\equiv 1 \bmod 4} \gamma_i\Big), \quad X_1 := f_\alpha^{-1}\Big(\bigcup_{i\equiv 3 \bmod 4} \gamma_i\Big),$$
$$Y_0 := f_\alpha^{-1}\Big(\bigcup_{i\equiv 0 \bmod 4} \gamma_i\Big), \quad Y_1 := f_\alpha^{-1}\Big(\bigcup_{i\equiv 2 \bmod 4} \gamma_i\Big).$$

Thus, if $k = 1$, then the sets X_0, X_1, Y_0, Y_1 are diffeomorphic to $\mathbb{T}^{n-1} \times [0, 1]$, while if $k > 1$ the sets are the unions of the same number of disjoint copies of $\mathbb{T}^{n-1} \times [0, 1]$.

3.4. General properties of bp_L and def_L. We list basic properties of bp_L and def_L that will be used further in the paper.

Homogeneity of def_L. def_L is positively homogeneous of degree -1,

$$def_L(c\alpha) = def_L(\alpha)/c$$

for any $c > 0$ and $\alpha \in H^1(L; \mathbb{R})$.

Question 3.8. *Is there an inequality/equality between $bp_L(k\alpha)$ and $kbp_L(\alpha)$ in case L is a Lagrangian torus and $k \in \mathbb{N}$?*

Semi-continuity of def_L. Let $L_j \subset (M, \omega)$, $j \in \mathbb{N}$, be a sequence of Lagrangian submanifolds Lagrangian isotopic to L and converging to $L \subset (M, \omega)$ in the C^1-topology. Then

$$def_L(\alpha) \le \liminf_{j\to+\infty} def_{L_j}(\alpha) \text{ for any } \alpha \in H^1(L; \mathbb{R}). \tag{3.6}$$

(Here we use the canonical isomorphism $H^1(L_j; \mathbb{R}) \cong H^1(L; \mathbb{R})$). The inequality follows from a parametric version of the Weinstein neighborhood theorem.

Now let us consider the general properties of pb_4^+. We will use the following notation: if $U \subset M$ (possibly $U = M$) is an open set containing an admissible quadruple X_0, X_1, Y_0, Y_1 we will denote by $pb_4^{U,+}(X_0, X_1, Y_0, Y_1)$ the Poisson bracket invariant defined using functions supported in U.

The following properties of pb_4^+ and bp_L follow easily from the definitions.

Monotonicity of pb_4^+. Assume $U \subset W$ are opens sets in M and $X_0', X_1', Y_0', Y_1' \subset U \subset W$ is an admissible quadruple. Let X_0, X_1, Y_0, Y_1 be another admissible quadruple such that $X_0 \subset X_0'$, $X_1 \subset X_1'$, $Y_0 \subset Y_0'$, $Y_1 \subset Y_1'$. Then

$$pb_4^{W,+}(X_0, X_1, Y_0, Y_1) \le pb_4^{U,+}(X_0', X_1', Y_0', Y_1'). \tag{3.7}$$

Semi-continuity of pb_4^+ and bp_L. Suppose that a sequence $X_0^{(j)}, X_1^{(j)}, Y_0^{(j)}, Y_1^{(j)}$, $j \in \mathbb{N}$, of ordered collections converges (in the sense of the Hausdorff distance between sets) to a collection X_0, X_1, Y_0, Y_1. Then

$$\limsup_{j\to+\infty} pb_4^+\left(X_0^{(j)}, X_1^{(j)}, Y_0^{(j)}, Y_1^{(j)}\right) \le pb_4^+(X_0, X_1, Y_0, Y_1).$$

Accordingly, if $L_j \subset (M,\omega)$, $j \in \mathbb{N}$, is a sequence of Lagrangian submanifolds Lagrangian isotopic to L and converging to $L \subset (M,\omega)$ in the C^1-topology. Then

$$bp_L(\alpha) \le \liminf_{j\to+\infty} bp_{L_j}(\alpha) \quad \text{for any } \alpha \in H^1(L;\mathbb{R}). \tag{3.8}$$

(Here we use the canonical isomorphism $H^1(L_j;\mathbb{R}) \cong H^1(L;\mathbb{R})$).

Behavior of pb_4^+ and bp_L under products. Suppose that M and N are connected symplectic manifolds. Equip $M \times N$ with the product symplectic form. Let $K \subset N$ be a compact subset. Then for every collection X_0, X_1, Y_0, Y_1 of compact subsets of M

$$pb_4^{M\times N,+}(X_0 \times K, X_1 \times K, Y_0 \times K, Y_1 \times K) \le pb_4^{M,+}(X_0, X_1, Y_0, Y_1). \tag{3.9}$$

The following product property of bp_L follows immediately from (3.9):

Proposition 3.9. *Assume $L_i \subset (M_i,\omega_i)$, $i = 1,2$, are Lagrangian tori and $\alpha \in H^1(L_1)$. Consider the Lagrangian submanifold $L_1 \times L_2 \subset (M_1 \times M_2, \omega_1 \oplus \omega_2)$ and the cohomology class $\alpha \times g \in H^1(L_1 \times L_2)$, where g is a generator of $H^0(L_2)$. Let $s_i : \mathbb{T}^{n_i} \to L_i$ be two isotopy classes of diffeomorphisms. Then*

$$bp^{s_1\times s_2}_{L_1\times L_2}(\alpha \times g) \ge bp^{s_1}_{L_1}(\alpha). \qquad \square$$

Behavior of pb_4^+ under symplectic reduction. The following property of pb_4^+ did not appear in [4, 14], but is proved similarly to (3.9).

Namely, let (M,ω) be a connected, not necessarily closed, symplectic manifold. Let $\Sigma \subset (M,\omega)$ be a coisotropic submanifold. We do not assume that Σ is a closed subset of M. Assume that the characteristic foliation of Σ defines a *proper* fibration $\pi : \Sigma \to (N,\eta)$ over a (not necessarily closed) symplectic manifold (N,η).

Let $X_0, X_1, Y_0, Y_1 \subset N$ be an admissible quadruple (in particular, the sets are compact). Assume

$$\hat{X}_0 \subset \pi^{-1}(X_0),\ \hat{X}_1 \subset \pi^{-1}(X_1),\ \hat{Y}_0 \subset \pi^{-1}(Y_0),\ \hat{Y}_1 \subset \pi^{-1}(Y_1)$$

are some compact sets in M. Then $\hat{X}_0, \hat{X}_1, \hat{Y}_0, \hat{Y}_1 \subset M$ is an admissible quadruple. Moreover,

$$pb_4^{M,+}\left(\hat{X}_0, \hat{X}_1, \hat{Y}_0, \hat{Y}_1\right) \le pb_4^{N,+}(X_0, X_1, Y_0, Y_1). \tag{3.10}$$

Indeed, let F, G be functions on N such that $(F, G) \in \mathcal{F}_N(X_0, X_1, Y_0, Y_1)$. Consider the functions $F \circ \pi$, $G \circ \pi$ on Σ. Since $\pi\colon \Sigma \to N$ is proper and F, G are compactly supported, so are $F \circ \pi$, $G \circ \pi$. Now cut off the functions $F \circ \pi$, $G \circ \pi$ in the isotropic direction normal to Σ. As a result we get functions $\chi(r^2)(F \circ \pi)$, $\chi(r^2)(G \circ \pi)$ with compact support lying in a tubular neighborhood U of Σ — here r is a radial function on the isotropic normal bundle to Σ with respect to a Riemannian metric on that bundle and $\chi\colon \mathbb{R} \to [0, 1]$ is a smooth function supported near 0 and satisfying $\chi(0) = 1$. Extend the two functions from U to M by zero and denote the resulting functions on M by $\widehat{F}, \widehat{G}$. One easily checks that $(\widehat{F}, \widehat{G}) \in \mathcal{F}_M(\widehat{X}_0, \widehat{X}_1, \widehat{Y}_0, \widehat{Y}_1)$ and $\max_M \{\widehat{F}, \widehat{G}\} = \max_N \{F, G\}$, which yields (3.10).

In the case when Σ is a fiber of the moment map of a Hamiltonian action of a Lie group H on M the reduced space $N = \Sigma/H$ may not be a smooth symplectic manifold but a symplectic orbifold, which brings us to the following discussion.

The Poisson bracket invariant for symplectic orbifolds. Recall (see e.g. [28]) that the notions of smooth functions, vector fields and differential forms can be extended to orbifolds. In particular, there is a well-defined notion of an orbifold symplectic form; an orbifold equipped with such a form is called a *symplectic orbifold*. Accordingly, there is a notion of the Poisson bracket of two smooth functions on a symplectic orbifold and the definition of pb_4^+ can be carried over literally to symplectic orbifolds.

It is easy to check that the proof of (3.10) goes through in the case when the reduced symplectic space N is an orbifold.

4. Lagrangian tori in symplectic surfaces: proofs

Proof of Theorem 2.2. Let us prove that

$$def_L(ke) \leq A_+/k.$$

(The inequality $def_L(-ke) \leq A_-/k$ is proved in the same way). If $A_+ = \infty$, the inequality is trivial, so let us assume that $A_+ < \infty$.

Fix $\epsilon > 0$. Let $\psi = \{\psi_t\colon L \to M\}_{0 \leq t \leq T}$ be a Lagrangian isotopy of L such that $Flux(\psi)_t = -tke$ for all $0 \leq t \leq T$. By definition of $def_L(ke)$ and since ϵ was chosen arbitrarily, it suffices to show that $T \leq A_+/k + \epsilon$.

There exists a compact surface K (possibly with boundary) which lies in M and contains the union $\cup_{0 \leq t \leq T} \psi_t(L)$ of all the Lagrangian submanifolds appearing in the isotopy. Cap off the boundary components of K lying in M_+ (if they exist) by disks and extend the symplectic form from K over the disks so that the total area of the disks is smaller than ϵ. Denote the resulting compact symplectic surface by $\widehat{K}$ and the symplectic form on it by $\widehat{\omega}$.

