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## A transfer principle and applications to eigenvalue estimates for graphs

Omid Amini and David Cohen-Steiner

**Abstract.** In this paper, we prove a variant of the Burger–Brooks transfer principle which, combined with recent eigenvalue bounds for surfaces, allows to obtain upper bounds on the eigenvalues of graphs as a function of their genus. More precisely, we show the existence of a universal constants  $C$  such that the  $k$ th eigenvalue  $\lambda_k^{\text{nr}}$  of the normalized Laplacian of a graph  $G$  of (geometric) genus  $g$  on  $n$  vertices satisfies

$$\lambda_k^{\text{nr}}(G) \leq C \frac{d_{\max}(g + k)}{n},$$

where  $d_{\max}$  denotes the maximum valence of vertices of the graph. Our result is tight up to a change in the value of the constant  $C$ , and improves recent results of Kelner, Lee, Price and Teng on bounded genus graphs.

To show that the transfer theorem might be of independent interest, we relate eigenvalues of the Laplacian on a metric graph to the eigenvalues of its simple graph models, and discuss an application to the mesh partitioning problem, extending results of Miller–Teng–Thurston–Vavasis and Spielman–Teng to arbitrary meshes.

**Mathematics Subject Classification (2010).** 05C10, 05C50, 35P15, 58J50, 65F99.

**Keywords.** Eigenvalues, graphs on surfaces, spectral geometry, metric graphs, mesh partitioning.

### 1. Introduction

The spectrum of the Laplacian of a finite graph reflects information about the structural properties of the graph and has been successfully used in a large variety of applications to other domains. In particular, the eigenvalues of a bounded degree graph provide information on the existence of good clusterings of that graph, see [1] for clustering in two classes and [20, 22] for  $k$ -way clusterings, whose optimal quality is shown to relate to the  $k$ th eigenvalue.

In particular, upper bounds on the eigenvalues of a class of graphs directly translate into efficient clustering algorithms with quality guarantees. This motivated a series of work, starting with Spielman and Teng [27], who gave an  $O(1/n)$  bound for

the Fiedler value of a bounded degree planar graph on  $n$  vertices, using a suitably centered circle packing representation of the graph. Kelner extended this result to an  $O((g + 1)/n)$  bound for (geometric) genus  $g$  graphs [16]. The argument uses Riemann–Roch theorem to find a circle packing representation of the graph. Recently, Kelner, Lee, Price and Teng proved an  $O((g + 1) \log(g + 1)^2 k/n)$  upper bound for the  $k$ th eigenvalue [17], using a multicommodity flow problem to suitably uniformize the graph metric.

The study of the spectrum of a finite graph is in many ways related to the spectral theory of Riemannian manifolds, and results in geometric analysis have been a source of inspiration to state and prove corresponding results concerning finite graphs. In particular, eigenvalue bounds for surfaces have a somewhat parallel history. Hersch [13] first proved an  $O(1/\text{vol}(M))$  bound for the Neumann value of the sphere  $\mathbb{S}^2$  equipped with a Riemannian metric. Yang and Yau [28] then showed that for genus  $g$  surfaces an  $O((g + 1)/\text{vol}(M))$  bound holds, and Li and Yau improved the latter result by replacing the genus with the finer conformal invariant they defined [21]. It is interesting to notice that these proofs are quite similar at a high level to the ones later used in the graph setting. Conformal uniformization was used in place of circle packing representations, but the very same topological argument for centering the packing in the discrete case was used in the manifold case as well. For higher eigenvalues, Korevaar [19] established an  $O((g + 1)k/\text{vol}(M))$  for genus  $g$  surfaces, and Hassannezhad [15] improved this to  $O((g + k)/\text{vol}(M))$ , by combining the two methods of constructing disjoint capacitors of Grigor’yan, Netrusov and Yau [14], and Colbois and Maerten [8].

While traditionally bounds on graph eigenvalues are used to prove bounds for Riemannian manifolds [3–6, 9, 10], it is intriguing to see that the spectral theory of Riemannian manifolds has not been much used so far to provide information on the spectral properties of general finite graphs.

Our aim in this paper is to show how eigenvalue bounds for surfaces combined with basic spectral theory of (singular) surfaces, and a suitable transfer principle, allows to obtain eigenvalue estimates for graphs in terms of their geometric genus. In this way, we are able to extend the above mentioned result of [15, 19, 28] for surfaces to the graph setting using a suitable variant of the Burger–Brooks transfer method, c.f. Theorem 1.2. Our results are tight and improve the recent results of Kelner, Lee, Price and Teng [17] on bounded genus graphs. In addition to providing a uniform arguably more conceptual proof of the results of [16, 17, 27], we hope that our method makes the above mentioned existing similarities between the methods used in the spectral theory of surfaces and graphs more transparent.

The transfer principle proved in this paper may be of independent interest. In fact, we shall show it can be used to provide uniform upper and lower bounds on the eigenvalues of metric graphs in terms of the eigenvalues of their simple graph models. Furthermore, it allows to generalize to completely arbitrary meshes the mesh partitioning results of Miller, Teng, Thurston, and Vavasis [24] and Spielman and Teng [27].

**1.1. Statement of the main theorem on eigenvalues of bounded genus graphs.**

Let  $G = (V, E)$  be a finite simple graph, that we assume connected all through the paper. For two vertices  $u, v \in V$ , we write  $u \sim v$  if the two vertices  $u$  and  $v$  are connected by an edge in  $G$ . The valence of a vertex  $v$  of  $G$  is denoted by  $d_v^G$ , or simply  $d_v$  if there is no risk of confusion and the graph  $G$  is understood from the context. We denote by  $d_{\max}$  the maximum degree of vertices of the graph, and by  $n$  the number of vertices. The geometric genus of  $G$  is by definition the minimum integer  $g$  such that  $G$  can be embedded with no crossing on the compact orientable surface of genus  $g$ .

Denote by  $C(G)$  the vector space of all real valued functions  $f$  defined on the set of vertices of  $G$ . The (discrete) Laplacian  $\Delta$  and the normalized Laplacian  $\mathcal{L}$  of  $G$  are defined as follows: the Laplacian  $\Delta: C(G) \rightarrow C(G)$  is the linear operator which sends a function  $f \in C(G)$  to  $\Delta(f) \in C(G)$  defined by

$$\forall v \in V(G), \quad \Delta(f)(v) = \sum_{u: u \sim v} f(v) - f(u).$$

Let  $S$  be the linear operator on  $C(G)$  whose matrix in the standard basis of  $C(G)$  is diagonal with entries the valences of the vertices of  $G$ , i.e. for any  $f \in C(G)$

$$\forall v \in V(G), \quad S(f)(v) = d_v f(v).$$

The normalized Laplacian is the operator  $S^{-1/2} \Delta S^{-1/2}$ .

We denote by

$$\lambda_0(G) = 0 < \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_{n-1}(G)$$

the set of eigenvalues of  $\Delta$ , which we call the standard spectrum of  $G$ , and by

$$\lambda_0^{\text{nr}}(G) = 0 < \lambda_1^{\text{nr}}(G) \leq \cdots \leq \lambda_{n-1}^{\text{nr}}(G)$$

the set of all eigenvalues of the normalized Laplacian  $\mathcal{L}$ , which we call the normalized spectrum. The standard and normalized spectrum of  $G$  are easily seen to satisfy the inequalities  $d_{\min} \lambda_k^{\text{nr}}(G) \leq \lambda_k(G) \leq d_{\max} \lambda_k^{\text{nr}}(G)$  for any  $k$ .

In this paper we prove the following theorem.

**Theorem 1.1.** *There exists a universal constant  $C$  such that the eigenvalues of the normalized Laplacian of any graph  $G$  on  $n$  vertices satisfy:*

$$\forall k \in \mathbb{N}, \quad \lambda_k^{\text{nr}}(G) \leq C \frac{d_{\max}(g + k)}{n},$$

where  $d_{\max}$  and  $g$  are the maximum valence and the geometric genus of  $G$ , respectively.

