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## Non-arithmetic ball quotients from a configuration of elliptic curves in an Abelian surface

Martin Deraux

**Abstract.** We construct some non-arithmetic ball quotients as branched covers of a quotient of an Abelian surface by a finite group, and compare them with lattices that previously appear in the literature. This gives an alternative construction, which is independent of the computer, of some lattices constructed by the author with Parker and Paupert.

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### 1. Introduction

Recall that finite groups generated by complex reflections have been classified by Shephard and Todd [25]. Such a group  $G$  comes with an isometric action on  $\mathbb{P}^n$  (for the Fubini–Study metric), and the quotient  $X = G \backslash \mathbb{P}^n$  turns out to be a weighted projective space. In fact the ring of invariant polynomials in  $n + 1$  variables is generated by homogeneous polynomials  $f_{d_0}, \dots, f_{d_n}$ , where  $f_{d_j}$  has degree  $d_j$ , hence the quotient, which is given by the projective spectrum of the ring of invariants, is the weighted projective plane  $\mathbb{P}(d_0, \dots, d_n)$ . Note that the weights can be computed from simple combinatorial data, since the degrees satisfy  $d_0 \cdot d_1 \cdots d_n = |G|$ , and  $\sum (d_j - 1)$  is equal to the number of reflections in the group (see [26], for instance).

An analogous classification has been produced for affine crystallographic complex reflection groups, see [23]. The basis for the classification of affine groups is the fact that the group of automorphisms of affine space  $\mathbb{C}^n$  is a semi-direct product  $V \rtimes GL(V)$ , where  $V$  is the vector space of translations in  $\mathbb{C}^n$ , which allows us to reduce the classification to the problem of classifying extensions of finite unitary reflection groups by a lattice in  $\mathbb{C}^n$ . In particular, if  $G$  is an affine crystallographic group, the quotient  $G \backslash \mathbb{C}^n$  is the quotient of a complex torus by a finite group. Note that in general,  $G$  is not the semi-direct product of its linear part and its translation subgroup (see [23, p. 57]).

It was observed by Bernstein and Schwarzman [4] that, at least in many cases, if  $G$  is generated by complex reflections, the quotient  $G \backslash \mathbb{C}^n$  is again a weighted projective plane. The heart of their proof is to construct suitable  $\Theta$ -functions that play the role of the homogeneous invariant polynomials in the Shephard–Todd–Chevalley theorem, which they managed to do only when the linear part of  $G$  is a real Coxeter group (in that case, the weights of the weighted projective space are given by the so-called *exponents* of the corresponding Coxeter group).

Some quotients  $G \backslash \mathbb{C}^2$ , where  $G$  is an affine crystallographic complex reflection group whose linear part is not a Coxeter group, were worked out by Kaneko, Tokunaga and Yoshida [18], building on the Bernstein–Schwarzman result. The corresponding quotients turn out to be explicit weighted projective planes, but their proof does not shed much light on the general case. Still, for a general affine crystallographic complex reflection group  $G$ , it is believed that the quotient  $G \backslash \mathbb{C}^n$  should be a weighted projective space (see [14, p. 17]).

In this note, we investigate a particular affine crystallographic complex reflection group  $G$ , whose linear part is the Shephard–Todd group  $G_{12}$ , and whose subgroup of translations is given by the lattice  $\Lambda = (\mathbb{Z} \oplus i\sqrt{2}\mathbb{Z})^2$ . In other words, there is an extension

$$1 \rightarrow \Lambda \rightarrow G \rightarrow G_{12} \rightarrow 1, \quad (1.1)$$

and one can think of the quotient  $G \backslash \mathbb{C}^2$  as the quotient of the Abelian surface  $A = \mathbb{C}^2/\Lambda$  by the group  $G_{12}$ . For concreteness, we mention that the group  $G_{12}$  has order 48, it is a central extension of the octahedral group, and it is also known to be isomorphic to  $GL(2, \mathbb{F}_3)$ .

Our group  $G$  is not a semi-direct product (equivalently the sequence (1.1) does not split), which characterizes it uniquely up to affine equivalence, according to [23]. Note also that the action of  $G_{12}$  has no global fixed point in the Abelian surface  $A$ , so our action is *not* the same as the action given by Birkenhake and Lange (see [5, Theorem 13.4.5])

Since  $G_{12}$  is not a Coxeter group, it is not in the list of groups treated by Bernstein and Schwarzman, and it is not in the list of groups treated by Kaneko, Tokunaga and Yoshida, so the structure of the quotient seems to be unknown.

We will show that the quotient  $X = G \backslash \mathbb{C}^2$  has two singular points of type  $\frac{1}{3}(1, 2)$  and  $\frac{1}{8}(1, 3)$  respectively, and that the map  $A \rightarrow X$  ramifies with order 2 along a (highly singular) rational curve in  $X$ , which is given by the image of the union of all mirrors of complex reflections in the group  $G$ . We refer to the branch locus as the discriminant curve, and denote it by  $M$ . It does not contain any of the singular points of  $X$ , and the curve  $M$  has four singular points, two ordinary cusps, a point with multiplicity four and another with multiplicity 6 (see Figure 1 for a schematic picture of the singularities of  $M$ ).

Assuming that  $X$  is indeed a weighted projective plane, the list of its singular points shows that it must be isomorphic to  $\mathbb{P}(1, 3, 8)$ . The curve  $M$  would then be an irreducible curve of homogeneous degree 24, whose explicit equation remains

elusive (see [10] for the analogous equation in the case of  $\mathbb{P}(2, 3, 7)$  in relation to the Klein quartic).

We can rephrase the preceding paragraphs as follows.

**Theorem 1.1.** *The pair  $(X, \frac{1}{2}M)$  is an orbifold which is uniformized by  $\mathbb{C}^2$ , and  $\pi_1^{\text{orb}}(X, \frac{1}{2}M)$  is the affine crystallographic complex reflection group  $G$ .*

Theorem 1.1 parallels Proposition 2 of [10], which says that  $\mathbb{P}(2, 3, 7)$ , with a specific curve with weight  $\frac{1}{2} = 1 - \frac{1}{2}$ , gives an orbifold uniformized by the positively curved complex space form  $\mathbb{P}^2$ . The main result in [10] is obtained by changing  $p = 2$  to higher integer values, i.e. changing the weight of the curve to be  $1 - \frac{1}{p}$  (in fact, for most values of  $p$ , more subtle modifications are needed).

It is then tempting to mimic the construction of [10], and to consider the pairs  $(X, (1 - \frac{1}{p})M)$  for integer values  $p > 2$  (in fact, it is convenient to allow also  $p = \infty$ ). The basic questions are the following.

- (i) When is the pair  $(X, (1 - \frac{1}{p})M)$  an orbifold? When it is an orbifold, is it modeled on a space form?
- (ii) When it is not an orbifold, is there a suitable model birational to it that is an orbifold? If so, is that orbifold modeled on a space form?

The main goal of the present paper is to show that, even though the answer to (i) (only for  $p = 2$ ) may seem disappointing, there is an affirmative answer to question (ii) for some other well-chosen values of  $p$ , namely  $p = 3, 4, 6$  or  $\infty$ . For these values, the universal cover is the 2-dimensional complex space form of curvature  $-1$ , which we denote by  $\mathbb{H}^2$  (for basic facts on the complex hyperbolic plane  $\mathbb{H}^2$  and lattices in its isometry group, see Section 2).

The precise statements are somewhat technical (they will only be given in Section 5), because on the one hand the birational modifications are not that easy to describe, and on the other only a proper open set turns out to be uniformized by  $\mathbb{H}^2$ . For now we suggest that the reader keeps in mind that the statements below roughly say that there is indeed a complex hyperbolic uniformization of suitable open sets in the pairs  $(X, (1 - \frac{1}{p})M)$ , for  $p = 3, 4, 6, \infty$ .

For  $p = 3$ , we will prove the following.