Note that L is a Lagrangian submanifold of $(\widehat{K}, \widehat{\omega})$, ψ defines a Lagrangian isotopy of L in $(\widehat{K}, \widehat{\omega})$ and the Lagrangian flux of the latter Lagrangian isotopy in $\widehat{K}$

is the same as that of the original Lagrangian isotopy in M. By our construction, L bounds in $\widehat{K}$ a domain *without boundary* of area $A < A_+ + \epsilon$. Therefore, $\partial(ke) = k[\widehat{\omega}]_L/A$ and $(\widehat{K}, \widehat{\omega})$ does not admit weakly exact Lagrangian submanifolds in the Lagrangian isotopy class of L (because any curve in $\widehat{K}$ isotopic to L bounds a domain of positive area). Thus we can apply Theorem 2.1 and get that $def_L(ke)$, for L viewed as a Lagrangian submanifold of $(\widehat{K}, \widehat{\omega})$, is not bigger than A/k. On the other hand, by definition,

$$T \leq def_L(ke).$$

Thus, $T \leq A/k < A_+/k + \epsilon$. Since this holds for any Lagrangian isotopy ψ as above and any $\epsilon > 0$, by the definition of $def_L(ke)$ for L viewed as a Lagrangian submanifold of (M, ω), we get that

$$def_L(ke) \leq A_+/k.$$

Since, by Theorem 3.5, $bp_L(\pm ke) \leq def_L(\pm ke)$, it remains to prove that

$$bp_L(\pm ke) \geq A_\pm/k.$$

We first consider a model situation. Let $\varepsilon, A > 0$ and denote by $(x, y) \in \mathbb{R}^2$ the coordinates and by $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ the projection onto the x-axis, $\pi(x, y) = x$. Define

$$Q(A) := [0, A] \times [0, 1] \subset Q_\varepsilon(A) := (-\varepsilon, A + \varepsilon) \times (-\varepsilon, 1 + \varepsilon) \subset \mathbb{R}^2.$$

Label three sides of $Q(A)$ as follows:

$$\widehat{X}_0 = [0, A] \times \{0\}, \quad \widehat{Y}_0 = \{0\} \times [0, 1], \quad \widehat{Y}_1 = \{A\} \times [0, 1].$$

For the remaining side $[0, A] \times \{1\}$ we choose a partition into $4k - 3$ closed intervals $\gamma_1, \ldots, \gamma_{4k-3}$, ordered from right to left, such that for $i \equiv 0, 2, 3 \bmod 4$ the intervals γ_i have length ε, the interval γ_1 has length A/k and the remaining intervals have length $\frac{A}{k} - 3\varepsilon$. Set

$$X_0 := \widehat{X}_0 \cup \bigcup_{i \equiv 0 \bmod 4} \gamma_i, \quad X_1 := \bigcup_{i \equiv 1 \bmod 4} \gamma_i,$$
$$Y_0 := \widehat{Y}_0 \cup \bigcup_{i \equiv 2 \bmod 4} \gamma_i, \quad Y_1 := \widehat{Y}_1 \cup \bigcup_{i \equiv 0 \bmod 4} \gamma_i.$$

Now choose a piecewise-linear function $G_\varepsilon\colon Q_\varepsilon(A) \to [0, 1]$ that satisfies:

(i) $G_\varepsilon|_{Y_0} = 0$, $G_\varepsilon|_{Y_1} = 1$,

(ii) G_ε only depends on x in $Q(A)$,

(iii) G_ε has compact support.

One can choose G_ε to satisfy

$$\partial_x G_\varepsilon \leq 1/\left(\tfrac{A}{k} - 3\varepsilon\right)$$

on $\pi(X_1) \times [0, 1] \subset Q(A)$; see for example Figure 5. Therefore there exists a smooth approximation G that satisfies (i), (ii), and (iii) and has slope

$$\partial_x G < 1/\left(\tfrac{A}{k} - 4\varepsilon\right)$$

on $\pi(X_1) \times [0, 1]$. This implies that all chords of G from X_0 to X_1 have time-length $T > \frac{A}{k} - 4\varepsilon$. Since $\varepsilon > 0$ can be chosen arbitrarily small, by the dynamical characterization of bp_L in Section 3.2, we have

$$1/pb_4^+(X_0, X_1, Y_0, Y_1) \geq \tfrac{A}{k}.$$

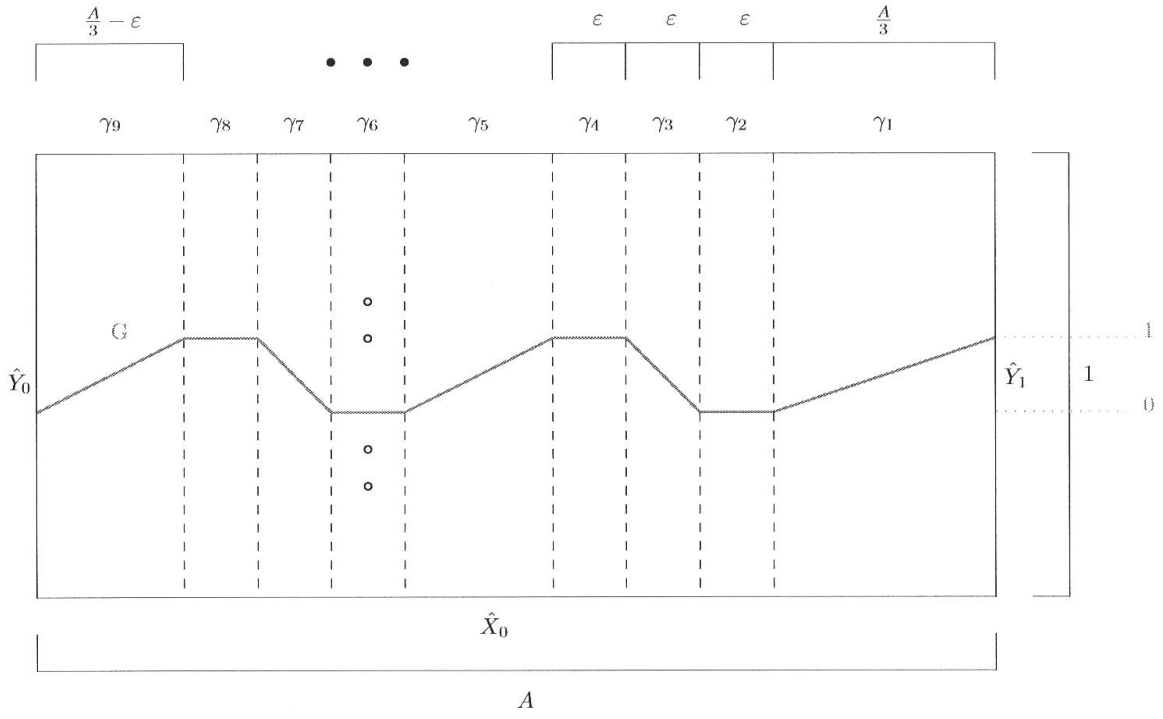

Figure 5. Partition and function in the case $k = 3$.

We now construct a specific neighborhood of $L \subset M$. We first treat the case where L is a boundary component of M_+. Note that the boundary orientation of L corresponds to $+e$.

For a small $\delta > 0$ choose $A = A_+ - \delta$. Let $D_1, \ldots, D_n \subset \operatorname{Int} Q(A)$ denote a non-intersecting (possibly empty) finite union of closed disks of total symplectic area less than δ. Then for a sufficiently small $\varepsilon > 0$ and for an appropriate choice of the disks D_i there exists an open neighborhood $U_+ \subset \operatorname{Int} M$ of L that can be symplectically identified with $Q_\varepsilon(A) \backslash D_1 \cup \cdots \cup D_n$ and so that L (with the

orientation corresponding to $+e$) gets mapped arbitrarily close to $\partial Q(A)$ with the standard boundary orientation. If $\delta > 0$ is chosen small enough, one can choose the disks D_i to lie in the region $\pi(Y_0) \times [0,1] \subset Q(A)$. Thus we can extend the function G from U_+, identified with $Q_\varepsilon(A) \setminus D_1 \cup \cdots \cup D_n$, to the whole symplectic manifold M by 0. Then all chords of G from X_0 to X_1 have time-length $T > \frac{A}{k} - 4\varepsilon$. Since $\varepsilon, \delta > 0$ can be chosen arbitrarily small, together with the semi-continuity and symplectic invariance properties of pb_4^+ (see Section 3.4) this implies that

$$bp_L(ke) \geq A_+/k.$$

When L with the orientation $-e$ is a boundary component of M_- a similar construction gives us the lower bound

$$bp_L(-ke) \geq A_-/k.$$

This completes the proof. □

Proof of Theorem 2.3. The case $k = \pm 1$ was shown in [33]. We generalize this construction to arbitrary $k \in \mathbb{Z}$.

By Remark 3.7, we see that an admissible quadruple associated to ke is given by dividing $L \simeq \mathbb{S}^1$ into $4k$ consecutive closed arcs and labeling them with X_0, Y_1, X_1, X_1 following the orientation of L in the case $k > 0$ and else following the opposite orientation in the case $k < 0$. Note that, by Proposition 3.3, $bp_L(ke)$ does not depend on the choice of subdivision as long as the order of the $4k$ arcs is preserved.

For $\varepsilon > 0$ consider the symplectic annulus,

$$\big(Z_\varepsilon = (-\varepsilon, \varepsilon) \times [-1,1]/\sim,\ dx \wedge dy\big),$$

where we identify $(x,-1) \sim (x,1)$. Partition the subset $(-\varepsilon,\varepsilon) \times \{0\}$ into $4k+1$ consecutive intervals $\gamma_1, \ldots, \gamma_{4k+1}$ respecting the standard orientation on $(-\varepsilon,\varepsilon)$. Set

$$\widehat{X}_0 := \bigcup_{i \equiv 1 \bmod 4} \gamma_i, \quad \widehat{X}_1 := \bigcup_{i \equiv 3 \bmod 4} \gamma_i,$$
$$\widehat{Y}_0 := \bigcup_{i \equiv 0 \bmod 4} \gamma_i, \quad \widehat{Y}_1 := \bigcup_{i \equiv 2 \bmod 4} \gamma_i.$$

One can easily construct functions $F, G \colon Z_\varepsilon \to [0,1]$ such that $(F,G) \in \mathcal{F}'(\widehat{X}_0, \widehat{X}_1, \widehat{Y}_0, \widehat{Y}_1)$ and $\{F,G\} \equiv 0$, namely choose F and G to only depend on the x-coordinate.

The Lagrangian $L \subset M$ is not a separating curve. Thus there exists an embedded loop $\gamma \subset M$ that intersects L exactly once. Furthermore, one can find a tubular neighborhood N of γ that is symplectomorphic to Z_ε for $\varepsilon > 0$ small enough. The symplectomorphism can be chosen to map γ to $\{0\} \times [-1,1]$ and $L \cap N$ to $(-\varepsilon,\varepsilon) \times \{0\}$.

We now make a specific choice of admissible quadruple X_0, X_1, Y_0, Y_1 on L. Namely choose $X_0 \subset L$ such that $L \setminus N \subset X_0$ and

$$X_0 \cap N = \widehat{X}_0, \quad X_1 \cap N = \widehat{X}_1, \quad Y_0 \cap N = \widehat{Y}_0, \quad \text{and} \quad Y_1 \cap N = \widehat{Y}_1$$

(or, if the order requires, use the reflection of $\widehat{X}_0, \widehat{X}_1, \widehat{Y}_0, \widehat{Y}_1 \subset Z_\varepsilon$ along the y-axis). Since F, G have compact support in Z_ε, we may pull back the functions to M and obtain functions on M that satisfy (3.1), have compact support and Poisson-commute. This proves the theorem. □

A lower bound on bp_L for orbifolds. The proof of the lower bound for $k \in \mathbb{N}$,

$$A_\pm / k \leq bp_L(\pm ke), \tag{4.1}$$

from the proof of Theorem 2.2 can also be modified for the orbifold case.

Indeed, the singular points of a 2-dimensional symplectic orbifold M form a discrete subset of M [18] and one can assume without loss of generality that they all lie in $M \setminus U_+$ where the function G vanishes identically (and thus can be extended over the singularities).

Symplectic reduction and bp_L. Let (M^{2n}, ω) be a connected, not necessarily closed, symplectic manifold. Let $\Sigma^{n+1} \subset (M, \omega)$ be a smooth coisotropic submanifold. We do not assume that Σ is a closed subset of M. Assume that the characteristic foliation of Σ defines a *proper* fibration $\pi\colon \Sigma \to (N, \eta)$ over a (not necessarily closed) 2-dimensional symplectic orbifold (N^2, η). Let $L \subset (M, \omega)$ be a Lagrangian torus lying in Σ and s an isotopy class of diffeomorphisms $\mathbb{T}^n \to L$. Assume that $\pi|_L\colon L \to \Gamma$ is a fiber bundle over a simple closed curve $\Gamma := \pi(L)$ lying in the non-singular part of N and dividing N into two domains of areas A_+ and A_-.

Equip Γ with an orientation induced by the orientation of the domain of area A_+. The orientation of Γ defines a positive generator $e \in H^1(\Gamma) \cong \mathbb{Z}$.

