Local rigidity of uniform lattices

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Abstract. We establish topological local rigidity for uniform lattices in compactly generated groups, extending the result of Weil from the realm of Lie groups. We generalize the classical local rigidity theorem of Selberg, Calabi and Weil to irreducible uniform lattices in $\mathrm{Isom}(X)$ where X is a proper CAT(0) space with no Euclidian factors, not isometric to the hyperbolic plane. We deduce an analog of Wang's finiteness theorem for certain non-positively curved metric spaces.

1. Introduction

Over the last fifty years, the study of rigidity phenomena played a central role in group theory and its interplay with number theory, geometry and dynamics. Three fundamental results of the theory are local rigidity, strong rigidity and super-rigidity.

In the last two decades there has been a comprehensive, and quite successful, effort to extend this theory from the classical setting of Lie groups and symmetric spaces to the much wider framework of locally compact groups and non-positively curved metric spaces. Much of the attempt has been dedicated towards generalising the latter two phenomena, namely Mostow strong rigidity and Margulis super-rigidity, see e.g. [3–5, 12, 13, 16, 24, 33, 40, 41]. The present paper focuses on *local rigidity*.

Overview. Let Γ be a uniform lattice in the locally compact group G. The lattice Γ is *topologically locally rigid* if every homomorphism from Γ to G which is sufficiently close to the inclusion mapping in the point-wise convergence topology is injective and its image is a uniform lattice in G. The lattice Γ is *locally rigid* if every such homomorphism is in fact conjugate in G to the inclusion mapping. See §2.1 for the precise definitions.

One main goal of this paper is to establish topological local rigidity for uniform lattices in general compactly generated groups — extending Weil's theorem [46] from

the realm of Lie groups to locally compact groups. Another main goal is to generalize the classical local rigidity theorems of Selberg, Calabi and Weil [14, 39, 47] to the general context of CAT(0) groups. Furthermore we study local rigidity from another perspective, in terms of the Chabauty topology.

We apply these results to obtain new finiteness statements on non-positively curved metric spaces in analogy with Wang's classical finiteness theorem.

Topological local rigidity. A. Weil proved that uniform lattices in connected Lie groups are topologically locally rigid [46]. We complete the picture by extending this result to all compactly generated groups.

Theorem 1.1. *Uniform lattices in compactly generated groups are topologically locally rigid.*

It is rather straightforward that a small deformation of a uniform subgroup remains uniform. The main point is in showing that it stays discrete and injective.

Compact generation of G is equivalent to finite generation of a uniform lattice Γ in G when the later exists. Without this requirement lattices need not be topologically locally rigid. For instance, let $G = \Gamma$ be an infinite rank free group and note that the identity map $i \colon \Gamma \to \Gamma$ is approximated in the representation space by non-injective maps sending all but finitely many generators to the trivial element.[1]

Let us recall that a topological group admitting a uniform lattice is automatically locally compact [34, 2.2].

Weil's proof relied on the differentiable structure of G as well as the connectedness and the existence of a simply connected cover in the classical case of Lie groups. These notions are not available in the general context under consideration. Instead, our approach is to make use of the action of G on an appropriate Rips complex.

The Rips complex is simply connected whenever G is compactly presented and for this reason we first prove Theorem 1.1 in §5 under this additional assumption and then explain in §6 how it can be removed. §3 is dedicated to several geometric preliminaries that are required for the proof. §4 contains an independent proof of topological local rigidity for certain compactly presented totally disconnected groups, having the advantage of being rather elementary while providing an insight into the more complicated general case.

Local rigidity. Local rigidity was first proved by Selberg [39] for uniform lattices in $SL_n(\mathbb{R})$, $n \geq 3$ and by Calabi [14] for uniform lattices in $PO(n, 1) = \text{Isom}(\mathbb{H}^n)$, $n \geq 3$. Weil [47] generalized these results to uniform irreducible lattices in any connected semisimple Lie group G, assuming that G is not locally isomorphic

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[1] Note that there are examples of topologically locally rigid lattices in non-compactly generated groups. For instance, the additive group $\mathbb{Z}\left[\frac{1}{2}\right]$ with the discrete topology is topologically locally rigid regarded as a lattice in itself.

to $SL_2(\mathbb{R})$, the isometry group of the hyperbolic plane $\mathbb{H}^2$. In the latter case lattices have many non-trivial deformations.

Quite surprisingly, it turns out that $\mathbb{H}^2$ is the only obstruction to local rigidity of uniform lattices in the much greater generality of irreducible CAT(0) spaces.

Theorem 1.2. *Let X be a proper geodesically complete CAT(0) space without Euclidean factors and with $\mathrm{Isom}(X)$ acting cocompactly. Let Γ be a uniform lattice in $\mathrm{Isom}(X)$. Assume that for every de Rham factor Y of X isometric to the hyperbolic plane the projection of Γ to $\mathrm{Isom}(Y)$ is non-discrete. Then Γ is locally rigid.*

As in the works of Selberg, Calabi and Weil topological local rigidity plays an important role in the proof of local rigidity. Theorem 1.1 allows us to apply the Caprace–Monod theory of CAT(0) isometry groups and their lattices [15, 16] reducing the question to an irreducible lattice in a product of a semisimple Lie group and a totally disconnected group. In the purely Lie group case we rely on the above mentioned classical results, while the purely totally disconnected case is treated in a somewhat more general context in §4. Finally, the case where both factors are non-trivial relies on the superrigidity theorem of Monod [33].

The Chabauty space and local rigidity. Consider the space of closed subgroups $\mathrm{Sub}(G)$ of a topological group G equipped with the Chabauty topology. This suggests a different approach to local rigidity — we say that a uniform lattice is *Chabauty locally rigid* if it admits a Chabauty neighborhood consisting of isomorphic uniform lattices. For Lie groups this notion was first considered by Macbeath [31]. We prove Chabauty local rigidity for a rather broad family of groups.

Theorem 1.3. *Let G be a compactly generated group without non-trivial compact normal subgroups. Then the collection of uniform lattices is Chabauty open. If moreover G is compactly presented then uniform lattices are Chabauty locally rigid.*

See Theorem 7.1 for a sharper statement. In view of Proposition 7.2 the notion of Chabauty local rigidity is stronger than topological local rigidity. The compact generation assumption is necessary — see Proposition 3.6. Isometry groups of certain CAT(0) spaces are a special case for which Theorem 1.3 applies.

Corollary 1.4. *Let X be a proper geodesically complete CAT(0) space with $\mathrm{Isom}(X)$ acting cocompactly. Then uniform lattices in $\mathrm{Isom}(X)$ are Chabauty locally rigid.*

Whenever the CAT(0) space X has no Euclidean factors and the lattice Γ projects non-discretely to the isometry group of every hyperbolic plane factor of X, Theorem 1.2 implies that the Chabauty neighborhood of Γ given in Corollary 1.4 consists of conjugates of Γ. This is discussed in Corollary 8.9 below. In that situation Γ is an *OCC* or *open conjugacy class subgroup* in the sense of [25].

Finiteness results for lattices of bounded covolume. A well-known classical result that relies on local rigidity is the finiteness theorem of Wang. It says that a

connected semisimple Lie group G not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$ has only finitely many conjugacy classes of irreducible lattices of co-volume $\leq v$ for every $v > 0$. We refer to Wang's original proof [44] and to [21, Section 13] for a treatment of the case where G has factors isomorphic to $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$.

The two main ingredients of Wang's proof are:

(1) Local rigidity of all lattices in G.

(2) The Kazhdan–Margulis theorem [26][2].

Indeed, (2) combined with the Mahler–Chabauty compactness criterion implies that the set of conjugacy classes of lattices of co-volume $\leq v$ is compact with respect to the Chabauty topology, while (1) implies that it is discrete. However, even if one restricts attention to uniform lattices only, it is crucial that (1) holds for all lattices[3] in G. This is illustrated in the case of $\mathrm{SL}(2,\mathbb{C})$ where uniform lattices are locally rigid while the finiteness statement fails. The issue is that a sequence of uniform lattices of bounded co-volume may converge to a non-uniform lattice which is not locally rigid.

We obtain a similar finiteness result for CAT(0) groups. However, since local rigidity in the general CAT(0) context is established for uniform lattices only, we need to make an additional assumption.

Definition 1.5. Let G be a locally compact group. A family $\mathcal{F}$ of lattices in G is *uniformly discrete* if there is an identity neighborhood $U \subset G$ such that $\Gamma^g \cap U = \{e\}$ holds for every lattice $\Gamma \in \mathcal{F}$ and every $g \in G$.

Lattices belonging to a uniformly discrete family[4] are all uniform [38, 1.12], and their co-volumes are clearly bounded from below.

Theorem 1.6. *Let X be a proper geodesically complete* CAT(0) *space without Euclidean factors and with* $\mathrm{Isom}(X)$ *acting cocompactly. Let $\mathcal{F}$ be a uniformly discrete family of lattices in* $\mathrm{Isom}(X)$ *so that every* $\Gamma \in \mathcal{F}$ *projects non-discretely to the isometry group of every hyperbolic plane factor. Then $\mathcal{F}$ admits only finitely many lattices up to conjugacy with co-vol $\leq v$ for every fixed $v > 0$.*

The proof follows rather immediately from the open conjugacy class property, Corollary 8.9, and is perhaps more conceptual than Wang's original argument.

Finiteness of lattices containing a given lattice. Prior to the finiteness theorem mentioned above, Wang proved a weaker result that applies in a more general situation. Namely, he showed that a lattice in a semisimple Lie group is contained in only finitely

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[2]See Definition 1.7 and the paragraph following it for a discussion of this theorem.

[3]Recall that when G is a semisimple Lie group not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$ then also the non-uniform irreducible lattices are locally rigid. For rank one groups this was proved by Garland and Raghunathan [20]. (An alternative geometric proof for the rank one case is given in [9].) For higher rank groups this is a consequence of Margulis' super-rigidity theorem [32].

[4]Uniform discreteness is not to be confused with a weaker notion of a *joint discreteness* given in Definition 9.1. See also Footnote 13 concerning this terminology.

many lattices [43]. This result relies on the Kazhdan–Margulis theorem as well. To obtain a suitable generalization to locally compact groups we introduce the following notion.

Definition 1.7. Let G be a locally compact group. A family $\mathcal{F}$ of lattices in G has property **(KM)** if there is an identity neighborhood $U \subset G$ such that for every lattice $\Gamma \in \mathcal{F}$ there is an element $g \in G$ satisfying $g\Gamma g^{-1} \cap U = \{e\}$. We will say that G has property **(KM)** if the family of all lattices in G has property **(KM)**.

Kazhdan and Margulis showed [26] that a semisimple Lie group with no compact factors has property **(KM)**. We refer to [22] for a short proof of the Kazhdan–Margulis theorem and for additional examples of groups with property **(KM)**.

Theorem 1.8. *Let G be a compactly generated locally compact group. Assume that every lattice in G has a trivial centralizer. Let $\mathcal{F}$ be a family of lattices in G with property **(KM)** and admitting some $\Gamma \in \mathcal{F}$ that is finitely generated and is a least element in $\mathcal{F}$ with respect to inclusion. Then $\mathcal{F}$ is finite.*

Examples of groups in which all lattices have trivial centralizers include second countable locally compact groups with a trivial amenable radical [17, 5.3] as well as the isometry groups of certain CAT(0) spaces. See §8.1 for more details.

In particular, it follows that a family of lattices $\mathcal{F}$ containing a given finitely generated lattice Γ and satisfying $\Gamma' \cap U = \{e\}$ for some identity neighborhood U and all $\Gamma' \in \mathcal{F}$ must be finite.

We would like to mention the work [7] that contains somewhat related finiteness results for uniform tree lattices. Recently Burger and Mozes have established the analogue of [43] for certain uniform lattices in products of trees by carefully studying discrete groups containing a given irreducible uniform lattice.

Remark 1.9. §9 dealing with these generalizations of [43] does not depend on local rigidity and is the only part of the current paper that extends without difficulty to non-uniform lattices.

Invariant random subgroups. Finally, we investigate several questions concerning Chabauty spaces in general. For instance, the *invariant random subgroup* corresponding to a uniform lattice Γ is shown to depend continuously on Γ in the Chabauty topology; see Proposition 11.2. This relies on the Chabauty-continuity of the co-volume of uniform lattices established in Proposition 11.4.

We extend our gratitude to Nir Lazarovich for helpful conversations in which some arguments of Proposition 3.5 were established.

2. Deformations of lattices

2.1. The representation space. We describe a topological space classifying the representations of a given discrete group into a target topological group. The notion of local rigidity for a lattice is then defined in terms of that space.

Definition 2.1. Let Γ be a discrete group and G a topological group. The *representation space* $\mathcal{R}(\Gamma, G)$ is the space of all group homomorphisms from Γ to G with the point-wise convergence topology.

Note that for any point $r \in \mathcal{R}(\Gamma, G)$ the group $r(\Gamma)$ is a quotient of Γ.

If Γ has a finite generating set Σ then $\mathcal{R}(\Gamma, G)$ is homeomorphic to a closed subspace of G^{Σ} with the product topology. This subspace is determined by closed conditions arising from the relations among the generators in Σ.

Given a discrete subgroup Γ of G the inclusion morphism $\Gamma \hookrightarrow G$ can naturally be regarded as a point of $\mathcal{R}(\Gamma, G)$. We will denote this point by r_0 and frequently refer to neighborhoods of r_0 in $\mathcal{R}(\Gamma, G)$ as *deformations* of Γ in G.

Recall that a discrete subgroup $\Gamma \leq G$ is a *uniform lattice* in G if there is a compact subset $K \subset G$ such that $G = \Gamma K$. Note that Γ is finitely generated if and only if G is compactly generated [18, 5.C.3].

Definition 2.2. The lattice Γ is *topologically locally rigid* if the inclusion morphism r_0 admits an open neighborhood $\mathcal{U}$ in $\mathcal{R}(\Gamma, G)$ such that for every $r \in \mathcal{U}$ the subgroup $r(\Gamma) \leq G$ is a uniform lattice and r is injective.

Less formally, a uniform lattice is topologically locally rigid if sufficiently small deformations are injective, discrete and co-compact.

Definition 2.3. The lattice Γ is *locally rigid* if sufficiently small deformations of it are given by conjugation in G.

Local rigidity inside a larger group. There is a more general notion of local rigidity requiring small deformations of Γ to arise from conjugation inside a larger group containing G. The following is one particular example of this.

Definition 2.4. Assume that G is a closed subgroup of $G^{\dagger}$. The lattice Γ in G is *locally rigid in $G^{\dagger}$* if sufficiently small deformations inside $\mathcal{R}(\Gamma, G)$ arise from conjugation by an element of $G^{\dagger}$.

If Γ is locally rigid in some larger group $G^{\dagger}$ then it is in particular topologically locally rigid. Moreover, if G is cocompact in $G^{\dagger}$ and all uniform lattices in $G^{\dagger}$ are known to be locally rigid then uniform lattices in G are locally rigid in $G^{\dagger}$ in the sense of Definition 2.4.

Proposition 2.5. *Let G_i be a closed and cocompact subgroup of the locally compact group $G_i^\dagger$ for $i = 1, 2$. Let Γ be a uniform lattice in $G_1 \times G_2$ projecting densely to both factors. If Γ is locally rigid in $G_1^\dagger \times G_2^\dagger$ then Γ is locally rigid in $\mathbf{N}_{G_1^\dagger \times G_2^\dagger}(G_1 \times G_2)$.*

Proof. To simplify notation let

$$G = G_1 \times G_2, \quad G^\dagger = G_1^\dagger \times G_2^\dagger.$$

Consider a deformation $r \in \mathcal{R}(\Gamma, G)$ which is sufficiently close to the inclusion so that $r(\gamma) = \gamma^g$ for some $g = (g_1, g_2) \in G_1^\dagger \times G_2^\dagger = G^\dagger$ and all $\gamma \in \Gamma$. We have that

$$\overline{\mathrm{proj}_i(r(\Gamma))} = \overline{\mathrm{proj}_i(g^{-1}\Gamma g)} = g_i^{-1}\overline{\mathrm{proj}_i(\Gamma)}g_i = g_i^{-1}G_i g_i$$

On the other hand $r(\Gamma) \subset G$ and therefore $\overline{\mathrm{proj}_i(r(\Gamma))} \leq G_i$. This means that $g \in G^\dagger$ conjugates G to a subgroup of itself.