The linear dependance in the maximum degree is clearly optimal, as can be seen by considering star graphs, which have lower bounded Fiedler value. The above result also implies a similar bound for the eigenvalues of the standard Laplacian, at the expense of an extra  $d_{\max}$  factor. We note that Kelner, Lee, Price and Teng [17] give a similar bound for the standard spectrum with a linear rather than quadratic dependence in  $d_{\max}$ . However, their bound has a  $gk \log(g+1)^2$  dependence instead of our  $(g+k)$  dependence. In addition to simplifying and improving the result of [17] for bounded genus graphs, we note that the dependence in  $g$  and  $k$  of our estimate is tight, at least when  $g$  is sufficiently high, see Remark 2.6.

Informally, the improvement over [17] means that the asymptotic behavior of graphs' eigenvalues do not depend on the (geometric) genus of the graph. This fact, which may be seen as a one-sided discrete form of Weyl's law for surfaces, is consistent with the intuition that at a small scale, bounded genus graphs behave like planar graphs. Finally, we note that the result in [17] also applies to graphs in any fixed proper minor-closed family (where the genus  $g$  is replaced with a parameter  $h$  depending on the family), while the stronger bounds of Theorem 1.1 cannot be extended to minor-closed classes, as we show by explicit examples in Remark 2.7.

**1.2. Two-fold covers and their associated discrete Laplacians.** Let  $M$  be a measured topological space, and denote by  $\mu$  the measure on  $M$ . A *2-fold cover* of  $M$  is a finite collection  $\mathcal{U} = (U_v)_{v \in V}$ , for a finite index set  $V$ , of open subsets  $U_v$  of non-zero measure such that almost every point in  $M$  is covered by exactly two subsets. To any 2-fold cover of a measured space we associate a discrete Laplacian as follows:

We first form a graph  $G = (V, E)$  on the set of vertices  $V$  and with edges  $\{u, v\} \in E$  for two vertices  $u, v$  such that  $\mu(U_v \cap U_u) \neq 0$ . We define a weight function  $\omega: E \rightarrow \mathbb{R}$  which to any edge  $e = \{u, v\}$  of  $G$ , associates the weight  $\omega(e) = \mu(U_u \cap U_v)$ . The weighted valence  $d_v^\omega$  of a vertex  $v$  of  $G$  is defined by

$$d_v^\omega = \sum_{u: u \sim v} \mu(U_u \cap U_v).$$

The discrete Laplacian associated to the 2-fold cover  $\mathcal{U}$  denoted by  $\mathcal{L}_{\mathcal{U}}$  is the normalized graph Laplacian associated to the weighted graph  $(G, \omega)$ . This is defined from the weighted Laplacian by normalizing using the weighted valence (as in the previous section). Formally, define the weighted Laplacian  $\Delta_{\mathcal{U}}: C(G) \rightarrow C(G)$  by

$$\forall v \in V, \quad \Delta_{\mathcal{U}}(f)(v) = \sum_{u: u \sim v} (f(v) - f(u)) \omega(\{u, v\}),$$

for any  $f \in C(G)$ . Let  $S_{\mathcal{U}}$  be the diagonal operator with entries the weighted valence  $d_v^\omega$  of vertices  $v \in V$ , i.e. for any  $f \in C(G)$ ,

$$\forall v \in V, \quad S_{\mathcal{U}}(f)(v) = d_v^\omega f(v).$$

Then we let  $\mathcal{L}_{\mathcal{U}} := S_{\mathcal{U}}^{-\frac{1}{2}} \Delta_{\mathcal{U}} S_{\mathcal{U}}^{-\frac{1}{2}}$ . Denote by  $\lambda_k(\mathcal{L}_{\mathcal{U}})$  the  $k$ th smallest eigenvalue of  $\mathcal{L}_{\mathcal{U}}$ .

When  $(M, \mu)$  carries a natural notion of Laplacian, it is possible to relate the eigenvalues of the Laplacian on  $M$  to the eigenvalues of  $\mathcal{L}_{\mathcal{U}}$  for any 2-fold cover  $\mathcal{U}$ . More precisely, let the measured space  $(M, \mu)$  belong to any of the following three classes:

- (C1) a smooth manifold with a smooth Riemannian metric  $g$ , and  $\mu$  the measure associated to the metric  $g$ ;
- (C2) a compact smooth surface with a conformal class of smooth Riemannian metrics  $g$ , and  $\mu$  a Radon measure absolutely continuous with respect to  $\mu_g$ , c.f. Section 2.1;
- (C3) a metric graph with  $\mu$  the Lebesgue measure.

In any of the above cases, we can define a Laplacian on  $(M, \mu)$  c.f. Section 2.1 and Section 3, and we denote by  $\lambda_k(M, \mu)$ , or simply  $\lambda_k(M)$  if there is no risk of confusion, the eigenvalues for the corresponding Laplacian.

Our transfer principle is stated as follows.

**Theorem 1.2.** *Let  $(M, \mu)$  be a measured space as in (C1), (C2), or (C3) above. Assume all the elements in a 2-fold cover  $\mathcal{U}$  of  $M$  have Neumann value at least  $\eta$ . Then for all positive integers  $k$  we have:*

$$\lambda_k(\mathcal{L}_{\mathcal{U}}) \leq 2 \frac{\lambda_k(M)}{\eta}.$$

The main difference with the classical versions of the transfer principle [3, 5, 23] is that we discretize the continuous Laplacian as a weighted normalized graph Laplacian instead of a combinatorial one, which allows for a closer connection between the two. Our variant here uses a different notion of graph approximation that involves particular weights. In addition, the above mentioned results take as input a partition of  $M$ , while our theorem is expressed in terms of two-fold covers, which adds more flexibility.

In order to prove Theorem 1.1, we apply the above theorem in the case where  $(M, \mu)$  is a measured surface equipped with a conformal class of smooth Riemannian metrics  $g$ . This version seems to be required to get our Theorem 1.1 on eigenvalues of bounded (geometric) genus graphs.

**1.3. Organization of the paper.** The necessarily background on Laplacian eigenvalues in measured surfaces is recalled in Section 2. The proof of Theorem 1.2 for measured surfaces (Case (C2) among the above three cases) is given in Section 2. The proof in the two other cases is similar and is thus omitted. Section 2 contains also the proof of Theorem 1.1. In Section 3, we apply Theorem 1.2 in the case of metric graphs with Lebesgue measure (C3), to obtain a uniform quantitative

complement to a theorem of X. Faber [11] on the spectral convergence of finite graphs to metric graphs. Moreover, we give in Section 4 an algorithmic application of the above theorem to mesh partitioning in numerical analysis, generalizing the results of Miller–Teng–Thurston–Vavasis [24] and Spielman–Teng [27] to anisotropic meshes.

## 2. Eigenvalues of bounded genus graphs

In this section we give the proofs of Theorem 1.2 and Theorem 1.1. We start by recalling the variational approach to study eigenvalue problems for surfaces with measures [18], which provides a setting to study eigenvalue problems for singular surfaces. This makes the statement of Theorem 1.2 precise in the case of a measured metric surface.

**2.1. Eigenvalues on measured surfaces.** Let  $M$  be a smooth compact surface, possibly with boundary, which we suppose equipped with a smooth Riemannian metric  $g$ . Denote by  $\mu_g$  the induced volume form on  $M$ . Let  $\mu$  be a Radon measure on  $M$  which we suppose absolutely continuous with respect to the measure  $\mu_g$ . For a  $C^\infty$ -smooth function  $f \in L^2(M, \mu)$ , the Rayleigh quotient  $R_{M_g}(f, \mu)$  is defined by

$$R_{M_g}(f, \mu) := \frac{\int_M |\nabla_g f|^2 d\mu_g}{\int_M f^2 d\mu}.$$

The eigenvalues of the measured metric surface  $(M_g, \mu)$  are defined by the variational formula:

$$\lambda_k(M_g, \mu) := \inf_{\Lambda_{k+1}} \sup_{f \in \Lambda_{k+1}^*} R_{M_g}(f, \mu), \quad (2.1)$$

where  $\Lambda_{k+1} \subset L^2(M, \mu)$  varies over subspaces of dimension  $k + 1$  which consist only of smooth functions on  $M$ , and  $\Lambda_{k+1}^* = \Lambda_{k+1} \setminus \{0\}$ . Note that in the case  $\mu = \mu_g$ , we recover the usual variational characterization of the eigenvalues of the Laplacian  $\Delta_g$  associated to the Riemannian surface  $M_g$ .