Assume that $k \in \mathbb{N}$ and $\alpha \in H^1(L)$ is a primitive class. Assume that there exists an orientation-preserving diffeomorphism $\mathbb{S}^1 \to \Gamma$ that identifies the fiber bundle $\pi|_L\colon L \to \Gamma$ with a fibration $f_\alpha\colon L \to \mathbb{S}^1$ associated to s and α; see Remark 3.7. Then an admissible quadruple associated to a fibration $f_{k\alpha}$ can be described in terms of f_α as follows (see Remark 3.7).

Divide $\mathbb{S}^1 \cong \Gamma$ in $4k$ consecutive closed arcs $\gamma_1, \ldots, \gamma_{4k}$ numbered in the counterclockwise order. Set

$$X_0 := \bigcup_{i \equiv 1 \bmod 4} \gamma_i, \quad X_1 := \bigcup_{i \equiv 3 \bmod 4} \gamma_i,$$
$$Y_0 := \bigcup_{i \equiv 0 \bmod 4} \gamma_i, \quad Y_1 := \bigcup_{i \equiv 2 \bmod 4} \gamma_i,$$

and furthermore

$$\widehat{X}_0 := \pi^{-1}(X_0) \cap L = f_\alpha^{-1}(X_0), \quad \widehat{X}_1 := \pi^{-1}(X_1) \cap L = f_\alpha^{-1}(X_1),$$
$$\widehat{Y}_0 := \pi^{-1}(Y_0) \cap L = f_\alpha^{-1}(Y_0), \quad \widehat{Y}_1 := \pi^{-1}(Y_1) \cap L = f_\alpha^{-1}(Y_1).$$

By Remark 3.7, $(\widehat{X}_0, \widehat{X}_1, \widehat{Y}_0, \widehat{Y}_1)$ is an admissible quadruple associated to $f_{k\alpha}$.

Combining this observation with (3.7) and (3.10) (in the orbifold case), we immediately get:

$$bp_L^s(k\alpha) = 1/pb_4^{M,+}\big(\widehat{X}_0, \widehat{X}_1, \widehat{Y}_0, \widehat{Y}_1\big) \geq 1/pb_4^{N,+}(X_0, X_1, Y_0, Y_1) = bp_\Gamma(ke).$$

In view of

$$bp_L^s(-k\alpha) = 1/pb_4^{M,+}\big(\widehat{X}_0, \widehat{X}_1, \widehat{Y}_1, \widehat{Y}_0\big)$$

(see Remark 3.7) and (4.1) (in the orbifold case) this yields the following claim.

Proposition 4.1. *With the setup as above, for any $k \in \mathbb{N}$ we have*

$$bp_L^s(k\alpha) \geq A_+/k \quad \textit{and} \quad bp_L^s(-k\alpha) \geq A_-/k. \qquad \square$$

5. Toric orbits in symplectic toric manifolds: proofs

Proof of Theorem 2.4. Set

$$k\alpha := (m_1, \ldots, m_n) \in (\mathbb{Z}^n)^* = H^1(\mathbb{T}^n),$$

where α is a primitive class and $k \in \mathbb{N}$.

Complete α to an integral basis $\alpha_1, \ldots, \alpha_{n-1}, \alpha$ of $H^1(\mathbb{T}^n) = (\mathbb{Z}^n)^*$ and let $\beta_1, \ldots, \beta_n$ be the dual integral basis of $\mathbb{Z}^n$. Thus, we have a splitting

$$Lie^*\,\mathbb{T}^n = \mathrm{Span}_{\mathbb{R}}\{\alpha_1, \ldots, \alpha_{n-1}\} \oplus \mathrm{Span}_{\mathbb{R}}\{\alpha\}.$$

Let $\pi_1 : Lie^*\,\mathbb{T}^n \to \mathrm{Span}_{\mathbb{R}}\{\alpha_1, \ldots, \alpha_{n-1}\}$ and $\pi_2 : Lie^*\,\mathbb{T}^n \to \mathrm{Span}_{\mathbb{R}}\{\alpha\}$ be the projections defined by the splitting.

Consider the $(n-1)$-dimensional subtorus $H \subset \mathbb{T}^n$ whose Lie algebra is

$$Lie\,H := \mathrm{Ker}\,\alpha = \mathrm{Span}_{\mathbb{R}}\{\beta_1, \ldots, \beta_{n-1}\} \subset \mathbb{R}^n = Lie\,\mathbb{T}^n.$$

The map $Lie^*\,\mathbb{T}^n \to Lie^*\,H$ dual to the inclusion $Lie\,H \to Lie\,\mathbb{T}^n$ can be identified with the projection π_1.

Since H is a subtorus of $\mathbb{T}^n$, there is a Hamiltonian action of H on (M, ω) whose moment map Φ_H can be described as the composition of Φ and π_1. Thus, $L_{\mathbf{x}} = \Phi^{-1}(\mathbf{x})$ lies in a fiber $\Sigma'_{\mathbf{x}}$ of Φ_H which is the union of the fibers $\Phi^{-1}(\mathbf{y})$ for all $\mathbf{y} \in \Delta$ such that $\pi_1(\mathbf{y}) = \pi_1(\mathbf{x})$. The set $\Sigma'_{\mathbf{x}}$ may be an orbifold, but its

smooth part $\Sigma_{\mathbf{x}} \subset \Sigma'_{\mathbf{x}}$, which is the union of the fibers $\Phi^{-1}(\mathbf{y})$ for $\mathbf{y} \in \operatorname{Int}\Delta$ with $\pi_1(\mathbf{y}) = \pi_1(\mathbf{x})$, is a smooth coisotropic $(n+1)$-dimensional submanifold of M.

The torus H acts on $\Sigma_{\mathbf{x}}$ and the reduced space $\Sigma_{\mathbf{x}}/H$ is a 2-dimensional symplectic orbifold $N_{\mathbf{x}}$. The natural projection $\pi_H\colon \Sigma_{\mathbf{x}} \to \Sigma_{\mathbf{x}}/H$ is proper and its fibers are exactly the leaves of the characteristic foliation of $\Sigma_{\mathbf{x}}$. The 1-dimensional torus $\mathbb{T}^n/H$ acts in a Hamiltonian way on $N_{\mathbf{x}}$. In fact, the torus $\mathbb{T}^n/H$ can be identified with a subtorus of $\mathbb{T}^n$ whose Lie algebra is $\operatorname{Span}_{\mathbb{R}}\{\beta_n\} \subset \mathit{Lie}\,\mathbb{T}^n$. Accordingly, $\mathit{Lie}^*(\mathbb{T}^n/H)$ is identified with $\operatorname{Span}_{\mathbb{R}}\{\alpha\} \subset \mathit{Lie}^*\,\mathbb{T}^n$. Thus, the moment map $\Phi_{\mathbb{T}^n/H}$ of the $\mathbb{T}^n/H$-action on $N_{\mathbf{x}}$ can be viewed as a map

$$\Phi_{\mathbb{T}^n/H}\colon N_{\mathbf{x}} \to \operatorname{Span}_{\mathbb{R}}\{\alpha\}.$$

A well-known property of Hamiltonian group actions (the so-called "reduction in stages", see e.g. [3, Exercise III.12]) implies that the orbits of the original $\mathbb{T}^n$-action on (M,ω) lying in $\Sigma_{\mathbf{x}}$ project under $\pi_H\colon \Sigma_{\mathbf{x}} \to N_{\mathbf{x}}$ to the orbits of the $\mathbb{T}^n/H$-action on $N_{\mathbf{x}}$ and for any $\mathbf{y} \in \mathit{Lie}^*\,\mathbb{T}^n \cap \Delta$ such that $\pi_1(\mathbf{x}) = \pi_1(\mathbf{y})$ we have

$$\pi_2 \circ \Phi = \Phi_{\mathbb{T}^n/H} \circ \pi_H \quad \text{on } \Phi^{-1}(\mathbf{y}).$$

Therefore the image of the moment map $\Phi_{\mathbb{T}^n/H}$ can be identified with the image under π_2 of the intersection of the line $\mathbf{x} - t\alpha$, $t \in \mathbb{R}$, in the affine space $(\mathbb{R}^n)^* = \mathit{Lie}^*\,\mathbb{T}^n$ with $\Delta = \operatorname{Im}\Phi$.

Another conclusion is that the torus $L_{\mathbf{x}} \subset \Sigma_{\mathbf{x}}$, which is an orbit of the $\mathbb{T}^n$-action on (M,ω), projects under $\pi_H\colon \Sigma_{\mathbf{x}} \to \Sigma_{\mathbf{x}}/H$ to a simple closed curve $\Gamma \subset N_{\mathbf{x}}$ that lies in the non-singular part of the orbifold $N_{\mathbf{x}}$. The curve Γ is an orbit of $\mathbb{T}^n/H$ and as such can be identified with $\mathbb{S}^1$ — this identification is unique up to a rotation of $\mathbb{S}^1$. Moreover, under this identification the map

$$\pi_H|_{L_{\mathbf{x}}}\colon L_{\mathbf{x}} \to \Gamma$$

becomes a fibration $f_\alpha\colon L_{\mathbf{x}} \to \mathbb{S}^1$ associated to α; see Remark 3.7. (Indeed, f_α can be viewed as the projection $L_{\mathbf{x}} \to L_{\mathbf{x}}/H = \mathbb{S}^1$).

The symplectic properties of Γ inside $N_{\mathbf{x}}$ are completely determined by the relative position of the point $\pi_2(\mathbf{x}) = \Phi_{\mathbb{T}^n/H}(\Gamma)$ in the image of $\Phi_{\mathbb{T}^n/H}$ or, equivalently, by the position of $\mathbf{x}$ in the intersection of the line $\mathbf{x} - t\alpha$, $t \in \mathbb{R}$, with Δ. Recall that $\mathcal{I}(\mathbf{x},\alpha)$ is defined as the open part of the intersection of the ray $\mathbf{x} - t\alpha$, $0 < t < +\infty$, with Δ. By a basic version of the Delzant theorem [9], the oriented curve Γ (it is oriented as an orbit of $\mathbb{T}^n/H$) is the oriented boundary of a domain D in $N_{\mathbf{x}}$ whose area is the rational length of $\mathcal{I}(\mathbf{x},\alpha)$, if $\mathcal{I}(\mathbf{x},\alpha)$ is an interval, or $+\infty$, if $\mathcal{I}(\mathbf{x},\alpha)$ is an infinite ray.

Thus, we can apply Proposition 4.1, which implies that

$$bp_{\mathbf{x}}(k\alpha) \geq l_{\mathbf{x}}(\alpha)/k$$

for $k \in \mathbb{N}$. □

6. Lagrangian tori in $\mathbb{C}^2$: proofs

Proof of Theorem 2.6. The original idea of the proof below belongs to E. Opshtein.

Consider a Lagrangian isotopy

$$\psi = \{\psi_t : L \to \mathbb{C}^2\}, \quad 0 \le t \le T, \ \psi_0 = \iota,$$

such that

$$[\omega]_t^{\psi} = [\omega]_L - t\,\partial\alpha,$$

with $L \subset \mathbb{C}^2$ and α as in the statement of the theorem.