Consider the Chabauty space $\mathrm{Sub}\left(G^\dagger\right)$ with the natural containment partial order. This partial order is semicontinuous on $\mathrm{Sub}\left(G^\dagger\right)$ in the sense that given $H_1 \leq H_2$ there are Chabauty open neighborhoods $H_1 \in \Omega_1$ and $H_2 \in \Omega_2$ so that

$$H_1' \leq H_2, \ \forall H_1' \in \Omega_1, \quad \text{and} \quad H_1 \leq H_2', \ \forall H_2' \in \Omega_2.$$

The $G^\dagger$-conjugacy class of G is a compact subset of the Chabauty space $\mathrm{Sub}\left(G^\dagger\right)$ and therefore it admits minimal and maximal elements with respect to containment [45]. In particular, every group in the $G^\dagger$-conjugacy class of G is both minimal and maximal so that $g^{-1}Gg = G$ and $g \in \mathbf{N}_{G^\dagger}(G)$ as required. $\quad\square$

Local rigidity and finite index subgroups. In certain situations, it is convenient to replace the group Γ by a finite index subgroup having additional properties. The following lemma allows this procedure.

Lemma 2.6. *Let G be a topological group, $\Gamma \leq G$ a finitely generated lattice and $\Delta \lhd \Gamma$ a normal subgroup of finite index. Assume that $\mathbf{C}_G(\Delta)$ is trivial. If Δ is locally rigid then Γ is locally rigid as well.*

Proof. Let $r \in \mathcal{R}(\Gamma, G)$ be a sufficiently small deformation of Γ so that $r(\delta) = g^{-1}\delta g$ for some $g \in G$ and all $\delta \in \Delta$. For every pair of elements $\gamma \in \Gamma$ and $\delta \in \Delta$ we have

$$r(\delta)^{r(\gamma)} = r(\delta^\gamma) = \delta^{\gamma g} = r(\delta)^{g^{-1}\gamma g}.$$

The group Δ is finitely generated, being a finite index subgroup of Γ. Applying the above argument with δ ranging over a finite generating set for Δ gives

$$g^{-1}\gamma g r(\gamma)^{-1} \in \mathbf{C}_G(r(\Delta)) = \mathbf{C}_G(\Delta)^g = \{e\},$$

which means that $\gamma^g = r(\gamma)$ for all $\gamma \in \Gamma$ as required. $\quad\square$

2.2. The Chabauty topology. There is another related viewpoint on local rigidity, expressed in terms of the Chabauty space of closed subgroups.

Definition 2.7. Given a locally compact group G let $\mathrm{Sub}\,(G)$ denote the space of closed subgroups of G equipped with the *Chabauty topology* generated by the sub-basis consisting of

$$\mathcal{O}_1(U) = \{H \leq G \text{ closed} : H \cap U \neq \emptyset\} \text{ for every open subset } U \subset G,$$

$$\mathcal{O}_2(K) = \{H \leq G \text{ closed} : H \cap K = \emptyset\} \text{ for every compact subset } K \subset G.$$

The space $\mathrm{Sub}\,(G)$ is Chabauty compact. A reference to the Chabauty topology is [10, VIII.§5]. One major difference between the representation space and the Chabauty topology is that the latter only takes into account the embedding of a given subgroup as a closed subspace of G, a priori irrespective of the algebraic structure of that subgroup.

Let $\Gamma \leq G$ be a uniform lattice. The following two notions are variants of Definitions 2.2 and 2.3 respectively, given in terms of the Chabauty topology.

Definition 2.8. Γ is *Chabauty locally rigid* if the point $\Gamma \in \mathrm{Sub}\,(G)$ admits a Chabauty neighborhood consisting of uniform lattices isomorphic to Γ.

Definition 2.9. Γ has the *open conjugacy class (OCC)* property if the point $\Gamma \in \mathrm{Sub}\,(G)$ admits a Chabauty neighborhood consisting of conjugates of Γ.

The term OCC was recently coined in [25].

The image of a deformation. Given any homomorphism $r : \Gamma \to G$ of some group Γ into a topological group G the closure of $r(\Gamma)$ is a closed subgroup of G.

Definition 2.10. Let Γ be a discrete group and G a locally compact group. The closure of the image of a representation is a well-defined map

$$\mathrm{C} : \mathcal{R}(\Gamma, G) \to \mathrm{Sub}\,(G), \quad \mathrm{C}(r) = \overline{r(\Gamma)}.$$

The map C relates the representation space of Γ in G to the Chabauty space of G. In fact, we show in §7.1 that C is continuous at the point r_0 corresponding to the inclusion $\Gamma \hookrightarrow G$ whenever G is compactly generated and Γ is uniform. In this sense Chabauty local rigidity is a stronger notion than topological local rigidity.

On the other hand, observe that if the map C is known to be open then topological local rigidity implies Chabauty local rigidity, and likewise local rigidity implies the OCC property. This is the path taken in §7.3 towards Theorem 7.1.

3. Geometric preliminaries

Our research on topological local rigidity has benefited greatly from the ideas and excellent exposition of the new book [18] by Cornulier and de la Harpe.

In this section we collect from [11, 18] a toolkit of geometric notions needed for the proof of Theorem 1.1. The central theme is the construction of a geometric action of G on the Rips complex, which is a certain simplicial complex associated to a choice of a metric on G. The notion of compact presentation and its implication to the simple connectedness of the Rips complex is discussed. In addition we discuss geometric actions and length spaces.

3.1. Geometric actions. Let G be a topological group acting on a pseudo-metric space (X, d). We emphasize that the action need not be continuous.

Definition 3.1. The action of G on X is:

(1) *cobounded* if there exists a bounded $F \subset X$ with $GF = X$;

(2) *locally bounded* if $\forall g \in G, x \in X$ there exists a neighborhood $g \in V$ with Vx bounded;

(3) *metrically proper* if $\forall x \in X$ the subset $\{g \in G : d(gx, x) \leq R\}$ is relatively compact for all $R > 0$;

(4) *geometric* if it is isometric, metrically proper, locally bounded and cobounded.

For example, if G is compactly generated and d is a left invariant, proper and locally bounded[5] pseudo-metric then the action of G on (G, d) is geometric.

Proposition 3.2. *Let G be a compactly generated locally compact group and $H \leq G$ a closed cocompact subgroup. If G acts on X geometrically then so does H.*

Proof. The only property that needs to be checked is coboundedness. There is a compact subset $K \subset G$ with $G = HK$ and a bounded subset $F \subset X$ with $GF = X$. Therefore $HKF = X$ and it remains to show that $KF \subset X$ is bounded.

Pick a point $x_0 \in X$. Then for any $k \in K$ there is a neighborhood $k \in V_k$ with $V_k x_0$ bounded. As K is compact, this implies that Kx_0 is bounded, and so KF is bounded as well.

$\square$

Lemma 3.3. *Let G be a locally compact group admitting a geometric action on (X, d). Let $H \leq G$ be a subgroup considered abstractly with the discrete topology. If the restricted action of H on (X, d) is geometric then H is a uniform lattice in G.*

Proof. If H was not discrete in the topology induced from G, there would be a sequence $h_n \in H$ with $h_n \neq e, h_n \to e \in G$. Pick a point $x_0 \in X$. As the G-action is locally bounded, there is an open neighborhood $V \subset G$ containing e such that Vx_0 is bounded. In particular the set $\{h_n x_0\}_{n \in \mathbb{N}} \subset X$ is bounded in contrast with assumption that the action of H is metrically proper.

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[5]A pseudo-metric d is *locally bounded* if every point has a neighborhood of bounded diameter. A left invariant, proper and locally-bounded pseudo-metric on a group is called *adapted* in [18].

H is acting coboundedly on X, and so there exists a bounded subset $F \subset X$ with $HF = X$. Pick a point $x_0 \in X$ and let $K = \overline{\{g \in G : gx_0 \in F\}}$. Because the action of G is metrically proper K is compact. Note that $KH = G$. $\qquad\square$

Recall the following coarse variant of connectedness [18, 3.B].

Definition 3.4. A pseudo-metric space (X, d) is *coarsely connected* if $\exists c > 0$ such that $\forall x, y \in X$ there is a sequence $x = x_0, x_1, \ldots, x_n = y$ of points $x_i \in X$ with $d(x_i, x_{i+1}) < c$.

The following two propositions are a first step towards the study of Chabauty neighborhoods of uniform lattices.

Proposition 3.5. *Let G be a locally compact group admitting a continuous geometric action on a coarsely connected pseudo-metric space (X, d). Let Γ be a uniform lattice in G. Then Γ admits a Chabauty neighborhood consisting of subgroups whose action on X is cobounded.*

Proof. The lattice Γ is acting on X geometrically according to Proposition 3.2. In particular, there is a point $x \in X$ and a radius $R > 0$ so that $\Gamma B_1 = X$ for $B_1 = B_x(R)$. We may assume without loss of generality that X is c-coarsely connected for some $c > 0$ and that $R > c$. Denote $B_2 = B_x(2R)$ and $B_3 = B_x(3R)$. Let $\Sigma = \{\gamma \in \Gamma : \gamma B_1 \cap B_3 \neq \emptyset\}$ so that $B_3 \subset \Sigma B_1$. Since the action of Γ on X is metrically proper and the subgroup Γ is discrete the set Σ is finite.

The continuity of the action of G on X implies that there is a symmetric identity neighborhood $U \subset \text{Isom}(X)$ so that $UB_1 \subset B_2$. In particular

$$B_3 \subset \bigcup_{\sigma \in \Sigma} \sigma u_\sigma B_2$$

for every choice of elements $u_\sigma \in U$ with $\sigma \in \Sigma$. We claim that the Chabauty neighborhood

$$\Omega = \bigcap_{\sigma \in \Sigma} \mathcal{O}_1(\sigma U)$$

of Γ is as required. Consider any closed subgroup $H \in \Omega$. There are elements $h_\sigma \in H \cap \sigma U$ for every $\sigma \in \Sigma$. We will show that in fact $H_0 B_2 = X$ so that already the finitely generated subgroup $H_0 = \langle h_\sigma \rangle_{\sigma \in \Sigma}$ of H is acting coboundedly on X.

Let $y \in X$ be any point. The c-coarse connectedness of X allows us to find a sequence of points $x_0, x_1, \ldots, x_n$ for some $n \in \mathbb{N}$, where $x_0 = x, x_n = y$ and $d(x_i, x_{i-1}) < c$ for every $1 \leq i \leq n$. Clearly $x_0 \in H_0 B_2$. Arguing by induction assume that $x_{i-1} \in h B_2$ for some $1 \leq i \leq n$ and $h \in H_0$. Therefore

$$d(x_i, hx) \leq d(x_i, x_{i-1}) + d(x_{i-1}, hx) \leq c + 2R \leq 3R.$$

In other words $x_i \in h B_3$. Note that $B_3 \subset \bigcup_{\sigma \in \Sigma} h_\sigma B_2$ so that $x_i \in h h_\sigma B_2$ for some $\sigma \in \Sigma$. In particular $x_i \in H_0 B_2$. We conclude by induction. $\qquad\square$

Proposition 3.6. *Let G be a locally compact group and Γ a uniform lattice in G. Then Γ admits a Chabauty neighborhood consisting of cocompact subgroups if and only if G is compactly generated.*

Recall that any topological group admitting a uniform lattice is locally compact.

Proof. If G is compactly generated then it admits a left-invariant, continuous, proper and coarsely connected pseudo-metric [18, 4.B.8]. Consider the action of G on itself equipped with such a pseudo-metric d by left multiplication. Proposition 3.5 implies that Γ admits a Chabauty neighborhood consisting of cobounded subgroups. Since d is proper a cobounded subgroup is cocompact.

Conversely, assume that Γ admits a Chabauty neighborhood Ω consisting of cocompact subgroups. For every finite subset $F \subset \Gamma$ let Γ_F denote the subgroup of Γ generated by F. Then $(\Gamma_F)_F$ is a net of discrete subgroups of G having Γ as an accumulation point. Therefore Γ_F belongs to Ω for some F. In other words G admits a finitely generated uniform lattice, so that G is compactly generated. $\square$

Remark 3.7. The proofs of Propositions 3.5 and 3.6 establish that a uniform lattice Γ in a compactly generated group G admits a Chabauty neighborhood Ω and a compact subset $K \subset G$ so that every closed subgroup $H \in \Omega$ satisfies $HK = G$.

3.2. Compact presentation.

We recall the notion of compact presentation.

Definition 3.8. A group G is *compactly presented* if it admits a compact generating set S and a presentation with relations of bounded length in S.

A compactly presented locally compact group is boundedly presented by all of its compact generating sets. See [18, Section 8] for additional details.

Remark 3.9. Connected Lie groups, as well as reductive algebraic groups over local fields, are compactly presented. There are however algebraic groups over local fields which are not compactly presented, such as the group $\mathrm{SL}_2(k) \ltimes k^2$ where k is a non-Archimedean local field [18, Section 8.A].

3.3. The Rips complex.

Let (X, d) be a pseudo-metric space and $c > 0$. Consider the 2-dimensional simplicial complex $\mathrm{R} = \mathrm{R}_c^2(X, d)$ whose vertices are the points of X. Two distinct points $x, y \in X$ span an edge if and only if $d(x, y) \le c$. Three distinct points $x, y, z \in X$ span a 2-simplex if and only if each pair spans an edge.

Note that the 0-skeleton of $\mathrm{R}_c^2(X, d)$ is naturally identified with X, and we frequently make this identification implicit.

Definition 3.10. The *Rips 2-complex* $\mathrm{R}_c^2(X, d)$ is the topological realization of the above simplicial complex.

$\mathrm{R}_c^2(X, d)$ is given the *weak topology*, namely each simplex is regarded with its natural Euclidean topology, and an arbitrary subset is open if and only if its intersection with each simplex is open.

Definition 3.11. (X, d) is λ-*length* if for all pairs $x, y \in X$ the distance $d(x, y)$ equals $\inf \sum_{i=0}^{n-1} d(x_i, x_{i+1})$, where the infimum is taken over all sequences $x = x_0, \ldots, x_n = y$ of points $x_i \in X$ with $d(x_i, x_{i+1}) < \lambda$.

Proposition 3.12. *Let G be a compactly presented locally compact group. Then G admits a left-invariant, continuous and proper pseudo-metric d so that $\mathrm{R}_c^2(G, d)$ is connected and simply connected for all c sufficiently large. Moreover we may take the pseudo-metric d to be λ-length for some $\lambda > 0$.*

Proof. According to Proposition 7.B.1 and Milestone 8.A.7 of [18], and as G is compactly presented, the Rips complex $\mathrm{R}_c^2(G, d)$ is connected and simply connected provided that the pseudo-metric d is left-invariant, proper, locally bounded and large-scale geodesic[6]. The existence of such a pseudo-metric which is moreover continuous follows from the implication (i) $\Rightarrow$ (vi) in Proposition 4.B.8 of [18]. The fact that d can be taken to be λ-length follows from the details of the proof, combined with statement (2) of Lemma 4.B.7 of [18]. $\qquad\square$

The Rips complex as a metric space. The Rips complex $\mathrm{R} = \mathrm{R}_c^2(G, d)$ can be viewed as a quotient space obtained by gluing together a family of zero, one and two-dimensional pieces. Assume that we declare edges to have unit length and 2-simplices to be isometric to equilateral Euclidean triangles. This defines a quotient pseudo-metric $\rho = \rho_{G,d}$ on R that is compatible with the weak topology on R, see [11, §I.5]. Metric simplicial complexes are considered in [11, §I.7], where a metric ρ as above is called the *intrinsic pseudo-metric*.

Theorem 3.13. *Let d be a pseudo-metric on G. If the Rips complex $\mathrm{R} = \mathrm{R}_c^2(G, d)$ is connected for some $c > 0$ then R endowed with the metric $\rho_{G,d}$ is a complete geodesic space.*

Proof. The simplicial complex R has finitely many isometry types of Euclidean simplices (in fact, a unique one in every dimension). Assuming that R is connected Theorem I.7.19 of [11] applies. $\qquad\square$

It is shown in Proposition 6.C.2 of [18] that for a pseudo-metric d as in Proposition 3.12, the inclusion of (G, d) into (R, ρ) is a quasi-isometry. This implies that the action of G on (R, ρ) is geometric.

Proposition 3.14. *Every two vertices $g, h \in \mathrm{R}_{(0)} = G$ are connected in R by a rectifiable path p of length $l(p) \leq 2\rho(g, h)$ whose image lies in the 1-skeleton of R.*

Proof. Let $g, h \in \mathrm{R}_{(0)}$ be any pair of vertices. Since R is a geodesic space there is a path $p' : [0, 1] \to \mathrm{R}$ with $p'(0) = g$, $p'(1) = h$ and length $l(p') = \rho(g, h)$.

