Rigidity and flexibility for handlebody groups

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Rigidity and flexibility for handlebody groups

Sebastian Hensel

Abstract. We show that finite index subgroups of the handlebody group are rigid in their ambient mapping class group: any injective map of a finite index subgroup of the genus g handlebody group into the genus g mapping class group is conjugation by a mapping class group element.

On the other hand, we construct an injection of the genus g handlebody group into a genus $h > g$ mapping class group which is not conjugate into a handlebody group.

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1. Introduction

Homomorphisms, and in particular injections, between mapping class groups have received considerable attention over the last years. See [4] for a survey, and e.g. [1,2, 12,15,17,20,24] for examples of results. A guiding theme in this subject is to try and imitate (super)rigidity results from the theory of lattices in Lie groups. For example, under suitable complexity bounds, the only injections between mapping class groups arise from "obvious" topological operations on surfaces.

In this article we investigate rigidity phenomena from a slightly different point of view. Namely, we let the mapping class group play the role of the "ambient Lie group", and study rigidity of subgroups. To be precise, by rigidity we here mean the following.

Definition 1.1. Let Γ be a subgroup of the mapping class group $\mathrm{Mcg}(\Sigma_g)$ of a closed genus g surface Σ_g. We say that Γ is *rigid in* $\mathrm{Mcg}(\Sigma_g)$ if every injective map $f\colon \Gamma \to \mathrm{Mcg}(\Sigma_g)$ is (the restriction of) an inner automorphism of $\mathrm{Mcg}(\Sigma_g)$.

We focus on an important, topologically motivated subgroup of $\mathrm{Mcg}(\Sigma_g)$, namely the the *handlebody group* $\mathcal{H}_g < \mathrm{Mcg}(\Sigma_g)$. It consists of all those mapping classes which extend to a given handlebody V with boundary Σ_g. We show.

Theorem 1.2 (Rigidity). *Suppose that* $\Gamma < \mathcal{H}_g$ *is a finite index subgroup. Then* Γ *is rigid in* $\mathrm{Mcg}(\Sigma_g)$.

As a consequence we also obtain the following, which improves the main theorem of [21].

Corollary 1.3. *The abstract commensurator of $\mathcal{H}_g$ is equal to $\mathcal{H}_g$.*

The mapping class group itself, and its finite index subgroups are rigid in all but a few exceptional low-complexity cases. These results have a long history, starting with Ivanov's study of the automorphism group and commensurator of the mapping class group [16, 19], whose methods were later greatly extended (see [5, 6, 17, 29] and the references therein). Rigidity is also known for the group generated by powers of Dehn twists [3]. In [7, 18] it is shown that the Johnson kernel is rigid inside the Torelli subgroup of the mapping class group.

We next study injections of $\mathcal{H}_g$ into higher genus mapping class groups. Here, the situation is drastically different.

Theorem 1.4 (Flexibility). *For any $g \geq 2$ there is an $h > g$, a finite index subgroup $\Gamma < \mathcal{H}_g$ and an injection $f: \Gamma \to \mathrm{Mcg}(\Sigma_h)$, so that the image of f is not conjugate into $\mathcal{H}_h$.*

The examples in Theorem 1.4 comes from a covering construction, and we can completely characterise rigidity and flexibility for such injections.

Theorem 1.5 (Covers). *Suppose that $\Sigma' \to \Sigma$ is a finite normal cover of a surface of genus $g \geq 3$. Let $\Gamma < \mathcal{H}_g$ be a finite index subgroup of mapping classes which lift to Σ'. Denote by Γ' a finite index subgroup of the lifts of elements in Γ.*

Then Γ' is conjugate into a handlebody group of Σ' if and only if $\Sigma' \to \Sigma$ can be extended to a cover of handlebodies.

The genus restriction in this theorem is likely not required, and an artefact of our proof.

In the course of the proof of Theorem 1.2 we show rigidity for a different group. The *twist group* $\mathcal{T}_g < \mathcal{H}_g$ is the subgroup generated by Dehn twists about meridians of a handlebody. It is known to be of infinite index, not finitely generatable, and with infinite rank first homology [26]. Nevertheless, rigidity holds:

Theorem 1.6. *Suppose that $\Gamma < \mathcal{T}_g$ is a finite index subgroup. Then Γ is rigid in $\mathrm{Mcg}(\Sigma_g)$. The commensurator of $\mathcal{T}_g$ is the handlebody group $\mathcal{H}_g$.*

The flexibility exhibited in Theorem 1.4, and restrictions for covering constructions as in Theorem 1.5 is also already true for $\mathcal{T}_g$.

Methods of proof. The argument which is used to show rigidity results on injections f between subgroups of mapping class groups goes back to Ivanov. It has by now become somewhat standard, and consists of three main steps. First, one shows that powers of Dehn twists map under f to (roots of) multi-twists. In this way one obtains a map between curve graphs (or related objects). Then, one uses rigidity results for maps between curve graphs to find a candidate conjugation map which f will be equal to. Checking this equality in a third step is then usually straightforward.

For finite index subgroups of $\mathrm{Mcg}(\Sigma)$, the first step is well known and due to Ivanov (see [6, 14] for well-written modern treatments of this argument). A key ingredient in his proof is that one can characterise (powers of) Dehn twists in the mapping class group via ranks of maximal Abelian subgroups of centralisers and centralisers of centralisers.

In Section 3, we develop a variant of Ivanov's argument, which may be of interest in studying the rigidity of other subgroups of the mapping class group. It bypasses an explicit identification of Dehn twists via their centralisers and also tries to avoid using maximal Abelian subgroups as much as possible. A reader experienced with arguments of this type who is only interested in the handlebody group may skip directly to Section 4.

We want to emphasise that there is an alternative approach to this first step due to Aramayona–Souto [4] which would work (with minor modifications) also for the handlebody group (but, to the knowledge of the author, not the twist group, since it has infinite rank first rational cohomology [26]).

The second and third steps of the proof require new arguments in the case of the handlebody group. In Section 4, we show that the disk graph of a handlebody is rigid inside the curve graph of the surface (compare also [3] for rigidity of subgraphs of the curve graph). This is then used to find the candidate conjugation, relying on the main result of [21].

In Section 6, we prove the Flexibility and Covering Theorems 1.4 and 1.5. The proofs rely on two main ingredients: on the one hand, a theorem of Oertel [28] characterises which multi-twists on the boundary of a handlebody extend to homeomorphisms of that handlebody. This allows to translate the condition of lifts being conjugate into $\mathcal{H}_g$ into a condition on lifts of meridians. Careful analysis of how intersection patterns between meridians behave under lifting is then used to show the results.

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2. Preliminaries

In this section we collect some well-known facts that we will use throughout. A few conventions: all *curves* will be simple, closed and essential. When not explicitly stated otherwise, we will identify curves with their isotopy classes. By *disjointness* of two curves we always mean disjointness up to homotopy. *Multicurves* are collections of disjoint curves, no two of which are freely homotopic.

2.1. Canonical reduction systems and centralisers. Let Σ be a surface of finite type, possibly with boundary and/or marked points. We let $\mathrm{Mcg}(\Sigma)$ denote the mapping class group, i.e. the group of homeomorphisms of Σ up to isotopy. Given a mapping class $\phi \in \mathrm{Mcg}(\Sigma)$ we say that ϕ is *reducible* if there is some multicurve on Σ which is (set-wise) preserved by ϕ. The mapping class ϕ is *pure* if there is a multicurve C so that ϕ preserves every component of C, and induces on each component of $S - C$ either the identity, or a pseudo-Anosov map. If ϕ is pure, then the *canonical reduction system* $C(\phi)$ is the (unique) smallest such multicurve. If ϕ is pseudo-Anosov, we set $C(\phi) = \emptyset$.

The following is due to Ivanov (compare e.g. [15, Theorem 1.2]).

Proposition 2.1. *There is a finite index subgroup* $\Gamma_p < \mathrm{Mcg}(\Sigma)$ *so that every reducible element in* Γ_p *is pure.*

Hence, we may define the canonical reduction system for any element ϕ to be the canonical reduction system of a suitably big, pure power of ϕ.