For simplicity let us consider a dual picture. Namely, for $0 \le t \le T$ there exists a family of compactly supported diffeomorphisms $\varphi_t : \mathbb{C}^2 \to \mathbb{C}^2$, $\varphi_0 = \mathrm{Id}$, such that $\varphi_t(L) = \psi_t(L)$. By pulling back our symplectic form $\varphi_t^*\omega =: \omega_t$ we may consider a fixed Lagrangian $L \subset (\mathbb{C}^2, \omega_t)$. We have $H_2(\mathbb{C}^2, L) \simeq \mathbb{Z}\langle A, B\rangle$ and

$$\omega_t(A) = \omega(A) - \sigma t \quad \text{and} \quad \omega_t(B) = \omega(B) - \rho t.$$

Let $\mathcal{J}_t$ be the space of almost complex structures on $\mathbb{C}^2$ compatible with the symplectic form ω_t. Given an almost complex structure $J \in \mathcal{J}_t$, by a *(parameterized) J-holomorphic disk* we always mean a smooth J-holomorphic map $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^2, L)$ (here $\mathbb{D} \subset \mathbb{C}$ is the standard closed unit disk). By the homology class of a J-holomorphic disk u we always mean the relative homology class $u_*([\mathbb{D}]) \in H_2(\mathbb{C}^2, L)$. Such a u is called *somewhere injective*, if $du(z) \neq 0$ for some $z \in \mathbb{D}$ such that $u^{-1}(u(z)) = \{z\}$.

Given a relative homology class $C \in H_2(\mathbb{C}^2, L)$ define $\widetilde{\mathcal{M}}(C, J)$ as the moduli space of somewhere injective (parametrized) J-holomorphic disks in the class C. Let $\widetilde{\mathcal{M}}(C, J) \times \partial\mathbb{D}$ be the moduli space of J-holomorphic disks with one marked point on the boundary. $PSL_2(\mathbb{R})$ is the group of biholomorphisms of $\mathbb{D}$ and we consider the quotient

$$\mathcal{M}_1(C, J) := \big(\widetilde{\mathcal{M}}(C, J) \times \partial\mathbb{D}\big)/PSL_2(\mathbb{R}),$$

where the action of $PSL_2(\mathbb{R})$ is defined as $g \cdot (u, x) = (u \circ g, g^{-1}(x))$. This space comes with an evaluation map

$$ev : \mathcal{M}_1(C, J) \to L$$

given by $ev([u, x]) := u(x)$ and for a chosen point $p \in L$ we define $\mathcal{M}_1(C, J, p) := ev^{-1}(p)$.

Given a family $\{J_t\}$, $0 \le t \le T$, of almost complex structures $J_t \in \mathcal{J}_t$ and a class $C \in H_2(\mathbb{C}^2, L)$, define $\mathcal{M}_1(C, \{J_t\})$ as the set of pairs (t, D), where $0 \le t \le T$ and $D \in \mathcal{M}_1(C, J_t)$. The set $\mathcal{M}_1(C, \{J_t\}, p) \subset \mathcal{M}_1(C, \{J_t\})$ is defined analogously with $D \in \mathcal{M}_1(C, J_t, p)$.

We will show that for A as in the assumption of the theorem and for a generic family $\{J_t\}$ the set $\mathcal{M}_1(A, \{J_t\}, p)$ is a smooth compact manifold of dimension 1.

We recall first some general facts about the moduli spaces.

We say that $J \in \mathcal{J}_t$, $t \in [0, T]$, is *regular*, if for all $C \in H_2(\mathbb{C}^2, L)$ the space $\mathcal{M}_1(C, J)$ is a (transversally cut out) smooth manifold of dimension

$$\dim \mathcal{M}_1(C, J) = \dim L + \mu(C) + 1 - \dim PSL_2(\mathbb{R}) = \mu(C).$$

An almost complex structure $J \in \mathcal{J}_t$ is called *regular with respect to* $p \in L$, if it is regular and, in addition, $\mathcal{M}_1(C, J, p)$ is a (transversally cut out) smooth manifold of dimension

$$\dim \mathcal{M}_1(C, J, p) = \dim \mathcal{M}_1(C, J) - \dim L = \mu(C) - 2.$$

We will say that a family $\{J_t\}$, $0 \leq t \leq T$, $J_t \in \mathcal{J}_t$, is *regular with respect to* p if

(1) for any $t \in [0, T]$ the spaces $\mathcal{M}_1(C, J_t)$ are empty for all C with $\mu(C) < 0$,

(2) $\mathcal{M}_1(C, \{J_t\}, p)$ is a (transversally cut out) smooth manifold of dimension

$$\dim \mathcal{M}_1(C, \{J_t\}, p) = \dim \mathcal{M}_1(C, J, p) + 1 = \mu(C) - 1$$

with boundary $\mathcal{M}_1(C, J_0, p) \cup \mathcal{M}_1(C, J_T, p)$.

Similarly, given a (regular) path $\gamma(s)$, $0 \leq s \leq 1$, in L, and $t \in [0, T]$, we say that a family $\{J_s\}_{0 \leq s \leq 1} \subset \mathcal{J}_t$ is *regular with respect to* γ, if

(1′) for any $s \in [0, 1]$ the spaces $\mathcal{M}_1(C, J_s)$ are empty for all C with $\mu(C) < 0$,

(2′) $\mathcal{M}_1(C, \{J_s\}, \gamma) := \cup_{0 \leq s \leq 1} \mathcal{M}_1(C, J_s, \gamma(s))$ is a smooth manifold of dimension $\mu(C) - 1$ with boundary $\mathcal{M}_1(C, J_0, \gamma(0)) \cup \mathcal{M}_1(C, J_1, \gamma(1))$.

It follows from standard regularity and transversality arguments (see e.g. [29,31], where the arguments are explained in detail for pseudo-holomorphic curves without boundary) that for any $p \in L$

- a generic $J \in \mathcal{J}_t$ is regular and moreover regular with respect to p,
- for any $p \in L$ and any $J_0 \in \mathcal{J}_0$, $J_T \in \mathcal{J}_T$ that are regular with respect to p, a generic family $\{J_t\}$, $0 \leq t \leq T$, $J_t \in \mathcal{J}_t$, connecting J_0 and J_T satisfies (2),
- for any γ as above, any $t \in [0, T]$ and any $J_0, J_1 \in \mathcal{J}_t$ regular, respectively, with respect to $\gamma(0)$ and $\gamma(1)$, a generic family $\{J_s\}_{0 \leq s \leq 1} \subset \mathcal{J}_t$ connecting J_0, J_1 satisfies (2′).

In order to show that condition (1) also holds for a generic family $\{J_t\}$ note that L is orientable and, accordingly, the Maslov index of *any* disk with boundary on L is even. Thus, if $\mu(C)$ is negative, then virtual $\dim \mathcal{M}_1(C, J) \leq -2$ for any $J \in \mathcal{J}_t$ and $t \in [0, T]$, meaning that the existence of *somewhere injective* J_t-holomorphic disks of negative Maslov index is a codimension-2 phenomenon and can be avoided

by choosing a generic 1-parametric family $\{J_t\}$. This shows that (1) holds for a generic family $\{J_t\}$. Similarly, one can show that (1′) holds for a generic family $\{J_s\}$ as above.

In fact, we claim that there are no J_t-holomorphic disks of negative Maslov index, somewhere injective or not. Indeed, let $\{J_t\}$ satisfy (1). By a result of Kwon–Oh [22] (cf. [26, 27]), any non-parameterized J_t-holomorphic disk in the class C with boundary in L, viewed as a subset of $\mathbb{C}^2$, is a finite union of non-parameterized somewhere injective J_t-holomorphic disks $\mathcal{D}_1, \ldots, \mathcal{D}_j$ and $C = k_1[\mathcal{D}_1] + \cdots + k_j[\mathcal{D}_j]$, where for each i $[\mathcal{D}_i]$ is the relative homology class of $\mathcal{D}_i$ and $k_i \in \mathbb{N}$. If $k_1\mu(\mathcal{D}_1) + \cdots + k_j\mu(\mathcal{D}_j) = \mu(C) < 0$, then $\mu(\mathcal{D}_i) < 0$ for some $i = 1, \ldots, j$, in contradiction to the non-existence of somewhere injective J_t-holomorphic disks of negative Maslov index, which proves the claim.

Lemma 6.1. *For any (possibly not even regular) $J \in \mathcal{J}_t$, $0 \leq t \leq T$, there are no non-constant J-holomorphic Maslov-0 disks of area less than $\omega_t(A)$, somewhere injective or not (recall that $A \in H_2(\mathbb{C}^2, L)$ is the class appearing in the hypothesis of the theorem).*

Proof of Lemma 6.1. Indeed, assume by contradiction that such a disk exists and denote its relative homology class by C. By the hypothesis of the theorem,

$$\omega(B) \geq (1+k)\omega(A) \quad \text{and} \quad (1+k)\sigma \geq \rho,$$

so that

$$\omega_t(B - kA) - \omega_t(A) = \omega(B) - (1+k)\omega(A) + ((1+k)\sigma - \rho)t \geq 0.$$

Moreover, $\omega_t(C) > 0$, since C is non-constant and J is compatible with ω_t. Thus

$$\omega_t(B - kA) \geq \omega_t(A) > \omega_t(C) > 0. \tag{6.1}$$

On the other hand, since $\mu(C) = 0$, by the hypothesis of the theorem, C is an integral multiple of $B - kA$, and therefore $\omega_t(C)$ has to be an integral multiple of $\omega_t(B - kA)$, which is impossible by (6.1). This proves the lemma. □

Continuing with the proof of the theorem, we now show that for any almost complex structures $J_0 \in \mathcal{J}_0$, $J_T \in \mathcal{J}_T$ that are regular with respect to $p \in L$ and any family $\{J_t\}$, $0 \leq t \leq T$, $J_t \in \mathcal{J}_t$, that connects J_0 and J_T and is regular with respect to p, the moduli space $\mathcal{M}_1(A, \{J_t\}, p)$ is compact and hence, in particular, so are $\mathcal{M}_1(A, J_0, p)$ and $\mathcal{M}_1(A, J_T, p)$.

It suffices to show that any sequence $\{D_i\}$ in $\mathcal{M}_1(A, J_{t_i}, p)$ with $\{t_i\} \to s$ has a subsequence converging to an element of $\mathcal{M}_1(A, J_s, p)$. To prove this claim note that, by Gromov compactness (see [16]), since $\pi_2(\mathbb{C}^2) = 0$, the sequence $\{D_i\}$ has a subsequence converging to a bubbling configuration of non-constant non-parameterized J_s-holomorphic disks $\mathcal{D}_1, \ldots, \mathcal{D}_l$ with boundary in L

whose homology classes $[\mathcal{D}_1], \ldots, [\mathcal{D}_l]$ add up to A:

$$[\mathcal{D}_1] + \cdots + [\mathcal{D}_l] = A.$$

Thus, $0 < \omega_s([D_i]) \leq \omega_s(A)$ for all $i = 1, \ldots, l$ and therefore, by Lemma 6.1, none of the disks $\mathcal{D}_i$ has Maslov index 0. Moreover, since the family $\{J_t\}$ is regular with respect to p, none of the disks $\mathcal{D}_i$ has a negative Maslov index. Since $\mu([\mathcal{D}_1]) + \cdots + \mu([\mathcal{D}_l]) = \mu(A) = 2$ and the Maslov indices of the $\mathcal{D}_i$'s are all even, this means that $l = 1$ – that is, there is only one disk in the bubbling configuration and its relative homology class is A.