---

[6] A left-invariant, proper, locally bounded and large-scale geodesic pseudo-metric on a group is called *geodesically adapted* in [18].

We may assume without loss of generality that whenever $p'([a, b]) \subset T$ for a 2-simplex $T \subset \mathbb{R}$ with $0 \leq a < b \leq 1$ and $p'(a), p'(b) \in \partial T$ then $p'_{|[a,b]}$ is a geodesic segment of T. Let p be the path obtained from p' by replacing, for any instance of $a < b$ and T as above, the geodesic segment $p'_{|[a,b]}$ with a shortest path connecting $p'(a)$ and $p'(b)$ along the boundary ∂T. As T is isometric to an equilateral Euclidean triangle, $l(p) \leq 2l(p')$. $\qquad\square$

3.4. Length spaces and covering maps.

Definition 3.15. A map $f: X \to Y$ of metric spaces is a *local isometry* if for every $x \in X$ there is $s = s(x)$ such that the restriction of f to $B_x(s)$ is an isometry onto $B_{f(x)}(s)$. The map f is an *s-local isometry* if s can be chosen independently of x.[7]

Lemma 3.16. *Let $f: X \to Y$ be an s-local isometry where Y is a connected metric space and $X \neq \emptyset$. Then f is a covering map.*

Proof. Denote by $Y' = f(X)$ the image of f. Since f is an s-local isometry, Y' has the property that if $y \in Y'$ then $B_y(s) \subset Y'$. This condition implies that Y' is both open and closed. Since Y is connected we deduce that $Y' = Y$ and f is surjective.

To establish the covering property, given a point $y \in Y$ consider the open neighborhoods $U_x = B_x(s/2) \subset X$ for all points $x \in f^{-1}(y)$ in the preimage of y. As f is an s-local isometry it restricts to a homeomorphism on each U_x. Moreover $U_{x_1} \cap U_{x_2} \neq \emptyset$ implies $0 < d_X(x_1, x_2) < s$ but $f(x_1) = f(x_2) = y$ which is absurd. So the open neighborhoods U_x are pairwise disjoint as required. $\qquad\square$

Pullback length metric. Recall that a metric is a *length metric* if the distance between any two points is equal to the infimum of the lengths of all rectifiable paths connecting these two points. A space equipped with a length metric is called a *length space*. Let $f: X \to Y$ be a local homeomorphism such that Y is a length space and X is Hausdorff. It is possible to pullback the length metric from Y to obtain a metric on X that is *induced by f* (see Definition I.3.24 of [11]).

Proposition 3.17. *Let $f: X \to Y$ be a local homeomorphism where Y is a length space and X is Hausdorff. Then the metric d_f on X induced by f is a length metric and f becomes a local isometry. Moreover d_f is the unique metric on X satisfying these two properties.*

In addition, assume that there is $s > 0$ such that every $x \in X$ has a neighborhood $x \in U$ with $f|_U$ being a homeomorphism onto its image and $B_{f(x)}(s) \subset f(U)$. Then f is an s'-local isometry for some $s' > 0$.

---

[7]Our definition is different from the usage by several authors, see e.g. Definition 3.4.1 of [36]. We require a somewhat stronger local condition, while [36] requires in addition that a local isometry is surjective.

Proof. The first part is Proposition I.3.25 of [11]. The statement concerning s-local isometries follows from the proof.   □

Compare Lemma 3.16 and Propositions 3.17 with the more general Proposition 3.28 of [11].

Proposition 3.18. *Let $f\colon X \to Y$ be a homeomorphism of length spaces that is a local isometry. Then f is an isometry.*

Proof. See Corollary 3.4.6 of [36].   □

4. Totally disconnected groups

We establish local rigidity for the isometry groups of connected and simply-connected proper length spaces assuming the action is smooth.

Definition 4.1. An action of a totally disconnected topological group G on a connected topological space X is *smooth* if compact subsets of X have open pointwise stabilizers in G.

Any continuous action of a locally compact group on a connected locally finite graph is smooth. In addition, the actions of totally disconnected isometry groups on certain CAT(0) spaces are smooth [15, Theorem 6.1].

Theorem 4.2. *Let X be a connected and simply connected proper length space and $G \le \mathrm{Isom}(X)$ a closed subgroup acting cocompactly and smoothly. Then uniform lattices of G are locally rigid in $\mathrm{Isom}(X)$.*

The conclusion means that a small deformation of a uniform lattice $\Gamma \le G$ is a conjugation by an element of $\mathrm{Isom}(X)$. This theorem, or rather its variant Theorem 4.5, is used in §8 as an ingredient in the proof of local rigidity for CAT(0) groups.

The following is yet another interesting application.

Corollary 4.3. *Let G be a compactly presented totally disconnected locally compact group without non-trivial compact normal subgroups. Then there is a locally compact group $G^\dagger$ admitting G as a closed cocompact subgroup such that uniform lattices in $G^\dagger$ are locally rigid.*

In particular, all uniform lattices in G as above are topologically locally rigid. Note that this conclusion follows immediately from our main Theorem 1.1. However the independent proof given in §4.2 and §4.3 is considerably simpler and yet provides an insight into the general case.

Corollary 4.3 can be improved in the special case where G as above is a direct product of two groups and the uniform lattice Γ has dense projections to the factors.

Corollary 4.4. *Let* $G = G_1 \times G_2$ *be a compactly presented totally disconnected locally compact group without non-trivial compact normal subgroups. Let* Γ *be a uniform lattice in* G *projecting densely to both factors. Then* Γ *is locally rigid in* $\mathrm{Aut}(G)$.

4.1. Isometry groups with smooth action.

Theorem 4.5. *Let* X *be a connected and simply connected proper length space. Assume that* X *splits as a direct product* $X = X_1 \times X_2$ *of metric spaces and let* $G_i \leq \mathrm{Isom}(X_i)$ *be closed subgroups acting cocompactly. Assume that* G_1 *acts smoothly.*

Let $\Gamma \leq G_1 \times G_2$ *be a uniform lattice. Then any sufficiently small deformation of* Γ *that is trivial on* G_2 *is a conjugation by an element of* $\mathrm{Isom}(X_1)$.

This somewhat cumbersome statement involving a product decomposition is needed in the proof of local rigidity for CAT(0) groups below.

Note that Theorem 4.2 is obtained as an immediate consequence of Theorem 4.5 in the special case where $X \cong X_1$, the factor X_2 is a reduced to a point, and a subgroup of $\mathrm{Isom}(X) \cong \mathrm{Isom}(X_1)$ is acting smoothly and cocompactly.

Proof of Theorem 4.5. Let $\mathcal{R}_1(\Gamma, G)$ denote the subspace of the representation space $\mathcal{R}(\Gamma, G)$ consisting of these r with $\mathrm{Proj}_2(r(\gamma)) = \mathrm{Proj}_2(\gamma)$ for all $\gamma \in \Gamma$. Let $r_0 \in \mathcal{R}_1(\Gamma, G)$ correspond to the inclusion morphism $r_0 \colon \Gamma \hookrightarrow G$ and $r \in \mathcal{R}_1(\Gamma, G)$ denote a point that should be understood as being close to r_0, the precise degree of closeness will be specified below.

According to Proposition 5.B.10 of [18] and as X is proper, G is acting geometrically. In particular we may apply Proposition 3.2 and find a compact subset $K \subset X$ such that $X = \Gamma K$. Denote $K_1 = \mathrm{Proj}_1(K)$ and let $O_K \leq G_1$ be the point-wise stabilizer of K_1. Since K_1 is compact, smoothness implies that O_K is an open subgroup of G_1.

Fix some arbitrary $s > 0$ and let $N_s(K)$ denote the s-neighborhood of K, so that $N_s(K)$ is relatively compact and open. Denote

$$\Delta_s = \{\gamma \in \Gamma : \gamma K \cap N_s(K) \neq \emptyset\},$$

with Δ_s being finite as Γ is discrete and its action is metrically proper.

We now construct a Γ-equivariant covering map $f \colon X \to X$, where Γ acts on the domain of f via r_0 and on the range of f via r, namely

$$f(\gamma x) = r(\gamma) f(x), \quad \forall \gamma \in \Gamma, \ x \in X.$$

We propose to define $f \colon X \to X$ piece-wise by

$$f_{|\gamma K} = r(\gamma)\gamma^{-1}, \quad \forall \gamma \in \Gamma$$

and care needs to be taken for this to be well-defined. Note that f depends on r implicitly. In any case, the above formula is equivariant in the sense that given $x \in \gamma K$ and $\gamma' \in \Gamma$ we have

$$f_{|\gamma'\gamma K}(\gamma' x) = \left(r(\gamma'\gamma)\gamma^{-1}\gamma'^{-1}\right)(\gamma' x) = r(\gamma')\left(r(\gamma)\gamma^{-1}\right)x = r(\gamma') f_{|\gamma K}(x).$$

An ambiguity in the definition of f arises whenever $\gamma K \cap \gamma' K \neq \emptyset$ for a pair of elements $\gamma, \gamma' \in \Gamma$. By the equivariance property we may assume $\gamma' = 1$, that is $\gamma K \cap K \neq \emptyset$ and so $\gamma \in \Delta_s$. To resolve the ambiguity we require that

$$f_{|K \cap \gamma K} = r(\gamma)\gamma^{-1}_{|K \cap \gamma K} = \mathrm{id}_{|K \cap \gamma K}.$$

This will certainly be the case given that $r(\gamma)\gamma^{-1}$ fixes the set K point-wise. Note that $\mathrm{Proj}_2(r(\gamma)\gamma^{-1}) = e_2 \in \mathrm{Isom}(X_2)$ and so it suffices to require

$$\mathrm{Proj}_1\left(r(\gamma)\gamma^{-1}\right) \in O_K \Leftrightarrow \mathrm{Proj}_1\left(r(\gamma)\right) \in O_K \mathrm{Proj}_1(\gamma).$$

This means that if we let $\Omega \subset R_1(\Gamma, G)$ be the open condition given by

$$\Omega = \{r \in R_1(\Gamma, G) : \mathrm{Proj}_1(r(\gamma)) \in O_K \mathrm{Proj}_1(\gamma), \ \forall \gamma \in \Delta_s\},$$

the function f is well-defined as long as $r \in \Omega$.

Next to ensure that f is a local isometry. We define an open neighborhood Ω_s of r_0 in $R_1(\Gamma, G)$ so that for every $f \in \Omega_s$ the function f is an s-local isometry. Let

$$\Omega_s = \{r \in R_1(\Gamma, G) : \mathrm{Proj}_1(r(\gamma)) \in \mathrm{Proj}_1(\gamma) O_K, \ \forall \gamma \in \Delta_s\}$$

and it is easy to see that for every $r \in \Omega_s$ and $\gamma \in \Delta_s$ the map $r(\gamma)\gamma^{-1}$ fixes point-wise the set γK. In particular $f_{|N_s(K)} = \mathrm{id}$ for all such $r \in \Omega_s$.

Given a point $x \in X$ we may by equivariance assume that $x \in K$ and so $B_s(x) \subset N_s(K)$. In particular $f_{|B_s(x)} = \mathrm{id}$ and f is certainly a s-local isometry at x.

We have shown that $f : X \to X$ is a Γ-equivariant s-local isometry. Therefore f is a covering map by Lemma 3.16. The connectedness and simple connectedness of X implies that f is an homeomorphism. A local isometry which is a homeomorphism between length spaces is in fact an isometry, see Proposition 3.18.

To conclude we may regard $f \in \mathrm{Isom}(X)$ and the equivariance formula implies that $f\Gamma f^{-1} = r(\Gamma)$ as long as $r \in \Omega \cap \Omega_s$ for some $s > 0$. Observe that $\mathrm{Proj}_2(f) = e_2 \in \mathrm{Isom}(X_2)$ and so in fact $f \in \mathrm{Isom}(X_1)$. $\square$

4.2. The Cayley–Abels graph.

In order to apply Theorem 4.2 towards Corollary 4.3 we need to construct a suitable space X associated with a totally disconnected compactly presented group G. With this aim in mind we recall the notion of a Cayley–Abels graph; see e.g. [29, 48] or [18, 2.E.10].

Proposition 4.6. *Let G be a compactly generated totally disconnected locally compact group. There is a locally finite connected graph X on which G acts transitively, continuously and with compact open stabilizers.*

Proof. By the van-Dantzig theorem there is a compact open subgroup $U \subset G$. We refer to Proposition 2.E.9 of [18] for the construction of the required graph X. In particular, the vertices of X are taken to be the cosets G/U.   □

Any graph satisfying the conditions of Proposition 4.6 is called a *Cayley–Abels* graph[8] for G. It is clear that the action of G on a corresponding Cayley–Abels graph is geometric. In particular all such graphs for G are quasi-isometric to G and therefore to each other, as follows from Theorem 4.C.5 of [18] (this fact is also proved in [29]).

Proposition 4.7. *Let G be a compactly presented totally disconnected locally compact group and let X be a Cayley–Abels graph for G. Then the Rips complex $R_c^2(X^{(0)})$ constructed with respect to the graph metric on the vertices of X is simply-connected for all c sufficiently large.*

Note that the Rips complex $R_c^2(X^{(0)})$ constructed on the vertices of X is a locally finite 2-dimensional simplicial complex.

Proof of Proposition 4.7. In terms of the machinery developed in [18] the proof is transparent. Namely, G is large-scale equivalent (i.e. quasi-isometric) to X. The fact that G is compactly presented is equivalent to G being coarsely simply connected[9] with respect to some geodesically adapted metric; [18, 8.A.3]. Hence X is coarsely simply connected, which implies that $R_c^2(X)$ is simply connected for all c sufficiently large by [18, 6.C.6].

Alternatively, a hands on proof can be obtained by recalling that the vertices of X correspond to G/U for some compact open subgroup U and verifying that the proof of [18, 7.B.1] goes through in this case as well.   □

4.3. Local rigidity of uniform lattices in totally disconnected groups.

Proof of Corollary 4.3. Let G be a compactly presented totally disconnected locally compact group without non-trivial compact normal subgroups. Let X be a Cayley–Abels graph for G and $R = R_c^2(X)$ the corresponding Rips complex for c sufficiently large so that R is connected and simply connected, as in Proposition 4.7. Denote $G^\dagger = \mathrm{Isom}(R)$ and let $\alpha \colon G \to G^\dagger$ be the natural homomorphism [18, 5.B]. The group $G^\dagger$ is locally compact, totally disconnected and compactly presented, and the homomorphism α is continuous, has compact kernel, and its image is closed

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[8]If G is a countable discrete group then a Cayley–Abels graph is simply a Cayley graph with respect to some generating set.

[9]The notion of *coarse simple connectedness* is discussed in [18, 6.A].

and cocompact in $G^\dagger$. Our assumptions imply that α is injective so that G can be identified with the subgroup $\alpha(G)$ of $G^\dagger$.

It is clear from the definition of a Cayley–Abels graph that the action of $G^\dagger$ on R is smooth. Moreover, R is a length space, as in Theorem 3.13. Being a locally finite simplicial complex with a single isometry type of simplices in each dimension, R is proper. The group $G^\dagger$ is acting transitively on the vertices of R and hence cocompactly on R. Having verified all the required conditions, local rigidity of uniform lattices in $G^\dagger$ follows immediately from Theorem 4.2. $\qquad\square$

Remark 4.8. Uniform tree lattices are locally rigid — let T be a locally finite tree and G a cocompact group of automorphisms of T. Every uniform lattice Γ in G admits a free Schottky subgroup Δ of finite index [30]. A Schottky subgroup is defined by an open condition and it is clear that Δ is topologically locally rigid. Since the centralizer of Γ in G is trivial, our Lemma 2.6 implies that Γ is topologically locally rigid as well.

Remark 4.9. The previous remark leads to a different and possibly simpler proof of Corollary 4.3. Let G be a compactly generated totally disconnected locally compact group. There is a locally compact group $\widetilde{G}$ admitting G as a quotient with discrete kernel, such that $\widetilde{G}$ is acting by automorphisms on the universal covering tree of some Cayley–Abels graph for G. Uniform lattices in $\widetilde{G}$ are topologically locally rigid by Remark 4.8. Topological local rigidity for uniform lattices in G follows relying on the methods of §6.