**Theorem 1.2.** *The pair  $(X, (1 - \frac{1}{3})M)$  is a compactification of a ball quotient. More precisely, there is a lattice  $\Gamma_3 \subset PU(2, 1)$  with one cusp, such that  $X_0 = \Gamma_3 \backslash \mathbb{H}^2$  has 1-point compactification isomorphic to  $X$ . Modulo this isomorphism, the quotient map  $\mathbb{H}^2 \rightarrow X_0$  branches with order 3 along  $M_0$ , which is obtained from  $M$  by removing its point with multiplicity 6.*

Note that the presence of a 1-dimensional branch locus for the quotient map  $\mathbb{H}^2 \rightarrow X$  says that the lattice  $\Gamma_3$  contains complex reflections, we will see later that it is actually generated by complex reflections. Observe also that the point that needs to be removed from  $X$  in order to get a ball quotient is characterized by the fact that it is the only point where the pair  $(X, (1 - \frac{1}{3})M)$  is not log-terminal, see Section 5.



The orbifold structure on  $X = A/G$  does not lift to an orbifold structure on the Abelian surface  $A$ , since the corresponding weights on the preimage of the discriminant curve would have to be equal to  $3/2$ , which is not an integer.

There are statements analogous to Theorem 1.2 for  $p = 4, 6$  or  $p = \infty$ , but then the pair  $(X, (1 - \frac{1}{p})M)$  is actually not an orbifold, and one needs to perform a suitable birational modification before it becomes one. After suitable modification, for each case  $p = 4, 6$  or  $p = \infty$ , one gets an orbifold which is uniformized by  $\mathbb{H}^2$ , with orbifold fundamental group given by a non-cocompact lattice  $\Gamma_p$ .

For now we simply give a rough statement.

**Theorem 1.3.** *There are lattices  $\Gamma_p \subset PU(2, 1)$ ,  $p = 4, 6, \infty$  such that  $\Gamma_p \backslash \mathbb{H}^2$  has a compactification birational to  $X$ . The groups  $\Gamma_p$  have one cusp for  $p = 4, \infty$ , two cusps for  $p = 6$ .*

The explicit birational transformation that yields the corresponding compactification will be given later in the paper (see Section 5, Theorem 5.5 in particular).

For the groups that appear in Theorem 1.3, the weight of the orbifold structure along (the strict transform of) the discriminant curve is even, so the orbifold structure on (the suitable surface birational to)  $X = A/G$  lifts to an orbifold structure on (a suitable blow up of) the Abelian surface  $A$ , with multiplicity  $2 = 4/2$ ,  $3 = 6/2$  or  $\infty$  respectively at a generic point of the union of mirrors of reflections of  $G$ . The fact that the orbifold structure lifts to  $A$  only when  $p$  is even has a similar incarnation in Deligne–Mostow theory, when passing from the Picard integrality condition INT to the condition  $\Sigma$ -INT (see [8, 9, 21]).

For  $p = 3, 4$  and  $6$ , the lattices  $\Gamma_p$  turn out to be conjugate to lattices constructed by the author in joint work with Parker and Paupert, see [12] and [11], namely the groups  $\mathcal{S}(p, \sigma_1)$ , generated by a complex reflection  $R_1$  of angle  $2\pi/p$ , and a regular elliptic element  $J$  of order 3 such that  $\text{tr}(R_1 J) = -1 + i\sqrt{2}$ . For basic notation on these groups, see section 3.

It was proved in [12] and [11] that  $\mathcal{S}(p, \sigma_1)$  is discrete if and only if  $p = 3, 4, 6$ , and in those cases it is a non-cocompact lattice. It has one cusp for  $p = 3, 4$ , two cusps for  $p = 6$ . Note also that the three groups can be checked to be generated by complex reflections, namely by  $R_1$ ,  $R_2 = JR_1J^{-1}$  and  $R_3 = J^{-1}R_1J$  (see Section 3).

We will prove the following.

**Theorem 1.4.** *For every  $p = 3, 4, 6$ , the group  $\Gamma_p$  is conjugate in  $PU(2, 1)$  to the group  $\mathcal{S}(p, \sigma_1)$ .*

In particular, because of the analysis in [11], we know that the  $\Gamma_p$ ,  $p = 3, 4, 6$  are non-arithmetic lattices. The group corresponding to  $p = \infty$  does not appear in [12], but it is in a sense less interesting since it turns out to be arithmetic.

Complex hyperbolic lattices have been previously constructed from configurations of elliptic curves on an Abelian surface. One important construction was worked out by Livne, see [20] (and also [9]), from a point of view that is fairly different from

ours. Another construction, closer in spirit to the results in this paper, appears in [16] (see also [13, 24, 28] for recent developments).

Just as in [10], the results of this paper give an alternative construction of certain non-arithmetic ball quotients, whose existence was known so far only by giving explicit matrix generators and constructing a fundamental domain for their action (see [12] and [11]).

The analysis in [10] shows that some of the non-arithmetic lattices in [11], even though they are not commensurable to Deligne–Mostow lattices (see [8, 21]), are commensurable to Couwenberg–Heckman–Looijenga lattices (see [7], which was inspired in part by [3]). For brevity, we refer to these two classes of lattices as DM and CHL lattices, respectively (note that DM lattices are special cases of CHL lattices). In fact, an analysis similar to the one in [10] shows the following (for notation of Sporadic and Thompson triangle groups, see Section 3).

**Theorem 1.5.** (1) *The group  $\mathcal{S}(2, \sigma_{10})$  is isomorphic to the Shephard–Todd group  $G_{23}$ . The lattices  $\mathcal{S}(p, \sigma_{10})$ ,  $p = 3, 4, 5, 10$  are conjugate to the corresponding CHL lattices of type  $H_3$ .*

(2) *The group  $\mathcal{S}(2, \overline{\sigma}_4)$  is isomorphic to a subgroup of index two in the Shephard–Todd group  $G_{24}$ , both groups having isomorphic projectivizations of order 168. The lattices  $\mathcal{S}(p, \overline{\sigma}_4)$ ,  $p = 3, 4, 5, 6, 8, 12$  are conjugate to the corresponding CHL lattices.*

(3) *The group  $\mathcal{T}(2, \mathbf{S}_2)$  is isomorphic to a subgroup of index two in the Shephard–Todd group  $G_{27}$ , both having isomorphic projectivization of order 360. The lattices  $\mathcal{T}(p, \mathbf{S}_2)$ ,  $p = 3, 4, 5$  are conjugate to the corresponding CHL lattices.*

The three families of lattices in Theorem 1.5, together with Deligne–Mostow lattices, exhaust the list of CHL lattices in  $PU(2, 1)$  (the other ones constructed in [7] are in  $PU(n, 1)$  for  $n > 2$ ).

In particular, we have the following.

**Theorem 1.6.** *The lattices  $\mathcal{S}(p, \sigma_1)$ ,  $p = 3, 4, 6$  are not commensurable to any CHL lattice (and in particular not to any DM lattice either).*

Some lattices in [11] are still not treated by the methods in [10] nor of the present paper, for instance the sporadic lattices  $\mathcal{S}(p, \sigma_5)$ . Indeed, in the family of  $\sigma_5$  groups, there seems to be no finite nor any crystallographic group.

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## 2. Basic complex hyperbolic geometry

Recall that the complex hyperbolic plane  $\mathbb{H}^2$  is the only complete, simply connected Kähler surface of holomorphic sectional curvature  $-1$ . It is biholomorphic to the

unit ball  $\mathbb{B}^2 \subset \mathbb{C}^2$ , and we equip it with the only metric that is invariant under the group of biholomorphisms of  $\mathbb{B}^2$  (normalized so that the holomorphic sectional curvature is  $-1$ ). In terms of Riemannian symmetric spaces,  $\mathbb{H}^2$  is the non-compact dual of  $\mathbb{P}^2$ . We summarize a few basic facts that we will use in this paper (see [15] for much more information).

Working in homogeneous coordinates for  $\mathbb{P}^2$  and seeing  $\mathbb{B}^2 \subset \mathbb{C}^2 \subset \mathbb{P}^2$  as sitting in an affine chart of the complex projective plane, one can see biholomorphisms of  $\mathbb{B}^2$  as induced by linear transformations of  $\mathbb{C}^3$  that preserve a Hermitian form of signature  $(2, 1)$ , say  $\langle Z, W \rangle = -Z_0 \bar{W}_0 + Z_1 \bar{W}_1 + Z_2 \bar{W}_2$ . The unit ball is then identified with the set of negative complex lines in  $\mathbb{C}^3$ , i.e. lines spanned by a vector  $V$  with  $\langle V, V \rangle < 0$ . This description gives an isomorphism  $\text{Bihol}(\mathbb{B}^2) \simeq PU(2, 1)$ , which produces almost all isometries of  $\mathbb{H}^2$  (the full group of isometries is generated by  $PU(2, 1)$  and the single isometry given by complex conjugation).