We need a version for subgroups as well. If $\Gamma < \mathrm{Mcg}(\Sigma)$ is a *pure subgroup*, i.e every reducible element in Γ is pure, then we define the canonical reduction system

$$C(\Gamma) = \bigcap_{\phi \in \Gamma} C(\phi)$$

to be the intersection of all canonical reduction systems $C(\phi)$ of every element $\phi \in \Gamma$. If Γ is not pure, we define $C(\Gamma)$ as the canonical reduction system of a finite index pure subgroup.

$C(\Gamma)$ has the property that for each complementary component Y of $C(\Gamma)$, either every pure element $\phi \in \Gamma$ restricts to the identity in Y, or there is an element in Γ which restricts to a pseudo-Anosov map in Y.

We also use the following standard results on (non-)commuting elements in the mapping class groups.

Proposition 2.2 ([25]). *Let ψ be a pseudo-Anosov. Then the cyclic group generated by ψ is finite index in the centraliser of ψ.*

In particular, if ψ is a pseudo-Anosov, then no Dehn twist commutes with ψ, and neither does an *independent* pseudo-Anosov (i.e. one which does not admit a common root, or alternatively, has different stable and unstable foliations).

The following facts on Dehn twists can e.g. be found in [9, Section 3.3].

Lemma 2.3. *Some powers of two Dehn twists T_α and T_β commute if and only if α and β are disjoint.*

Lemma 2.4. *Two powers T_α^n and T_β^m of Dehn twists are equal if and only if $n = m$, $\alpha = \beta$.*

2.2. Meridians and handlebody groups. Let V be a handlebody of genus g. Identify the boundary ∂V of V with a surface Σ of genus g. A *meridian* for V is a curve α on Σ which bounds a disk in V.

Suppose that α, β are two meridians. A *wave of α with respect to β* is a (closed) subarc $a \subset \alpha$ with the following properties:

(i) a intersects β exactly in its endpoints.

(ii) a intersects β on the same side at both endpoints. More formally, there is a regular neighbourhood $\mathcal{U}$ of β so that the intersection of a with $\mathcal{U} \setminus \beta$ consists of two arcs which are contained in the same component of $\mathcal{U} \setminus \beta$.

(iii) There is a subarc $b \subset \beta$ so that $a \cup b$ is a meridian[1].

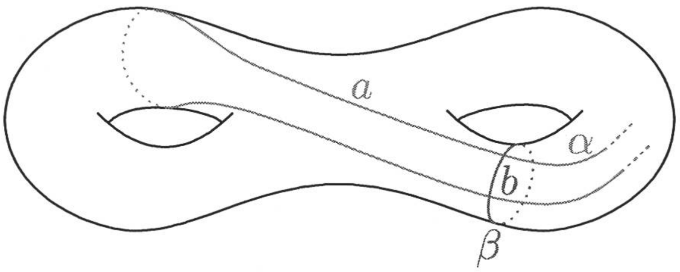

Figure 1. A wave.

The importance of waves stems form the following standard observation, whose proof we sketch for completeness. For more details of this argument, compare e.g. the proof of Theorem 5.3 in [27] (note that this source does not use the term "wave", but does construct them in the process of explaining surgery of disks).

Lemma 2.5. *Suppose that A, B are two multicurves consisting of meridians. Then either A and B are disjoint (up to isotopy), or there are $\alpha \in A, \beta \in B$ and a wave of α with respect to β, which only intersects B in its endpoints.*

Proof. If the geometric intersection number $i(A, B)$ is zero, then there is nothing to prove. Otherwise, let $A = \{\alpha_1, \dots, \alpha_r\}$, $B = \{\beta_1, \dots, \beta_s\}$. Up to isotopy, we can find properly embedded disks $D_1, \dots, D_r, D'_1, \dots, D'_s$ in the handlebody so that $\partial D_i = \alpha_i$, $1 \le i \le r$, $\partial D'_j = \beta_j$, $1 \le j \le s$, and additionally the intersection of $(D_1 \cup \dots \cup D_r) \cap (D'_1 \cup \dots \cup D'_s)$ consists of $\frac{1}{2} i(A, B)$ arcs (compare e.g. Lemma 5.1 of [27]). In particular, the multi-curves A and B are in minimal position.

Since A and B intersect, there is some i so that the intersection $D_i \cap (D'_1 \cup \dots \cup D'_s)$ is nonempty. Let b' be an *outermost* component in this intersection. By this we mean that $b' \subset D_i \cap (D'_1 \cup \dots \cup D'_s)$ is an arc which bounds, together with an arc $a \subset \partial D_i$, a subdisk $C \subset D_i$ so that the interior of C (in D_i) is disjoint from all D'_j. The arc a then clearly fulfills properties (i) and (ii) from the definition of a wave. If b

[1] We warn the reader that requirement (iii) is not standard everywhere "wave" is used in the literature.

is any subarc of a component of B with the same endpoints as b', then a and b also satisfy (iii). Namely, $a \cup b$ is embedded (because of property (ii)), and it is an essential curve (because A and B are in minimal position). Note that b and b' are homotopic relative to their endpoints in the handlebody, since they are both contained in one of the disks D'_j. Thus $a \cup b$ is homotopic to $a \cup b'$, which bounds the disk C, and is therefore nullhomotopic in the handlebody. This shows (iii). □

The restriction map induces a homomorphism

$$\mathrm{Mcg}(V) \to \mathrm{Mcg}(\Sigma)$$

whose image $\mathcal{H}_g$ we call the *handlebody group* of V. Up to conjugation, $\mathcal{H}_g$ is independent of the identification of ∂V with Σ. Usually, we will not need to distinguish between different conjugates, and fix some handlebody group $\mathcal{H}_g$. In any case, the statement that some group is conjugate into $\mathcal{H}_g$ is well-defined without choices.

A *reduced disk system for V* is a multicurve $\alpha_1, \dots, \alpha_g$ consisting of meridians so that $\Sigma - (\alpha_1 \cup \dots \cup \alpha_g)$ is connected. Note that every simple closed curve which is disjoint from a reduced disk system is a meridian. This is due to the fact that any curve on the boundary of a ball bounds a disk in the ball. The following is standard, and an immediate consequence of the fact that any homeomorphism of a sphere extends to the ball it bounds.

Lemma 2.6. *Suppose that $\phi \in \mathrm{Mcg}(\Sigma)$ is such that $\phi(C)$ is a reduced disk system for V for some reduced disk system C for V. Then $\phi \in \mathcal{H}_g$.*

The following lemma describes a well-known method to transform one reduced disk system into another. See [11, Lemma 5.4] for a formulation and proof very close to our language, but the result also follows from the arguments of e.g. [13, Lemma 1.3] or [23, Lemma 1.1].

Lemma 2.7. *Let C, C' be reduced disk systems for V. Then there is a sequence*

$$C = C_1, C_2, \dots, C_n = C'$$

of reduced disk systems for V so that C_i, C_{i+1} are disjoint for all i.

We need the following criterion for a multitwist to be an element of $\mathcal{H}_g$, which relies on [28, Theorem 1.11].

Theorem 2.8. *Let $\phi = T_{\alpha_1} \cdots T_{\alpha_n}$ be a product of Dehn twists about disjoint curves α_i and suppose that ϕ is an element of $\mathcal{H}_g$.*

Then, up to relabeling, there is a $l > 0$ and a bijection $k\colon \{l+1, \dots, n\} \to \{l+1, \dots, n\}$ so that

(i) *α_i is a meridian for all $i \leq l$.*

(ii) *α_i and $\alpha_{k(i)}$ are joined in V by a properly embedded annulus for all $i > l$.*

(iii) *If $i > l$, then T_{α_i} and $T_{\alpha_{k(i)}}$ are not both left or both right Dehn twists.*

Proof. Theorem 1.11 of [28] implies that ϕ is the restriction to ∂V of a homeomorphism $F: V \to V$, which is a product of twists about disjoint disks and annuli in the handlebody. A twist about a disk in V restricts to a Dehn twist about a meridian α_i on ∂V. A twist about an annulus A with boundary $\partial A = \alpha_i \cup \alpha_{k(i)}$ restricts to the product of a left and a right Dehn twist about α_i and $\alpha_{k(i)}$ (or vice versa). This shows the theorem. □

Corollary 2.9. *Suppose that $\alpha_1, \dots, \alpha_k$ are disjoint simple closed curves. Then the product of left Dehn twists $T_{\alpha_1} \cdots T_{\alpha_k}$ is an element of $\mathcal{H}_g$ if and only if all α_i are meridians.*

3. Full and abundant subsurfaces

In this section we discuss the first step of the rigidity proof outline given in the introduction.