Proof of Corollary 4.4. Let $G = G_1 \times G_2$ be a product of two compactly presented totally disconnected locally compact groups without non-trivial compact normal subgroups and Γ a uniform lattice in G projecting densely to both factors. Let X be the product of the two Rips complexes R_1 and R_2 associated to the two factors by Proposition 4.7 and as in the proof of Corollary 4.3. We consider X with the product L^2 metric. The de Rham decomposition theorem of Foertsch–Lytchak [19] implies that $G^\dagger = \mathrm{Isom}(R_1) \times \mathrm{Isom}(R_2)$ is an open finite index subgroup of $\mathrm{Isom}(X)$. In particular the uniform lattice Γ is locally rigid in $G^\dagger$ by Theorem 4.2. We conclude relying on Proposition 2.5. $\qquad\square$

5. Compactly presented groups

In this section we prove Theorem 1.1 with the additional assumption of compact presentation. Let G be a compactly presented group and $\Gamma \le G$ a uniform lattice. We show that a small deformation $r(\Gamma)$ of Γ is a uniform lattice isomorphic to it.

Overview. The strategy of the proof is as follows. Consider the Rips complex R associated to G. Then R is connected and simply connected and G is acting on R

geometrically. The main ingredient of the proof is the construction of a connected space Y and a covering map $f: Y \to R$ depending on the deformation r. Such a map f must of course be a homeomorphism.

We consider certain open subsets of R called *bubbles*. The space Y is constructed by gluing together bubbles indexed by elements $\gamma \in \Gamma$. The map f is naturally associated to this gluing and is a local homeomorphism. This allows us to define a pull-back length metric on Y. We then promote f to an s-local isometry using a certain "Lebesgue number" argument with respect to the cover of Y by bubbles.

There is an obvious action of Γ on Y by homeomorphisms, coming from the indexing of the bubbles. The map f becomes Γ-equivariant with respect to this action, with Γ acting on R via r. Finally, we show that Γ is acting on Y geometrically with respect to the induced length metric and deduce from this Theorem 1.1.

Standing notations. Recall that Γ is a uniform lattice in the compactly presented group G. Let $r_0 \in \mathscr{R}(\Gamma, G)$ correspond to the inclusion and $r \in \mathscr{R}(\Gamma, G)$ be a small deformation of r_0.

Let $R = R_c^2(G, d)$ be the associated Rips complex where d is some left invariant, continuous and proper pseudo-metric on G, see Proposition 3.12. We assume that d is λ-length and that $c > \lambda$ is sufficiently large so that R is connected and simply-connected. Moreover ρ is the intrinsic metric on R, and $\bar{\rho}$ will denote the induced length metric on the space Y.

We will use the notation $U = \mathscr{B}(e, D)$, where $\mathscr{B}$ denotes a bubble as in Definition 5.2. Here $D > 0$ is a fixed radius. An additional radius L (the "Lebesgue number") is introduced in Lemma 5.14 and it is assumed that $L > c > \lambda$ and $D > D(L)$. See Lemma 5.14 for the meaning of the parameter $D(L)$.

Finally, Σ will denote a certain generating set for Γ depending on the choice of the bubble U.

5.1. Bubbles.

We consider certain open subsets of R that share many of the properties of metric balls.

Definition 5.1. Let R be a simplicial complex and $A \subset R_{(0)}$ a subset of its 0-skeleton. The *span* $\mathscr{S}(A) \subset R$ of A is the union of those simplices in R all of whose vertices belong to A.

Definition 5.2. Let $(R, \rho) = R_c^2(G, d)$ be the Rips complex of (G, d) with its associated metric ρ. Let $0 < \varepsilon < \sqrt{3}/4$ be fixed. The *bubble* at a vertex $g \in G = R_{(0)}$ of radius $D > 0$ is

$$\mathscr{B}(g, D, \varepsilon) = \mathscr{N}_\varepsilon(\mathscr{S}(B)) = \{x \in R : \rho(x, \mathscr{S}(B)) < \varepsilon\},$$

where

$$B = B_{(G,d)}(g, D).$$

The constant $0 < \varepsilon < \sqrt{3}/4$ will be kept fixed in what follows and its exact value will not affect the arguments in any way. For this reason we will sometimes suppress ε from the notation for $\mathcal{B}$.

We collect basic properties of bubbles.

Proposition 5.3. *Let* $\mathcal{B} = \mathcal{B}(x, D, \varepsilon)$ *be a bubble.*

(1) $\mathcal{B}$ *is open in* R.

(2) *If* d *is* λ*-length,* $\mathrm{R} = \mathrm{R}_c^2(G, d)$ *and* $D > c > \lambda$ *then* $\mathcal{B}$ *is connected.*

(3) *Given* $g \in G$ *we have* $g\mathcal{B} = \mathcal{B}(gx, D, \varepsilon)$.

(4) $x \in \mathcal{B} = \mathcal{B}(x, D, \varepsilon)$.

Let $\mathcal{B}' = \mathcal{B}(x', D', \varepsilon)$ *be an additional bubble.*

(5) *If* $d(x, x') + D' < D$ *then* $\mathcal{B}' \subset \mathcal{B}$.

Let $\mathcal{B}_i = \mathcal{B}(x_i, D_i, \varepsilon)$ *be a collection of bubbles for* $i \in I$ *with* $|I| < \infty$.

(6) *If* $\cap_{i \in I} \mathcal{B}_i \neq \emptyset$ *then this intersection contains a vertex of* R.

Proof. Item (1) is obvious.

To establish (2), consider a bubble at the vertex $x = g \in G$. Theorem 3.13 implies that ρ-balls are path connected in R, and so it suffices to show that $\mathcal{S}(B)$ is path connected. Any point of $\mathcal{S}(B)$ can in turn be connected by a segment to one of the vertices in B. It remains to show that any vertex $h \in B$ can be connected to g within $\mathcal{S}(B)$. As $d(g, h) < D$ and d is λ-length there is a sequence of points $x_0 = g, x_1, \ldots, x_n = h$ with $d(x_i, x_{i+1}) < \lambda < c$ and $d(g, x_i) < D$. The corresponding path in R connects g to h and is contained in $\mathcal{S}(B)$ as required.

Statement (3) follows immediately from Definition 5.2 and the left G-invariance of the pseudo-metric d and metric ρ, and statements (4) and (5) are clear.

To observe (6) we make an auxiliary construction. Let R$'$ be the simplicial complex obtained by subdividing every 2-cell of R in the obvious way into four 2-cells isometric to an equilateral triangle of side length $1/2$. Every edge of R is subdivided into two new edges in R$'$ of length $1/2$. Clearly the underlying metric spaces of R and R$'$ are isometric. Given a subcomplex $Q \subset$ R let $\mathcal{N}'(Q) \subset$ R$'$ denote the union of the interiors of those cells of R$'$ whose closure has non-empty intersection with Q. Since $\varepsilon < \sqrt{3}/4$ we see that

$$\mathcal{B}_i = \mathcal{N}_\varepsilon(\mathcal{S}(B_i)) \subset \mathcal{N}'(\mathcal{S}(B_i)),$$

where $B_i = B_{(G,d)}(x_i, D_i)$. However, note that because of the definition of R$'$

$$\mathcal{N}'(Q_1 \cap Q_2) = \mathcal{N}'(Q_1) \cap \mathcal{N}'(Q_2)$$

for every pair of subcomplexes $Q_1, Q_2 \subset$ R. Moreover, $\mathcal{S}(B_i) \cap \mathcal{S}(B_j) = \mathcal{S}(B_i \cap B_j)$. Statement (6) follows. $\qquad\square$

Bubbles provide us with a system of nicely behaved open subsets of R. However, the "center points" of bubbles belong by definition to the 0-skeleton G of R. This will not pose any serious difficulty since any point of R is at most a unit distance from G.

Proposition 5.4. *Let* $(R, \rho) = R_c^2(G, d)$ *be a Rips complex associated to* G. *Any* ρ-*bounded subset is contained in the bubble* $\mathcal{B}(e, D)$ *for all* D *sufficiently large.*

Proof. Let $F \subset R$ be a ρ-bounded subset. Let V be the union of all vertices belonging to simplices of R which intersect F non-trivially. Then V is bounded as well and in particular $V \subset B_\rho(e, N) \subset R$ for some $N > 0$ sufficiently large. Regarded as a subset of G, this implies $V \subset B_d(e, cN)$. We obtain $F \subset \mathcal{S}(V) \subset \mathcal{B}(e, D)$ for all $D > cN$. $\qquad\square$

Essentially, Proposition 5.4 follows immediately from the fact that (G, d) and (R, ρ) are quasi-isometric.

5.2. The nerve of a deformation.

Definition 5.5. Let $\Delta \subset \Gamma$ be any subset, $U \subset R$ an open set and $r \in \mathcal{R}(\Gamma, G)$. The *nerve* $N(\Delta, U, r)$ consists of all subsets $A \subset \Delta$ such that $\bigcap_{\sigma \in A} r(\sigma)U$ is non-empty.

For example, the condition $\{\sigma_1, \sigma_2\} \in N(\Delta, U, r)$ implies that $r(\sigma_1)u_1 = r(\sigma_2)u_2$ with u_1 and u_2 ranging over certain non-empty open subsets of U.

Proposition 5.6. *Let* $\Delta \subset \Gamma$ *be any finite subset and* $U = \mathcal{B}(g, D) \subset R$ *a bubble at a vertex* $g \in G = R_{(0)}$. *Then, denoting* $U' = \mathcal{B}(g, D - \alpha)$ *the equality*

$$N(\Delta, U, r_0) = N(\Delta, U', r_0) = N(\Delta, U', r)$$

holds for every $r \in \mathcal{R}$ *sufficiently close to* r_0 *and* $\alpha > 0$ *sufficiently small.*

Proof. By definition $A = \{\delta_1, \ldots, \delta_m\} \in N(\Delta, U, r)$ if and only if the bubbles $r(\delta_1)U, \ldots, r(\delta_m)U$ intersect non-trivially. By Items (3) and (6) of Proposition 5.3 this happens if and only if there exists $g_A \in G$ that satisfies

$$d(r(\delta)g, g_A) < D, \quad \forall \delta \in A.$$

Since the pseudo-metric d is continuous and Δ is finite, for r sufficiently close to r_0 and α sufficiently small the conditions

$$N(\Delta, U, r_0) = N(\Delta, U', r_0) \subset N(\Delta, U', r)$$

are satisfied. Note that $N(\Delta, U, r_0) \supset N(\Delta, U', r_0)$ follows trivially from the definitions.

To obtain the opposite inclusion, first choose $\alpha > 0$ sufficiently small as required above. It is then clear from Item (5) of Proposition 5.3 that $r(\delta)U' \subset \delta U$ for all $\delta \in \Delta$ and r sufficiently close to r_0. In particular $N(\Delta, U, r_0) \supset N(\Delta, U', r)$ and equality holds throughout. $\qquad\square$

We remark that for our purposes it will only be relevant to consider elements of the nerve of size at most three.

Remark 5.7. In general, let G be a group acting by homeomorphisms on a connected path-connected space X and admitting an open path-connected subset $U \subset G$ with $GU = X$. Then one can derive a presentation for G from knowledge of the nerve $N(G, U)$. For details see [31] or [11, I.8.10].

5.3. The space Y. Consider the action of Γ on the Rips complex R. By Proposition 3.2 and the remark following Theorem 3.13 this action is geometric, and in particular cobounded. By Proposition 5.4 there is some $D > 0$ such that $\Gamma U = $ R where U is the bubble

$$U = \mathcal{B}(e, D)$$

and $e \in G$ is regarded as a vertex of R. The bubble U is open and connected according to Proposition 5.3. In addition, denote

$$\Sigma = \{\gamma \in \Gamma : \gamma U \cap U \neq \emptyset\}$$

so that Σ is a finite symmetric generating set for Γ, containing the identity element.

Remark 5.8. Using our notation for nerves, Σ can be defined alternatively as consisting of those $\sigma \in \Gamma$ such that $\{1, \sigma\} \in N(\Gamma, U, r_0)$.

Construction of Y as a quotient space. Consider the space $\Gamma \times U$ with the product topology, Γ being discrete. We introduce a certain equivalence relation $\mathcal{E}_r$ on $\Gamma \times U$.

Definition 5.9. $\mathcal{E}_r$ is the equivalence relation generated by

$$(\gamma, u) \sim_{\mathcal{E}_r} (\gamma', u')$$

if $\gamma' = \gamma\sigma$ for some $\sigma \in \Sigma$ and $r(\gamma)u = r(\gamma')u'$.

We emphasize that the element σ in the above definition must belong to Σ.

Definition 5.10. Let Y be the quotient space $(\Gamma \times U)/\mathcal{E}_r$. Let $[\gamma, u] \in Y$ denote the $\mathcal{E}_r$-class of the point (γ, u). For a subset $V \subset U$ let $[\gamma, V] = \{[\gamma, v] : v \in V\}$.

The map

$$f' : \Gamma \times U \to \text{R}, \quad (\gamma, u) \mapsto r(\gamma)u$$

factors through a well-defined map

$$f : Y \to \text{R}.$$

Note that the space Y, as well as the map f, implicitly depend on the choice of $r \in $ R, that has not yet been specified but should be thought of as close to r_0.

The following propositions establish several useful properties of the space Y and the map $f : Y \to $ R.

Proposition 5.11. *The space Y is connected, assuming that $r \in \mathcal{R}$ is sufficiently close to r_0.*

Proof. Since U is path-connected so is $[\gamma, U]$ for every $\gamma \in \Gamma$; see Proposition 5.3(2). For $r = r_0$ the proposition follows from the definition of Σ and the fact that $\Gamma = \langle \Sigma \rangle$. The same argument shows that Y is connected as long as r is sufficiently close to r_0 so that Proposition 5.6 applies with the data $\Delta = \Sigma$ and U. $\square$

Proposition 5.12. *The $\mathcal{E}_r$-quotient map is a local homeomorphism onto Y, and in particular Y is Hausdorff. The map f is a local homeomorphism, namely f restricts to a homeomorphism on $[\gamma, U]$, $\forall \gamma \in \Gamma$.*

Proof. Let $\pi : \Gamma \times U \to Y$ denote the quotient map with respect to $\mathcal{E}_r$ and denote $f' = f \circ \pi$. Note that

$$u_1 \neq u_2 \Rightarrow [\gamma, u_1] \neq [\gamma, u_2], \quad \forall \gamma \in \Gamma, \ \forall u_1, u_2 \in U$$

which implies that the restriction of π to each $\{\gamma\} \times U$ is a homeomorphism and π is a local homeomorphism. Similarly, Proposition 5.3(1) shows that f' is a local homeomorphism.

The facts that R is Hausdorff and π a local homeomorphism imply that Y is Hausdorff. Since π and f' are both local homeomorphisms and π is surjective, it follows that f is a local homeomorphism as well. $\square$

Proposition 5.13. *The $\mathcal{E}_r$-closure of $\{e\} \times U$ is contained in $\Sigma \times U$, as long as $r \in \mathcal{R}$ is sufficiently close to r_0.*

Proof. Assume that $D > 0$ is a priori chosen so that $U' = \mathcal{B}(x_0, D - \alpha)$ satisfies $R = \Gamma U'$, for $\alpha > 0$ sufficiently small. Using Proposition 5.6, up to replacing U by U' if necessary and without loss of generality, we may assume that $N(\Sigma^2, U, r) = N(\Sigma^2, U, r_0)$ for all r sufficiently close to r_0. Denote this nerve by N. In particular, an element $\gamma \in \Sigma^2$ satisfies $\{1, \gamma\} \in N$ if and only if $\gamma \in \Sigma$.

Consider an $\mathcal{E}_r$-equivalent pair of points $(e, u), (\gamma, u') \in \Gamma \times U$. We claim that in fact $\gamma \in \Sigma$. By Definition 5.9 we may write $\gamma = \sigma_1 \cdots \sigma_n$ where $\sigma_i \in \Sigma$ (possibly with repetitions), and there are points $u_1, \ldots, u_n \in U$ with $u_n = u'$ and

$$u = r(\sigma_1) \cdots r(\sigma_m) u_m, \quad \text{for } 1 \leq m \leq n.$$

We use induction to show that $\sigma_1 \cdots \sigma_m \in \Sigma$, for $1 \leq m \leq n$. The base case $m = 1$ is clear. Next assume that $\sigma = \sigma_1 \cdots \sigma_{m-1} \in \Sigma$. Since

$$u \in U \cap r(\sigma_1 \cdots \sigma_{m-1}) r(\sigma_m) U = U \cap r(\sigma) r(\sigma_m) U$$

this intersection is non-empty. Namely $\{1, \sigma \sigma_m\} \in N$ and by the above $\sigma \sigma_m \in \Sigma$.

