Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 92 (2017)

Heft: 4

Artikel: Counting periodic orbits of Anosov flows in free homotopy classes

Autor: Barthelmé, Thomas / Fenley, Sergio R.

DOI: https://doi.org/10.5169/seals-730825

Comment. Math. Helv. 92 (2017), 641–714
DOI 10.4171/CMH/421

Commentarii Mathematici Helvetici

Counting periodic orbits of Anosov flows in free homotopy classes

Thomas Barthelmé and Sergio R. Fenley*

Abstract. The main result of this article is that if a 3-manifold M supports an Anosov flow, then the number of conjugacy classes in the fundamental group of M grows exponentially fast with the length of the shortest orbit representative, hereby answering a question raised by Plante and Thurston in 1972. In fact we show that, when the flow is transitive, the exponential growth rate is exactly the topological entropy of the flow. We also show that taking *only* the shortest orbit representatives in each conjugacy classes still yields Bowen's version of the measure of maximal entropy. These results are achieved by obtaining counting results on the growth rate of the number of periodic orbits inside a *free homotopy class*. In the first part of the article, we also construct many examples of Anosov flows having some finite and some infinite free homotopy classes of periodic orbits, and we also give a characterization of algebraic Anosov flows as the only $\mathbb{R}$-covered Anosov flows up to orbit equivalence and finite lifts that do not admit at least one infinite free homotopy class of periodic orbits.

Mathematics Subject Classification (2010). 37D20, 37C27; 57M50, 57R30, 37C15, 37D5.

Keywords. Anosov flows, counting orbits.

1. Introduction

A classical and fundamental problem in dynamical systems is to count the number of closed orbits of the system with respect to the period. For Anosov flows, Margulis in his thesis [41], and independently (and, more generally, for Axiom A flows) Bowen [14] showed that the number of orbits grows exponentially with the period. In fact, Margulis gave an asymptotic formula for the growth of the number of closed orbits as a function of the period for weak-mixing flows and later Parry and Policott [42] gave a formula for the general case. In particular, the topological entropy, an ubiquitous quantity in dynamical systems that measures the complexity of the flow, appears in the asymptotics of the counting function [14,41]. Moreover, Bowen [14] showed that the unique invariant measure of maximal entropy is supported by periodic orbits, that is, it can be obtained as the normalization of the sum of the Lebesgue probability measures supported on periodic orbits. Bowen's and Margulis' work have been essential in the theory of hyperbolic dynamical systems.

*Research by the second author was partially supported by a grant from the Simons Foundation.

At the same time that Bowen's work appeared, Plante and Thurston [46] proved that if a manifold M supports a *codimension one* Anosov flow (i.e. such that one of the strong foliations of the flow is of dimension one), then $\pi_1(M)$ has exponential growth. In that paper, they also asked the following question:

Question 1 (Plante, Thurston [46]). Suppose that M supports a codimension one Anosov flow. Does the number of *conjugacy classes* in $\pi_1(M)$ grow exponentially fast with the length of a shortest orbit representative?

Given Bowen's result, Plante and Thurston remark that a positive answer to this question can be obtained by giving a low upper bound on the growth of the number of orbits of period less than t as a function of t *inside a free homotopy class*.

To the best of our knowledge, it appears that no one has yet managed to answer Question 1 in any setting, nor obtained any results on the number of orbits inside a free homotopy class.

The main goal of this article is to give a positive answer to Plante and Thurston's question in the case of 3-manifolds. In fact we will obtain more information since we also get a (coarse) estimate of the growth rate of the number of conjugacy classes, as well as an equidistribution property of a shortest orbit representative (see Theorem A below).