Throughout, Σ will be a finite type surface, possibly with boundary or cusps. A subsurface $S \subset \Sigma$ is *essential* if every component of ∂S is an essential simple closed curve on Σ. If $\Gamma < \mathrm{Mcg}(\Sigma)$ is any subgroup, we denote by $\mathrm{Stab}_\Gamma(S)$ the subgroup of Γ consisting of all elements which preserve S (up to isotopy).

If S is a surface with a specified collection of boundary components B, we denote by $\widehat{S}$ the surface obtained from S by gluing punctured disks to each boundary component of S in B. We say that $\widehat{S}$ is obtained from S by *cusping off the boundaries* B. There is a homomorphism

$$r_S: \mathrm{Mcg}(S) \to \mathrm{Mcg}(\widehat{S})$$

To ease notation, we will often say that $\phi \in \Gamma$ has a property *when viewed as a mapping class of* $\widehat{S}$ if $r_S(\phi)$ has this property. Also note that the kernel of r_S consists of Dehn twists about the boundary components B. See [9, Section 4.2] for this, and related background on mapping class groups.

If $S \subset \Sigma$ is an essential subsurface, then we will always denote by $\widehat{S}$ the surface obtained by cusping off all boundary components which are not contained in the boundary of Σ.

Definition 3.1. Let $S \subset \Sigma$ be an essential subsurface. A subgroup $\Gamma < \mathrm{Mcg}(\Sigma)$ is

- *full in* S: if there are elements ϕ_1, ϕ_2 which are independent pseudo-Anosov elements when viewed as mapping classes of $\widehat{S}$.
- *abundant in* S: if additionally there is a pants decomposition $\{\alpha_1, \dots, \alpha_k\}$ of $\widehat{S}$ and $T_1, \dots, T_k \in \mathrm{Stab}_\Gamma(S)$ so that T_i is a power of Dehn twist about α_i (viewed as a mapping class of $\widehat{S}$).

Remark 3.2. If $\Gamma' < \Gamma$ is finite index, and Γ is full or abundant in S, then so is Γ'.

Lemma 3.3. *Suppose that $S \subset \Sigma$ is an essential subsurface and that $\Gamma < \mathrm{Mcg}(\Sigma)$ is full in S. Then every element in the centre of* $\mathrm{Stab}_\Gamma(S)$ *has a power which is a multitwist about ∂S.*

Proof. Consider the induced map

$$r_S \colon \mathrm{Stab}_\Gamma(S) \to \mathrm{Mcg}(\widehat{S})$$

If the centre of $\mathrm{Stab}_\Gamma(S)$ contains an element none of whose powers are multitwists about ∂S, then its image is an infinite order element in the centre of $r_S(\mathrm{Stab}_\Gamma(S))$. However, since we assume that Γ is full in S, the group $r_S(\mathrm{Stab}_\Gamma(S))$ contains two independent pseudo-Anosov elements. By Proposition 2.2 this is impossible. □

Proposition 3.4. *Suppose that $S \subset \Sigma$ is an essential subsurface and that $\Gamma <$ $\mathrm{Mcg}(\Sigma)$ is full in S. Suppose $f \colon \Gamma \to G$ is a homomorphism.*

If $\ker(f) \cap \mathrm{Stab}_\Gamma(S)$ *contains an element none of whose powers are multitwists about ∂S, then* $\ker(f) \cap \mathrm{Stab}_\Gamma(S)$ *contains an element which is pseudo-Anosov when viewed as a mapping class of $\widehat{S}$.*

Proof. Since Γ is full in S, there is an element $\psi \in \mathrm{Stab}_\Gamma(S)$ which is pseudo-Anosov as a mapping class on $\widehat{S}$. Furthermore, by assumption, there is an element $\phi \in \ker(f) \cap \mathrm{Stab}_\Gamma(S)$, none of whose powers are multitwists about ∂S. Thus, ϕ defines an infinite order mapping class on $\widehat{S}$. If ϕ or any power of it is pseudo-Anosov, we are already done.

Otherwise, consider $\phi\psi^n\phi^{-1}$. As a mapping class of $\widehat{S}$ this is pseudo-Anosov. In fact, it is an independent pseudo-Anosov to ψ: by Lemma 2.2, any infinite order element in the centraliser of a pseudo-Anosov has a power which is a pseudo-Anosov itself.

Thus, for any $n > 0$, the element $\phi\psi^n\phi^{-1}\psi^{-n}$ is contained in $\ker(f)$ since the latter is normal. Once n is large enough, it will also be pseudo-Anosov, since large powers of independent pseudo-Anosovs on $\widehat{S}$ generate a purely pseudo-Anosov group (compare [10]). □

The following is the core technical result of this section.

Theorem 3.5. *Let $S \subset \Sigma$ be an essential subsurface and suppose that $\Gamma < \mathrm{Mcg}(\Sigma)$ is full in S. Let $f \colon \Gamma \to \mathrm{Mcg}(\Sigma')$ be an injection into another mapping class group.*

Then there is a complementary component Y of the canonical reduction system $C(f(\mathrm{Stab}_\Gamma(S)))$ and a finite index subgroup $\Gamma' < \mathrm{Stab}_\Gamma(S)$, so that the induced map

$$f \colon \Gamma' \to \mathrm{Mcg}(Y)$$

is an injection. The kernel of the induced map $\hat{f} \colon \Gamma' \to \mathrm{Mcg}(\hat{Y})$ consists of multitwists about the boundary of S.

Proof. We may assume that $f(\Gamma)$ is pure (otherwise, pass to a suitable finite index subgroup Γ'). Put $C = C(f(\mathrm{Stab}_\Gamma(S)))$. If there is only one complementary component of C, there is nothing to prove for the first claim. The second claim follows since any ϕ so that $\widehat{f}(\phi) = 1$ is contained in the centre of $\mathrm{Stab}_\Gamma(S)$.

Hence, suppose that there is more than one complementary component of C; let Σ_1, Σ_2 be two disjoint nonempty unions of complementary components of C whose union is all of Σ'. Denote by $\widehat{\Sigma_i}$ the surface obtained from Σ_i by cusping off the boundary components corresponding to curves in C.

The injection f induces maps

$$f_i\colon \mathrm{Stab}_\Gamma(S) \to \mathrm{Mcg}(\Sigma_i), \quad i = 1, 2$$

and

$$\widehat{f_i}\colon \mathrm{Stab}_\Gamma(S) \to \mathrm{Mcg}(\widehat{\Sigma_i}), \quad i = 1, 2.$$

By induction, it suffices to show that one of the f_i is injective.

We define the product map

$$\widehat{f} = \widehat{f_1} \times \widehat{f_2}\colon \mathrm{Stab}_\Gamma(S) \to \mathrm{Mcg}(\widehat{\Sigma_1}) \times \mathrm{Mcg}(\widehat{\Sigma_2}).$$

Suppose that ϕ is such that $\widehat{f}(\phi) = 1$. Then $f(\phi)$ is a multitwist about C, and thus commutes with every element in $f(\mathrm{Stab}_\Gamma(S))$. As f is injective, this implies that ϕ commutes with every element of $\mathrm{Stab}_\Gamma(S)$, and thus ϕ has a power which is a multitwist about the boundary of S by Lemma 3.3.

Suppose now that $\ker(\widehat{f_i})$ contains elements which do not have powers which are multitwists about ∂S for both $i = 1, 2$. We let ϕ_1, ϕ_2 be mapping classes so that

(1) $\phi_i \in \ker(\widehat{f_i})$;

(2) ϕ_i is pseudo-Anosov on S;

whose existence is guaranteed by Proposition 3.4.

Then, $f_2(\phi_1)$ is a multitwist about $\partial\Sigma_2$ and $f_1(\phi_2)$ is a multitwist about $\partial\Sigma_1$ and therefore

$$1 = [f(\phi_1), f(\phi_2)] = f\big([\phi_1, \phi_2]\big).$$

Since f is injective this implies that ϕ_1 and ϕ_2 commute. Thus, ϕ_1, ϕ_2, seen as mapping classes of $\widehat{S}$, are commuting pseudo-Anosovs and thus powers of a common pseudo-Anosov of $\widehat{S}$. By passing to powers we may therefore assume that $\phi_1 = \psi T_1$, $\phi_2 = \psi T_2$ for some ψ pseudo-Anosov on $\widehat{S}$ and T_1, T_2 multitwists about ∂S. In other words, $\phi_2 = \phi_1 m$ for some m in the centre of $\mathrm{Stab}_\Gamma(S)$.