To see the point of introducing this formalism, assume that the two metrics  $g$  and  $h$  on  $M$  are conformally equivalent. From the conformal invariance of the Dirichlet integral, we see that  $R_{M_g}(f, \mu) = R_{M_h}(f, \mu)$ . In particular, letting  $\mu = \mu_h$ , we see that the spectra of the metric  $h$  within the conformal class of  $M_g$  coincides with the spectra of measured surface  $(M_g, \mu)$  for an appropriate Radon measure  $\mu$ . Now, if  $h$  is a metric with conical singularities, it is a classical fact that  $M$  is conformally equivalent to a constant curvature metric  $g$ , the conformal factor being square integrable with respect to the corresponding area form  $\mu_g$ . Thus the framework of measured metric surfaces allows in particular to define spectra



of surfaces with conical singularities as the spectra of the measured metric surface  $(M_g, \mu_h)$  [18].

Let  $U$  be an open subset of  $M$ , and denote by  $\bar{U}$  the topological closure of  $U$  in  $M$ . The Neumann value  $\lambda(U)$  of  $U$  is defined as the infimum of the Rayleigh ratio  $\int_U |\nabla_g f|^2 d\mu_g / \int_U f^2 d\mu$  over all smooth functions  $f$  on  $U$  which extend continuously to  $\bar{U}$  and satisfy  $\int_U f d\mu = 0$ ,

$$\lambda(U) := \inf_{f: \int_U f d\mu = 0} \frac{\int_U \|\nabla_g f\|^2 d\mu_g}{\int_U f^2 d\mu}.$$

**2.2. Proof of Theorem 1.2.** We suppose  $M, \mu, g$ , and  $\mu_g$  as above, and consider a 2-fold cover  $\mathcal{U} = (U_v)_{v \in V}$  of  $M$ . Denote by  $\eta$  the minimum of  $\lambda(U_v)$  for  $v \in V$ . Let  $G$  be the associated weighted graph with vertex set  $V$  and weight matrix  $W_{\mathcal{U}} = [\omega(\{u, v\})]_{u, v}$ , where  $\omega(\{u, v\}) = \mu(U_u \cap U_v)$  for  $u \neq v$ . Let  $\mathcal{L}_{\mathcal{U}}$  be the matrix of the associated normalized graph Laplacian. We have  $\mathcal{L}_{\mathcal{U}} = I - S_{\mathcal{U}}^{-1/2} W_{\mathcal{U}} S_{\mathcal{U}}^{-1/2}$ , where the matrix  $S_{\mathcal{U}}$  is diagonal with entries given by the weighted valences of the vertices  $d_v^\omega = \sum_{u: u \sim v} \omega(\{u, v\})$ .

*Proof of Theorem 1.2.* Let  $v \in V$  and  $f$  any smooth function on  $M$ . By restricting  $f$  to  $U_v$  and subtracting the mean over  $U_v$ , we get:

$$\begin{aligned} \int_{U_v} \|\nabla_g f\|^2 d\mu_g &\geq \lambda(U_v) \int_{U_v} \left(f - \frac{1}{\mu(U_v)} \int_{U_v} f d\mu\right)^2 d\mu \\ &\geq \eta \left( \int_{U_v} f^2 d\mu - \frac{1}{\mu(U_v)} \left( \int_{U_v} f d\mu \right)^2 \right). \end{aligned}$$

Summing the last inequalities over  $v \in V$  yields:

$$\frac{2}{\eta} \int_M \|\nabla_g f\|^2 d\mu_g \geq 2\|f\|_2^2 - \sum_v \frac{1}{\mu(U_v)} \left( \int_{U_v} f d\mu \right)^2, \quad (2.2)$$

where the  $L^2$  norm  $\|\cdot\|_2$  is with respect to the measure  $\mu$ . Denote by  $\mathbf{1}_{U_v}$  the characteristic function of the open set  $U_v$ , and let  $\phi_v = \mu(U_v)^{-1/2} \mathbf{1}_{U_v}$ . Define  $\Phi: L^2(M) \rightarrow C(G)$ , by

$$\Phi(f)(v) := \int_M f \phi_v,$$

on any vertex  $v$  of  $G$ . We see that the quadratic form in  $f$  in the right hand side of Equation (2.2) is given by  $2\|f\|_2^2 - \|\Phi f\|_2^2$ .

Let  $\epsilon > 0$ , and denote by  $\Lambda_{k+1}^\epsilon$  a  $(k+1)$ -dimensional space of smooth functions on  $M$  such that for any  $f \in \Lambda_{k+1}^\epsilon \setminus \{0\}$ , we have

$$\frac{\int_M \|\nabla_g f\|^2 d\mu_g}{\int_M f^2 d\mu} \leq (1 + \epsilon) \lambda_k(M).$$



Note that by the variational characterization of the eigenvalues (see e.g. (2.1) and (3.1)), such a space exists. For any  $f \in \Lambda_{k+1}^\epsilon$ , by inequality (2.2), we have:

$$\frac{2(1+\epsilon)\lambda_k(M)}{\eta} \|f\|_2^2 \geq \frac{2}{\eta} \int_M \|\nabla_g f\|^2 d\mu_g \geq 2\|f\|_2^2 - \|\Phi f\|_2^2.$$

That is:

$$\|\Phi f\|_2^2 \geq 2\left(1 - \frac{(1+\epsilon)\lambda_k(M)}{\eta}\right) \|f\|_2^2.$$

Let  $\Phi^*$  denote the adjoint of the operator  $\Phi: L^2(M) \rightarrow C(G)$ . From the variational characterization of the eigenvalues, this implies that the compact self-adjoint operator  $\Phi^*\Phi$  on  $L^2(M)$  has at least  $k+1$  eigenvalues greater than or equal to  $2(1 - (1+\epsilon)\lambda_k(M)/\eta)$ . We can assume that this latter quantity is positive, otherwise there is nothing to prove since all the eigenvalues of the normalized Laplacian are at most 2. Since the non zero eigenvalues of  $\Phi^*\Phi$  are the same as the non zero eigenvalues of  $\Phi\Phi^*$ , we thus deduce that

$$\lambda_k(2I - \Phi\Phi^*) \leq \frac{2(1+\epsilon)\lambda_k(M)}{\eta}. \quad (2.3)$$

To conclude the proof, it suffices to notice that

$$[\Phi\Phi^*]_{u,v} = \int_M \phi_u \phi_v = \frac{\mu(U_u \cap U_v)}{(\mu(U_u)\mu(U_v))^{1/2}}.$$

Because  $\mathcal{U}$  is a 2-fold cover, the  $v$ th entry of the diagonal matrix  $D_{\mathcal{U}}$  is equal to  $\mu(U_v)$ . We thus easily check that  $2I - \Phi\Phi^* = \mathcal{L}_{\mathcal{U}}$ . Therefore, inequality (2.3) gives

$$\lambda_k(\mathcal{L}_{\mathcal{U}}) \leq \frac{2(1+\epsilon)\lambda_k(M)}{\eta}.$$

Since this holds for any  $\epsilon > 0$ , the theorem follows.  $\square$

**2.3. Proof of Theorem 1.1.** We first give some background about graphs embedded in a surface, and refer to [25] for more details. We assume that all surfaces are compact, orientable and without boundary. An *embedding* of a graph  $G$  in a surface  $M$  is a drawing of  $G$  on  $M$  so that all vertices of  $G$  are distinct on  $M$ , and every edge of  $G$  form a simple arc on  $M$  connecting its two endpoint vertices. Interior of edges and vertices are assumed to be pairwise disjoint. A *face* of an embedding, or simply a face of  $G$  if the embedding is clear from the context, is a connected component of the complementary of  $G$  in  $M$ .