Let us denote this (non-parameterized) disk by D. By the result of Kwon–Oh [22] (cf. [26, 27]) mentioned above, the non-parameterized disk D, viewed as a subset of $\mathbb{C}^2$, is a finite union of non-parameterized somewhere injective disks $D^{(1)}, \ldots, D^{(m)}$ and A is a linear combination with positive integral coefficients of the homology classes of $D^{(1)}, \ldots, D^{(m)}$. The ω_s-areas of $D^{(1)}, \ldots, D^{(m)}$ are all positive numbers smaller than $\omega_s(A)$. Therefore, arguing as above, we get that all $D^{(1)}, \ldots, D^{(m)}$ must have positive even Maslov indices, meaning that $m = 1$ and $D = D^{(1)} \in \mathcal{M}_1(A, J_s, p)$. Hence the sequence $\{D_i\}$ has a subsequence converging to an element of $\mathcal{M}_1(A, J_s, p)$. This finishes the proof that the smooth manifold $\mathcal{M}_1(A, \{J_t\}, p)$ is compact.

Thus the moduli space $\mathcal{M}_1(A, \{J_t\}, p)$ is a compact smooth 1-dimensional cobordism between the compact 0-dimensional manifolds $\mathcal{M}_1(A, J_0, p)$ and $\mathcal{M}_1(A, J_T, p)$.

In a similar way one can show that given a (regular) path $\gamma(s)$, $0 \leq s \leq 1$, in L, $t \in [0, T]$ and a family $\{J_s\} \subset \mathcal{J}_t$ regular with respect to γ, the moduli space $\mathcal{M}_1(A, \{J_s\}, \gamma)$ is a compact smooth 1-dimensional cobordism between the compact 0-dimensional manifolds $\mathcal{M}_1(A, J_0, \gamma(0))$ and $\mathcal{M}_1(A, J_1, \gamma(1))$.

Let us summarize: for any $t \in [0, T]$ and any $J \in \mathcal{J}_t$ regular with respect to p the moduli space $\mathcal{M}_1(A, J, p)$ is a compact 0-dimensional manifold. The number $n_A(p, J) := \#\mathcal{M}_1(A, J, p) \bmod 2$ — that is, the mod-2 number of non-parameterized J-holomorphic disks with one marked point that represent A and pass through p — is independent of p and J. Indeed, for a different $p' \in L$ and an almost complex structure $J' \in \mathcal{J}_t$ regular with respect to p' the manifolds $\mathcal{M}_1(A, J, p)$ and $\mathcal{M}_1(A, J', p')$ are cobordant and therefore $n_A(p, J) = n_A(p', J')$. This proves part (A) of the theorem.

Moreover, for any $J_0 \in \mathcal{J}_0$ and $J_T \in \mathcal{J}_T$ regular with respect to p we have $n_A(p, J_0) = n_A(p, J_T)$, since $\mathcal{M}_1(A, J_0, p)$ and $\mathcal{M}_1(A, J_T, p)$ are cobordant compact 0-dimensional manifolds.

In view of the above, if $n_A(p, J_0)$ is non-zero, then so is $n_A(p, J_T)$ for $J_0 \in \mathcal{J}_0$ and $J_T \in \mathcal{J}_T$ regular with respect to p. In particular, there exists a J_T-holomorphic disk in $\mathbb{C}^2$ with boundary in L in the relative homology class A and therefore

$$\omega_T(A) = \omega(A) - \sigma T > 0,$$

and hence

$$T < \omega(A)/\sigma.$$

Since this holds for any Lagrangian isotopy $\{\psi_t\}_{0\le t\le T}$ as above, we get that $def_L(\alpha)$, which is the supremum of such T, is less or equal to $\omega(A)/\sigma$:

$$def_L(\alpha) \le \omega(A)/\sigma.$$

This proves part (B) of the theorem. □

6.1. Proof of Theorem 2.10. We calculate lower bounds for the *bp*-invariants of Chekanov tori Θ_a. The proofs of both parts rely on an increasing sequence of neighborhoods of Θ_a in $\mathbb{C}^2$. We first explain this construction.

For $r \in \mathbb{R}_{>0}$ let $D(r) \subset \mathbb{C}$ denote the standard open symplectic disk of area r and $T^*_r\mathbb{S}^1 \subset T^*\mathbb{S}^1$ be the subset of covectors of norm $< r$ (here we choose the flat metric on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$). Let λ be the standard Liouville form on $T^*\mathbb{S}^1$.

Proposition 6.2. *For every $r > a$ there exists a neighborhood $U(r) \subset \mathbb{C}^2$ of Θ_a such that $U(r)$ is symplectomorphic to*

$$\big(D(r) \times T^*_r\mathbb{S}^1, dx \wedge dy + d\lambda\big)$$

and Θ_a is identified with $\partial D(a) \times \{0\text{-section}\}$.

Proof. Let η be the simple closed oriented curve in the open first quadrant $Q \subset \mathbb{C}$ used in the construction of the Chekanov torus Θ_a. The curve bounds a disk of area a which is contained in a larger open disk $\mathcal{D} \subset Q$ of area $a + \delta$ for some small $\delta > 0$. The map $\Xi\colon \mathbb{S}^1 \times \mathcal{D} \to \mathbb{C}^2$ given by

$$\Xi\big(e^{2\pi it}, z\big) = \frac{1}{\sqrt{2}}\big(e^{2\pi it}z, e^{-2\pi it}z\big)$$

is an embedding and its image contains Θ_a as well as the Chekanov tori $\Theta_{a'}$ for every $0 < a' \le a$. Denote the image of Ξ by N. We see that Ξ preserves the symplectic structure on $\mathcal{D}$ and that N is a coisotropic submanifold with characteristic foliation generated by the $\mathbb{S}^1$-action. By the neighborhood theorem for coisotropic submanifolds [17], there exists a neighborhood U of N in $\mathbb{C}^2$ that is symplectomorphic to a neighborhood of the zero-section in E^*, where $E \subset TN$ is the characteristic bundle of N. One sees that the characteristic bundle of N is trivial, hence

$$E^* \simeq N \times \mathbb{R} \simeq \mathcal{D} \times \mathbb{S}^1 \times \mathbb{R}.$$

Since the disk $\mathcal{D}$ is symplectic, the neighborhood U is symplectomorphic to a neighborhood V of $D(a + \delta) \times \{0\text{-section}\}$ in $D(a + \delta) \times T^*\mathbb{S}^1$ with its standard split symplectic form $dx \wedge dy + d\lambda$. By choosing a smaller neighborhood U if

necessary, we can assume that U is symplectomorphic to $V = D(a+\delta) \times T^*_\varepsilon \mathbb{S}^1$ for an $\varepsilon > 0$ so that $\Theta_a \subset U$ is identified with $\partial D(a) \times \{0\text{-section}\} \subset V$.

For $c > 0$ the map $\varphi_c \colon \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$(z_1, z_2) \mapsto \sqrt{c}(z_1, z_2)$$

is a conformal symplectomorphism. Recall that the neighborhood U contains all Chekanov tori $\Theta_{a'}$ for $0 < a' \leq a$. Now let $0 < a' < a$ and $c = a/a'$, then the image $\varphi_c(\Theta_{a'})$ is a Lagrangian torus that is Hamiltonian isotopic to Θ_a. The image of the neighborhood, $\varphi_c(U)$, is then a neighborhood of $\varphi_c(\Theta_{a'})$. The neighborhood $\varphi_c(U)$ is symplectomorphic to $V' = D(c(a+\delta)) \times T^*_{c\varepsilon}\mathbb{S}^1$ (recall that $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and $\lambda = p\, d\theta$ is the standard Liouville form).

Let us now construct the wanted neighborhoods $U(r)$ for $r > a$. For a given $r > a$ we choose $a' > 0$ so that for $c = a/a'$ we have $c \int dx \wedge dy(D) > r$ and $c\,\varepsilon > r$. Then $D(r) \times T^*_r \mathbb{S}^1 \subset V'$ and this gives us a neighborhood $U(r)$. Since $a' > 0$ can be chosen arbitrarily small, this provides us with a neighborhood $U(r)$ for all $r > a$. The statement follows from this. □

(A) We return to the proof of part (A). For the lower bound $bp_a(m,n) \geq a/m$, following the dynamical characterization of bp_L in Section 3.2, we use a neighborhood that we obtain from Proposition 6.2 to construct for any $\varepsilon > 0$ a complete Hamiltonian H that has no chords from X_0 to X_1 of time-length $< \frac{a}{m} - \varepsilon$.

For $m \geq 1$ and $n \in \mathbb{Z}$ fixed we choose $k \in \mathbb{N}$ such that $k > a$ and $k > C := 4a|n|$. Now let $U(k)$ be a neighborhood of Θ_a as in Proposition 6.2 and we identify

$$U(k) \simeq D(k) \times T^*_k \mathbb{S}^1.$$

Recall that Θ_a is mapped to $\partial D(a) \times \{0\text{-section}\}$ under this identification. Furthermore, we identify $\mathbb{C} \simeq \mathbb{R}^2$ and write the coordinates of $D(k) \times T^*_k \mathbb{S}^1$ as (x, y, θ, p).

First we define a partition of $\partial D(a)$. Namely, partition $\mathbb{S}^1$ into $4m$ closed arcs by setting

$$\gamma_j := \left\{ e^{2\pi i \xi} \in \mathbb{S}^1 \;\middle|\; \xi \in \left[\frac{j-1}{4m}, \frac{j}{4m} \right] \right\}, \quad j = 1, \ldots, 4m.$$

Using the identification $\partial D(a) \simeq \mathbb{S}^1$ we consider the γ_j to be arcs of $\partial D(a)$. For $\theta \in \mathbb{R}/\mathbb{Z}$ denote by $R(\theta)\colon \mathbb{R}^2 \to \mathbb{R}^2$ the rotation by angle $2\pi\theta$. In $D(k) \times T_k^*\mathbb{S}^1$ we define the four sets,

$$\begin{aligned}
X_0 &= \bigcup_{j\equiv 1 \bmod 4} \{(R(n\theta)(x,y),\theta,0) \mid (x,y) \in \gamma_j,\ \theta \in \mathbb{R}/\mathbb{Z}\},\\
X_1 &= \bigcup_{j\equiv 3 \bmod 4} \{(R(n\theta)(x,y),\theta,0) \mid (x,y) \in \gamma_j,\ \theta \in \mathbb{R}/\mathbb{Z}\},\\
Y_0 &= \bigcup_{j\equiv 0 \bmod 4} \{(R(n\theta)(x,y),\theta,0) \mid (x,y) \in \gamma_j,\ \theta \in \mathbb{R}/\mathbb{Z}\},\\
Y_1 &= \bigcup_{j\equiv 2 \bmod 4} \{(R(n\theta)(x,y),\theta,0) \mid (x,y) \in \gamma_j,\ \theta \in \mathbb{R}/\mathbb{Z}\}.
\end{aligned} \tag{6.2}$$

A brief calculation reveals that under the identification $U(k) \simeq D(k) \times T_k^*\mathbb{S}^1$ the quadruple is mapped to an admissible quadruple associated to the fibration $f_\alpha\colon \Theta_a \to \mathbb{S}^1$, where $\alpha = m\widehat{\Gamma} - n\widehat{\gamma} \in H^1(\Theta_a)$.