If $\phi_1 = \phi_2$, then $\widehat{f}(\phi_1) = 1$ and hence $f(\phi_1)$ is central. By injectivity of f this would imply that ϕ_1 is central, which is impossible by Lemma 3.3.

Otherwise, $\phi_1^{-1}\phi_2 = m \neq 1$ is a nontrivial central element. Since Γ is full in S, there is an element ρ so that

$$[\phi_1, \rho] \neq 1$$

(there are independent pseudo-Anosovs, hence not every element can commute with ϕ_1). By injectivity of f, we therefore have that

$$[f(\phi_1), f(\rho)] \neq 1.$$

Since the kernel of $\widehat{f}$ is central, this implies that

$$[\widehat{f}(\phi_1), \widehat{f}(\rho)] \neq 1.$$

As $\widehat{f_1}(\phi_1) = 1$ this means that

$$[\widehat{f_2}(\phi_1), \widehat{f_2}(\rho)] \neq 1.$$

But since $\widehat{f_2}(\phi_1) = \widehat{f_2}(m^{-1})$, this would imply that $\widehat{f_2}(m)$ and $\widehat{f_2}(\rho)$ do not commute, contradicting the fact that m is central.

This contradiction shows that we may assume (up to relabeling) that $\ker(\widehat{f_1})$ contains only contains elements which do have a power which is a multitwist about ∂S.

Suppose now that m is an element of $\ker(f_1)$. By the above, it has a power m^k which is a multitwist about ∂S, and therefore central in $\mathrm{Stab}_\Gamma(S)$. Hence, either m is finite order, or $f(m^k)$ is nontrivial multitwist about $C(f(\mathrm{Stab}_\Gamma(S)))$. In the latter case $f_1(m) \neq 1$. Taking Γ' so that $\mathrm{Stab}_\Gamma(S)$ is torsion-free therefore shows the theorem. □

Using this result we can show that, under suitable assumptions, images of Dehn twists are roots of Dehn twists. In the proof we require the notion of the *complexity* $\xi(F)$ of a finite type surface. Namely, $\xi(F) = 3g(F) + 2b(F) - 3$ is the number of curves in a pants decomposition for F, where $g(F)$ is the genus and $b(F)$ is the number of boundary components and cusps.

Corollary 3.6. *Let γ be a non-separating simple closed curve on Σ, and let S be the complement of γ. Suppose that Γ is abundant in S, and that $f\colon \Gamma \to \mathrm{Mcg}(\Sigma)$ is any injection. Then $f(T_\gamma)$ has a power which is a Dehn twist about some non-separating curve δ.*

Proof. Using that Γ is full in S, we can replace Γ with the finite index subgroup Γ' from Theorem 3.5 and there is then a complementary component $Y \subset \Sigma$ and an injective map

$$\mathrm{Stab}_\Gamma(S) \to \mathrm{Mcg}(Y)$$

so that the induced map

$$\mathrm{Stab}_\Gamma(S) \to \mathrm{Mcg}(\hat{Y})$$

has a kernel consisting only of twists about γ. Using that Γ is abundant in S, there is a free Abelian group of rank $\xi(\Sigma) - 1$ in $\mathrm{Stab}_\Gamma(S)$ which does not contain any twist about γ, and therefore there is such an Abelian group in in $\mathrm{Mcg}(\hat{Y})$. Thus, $\hat{Y}$ has at least complexity $\xi(\Sigma) - 1$, but is obtained by taking a subsurface of Σ and cusping off boundaries. This is only possible if Y is the complement of a single non-separating curve δ, and thus the canonical reduction system of $f(\mathrm{Stab}_\Gamma(S))$ is a single curve. This implies that some power of T_γ, which maps to a central element in $f(\mathrm{Stab}_\Gamma(S))$, is a Dehn twist about δ. □

In fact, by induction, we also obtain the following result. For its statement, recall that a *cut system* is a multicurve $\alpha_1, \ldots, \alpha_g$ on a surface so that the complement $\Sigma - (\alpha_1 \cup \cdots \cup \alpha_g)$ is connected and has genus 0.

Corollary 3.7. *Suppose that $\alpha_1, \ldots, \alpha_g$ is a cut system for Σ. Assume that Γ is abundant in $\Sigma - (\alpha_1 \cup \cdots \cup \alpha_k)$ for all $1 \leq k \leq g$.*

Let $f: \Gamma \to \mathrm{Mcg}(\Sigma)$ be injective for some $\Gamma < \mathrm{Mcg}(\Sigma)$. Then $f(T_{\alpha_i})$ have powers which are Dehn twists about a cut system in Σ.

Proof. By the previous Corollary 3.6, $f(T_{\alpha_1})$ has a power which is a Dehn twist about some non-separating δ_1, and furthermore f induces an injection of $\mathrm{Stab}_\Gamma(\Sigma - \alpha_1)$ into $\mathrm{Mcg}(\Sigma - \delta_1)$. By the assumption on abundance, we can continue the argument inductively. □

4. Rigidity of the disk graph

Recall that V is a handlebody of genus g, and we have identified the boundary ∂V of V with a surface Σ of genus g.

The *curve graph of* Σ is the simplicial complex $\mathcal{C}(\Sigma)$ whose k-simplices correspond to multicurves with $k+1$ components. The *disk graph $\mathcal{D}(V)$ of V* is the full sub-complex of the curve graph $\mathcal{C}(\Sigma)$ spanned by the meridians for V. Explicitly, k-simplices of $\mathcal{D}(V)$ correspond to multicurves with $k+1$ components, each of which is a meridian. We will usually identify a multicurve with the simplex of $\mathcal{C}(\Sigma)$ or $\mathcal{D}(\Sigma)$ that it defines. The following is obvious from the definitions.

Lemma 4.1. *Let $\alpha_1, \ldots, \alpha_k$ be a multicurve on Σ, and let $Y_1, \ldots, Y_l$ be its complementary components. Then the link of the simplex Δ defined by $\alpha_1, \ldots, \alpha_k$*

$$\mathrm{lk}(\Delta) = \mathcal{C}(Y_1) * \cdots * \mathcal{C}(Y_k)$$

is the join of the curve graphs of the Y_i.

In this section we show a combinatorial rigidity for the disk graph inside the curve graph (compare also [3] for a stronger definition of rigid subgraph). Recall from [14] that a *superinjective* map between (subgraphs) of curve graphs is a simplicial map ι with the property that $\iota(\alpha)$ and $\iota(\beta)$ are joined by an edge if and only if α and β are joined by an edge.

Theorem 4.2. *Let $\iota: \mathcal{D}(V) \to \mathcal{C}(\Sigma)$ be a superinjective simplicial map. Then ι is induced by a mapping class of Σ: there is a mapping class $\phi \in \mathrm{Mcg}(\Sigma)$ so that $\iota(\alpha) = \phi(\alpha)$ for all simple closed curves α.*

We expect that the result is also true for injective simplicial maps ι, but have not explored this (since it is not used in the sequel).

Proof. The proof has various stages. In each stage, we might modify ι by a mapping class to ensure additional properties.

Reduced disk systems map to cut systems. Fix a reduced disk system $\alpha_1, \dots, \alpha_g$ for V. This defines a $(g-1)$–dimensional simplex Δ of $\mathcal{D}(V) \subset \mathcal{C}(\Sigma)$ whose link in $\mathcal{C}(\Sigma)$ is completely contained in $\mathcal{D}(V)$, and is isomorphic to the curve graph $\mathcal{C}(\Sigma_{0,2g})$ of a $2g$–holed sphere.