An embedding is called *cellular* if every face is homeomorphic to an open disk in  $\mathbb{R}^2$ .

The genus  $g(G)$  is the minimum integer  $g$  such that  $G$  has an embedding in a surface  $M$  of genus  $g$ . The following result will allow us to suppose that a graph  $G$  with a given genus  $g(G)$  is embedded in a cellular way.

**Proposition 2.1** ([25, Proposition 3.4.1]). *Every embedding of a connected graph  $G$  in a surface of genus  $g(G)$  is cellular.*

Suppose from now on that the connected graph  $G$  is embedded in a cellular way in a surface  $M$  of genus  $g$ , so that every face  $F$  is homeomorphic to an open disk  $D_F$  in  $\mathbb{R}^2$ . The boundary of the face  $F$  in  $M$  is the image of the boundary  $\partial D_F \simeq S^1$  under a continuous map, which is locally a homeomorphism away from the preimage of the vertices. We denote by  $\mathcal{F}$  the set of all faces of  $G$ . For any face  $F \in \mathcal{F}$ , we define a *boundary walk* of  $F$  to be any walk in the graph  $G$  consisting of vertices and edges as they are encountered when walking along the whole boundary of  $F$ , following the circle  $\partial D_F$ , and starting at some vertex. Note that some edges may appear more than once in a boundary walk. The *degree* of a face  $F \in \mathcal{F}$ , denoted  $\deg(F)$ , is the number of edges on any boundary walk of  $F$ .

We define a new multigraph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  embedded in  $M$ , and containing  $G$  as an induced subgraph, by coning over boundary walks of faces as follows. The vertex set  $\mathcal{V}$  of  $\mathcal{G}$  consists of the vertices in  $G$  and a new vertex  $v_F$  for each face  $F$  of  $\mathcal{F}$ , i.e.  $\mathcal{V} = V \sqcup \{v_F\}_{F \in \mathcal{F}}$ . For each face  $F$  of  $\mathcal{F}$ , let  $v_1, \dots, v_{\deg(F)}$  be the vertices of  $G$  which appear in this order in a boundary walk of  $F$ . Note that a vertex might appear more than once. The edge set  $\mathcal{E}$  of  $\mathcal{G}$  consists of the edges in  $E$ , and new edges  $\{v_F, v_i\}$ , for  $i = 1, \dots, \deg(F)$ . The embedding of  $\mathcal{G}$  in  $M$  is obtained in the following natural way: each face  $F \in \mathcal{F}$  is homeomorphic to a disk  $D_F$  in  $\mathbb{R}^2$ , and under this homeomorphism, the vertices  $v_1, \dots, v_{\deg(F)}$  in the boundary walk of  $F$ , appear in this cyclic order on the boundary of  $D_F$ . Choose the center of  $D_F$  as the image of  $v_F$  and the rays from  $v_F$  to  $v_i$  as the image of the edges  $\{v_F, v_i\}$ . We refer to all the new edges  $\{v_F, v_i\}$  added in the process as *cone edges* of  $\mathcal{G}$ .

Call an embedding of a graph in  $M$  a *weak triangulation* if the degree of any face of the embedding is three. We use this terminology since it can happen that two different faces of the embedding share more than one edge, in which case we do not have a triangulation.

The embedded (multi)graph  $\mathcal{G}$  constructed above has the following properties.

**Proposition 2.2.** *The embedding of  $\mathcal{G}$  in  $M$  is a weak triangulation, and each face of  $\mathcal{G}$  is incident to exactly one edge of  $G$ . Moreover, for any vertex  $v$  of  $G$ , we have  $d_v^{\mathcal{G}} = 2d_v^G$ , where  $d_v^{\mathcal{G}}$  and  $d_v^G$  denote the valence of  $v$  in  $\mathcal{G}$  and  $G$ , respectively.*

*Proof.* By definition of the embedding, each face of  $\mathcal{G}$  consists of two cone edges and an edge of  $G$ , which proves the first assertion. To prove the second statement, let  $F_1, \dots, F_k \in \mathcal{F}$  be all the faces of  $G$  which are incident to the vertex  $v \in V$ . For each  $i = 1, \dots, k$ , the number of edges  $\{v_{F_i}, v\}$  in  $\mathcal{E}$  is half the number of edges of  $G$  in a boundary walk of  $F_i$ . Each edge  $e \in E$  incident to  $v$  appears precisely twice in the union of the edges of the boundary walks of  $F_1, \dots, F_k$ . This shows that the total number of edges of  $\mathcal{E}$  of the form  $\{v_{F_i}, v\}$ , for  $i = 1, \dots, k$ , is equal to  $d_v^G$ , which proves the claim.  $\square$

**Definition 2.3** (Open star). For each vertex  $v$  of  $G$ , we define the *open star* of  $v$  in  $M$  with respect to the embedding of  $\mathcal{G}$ , or simply the open star of  $v$ , denoted by  $\mathcal{S}_v$ , as the interior of the union of all the faces of  $\mathcal{G}$  which contain  $v$  in their boundaries.

Let now  $(M, g, \mu)$  be any surface as in Section 2.1, so the measure  $\mu$  is absolutely continuous with respect to the volume form  $\mu_g$  of the smooth Riemannian metric  $g$ .

**Proposition 2.4.** *The open stars of vertices of  $G$  form a 2-fold cover of  $M$ .*

*Proof.* By Proposition 2.2, the boundary walk of each face of  $\mathcal{G}$  is a triangle which has exactly two vertices in  $G$ . It follows that every point of  $M \setminus \mathcal{G}$  appears in exactly two open stars, which proves the claim by absolute continuity of  $\mu$  with respect to  $\mu_g$ .  $\square$

We now introduce a metric  $h$  on  $M$  with conical singularities (and will later assume  $\mu = \mu_h$ ). For reasons that will soon become clearer, we set the length of each edge of  $E \subset \mathcal{E}$  to be equal to one, and the length of each cone edge in  $\mathcal{E}$  to be  $\cos(\pi/(2d_{\max}))^{-1}/2$ . We equip  $M$  with the natural metric  $h$  such the triangles have the Euclidean metric induced by their edge lengths. Note that for any triangle  $T$  of  $\mathcal{G}$ , the angle of  $T$  at any of its vertices that belongs to the graph  $G$  is equal to  $\pi/(2d_{\max})$ . The metric  $h$  has only conical singularities, and we denote by  $\lambda_k(M) = \lambda_k(M_h)$ , the eigenvalues of the surface  $M$  as defined in Section 2.1. Thus,  $\lambda_k(M)$  is the  $k$ th eigenvalue of the measured metric surface  $(M_g, \mu = \mu_h)$ , where  $g$  is a metric of constant curvature in the conformal class of  $h$ .

Using Theorem 1.2 we can relate the eigenvalues of  $M$  to those of  $G$ . Denote by  $\mathcal{U}$  the 2-fold cover of  $M$  given by the open stars  $\mathcal{S}_v$  of vertices of  $V$ , c.f. Proposition 2.4. The intersection of two distinct open stars  $\mathcal{S}_u$  and  $\mathcal{S}_v$ , for two vertices  $u$  and  $v$  of  $G$ , has non zero measure if and only if  $u$  and  $v$  are neighbors in  $G$ . Moreover, all non-empty intersections have the same measure, equal to the area of two triangles in  $M$ . Therefore, the normalized Laplacian  $\mathcal{L}_{\mathcal{U}}$  equals the normalized Laplacian of  $G$ . Hence, in order to apply the transfer result Theorem 1.2, we only need to lower bound the Neumann value of the open stars  $\mathcal{S}_v$  of the vertices of  $G$ .