Before stating our results, we review what is known about freely homotopic periodic orbits. A *free homotopy class of periodic orbits* is a maximal collection of closed orbits that are pairwise freely homotopic to each other. Since we only talk about periodic orbits in this article, we will henceforth just call it a *free homotopy class*. First, it is easy to note that, in the case of algebraic flows, or more generally flows that are orbit equivalent to either a suspension of an Anosov diffeomorphism or the geodesic flow of a negatively curved metric, then the answer to Plante and Thurston's question is clearly yes. Indeed, in that case, there is at most one periodic orbit in each free homotopy class (or two in the case of geodesic flows if one takes our definition of free homotopy that forgets about the direction of an orbit, see Convention 3). Hence Bowen's or Margulis' work directly implies the result.

Plante and Thurston knew of the existence of Anosov flows admitting distinct orbits in the same free homotopy class. But non trivial explicit examples were not constructed for more than ten years afterwards. The "trivial" examples are obtained as finite lifts of the geodesic flow on the unit tangent bundle of a hyperbolic surface. These manifolds are Seifert fibered (see definition in the Section 2) and finite covers of any order can be obtained by unrolling the Seifert fibers. Then for each natural number n one can obtain examples where every free homotopy class has $2n$ elements. These examples are in some sense artificial, for example all orbits in a given free homotopy class have exactly the same length in the lifted metric. Plante and Thurston did not know whether there is an upper bound on the number of orbits in an arbitrary free homotopy class. In 1994, the second author [19] constructed examples of Anosov flows in 3-manifolds such that *every* periodic orbit is freely homotopic to *infinitely* many distinct orbits. In particular, for any enumeration of the orbits in the free homotopy class, the lengths of the orbits diverge to infinity. It follows that counting orbits inside a free homotopy class is a natural, and also non-trivial question at least for some Anosov flows.

One goal of this article is to show that infinite free homotopy classes are very common amongst Anosov flows. An Anosov flow in a 3-manifold is called $\mathbb{R}$-*covered* if the stable (or equivalently the unstable) foliation lifts to a foliation in the universal cover that has leaf space homeomorphic to the reals. A vast amount of such flows exists [5,19]. We will show in this article that, when one considers $\mathbb{R}$-covered Anosov flows then a flow is either orbit equivalent to a finite cover of an algebraic Anosov flow, or it admits an enormous amount of free homotopy classes with infinitely many distinct orbits (see Theorem B). For general Anosov flows, even if the same result does not hold, it is not hard to construct examples with some infinite free homotopy classes. So Plante and Thurston's question is not trivial for "most" Anosov flows on 3-manifolds.

Before stating more precisely our results, there are a few more remarks that one should make.

First, conjecturally, Plante and Thurston's question is trivial in dimension at least 4. Indeed, Verjovsky's conjecture [54] states that any *codimension one* Anosov flow in dimension at least 4 is orbit equivalent to a suspension of an Anosov diffeomorphism, so a free homotopy class contains at most one orbit. Verjovsky's conjecture is still open in full generality, but it has been proven if the fundamental group is solvable [45], or given some smoothness conditions on the Anosov splitting [32,51].

Second, consider the question of counting periodic orbits inside a free homotopy class for a generic Anosov flow, that is for an Anosov flow in higher codimension. One must note that nothing is known about the topology of manifolds admitting Anosov flows in higher codimension. For instance, in higher codimension, we do not know whether a periodic orbit has to be homotopically non trivial. In particular, no one even knows whether any $\mathbb{S}^n$, when $n \geq 6$, supports (or, presumably, does not) an Anosov flow. Not knowing the answer to this most basic question does not bode well for trying to understand fine properties of free homotopy classes. More explicitly all the techniques used in this article for Anosov flows in 3-manifolds completely break down in higher codimension, because we do not yet have any of the understanding of free homotopy classes that we have for 3-manifolds.