Consider the image $\iota(\alpha_1), \dots, \iota(\alpha_g)$. This is a $(g-1)$–dimensional simplex $\iota(\Delta)$ in $\mathcal{C}(\Sigma)$. We claim that $\iota(\alpha_1), \dots, \iota(\alpha_g)$ is non-separating, hence a cut system. Namely, suppose that the complement had components $Y_1, \dots, Y_k$. Choose some curve δ disjoint from $\alpha_1, \dots, \alpha_g$, and assume that $\iota(\delta)$ is a curve contained in Y_1. Then, any δ' with $\delta \cap \delta' \neq \emptyset$ satisfies $\iota(\delta) \subset Y_1$ as well (as otherwise, $\iota(\delta), \iota(\delta')$ would be disjoint, violating superinjectivity). This shows that the sub-complex of $\mathrm{lk}(\Delta)$ spanned by every vertex not contained in the star $\mathrm{st}(\delta)$ is mapped into $\mathcal{C}(Y_1)$ (under the identification given by Lemma 4.1).

Next, choose some δ' to be distance at least 3 from δ in $\mathcal{C}(\Sigma_{0,2g})$, and repeat the argument with δ' in place of δ, to see that $\mathrm{st}(\delta) \cap \mathrm{lk}(\Delta)$ is also mapped into $\mathcal{C}(Y_1)$.

Therefore, ι induces a superinjective simplicial map $\mathcal{C}(\Sigma_{0,2g}) \cong \mathrm{lk}(\Delta) \to \mathcal{C}(Y_1)$. Since the dimension of the curve graph is one less than the complexity of the surface, and $Y_1 \subset \Sigma$, this is only possible if Y_1 is the only complementary component of $\iota(\alpha_1), \dots, \iota(\alpha_g)$.

Since the mapping class group of Σ acts transitively on the set of cut systems, up to modifying ι by a mapping class, we may assume that $\iota(\alpha_1) = \alpha_1, \dots, \iota(\alpha_g) = \alpha_g$ and therefore $\iota(\alpha_1), \dots, \iota(\alpha_g)$ is a reduced disk system for V.

Reduced disk systems map to reduced disk systems. Let $\beta_1, \dots, \beta_g$ be a reduced disk system for V, which is disjoint from $\alpha_1, \dots, \alpha_g$. Then, $\iota(\beta_1), \dots, \iota(\beta_g)$ is a cut system, which is disjoint from the reduced disk system $\iota(\alpha_1), \dots, \iota(\alpha_g)$.

Next, note that any curve disjoint from a reduced disk system for V is also a meridian for V. This is simply a consequence of the fact that any simple closed curve on the sphere bounds a disk in the ball. Hence, $\iota(\beta_1), \dots, \iota(\beta_g)$ is a reduced disk system as well.

By Lemma 2.7, this inductively implies that $\iota(\beta_1), \dots, \iota(\beta_g)$ is in fact a reduced disk system for any reduced disk system $\beta_1, \dots, \beta_g$ for V.

Meridians map to meridians. Note that any meridian δ is disjoint from some reduced disk system $\beta_1, \dots, \beta_g$, and hence $\iota(\delta)$ is curve disjoint from the reduced disk system $\iota(\beta_1), \dots, \iota(\beta_g)$; and hence a meridian (by the same argument as above). This implies that ι is now a superinjective self-map of the disk graph $\mathcal{D}(V)$.

Surjectivity of ι. As a first step, we prove that ι is locally surjective in the following sense. Suppose that $\alpha_1, \dots, \alpha_g$ is a reduced disk system for V. By the previous steps, $\iota(\alpha_1), \dots, \iota(\alpha_g)$ is also a reduced disk system for V. Arguing as above, ι induces a superinjective map between links, which can be interpreted as a superinjective map

$$\mathcal{C}(\Sigma_{0,2g}) \to \mathcal{C}(\Sigma_{0,2g}).$$

By Theorem 2 of [6] such a map is induced by a mapping class, and thus in particular surjective. This implies that every curve which is disjoint from $\iota(\alpha_1), \dots, \iota(\alpha_g)$ lies in the image of ι. By Lemma 2.7 and induction, this first implies that every reduced disk system for V is the image of a reduced disk system under ι. Since every meridian is disjoint from some reduced disk system, ι is in fact surjective.

Rigidity. At this point, ι is a superinjective, surjective self-map of the disk graph, and therefore in particular a simplicial automorphism. By the main result of [21], it is therefore induced by a handlebody group element, finishing the proof. □

5. Rigidity for the twist and handlebody groups

Recall that $\mathcal{T}_g < \mathcal{H}_g$ is defined to be the subgroup of $\mathcal{H}_g$ generated by Dehn twist about meridians. By Luft's theorem [22], $\mathcal{T}_g$ agrees with the kernel of the canonical map $\mathcal{H}_g \to \mathrm{Out}(\pi_1(V))$ induced by the action of homeomorphisms of V on the fundamental group of V.

We begin with some generalities on the handlebody and twist groups.

Lemma 5.1. *Let $\Gamma < \mathcal{T}_g$ be finite index. Suppose that S is a subsurface of Σ whose boundary consists of meridians, and so that $\xi(S) > 0$. Then Γ is abundant on S.*

Proof. We begin by noting that a Dehn twist about a meridian is an element of $\mathcal{H}_g$. If $\xi(S) > 0$, and S is bounded by meridians, then there are two meridians α_1, α_2 in S which fill S. Since the product $T_{\alpha_1} T_{\alpha_2} \in \mathcal{T}_g$ of Dehn twists about such curves is pseudo-Anosov, and supported in S, there is an element ϕ in $\mathrm{Stab}_\Gamma(S)$ which defines a pseudo-Anosov in $\widehat{S}$. Since some power of T_{α_1} also lies in $\mathrm{Stab}_\Gamma(S)$, and does not commute with ϕ, by Proposition 2.2, Γ is full for S.

Also, if S is bounded by meridians, there is a pants decomposition of S consisting of meridians. This shows abundance. □

Lemma 5.2. *Suppose that $\phi \in \mathcal{H}_g$ is such that $\phi(\delta) = \delta$ for every meridian δ. Then $\phi = \mathrm{id}$.*

Proof. Consider a reduced disk system $\delta_1, \dots, \delta_g$ for the handlebody V. Denote by S the subsurface obtained as the complement of the δ_i. By assumption, ϕ preserves all δ_i, and thus S. Every simple closed curve in S is a meridian, and thus ϕ induces the identity seen as a mapping class of $\widehat{S}$. This implies that ϕ is a multitwist about the δ_i. Since for each i there is a meridian crossing δ_i, it is in fact the trivial multitwist. □

Now fix a finite index subgroup $\Gamma < \mathcal{T}_g$ or $\Gamma < \mathcal{H}_g$. In fact, the only property of Γ we use is that for any twist T_α about some meridian, a suitable power T_α^n is contained in Γ. Furthermore, assume that $f\colon \Gamma \to \mathrm{Mod}(\Sigma_g)$ is a given injection. We now follow the strategy outlined in the introduction to show that f is in fact a suitable conjugation.

In this setting, Corollary 3.6 implies that $f(T_\alpha^n)$ is the power of a non-separating twist for any non-separating meridian α and n big enough. First, we note that this conclusion also holds for separating meridians.

Lemma 5.3. *In the setting as above, if δ is any meridian, there is some $n > 0$ so that $f(T_\delta^n)$ is the power of some Dehn twist.*

Proof. Let δ be arbitrary. Choose a reduced disk system $\alpha_1, \dots, \alpha_g$ disjoint from δ. Then, by Corollary 3.7, the twists about α_i map to twists about a non-separating multicurve $\beta_1, \dots, \beta_g$. Thus, f induces an injective homomorphism

$$\hat{f}\colon \Gamma' \to \mathrm{Mod}(S_{0,2g})$$

whose domain Γ' has the property that some power of any Dehn twist is contained in Γ'. By Corollary 2 of [3] such a map is induced by a surface diffeomorphism, and in particular maps Dehn twists to Dehn twists. □

Theorem 5.4. *Suppose that $\Gamma < \mathcal{T}_g$ or $\Gamma < \mathcal{H}_g$ is any finite index subgroup, and let $f\colon \Gamma \to \mathrm{Mcg}(\Sigma_g)$ be injective. Then f is the restriction of a conjugation by an element in the mapping class group* $\mathrm{Mod}(\Sigma)$.