The last induction step $m = n$ shows that $\gamma \in \Sigma$ as required. $\square$

The action of Γ on Y. Γ has a natural action on $\Gamma \times U$, which descends to a Γ-action on Y by homeomorphisms. Note that f is Γ-equivariant with respect to this action on Y and the action via r on X, namely

$$f\left(\gamma[\gamma', u]\right) = r(\gamma)r(\gamma')u = r(\gamma\gamma')u, \quad \forall u \in U, \ \gamma, \gamma' \in \Gamma.$$

5.4. f is a covering map. We have constructed a local homeomorphism $f: Y \to \mathrm{R}$, depending on $r \in \mathcal{R}$. The Rips complex R is a length space by Theorem 3.13 and Y is Hausdorff by Proposition 5.12. Therefore we may use f to induce a length metric on Y. Let $\bar{\rho}$ be the pullback length metric on Y induced by $f: Y \to (\mathrm{R}, \rho)$.

This makes f a local isometry according to Proposition 3.17. However, we would like to apply the additional clause of that proposition to deduce that f is actually an s-local isometry for some $s > 0$. Towards this aim we make the following key observation, which is somewhat reminiscent of the existence of a Lebesgue number[10].

Lemma 5.14. *Consider* $\mathrm{R} = \mathrm{R}_c^2(G, d)$ *and* $r \in \mathcal{R}$ *sufficiently close to* r_0. *For every* $L > 0$ *a radius* $D = D(L) > 0$ *may be chosen sufficiently large, so that for any vertex* $x \in f(Y) \cap G \subset \mathrm{R}_{(0)}$ *the inclusions*

$$\mathcal{B}(x, L) \subset f\left([\gamma, U]\right) \subset f(Y) \subset \mathrm{R}$$

hold for some $\gamma \in \Gamma$.

Here D is taken to be radius of the bubble $U = \mathcal{B}(e, D)$ used in Definition 5.10 to construct the space Y. In particular, the choice of the generating set Σ depends on D. It is understood that in our discussion we first specify L and then U and Σ are chosen accordingly.

Proof. Let $L > 0$ be arbitrary and choose $D = D(L)$ sufficiently large so that

$$U' = \mathcal{B}(e, D - L - \eta)$$

satisfies $\mathrm{R} = \Gamma U'$ for some $\eta > 0$. Say $x \in r(\gamma)U = f([\gamma, U]) \subset f(Y)$ with $\gamma \in \Gamma$. As f is Γ-equivariant, we may assume by applying γ^{-1} that $x \in U$ and it then suffices to show that $\mathcal{B}(x, L) \subset r(\sigma)U$ for some $\sigma \in \Sigma$.

By the assumption on U' we know that $x \in \gamma U'$ for some $\gamma \in \Gamma$. So $x \in U \cap \gamma U' \subset U \cap \gamma U$ which implies $\gamma = \sigma \in \Sigma$. Applying Proposition 5.3 we obtain

$$\mathcal{B}(x, L) \subset \sigma \mathcal{B}(e, D - \eta) = \mathcal{B}\left(r_0(\sigma), D - \eta\right).$$

If r is sufficiently close to r_0 so that $d(\sigma, r(\sigma)) < \eta$ for all $\sigma \in \Sigma$ the above conclusion can be modified to read

$$\mathcal{B}(x, L) \subset \mathcal{B}\left(r(\sigma), D\right) = r(\sigma)U = f\left([\sigma, U]\right) \subset f(Y)$$

as required. $\square$

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[10]Unlike the classical Lebesgue number, no compactness argument is used in Lemma 5.14.

Corollary 5.15. *Assume that* $R = R_c^2(G, d)$ *and* $U = \mathcal{B}(e, D)$ *for* $c, D > 0$ *sufficiently large, and that* r *is sufficiently close to* r_0 *in* $\mathcal{R}(\Gamma, G)$. *Then* f *is a homeomorphism.*

Proof. Fix a "Lebesgue constant" $L > c$ and choose $D > 0$ as provided by Lemma 5.14. This in turn determines the bubble $U = \mathcal{B}(e, D)$ and the generating set $\Sigma \subset \Gamma$.

It has already been established that $f : Y \to R$ is a local isometry. Lemma 5.14 provides the additional assumption needed in Proposition 3.17 to deduce that f is an s-local isometry for some $s > 0$. Lemma 3.16 shows that f is a covering map.

With Y being connected by Proposition 5.11 and R being connected and simply-connected the fact that f is a homeomorphism follows[11]. We assume throughout that r is sufficiently close to r_0 as required.   □

5.5. A geometric action of Γ on Y. We have established that $f : Y \to R$ is a Γ-equivariant homeomorphism. According to Proposition 3.18 the map f is in fact an isometry. In other words, $(Y, \bar{\rho})$ is simply an isometric copy of the Rips complex with an action of Γ via the representation r.

Proposition 5.16. Γ *is acting on* $(Y, \bar{\rho})$ *geometrically.*

Proof. The action is clearly by isometries. Moreover, as $Y = \Gamma[e, U]$ it is cobounded. As Γ is discrete the action is locally bounded.

It remains to verify that the action of Γ is metrically proper. Let d_Σ be the word metric on Γ. It suffices to show that $d_\Sigma(\gamma_n, e) \to \infty$ implies $\bar{\rho}([\gamma_n, e], [e, e]) \to \infty$, with e being regarded both as an element of Γ and as a vertex in the 0-skeleton of $U \subset R$. Let $\gamma \in \Gamma$ be an arbitrary element and consider the two points

$$y_e = [e, e], \quad y_\gamma = [\gamma, e]$$

of Y. According to Proposition 3.14 there is a path $p : [0, 1] \to Y$ with

$$p(0) = y_e, \ p(1) = y_\gamma, \ \text{of length } l(p) \le 2 \cdot \bar{\rho}(y_e, y_\gamma)$$

so that the image of p lies in the 1-skeleton $Y_{(1)}$.

We claim that

$$l(p) \ge d_\Sigma(\gamma, e) - 2$$

recalling that the length $l(p)$ equals the number of edges along p. To prove this claim let $v_0, v_1, \ldots, v_{l(p)}$ denote the vertices of $Y_{(0)}$ that appear along the path p. For each $0 \le m \le l(p)$ choose $\delta_m \in \Gamma$ as provided by Lemma 5.14 so that $\mathcal{B}(v_m, L) \subset [\delta_m, U]$. Since $d(v_{m-1}, v_m) < c$ and L was chosen so that $L > c$, this implies $v_m \in \mathcal{B}(v_{m-1}, L)$ for all $1 \le m \le l(p)$. In particular, by Proposition 5.13

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[11]It is for this argument that we crucially rely on G being compactly presented.

and assuming r is sufficiently close to r_0 we see that $\delta_m \in \delta_{m-1}\Sigma$ for $1 \leq m \leq l(p)$. But the same reasoning shows $\delta_0 \in \Sigma$ and $\gamma \in \delta_m\Sigma$ and the required estimate on $l(p)$ follows.

Putting together this claim and the lower bound on $\bar{\rho}(y_e, y_\gamma)$ in terms of $l(p)$ we conclude that the action of Γ on $(Y, \bar{\rho})$ is indeed metrically proper. $\square$

Proposition 5.17. Γ *is acting on* $(Y, \bar{\rho})$ *faithfully.*

Proof. Given an element $\gamma \in \Gamma$ and a point $[\gamma', x] \in Y$ for some $\gamma' \in \Gamma$ and $x \in \mathcal{B}$, recall that $\gamma[\gamma', x] = [\gamma\gamma', x]$. Therefore an element γ which is not in Σ certainly acts non-trivially by Proposition 5.13. Consider the remaining case of an element $\gamma = \sigma \in \Sigma$. Then

$$\sigma[e, e] = [e, e] \iff r(\sigma) = e$$

and so σ is acting non-trivially on Y as long as $r(\sigma) \neq e \in G$. $\square$

As is commonly the case throughout this section, we implicitly assume in both propositions that $r \colon \Gamma \to G$ is sufficiently close to r_0.

5.6. Proof assuming compact presentation. Putting together the results of this section we complete the proof of Theorem 1.1 in the compactly presented case, showing that a small deformation r of Γ is injective, discrete and uniform.

Proof of Theorem 1.1 assuming G is compactly presented. Recall the notations introduced in the beginning of this section. In particular, fix $L > c$ arbitrary and let $D = D(L)$ be as required for Lemma 5.14 to hold. Consider the bubble

$$U = \mathcal{B}(e, D) \subset \mathrm{R}$$

so that $\mathrm{R} = \Gamma U$. Let $\Sigma \subset \Gamma$ be the finite generating set corresponding to U.

We are now in a situation to construct the space Y and the map $f \colon Y \to \mathrm{R}$ depending on r, as in §5.3. Corollary 5.15 and Proposition 3.18 imply that f is a Γ-equivariant isometry with Γ acting on R via r. It was established in Propositions 5.16 and 5.17 that the Γ-action on Y equipped with the induced length metric is geometric and faithful.

We conclude that $r \colon \Gamma \to r(\Gamma)$ is an isomorphism. Therefore the subgroup $r(\Gamma) \leq G$ considered abstractly with the discrete topology is acting geometrically on R. In particular $r(\Gamma)$ is in fact a uniform lattice in G according to Lemma 3.3. $\square$

We extract from the above proof two corollaries that will become useful in the study of Chabauty local rigidity in §7. As before, let Γ be a uniform lattice in the compactly presented group G.

Corollary 5.18. *There is a compact subset* $K \subset G$ *so that* $r(\Gamma)K = G$ *for every deformation* r *sufficiently close to the inclusion mapping.*

Proof. The fact that the Γ-equivariant map $f \colon Y \to X$ is an isometry implies in particular that $r(\Gamma)U = R$. We may take K to be the closed ball $\bar{B}_{(G,d)}(e, D)$, or equivalently the closure in G of the vertex set of the bubble $U = \mathcal{B}(e, D)$. Note that K is compact since the pseudo-metric d is proper. $\square$

Corollary 5.19. *Let $Q \subset G$ be a compact subset so that $\Gamma \cap Q = \emptyset$. Then $r(\Gamma) \cap Q = \emptyset$ provided that the deformation r is sufficiently small.*

Proof. We proceed as in the above proof of topological local rigidity for the compactly presented case. Choose the radius $D > 0$ to be sufficiently large for the purposes of that proof and so that moreover $Q \subset \mathcal{B} = \mathcal{B}(e, D)$ where $\mathcal{B}$ denotes a bubble at e of radius D, see Proposition 5.4.

Note that $r(\gamma) \in Q$ implies in particular $\mathcal{B} \cap r(\gamma)\mathcal{B} \neq \emptyset$. This last condition implies that $\gamma \in \Sigma$ for some finite subset Σ of Γ and for every sufficiently small deformation r, see Proposition 5.13. The requirement that $r(\sigma) \notin Q$ for every $\sigma \in \Sigma$ is an open neighborhood of the inclusion in $\mathcal{R}(\Gamma, G)$. $\square$

6. From compactly presented to compactly generated groups

We now discuss Theorem 1.1 in the general case where $\Gamma \leq G$ is a uniform lattice and G is compactly generated. The main idea is to reduce the question back to the compactly presented situation, using the following result.

Proposition 6.1. *Let G be a compactly generated locally compact group. Then there is a compactly presented locally compact group $\widetilde{G}$ admitting a discrete normal subgroup $N \lhd \widetilde{G}$ so that $\widetilde{G}/N \cong G$ as topological groups.*

Proof. See Proposition 8.A.13 and Corollary 8.A.15 of [18] $\square$

Consider the compactly presented group $\widetilde{G}$ given by Proposition 6.1 and let

$$p \colon \widetilde{G} \to G, \quad \ker p = N \lhd \widetilde{G}$$

be the associated homomorphism and its kernel. For a subgroup $H \leq G$ denote $\widetilde{H} = p^{-1}(H) \leq \widetilde{G}$.

In this situation we have two representation spaces $\mathcal{R} = \mathcal{R}(\Gamma, G)$ and $\widetilde{\mathcal{R}} = \mathcal{R}(\widetilde{\Gamma}, \widetilde{G})$ with the points $r_0 \in \mathcal{R}$ and $\widetilde{r}_0 \in \widetilde{\mathcal{R}}$ corresponding to the inclusions $\Gamma \leq G$ and $\widetilde{\Gamma} \leq \widetilde{G}$ respectively.

Lemma 6.2. *If $\Gamma \leq G$ is a uniform lattice then so is $\widetilde{\Gamma} \leq \widetilde{G}$. Conversely if $\widetilde{\Gamma} \leq \widetilde{G}$ is a uniform lattice containing N, then $\Gamma = p(\widetilde{\Gamma}) \leq G$ is a uniform lattice.*

Proof. Let $\Gamma \leq G$ be a uniform lattice with $G = \Gamma K$ for some compact set $K \subset G$. There is an open set $U \subset G$ such that $U \cap \Gamma = \{e\}$ and so $p^{-1}(U) \cap \widetilde{\Gamma} = N$. Let $V \subset \widetilde{G}$ be an open set such that $V \cap N = \{\widetilde{e}\} \subset \widetilde{G}$. Then $\widetilde{\Gamma} \cap (p^{-1}(U) \cap V) = \{\widetilde{e}\}$

and hence $\widetilde{\Gamma} \leq \widetilde{G}$ is discrete. Using the fact that $\widetilde{G}$ is locally compact and by Lemma 2.C.9 of [18] there exists a compact subset $\widetilde{K} \subset \widetilde{G}$ with $p(\widetilde{K}) = K$. It follows that $\widetilde{\Gamma}\widetilde{K} = \widetilde{G}$ and $\widetilde{\Gamma}$ is a uniform lattice.

Conversely let $N \leq \widetilde{\Gamma} \leq \widetilde{G}$ be a uniform lattice with $\widetilde{G} = \widetilde{\Gamma}\widetilde{K}$ for $\widetilde{K} \subset \widetilde{G}$ compact. The discreteness of $N = \ker p$ implies that p is a local homeomorphism (see e.g. [37, p. 81]). Take $U \subset \widetilde{G}$ open and sufficiently small so that $\widetilde{\Gamma} \cap U = \{\widetilde{e}\}$ and $p(U) \cong U$. Since $N \leq \widetilde{\Gamma}$ we also have $p(U) \cap \Gamma = \{e\}$, hence Γ is discrete. Since $G = \Gamma p(\widetilde{K})$, Γ is a uniform lattice. $\qquad\square$

Definition 6.3. The space $\widetilde{\mathcal{R}}$ admits a closed subspace $\widetilde{\mathcal{R}}_N$ given by

$$\widetilde{\mathcal{R}}_N = \{\widetilde{r} : \widetilde{\Gamma} \to \widetilde{G} : \widetilde{r}(n) = \widetilde{r}_0(n) = n, \ \forall n \in N\}.$$

Associated to the quotient map p there is a well-defined map

$$p_* : \widetilde{\mathcal{R}}_N \to \mathcal{R}, \quad p_*r(\gamma) = p \circ r(\widetilde{\gamma}), \quad \forall \gamma \in \Gamma$$

where $\widetilde{\gamma} \in \widetilde{\Gamma}$ in any element satisfying $p(\widetilde{\gamma}) = \gamma$.

Note that clearly $\widetilde{r}_0 \in \widetilde{\mathcal{R}}_N$. The map p_* is not injective in general.

Proposition 6.4. *The map $p_* : \widetilde{\mathcal{R}}_N \to \mathcal{R}$ is a local homeomorphism.*

Proof. Recall that $\mathcal{R}(\Gamma, G)$ and $\widetilde{\mathcal{R}}_N$ may be identified with closed subsets of the two spaces G^Γ and $\widetilde{G}^{\widetilde{\Gamma}}$, respectively, equipped with the product topology. Let $\gamma \in \Gamma$ be any element and $\widetilde{\gamma} \in \widetilde{\Gamma}$ be such that $p(\widetilde{\gamma}) = \gamma$. The equation $(p_*r)(\gamma) = p(r(\widetilde{\gamma}))$ implies that p_* is both continuous and open.

Let $\widetilde{\Sigma}$ be a finite generating set for $\widetilde{\Gamma}$ and $\widetilde{U} \subset \widetilde{G}$ a symmetric open identity neighborhood so that $\widetilde{U}^2 \cap N = \{e\}$. For every point $\widetilde{r} \in \widetilde{\mathcal{R}}_N$ consider the open neighborhood $\widetilde{r} \in \widetilde{\Omega} \subset \widetilde{\mathcal{R}}_N$ determined by the condition that $\widetilde{s}(\widetilde{\sigma}) \in \widetilde{r}(\widetilde{\sigma})\widetilde{U}$ for every $\widetilde{\sigma} \in \widetilde{\Sigma}$ and every $\widetilde{s} \in \widetilde{\Omega}$. This clearly implies that p_* is injective on $\widetilde{\Omega}$. We conclude that p_* is a local homeomorphism. $\qquad\square$

Proof of Theorem 1.1 for compactly generated groups. Let G be a compactly generated group and $\Gamma \leq G$ a uniform lattice. Let $\widetilde{G}, p, N$ be as in Proposition 6.1. Lemma 6.2 implies that $\widetilde{\Gamma} = p^{-1}(\Gamma)$ is a uniform lattice in $\widetilde{G}$.