We will use the classification of (non-trivial) isometries of  $\mathbb{H}^2$  into elliptic, parabolic and loxodromic elements (see [6] for instance). Elliptic isometries are characterized by the fact that they fix at least one point in  $\mathbb{H}^2$ . Parabolic elements have unique fixed point at infinity, i.e. in  $\partial_\infty \mathbb{H}^2 \simeq S^1$ . Loxodromic elements have precisely two fixed points at infinity.

Elliptic isometries whose matrix representatives have distinct eigenvalues are called regular elliptic isometries. Among non-regular elliptic isometries, an important class is given by complex reflections, that fix pointwise the intersection with  $\mathbb{B}^2$  of an affine complex line in  $\mathbb{C}^2$ . These are characterized in terms of their matrix representative in  $U(2, 1)$  by the fact that they have a double eigenvalue, and that the simple eigenvalue eigenspace is spanned by a vector with positive square norm.

A lattice  $\Gamma \subset PU(2, 1)$  is a discrete subgroup such that  $\Gamma \backslash PU(2, 1)$  has finite Haar measure. Equivalently, the quotient  $\Gamma \backslash \mathbb{H}^2$  has finite volume for the Riemannian volume form on  $\mathbb{H}^2$ .  $\Gamma$  is called co-compact (or uniform) if the quotient  $\Gamma \backslash \mathbb{H}^2$  is compact. If it is not, there are finitely many conjugacy classes of maximal parabolic subgroups in  $\Gamma$ , and the quotient decomposes as a disjoint union of a compact part and finitely many cusps (a cusp is the quotient of a sufficiently small horoball centered at the fixed point of one of the parabolic subgroups). We say  $\Gamma$  has  $n$  cusps if the quotient has  $n$  cusps, equivalently if there are  $n$  conjugacy classes of maximal parabolic subgroups in  $\Gamma$ .

### 3. Complex hyperbolic lattice triangle groups

**3.1. Sporadic triangle groups.** In this section we briefly review some of the basic facts and notation in [11] (and also previous papers cited there). In the following statement, we write  $\omega = (-1 + i\sqrt{3})/2$ .

**Proposition 3.1.** *Let  $p \in \mathbb{R}^*$ ,  $u = e^{2\pi i/3p}$ ,  $\tau, \tau' \in \mathbb{C}$ . Up to conjugacy in  $SL(3, \mathbb{C})$ , there is a unique pair  $(R_1, J)$  of matrices such that*

- $R_1$  has eigenvalues  $u^2, \bar{u}, \bar{u}$ ;
- $J$  has eigenvalues  $1, \omega, \bar{\omega}$ ;
- $\text{tr}(R_1 J) = \tau$  and  $\text{tr}(R_1 J^{-1}) = \tau'$ .

*The group generated by  $R_1$  and  $J$  preserves a non-zero Hermitian form if and only if  $\tau' = -u\bar{\tau}$ , and in that case the form is unique up to scaling.*

Choosing the basis of  $\mathbb{C}^3$  given by  $e_1, e_2 = Je_1, e_3 = J^{-1}e_1$ , we can write

$$R_1 = \begin{pmatrix} u^2 & \tau & \tau' \\ 0 & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (3.1)$$

In the Hermitian case, i.e. when  $\tau' = -u\bar{\tau}$ , and assuming moreover that  $u^3 \neq 1$ , the invariant Hermitian form is given (up to a nonzero scalar) by

$$\begin{pmatrix} \alpha & \beta & \bar{\beta} \\ \bar{\beta} & \alpha & \beta \\ \beta & \bar{\beta} & \alpha \end{pmatrix}, \quad (3.2)$$

where  $\alpha = 2 - u^3 - \bar{u}^3$ ,  $\beta = (\bar{u}^2 - u)\tau$ . Note that this matrix tends to 0 when  $p \rightarrow +\infty$  (recall  $u = e^{2\pi i/3p}$ ), but after rescaling it by  $1/\sqrt{2 - u^3 - \bar{u}^3}$ , it converges to

$$\begin{pmatrix} 0 & -i\tau & i\bar{\tau} \\ i\bar{\tau} & 0 & -i\tau \\ -i\tau & i\bar{\tau} & 0 \end{pmatrix},$$

which gives the invariant Hermitian form when  $u^3 = 1$ .

**Definition 3.2.** We denote by  $\mathcal{S}(p, \tau)$  the group generated by  $R_1$  and  $J$  as in (3.1), where  $\tau' = -u\bar{\tau}$ , and refer to it as a *sporadic triangle group* with trace parameter  $\tau$ .

In such a sporadic triangle group, it is natural to consider

$$R_2 = JR_1J^{-1}, \quad R_3 = J^{-1}R_1J.$$

The groups are constructed so that  $R_1J$  has finite order (its order is actually independent of  $p$ ). When that order is not a multiple of 3, the group generated by  $R_1, R_2$  and  $R_3$  is actually the same as the group generated by  $R_1$  and  $J$  (in particular, in those cases,  $\mathcal{S}(p, \tau)$  is generated by complex reflections).

The main groups of interest in this paper will be the groups  $\mathcal{S}(p, \sigma_1)$ , for suitable integer values of  $p$ , and

$$\sigma_1 = -1 + i\sqrt{2}.$$

The following is easily obtained using (3.2) and the above discussion.

**Proposition 3.3.** *For  $p \geq 2$  an integer, the Hermitian form preserved by  $\mathcal{S}(p, \sigma_1)$  is definite if and only if  $p = 2$ , and it has signature  $(2, 1)$  for all  $p > 2$ . For every  $p$ ,  $R_1 J$  has order 8, and the group  $\mathcal{S}(p, \sigma_1)$  is generated by  $R_1$ ,  $R_2$  and  $R_3$ .*

Note that  $R_1 J$  having order 8 is easily seen to imply that  $J = R_1 R_2 R_3 R_1 R_2 R_3 R_1 R_2$ .

**3.2. Thompson triangle groups.** The groups  $\mathcal{T}(p, \mathbf{T})$  are analogs of the sporadic groups that were constructed in James Thompson's Ph.D. thesis [29]. They are generated by three complex reflections  $R_1$ ,  $R_2$  and  $R_3$ , that have the same rotation angles, but are not cyclically conjugated by any element of  $J$  order 3. Here  $\mathbf{T} = (\rho, \sigma, \tau)$  is a triple of complex numbers that generalizes the trace parameter of sporadic triangle groups, related to traces of  $R_j R_k$ . Since they are not central to this paper, we omit the detailed description of these groups and simply refer to [11].

The Thompson triangle groups that appear in Theorem 1.5 are the groups  $\mathcal{T}(p, \mathbf{S}_2)$ , with trace parameter triple  $\mathbf{S}_2 = (1 + \omega \frac{1+\sqrt{5}}{2}, 1, 1)$ .

We will also use the description of  $\mathcal{S}(p, \sigma_1)$  as 3,3,4;6 triangle groups, in other words, in the terminology of [11], as  $\mathcal{T}(p, \mathbf{E}_1)$ , where  $\mathbf{E}_1 = (i\sqrt{2}, 1, 1)$ .

Recall that the integers in 3,3,4;6 stand for specific braid lengths  $\text{br}(a, b)$ , namely  $\text{br}(R_2, R_3) = 3$ ,  $\text{br}(R_3, R_1) = 3$ ,  $\text{br}(R_1, R_2) = 4$ ,  $\text{br}(R_1, R_3^{-1} R_2 R_3) = 6$ , and  $\text{br}(a, b) = k$  means

$$(ab)^{k/2} = (ba)^{k/2},$$

but  $(ab)^{n/2} \neq (ba)^{n/2}$  for every  $n < k$ .