Consider now the sector $\mathcal{S} \subset D(k)$ given by

$$\mathcal{S} := D(k) \cap \left\{ re^{2\pi i\xi} \;\middle|\; r \geq 0,\ \xi \in \left[\frac{-1}{8m}, \frac{7}{8m}\right] \right\}.$$

The intersection $\mathcal{S} \cap D(a)$ has area a/m and $\partial\mathcal{S}$ intersects $\partial D(a)$ in the arcs γ_4 and γ_{4m}. We consider a Hamiltonian $G\colon \mathcal{S} \to \mathbb{R}$ that satisfies the following:

(i) G has compact support in $\mathrm{Int}(\mathcal{S})$,

(ii) $G \equiv 0$ in a neighborhood of $(\gamma_{4m} \cup \gamma_4) \cap \mathcal{S}$,

(iii) $G \equiv 1$ in a neighborhood of γ_2,

(iv) all chords of G from γ_1 to γ_3 have time-length $T > \frac{a}{m} - \varepsilon$,

(v) on $\mathcal{S} \cap D(a)$ the Euclidean norm of the gradient of G is bounded,

$$|(\partial_x G, \partial_y G)| \leq \frac{4m}{\sqrt{a}}.$$

The existence of such a Hamiltonian follows easily. See Figure 6 for an illustration. Namely, one can choose a piecewise-linear function that satisfies all conditions except (iv) and then find a smooth approximation that satisfies all conditions.

Extend G to the whole disk $D(k)$ such that G is periodic under the rotation $R(1/m)$. Since G has support in $\mathrm{Int}(\mathcal{S})$, this extension is smooth and has compact support. By abuse of notation we denote the extension by G. Now choose a non-negative function $\beta\colon T_k^*\mathbb{S}^1 \to \mathbb{R}$ that is constant equal to 1 for $|p| \leq C$ and has compact support in $T_k^*\mathbb{S}^1$.

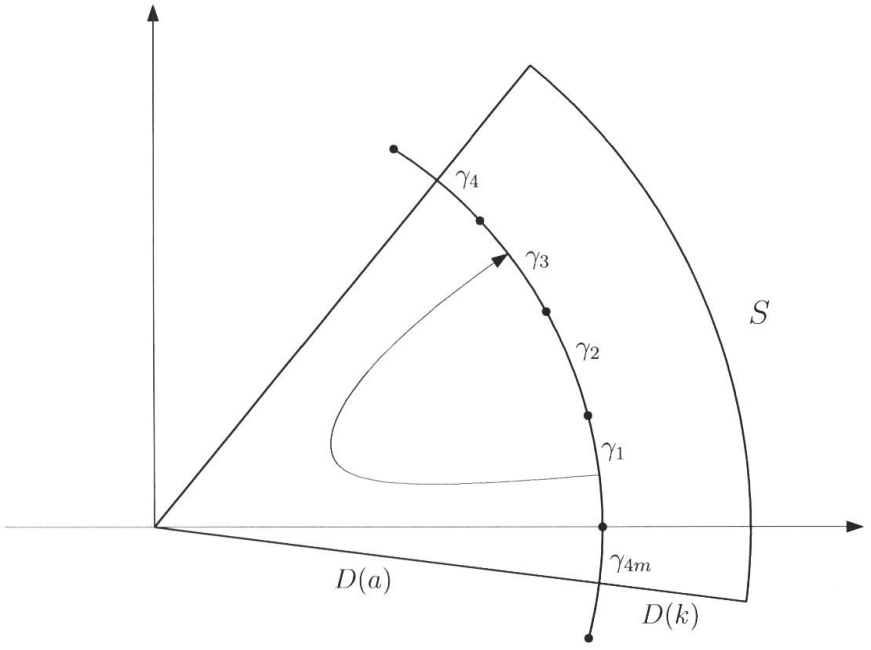

Figure 6. Hamiltonian chord of G from γ_1 to γ_3 in $\mathcal{S}$.

Define the Hamiltonian $H\colon D(k) \times T_k^*\mathbb{S}^1 \to \mathbb{R}$ via

$$H(x, y, \theta, p) = G\big(R(n\theta)(x, y)\big)\beta(p).$$

By construction H is complete and satisfies $H \equiv 0$ on Y_0 and $H \equiv 1$ on Y_1. Since $\beta(p)$ is constant in $\{|p| \leq C\}$, we see that $\partial_p H \equiv 0$ in this region. This implies that the Hamiltonian vector field of H in $\{|p| \leq C\}$ is tangent to the fibers $\{\theta = \text{constant}\}$. Thus all Hamiltonian chords of H starting on $\partial D(a) \times \{0\text{-section}\}$ and contained in $\{|p| \leq C\}$ will project to the fibers.

Now for $\{|p| \leq C\}$ and $(x, y) \in D(a)$ we estimate the Euclidean norm of the differential,

$$|\partial_\theta H| \leq |(\partial_x G, \partial_y G)| \cdot |n| \cdot |(x, y)| \leq \frac{4m|n|}{\sqrt{\pi}}.$$

This implies that all Hamiltonian chords of H starting in $\partial D(a) \times \{0 - \text{section}\}$ remain in the region $\{|p| \leq C\}$ for times $t \in [0, a/m]$ by our choice of C. Therefore any chord of time-length $t \leq a/m$ is contained in the fiber $\{\theta = \text{constant}\}$ and under the projection $D(k) \times T_k^*\mathbb{S}^1 \to D(k)$ these chords project to rotated chords of G. This proves that any chord from X_0 to X_1 must have time-length $T > a/m - \varepsilon$. This completes the proof of part (A).

(B) We prove part (B). We construct a sequence of Hamiltonians that have chords from X_0 to X_1 with time-length increasing to $+\infty$.

We consider the admissible quadruple associated to the fibration $f_\alpha\colon \Theta_a \to \mathbb{S}^1$, where $\alpha = m\widehat{\Gamma} - n\widehat{\gamma}$ with $m < 0$ and $n \in \mathbb{Z}$. Following Remark 3.7 we see

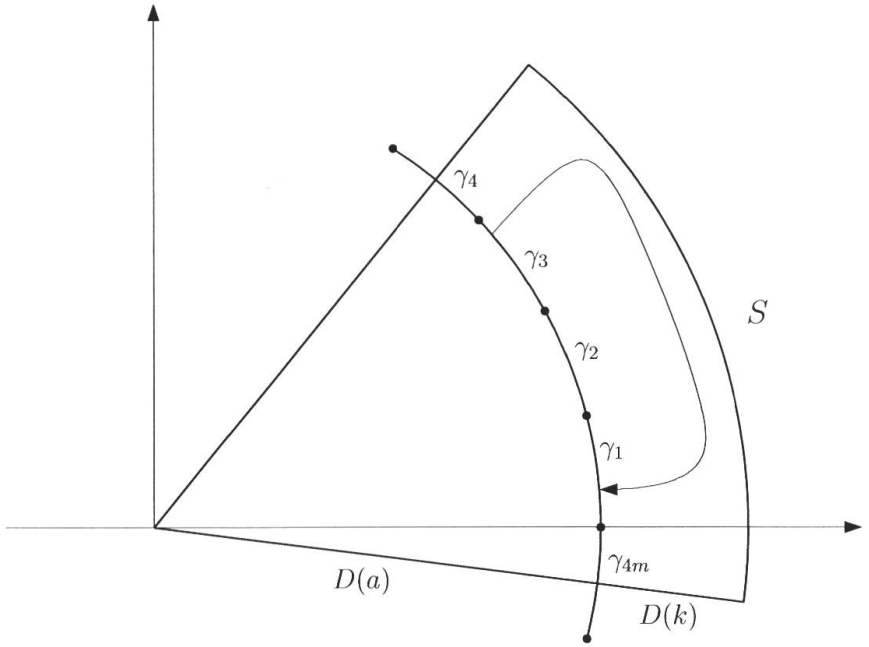

Figure 7. Hamiltonian chord of G from γ_3 to γ_1 in $\mathcal{S}$.

that if (X_0, X_1, Y_0, Y_1) is an admissible quadruple for α, then (X_1, X_0, Y_0, Y_1) is an admissible quadruple for $-\alpha$. Thus we may take the admissible quadruple associated to $-m\widehat{\Gamma} + n\widehat{\gamma}$ as constructed in (6.2) and interchange X_0 and X_1 to obtain an admissible quadruple associated to $m\widehat{\Gamma} - n\widehat{\gamma}$.

Following the construction in part (A), we find a Hamiltonian $G\colon \mathcal{S} \to \mathbb{R}$ that satisfies $G \equiv 0$ on $\gamma_{4m} \cup \gamma_4$ and $G \equiv 1$ on γ_2 and consider its chords from γ_3 to γ_1; see Figure 7. In our sequence of symplectic neighborhoods we have $dx \wedge dy(D(k)/D(a)) \to +\infty$. Hence we can choose for all $k > a$ a sequence of Hamiltonians G_k for which the time-length T of chords from γ_3 to γ_1 goes to $+\infty$. Repeating the construction as in part (A) then gives us a sequence of Hamiltonians H_k in $\mathbb{C}^2$ with the desired properties. □

7. Lagrangian tori in $\mathbb{C}^n$: proofs

7.1. Basic properties of $bp_{\mathbf{x}}$, $def_{\mathbf{x}}$ for split Lagrangian tori in $\mathbb{C}^n$.

Proposition 7.1.

(A) Permutation invariance: *Let* $\mathbf{x}_\sigma \in \mathbb{R}^n$, $\alpha_\sigma \in (\mathbb{R}^n)^*$ *be the vectors obtained, respectively, from* $\mathbf{x} \in \mathbb{R}^n$, $\alpha \in (\mathbb{R}^n)^*$ *by a permutation* σ *of the coordinates. Then for any* α

$$def_{\mathbf{x}_\sigma}(\alpha_\sigma) = def_{\mathbf{x}}(\alpha), \quad bp_{\mathbf{x}_\sigma}(\alpha_\sigma) = bp_{\mathbf{x}}(\alpha). \tag{7.1}$$

(B) Homogeneity in $\mathbf{x}$: *For any* $c > 0$

$$def_{c\mathbf{x}} = \frac{1}{c}\, def_{\mathbf{x}}, \quad bp_{c\mathbf{x}} = \frac{1}{c} bp_{\mathbf{x}}.$$

(C) Semi-continuity with respect to $\mathbf{x}$:

$$\{\mathbf{x}_i\} \to \mathbf{x} \implies \forall\, \alpha \in (\mathbb{Z}^n)^* : \ bp_{\mathbf{x}}(\alpha) \le \liminf_{\mathbf{x}_i} bp_{\mathbf{x}_i}(\alpha),$$
$$\{\mathbf{x}_i\} \to \mathbf{x} \implies \forall\, \alpha \in (\mathbb{R}^n)^* : \ def_{\mathbf{x}}(\alpha) \le \liminf_{\mathbf{x}_i} def_{\mathbf{x}_i}(\alpha).$$

(D) The product property: *For any* α

$$def_{(\mathbf{x},x_{n+1},\dots,x_N)}(\alpha, 0, \dots, 0) \ge def_{\mathbf{x}}(\alpha),$$
$$bp_{(\mathbf{x},x_{n+1},\dots,x_N)}(\alpha, 0, \dots, 0) \ge bp_{\mathbf{x}}(\alpha).$$

Proof. (A) Claim (A) holds, since any split torus obtained from

$$T^n(\mathbf{x}) = T^1(x_1) \times \cdots \times T^1(x_n)$$

by a permutation of the T^1-factors is Hamiltonian isotopic to $T^n(\mathbf{x})$.

(B) Claim (B) follows easily from the fact that $(p, q) \mapsto \sqrt{c}(p, q)$, $c > 0$, is a conformal symplectomorphism.

(C) Claim (C) follows from the semi-continuity property of bp_L (see (3.8)) and def_L (see (3.6)).

(D) Claim (D) follows from the product property of pb_4^+ (see (3.9)). □

7.2. Proof of Theorem 2.11. We now prove that for certain Lagrangian tori $L \subset \mathbb{C}^n$ and specific cohomology classes $\alpha \in H^1(L; \mathbb{R})$ there are upper bounds for the associated Lagrangian isotopies. This proof follows the same route as in the proof of Theorem 2.6. We indicate the changes here and refer the reader to the proof of Theorem 2.6 for more details.