Let $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ be the associated representation spaces, as in the above discussion. Applying Theorem 1.1 for compactly presented groups (see §5.6) we deduce the existence of an open neighborhood $\widetilde{r}_0 \in \widetilde{\Omega} \subset \widetilde{\mathcal{R}}$ that satisfies the conclusion of the theorem for the uniform lattice $\widetilde{\Gamma} \leq \widetilde{G}$. Next consider the subset

$$\Omega = p_*(\widetilde{\Omega} \cap \widetilde{\mathcal{R}}_N) \subset \mathcal{R}$$

and it follows from Proposition 6.4 that Ω is an open neighborhood of r_0 in $\mathcal{R}$.

We claim that Theorem 1.1 holds for $\Gamma \leq G$ with respect to the neighborhood $\Omega \subset \mathcal{R}$. To see this, consider a representation $r \in \Omega$ and let $\widetilde{r} \in \widetilde{\Omega} \cap \widetilde{\mathcal{R}}_N$ be such

that $p_* \tilde{r} = r$. Then $\tilde{r}$ is injective and $\tilde{r}(\tilde{\Gamma})$ is a uniform lattice in $\tilde{G}$ containing N and satisfying $\tilde{\Gamma} \cong \tilde{r}(\tilde{\Gamma})$. From Lemma 6.2 it follows that $r(\Gamma) = p(\tilde{r}(\tilde{\Gamma}))$ is a uniform lattice in G. The injectivity of $\tilde{r}$ and the fact that $\tilde{r}(n) = n$ for all elements $n \in N \leq \tilde{\Gamma}$ implies that r is injective as well.   □

7. Chabauty local rigidity

We study uniform lattices regarded as points in the Chabauty space, establishing Chabauty local rigidity and the open conjugacy class property. Recall the *image closure* map associated to a uniform lattice Γ in a topological group G, namely

$$\mathrm{C} \colon \mathcal{R}(\Gamma, G) \to \mathrm{Sub}\,(G), \quad \mathcal{R}(\Gamma, G) \ni r \mapsto \overline{r(\Gamma)} \in \mathrm{Sub}\,(G)$$

Our current goal is the following result.

Theorem 7.1. *Let G be a topological group without non-trivial compact normal subgroups. Let Γ be a uniform lattice in G and $r_0 \in \mathcal{R}(\Gamma, G)$ the inclusion mapping.*

(1) *If G is compactly generated then Γ has a Chabauty neighborhood consisting of uniform lattices.*

(2) *If G is compactly presented then C is a local homeomorphism at r_0 and Γ has a Chabauty neighborhood consisting of isomorphic uniform lattices.*

Theorem 1.3 of the introduction is essentially a restatement of Theorem 7.1.

7.1. The map C is continuous for compactly generated groups.

Proposition 7.2. *Let Γ be a uniform lattice in the compactly generated group G and r_0 the inclusion mapping. Then C is continuous at r_0.*

In proving the above proposition we may assume without loss of generality that G is compactly presented. To see this, let $\tilde{G}$ be a compactly presented group admitting a quotient map $p \colon \tilde{G} \to G$ with discrete kernel $N = \ker p$, as in Proposition 6.1. Consider the commutative square

$$
\begin{array}{ccc}
\tilde{\mathcal{R}}_N & \xrightarrow{\ \tilde{\mathrm{C}}\ } & \mathrm{Sub}_N\,(\tilde{G}) \\
{\scriptstyle p_*}\big\downarrow & & \big\downarrow{\scriptstyle p_*} \\
\mathcal{R}(\Gamma, G) & \xrightarrow{\ \mathrm{C}\ } & \mathrm{Sub}\,(G).
\end{array}
$$

The representation space $\tilde{\mathcal{R}}_N$ and the map p_* along the left vertical arrow are described in Definition 6.3. Here $\mathrm{Sub}_N\,(\tilde{G})$ is the space of closed subgroups of $\tilde{G}$ containing N and the map p_* from that space to $\mathrm{Sub}\,(G)$ is the projection modulo N. The map $p_* \colon \tilde{\mathcal{R}}_N \to \mathcal{R}$ is a local homeomorphism by Proposition 6.4. Now observe that the continuity of C follows from that of the map $\tilde{\mathrm{C}}$.

Proof of Proposition 7.2. Recall the sub-basis sets generating the Chabauty topology given in Definition 2.7. It suffices to verify that the preimage under C of every such sub-basis set containing Γ contains a neighbourhood of r_0. This is straightforward for sets of the form $C^{-1}(\mathcal{O}_1(U))$ with U being open in G. For sets of the form $C^{-1}(\mathcal{O}_2(K))$ with K being a compact subset of G we rely on Corollary 5.19. $\quad\square$

7.2. A Chabauty neighborhood consisting of discrete subgroups. As a preliminary step towards proving Theorem 7.1 we study Chabauty neighborhoods of uniform lattices in topological groups without non-trivial compact normal subgroups. The main result is Proposition 7.4 below.

Lemma 7.3. *Let G be a locally compact group and $\Gamma \le G$ a discrete subgroup. Let U be a symmetric relatively compact identity neighborhood in G with $\Gamma \cap U = \{e\}$. Then every Chabauty neighborhood Ω of Γ has a Chabauty sub-neighborhood Ω_U with $\Gamma \in \Omega_U \subset \Omega$ such that every closed subgroup $H \in \Omega_U$ satisfies*

(1) $H \cap U$ *is a subgroup of G, and*

(2) *There is a closed subgroup $L \le H$ so that $L \in \Omega$ and L normalizes $H \cap U$.*

Proof. We may assume, up to passing to a smaller neighborhood Ω if needed, that there are open relatively compact subsets $V_1, \ldots, V_n \subset G$ and a compact subset $K \subset G$ so that

$$\Omega = \bigcap_{i=1}^{n} \mathcal{O}_1(V_i) \cap \mathcal{O}_2(K)$$

in terms of the notations of Definition 2.7. Let

$$V = \bigcap_{i=1}^{n} \bigcap_{v \in V_i} U^v$$

so that V is an identity neighbourhood in G by [34, Theorem 2.4]. In addition let W be an open symmetric identity neighbourhood satisfying $W^2 \subset U \cap V$.

Consider the Chabauty neighborhood Ω_U, given by

$$\Omega_U = \Omega \cap \mathcal{O}_2(\bar{U} \setminus W).$$

Let $H \in \Omega_U$ be any closed subgroup of G. Observe that the facts $H \cap W = H \cap U$, $W^2 \subset U$ and $W = W^{-1}$ taken together imply that the intersection $H \cap U$ is a subgroup, establishing (1). Choose elements $h_i \in H \cap V_i$ for every $i = 1, \ldots, n$. It remains to show that the closed subgroup $L = \overline{\langle h_1, \ldots, h_n \rangle} \le H$ satisfies (2), that is L normalizes $H \cap U$. Note that

$$(H \cap U)^{h_i} = (H \cap V)^{h_i} = H \cap V^{h_i} \subset H \cap U$$

and so $H \cap U$ is indeed normalized by $\langle h_1, \ldots, h_n \rangle$ and hence by L, as required. $\quad\square$

Proposition 7.4. *Let G be a compactly generated group without non-trivial compact normal subgroup. Let Γ be a uniform lattice in G satisfying $\Gamma \cap U = \{e\}$ for some relatively compact symmetric open identity neighborhood U in G. Then Γ admits a Chabauty neighborhood consisting of uniform lattices Δ with $\Delta \cap U = \{e\}$.*

Proof. The uniform lattice Γ admits a Chabauty neighborhood Ω consisting of cocompact subgroups according to Proposition 3.6. Let U be a relatively compact symmetric open identity neighborhood in G so that $\Gamma \cap U = \{e\}$. Relying on Lemma 7.3 we find a smaller Chabauty neighborhood Ω_U of Γ with $\Omega_U \subset \Omega$ and satisfying Conditions (1) and (2) of Lemma 7.3.

Let H be any closed subgroup in Ω_U. Since $\Omega \subset \Omega_U$ it follows that H is cocompact in G. It remains to show that the intersection $V = H \cap U$ is the trivial subgroup. Note that V is a compact subgroup of G. Moreover there is a closed subgroup $L \leq H$ so that L normalizes V and so that L is cocompact in G.

Let $v \in V$ be an arbitrary element. Since V is normalized by L the set $v^L = \{v^l : l \in L\}$ is relatively compact. Additionally since L is cocompact in G it follows that the entire conjugacy class $v^G = \{v^g : g \in G\}$ of v is relatively compact. Let J be the closed subgroup of G generated by v^G. In particular J in normal in G and it admits a dense subset consisting of elements with a relatively compact conjugacy class in G. Observe that J is compactly generated since the compact set $\overline{v^G}$ generates a dense subgroup of J.

It follows from [28, Lemma 7] and [42] that J admits a unique largest compact subgroup $K \lhd J$ and that J/K is abelian without non-trivial compact subgroups. This implies that K is characteristic in J and therefore $K \lhd G$. By the assumption K is trivial and J is abelian without non-trivial compact subgroups. Recall that $v \in V \cap J$ and therefore v generates a compact subgroup of J. However by the above the element v must be trivial. We conclude that $V = H \cap U = \{e\}$ as required. $\qquad\square$

7.3. Proof of Theorem 7.1.

Let G be a compactly generated group without non-trivial compact normal subgroups and $\Gamma \leq G$ a uniform lattice. It follows from Proposition 7.4 that Γ admits a Chabauty neighborhood Ω consisting of uniform lattices Δ satisfying $\Delta \cap U = \{e\}$ for some relatively compact symmetric open identity neighborhood U in G. This completes the first part of Theorem 7.1.

Assume that G is moreover compactly presented. Recall that the continuity of C at r_0 was given in Proposition 7.2 above. It remains to verify that C is open at r_0. In other words, we would like to show that every $H \in \Omega$ is actually equal to the image of some small deformation of Γ. It would then follows from Theorem 1.1 that Γ admits a Chabauty neighborhood consisting of *isomorphic* uniform lattices.

The map C is open at the point r_0. We rely on the fact that Γ is finitely presented [18, 1.D.4]. Let Σ be a finite generating set for Γ and let V be an identity neighborhood

in G. Up do passing to a smaller Chabauty neighborhood Ω, we may choose elements $h_\sigma \in H \cap V\sigma$ for every $\sigma \in \Sigma$. We propose to construct a map $r_H : \Gamma \to H$ by defining it on generators[12], declaring $r_H(\sigma) = h_\sigma$ for all $\sigma \in \Sigma$. In order that r_H extends to a well-defined homomorphism, it must send all the relations of Γ to the identity. Since Γ is finitely presented, and in view of the fact that $H \cap U = \{e\}$, the map r_H is indeed well-defined provided that the neighborhood V is sufficiently small. Therefore $r_H(\Gamma) \leq H$ and it remains to show that $r_H(\Gamma)$ is equal to H, possibly up to passing once more to a smaller Chabauty neighborhood.

There is a symmetric compact subset $K \subset G$ so that $r_H(\Gamma)K = G$ for every closed subgroup H in Ω provided that the neighborhood V is sufficiently small, see Corollary 5.18. In particular every element $h \in H$ can be written as $h = r_H(\gamma)k_h$ for some elements $\gamma \in \Gamma$ and $k_h \in K \cap H$. The equality $r_H(\Gamma) = H$ will follow as soon as we show that $H \cap K \subset r_H(\Gamma)$ whenever r_H is a sufficiently small deformation.

Let W be a symmetric open identity neighborhood satisfying $W^2 \subset U$. Let Δ be the finite collection of all elements $\delta \in \Gamma$ such that $K \cap W\delta \neq \emptyset$. Consider the compact subset $Q = K \setminus W\Delta$. Every closed subgroup H of G belonging to the Chabauty neighborhood Ω_1

$$\Omega_1 = \Omega \cap \bigcap_{\delta \in \Delta} \mathcal{O}_1(W\delta) \cap \mathcal{O}_2(Q) \subset \mathrm{Sub}\,(G)$$

intersects each translate $W\delta$ exactly once. Such a subgroup H therefore satisfies

$$H \cap K \subset H \cap W\Delta \subset r_H(\Gamma)$$

and in particular $r_H(\Gamma) = H$ by the above discussion. This completes the proof of the fact that C is open at r_0 as well as of Theorem 7.1. $\square$

8. Local rigidity of CAT(0) groups

We turn to the question of local rigidity for uniform lattices in the isometry groups of CAT(0) spaces. Our goal is to prove Theorem 1.2 of the introduction.

In this section we rely to a large extent on the work of Caprace and Monod concerning the structure theory and discrete subgroups of isometry groups of CAT(0) spaces [15, 16]. For additional general information concerning these spaces we refer the reader to the books [11] and [36].

8.1. CAT(0) groups. We now collect several relevant facts about CAT(0) spaces and their lattices to be used below. These are recalled for the reader's convenience.

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[12]The notation r_H is a slight abuse of notation, since the map r_H depends not only on the group H but also on the choices of the elements h_σ.

Let X be a proper CAT(0) space with a cocompact isometry group Isom(X). Then Isom(X) is locally compact, second countable and is acting on X geometrically. Moreover as X is geodesic and simply-connected Isom(X) is compactly presented; see 5.B.5, 5.B.10, 6.A.10 and 8.A.8 of [18].

Definition 8.1. A CAT(0) *lattice* is a lattice in Isom(X). A uniform CAT(0) lattice is called a CAT(0) *group*.

The mere fact that the space X admits a CAT(0) group already has some significant implications. Assume that X is proper and *geodesically complete*, that is every geodesic segment can be extended to a bi-infinite geodesic line, and that $G \leq$ Isom(X) is a closed subgroup acting cocompactly and admitting a uniform lattice $\Gamma \leq G$. Assume moreover that X has no Euclidean factors. Then:

– The boundary ∂X has finite geometric dimension ([27, Theorem C]).

– Γ is acting minimally and without a fixed point in ∂X (geometric Borel density [16, Theorem 3.14]). In particular, G acts minimally and admits no fixed point at infinity as well.

– The centralizer of a lattice in Isom(X) is trivial [16, 2.7].

8.2. Irreducibility and splitting. We recall two notions of irreducibility developed in Section 4 of [16]. In particular, we show that a CAT(0)-group virtually decomposes into a product of abstractly irreducible groups.

Definition 8.2. A discrete group is *abstractly irreducible* if no finite index subgroup splits nontrivially as a direct product.

Definition 8.3. A lattice $\Gamma \leq G_1 \times \cdots \times G_n$ in a product of locally compact groups is *irreducible* if its projection to every proper subproduct of the G_i's is dense and each G_i is non-discrete.

In this terminology, an abstractly irreducible CAT(0) lattice becomes an irreducible lattice in the product of the closures of its projections ([16, Remark 4.1]).

In the case of a uniform lattice $\Gamma \leq G$ such that G is a cocompact subgroup of Isom(X) and X a proper CAT(0) space, these two notions of irreducibility turn out to be related. Namely, Γ is abstractly irreducible if and only if for every finite index subgroup $\Gamma' \leq \Gamma$ and a decomposition $X = X_1 \times X_2$, the projections of Γ' to both Isom(X_i) are non-discrete (see Theorem 4.2 of [16]).

Remark 8.4. As abstract irreducibility is an algebraic condition, it follows from Theorem 1.1 that in the setting of the previous paragraph the property of having non-discrete projections is stable under small deformations.

Lemma 8.5. *Let X be a proper* CAT(0) *space without Euclidean factors with* Isom(X) *acting cocompactly and* $\Gamma \leq G$ *a uniform lattice. Then there is a finite index normal subgroup* $\Gamma' \lhd \Gamma$ *and product decompositions*

$$\Gamma' = \Gamma_1 \times \cdots \times \Gamma_k, \quad X = X_1 \times \cdots \times X_k,$$

such that the $\Gamma_i \leq$ Isom(X_i) *are abstractly irreducible uniform lattices.*

We provide a proof for this lemma as we could not find one in the literature.