Proof. By the lemma above, for any meridian δ and $n > 0$ big enough, $f(T_\delta^n)$ is the power of a Dehn twist about some curve $\iota(\delta)$. Furthermore, if δ, δ' are disjoint, then $T_\delta^n, T_{\delta'}^n$ commute, thus so do the twist powers about $\iota(\delta), \iota(\delta')$; hence they are disjoint (Lemma 2.3). In other words, ι defines a simplicial map

$$\iota\colon \mathcal{D}(V) \to \mathcal{C}(\Sigma).$$

Since f is injective, this map ι is superinjective: if δ, δ' are not disjoint, then $T_\delta^n, T_{\delta'}^n$ and hence $f(T_\delta^n), f(T_{\delta'}^n)$ do not commute; hence $\iota(\delta), \iota(\delta')$ are not disjoint.

By Theorem 4.2, ι is therefore induced by some mapping class of Σ. Changing f by a conjugation we may therefore assume that $f(T_\delta^{n(\delta)}) = T_\delta^{m(\delta)}$ for every δ.

Now, let $\phi \in \Gamma$ be arbitrary. Note that for any meridian δ

$$T_{\phi(\delta)}^{m(\phi\delta)} = f\big(T_{\phi(\delta)}^{n(\phi\delta)}\big) = f\big(\phi T_\delta^{n(\phi\delta)} \phi^{-1}\big) = f(\phi) f\big(T_\delta^{n(\phi\delta)}\big) f(\phi)^{-1}$$

and therefore

$$T_{\phi(\delta)}^{m(\phi\delta)n(\delta)} = f(\phi) T_\delta^{m(\delta)n(\phi\delta)} f(\phi)^{-1} = T_{f(\phi)(\delta)}^{m(\delta)n(\phi\delta)}$$

and therefore $f(\phi)(\delta) = \phi(\delta)$ for all meridians δ (by Lemma 2.4). This implies $f(\phi) = \phi$ by Lemma 5.2. □

We are now ready to prove the second main result stated in the introduction. Recall that if G is a group, the *(abstract) commensurator of* G is the group of isomorphisms $f: H_1 \to H_2$ where $H_1, H_2 < G$ are finite index, up to the equivalence relation which identifies isomorphisms which agree on a finite index subgroup. Via conjugation, G is always a subgroup of the commensurator of G if G has trivial centre. If G is additionally a normal subgroup of a group G', then G' is contained in the commensurator of G (again via the conjugation action). The next corollary therefore shows that for the handlebody and twist groups, the commensurators are as small as possible.

Corollary 5.5. *The abstract commensurator of $\mathcal{H}_g$ is $\mathcal{H}_g$. The abstract commensurator of $\mathcal{T}_g$ is $\mathcal{H}_g$.*

Proof. In light of Theorem 5.4 the only claim to prove is the following: suppose that $\Gamma < \mathcal{H}_g$ or $\mathcal{T}_g$ is finite index, and ϕ is a mapping class such that $\phi\Gamma\phi^{-1} < \mathcal{H}_g$, then $\phi \in \mathcal{H}_g$. To show this, let δ be any meridian, and $n > 0$ be such that $T_\delta^n \in \Gamma$. Then by assumption $T_{\phi(\delta)}^n = \phi T_\delta^n \phi^{-1} \in \mathcal{H}_g$, and therefore $\phi(\delta)$ is a meridian. Hence, ϕ is a mapping class which preserves the set of meridians for V, and therefore contained in $\mathcal{H}_g$. □

6. Flexibility for the handlebody group

The first goal of this section is to prove the following.

Theorem 6.1. *There is a finite index subgroup $\Gamma < \mathcal{H}_g$ and an inclusion $f: \Gamma \to$ $\mathrm{Mcg}(\Sigma_h)$ whose image is not conjugate into $\mathcal{H}_h$.*

The construction is very explicit and uses finite covers. The strategy is to consider a cover in which meridians lift to curves whose intersection pattern is incompatible with being meridians (or even boundaries of annuli). For an example in genus 2, consider Figure 2. While Theorem 6.1 formally can be concluded quickly from Theorem 6.3, it is instructive to consider the (simpler) setting of Theorem 6.1 first, to understand the argument involved.

Proof. Let δ_0 be a meridian, and α a curve intersecting δ once. The map $\pi_1(\Sigma) \to \mathbb{Z}/3\mathbb{Z}$ defined by algebraic intersection number (mod 3) with α defines a cover $\Sigma' \to \Sigma$ of degree 3.

Let δ_1 be a meridian disjoint from δ, α, and let δ_2 be a meridian which intersects δ_1 in two points and α in two points, with algebraic intersection number 0.

Hence, the Dehn twists $T_{\delta_1}, T_{\delta_2}$ are in $\mathcal{H}_g$ and lift to Σ'. The lift of T_{δ_i} is the product of (left) Dehn twists about the three lifts $\delta_i^{(j)}$, $j = 1, 2, 3$ of δ_i to Σ'.

By construction, each $\delta_1^{(j)}$ intersects exactly two $\delta_2^{(k)}$; each in one point.

Suppose that both the lifts $\widetilde{T_{\delta_1}}$ and $\widetilde{T_{\delta_2}}$ would be conjugate into the same handlebody group. By Theorem 2.8, the multitwists $T_{\delta_i^{(1)}} T_{\delta_i^{(2)}} T_{\delta_i^{(3)}}$ are then products of twists about meridians and twists about annuli.

Since 3 is odd and twist curves for annulus twists come in pairs, at least one of each of the curves involved is a meridian. On the other hand, as every $\delta_1^{(j)}$ intersects some $\delta_2^{(k)}$ in one point, it is impossible that all three curves are meridians.

Hence, we may assume that we have the following situation:

- $\delta_1^{(1)}$ is a meridian.
- $\delta_1^{(1)}$ intersects $\delta_2^{(1)}$ $\delta_2^{(2)}$, and the latter two are connected by an annulus in the handlebody.
- $\delta_2^{(3)}$ is a meridian.

However, in such a situation the product of left Dehn twists about $\delta_2^{(1)}$ and $\delta_2^{(2)}$ is not in the handlebody group, leading to a contradiction.

Thus, it is impossible that $\widetilde{T_{\delta_1}}$ and $\widetilde{T_{\delta_2}}$ are conjugate into the same handlebody subgroup of $\mathrm{Mcg}(\Sigma_h)$. Taking Γ to be a subgroup which lifts to Σ' yields the desired inclusion. □

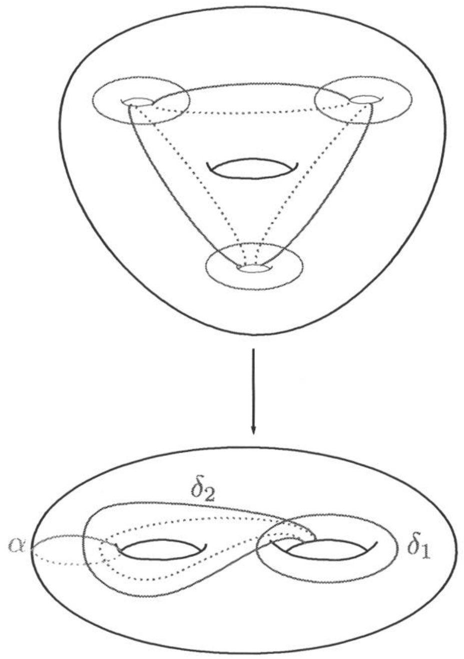

Figure 2. A 3-fold cover inducing odd intersections of meridians. The handlebody for the bottom surface is the "outside" handlebody of the standard Heegaard splitting of S^3: curves are meridians if they can be contracted in the non-compact region of the page bounded by the surface.

Denote by $\mathcal{C}^*(\Sigma_h)$ the *multicurve graph of* Σ_h, i.e. the graph whose vertices correspond to (isotopy classes of) multicurves, and edges correspond to disjointness.

We warn the reader that this graph is different from other multicurve graphs considered in the literature. Namely, we allow the number of elements in the multicurves to vary, and more importantly, adjacency does not correspond to basic moves (e.g. exchange one curve). The graph $\mathcal{C}_h^*$ is however the natural object when considering covering constructions. While there are strong restrictions for simplicial injections between n–multicurve graphs (see e.g. [8]), any covering induces interesting simplicial injections between multicurve graphs in our sense. The construction employed in the proof of the previous result immediately implies the following result on flexible inclusions of disk graphs.

Corollary 6.2. *There is a map $\mathcal{D}(V_g) \to \mathcal{C}^*(\Sigma_h)$ such that the images is not conjugate into any sub-graph where all vertices correspond to multi-meridians. The same remains true even if we allow that vertices map to multicurves which are boundaries of annuli in the handlebody.*

For covering constructions one can analyse the situation completely. The goal is the following theorem, whose proof will occupy the rest of this section.