Consider a Lagrangian isotopy

$$\psi = \{\psi_t : L \to \mathbb{C}^n\}, \quad 0 \le t \le T, \ \psi_0 = \iota,$$

such that

$$[\omega]_t^{\psi} = [\omega]_L - t\, \partial\alpha,$$

with $L \subset \mathbb{C}^n$ and α as in the statement of the theorem.

For simplicity we consider the dual picture. Namely, for $0 \le t \le T$ there exists a family of compactly supported diffeomorphisms $\varphi_t : \mathbb{C}^n \to \mathbb{C}^n$, $\varphi_0 = \mathrm{Id}$, such that

$\varphi_t(L) = \psi_t(L)$. By pulling back our symplectic form $\varphi_t^*\omega =: \omega_t$ we may consider a fixed Lagrangian $L \subset (\mathbb{C}^n, \omega_t)$. We have

$$H_2(\mathbb{C}^n, L) \simeq \mathbb{Z}\langle A_1, \ldots, A_n\rangle$$

and

$$\omega_t(A_1) = a - \sigma t \quad \text{and} \quad \omega_t(A_i) = b - \rho t$$

for $i = 2, \ldots, n$.

Let $\mathcal{J}_t$, $t \in [0, T]$, be the space of almost complex structures on $\mathbb{C}^n$ compatible with the symplectic form ω_t and let $C \in H_2(\mathbb{C}^n, L)$ be a relative homology class. Choose a point $p \in L$. We define the moduli spaces $\mathcal{M}_1(C, J)$, $\mathcal{M}_1(C, J, p)$ and $\mathcal{M}_1(C, \{J_t\}, p)$ as in the proof of Theorem 2.6. An almost complex structure $J \in \mathcal{J}_t$ is called *regular*, if $\mathcal{M}_1(C, J)$ is a (transversally cut out) smooth manifold of dimension

$$\dim \mathcal{M}_1(C, J) = \dim L + \mu(C) - 2,$$

and *regular with respect to* p, if the space $\mathcal{M}_1(C, J, p)$ is a (transversally cut out) smooth manifold of dimension

$$\dim \mathcal{M}_1(C, J, p) = \mu(C) - 2.$$

Further on we will assume $n = \dim L$ is even – the arguments for n odd are similar.

Let $S \in (0, T]$. We will say that a family $\{J_t\}$, $0 \le t \le S$, $J_t \in \mathcal{J}_t$, is *regular with respect to* p if

(1) for any $t \in [0, S]$ the spaces $\mathcal{M}_1(C, J_t)$ are empty for all C with $\mu(C) < 2 - n$,

(2) $\mathcal{M}_1(C, \{J_t\}, p)$ is a (transversally cut out) smooth manifold of dimension

$$\dim \mathcal{M}_1(C, \{J_t\}, p) = \dim \mathcal{M}_1(C, J, p) + 1 = \mu(C) - 1$$

with boundary $\mathcal{M}_1(C, J_0, p) \cup \mathcal{M}_1(C, J_S, p)$.

Again, by the same standard regularity and transversality arguments, used in the proof of Theorem 2.6, that for any $J_0 \in \mathcal{J}_0$ and $J_S \in \mathcal{J}_S$, that are regular with respect to p, a generic family $\{J_t\}$, $0 \le t \le S$, $J_t \in \mathcal{J}_t$, connecting J_0 and J_S satisfies (2). In order to show that condition (1) also holds for a generic family $\{J_t\}$ note that since L is orientable, the Maslov index of any disk is even and therefore, since $\dim L$ is even, $\dim \mathcal{M}_1(C, J)$ is even. Thus, if $\mu(C) < 2 - n$, then virtual $\dim \mathcal{M}_1(C, J) \le -2$ for any $J \in \mathcal{J}_t$ and $t \in [0, S]$, meaning that the existence of *somewhere injective* J_t-holomorphic disks of such Maslov indices is a codimension-2 phenomenon and can be avoided by a generic choice of $\{J_t\}$.

We remark the following difference to the proof of Theorem 2.6. For a general $C \in H_2(\mathbb{C}^n, L)$ with $\mu(C) < 2 - n$, we cannot a priori exclude the existence of *not somewhere injective* J_t-holomorphic disks in the class C. However, we will show

now that *for* $t \in [0, a/\sigma)$ all J_t-holomorphic disks of Maslov index lying in $[2-n, 0]$, somewhere injective or not, can be excluded using our assumptions on $A_1, \ldots, A_n$ and α.

Set $a(t) := a - \sigma t$ and $b(t) := b - \rho t$.

Lemma 7.2. *For* $t \in [0, a/\sigma)$ *there is no* J_t*-holomorphic disk with boundary in* L *with Maslov index equal to* $-2l$ *for* $l \in \{0, 1, \ldots, (n-2)/2\}$ *that has positive symplectic area smaller than* $\omega_t(A_1) = a(t)$.

Proof of Lemma 7.2. Let $t \in [0, a/\sigma)$. For $l \in \{0, 1, \ldots, (n-2)/2\}$ let $D \in H_2(\mathbb{C}^n, L)$ be a class such that $\mu(D) = -2l$ and D has positive symplectic area.

A brief calculation shows that $b(t) - a(t) > 0$ and $a(t) > 0$, since $t \in [0, a/\sigma)$ and

$$\rho \leq \frac{\sigma(n+2)}{2} \leq \frac{b\sigma}{a}.$$

Hence, we also have that $b(t) > 0$.

A calculation of the symplectic area of D reveals that, since

$$b(t) > 0, \quad b(t) - a(t) > 0 \quad \text{and} \quad \omega_t(D) > 0,$$

for each $t \in [0, a/\sigma)$ we have

$$\omega_t(D) = k(b(t) - a(t)) - lb(t)$$

for some $k \in \mathbb{Z}$ such that

$$k > \frac{lb(t)}{b(t) - a(t)} > l,$$

where the last inequality actually means that

$$k \geq l + 1,$$

because $k, l \in \mathbb{Z}$.

Hence,

$$\omega_t(D) \geq (l+1)(b(t) - a(t)) - lb(t) = b(t) - (l+1)a(t) \geq a(t). \tag{7.2}$$

Here the last inequality in (7.2) can be deduced from the assumptions

$$l \in \{0, 1, \ldots, (n-2)/2\}, \quad \rho/\sigma \leq (n+2)/2 \leq b/a.$$

Inequality (7.2) implies the lemma. □

Continuing with the proof of the theorem, assume $0 < S < a/\sigma$. We will now show that for any almost complex structures $J_0 \in \mathcal{J}_0$ and $J_S \in \mathcal{J}_s$ regular with respect to p and any family $\{J_t\}$, $0 \leq t \leq S$, $J_t \in \mathcal{J}_t$, that connects J_0 and J_S and is

regular with respect to p, the moduli space $\mathcal{M}_1(A_1, \{J_t\}, p)$ is compact and hence, in particular, the moduli spaces $\mathcal{M}_1(A_1, J_0, p)$ and $\mathcal{M}_1(A_1, J_S, p)$ are compact.

As in the proof of Theorem 2.6 it suffices to consider a sequence $\{D_i\}$ in $\mathcal{M}_1(A_1, J_{t_i}, p)$ with $\{t_i\} \to s \in [0, S]$ and show that it has a subsequence converging to an element of $\mathcal{M}_1(A_1, J_s, p)$. By Gromov compactness (see [19]), since $\pi_2(\mathbb{C}^n) = 0$, the sequence $\{D_i\}$ has a subsequence converging to a bubbling configuration of non-constant non-parameterized J_s-holomorphic disks $\mathcal{D}_1, \ldots, \mathcal{D}_l$ with boundary in L whose homology classes $[\mathcal{D}_1], \ldots, [\mathcal{D}_l]$ add up to A_1:

$$[\mathcal{D}_1] + \cdots + [\mathcal{D}_l] = A_1.$$

By the result of Kwon–Oh [22], each disk $\mathcal{D}_i$, viewed as a subset of $\mathbb{C}^n$, is a finite union of non-parameterized somewhere injective J_s-holomorphic disks $\mathcal{D}_i^{(1)}, \ldots, \mathcal{D}_i^{(i_n)}$ so that

$$\mu(\mathcal{D}_i) = k_1 \mu\big(\mathcal{D}_i^{(1)}\big) + \cdots + k_{i_n} \mu\big(\mathcal{D}_i^{(i_n)}\big)$$

where the coefficients k_i are positive integers. Now $0 < \omega_s([\mathcal{D}_i]) \leq \omega_s(A_1)$ for all $i = 1, \ldots, l$ and this implies that

$$0 < \omega_s\big([\mathcal{D}_i^{(j)}]\big) \leq \omega_s(A_1)$$

for all somewhere injective disks. By Lemma 7.2, this means that

$$\mu(\mathcal{D}_i^{(j)}) \notin \{2 - n, \ldots, -2, 0\}.$$

Since the family $\{J_t\}$ is regular with respect to p, somewhere injective J_s-holomorphic disks of Maslov index $< 2 - n$ do not exist. Putting everything together we conclude that for all i, j $\mu(\mathcal{D}_i^{(j)}) > 0$, which, in fact, means that $\mu(\mathcal{D}_i^{(j)}) \leq 2$ (recall that L is orientable and therefore the Maslov indices of disks with boundary in L are all even). Therefore $A_1 = [\mathcal{D}_1]$, the disk $\mathcal{D}_1$ lies in $\mathcal{M}_1(A_1, J_s, p)$ and the sequence $\{D_i\}$ has a subsequence converging to an element of $\mathcal{M}_1(A_1, J_s, p)$. This finishes the proof that the smooth manifold $\mathcal{M}_1(A_1, \{J_t\}, p)$ is compact.

Thus for any $J_0 \in \mathcal{J}_0$, $J_S \in \mathcal{J}_S$ that are regular with respect to p and a family $\{J_t\}_{0 \leq t \leq S}$ that connects J_0 and J_S and is regular with respect to p the moduli space $\mathcal{M}_1(A_1, \{J_t\}, p)$ is a compact smooth 1-dimensional cobordism between the compact 0-dimensional manifolds $\mathcal{M}_1(A_1, J_0, p)$ and $\mathcal{M}_1(A_1, J_S, p)$, as long as $0 < S < a/\sigma$. (Note the difference with the proof of Theorem 2.6 where a similar claim was proved for *any* S such that ω_t is symplectic for all $t \in [0, S]$). This implies that $n_{A_1}(J_0, p) = n_{A_1}(J_S, p)$.

In a similar way one can show that given a (regular) path $\gamma(s)$, $0 \leq s \leq 1$, in L, $t \in [0, S]$ and a family $\{J_s\} \subset \mathcal{J}_t$ regular with respect to γ, the moduli space $\mathcal{M}_1(A, \{J_s\}, \gamma)$ is a compact smooth 1-dimensional cobordism between the compact 0-dimensional manifolds $\mathcal{M}_1(A, J_0, \gamma(0))$ and $\mathcal{M}_1(A, J_1, \gamma(1))$, and

therefore $n_{A_1}(J_0, \gamma(0)) = n_{A_1}(J_1, \gamma(1))$. The latter claim proves part (A) of the theorem for n even. The case for n odd is similar.