Proof of Lemma 8.5. Note that every finite index subgroup Γ_0 of Γ is a uniform lattice and so acts minimally and without a fixed point at infinity ([16, Theorem 3.14]). By the splitting theorem (see Theorem 9, and in particular Corollary 10, in [33]) every direct product decomposition of such a Γ_0 into finitely many factors gives rise to an associated product decomposition of X with the same number of factors.

However, there is an a priori canonical maximal isometric splitting of the space X into a product of finitely many irreducible spaces (see Theorem 1.9 of [15]). Therefore the process of passing to a further finite index subgroup of Γ and writing an abstractly reducible group as a direct product must terminate after finitely many iterations. This gives a finite index subgroup $\Gamma' \leq \Gamma$ that decomposes as the direct product of k abstractly irreducible factors Γ_i. The associated decomposition of X is obtained as above, and it is clear that every Γ_i is a uniform lattice in $\mathrm{Isom}(X_i)$.

It remains to show that Γ' can be taken to be normal in Γ. Indeed Γ preserves the decomposition of X upon possibly permuting isometric factors, as do all elements of $\mathrm{Isom}(X)$. Therefore every conjugate $(\Gamma')^\gamma$ with $\gamma \in \Gamma$ decomposes as a direct product in a manner similar to Γ'. The intersection of such conjugates with γ running over finitely many coset representatives for Γ / Γ' is as required. $\square$

8.3. Totally disconnected isometry groups.

We first deal with the special case of Theorem 1.2 where G is totally disconnected. This relies on the results of §4 and in particular Theorem 4.2.

Theorem 8.6. *Let X be a proper geodesically complete* CAT(0) *space with $G \leq$* $\mathrm{Isom}(X)$ *totally disconnected and acting cocompactly. Then uniform lattices in G are locally rigid in* $\mathrm{Isom}(X)$.

Namely, a small deformation within G of a uniform lattice $\Gamma \leq G$ is a conjugation by an element of $\mathrm{Isom}(X)$.

Proof. According to Theorem 6.1 of [15] and as G is totally disconnected, G is acting on X smoothly in the sense of Definition 4.1. As X is proper and G is acting cocompactly by the assumptions, Theorem 4.2 applies in the present situation and uniform lattices are indeed locally rigid in $\mathrm{Isom}(X)$. $\square$

8.4. Proof of Theorem 1.2.

We proceed to establish the local rigidity of lattices in proper geodesically complete CAT(0) spaces without Euclidean factors. The proof will rely on the following important result, established by Caprace and Monod. Namely, the CAT(0) space X admits a splitting

$$X = M \times Y,$$

where M is a symmetric space of non-compact type. Correspondingly, $\mathrm{Isom}(X)$ has a finite index open characteristic subgroup $\mathrm{Isom}(X)^*$ that decomposes as

$$\mathrm{Isom}(X)^* = G_{\mathrm{c}} \times G_{\mathrm{td}}, \quad G_{\mathrm{c}} = \mathrm{Isom}(M),\, G_{\mathrm{td}} = \mathrm{Isom}(Y)$$

where G_{c} is a product of almost-connected non-compact center-free simple Lie groups and G_{td} is totally disconnected. See Theorems 1.1, 1.6, 1.8 of [15].

Proof of Theorem 1.2. In light of Lemma 2.6 it suffices therefore to prove that Γ admits a locally rigid normal subgroup of finite index.

Let $\Gamma \le \mathrm{Isom}(X)$ be a uniform lattice. Up to passing to a finite index normal subgroup and relying on Lemma 8.5 we obtain a splitting of Γ as a direct product of abstractly irreducible subgroups and an associated splitting of X. Treating each factor of X separately and without loss of generality we may assume that Γ is abstractly irreducible to begin with.

Recall the decomposition $\mathrm{Isom}(X)^* = G_{\mathrm{c}} \times G_{\mathrm{td}}$ mentioned above. Up to passing to a further finite index normal subgroup of Γ we may assume, to begin with, that $\Gamma \le G_{\mathrm{c}}^{\circ} \times G_{\mathrm{td}}$ where G_{c}° denotes the connected component of the identity of G_{c}.

We treat separately three possible cases. If G_{c}° is trivial the conclusion follows from Theorem 8.6. On the other hand, if G_{td} is trivial we may apply the classical result of Weil. Namely, it is shown in [47] that uniform lattices in connected semisimple Lie groups without compact factors are locally rigid, provided that the projection to every factor locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ is non-discrete. This additional condition is ensured by our assumptions.

It remains to deal with the third case where both factors $G_{\mathrm{c}}, G_{\mathrm{td}}$ are non-trivial and Γ is abstractly irreducible. We will rely on the superrigidity theorem of Monod [33] for irreducible uniform lattices in products of locally compact groups. See Theorem 8.4 of [15] for a variant of that theorem adapted to our situation (we also refer to [24] for more general results).

Consider any simple factor S of G_{c}°. As Γ is abstractly irreducible the projection of Γ to S is non-discrete. It follows from the Borel density theorem (see e.g. Chapter 5 of [38]) that this projection is in fact dense. Similarly, Γ projects non-discretely to every factor of G_{td} corresponding to an irreducible factor of Y. Let $D \le G_{\mathrm{td}}$ denote the product of the closures of these projections. It follows that Γ is an irreducible uniform lattice in $G_{\mathrm{c}}^{\circ} \times D$. Moreover as $\Gamma \le \mathrm{Isom}(X)$ is uniform the subgroup D is acting cocompactly on Y.

Let $r \in \mathcal{R}(\Gamma, \mathrm{Isom}(X))$ be a sufficiently small deformation of Γ so that $r(\Gamma)$ is a uniform lattice in $\mathrm{Isom}(X)$, as guaranteed by Theorem 1.1. The action of $r(\Gamma)$ on X is minimal and without fixed points at infinity, as follows from the results of Caprace and Monod discussed in §8.1. In particular, for every simple factor S of G_{c}° the action of Γ on the corresponding irreducible symmetric space M_S given by the projection of $r(\Gamma)$ into S is clearly minimal and fixes no point of ∂M_S.

Regarding Γ as an irreducible lattice in $G_c^\circ \times D$ we are now precisely in a situation to apply Theorem 8.4 of [15], for every simple subgroup S of G_c° and with respect to the given action of Γ on M_S. Therefore this Γ-action extends to a continuous $G_c^\circ \times D$-action by isometries. In other words there is a continuous homomorphism $\alpha: S \to \mathrm{Isom}(M_S)^\circ = S$ such that

$$\pi_S \circ r = \alpha \circ \pi_S \in \mathcal{R}(\Gamma, S),$$

where $\pi_S: G_c^\circ \times D \to S$ is the projection. Since S is simple and α is a Lie group homomorphism with a dense image, it must be an automorphism of S relying on [35] or [6, Corollary 1.4]. According to Lemma 8.8 below, if r is a sufficiently small deformation α must be inner.

We have showed that up to conjugation by an element of G_c° a sufficiently small deformation of Γ may be assumed to be trivial on the G_c° factor. The result now follows from Theorem 4.5 applied to the isometry group $\mathrm{Isom}(X)^* = G_c \times G_{td}$. $\square$

Remark 8.7. Assume that the space X is reducible and that the lattice Γ is abstractly irreducible. Let $G \leq \mathrm{Isom}(X)$ denote the product of the closures of the projections of Γ to the different factors of $\mathrm{Isom}(X)$. Therefore Γ can be regarded as an irreducible lattice in G. In this situation Theorem 1.2 holds in a slightly stronger sense. Indeed, relying on Proposition 2.5 we deduce that Γ regarded as a lattice in G is locally rigid in $N_{\mathrm{Isom}(X)}(G)$.

The following result concerning automorphisms of simple Lie groups was used in the proof of Theorem 1.2 to show that a certain automorphism of a Lie group is inner.

Lemma 8.8. *Let S be a connected center-free simple Lie group and $\Sigma \leq S$ be a finite set generating a dense subgroup. There is an identity neighborhood $U \subset S$ such that if T is an automorphism of S satisfying $T(\sigma)\sigma^{-1} \in U$ for all $\sigma \in \Sigma$, then T is inner.*

Proof. Recall that $\mathrm{Aut}(S)^\circ = \mathrm{Inn}(S) \cong \mathrm{Ad}(S) \cong S$ and that $\mathrm{Aut}(S)$ is canonically isomorphic to $\mathrm{Aut}(\mathrm{Lie}(S))$. Moreover, the analog statement for the Lie algebra, obtained by replacing Σ with a spanning set of $\mathrm{Lie}(S)$ and U with a neighborhood of 0 in the vector space $\mathrm{Lie}(S)$, is obvious by the definition of the topology of $\mathrm{Aut}(\mathrm{Lie}(S))$, since $\mathrm{Aut}(\mathrm{Lie}(S))$ is a Lie group and $\mathrm{Aut}(\mathrm{Lie}(S))^\circ$ is open subgroup. Now since $\langle \Sigma \rangle$ is dense in S, we can find $d = \dim(S)$ words $W_1, \ldots, W_d$ in the generators Σ which fall sufficiently close to the identity element so that $\log(W_i)$, $i = 1, \ldots, d$ are well defined, and such that $\log(W_i)$, $i = 1, \ldots, d$ span $\mathrm{Lie}(S)$. This implies the lemma. $\square$

8.5. Chabauty local rigidity for CAT(0) groups. Let X be a proper geodesically complete CAT(0) space with a cocompact group of isometries $\mathrm{Isom}(X)$. The group $\mathrm{Isom}(X)$ is compactly presented, as was mentioned in §8.1 above. Moreover any

compact normal subgroup $N \lhd \text{Isom}(X)$ is trivial. Indeed, the subspace X^N of N-fixed points is $\text{Isom}(X)$-invariant, convex and non-empty. This discussion shows that Corollary 1.4 immediately follows from Theorem 1.3.

The following statement is a direct consequence of local rigidity for CAT(0) groups as in Theorem 1.2 and of Chabauty local rigidity as in Theorem 7.1.

Corollary 8.9. *Let X be a proper geodesically complete CAT(0) space without Euclidean factors and with $\text{Isom}(X)$ acting cocompactly. Let $\Gamma \leq \text{Isom}(X)$ be a uniform lattice and assume that for every de Rham factor Y of X isometric to the hyperbolic plane the projection of Γ to $\text{Isom}(Y)$ is non-discrete. Then Γ has the open conjugacy class property.*

Proof. The image closure map $C \colon \mathcal{R}(\Gamma, \text{Isom}(X)) \to \text{Sub}\,(\text{Isom}(X))$ is a local homeomorphism at the point $r_0 \in \mathcal{R}(\Gamma, \text{Isom}(X))$ corresponding to the inclusion mapping $r_0 \colon \Gamma \hookrightarrow \text{Isom}(X)$. Theorem 1.2 provides us with an open neighborhood Ω of r_0 in $\mathcal{R}(\Gamma, \text{Isom}(X))$ so that for every $r \in \Omega$ the subgroup $r(\Gamma)$ is conjugate to Γ in G. Therefore the Chabauty neighborhood $C(\Omega)$ of Γ consists of conjugates. $\qquad\square$

9. First Wang's finiteness — lattices containing a given lattice

In this section we prove Theorem 1.8 which is a finiteness result for the number of lattices that contain a given lattice. This is a generalization of a theorem of H.C. Wang [43], and the proof that we give is inspired by Wang's original proof; see also Chapter 9 of [38].

Definition 9.1. Let G be a locally compact group. A family of lattices in G is *jointly discrete*[13] if there is an identity neighborhood $U \subset G$ such that $\Gamma \cap U = \{e\}$ for every lattice Γ in that family.

Recall the notion of property (**KM**) defined in the introduction. The following lemma relates this property to joint discreteness.

Lemma 9.2. *Let G be group and $\mathcal{F}$ a family of lattices in G with property (**KM**). If $\mathcal{F}$ has a least element then it is jointly discrete.*

Proof. Let $U \subset G$ be an open neighborhood as in Definition 1.7 and $\Gamma \leq G$ be a least element of $\mathcal{F}$. Since Γ is of co-finite volume, the set

$$\{g\Gamma \in G/\Gamma : g\Gamma g^{-1} \cap U = \{e\}\}$$

is relatively compact and nonempty. Thus, we can construct a Borel fundamental domain $\mathcal{D}$ for Γ in G such that

$$\mathcal{D}_U := \{g \in \mathcal{D} : g\Gamma g^{-1} \cap U = \{e\}\}$$

is relatively compact.

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[13]In the older literature such a family is called *uniformly discrete*. However, in the more recent literature, the notion of uniform discreteness is often used for the stronger property given in Definition 1.5.

Let $\Gamma' \leq G$ be any lattice in the family $\mathcal{F}$, so that $\Gamma \leq \Gamma'$. Then Γ' admits a fundamental domain $\mathcal{D}'$ which is contained in $\mathcal{D}$. Because of property **(KM)**, there is an element $g \in \mathcal{D}'$ such that $g\Gamma'g^{-1} \cap U = \{e\}$. As $\Gamma \leq \Gamma'$ we have that $g \in \mathcal{D}_U$. Finally since G is locally compact and $\mathcal{D}_U$ is relatively compact

$$U_\Gamma := \bigcap_{g \in \mathcal{D}_U} g^{-1}Ug$$

is a neighborhood of the identity in G; see e.g. [34, Theorem 2.4]. By construction the neighborhood U_Γ intersects trivially every lattice from the family $\mathcal{F}$. $\square$

Theorem 1.8 is a consequence of Lemma 9.2 and the following result, which is of independent interest.

Theorem 9.3. *Let G be a compactly generated locally compact group. Assume that every lattice in G has a trivial centralizer and let $\Gamma \leq G$ be a finitely generated lattice. Then every jointly discrete family of lattices in G all containing Γ is finite.*

In the proof we rely on the notion of Chabauty topology as well as the Mahler–Chabauty compactness criterion, see §7. The discussion of these notions in [38, Chapter 1] required separability, but that assumption is in fact redundant.

Proof of Theorem 9.3. Observe that the co-volume of the lattices Γ' with $\Gamma \leq \Gamma' \leq G$ and $\Gamma' \cap U = \{e\}$ is bounded from below, and since

$$\text{co-vol}(\Gamma) = [\Gamma' : \Gamma] \cdot \text{co-vol}(\Gamma'),$$

the index $[\Gamma' : \Gamma]$ is bounded from above. We may therefore restrict our attention to lattices Γ' as above with $[\Gamma' : \Gamma] = d$, where $d \in \mathbb{N}$ is fixed. Assume that there exists a sequence Γ_n of pairwise distinct lattices such that

$$\Gamma \leq \Gamma_n \leq G, \quad [\Gamma_n : \Gamma] = d, \quad \text{and} \quad \Gamma_n \cap U = \{e\}.$$

We will arrive at a contradiction by showing that Γ_n has an infinite repetition.

Consider the actions of Γ_n on the finite sets Γ_n/Γ given by the natural homomorphisms $\varphi_n\colon \Gamma_n \to \text{Sym}(\Gamma_n/\Gamma) \cong S_d$ for every $n \in \mathbb{N}$. Denote

$$\Gamma'_n = \ker \varphi_n \lhd \Gamma_n$$

so that

$$\Gamma'_n \leq \Gamma \quad \text{and} \quad [\Gamma : \Gamma'_n] = \frac{[\Gamma_n : \Gamma'_n]}{[\Gamma_n : \Gamma]} \leq (d-1)!$$

Note that since Γ is finitely generated it has finitely many subgroups of every given finite index. We now consider the group $\Delta \leq \Gamma$ given by

$$\Delta = \bigcap_n \Gamma'_n$$

and deduce from the above discussion that Δ is a finite index subgroup of Γ. In particular Δ is finitely generated as well. In addition, for every pair of elements $\sigma \in \Delta$ and $\delta_n \in \Gamma_n$ we have that

$$\delta_n \sigma \delta_n^{-1} \in \delta_n \Delta \delta_n^{-1} \leq \delta_n \Gamma_n' \delta_n^{-1} = \Gamma_n' \leq \Gamma.$$

We now apply the Mahler–Chabauty compactness criterion. Namely, from the assumption $\Gamma_n \cap U = \{e\}$ it follows that some subnet Γ_{n_α} converges in the Chabauty topology to a discrete subgroup $\Lambda \leq G$ containing Γ and such that

$$\text{co-vol}(\Lambda) \leq \lim_\alpha \text{co-vol}(\Gamma_{n_\alpha}) = d^{-1} \cdot \text{co-vol}(\Gamma).$$

In particular, it follows that $[\Lambda : \Gamma] \geq d$. On the other hand Λ is a lattice containing Γ and hence Λ is finitely generated.