Theorem 6.3. *Suppose that $\Sigma' \to \Sigma$ is a finite normal cover, where Σ is closed of genus $g \geq 3$. Let $\Gamma < \mathcal{H}_g$ be a finite index subgroup of mapping classes which lift to Σ'. Denote by Γ' a finite index subgroup of the lifts of elements in Γ.*

Then Γ' is conjugate into a handlebody group of Σ' if and only if $\Sigma' \to \Sigma$ can be extended to a cover of the handlebody V corresponding to $\mathcal{H}_g$.

One direction is easy: suppose $V' \to V$ is a cover of handlebodies, and $\partial V' = \Sigma' \to \Sigma = \partial V$ its boundary cover. If $F\colon V \to V$ is a homeomorphism whose restriction ϕ to the boundary lifts to Σ', then F lifts to a homeomorphism of V'. Hence, any group Γ' as in the statement is conjugate into the handlebody group defined by V'.

The other direction is more involved. We begin with the following, which is a restatement of the final argument employed in proof of Theorem 6.1.

Proposition 6.4. *Suppose that $\Sigma' \to \Sigma$ is a finite cover. Let $\Gamma < \mathcal{H}_g$ be a finite index subgroup of mapping classes which lift to Σ'. Denote by Γ' a finite index subgroup of the lifts of elements in Γ.*

If Γ' is conjugate into a handlebody group of Σ' then Σ' can be identified with the boundary of a handlebody V' so that for every meridian δ, each component of the preimage of δ in Σ' is a meridian for V'.

Proof. Consider the left Dehn twist T_δ about any meridian, and consider a lift of T_δ^n, where n is big enough to ensure that δ^n lifts to a closed curve. The lift of T_δ^n is then product of left Dehn twists about the preimages $\delta_1, \dots, \delta_k$ of δ^n. By Corollary 2.9 this element is contained in the handlebody group of Σ' if and only if all δ_i are meridians. □

To use this, we note the following standard lemma.

Lemma 6.5. *Let $\Sigma' \to \Sigma$ be a finite cover, and suppose that $\Sigma = \partial V$. Then $\Sigma' \to \Sigma$ extends to a cover of handlebodies (with base V) if and only if every meridian for V lifts to Σ' with degree* 1.

Proof. If the cover $\Sigma' \to \Sigma$ extends to $V' \to V$, then every meridian lifts with degree 1. Namely, if α is a meridian, let $D \subset V$ be a disk bounding α. Being simply connected, D lifts to a disk $D' \subset V'$, whose boundary will be a (degree 1) lift of α.

Conversely, suppose that $\Sigma' \to \Sigma$ is a cover with the property that every meridian lifts with degree 1. Define

$$K = \ker\big(\pi_1(\Sigma) \to \pi_1(V)\big),$$

where the map is induced by the inclusion of the boundary. Since K is generated by meridians, we have $K < \pi_1(\Sigma')$, as we assume that all meridians lift to Σ' with degree 1. Hence, $\pi_1(\Sigma')/K < \pi_1(V)$ defines a finite index subgroup, which in term determines a cover $V' \to V$ of handlebodies, with $\pi_1(V') = \pi_1(\Sigma')/K$. By construction, its boundary will have $\pi_1(\partial V') = \pi_1(\Sigma')$ (as subgroups of $\pi_1(\Sigma)$), and hence $V' \to V$ extends $\Sigma' \to \Sigma$ as claimed. □

Thus, Theorem 6.3 will follow, once we can prove the following. In its formulation, an *elevation* of a simple closed curve α on Σ (with respect to a cover $p\colon \Sigma' \to \Sigma$) is any connected component of $p^{-1}(\alpha)$.

Proposition 6.6. *Suppose that $\Sigma' \to \Sigma$ is a regular finite cover, and $\Sigma = \partial V$, $\Sigma' = \partial V'$. Assume that any elevation of a meridian for V is a meridian for V'. Then every meridian of V lifts with degree* 1.

For the rest of the section, we fix the cover $\Sigma' \to \Sigma$ and assume that it is given by a surjection

$$\pi\colon \pi_1(\Sigma, p) \to G$$

to some finite group G. The core tool we use is the existence of waves (compare Lemma 2.5).

The first part of the proof involves trying to construct a pair of meridians whose elevations intersect in a manner incompatible with being meridians. Namely, we have the following.

Lemma 6.7. *Suppose that there is a pair of meridians α, β intersecting only in the basepoint p such that $\pi(\beta)$ is not equal to a power of $\pi(\alpha)$ in G. Then there is a meridian δ so that elevations of α and δ cannot be simultaneously be meridians (for any handlebody).*

Proof. First note that we may assume that β is non-separating in $\Sigma - \alpha$, since non-separating simple meridians generate the kernel $\ker(\pi_1(\partial W) \to \pi_1(W))$ for any handlebody W. Note that $\beta' = \beta\alpha$ (or $\alpha\beta$) also has the property that $\pi(\beta')$ is not equal to a power of $\pi(\alpha)$ in G. This means that (non-closed) lifts of β and β' connect different elevations of α in Σ'.

Next, choose a curve ρ intersecting α, β, β' in a single point, and transversely intersecting a meridian α' disjoint from α, β in a single point (this is where we use genus $g \geq 3$ to ensure the existence of the desired α').

The desired curve is
$$\delta = \beta * \rho * \beta'^{-1} * \rho^{-1}$$
which is a simple meridian (compare Figure 3). Consider an elevation $\widetilde{\delta}$ of δ. By our choices, consecutive intersection points of $\widetilde{\delta}$ with components of the preimage of $\alpha \cup \alpha'$ are never on the same component of $\alpha \cup \alpha'$.

However, if $\widetilde{\delta}$ and all elevations of α, α' are meridians, this is a contradiction, since by Lemma 2.5 the meridian $\widetilde{\delta}$ should have a wave with respect to the preimage of $\alpha \cup \alpha'$. □

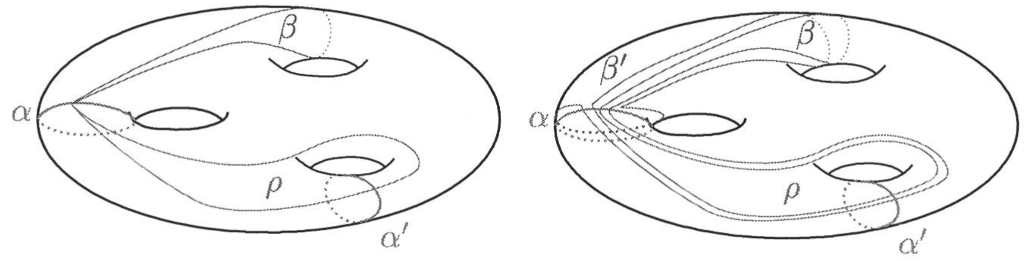

Figure 3. The construction in the proof of Lemma 6.7. The left figure shows the setup; the right one the curve constructed in the proof.

Corollary 6.8. *Assume that all meridians for V elevate to meridians for V'. Suppose that some meridian for V lifts to Σ' with degree* 1. *Then all meridians for V lift to Σ' with degree* 1.

Proof. First, suppose that there is a non-separating meridian δ which lifts with degree 1. We claim that then any meridian disjoint from δ also lifts with degree 1.

Namely, consider any meridian δ' which is disjoint from δ and suppose that δ' does not lift with degree 1. Take a basepoint p on δ, and homotope δ' so that it intersects δ only in p. Then Lemma 6.7 applies for $\alpha = \delta$, $\beta = \delta'$, since $\pi(\alpha) = 1$ (as δ lifts with degree 1) and $\pi(\beta) \neq 1$ (as δ' does not), and yields the contradiction that δ and δ' cannot both have elevations which are meridians.

Now suppose that δ' is any non-separating meridian. Since the non-separating disk graph is connected (this follows e.g. from Lemma 2.7), there is a sequence $\delta = \delta_1, \delta_2, \ldots, \delta_n = \delta'$ of non-separating meridians so that for any $i = 1, \ldots, n-1$ the curves δ_i, δ_{i+1} are disjoint. Inductively applying the above claim then yields that δ' lifts with degree 1. Hence, any non-separating meridian lifts with degree 1. Since any meridian is disjoint from some non-separating meridian, the corollary follows using the claim from above once again.