In other words,  $\mathcal{T}(p, \mathbf{E}_1)$  is a group generated by three reflections  $R_1$ ,  $R_2$ ,  $R_3$  of the same order  $p$ , such that

$$\begin{aligned} (R_1 R_2)^2 &= (R_2 R_1)^2, & R_2 R_3 R_2 &= R_3 R_2 R_3, & R_3 R_1 R_3 &= R_1 R_3 R_1, \\ (R_1 \cdot R_3^{-1} R_2 R_3)^3 &= (R_3^{-1} R_2 R_3 \cdot R_1)^3. \end{aligned}$$

The fact that  $\mathcal{S}(p, \sigma_1)$  is conjugate to  $\mathcal{T}(p, \mathbf{E}_1)$  follows from a change of generators along the same lines as in [17] (for details, see [11, Section 7.1]). Explicitly, if  $M_1$ ,  $M_2 = J M_1 J^{-1}$ ,  $M_3 = J^{-1} M_1 J$  denote standard generators for  $\mathcal{S}(p, \sigma_1)$ , then the matrices

$$\begin{aligned} R_1 &= (M_3 M_1 M_2 M_1^{-1}) M_3 (M_3 M_1 M_2 M_1^{-1})^{-1}, \\ R_2 &= (M_3 M_1) M_2 (M_3 M_1)^{-1}, & R_3 &= M_1 \end{aligned}$$

give another generating set, which exhibits an isomorphism with  $\mathcal{T}(p, \mathbf{E}_1)$ .

#### 4. The affine crystallographic reflection group

We start by describing the relevant affine crystallographic group. One way to write it is to use the matrices given in [12], which would give slightly complicated matrices, and

then to diagonalize the corresponding Hermitian form by a suitable coordinate change. Here we only give the matrices in a nice basis. For computational convenience, rather than choosing the generators to have determinant one as we did in [11], we adjust the repeated eigenvalue to be equal to 1 (this amounts to multiplying the generators by a suitable root of unity).

**Definition 4.1.** Let  $G$  be the group generated by the matrices  $R_1$ ,  $R_2$  and  $R_3$  given below

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1-i\sqrt{2} & -1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1+i\sqrt{2} & 2 \\ 0 & 1+i\sqrt{2} & 1-i\sqrt{2} \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1+i\sqrt{2}}{2} & 1 & -1-i\sqrt{2} \\ 1 & 0 & -1 \end{pmatrix},$$

First observe that  $G$  can be thought of as a subgroup of the semi-direct product  $\mathbb{C}^2 \rtimes U(2)$ . To see this, we write  $(z_0, z_1, z_2)$  for the coordinates in  $\mathbb{C}^3$ , and denote by  $\pi: \mathbb{C}^3 \rightarrow \mathbb{C}^2$  the projection onto the last two coordinates. Note that the group  $G$  clearly preserves every hyperplane  $z_0 = \lambda$ ,  $\lambda \in \mathbb{C}$ . We will study the affine action of  $G$  on  $\mathbb{C}^2$  given by

$$B \cdot (z_1, z_2) = \pi(B(1, z_1, z_2)).$$

Concretely, we think of the linear part of  $B$  as being given by the lower right  $2 \times 2$  block of the  $3 \times 3$  matrix  $B$ , and the translation part by the lower left  $2 \times 1$  block. We denote by  $\psi: G \rightarrow GL_2(\mathbb{C})$  the corresponding homomorphism. Note that the image of  $\psi$  preserves a positive definite Hermitian form, namely  $\langle z, w \rangle = w^* H z$  where

$$H = \begin{pmatrix} 1 & \frac{-1-i\sqrt{2}}{2} \\ \frac{-1+i\sqrt{2}}{2} & 1 \end{pmatrix}.$$

The unitary group  $U(H)$  is isomorphic to  $U(2)$  since the matrix  $H$  has eigenvalues  $2 \pm \sqrt{3}$ , which are both positive. One checks that the matrices  $\psi(R_j)$  are complex reflections of order 2 (i.e. each has eigenvalues 1 and -1), so  $G$  is an affine group generated by complex reflections. Next, we show that this group is crystallographic, i.e. it is discrete, and the quotient of  $\mathbb{C}^2$  by its action is cocompact. This follows from Propositions 4.2 and 4.3 below.

**Proposition 4.2.** *The linear part  $\psi(G)$  of  $G$  is isomorphic to the Shephard–Todd group  $G_{12}$ , which is isomorphic to  $GL(2, \mathbb{F}_3)$ .*

*Proof.* One easily checks that the three matrices  $A_1 = \psi(R_1)$ ,  $A_2 = \psi(R_2)$  and  $A_3 = \psi(R_3)$  generate a group of order 48, and that they satisfy the relations in a presentation for  $G_{12}$ , see [25], namely

$$A_1^2 = A_2^2 = A_3^2 = (A_1 A_2)^4 = (A_2 A_3)^3 = (A_3 A_1)^3 = \text{Id}, \quad (4.1)$$

and the element  $(A_1 A_2)^2$  is central of order 2.  $\square$



We refer to the matrix

$$T_v = \begin{pmatrix} 1 & 0 & 0 \\ v_1 & 1 & 0 \\ v_2 & 0 & 1 \end{pmatrix}$$

as a translation with vector  $v = (v_1, v_2)$ . Let  $K$  denote the kernel of  $\psi$ , and let  $T_\Lambda$  denote the group of translations  $T_v$ , where  $v \in \Lambda$  is a lattice vector.

**Proposition 4.3.** *The groups  $K$  and  $T_\Lambda$  are equal.*

*Proof.* One verifies that

$$\begin{aligned} (R_3 R_1 R_2 R_1)^2 R_3 R_2 &= T_{(i\sqrt{2}, 1)}, & (R_3 R_2 R_1 R_2)^2 R_3 R_1 &= T_{(0, 1)}, \\ R_2 [(R_2 R_1)^2, R_3] R_2 &= T_{(-1, -1)}, & (R_2 R_1 R_3 R_1)^2 R_1 R_2 R_3 R_1 &= T_{(-1, i\sqrt{2})}. \end{aligned}$$

These four translations generate  $T_\Lambda$ , so we have  $T_\Lambda \subset K$ .

In order to show the other inclusion, we observe that

$$R_1^2 = R_2^2 = R_3^2 = (R_1 R_2)^4 = (R_2 R_3)^3 = (R_3 R_1)^3 = \text{Id}, \quad (4.2)$$

and the element  $Z = (R_2 R_1)^2$  commutes with  $R_1$  and  $R_2$ , and it has order 2. The commutator  $Z R_3 Z^{-1} R_3^{-1} = (Z R_3)^2$  is given by the translation  $T_v$ , where  $v = (-1 - i\sqrt{2}, -2)$ . Once again, using the Shephard–Todd presentation for  $G_{12}$ , we get that  $G/T_\Lambda$  is a quotient of  $G_{12}$ , but since  $T_\Lambda \subset K$  and  $G/K$  has order 48, both quotients of  $G$  must have order 48.  $\square$

In what follows, we denote by  $F$  the finite group  $G_{12}$ . We denote by  $A$  the Abelian variety  $\mathbb{C}^2/\Lambda$ , and by  $X$  the quotient of  $A$  by the action of  $F = G/T_\Lambda$ .

The following two propositions follow from painful (but not particularly difficult) computation and bookkeeping.

**Proposition 4.4.** *The group  $F$  contains precisely 12 reflections, all of order 2, whose fixed point sets are elliptic curves in  $A$ . The group  $F$  acts transitively on the set of these 12 elliptic curves.*

For completeness, we list equations for these elliptic curves in Table 1.

We denote by  $\tilde{M}$  the union of the mirrors in  $A$ , and by  $M$  its image in  $X$ . By transitivity of the action,  $M$  is an irreducible curve in  $X$ .