In order to prove part (B) we need to show that $T \leq a/\sigma$. Let us assume by contradiction that $a/\sigma < T$. As we have shown above, $n_{A_1}(J_0, p) = n_{A_1}(J_S, p)$ for any $S \in (0, a/\sigma)$. Thus if $n_{A_1}(J_0, p)$ is non-zero, then so is $n_{A_1}(J_S, p)$. This implies that for every $S < a/\sigma$ there exists a J_S-holomorphic disk in $\mathbb{C}^n$ with boundary on L in the relative homology class A_1. Passing to the limit $S \to a/\sigma$ and applying again the Gromov compactness, we see that there must exist a configuration of $J_{a/\sigma}$-holomorphic disks whose total homology class is A_1 (and hence not all disks in the configuration are constant) and whose total $\omega_{a/\sigma}$-area is $\omega_{a/\sigma}(A_1) = 0$, which is impossible since the area of each non-constant $J_{a/\sigma}$-holomorphic disk has to be positive. Hence, we obtain a contradiction and this proves (B). □

Remark 7.3. We remark that the proof of Theorem 2.11 does not generalize to the settings where $\omega(A_i) \neq \omega(A_j)$ for some $2 \leq i, j \leq n$, $i \neq j$. The reason for this comes from the fact that if in the Lagrangian isotopy there exist two Maslov-0 disks with positive symplectic areas that are rationally independent (e.g. $A_2 - A_1$ and $A_3 - A_2$), then one can always find a Maslov-0 disk with positive symplectic area arbitrarily close to 0. Hence one can not exclude Maslov-0 disks from any bubbling configuration.

7.3. Proof of Theorem 2.14.

(A) The tori $T^n(\mathbf{x})$ are exactly the regular orbits of the standard Hamiltonian $\mathbb{T}^n$-action on $\mathbb{C}^n$ and the regular level sets of its moment map $\Phi\colon \mathbb{C}^n \to (\mathbb{R}^n)^*$ whose image is the non-negative quadrant $\Delta \subset (\mathbb{R}^n)^*$.

For $\alpha = (m_1, \ldots, m_n)$ one readily sees that $\mathcal{I}(\mathbf{x}, \alpha)$, which is the open part of the intersection of the ray $\mathbf{x} - t\alpha$, $0 < t < +\infty$, and Δ, is an infinite ray if and only if all m_i are non-positive. Otherwise, $\mathcal{I}(\mathbf{x}, \alpha)$ is an interval and its rational length is given by $\min_{i, m_i > 0} x_i/m_i$. By Theorem 2.4, this proves part (A) of the theorem. □

(B) The upper bound $def_{\mathbf{x}}(k, \ldots, k) \leq x/k$ for $k \in \mathbb{N}$ follows from Theorem 2.1, since $\mathbb{C}^n$ does not admit weakly exact closed Lagrangian submanifolds by a famous result of Gromov [19].

The lower bound $bp_{\mathbf{x}}(k, \ldots, k) \geq x/k$ follows immediately from part (A) of the theorem. □

(C) Let $p_1, \ldots, p_n, q_1, \ldots, q_n$ be the standard Darboux coordinates on $\mathbb{R}^{2n} = \mathbb{C}^n$ so that $z_j = p_j + iq_j$, $j = 1, \ldots, n$.

According to the assumption, $x_i/k > x_{\min}$, that is, $x_i/k > x_j$ for some $j \neq i$. By (7.1), we may assume without loss of generality, that $i = n$, $j = 1$, that is, $x_n/k > x_1$. Let us show that $bp_{\mathbf{x}}(ke_n) = +\infty$.

Observe that a circle bounding a round disk of a certain area in $\mathbb{R}^2$ can be mapped by an area-preserving map arbitrarily close to the boundary of a square of the same

area. Together with the semi-continuity, product and symplectic invariance properties of pb_4^+ (see Section 3.4) this easily implies that it is enough to prove

$$pb_4^+(X_0, X_1, Y_0, Y_1) = 0, \tag{7.3}$$

where the admissible quadruple X_0, X_1, Y_0, Y_1 is defined as follows: for $i = 1, \dots, n-1$ denote by Π_i the boundary of the square

$$[0, \sqrt{x_i}] \times [0, \sqrt{x_i}] \subset \big(\mathbb{R}^2(p_i, q_i), dp_i \wedge dq_i\big).$$

For $i = n$ consider the rectangle

$$[0, x_n] \times [0, 1] \subset \big(\mathbb{R}^2(p_n, q_n), dp_n \wedge dq_n\big).$$

Now choose $\varepsilon > 0$ such that

$$\frac{x_n}{k} - 4\varepsilon > x_1. \tag{7.4}$$

As in the proof of Theorem 2.2 we choose a partition of $[0, x_n] \times \{1\}$ into $4k-3$ intervals $\gamma_1, \dots, \gamma_{4k-3}$, ordered from right to left, such that for $i \equiv 0, 2, 3 \mod 4$ the intervals γ_i have length ε, the interval γ_1 has length $\frac{x_n}{k}$ and the remaining intervals have length $\frac{x_n}{k} - 3\varepsilon$. Note that if $k = 1$ then $\gamma_1 = [0, x_n] \times \{1\}$. We define

$$\widehat{X}_0 = [0, x_n] \times \{0\} \cup \bigcup_{i \equiv 3 \mod 4} \gamma_i, \quad \widehat{X}_1 = \bigcup_{i \equiv 1 \mod 4} \gamma_i,$$

$$\widehat{Y}_0 = [0, 1] \times \{0\} \cup \bigcup_{i \equiv 2 \mod 4} \gamma_i, \quad \widehat{Y}_1 = [0, 1] \times \{x_n\} \cup \bigcup_{i \equiv 0 \mod 4} \gamma_i.$$

We then set

$$\begin{aligned} X_0 &:= \Pi_1 \times \cdots \times \Pi_{n-1} \times \widehat{X}_0, \\ X_1 &:= \Pi_1 \times \cdots \times \Pi_{n-1} \times \widehat{X}_1, \\ Y_0 &:= \Pi_1 \times \cdots \times \Pi_{n-1} \times \widehat{Y}_0, \\ Y_1 &:= \Pi_1 \times \cdots \times \Pi_{n-1} \times \widehat{Y}_1, \end{aligned}$$

which gives us our admissible quadruple.

Now choose a constant C that satisfies

$$\left(\frac{x_n}{k} - 4\varepsilon\right) \frac{1}{\sqrt{x_1}} > C > \sqrt{x_1}. \tag{7.5}$$

Such a constant exists by inequality (7.4). On $\mathbb{R}^2(p_n, q_n)$ we define a piecewise-linear function $G_\varepsilon \colon \mathbb{R}^2 \to \mathbb{R}$ that satisfies:

(i) $G_\varepsilon \equiv 0$ on $\widehat{Y}_0$ and $G_\varepsilon \equiv C$ on $\widehat{Y}_1$,

(ii) G_ε only depends on p_n.

One can choose G_ε to satisfy

$$\partial_x G_\varepsilon \leq C/\Big(\frac{x_n}{k} - 3\varepsilon\Big)$$

on $\pi(\widehat{X}_1) \times \mathbb{R}$, where $\pi(p_n, q_n) = p_n$ is the projection. Therefore there exists a smooth approximation $G\colon \mathbb{R}^2 \to \mathbb{R}$ that is complete, satisfies (i) and (ii) and has slope

$$\partial_x G < C/\Big(\frac{x_n}{k} - 4\varepsilon\Big)$$

on $\pi(\widehat{X}_1) \times \mathbb{R}$. This implies that all chords of G from $\widehat{X}_0$ to $\widehat{X}_1$ have time-length

$$T > \Big(\frac{x_n}{k} - 4\varepsilon\Big)/C.$$

Now consider the Hamiltonian $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ defined by

$$H(p, q) := p_1 + G(p_n, q_n).$$

H is complete and satisfies

$$\min_{Y_1} H = C, \quad \max_{Y_0} H = \sqrt{x_1},$$

and thus, by inequality (7.5),

$$\min_{Y_1} H - \max_{Y_0} H > 0.$$

Again by inequality (7.5) we have

$$\Big(\frac{x_n}{k} - 4\varepsilon\Big)\frac{1}{C} > \sqrt{x_1}.$$

Under the projection $\mathbb{R}^{2n} \to \mathbb{R}^2(p_n, q_n)$ the chords of H map to the chords of G. All chords of G from $\widehat{X}_0$ to $\widehat{X}_1$ have time-length

$$T > \Big(\frac{x_n}{k} - 4\varepsilon\Big)/C.$$

Now under the projection $\mathbb{R}^{2n} \to \mathbb{R}^2(p_1, q_1)$ the chords of H map to vertical lines. We see that for any $T > \sqrt{x_1}$ the image of $\Pi_1 \subset \mathbb{R}^2(p_1, q_1)$ under the Hamiltonian flow at time T does not intersect Π_1. Thus there are no chords of H from X_0 to X_1, which, by the dynamical characterization of pb_4^+, proves (7.3) and part (C). □

8. Lagrangian tori in $\mathbb{C}P^n$ and $\mathbb{S}^2 \times \mathbb{S}^2$: proofs

We now prove that for Lagrangian tori L in $\mathbb{C}P^n$ or $\mathbb{S}^2 \times \mathbb{S}^2$ and certain cohomology classes $\alpha \in H^1(L; \mathbb{R})$ there are upper bounds for $def_L(\alpha)$.

8.1. Proof of Theorem 2.16. Consider a Lagrangian isotopy

$$\psi = \{\psi_t \colon L \to \mathbb{C}P^n\}, \quad 0 \le t \le T, \ \psi_0 = \iota,$$

such that

$$[\omega]_t^{\psi} = [\omega]_L - t\,\partial\alpha.$$

By contradiction assume $T > C$ and consider $L_C := \psi_C(L)$. By the assumption of the theorem, we have $[\omega]_C^{\psi} \in H^2(\mathbb{C}P^n, L; \frac{1}{n}\mathbb{Z})$ and therefore

$$\omega\big(H^2(\mathbb{C}P^n, L_C)\big) \subset \frac{1}{n}\mathbb{Z}.$$

Since L is a torus the group $\omega(H^2(\mathbb{C}P^n, L_C))$ has $n+1$ generators $y_1, \dots, y_n, 1$. Denote by $c \in [0, +\infty) \cap \frac{1}{n}\mathbb{Z}$ its positive generator. However, by a theorem of Cieliebak and Mohnke [7],

$$c \le \frac{1}{n+1}.$$

We obtain a contradiction and therefore $T < C$. □

8.2. Proof of Theorem 2.19. Consider a Lagrangian isotopy

$$\psi = \{\psi_t \colon L \to \mathbb{S}^2 \times \mathbb{S}^2\}, \quad 0 \le t \le T, \ \psi_0 = \iota,$$

such that

$$[\omega]_t^{\psi} = [\omega]_L - t\,\partial\alpha.$$

By contradiction, assume $T > C$ and consider $L_C := \psi_C(L)$. By the assumption of the theorem we have $[\omega]_C^{\psi} \in H^2(\mathbb{S}^2 \times \mathbb{S}^2, L; \mathbb{Z})$ and therefore

$$\omega\big(H^2(\mathbb{S}^2 \times \mathbb{S}^2, L_C)\big) \subset \mathbb{Z}.$$

Therefore the group $\omega(H^2(\mathbb{S}^2 \times \mathbb{S}^2, L_C))$ is generated by 1. However, by a theorem of Dimitroglou Rizell, Goodmann, and Ivrii [10, Proposition 5.3], for any torus $L \subset \mathbb{S}^2 \times \mathbb{S}^2$ the positive generator c of $\omega(H^2(\mathbb{S}^2 \times \mathbb{S}^2, L))$ satisfies

$$c \le \frac{1}{2}.$$

We obtain a contradiction and therefore $T < C$. □

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