We are thus left with the case that no non-separating meridian lifts with degree 1, but there is a separating meridian δ which does. In this case, we can take (non-separating) meridians δ_1, δ_2 in the two complementary components of δ which do not lift with degree 1. Concatenating δ_1 and δ_2 yields a meridian δ', whose elevation intersects the preimages of δ without waves (arguing exactly as in the proof of Lemma 6.7), and therefore its elevation and the preimages of δ cannot both be meridians. This contradicts the assumption of the corollary. □

Hence, for the rest of the section, we can make the following assumption

(AR) For any non-separating meridian α through p , and any loop ρ which recurs to the same side of α, and is a meridian for V, $\pi(\rho)$ is a power of $\pi(\alpha)$ in G.

Note that this first implies that any conjugate of α also maps to a power of $p(\alpha)$ (conjugations by all of the standard generators of $\pi_1(\Sigma)$ lie in the complement of some meridian ρ as in AR). Since any two meridians can be joined by a sequence of pairwise disjoint meridians, this implies that the image of the kernel

$$K = \ker\big(\pi_1(\Sigma) \to \pi_1(V)\big)$$

in G is cyclic, generated by some element m. As K is normal, and $\pi_1(\Sigma) \to G$ surjective, the subgroup $\langle m\rangle$ is therefore normal in G. We thus have a tower of regular coverings

$$\Sigma' \to \Sigma'/\langle m\rangle \to \Sigma.$$

By construction, and the fact that every cyclic cover of a surface is defined by algebraic intersection with some curve, we therefore know:

(i) Every meridian for V lifts with degree 1 to $\Sigma'/\langle m\rangle$.

(ii) There is $n > 0$ and a simple closed curve $\alpha \subset \Sigma'/\langle m\rangle$ so that a curve $\beta \subset \Sigma'/\langle m\rangle$ lifts to Σ' with degree 1 if and only if $i(\beta,\alpha) = 0 \bmod n$.

Lemma 6.9. *Let $H = G/\langle m\rangle$ denote the deck group of $\Sigma'/\langle m\rangle \to \Sigma$, and let α be as in* (ii)*. Then, for any $h \in H$, we have*

$$h[\alpha] = \pm[\alpha] \in H_1\big(\Sigma'/\langle m\rangle\big)$$

Proof. Since the cover $\Sigma' \to \Sigma$ is normal, we have that a loop $\beta \subset \Sigma'/\langle m\rangle$ lifts to Σ' with degree 1 if and only if this is true for $h^{-1}\beta$, for every $h \in H$. In other words,

$$i(\beta,\alpha) = 0 \quad \Leftrightarrow \quad i(h\beta,\alpha) = i(\beta,h\alpha) = 0.$$

Thus, $h\alpha$ has algebraic intersection number 0 with exactly the same loops as α. This implies that $h[\alpha]$ is a multiple of $[\alpha]$, and the multiple is ± 1 as H is finite. □

Lemma 6.10. *Either some meridian for V lifts to Σ' with degree* 1*, or the following is true: Let $\delta_1,\dots,\delta_g$ be any reduced disk system for V, and let Y be the complementary* *subsurface. Choose orientations on δ_i. Let $Y' \subset \Sigma'/\langle m\rangle$ be a (homeomorphic) lift of Y, and let δ_i^+, δ_i^- be lifts of δ_i on the boundary of Y', oriented compatibly with the orientation of δ_i. Then*

$$i(\delta_i^+, \alpha) = -i(\delta_i^-, \alpha)$$

Proof. Since δ_i^+, δ_i^- are (compatibly oriented) lifts of the same meridian, there is some element $h \in H$ so that $\delta_i^- = h\delta_i^+$. This already implies

$$i(\delta_i^+, \alpha) = \pm i(\delta_i^-, \alpha)$$

by the above. We have to exclude the positive sign. However, note that there is a simple closed meridian μ in Y' which is homologous to $[\delta_i^+] - [\delta_i^-]$ (compare Figure 4). If in the previous equation the sign is positive, this meridian has algebraic intersection number 0 with α, therefore lifts with degree 1 to Σ'. Being a meridian in Y' it is also a degree 1 lift of a simple closed meridian for V. This shows the lemma. □

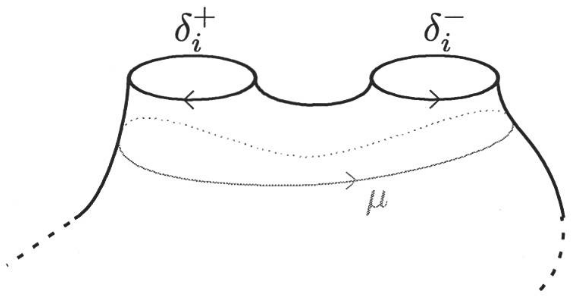

Figure 4. The construction in the proof of Lemma 6.10.

Lemma 6.11. *There is a meridian for V which lifts with degree 1 to Σ'.*

Proof. Let $\delta_1, \dots, \delta_g$ be a reduced disk system for V, and let Y be the complementary subsurface. Choose orientations on δ_i. Let $Y' \subset \Sigma'/\langle m\rangle$ be a (homeomorphic) lift of Y, and let δ_i^+, δ_i^- be lifts of δ_i on the boundary of Y', oriented compatibly with the orientation of δ_i. If any $i(\delta_i^\pm, \alpha) = 0$, we are done. Otherwise, assume that $i(\delta_1^+, \alpha) > 0$ and minimal among all $\delta_i^\pm$.

Now, note that (up to possibly swapping orientation of δ_2) there is a curve μ' in Y' which is homologous to $[\delta_2^+] + [\delta_1^+]$. This curve μ' is a degree 1 lift of a meridian μ for V, and we have

$$i(\mu, \alpha) = i(\delta_2^+, \alpha) + i(\delta_1^+, \alpha).$$

Perform an disk system exchange move, replacing δ_2 by μ. Similarly, we modify Y' by removing the pair of pants bounded by $\delta_1^+, \delta_2^+, \mu'$ and adding a pair of pants at δ_2^-, whose boundary components are other lifts $h_1\delta_1^+, h_2\mu'$ of δ_1, μ. Applying Lemma 6.10 twice, both for Y' and its modification, we have

$$i(\delta_1^+, \alpha) = -i(\delta_1^-, \alpha) = i(h_1\delta_1^+, \alpha)$$

Thus, repeating the argument, with μ in place of δ_2, we can find a meridian ν, with lift ν' so that

$$i(\nu, \alpha) = i(\delta_2^+, \alpha) + 2(\delta_1^+, \alpha).$$

By induction, and since δ_1 was chosen to minimise intersection with α, after finitely many steps we will have found a meridian with $i(\nu, \alpha) = 0$, which is the desired one. □

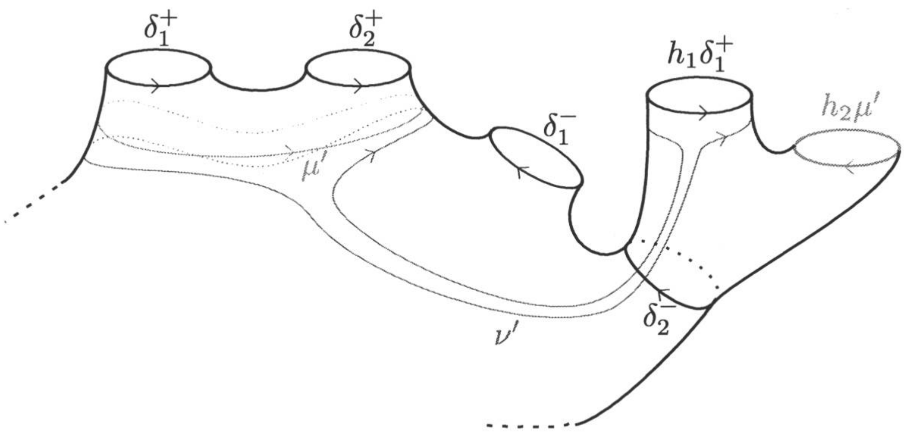

Figure 5. The construction in the proof of Lemma 6.11.

With Corollary 6.8, this is enough to finish the proof of Theorem 6.3.

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