**Proposition 4.5.** *The action of  $F$  on  $A$  has precisely two orbits of fixed points in  $A \setminus \tilde{M}$ , one with isotropy group of order 3, the other with isotropy group of order 8, as in Table 2. The isotropy groups of points in  $\tilde{M}$  are all generated by complex reflections, the generic point having isotropy of order 2. The points in  $\tilde{M}$  with isotropy of order larger than 2 consist of two orbits of points with isotropy group of order 6, one orbit of points with isotropy of order 8, and one orbit of points with isotropy of order 12, see Table 3.*



1	$z_2 = \frac{1-i\sqrt{2}}{2}z_1$
2	$z_2 = \frac{2-i\sqrt{2}}{2}z_1$
3	$z_2 = \frac{1}{2}$
121	$z_2 = \frac{-i\sqrt{2}}{2}z_1$
131	$z_2 = (1 - i\sqrt{2})z_1 - \frac{1}{2}$
212	$z_1 = 0$
232	$z_1 = \frac{1+2i\sqrt{2}}{3}z_2 + \frac{1-i\sqrt{2}}{6}$
32121	$z_1 = \frac{1+i\sqrt{2}}{2}z_2 + \frac{1+i\sqrt{2}}{4}$
23121	$z_1 = z_2 + \frac{1+i\sqrt{2}}{2}$
21321	$z_1 = \frac{i\sqrt{2}}{2}z_2 + \frac{2+i\sqrt{2}}{4}$
12321	$z_1 = \frac{2+i\sqrt{2}}{2}z_2 + \frac{i\sqrt{2}}{4}$
21231	$z_1 = (1 + i\sqrt{2})z_2 + \frac{1+i\sqrt{2}}{2}$

Table 1. Equations in  $\mathbb{C}^2$  of (representatives of) the 12 mirrors of reflections in  $A = \mathbb{C}^2/\Lambda$ .

Group	Order	eigenvalues	coords
$\langle R_1 R_2 R_3 \rangle$	8	$\zeta_8, \zeta_8^3$	$(\frac{1}{2}, \frac{1+i\sqrt{2}}{2})$
$\langle R_1 R_3 \rangle$	3	$\omega, \bar{\omega}$	$(\frac{1+i\sqrt{2}}{3}, \frac{1+2i\sqrt{2}}{6})$

Table 2. Representatives of the orbits of points with non-reflection stabilizer (these produce singular points of the quotient).

Generators	Order	ST-group	Sing. of $M$	notation	coords
$R_1, R_3$	6	$G(3, 3, 2)$	$z_1^3 = z_2^2$	$p_{13}$	$(\frac{1+i\sqrt{2}}{3}, \frac{1}{2})$
$R_2, R_3$	6	$G(3, 3, 2)$	$z_1^3 = z_2^2$	$p_{23}$	$(\frac{-2-i\sqrt{2}}{6}, \frac{1}{2})$
$R_1, R_2$	8	$G(2, 1, 2)$	$z_1^4 = z_2^2$	$p_{12}$	$(0, 0)$
$R_1, R_3(R_2 R_1)^2$	12	$G(6, 6, 2)$	$z_1^6 = z_2^2$	$p_{13(21)^2}$	$(0, \frac{1}{2})$

Table 3. Representatives of the orbits of points whose stabilizer is a reflection group (these produce smooth points of the quotient).

The results in Proposition 4.5 follow by explicit calculations. For a definition of the groups  $G(m, p, n)$ , see [25], and also [18, §1], for instance. The local analytic structure of the branch locus of the quotient (fourth column in Table 3) can be obtained by computing explicit invariant polynomials for the group, the results are tabulated in [2].

It follows from Proposition 4.5 that  $X$  has exactly two singular points. Let  $V$  denote the subset of  $A$  of points with trivial isotropy for the  $F$ -action, and let  $U$  denote its image in  $X$ .

**Proposition 4.6.** *We have  $\chi(V) = 48$ , hence  $\chi(U) = 1$ .*

*Proof.* There are  $48/3=16$  points above the isolated singularity of order 3,  $48/8=6$  points above the isolated singularity of order 8. There are  $2 \cdot (48/6) + (48/8) + (48/12) = 16 + 6 + 4 = 26$  points with reflection isotropy of order  $> 2$ . This gives 48 points.

There are also 12 mirrors, each being an elliptic curve and containing 8 special points. The Euler characteristic of the generic stratum of each mirror is then  $-8$ , so we get

$$0 = \chi(A) = 48 + 12 \cdot (-8) + \chi(V),$$

hence  $\chi(V) = 48$ , and  $\chi(U) = \chi(V)/48 = 1$ , since  $F$  has order 48.  $\square$

We will also need to study the stabilizer of a mirror of reflections.

**Proposition 4.7.** *Each mirror in the group contains precisely 8 points with special isotropy (i.e. stabilizer of order strictly larger than 2). The curve  $M$  is a  $\mathbb{P}^1$  with two pairs of points identified, and the map from each irreducible component of  $\tilde{M}$  to  $M$  is a branched cover of degree 2.*

*Proof.* In the coordinates we used above, the mirror of  $R_2 R_1 R_2$  corresponds to the elliptic curve  $z_1 = 0$ . The intersections with the other mirrors can be computed explicitly from the equations in Table 1, they are listed in Table 4.

Mirrors	$z_2$
1, 2, 121, 212	$0, -i\sqrt{2}/2$
212, 232, 12321	$\pm(1 + i\sqrt{2})/6$
212, 32121, 21231	$\pm(1 + 2i\sqrt{2})/6$
1, 3, 131, 212, 32121, 21231	$1/2$
1, 212, 232, 23121, 21321, 12321	$(1 + i\sqrt{2})/2$

Table 4. Special points on the mirror of  $R_2 R_1 R_2$ . We list the corresponding reflections whose mirrors meet at that point.

One verifies that the only reflection that stabilizes the mirror of  $R_2 R_1 R_2$  is  $R_1$  (note that  $R_1$  commutes with  $R_2 R_1 R_2$ , since  $R_1 R_2$  has order 4). Now  $R_1$  acts on  $z_1 = 0$  by  $z_2 \mapsto -z_2$ . Among the points listed in Table 4, the two points  $\pm(1 + i\sqrt{2})/6$  get identified, and so do the points  $\pm(1 + 2i\sqrt{2})/6$ . The other four points are fixed by the action of  $R_1$ .  $\square$

## 5. Statement of the main result

Recall that  $X$  denotes the quotient  $A/F$ , where  $A$  is the Abelian variety  $\mathbb{C}^2/\Lambda$ , and  $F$  is a specific group of order 48, isomorphic to the Shephard–Todd group  $G_{12}$ . As above, we denote by  $M \subset X$  the curve which is the image of the set of mirrors in  $A$  of reflections of  $F$ .

We denote by  $p_{12}$  the image in  $X$  of the fixed point of  $R_1 R_2$ , etc (see Table 3). As in [10], in order to produce orbifolds uniformized by the ball, we will need to perform suitable blow-ups on  $X$ .

The curve  $M$  has a local analytic equation of the form  $(z_1^3 - z_2)(z_1^3 + z_2) = 0$  near  $p_{13(21)^2}$  (see Table 3), so locally there are two tangent components. The space  $\widehat{Y}$  is obtained from  $X$  by blowing up  $p_{13(21)^2}$  three times (the first blow-up preserves the tangency, the second makes the intersection transverse, the third makes the two local components disjoint). The exceptional locus of  $\pi: \widehat{Y} \rightarrow X$  is a chain of projective lines with self-intersections  $-1, -2, -2$ .

**Definition 5.1.** The space  $Y$  is obtained from  $\widehat{Y}$  by contracting the two  $-2$  curves in the exceptional locus of  $\pi: \widehat{Y} \rightarrow X$ . We denote by  $\gamma: \widehat{Y} \rightarrow Y$  the contraction, and  $\varphi: Y \rightarrow X$  the corresponding birational transformation. We denote the exceptional locus of  $\varphi$  by  $E$ .

Similarly, the space  $\widehat{Z}$  is obtained from  $X$  by blowing up both points  $p_{13(21)^2}$  and  $p_{12}$ . Near the first one, the modification is the same as in the construction of  $Y$ . Near  $p_{12}$ , the curve  $M$  has a local equation of the form  $(z_1^2 - z_2)(z_1^2 + z_2) = 0$  (see Table 3 again). At that point, we perform two successive blow-ups (the first one makes the two tangent local components transverse, the second makes them disjoint), which produces a chain of two projective lines with self-intersection  $-1, -2$ .

**Definition 5.2.** The space  $Z$  is obtained from  $\widehat{Z}$  by contracting the two  $-2$  curves above  $p_{13(21)^2}$ , and the  $(-2)$ -curve above  $p_{12}$ . With a slight abuse of notation, we still denote by  $\gamma: \widehat{Z} \rightarrow Z$  the contraction, and  $\varphi: Z \rightarrow X$  the corresponding birational transformation. We denote the exceptional lines by  $E$  and  $F$ , above  $p_{13(21)^2}$  and  $p_{12}$  respectively.