We do so by again applying the transfer result to a specific 2-fold cover of each open star  $\mathcal{S}_v$ , for  $v \in V$ . Thanks to the choice of edge lengths, the vertices of  $G$  in  $(M, h)$  have non negative curvature. It follows that by cutting  $\mathcal{S}_v$  along an arbitrary cone edge, we can unfold  $\mathcal{S}_v$  to the plane without overlap. Denote by  $\mathcal{S}'_v$  the unfolded star of  $v$  as isometrically embedded in  $\mathbb{R}^2$ . The cutting operation can only decrease the Neumann value so it is sufficient to bound from below the Neumann value of the unfolded open subset  $\mathcal{S}'_v$  of  $\mathbb{R}^2$ .

We call a *kite* in  $\mathcal{S}'_v$  the union of two triangles in  $\mathcal{S}'_v$  which share an edge of  $G$ . So for any edge  $e = \{v, u\} \in E$ , there is a kite  $K_e$ , and the union of the kites  $K_e$  for  $e$  incident to  $v$  is equal to the planar set  $\mathcal{S}'_v$ . For any edge  $e \in E$  incident to  $v$ , the kite  $K_e$  has two diagonals composed of the edge  $e$  and the diagonal opposite to  $e$ , that we denote by  $\text{diag}_e^{\text{op}}$ . Cut  $\mathcal{S}'_v$  along all the opposite diagonals  $\text{diag}_e^{\text{op}}$  for  $e \in E$  incident

to  $v$ . This cuts  $\mathcal{S}'_v$  into an open region  $P_v$  with polygonal boundary containing the vertex  $v$ , together with one triangle  $T_e$  for each kite  $K_e$  for  $e \in E$  incident to  $v$ . Define

$$\mathcal{U}_v := \{K_e\}_{e=\{v,u\} \in E} \cup \{T_e\}_{e=\{v,u\} \in E} \cup \{P_v\}.$$

The cover  $\mathcal{U}_v$  of  $\mathcal{S}'_v$  is a 2-fold cover.

**Proposition 2.5.** *Any  $X \in \mathcal{U}_v$  is a convex set of diameter at most two.*

*Proof.* The triangle  $T_e$ , for  $e = \{u, v\} \in E$ , is obviously convex of diameter one, and so is the kite  $K_e$ . As for the region  $P_v$ , to prove the convexity of  $P$ , it will be enough to show that the angle of  $P$  at  $v$  is at most  $\pi$ . As we previously observed, by the choice of the edge lengths, all the triangles of  $\mathcal{S}$  has angle  $\pi/2d_{\max}$  at any of their vertices which belong to  $G$ . The number of triangles of  $\mathcal{S}'_v$  is at most  $2d_v^G$ , since  $d_v^G \leq d_{\max}$ , it follows that the angle at  $v$  of  $P$  is at most  $\pi$ , and the convexity follows. The claim on the diameter follows from the fact that all the edges of  $G$  have length equal to one.  $\square$

Since any element of  $\mathcal{U}_v$  is a planar convex sets of diameter at most two, there exists a universal constant  $C_1 > 0$  such that for any  $X \in \mathcal{U}_v$ , we have  $\lambda(X) \geq C_1$  for  $\lambda(X)$  the Neumann value of  $X$  [26]. For the 2-fold cover of  $\mathcal{S}'_v$ , the non-zero element  $\mu_h(X \cap Y)$  for  $X \neq Y \in \mathcal{U}_v$  have the same value, equal to the area of a triangle in  $M$ . Therefore, the normalized Laplacian  $\mathcal{L}_{\mathcal{U}_v}$  equals the normalized Laplacian of the graph whose edges are the pairs  $X \neq Y \in \mathcal{U}_v$  whose intersection has positive measure. This graph  $S_d^1$  is obtained from the star graph with  $d$  edges by inserting a new vertex in the middle of each edge. (Recall that a star graph with  $d$  edges has a central vertex connected to  $d$  other vertices.) The Neumann value of  $S_d^1$  is lower bounded by an absolute constant  $C_2 > 0$  independent of  $d$ . Hence, applying the transfer theorem 1.2 to the 2-fold cover  $\mathcal{U}_v$ , it follows that there exists a universal constant  $C_3 = C_1.C_2/2 > 0$  such that the Neumann value of  $\mathcal{S}'_v$  is bounded from below by  $C_3$ , i.e.  $\lambda(\mathcal{S}'_v) \geq C_3$ . This gives  $\lambda(\mathcal{S}_v) \geq C_3$ .

We get from these observations, and Theorem 1.2 applied to the 2-fold cover of  $\mathcal{U}$ , that

$$\lambda_k^{\text{nr}} = \lambda_k(\mathcal{L}_{\mathcal{U}}) \leq 2 \frac{\lambda_k(M)}{C_3}. \quad (2.4)$$

A result of Hassannezhad [15] states now that there is a universal constant  $A$  such that for each  $k$ :

$$\lambda_k(M)\mu(M) \leq A(g + k). \quad (2.5)$$

Note that this result is not explicitly stated in the framework of measured metric surfaces in [15], however the proof given in [15] works also in this setting and gives the above statement. Putting Equations (2.4) and (2.5) together, and observing

that  $\mu(M) \geq C_4 n / d_{\max}$  for some constant  $C_4$ , we conclude that for  $C = 2A / (C_3 C_4)$ , we have

$$\lambda^{\text{nr}}(G) \leq C \frac{d_{\max}(g + k)}{n},$$

which is the statement of Theorem 1.1.  $\square$

**Remark 2.6.** It is shown in [7] that for large  $g$ , there are area one and genus  $g$  Riemannian surfaces  $S$  with

$$\lambda_k(S) \geq \frac{4\pi}{5}(g - 1) + 8\pi(k - 1) - \epsilon$$

for any  $\epsilon > 0$ . Now, the classical Brooks-Burger method implies the existence of a bounded degree genus  $g$  graph  $G$  with  $n$  vertices such that  $\lambda_k(G) \geq C \lambda_k(S) / n$ . Hence, at least for large enough  $n$  and  $g$ , there are graphs whose eigenvalues match the behaviour of the estimate in Theorem 1.1.

**Remark 2.7.** The following example shows that the strong estimates as in Theorem 1.1 cannot hold for more general classes of graphs closed under taking minor.

Recall that the Cartesian product  $G_1 \square G_2$  of two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  has vertex set  $V_1 \times V_2$  and there is an edge between  $(v_1, v_2)$  and  $(u_1, u_2)$  in  $V_1 \times V_2$  if either  $u_1 = v_1$  and  $\{u_2, v_2\} \in E_2$ , or  $u_2 = v_2$  and  $\{u_1, v_1\} \in E_1$ . The Laplacian eigenvalues of  $G_1 \square G_2$  are of the form  $\lambda_i(G_1) + \lambda_j(G_2)$  for  $i = 1, \dots, |V_1|$  and  $j = 1, \dots, |V_2|$ .

Let  $d$  be a fixed large enough integer, and for any  $\ell \in \mathbb{N}$ , consider the Cartesian product  $C_{2\ell} \square G$  of a cycle  $C_{2\ell}$  of length  $2\ell$  with a  $d$ -regular graph  $G$  on  $t$  vertices, for an integer  $t \in \mathbb{N}$ .

For any fixed  $t \in \mathbb{N}$ , we get in this way a family of graphs by varying  $\ell$  and  $G$ . All these graph are of treewidth bounded by some  $f(t)$  for a (linear) function  $f$  of  $t$ . Bounded treewidth graphs form a minor-closed family, so all these graphs belong to a fixed proper minor-closed family  $\mathcal{F}_t$ . For  $G$  a random  $d$ -regular graph on  $t$  vertices, and for the  $l$ th eigenvalue of  $C_{2\ell} \square G \in \mathcal{F}_t$ , for  $l \in \mathbb{N}$ , we have  $\lambda_l(C_{2\ell} \square G) = \Omega(\frac{t\ell}{|C_{2\ell} \square G|})$  with high probability as  $t$  tends to infinity. This shows that there do not exist in general constants  $h = h(\mathcal{F}_t)$  and  $C = C(\mathcal{F}_t)$  associated to  $\mathcal{F}_t$  ensuring that the inequality  $\lambda_k(G) \leq C d_{\max}^2 (g_t + k) / n$  hold for any graph  $G \in \mathcal{F}_t$  on  $n$  vertices, and for any  $k \in \mathbb{N}$  (unlike what happens for the class of bounded genus graphs). In particular, the strong estimates as in Theorem 1.1 cannot hold for general minor-closed classes of graphs.