By the definition of Chabauty convergence, for every $\lambda \in \Lambda$ there is a $\delta_{n_\alpha} \in \Gamma_{n_\alpha}$ with $\delta_{n_\alpha} \to \lambda$. We obtain the following expression for every $\sigma \in \Delta$

$$\lim_\alpha \delta_{n_\alpha} \sigma \delta_{n_\alpha}^{-1} = \lambda \sigma \lambda^{-1}.$$

Since the elements $\delta_{n_\alpha} \sigma \delta_{n_\alpha}^{-1}$ all belong to the discrete group Γ this converging net must eventually stabilize. It other words

$$\delta_{n_\alpha} \sigma \delta_{n_\alpha}^{-1} = \lambda \sigma \lambda^{-1}$$

holds for all $\alpha \geq \alpha_\sigma$. Applying this argument with σ ranging over a finite generating set for Δ and in light of the fact that $C_G(\Delta)$ is trivial we see that $\delta_{n_\alpha} = \lambda$ for all α sufficiently large.

By the above argument, every generator of Λ belongs to Γ_{n_α} and in particular $\Lambda \leq \Gamma_{n_\alpha}$ for all α sufficiently large. However the fact that $[\Lambda : \Gamma] \geq [\Gamma_{n_\alpha} : \Gamma]$ shows that the net Γ_{n_α} must eventually stabilize, producing the required infinite repetition. $\qquad\square$

10. Second Wang's finiteness — lattices of bounded covolume

The classical finiteness theorem of Wang [44] states that a connected semisimple Lie group without factors locally isomorphic to $SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$ admits only finitely many conjugacy classes of irreducible lattices of volume bounded by v for any $v > 0$.

Recall Theorem 1.6 of the introduction which is a suitable generalization of Wang's finiteness to the CAT(0) context, restated here for the reader's convenience.

Theorem. *Let X be a proper geodesically complete CAT(0) space without Euclidean factors and with $\text{Isom}(X)$ acting cocompactly. Let $\mathcal{F}$ be a uniformly discrete family of lattices in $\text{Isom}(X)$ so that every $\Gamma \in \mathcal{F}$ projects non-discretely to the isometry group of every hyperbolic plane factor. Then $\mathcal{F}$ admits only finitely many conjugacy classes of lattices with co-vol $\leq v$ for every fixed $v > 0$.*

The proof follows rather immediately from a combination of Mahler–Chabauty compactness criterion and the open conjugacy class property for CAT(0) groups established in §7.

Proof of Theorem 1.6. Assume without loss of generality that the family $\mathcal{F}$ is closed under conjugation by elements of $\mathrm{Isom}(X)$. In particular $\mathcal{F}$ is jointly discrete in the sense of Definition 9.1, so that there is an identity neighborhood $U \subset G$ with $\Gamma \cap U = \{e\}$ for all $\Gamma \in \mathcal{F}$.

Consider some fixed $v > 0$. The Mahler–Chabauty compactness criterion [38, 1.20] says that the set of lattices in $\mathrm{Sub}\,(G)$ of co-volume at most v and intersecting U only at the identity element is compact. In particular $\mathcal{F}$ is relatively compact.

On the other hand, every lattice in the Chabauty closure of $\mathcal{F}$ is uniform by [38, 1.12] and therefore has the open conjugacy class property by Corollary 8.9. The required finiteness result follows. $\square$

The following consequence of Theorem 1.6 covers many classical examples:

Corollary 10.1. *Let X and $\mathrm{Isom}(X)$ be as in Theorem 1.6. Let $G \le \mathrm{Isom}(X)$ be a cocompact subgroup and $\mathcal{F}$ a uniformly discrete family of lattices in G. Then $\mathcal{F}$ admits only finitely many $\mathrm{Isom}(X)$-conjugacy classes of lattices with co-vol $\le v$ for every fixed $v > 0$.*

Since G is cocompact in $\mathrm{Isom}(X)$, for a lattice $\Gamma \le G$ the covolumes of Γ in G and in $\mathrm{Isom}(X)$ are proportional. Therefore to deduce Corollary 10.1 from Theorem 1.6 it suffices to show that $\mathcal{F}$ is uniformly discrete in $\mathrm{Isom}(X)$. This follows from:

Lemma 10.2. *Let H be a locally compact group, $G \le H$ a cocompact subgroup and $\mathcal{F}$ a family of subgroups which is uniformly discrete in G. Then $\mathcal{F}$ is uniformly discrete regarded as a family of lattices in H.*

Proof. Suppose that $\mathcal{F}$ is uniformly discrete in G with respect to the open subset $U \subset G$. Say $U = G \cap V$ with $V \subset H$ open. Moreover let $K \subset H$ be a compact symmetric subset so that $H = GK$ and choose an open subset $W \subset V$ so that $k^{-1}Wk \subset V$ for every $k \in K$, using e.g. [34, Theorem 2.4]. Then $\mathcal{F}$ is uniformly discrete in H with respect to W. Indeed, for $\Gamma \in \mathcal{F}$ and an element $h = gk$ with $h \in H$, $g \in G$ and $k \in K$ we have the equation $\Gamma^h = (\Gamma^g)^k$. $\square$

10.1. Linear groups over local fields. Corollary 10.1 immediately implies a Wang-type finiteness result for uniform lattices in simple Chevalley groups over non-Archimedean local fields. Such groups act on the associated Bruhat–Tits building X which is a CAT(0) space; see e.g. [11, II.10.A].

Corollary 10.3. *Let G be a k-simple algebraic k-group with $\mathrm{rank}_k(G) > 0$ where k is a non-Archimedean local field. Let X be the Bruhat–Tits building associated to G. Fix an identity neighborhood $U \subset G$ and some $v > 0$. Then there are only*

finitely many Isom(X)-*conjugacy classes of lattices* Γ *in* G *with co-vol*(Γ) $< v$ *and* $\Gamma^g \cap U = \{e\}$ *for every* $g \in G$.

Remark 10.4. If rank$_k(G) \geq 2$ then Isom(X) = Aut(G) and the statement of Corollary 10.3 can be somewhat simplified. See [2, Proposition C.1].

11. Invariant random subgroups

Let G be a locally compact group. It is acting on its space of closed subgroups Sub (G) by conjugation and this action is continuous.

Definition 11.1. An *invariant random subgroup* of G is a G-invariant Borel probability measure on Sub (G). Let IRS (G) denote the space of all invariant random subgroups of G equipped with the weak-$*$ topology.

The space IRS (G) is compact. For a further discussion of invariant random subgroups the reader is referred to [1, 23].

Let ULat(G) denote the space of all the uniform lattices in G with the induced topology from Sub (G). There is a natural map

$$\mathrm{I} \colon \mathrm{ULat}(G) \to \mathrm{IRS}\,(G), \quad \mathrm{I}(\Gamma) = \mu_\Gamma,$$

where the invariant random subgroup $\mu_\Gamma \in$ IRS (G) is obtained by pushing forward the G-invariant probability measure from G/Γ to Sub (G) via the map $g\Gamma \mapsto g\Gamma g^{-1}$.

Proposition 11.2. *If* G *is compactly generated then the map* $\mathrm{I} \colon \mathrm{ULat}(G) \to \mathrm{IRS}(G)$ *is continuous.*

The proof of this proposition depends on showing that the covolume function on the space ULat(G) is continuous, see Proposition 11.4 below.

Recall that a uniform lattice Γ in G is topologically locally rigid. Therefore, Proposition 7.2 implies that there exists a neighborhood Ω of the inclusion homomorphism r_0 in $\mathcal{R}(\Gamma, G)$ on which is image closure map C is continuous and takes values in ULat(G).

Corollary 11.3. *Let* Γ *be a uniform lattice in* G. *If* G *is compactly generated then there is a neighborhood* $\Omega \subset \mathcal{R}(\Gamma, G)$ *of the inclusion homomorphism* r_0 *so that the composition*

$$\Omega \xrightarrow{C} \mathrm{ULat}(G) \xrightarrow{\mathrm{I}} \mathrm{IRS}(G)$$

is continuous.

Note that local rigidity of Γ is equivalent to I $\circ$ C being locally constant at r_0.

11.1. Co-volumes of lattices. Let G be any locally compact group with a choice μ of a Haar measure. The co-volume function is well-defined on the space $\mathrm{ULat}(G)$ of uniform lattices in G.

Proposition 11.4. *If G is compactly generated then the co-volume function is continuous on $\mathrm{ULat}(G)$.*

The proof of Proposition 11.4 will rely on the following lemma, which is related to Serre's geometric volume formula [8, Chapter 1.5].

Lemma 11.5. *Let $K \subset G$ be a fixed compact set. Given a uniform lattice $\Gamma \in \mathrm{ULat}(G)$ let $\pi_\Gamma \colon G \to G/\Gamma$ denote the projection and let μ_Γ denote the induced measure on G/Γ. Then the following function*

$$v_K \colon \mathrm{ULat}(G) \to \mathbb{R}_{\geq 0}, \quad v_K(\Gamma) = \mu_\Gamma(\pi_\Gamma(K)).$$

is continuous.

Proof. Given a compact subset Q of G, consider the function

$$n_Q \colon G \times \mathrm{ULat}(G) \to \mathbb{N}_{\geq 0}, \quad n_Q(x, \Gamma) = |\{\gamma \in \Gamma : x\gamma \in Q\}|.$$

Note that $n_Q(\cdot, \Gamma)$ is Γ-invariant from the right, and so descends to a well defined function on G/Γ.

In terms of the function n_Q we obtain the formula:

$$
\begin{aligned}
v_Q(\Gamma) = \mu_\Gamma(\pi_\Gamma(Q)) &= \int_{G/\Gamma} \mathbb{1}_{\pi_\Gamma(Q)}(x)\, d\mu_\Gamma(x) \\
&= \int_{G/\Gamma} \frac{1}{n_Q(x, \Gamma)} \left(\sum_{\gamma \in \Gamma} \mathbb{1}_Q(\tilde{x}\gamma) \right) d\mu_\Gamma(x) \\
&= \int_G \frac{1}{n_Q(x, \Gamma)} \mathbb{1}_Q(x)\, d\mu(x) \\
&= \int_Q \frac{1}{n_Q(x, \Gamma)}\, d\mu(x),
\end{aligned}
$$

where $\tilde{x}$ is an arbitrary lift of x.

Let $\Gamma_0 \in \mathrm{ULat}(G)$ be some fixed uniform lattice. The following two facts[14] are consequences of the definition of the Chabauty topology:

(1) $\lim_{\Gamma \to \Gamma_0} n_K(x, \Gamma) \leq n_K(x, \Gamma_0)$, for all $x \in G$.

(2) If Q is a compact set containing K in its interior, then

$$\lim_{\Gamma \to \Gamma_0} n_Q(x, \Gamma) \geq n_K(x, \Gamma_0)$$

for all $x \in G$.

---

[14]The convergence $\Gamma \to \Gamma_0$ is understood in the sense of nets. If G is metrizable then it is enough to consider sequences.

Since $n_K(x, \Gamma) \geq 1$ for all $x \in K$, it follows from Item (1), the above volume formula and the dominated convergence theorem that $v_K(\Gamma_0) \leq \lim_{\Gamma \to \Gamma_0} v_K(\Gamma)$.

To obtain the reverse inequality, let $\varepsilon > 0$ and pick a compact set Q containing K in its interior and satisfying $\mu(Q \setminus K) < \varepsilon$. Note that this implies

$$\int_{Q \setminus K} \frac{1}{n_Q(x, \Gamma)} \, d\mu(x) < \epsilon$$

for all $\Gamma \in \mathrm{ULat}(G)$. It follows from Item (2) and the volume formula that

$$v_K(\Gamma_0) = \int_K \frac{1}{n_K(x, \Gamma_0)} \, d\mu(x) \geq \lim_{\Gamma \to \Gamma_0} \int_K \frac{1}{n_Q(x, \Gamma)} \, d\mu(x).$$

Combining these two last inequalities, we get

$$v_K(\Gamma_0) + \epsilon > \lim_{\Gamma \to \Gamma_0} \int_Q \frac{1}{n_Q(x, \Gamma)} \, d\mu(x) = \lim_{\Gamma \to \Gamma_0} v_Q(\Gamma) \geq \lim_{\Gamma \to \Gamma_0} v_K(\Gamma).$$

The continuity of the map v_K at the point Γ_0 follows by letting $\varepsilon \to 0$. $\square$

Proof of Proposition 11.4. Let Γ be a uniform lattice in G. Since G is compactly generated it follows from Proposition 3.6 and Remark 3.7 that there is a Chabauty neighborhood $\Gamma \in \Omega \subset \mathrm{ULat}(G)$ and a compact subset $K \subset G$ so that $\Gamma_0 K = G$ for every $\Gamma_0 \in \Omega$. Taking into account Lemma 11.5 and as soon as $\Gamma' \in \Omega$ we obtain

$$\text{co-vol}(G/\Gamma) = v_K(\Gamma) = \lim_{\Gamma' \to \Gamma} v_K(\Gamma') = \lim_{\Gamma' \to \Gamma} \text{co-vol}(G/\Gamma')$$

as required. $\square$

Remark 11.6. If $(\Gamma_\alpha)_\alpha$ is a net of lattices in G converging in the Chabauty topology to the lattice Γ then

$$\text{co-vol}(G/\Gamma) \leq \liminf_\alpha \text{co-vol}(G/\Gamma_\alpha).$$

This does not require compact generation or co-compactness [38, I.1.20]. Proposition 11.4 can be regarded as a refinement of this fact.

Proof of Proposition 11.2. Let G be a compactly generated group and $\Gamma_\alpha \in \mathrm{ULat}(G)$ be a net[15] of uniform lattices in G indexed by $\alpha \in A$ and converging to $\Gamma \in \mathrm{ULat}(G)$.

Let $\mathcal{D}$ be a Borel fundamental domain for Γ in G. There is an increasing sequence of compact subsets $K_m \subset \mathcal{D}$ so that the Haar measure of $\mathcal{D} \setminus K_m$ goes to zero as $m \to \infty$. Moreover the argument on p. 28 of [38] shows that every K_m injects into G/Γ_α for all $\alpha > \beta_1 = \beta_1(m)$ sufficiently large in A.

---

[15]Once more, if G is metrizable we may speak of sequences instead of nets.

Let $\varepsilon > 0$ be arbitrary and take $l = l(\varepsilon)$ so that $\mu(\mathcal{D} \setminus K_l) < \varepsilon$. Let $\mathcal{D}_\alpha$ be a Borel fundamental domain for Γ_α in G for every $\alpha \in A$. The subsets $\mathcal{D}_\alpha$ can be chosen in such a way that $K_l \subset \mathcal{D}_\alpha$ for all $\alpha > \beta_1(l)$.

Let μ be the Haar measure on G and $\mu_{|\mathcal{D}}$ its restriction to $\mathcal{D}$. Then $\mu_{|\mathcal{D}}$ projects to the G-invariant probability measure on G/Γ. Similarly let $\mu_{|\mathcal{D}_\alpha}$ denote the restrictions of μ to $\mathcal{D}_\alpha$ for $\alpha \in A$. Write

$$\mu_{|\mathcal{D}} = \mu_{|K_l} + \mu_{|\mathcal{D} \setminus K_l}, \quad \mu_{|\mathcal{D}_\alpha} = \mu_{|K_l} + \mu_{|\mathcal{D}_\alpha \setminus K_l}.$$

Recall that $\mu(\mathcal{D} \setminus K_l) < \varepsilon$ by the choice of l. In addition, the covolume function is continuous at the point $\Gamma \in \mathrm{ULat}(G)$ according to Proposition 11.4 and so

$$\mu(\mathcal{D}_\alpha) = \text{co-vol}(G/\Gamma_\alpha) \xrightarrow{\alpha \in A} \text{co-vol}(G/\Gamma) = \mu(\mathcal{D}).$$

This implies that $\mu(\mathcal{D}_\alpha \setminus K_l) < 2\varepsilon$ for all $\alpha > \beta_2 = \beta_2(\varepsilon)$ in A.

Consider the maps φ_α defined for every $\alpha \in A$

$$\varphi_\alpha \colon K_l \to \mathrm{Sub}\,(G), \quad \varphi_\alpha(k) = k\Gamma_\alpha k^{-1}.$$

Since K_l is compact, the functions φ_α converge uniformly to the function $\varphi \colon K_l \to \mathrm{Sub}\,(G)$ where $\varphi(k) = k\Gamma k^{-1}$. In particular

$$(\varphi_\alpha)_* \mu_{|K_l} \xrightarrow{\alpha \in A} (\varphi)_* \mu_{|K_l}.$$

Taking $\varepsilon \to 0$ in the above argument gives the continuity of I at the point Γ. $\square$

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