Finally, we consider  $\widehat{W}$ , which is obtained from  $X$  by blowing up all points  $p_{13(21)^2}$  and  $p_{12}$ ,  $p_{13}$  and  $p_{23}$ . Near each point  $p_{13}$  and  $p_{23}$ , we need to perform three successive blow-ups, producing a chain of three  $\mathbb{P}^1$  with self-intersections  $-2, -1, -3$  (see [10]).

**Definition 5.3.** The space  $W$  is obtained from  $\widehat{W}$  by contracting the two  $-2$  curves above  $p_{13(21)^2}$ , and the  $(-2)$ -curve above  $p_{12}$ , the  $(-2)$  and the  $(-3)$ -curves above  $p_{13}$  and  $p_{23}$ . We still denote by  $\gamma: \widehat{W} \rightarrow W$  the contraction, and  $\varphi: W \rightarrow X$  the corresponding birational transformation. We denote the exceptional lines by  $E, F, G, H$ , above  $p_{13(21)^2}, p_{12}, p_{13}$  and  $p_{23}$  respectively.

**Remark 5.4.** Whenever a point of  $x \in X$  is not blown-up in order to get  $Y$  (resp.,  $Z$ ), we will use the same notation for its proper transform in  $Y, \widehat{Y}, Z$  or  $\widehat{Z}$ .

**Theorem 5.5.** (1) *The pair  $(X', \frac{2}{3}M')$  is a ball quotient orbifold with one cusp, where  $X' = X \setminus \{p_{13(21)^2}\}$  and  $M' = M \cap X'$ .*

(2) *The pair  $(Y', \frac{3}{4}M' + \frac{3}{4}E)$  is a ball quotient orbifold with one cusp, where  $Y' = Y \setminus \{p_{12}\}$ , and  $M'$  denotes the intersection with  $Y'$  of the strict transform of  $M$  in  $Y$ .*

(3) *The pair  $(Z', \frac{5}{6}M' + \frac{1}{2}E + \frac{5}{6}F)$  is a ball quotient orbifold with two cusps, where  $Z' = Z \setminus \{p_{13}, p_{23}\}$ , and  $M'$  denotes the intersection with  $Z'$  of the strict transform of  $M$  in  $Z$ .*

(4) *The pair  $(W', \frac{1}{2}F + \frac{1}{2}G + \frac{1}{2}H)$  is a ball quotient orbifold with one cusp, where  $W' = W \setminus M'$ , and  $M'$  denotes strict transform of  $M$  in  $W$ .*

## 6. Proof of the main result

The basis of the proof, like in [10], will be a detailed study of the pairs  $(X^{(p)}, D^{(p)})$ , where

$$X^{(3)} = X, \quad X^{(4)} = Y, \quad X^{(6)} = Z, \quad X^{(\infty)} = W$$

and the  $D^{(p)}$ 's are  $\mathbb{Q}$ -divisors given by

$$D^{(3)} = \frac{2}{3}M, \quad D^{(4)} = \frac{3}{4}\widetilde{M} + \frac{3}{4}E, \\ D^{(6)} = \frac{5}{6}\widetilde{M} + \frac{1}{2}E + \frac{5}{6}F, \quad \text{and} \quad D^{(\infty)} = \widetilde{M} + \frac{1}{2}F + \frac{1}{2}G + \frac{1}{2}H.$$

**Proposition 6.1.** *For each  $p$  as above,*

- (1) *the pair  $(X^{(p)}, D^{(p)})$  has at worst log canonical singularities;*
- (2) *the log-canonical divisor  $K_{X^{(p)}} + D^{(p)}$  is ample, i.e. the pair  $(X^{(p)}, D^{(p)})$  is its own canonical model;*
- (3)  *$c_1^2(X'^{(p)}, D'^{(p)}) = 3c_2(X'^{(p)}, D'^{(p)})$ , where  $X'^{(p)}$  denotes the log-terminal locus (obtained from  $X^{(p)}$  by removing the points where the pair is not log-terminal), and  $D'(p) = D^{(p)} \cap X'(p)$ .*

By a theorem of Kobayashi, Nakamura, and Sakai [19], Proposition 6.1 implies Theorem 5.5.

A schematic picture of the spaces  $X$ ,  $Y$ ,  $Z$ ,  $W$  showing the combinatorics/singularities of the relevant  $\mathbb{Q}$ -divisors is given in Figure 1. Note that all these spaces map to  $X$ , and these maps are isomorphisms over  $X \setminus M$ , where  $X$  (hence  $Y$ ,  $Z$ ,  $W$  as well) has two isolated singularities, of type  $\frac{1}{3}(1, 2)$  and  $\frac{1}{8}(1, 3)$ .

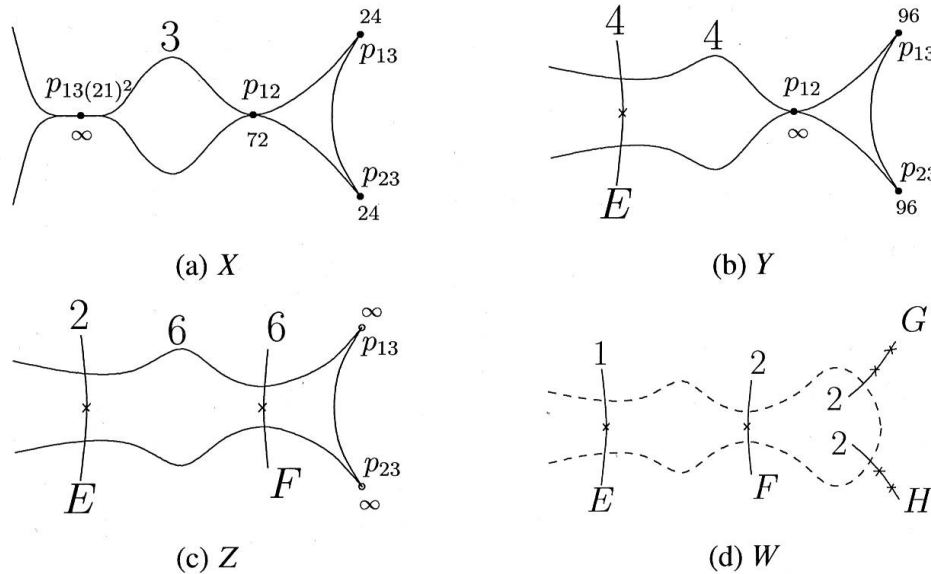


Figure 1. Schematic picture of the orbifold structure on  $X = A/F$ , and of the relevant birational surfaces  $Y$ ,  $Z$  and  $W$ . We label each curve with the relevant orbifold weight (in the case of  $W$ , the dotted curve is removed).

**6.1. Log-canonical singularities.** For part (1) of Proposition 6.1, the only point to consider is the point  $p_{13(21)^2}$ , since the others have local descriptions that were handled in [10]. In what follows, to simplify notation, we write  $q = p_{13(21)^2}$ .

At the point  $q$ , we denote by  $\widehat{X}$  the minimal resolution of the pair  $(X, \lambda M)$ , which is given by  $\pi: \widehat{X} \rightarrow X$ , and has exceptional locus a  $-1, -2, -2$  chain of projective lines, denoted by  $E_1$ ,  $E_2$  and  $E_3$  (note that  $E_1$  intersects the proper transform  $\widehat{M}$  twice, but  $E_2$  and  $E_3$  do not intersect  $\widehat{M}$ ). One checks that

$$K_{\widehat{X}} = \pi^* K_X + E_1 + 2E_2 + 3E_3,$$

and

$$\pi^* M = \widehat{M} + 2E_1 + 4E_2 + 6E_3.$$

This gives

$$K_{\widehat{X}} + \lambda \widehat{M} = \pi^*(K_X + \lambda M) + (1 - 2\lambda)E_1 + (2 - 4\lambda)E_2 + (3 - 6\lambda)E_3,$$

hence the pair  $(X, \lambda M)$  is log-canonical at  $q$  if and only if  $\lambda \leq 2/3$ . For  $\lambda = 1 - 1/p$ , this means  $p \leq 3$ . For  $p = 3$  the pair is *not* log-terminal at  $q$ .