### 3. Eigenvalues of the Laplacian on metric graphs

We briefly review the basic definitions concerning the spectral theory of metric graphs, and refer e.g. to [2, 29] for more details.

Let  $G = (V, E)$  be a finite connected graph and let  $\ell: E \rightarrow \mathbb{R}_{>0}$  be a (length) function on the edges of  $G$ . The length of  $e$  is denoted by  $\ell_e$ . We define the *metric realization* of  $(G, \ell)$  as follows: for each edge  $e = uv$  of  $G$  take a closed interval  $I_e \subset \mathbb{R}$  of length  $\ell_e$ , and a surjection  $\pi_e: \partial I_e \rightarrow \{u, v\}$  (which identifies the two extremities of  $I_e$  with the vertices of  $G$  in  $e$ ). Define the topological space (with the quotient topology)

$$\Gamma := (V \sqcup \bigsqcup_e I_e) / \{x = \pi_e(x) \quad \forall e \in E \text{ \& } x \in \partial I_e\}.$$

The space  $\Gamma$  has a natural metric, the shortest path metric induced by piecewise isometric paths between points, see e.g. [2]. We call a *metric graph* any metric space  $\Gamma$  isometric to a metric realization of a pair  $(G, \ell)$ , as above. The pair  $(G, \ell)$  is called a *model* of  $\Gamma$ ; when  $G$  is a simple graph, the model is called simple. Note that there are plenty of models for a metric graph  $\Gamma$ , e.g. any finite subset of points of  $\Gamma$  can be part of a simple model of  $\Gamma$ .

For any point  $p \in \Gamma$ , we denote by  $T_p^1 \Gamma$  the set of *unit tangent vectors* to  $\Gamma$  at  $p$ . For an interval  $I = [a, b]$ , in  $\mathbb{R}$ , we define  $T_a^1 I = \{\vec{1}\}$ , with  $\vec{1}$  the unit vector in  $\mathbb{R}$ . For a metric graph  $\Gamma$  and a point  $p \in \Gamma$ , let  $(G, \ell)$  be a simple model of  $\Gamma$  with  $p \in V(G)$ , and let  $e_1, \dots, e_d$  be the edges of  $G$  incident to  $v$ . Define  $T_p^1 \Gamma$  as the set of all unit tangent vectors at  $p$  of the intervals  $I_{e_j}$ , as above. Let  $\vec{u} \in T_p^1 \Gamma$  be a unit tangent vector, and let  $I = I_e$  be the corresponding interval (corresponding to the edge  $e$  of a simple graph model  $(G, \ell)$ ). For  $\epsilon > 0$  sufficiently small, we denote by  $p + \epsilon \vec{u}$  the unique point in  $I$  at distance  $\epsilon$  from  $p$  on  $I$ . A function  $f: \Gamma \rightarrow \mathbb{R}$  is *piecewise smooth* if there exists a simple graph model  $G = (V, E)$  of  $\Gamma$  such that the restriction of  $f$  to the intervals  $I_e$ , for  $e \in E$ , are of class  $C^2$ . The space of piecewise smooth function on  $\Gamma$  is denoted by  $S(\Gamma)$ . Let  $f: \Gamma \rightarrow \mathbb{R}$  be a piecewise smooth function on a metric graph  $\Gamma$ . Let  $p \in \Gamma$  and  $\vec{u} \in T_x^1 \Gamma$  a unit tangent vector to at  $x$ . The (*outgoing*) *slope* of  $f$  along  $\vec{u}$  denoted by  $d_{\vec{u}}(f)$  is defined by

$$d_{\vec{u}}(f) := \lim_{\epsilon \rightarrow 0^+} \frac{f(p + \epsilon \vec{u}) - f(p)}{\epsilon}.$$

For a point  $p \in \Gamma$ , we define  $\sigma_p$  as the sum of the slopes of  $f$  along unit tangents:

$$\sigma_p := \sum_{\vec{u} \in T_p^1 \Gamma} d_{\vec{u}} f(p),$$

Note that for all but at most a finite number of points  $p \in \Gamma$ , we have  $\sigma_p = 0$ . A metric graph  $\Gamma$  has a natural Lebesgue measure denoted by  $dx$ . The Laplacian of  $\Gamma$  is the (measure valued) operator  $\Delta$  on  $\Gamma$  which to a function  $f \in S(\Gamma)$  associates the measure

$$\Delta(f) := -f'' dx - \sum_{p \in \Gamma} \sigma_p \delta_p.$$



Define the *Zhang space*  $\text{Zh}(\Gamma)$  as the space of all functions  $f \in S(\Gamma)$  such that  $f'' \in L^1(\Gamma, dx)$ . The inner product  $(\cdot, \cdot)$  and the Dirichlet pairing  $(\cdot, \cdot)_{\text{Dir}}$  on  $\text{Zh}(\Gamma)$  are defined by

$$\begin{aligned} \forall f, g \in \text{Zh}(\Gamma), \quad (f, g) &:= \int_{\Gamma} fg \, dx, \quad \text{and} \\ (f, g)_{\text{Dir}} &:= \int_{\Gamma} f \Delta(g) = \int_{\Gamma} g \Delta(f) = \int_{\Gamma} f' g' \, dx = (f', g'). \end{aligned}$$

A function  $f$  in  $\text{Zh}(\Gamma)$  is an eigenfunction of the Laplacian on  $\Gamma$  with eigenvalue  $\lambda$  if for any function  $g \in \text{Zh}(\Gamma)$ , we have  $(f, g)_{\text{Dir}} = \lambda(f, g)$ . The eigenvalues of  $\Delta$  are all nonnegative and, assuming  $\Gamma$  is connected, they form a discrete subset  $0 = \lambda_0(\Gamma) < \lambda_1(\Gamma) < \lambda_2(\Gamma) < \dots < \lambda_n(\Gamma) < \dots$  of  $\mathbb{R}$ . In addition,  $\lambda_k(\Gamma)$  has the following (usual) variational characterization:

$$\lambda_k(\Gamma) = \inf_{\substack{\Lambda_{k+1} \subset \text{Zh}(\Gamma) \\ \dim(\Lambda_{k+1}) = k+1}} \sup_{f \in \Lambda_{k+1}} \frac{(f, f)_{\text{Dir}}}{(f, f)}. \quad (3.1)$$

**Definition 3.1** (Dilation of a metric graph). Let  $\Gamma$  be a metric graph with a simple graph model  $(G, \ell)$ , and  $\beta \in \mathbb{R}_{>0}$ . The metric graph  $\beta\Gamma$  is defined as the metric realization of the pair  $(G, \beta\ell)$ .

The following proposition is straightforward, see e.g. [2].

**Proposition 3.2.** *Let  $\Gamma$  be a metric graph and  $\beta > 0$  a real. For any integer  $k \geq 0$ , we have  $\lambda_k(\beta\Gamma) = \frac{1}{\beta^2} \lambda_k(\Gamma)$ .*

By a *metric star*  $\mathcal{S}$  we mean the metric realization of a pair  $(S_d, \ell)$  with  $S_d = K_{1,d}$  a star graph of arbitrary valence  $d$ , and  $\ell$  a length function on  $E(S_d)$ . For such a metric star, define  $\ell_{\max}(\mathcal{S}) := \max_{e \in E(S_d)} \ell(e)$ .

**Lemma 3.3.** *For any metric star  $\mathcal{S}$ , we have:*

$$\lambda_1(\mathcal{S}) \geq \frac{\pi^2}{4\ell_{\max}^2(\mathcal{S})}.$$

*Proof.* Assume that  $\mathcal{S}$  is the metric realization of a pair  $(S_d, \ell)$  with  $d \in \mathbb{N}$ . We adapt the argument in [12, Example 3] to the case where the branches of  $\mathcal{S}$  have non-necessary equal lengths. Let us parametrize each edge  $e$  of  $S_d$  with the interval  $[0, \ell_e]$  starting from the leaf vertex towards the central vertex of  $S_d$ . In this parametrization, an eigenfunction  $\phi$  of the Laplacian, with corresponding eigenvalue  $\lambda$ , must be of the form  $a_e \cos(\sqrt{\lambda}x_e)$ , where  $x_e$  is the length parameter of the edge  $e$  in  $\mathcal{S}$ , for  $e \in E(S_d)$ . This follows in particular from the fact that the slope of an eigenfunction must be zero at leaves. Now let  $a$  be the value of the eigenfunction  $\phi$  at the center of  $\mathcal{S}$ . If  $a = 0$ , we get that  $\sqrt{\lambda}\ell_e \in \pi/2 + \mathbb{N}$ , for any edge  $e$ , which implies the claim. If  $a$  is non zero, then we use the fact that at



the center of  $\mathcal{S}$ , the sum of the (out-going) slopes of  $\phi$  along the branches must be zero [2, Proposition 15.1], which gives

$$\sum_{e \in E(S_d)} a_e \sin(\sqrt{\lambda} \ell_e) = 0.$$

Since  $a_e \cos(\sqrt{\lambda} \ell_e) = a$  for any edge  $e$  of  $S_d$ , this implies

$$\sum_{e \in E(S_d)} \tan(\sqrt{\lambda} \ell_e) = 0,$$

and so, again, at least one of the arguments in the tangents must be at least  $\pi/2$ , and the lemma follows.  $\square$

For a simple graph  $G$  and a vertex  $v \in V(G)$ , we denote by  $\Sigma_G(v)$  the star subgraph of  $G$  with central vertex  $v$  and with the edge set all the incident edges to  $v$ . Let  $(G, \ell)$  be a simple graph model of a metric graph  $\Gamma$ . For any  $v \in V(G)$ , we define the *metric star with center  $v$*  (with respect to  $G$ ) of  $\Gamma$  denoted by  $\mathcal{S}_G(v)$ , or simply  $\mathcal{S}_v$  if there is no risk of confusion, as the subset of  $\Gamma$  isometric to the metric realization of  $\Sigma_G(v)$  with length function given by  $\ell$ . Denote by  $\ell_{\max, G}$  the maximum length of edges in  $G$ , and note that  $\ell_{\max, G} = \max_{v \in V(G)} \{\ell_{\max}(\mathcal{S}_v)\}$ .

Given a simple graph model  $(G, \ell)$  of a metric graph  $\Gamma$ , the family of all the metric stars  $\mathcal{S}_v$ , for  $v \in V(G)$ , forms a 2-fold cover  $\mathcal{S}$  of  $\Gamma$ . Denote by  $\lambda_k^{\text{nr}}(G, \ell)$  the  $k$ th eigenvalue of  $\mathcal{L}_{\mathcal{S}}$ .

Lemma 3.3 together with Theorem 1.2 yields the following bound:

**Theorem 3.4.** *Let  $\Gamma$  be a metric graph with a simple graph model  $(G, \ell)$ . For any  $k \in \mathbb{N}$ , we have*

$$\lambda_k(\Gamma) \geq \frac{\pi^2}{8\ell_{\max, G}^2} \lambda_k^{\text{nr}}(G, \ell).$$

We now show that under certain natural conditions, it is possible to achieve eigenvalue upper bounds closely matching the lower bounds of the above corollary. For a simple graph model  $(G, \ell)$  of  $\Gamma$  denote by  $\ell_{\min, G}$  the minimum length of edges  $e$  in  $E(G)$ .

**Definition 3.5.** A simple graph model of a metric graph  $\Gamma$  is called *length-balanced* if for any edge  $e \in E(G)$ , we have  $\ell_e \leq 2\ell_{\min, G}$ .

We have the following theorem.

**Theorem 3.6.** *There are absolute constants  $c_1, c_2 > 0$  such that for any length-balanced simple graph model  $(G, \ell)$  of  $\Gamma$  on  $n$  vertices, and for any non-negative integer  $k \leq n - 1$ , we have*

$$\frac{c_2}{d_{\max}} \ell_{\min, G}^2 \lambda_k(\Gamma) \leq \lambda_k^{\text{nr}}(G, \ell) \leq c_1 \ell_{\min, G}^2 \lambda_k(\Gamma).$$

Before giving the proof, we state an interesting corollary of the above theorem. We first need the following definition.

**Definition 3.7.** Let  $\Gamma$  be a metric graph. Define  $\ell_{\min}$  as the supremum of  $\ell_{\min, G}$  over all length-balanced simple graph models  $(G, \ell)$  of  $\Gamma$ .

It is easy to see that there is a length-balanced simple graph model  $G$  of  $\Gamma$  such that  $\ell_{\min} = \ell_{\min, G}$ . For such a simple graph model  $(G, \ell)$ , define the model  $(G_k, \ell)$  as the  $k$ th subdivision of  $G$  where each edge  $e$  is subdivided into  $k$  edges of equal lengths  $\ell_e/k$ . Note that  $G_k$  is length-balanced, has at least  $k + 1$  vertices, and has minimum edge length equal to  $\ell_{\min}/k$ . Thus as a consequence of Theorem 3.6, we get

**Corollary 3.8.** *With the notations as above, there are absolute constants  $c_1$  and  $c_2$  such that for any metric graph  $\Gamma$ , we have*

$$\frac{c_2}{d_{\max}} \ell_{\min}^2 \lambda_k(\Gamma) \leq k^2 \lambda_k^{\text{nr}}(G_k, \ell) \leq c_1 \ell_{\min}^2 \lambda_k(\Gamma).$$

Our results, especially corollary 3.8 above, should be viewed as a quantitative complement to a theorem of X. Faber [11] on the spectral convergence of finite graphs to metric graphs, in the sense that they provide uniform upper and lower bounds on the eigenvalues of  $\Gamma$  in terms of eigenvalues of simple graph models of  $\Gamma$ .

*Proof of Theorem 3.6.* First note that since

$$\lambda_k(\beta\Gamma) = \frac{1}{\beta^2} \lambda_k(\Gamma)$$

and since

$$\lambda_k^{\text{nr}}(G, \beta\ell) = \lambda_k^{\text{nr}}(G, \ell),$$

by the very definition, it will be enough to prove the theorem for  $\ell_{\min} = 1$ .

The right hand side inequality follows from Theorem 3.4, and the well-balanced property of the simple graph model  $G$  of  $\Gamma$ . We now prove the other inequality, namely the existence of  $c_2$  such that for any  $k \leq n - 1$ ,  $c_2 \lambda_k(\Gamma) \leq d_{\max} \lambda_k^{\text{nr}}(G, \ell)$  (still under the assumption that  $\ell_{\min} = 1$  and the length-balanced property of the model  $(G, \ell)$ ). Since the lengths of all edges are between 1 and 2, we get  $\lambda_k^{\text{nr}}(G, \ell) \geq \frac{1}{2d_{\max}} \lambda_k(G)$ . Indeed, letting  $g = D_{\mathcal{G}}^{1/2} f$ , we have the following expression for the Rayleigh quotient

$$\begin{aligned} \frac{(g, \mathcal{L}_{\mathcal{G}} g)}{(g, g)} &= \frac{\sum_{e=\{u,v\} \in E} \ell(e) (f(u) - f(v))^2}{\sum_v d_v^{\ell} f(v)^2} \\ &\geq \frac{1}{2d_{\max}} \frac{\sum_{e=\{u,v\} \in E} (f(u) - f(v))^2}{\sum_v f(v)^2} \end{aligned}$$