Near  $p_{12}$ , the pair  $(X, M)$  is log-canonical for  $p \leq 4$ , and log-terminal for  $p < 4$ ; at  $p_{13}$  and  $p_{23}$ , it is log-canonical for  $p \leq 6$ , log-terminal for  $p < 6$  (see [10] for more details, where the same type of singularities of the pair occur).

**6.2. Miyaoka–Yau equality.** The formulas for  $c_1^2(X^{(p)}, D^{(p)})$  are very similar to those in [10]. If we knew that  $X$  was a weighted projective plane, the formulas below would be obtained from those in [10] by replacing 2,3,7 by 1,3,8. We give a slightly different argument, that relies on the fact that  $X = A/F$ , where  $F$  is a specific group of order 48. In other words, there is a map  $f: A \rightarrow X$  of degree 48, that ramifies with order 2 along the union  $\cup E_j$  of 12 elliptic curves.

It follows from the discussion in Section 4 that for every  $k$ ,  $E_k \cdot \sum_{j=1}^{12} E_j = 24$  (more specifically, see Table 4). From this, it follows that

$$M^2 = \frac{1}{48} \left( 2 \sum E_j \right)^2 = \frac{1}{48} \cdot 4 \cdot 12 \cdot 24 = 24.$$

Note also that  $f^*(K_X + \frac{1}{2}M) = K_A$ , so

$$K_X \cdot M = \frac{1}{48} \left( K_A \cdot f^*M - \frac{1}{2}(f^*M)^2 \right) = -12,$$

where we have used the adjunction formula and the fact that  $\chi(E_j) = 0$ .

Finally, note that  $(K_X + \frac{1}{2}M)^2 = 0$  (since  $K_A$  is trivial), hence

$$K_X^2 = -K_X \cdot M - \frac{1}{4}M^2 = 6.$$

In particular, we get for any  $\lambda = 1 - 1/p$ , that

$$(K_X + \lambda M)^2 = 6(-1 + 2\lambda)^2 = \frac{1}{24}(-12 + 24\lambda)^2.$$

The last expression is written so as to resemble the formula in [10].

We now write

$$\begin{array}{ll} D = \lambda M & \text{on } X, \\ \lambda \tilde{M} + \mu E & \text{on } Y, \\ \lambda \tilde{M} + \mu E + \nu F & \text{on } Z, \\ \lambda \tilde{M} + \mu E + \nu F + \sigma G + \sigma H & \text{on } W \end{array}$$

(recall that the coefficient of each divisor has the form  $1 - 1/k$ , where  $k$  is an integer or  $\infty$ ).

We get for  $p = 3$ ,

$$(K_X + \lambda M)^2 = \frac{1}{24} \left( -12 + 24 \frac{2}{3} \right)^2 = \frac{2}{3}.$$

For  $p = 4$ , we take  $\lambda = \mu = 1 - 1/4$ , and get

$$(K_Y + \lambda \tilde{M} + \mu E)^2 = \frac{1}{24}(-12 + 24\lambda)^2 - \frac{1}{3}(3 - 6\lambda + \mu)^2 = \frac{21}{16}.$$

For  $p = 6$ , we take  $\lambda = \nu = 1 - 1/6$ ,  $\mu = 1 - 1/2$ , and get

$$(K_Z + \lambda \tilde{M} + \mu E + \nu F)^2 = \frac{1}{24}(-12 + 24\lambda)^2 - \frac{1}{3}(3 - 6\lambda + \mu)^2 - \frac{1}{2}(2 - 4\lambda + \nu)^2 = \frac{43}{24}.$$

For  $p = \infty$ , we take  $\lambda = 1$ ,  $\mu = 1 - 1 = 0$ ,  $\nu = \sigma = \tau = 1 - 1/2$ , and get

$$\begin{aligned} (K_W + \lambda \tilde{M} + \mu E + \nu F + \sigma G + \sigma H)^2 &= \frac{1}{24}(-12 + 24\lambda)^2 - \frac{1}{3}(3 - 6\lambda + \mu)^2 \\ &\quad - \frac{1}{2}(2 - 4\lambda + \nu)^2 - 2 \cdot \frac{1}{6}(4 - 6\lambda + \sigma)^2 = \frac{9}{8}. \end{aligned}$$

The orbifold Euler characteristics are given by the following. For  $p = 3$ , we get

$$\chi^{\text{orb}}(X, D) = \frac{1}{3} + \frac{1}{8} + \frac{1}{72} + 2\frac{1}{24} + \frac{-4}{3} + 1 = \frac{2}{9}.$$

For  $p = 4$ , we get

$$\chi^{\text{orb}}(X, D) = \frac{1}{3} + \frac{1}{8} + 2\frac{1}{4 \cdot 4} + \frac{1}{3 \cdot 4} + 2\frac{1}{96} + \frac{-4}{4} + \frac{-1}{4} + 1 = \frac{7}{16}.$$

For  $p = 6$ , we get

$$\chi^{\text{orb}}(X, D) = \frac{1}{3} + \frac{1}{8} + 2\frac{1}{2 \cdot 6} + 2\frac{1}{6 \cdot 6} + \frac{1}{3 \cdot 2} + \frac{1}{2 \cdot 6} + \frac{-4}{6} + \frac{-1}{2} + \frac{-1}{6} + 1 = \frac{43}{72}.$$

For  $p = \infty$ , we get

$$\chi^{\text{orb}}(X, D) = \frac{1}{3} + \frac{1}{8} + \frac{1}{3} + \frac{1}{2 \cdot 2} + 2\frac{1}{2 \cdot 2} + 2\frac{1}{2 \cdot 3} + \frac{-1}{1} + \frac{-1}{2} + 2\frac{-1}{2} + 1 = \frac{3}{8}.$$

Putting this together, we get that  $c_1^2 = 3c_2$  for all relevant values of  $p$ .

**6.3. Ampleness.** Our argument relies in part on the following fact, which would be obvious if we knew  $X$  to be a weighted projective plane.

**Proposition 6.2.** *Let  $X = \mathbb{C}^2/G = A/F$  be as above. Then  $\chi(X) = 3$ , and  $\text{Pic}(X) = \mathbb{Z}$ .*

*Proof.* The fact that  $\chi(X) = 3$  follows from the arguments in Proposition 4.6. Indeed, we use the stratification of  $X$  corresponding to isotropy groups, there are 2 isolated singularities, 4 points with non-cyclic reflection stabilizers, 1 6-punctured projective line, and the open part that has Euler characteristic 1. We then have

$$\chi(X) = 6 - 4 + 1 = 3.$$



We then use the fact that  $X$  is simply connected, because its orbifold fundamental group is generated by point stabilizers (this follows from a theorem of Armstrong, see [1]). This gives  $b_1(X) = 0$ . Since  $X$  has quotient singularities, it satisfies Poincaré duality (see [27, Theorem 1.13]), hence  $\chi(X) = 3$  gives  $b_2(X) = 1$ .

The fact that the Picard number is one then follows, see the proof of Proposition 4.20 of [22], for instance.  $\square$

From this and the analysis in the beginning of Section 6.2, it follows that  $K_X$  is numerically equivalent to  $-\frac{1}{2}M$ .

We want to check whether the log-canonical divisors

$$\begin{aligned} K_X + \frac{2}{3}\tilde{M} & \quad (\text{case } p = 3), \\ K_Y + \frac{3}{4}\tilde{M} + \frac{3}{4}E & \quad (\text{case } p = 4), \\ K_Z + \frac{5}{6}\tilde{M} + \frac{1}{2}E + \frac{5}{6}F & \quad (\text{case } p = 6), \\ K_W + \tilde{M} + \frac{1}{2}F + \frac{1}{2}G + \frac{1}{2}H & \quad (\text{case } p = \infty) \end{aligned}$$

are ample.

For the case  $p = 3$ , we simply have  $K_X + \frac{2}{3}M \equiv \frac{1}{6}M$ , which is clearly ample (for instance, by the Nakai–Moishezon it is enough to show that its intersection with  $M$  is  $> 0$ , but  $M^2 = 24 > 0$ ).