(where  $d_v^\ell = \sum_{u:u \sim v} \ell(\{u, v\})$ ), which using the variational characterization of the eigenvalues proves the claim. So it will be enough to show the existence of a constant  $c'_2$  such that

$$c'_2 \lambda_k(\Gamma) \leq \lambda_k(G).$$

Consider  $W_{k+1}$  the vector space of dimension  $k+1$  generated by the first  $k+1$  eigenfunctions  $g_0, \dots, g_k \in C(G)$  associated to  $\lambda_i(G)$ , for  $i = 0, \dots, k$ . Note that in particular

$$\lambda_k(G) \geq \sum_{\substack{u, v \in V(G) \\ u \sim v}} \frac{(g(u) - g(v))^2}{\sum_v g(v)^2}$$

for any  $g \in W_{k+1} \setminus \{0\}$ . We construct an injective linear map  $\Psi: C(G) \rightarrow \text{Zh}(\Gamma)$  such that for any  $g \in C(G) \setminus \{0\}$ , we have

$$\frac{(\Psi(g), \Psi(g))_{\text{Dir}}}{(\Psi(g), \Psi(g))} \leq 8 \sum_{\substack{u, v \in V(G) \\ u \sim v}} \frac{(g(u) - g(v))^2}{\sum_v g(v)^2}.$$

Applying the variational characterization of  $\lambda_k(\Gamma)$ , given in Equation (3.1), to the test space  $\Psi(W_k)$ , for  $k \leq n-1$ , will then give the result.

Consider an edge  $e = \{u, v\}$  of  $G$ , and denote by  $u_e$  and  $v_e$  the two points at distance  $\frac{1}{4d_u}$  and  $\frac{1}{4d_v}$  from  $u$  and  $v$  on  $e$ , respectively, where  $d_u$  and  $d_v$  denote the valence of the vertices  $u$  and  $v$  in  $G$ , respectively. Note that the length of each segment  $[u_e, v_e]$  in  $\Gamma$  is at least  $\frac{1}{2}$ .

For any vertex  $v$  of  $G$ , denote by  $B_v$  the union of all segments  $[v, v_e]$  on the edges  $e$  adjacent to  $v$  in  $G$  (i.e.  $B_v$  is the ball of radius  $\frac{1}{4d_v}$  around  $v$  in  $\Gamma$ ). For any function  $g \in C(G)$ , defined on the set of vertices of  $G$ , let  $\Psi(g)$  be the function on  $\Gamma$  which takes value equal to  $g(v)$  on each ball  $B_v$ , and which is affine linear of slope  $(g(v) - g(u))/\ell([u_e, v_e])$  on each segment  $[u_e, v_e]$ , for any edge  $e \in E(G)$ . Obviously,  $\Psi$  is an injective linear map from  $C(G)$  to  $\text{Zh}(\Gamma)$ .

Let now  $g \in C(G) \setminus \{0\}$  and denote  $f = \Psi(g)$ . We have

$$\begin{aligned} (f, f)_{\text{Dir}} &= \int_{\Gamma} f'^2 dx = \sum_{e=\{u,v\} \in E(G)} \frac{1}{\ell([u_e, v_e])} (g(u) - g(v))^2 \\ &\leq 2 \sum_{\{u,v\} \in E(G)} (g(u) - g(v))^2. \end{aligned}$$

Denote by  $B$  the union  $\cup_{v \in V(G)} B_v$ . Since each ball  $B_v$  has total length equal to  $1/4$ , we have

$$\int_{\Gamma} f^2 dx \geq \int_B f^2 dx = \frac{1}{4} \sum_{v \in V(G)} g(v)^2.$$

It thus follows from the two above estimates that for any  $g \in C(G) \setminus \{0\}$ , we have

$$\frac{(\Psi(g), \Psi(g))_{\text{Dir}}}{(\Psi(g), \Psi(g))} \leq 8 \sum_{\substack{u, v \in V(G) \\ u \sim v}} \frac{(g(u) - g(v))^2}{\sum_v g(v)^2},$$

and the theorem follows.  $\square$

#### 4. Anisotropic mesh partitioning

In this final section we discuss a practical application of our transfer theorem to the mesh partitioning problem in scientific computing. Parallelizing finite elements computations requires to split the base mesh in such a way that communication between different pieces is minimized. This is naturally formalized as a (possibly multi-way) sparsest cut problem, which we may want to solve using spectral clustering. Guarantees for such methods in this setting were proved by Miller–Teng–Thurston–Vavasis and Spielman–Teng [24, 27]. More precisely, these papers show that spectral partitioning provides good cuts for meshes in  $d$ -dimensional Euclidean space provided that all  $d$ -simplices in the mesh are well-shaped, i.e. not too far from being equilateral.

It is not hard to design a 2-fold cover of a general mesh such that our transfer result provides guarantees for spectral clustering applied to anisotropic meshes. Specifically, let  $T$  be a triangulation of a domain  $D \subset \mathbb{R}^d$ . Performing a barycentric subdivision of all  $d$ -simplices gives a triangulation  $T'$ . For a  $d$ -simplex  $\sigma$  of  $T$ , let now  $U_\sigma$  be the interior of the union of  $\sigma$  with the  $d + 1$   $d$ -simplices of  $T'$  that share a facet with  $\sigma$ . The collection of  $U_\sigma$  forms a 2-fold cover  $\mathcal{U}$  of the domain, and the corresponding Laplacian  $\mathcal{L}_\mathcal{U}$  is defined using weights  $w_{\sigma_1, \sigma_2}$  that are proportional to the sum of the volumes of  $\sigma_1$  and  $\sigma_2$ . Hence, assuming that neighboring  $d$ -simplices in  $T$  have volumes within a ratio of  $\kappa > 1$ , we see that the eigenvalues of  $\mathcal{L}_\mathcal{U}$  and those of the normalized Laplacian of the dual graph of  $T$  are also within a ratio of  $\kappa$ .

**Proposition 4.1.** *The Neumann value of  $U_\sigma$  is at least  $C^{-1} \kappa^{-1} \epsilon^{-2}$  for some universal constant  $C > 0$ , where  $\epsilon$  is the maximum diameter of simplices in  $T$ .*

*Proof.* Let  $\tau_i$ ,  $i = 1 \dots d + 1$ , be the  $d$ -simplices in  $T'$  that share a facet with  $\sigma$ , and  $\sigma_i$  be the  $d$ -simplex in  $T'$  that is included in  $\sigma$  and shares a facet with  $\tau_i$ . The interiors of  $\sigma$ ,  $\tau_i$ , and of  $\tau_i \cup \sigma_i$  form a 2-fold cover of  $U_\sigma$ . The entries of the corresponding Laplacian are within a factor  $\kappa$  of the those of the normalized Laplacian of the intersection graph of the elements of the cover, which is a once subdivided star graph. Such a star graph has Fiedler value lower bounded by a constant. Now each element in the cover is a convex set with diameter at most  $2\epsilon$ , so by [26] their Neumann value is lower bounded by a constant times  $\epsilon^{-2}$ . The claim then follows from Theorem 1.2.  $\square$

Therefore, Theorem 1.2 applied to the cover  $\mathcal{U}$  yields that the Fiedler value of the dual graph of  $T$  is at most  $2C\kappa^2\lambda_1(D)\epsilon^2$ . By Cheeger's inequality, a suitable spectral partitioning algorithm gives a balanced cut of size at most  $\kappa C'\sqrt{\lambda_1(D)}/\epsilon$ , for some constant  $C'$ . We note that if  $d$ -simplices in  $T$  are nearly equilateral, then  $\epsilon \simeq (\text{vol}(D)/n)^{1/d}$ , where  $n$  is the number of simplices in  $T$ . Hence in this case we recover the  $n^{1/d}$  behaviour proved in [24, 27] for the size of the cut, since the assumption that simplices are well-shaped implies an upper bound on  $\kappa$ . However, the methods used in those works do not seem to apply to the case of general anisotropic meshes.

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