For  $p = 4$ , we have

$$K_Y + \lambda\tilde{M} + \mu E = \varphi^*(K_X + \lambda M) + (3 - 6\lambda + \mu)E,$$

where  $\varphi$  is as in Definition 5.1. Since  $\varphi^*M \equiv \tilde{M} + 6E$ , the right hand side is linearly equivalent to

$$\left(\lambda - \frac{1}{2}\right)(\tilde{M} + 6E) + (3 - 6\lambda + \mu)E = \left(\lambda - \frac{1}{2}\right)\tilde{M} + \mu E.$$

We check that the latter divisor is ample by the Nakai–Moishezon criterion. As explained in Section 3.3 of [10], since  $\text{Pic}(X) = \mathbb{Z}$ , it is enough to check that its intersection with  $\tilde{M}$  and with  $E$  is  $> 0$ .

Now we recall that  $E^2 = -1/3$ , and compute

$$\begin{aligned} & \left(\varphi^*\left(\lambda - \frac{1}{2}\right)M + (3 - 6\lambda + \mu)E\right) \cdot \tilde{M} \\ &= \left(\varphi^*\left(\lambda - \frac{1}{2}\right)M + (3 - 6\lambda + \mu)E\right) \cdot (\varphi^*M - 6E) \\ &= \left(\lambda - \frac{1}{2}\right)M^2 - 6(3 - 6\lambda + \mu)E^2 = \frac{9}{2} > 0, \end{aligned}$$

and

$$\left(\varphi^*\left(\lambda - \frac{1}{2}\right)M + (3 - 6\lambda + \mu)E\right) \cdot E = (3 - 6\lambda + \mu)E^2 = \frac{1}{4} > 0.$$

The cases  $p = 6$ ,  $p = \infty$  are similar, simply with slightly longer computations. The basis of the computation is

$$K_Z + \lambda \tilde{M} + \lambda E + \mu F \equiv \left(\lambda - \frac{1}{2}\right) \varphi^* M + (3 - 6\lambda + \mu)E + (2 - 4\lambda + \nu)F,$$

and

$$\begin{aligned} K_Z + \lambda \tilde{M} + \lambda E + \mu F + \sigma G + \sigma H &\equiv \left(\lambda - \frac{1}{2}\right) \varphi^* M + (3 - 6\lambda + \mu)E \\ &\quad + (2 - 4\lambda + \nu)F + (4 - 6\lambda + \sigma)(G + H). \end{aligned}$$

Also, for  $\varphi: Z \rightarrow X$ , we have

$$\varphi^* M = \tilde{M} + 6E + 4F,$$

and for  $\varphi: W \rightarrow X$ , we have

$$\varphi^* M = \tilde{M} + 6E + 4F + 6(G + H).$$

**6.4. Identifying the groups.** In this section we briefly explain why the holonomy group of the complex hyperbolic structures constructed by uniformization (using the Kobayashi–Nakamura–Sakai version of equality case in the Miyaoka–Yau inequality) is isomorphic to the relevant sporadic triangle groups.

**Theorem 6.3.** *Let  $\Gamma_p$  be the group obtained from the statement of Theorem 5.5 for  $p = 3, 4$ , or  $6$ . Then  $\Gamma_p$  is conjugate to the triangle sporadic group  $\mathcal{S}(p, \sigma_1)$ .*

*Proof.* This follows from the description of orbifold fundamental group  $\Gamma_2$ , which is generated by three complex reflections  $R_j$ ,  $j = 1, 2, 3$  of order 2, such that:

- $\text{br}(R_1, R_2) = 4$ ,  $\text{br}(R_2, R_3) = 3$ ,  $\text{br}(R_3, R_1) = 3$ .
- $R_1 R_2 R_3$  (has linear part which) is regular elliptic of order 8.

Given how the orbifold structure with holonomy  $\Gamma_p$  is constructed, these same properties will hold, with complex reflections of order  $p$  instead of order 2, except that in the cases  $p > 2$ , the isometry  $R_1 R_2 R_3$  is regular elliptic of order 8 (the analogue of taking the linear part is then simply to view it as an element of the stabilizer of its fixed point, which is isomorphic to  $U(2)$ ).

The result then follows from Proposition 6.4, stated and proved below.  $\square$

**Proposition 6.4.** *Let  $\Gamma$  be a lattice generated by three complex reflections  $R_j$ ,  $j = 1, 2, 3$  such that  $\text{br}(R_1, R_2) = 4$ ,  $\text{br}(R_2, R_3) = 3$ ,  $\text{br}(R_3, R_1) = 3$  and  $R_1 R_2 R_3$  is regular elliptic of order 8. Then  $\Gamma$  is conjugate to the Thompson group  $\mathcal{T}(p, \mathbf{E}_1)$ , which is isomorphic to the sporadic triangle group  $\mathcal{S}(p, \sigma_1)$ .*

*Proof.* First, the fact that  $\mathcal{T}(p, \mathbf{E}_1)$  and  $\mathcal{S}(p, \sigma_1)$  are conjugate follows from a suitable change of generators, in the same vein as in [17]; the details are given in Proposition 7.1 of [11].

Any group as above must be conjugate to  $\mathcal{S}(p, \sigma_1)$  or  $\mathcal{S}(p, \bar{\sigma}_1)$ , but the last group is not discrete if  $p = 3$  or  $6$  (see section Section 9.4 of [12]).

One checks that  $\mathcal{S}(4, \bar{\sigma}_1)$  is not discrete either, for instance by showing that  $M = R_3 R_1 R_2 J$  is regular elliptic but has infinite order (here  $R_1, R_2$  and  $R_3$  stand for the standard generators of  $\mathcal{S}(4, \bar{\sigma}_1)$ , and  $J$  stands for the regular elliptic element of order 3 that conjugates  $R_j$  into  $R_{j+1}$ ). Indeed, one checks that

$$\mathrm{tr}(M) = (\sqrt{3} + i)(i - (1 + i)\sqrt{2})/2 := \tau,$$

and

$$|\tau|^4 - 8\Re(\tau^3) + 18|\tau|^2 - 27 = 88 - 64\sqrt{2} < 0,$$

so  $M$  is regular elliptic (see [15, Section 6.2.3]).

The characteristic polynomial of  $M$  is equal to  $\lambda^3 - \tau\lambda^2 + \bar{\tau}\lambda - 1$ , and one verifies that only one of its roots is a root of unity (namely  $-(i + \sqrt{3})/2$ ). Indeed, the other two roots have a minimal polynomial of degree 16, that is not cyclotomic.

Note that in  $\mathcal{S}(4, \bar{\sigma}_1)$ , the element  $R_3 R_1 R_2 J$  is loxodromic (and indeed, we know that this is a lattice, see [11]).  $\square$

**Remark 6.5.** The argument we just gave provides a short proof that  $\mathcal{S}(p, \sigma_1)$  is indeed a lattice for  $p = 3, 4, 6$ , a fact which was proved using heavy computer power in [11].

For  $p = \infty$ , Proposition 6.4 has the following analogue (recall that unipotent elements are isometries whose matrix representative has a single eigenvalue of multiplicity 3), which can be proved with very similar methods as in [12]. We omit the details because the corresponding group turns out to be arithmetic.

**Proposition 6.6.** *Let  $\Gamma$  be a lattice generated by unipotent elements  $R_j$ ,  $j = 1, 2, 3$  such that  $\mathrm{br}(R_1, R_2) = 4$ ,  $\mathrm{br}(R_2, R_3) = 3$ ,  $\mathrm{br}(R_3, R_1) = 3$  and  $R_1 R_2 R_3$  is regular elliptic of order 8. Then  $\Gamma$  is conjugate to Thompson group  $\mathcal{T}(\infty, \mathbf{E}_1)$ , which is isomorphic to the sporadic triangle group  $\mathcal{S}(\infty, \sigma_1)$ .*

The arithmeticity of the group  $\mathcal{S}(\infty, \sigma_1)$  is fairly obvious from the description given in Section 3, where we give a generating set with entries in  $\mathbb{Z} + i\sqrt{2}\mathbb{Z}$ . From this it follows that the adjoint trace field is  $\mathbb{Q}$ , hence the group is indeed arithmetic (see [11], for instance).

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M. Deraux, Université Grenoble Alpes, Institut Fourier,  
100 rue des maths, 38610 Gières, France  
E-mail: martin.deraux@univ-grenoble-alpes.fr