Finally, a problem that attracted a lot of attention in the past was to give counting results for the number of periodic orbits of an Anosov flow inside a fixed *homology* class. Amongst others, Katsuda and Sunada [38] (in the case of a surface with an hyperbolic metric), Phillips and Sarnak [44] (for geodesic flows in negative sectional curvature), Sharp [50], and Babillot and Ledrappier [2] gave precise estimates for the asymptotic of the number of periodic orbits of period less than a given real number in a homology class. Sharp's and Babillot–Ledrappier's asymptotics holds for Anosov flows on any manifold, provided the flow is homologically full (i.e. if every homology class admits at least one periodic orbit) or a suspension. In particular, there are no assumptions on the dimension of the manifold.

Given all the results on counting closed orbits inside a homology class, the lack of counting results inside free homotopy classes seems even more surprising. However, it must be mentioned that the tools used in the homological setting rely in part on deep number theoretical results, whereas our tools are only topological and geometric in nature. This difference in available tools also affects the results: while they obtained precise asymptotics in the homological setting, we only get relatively coarse upper and lower bounds.

We can now present more carefully the results of this article. From now on, we will always be in a 3-manifold setting.

1.1. Statement of results. Our main result about the growth of conjugacy classes and the equidistribution of a shortest orbit in a conjugacy class is the following (see Theorem 7.4 for a more precise version):

Theorem A. *Let ϕ^t be an Anosov flow on M^3. Then the number of conjugacy classes in $\pi_1(M)$ grows exponentially fast with the length of a shortest representative closed orbit in the conjugacy class.*

Moreover, if the flow is transitive, then the exponential growth rate is given by the topological entropy of the flow. That is, if we write $\mathrm{Cl}(h)$ *for the conjugacy class of an element $h \in \pi_1(M)$, $\alpha_{\mathrm{Cl}(h)}$ for a periodic orbit in the conjugacy class* $\mathrm{Cl}(h)$ *with smallest period (if such a periodic orbit exists), and*

$$\mathrm{CCl}(t) := \left\{\mathrm{Cl}(h) \mid h \in \pi_1(M),\ T\left(\alpha_{\mathrm{Cl}(h)}\right) < t\right\},$$

where $T\left(\alpha_{\mathrm{Cl}(h)}\right)$ is the period of $\alpha_{\mathrm{Cl}(h)}$, then we have

$$\limsup_{t\to+\infty} \frac{1}{t} \log \sharp\mathrm{CCl}(t) = h_{\mathrm{top}},$$

where h_{top} is the topological entropy of the flow.

Furthermore, the Bowen–Margulis measure μ_{BM} of the transitive flow ϕ^t (i.e. measure of maximal entropy) can be obtained as

$$\mu_{BM} = \lim_{t\to+\infty} \frac{1}{\sharp\mathrm{CCl}(t)} \sum_{\mathrm{Cl}(h)\in\mathrm{CCl}(t)} \delta_{\alpha_{\mathrm{Cl}(h)}},$$

where $\delta_{\alpha_{\mathrm{Cl}(h)}}$ is the Lebesgue probability measure supported on $\alpha_{\mathrm{Cl}(h)}$.

As we will explain later, we obtain this result by counting orbits inside a free homotopy class. But in order to do that, we first need to understand free homotopy classes as well as possible. This is what the first part of this article is concerned with, and it owes a lot to the good understanding of the topology of Anosov flows in 3-manifolds obtained through the cumulative work of Barbot and the second author [3–5,7,9,10,19–22]. We first obtain a classification of $\mathbb{R}$-covered flows such that all of their free homotopy classes are finite:

Theorem B. *Let ϕ^t be an $\mathbb{R}$-covered Anosov flow on a closed* 3*-manifold M. Suppose that every periodic orbit of ϕ^t is freely homotopic to* at most *a finite number of other periodic orbits. Then either ϕ^t is orbit equivalent to a suspension or, up to finite cover, ϕ^t is orbit equivalent to the geodesic flow of a negatively curved surface.*

Most of the pieces of the proof of this result were essentially known, thanks to the cumulative work of (mainly) Verjovsky, Ghys, Barbot and the second author. The relevance of this result is that it shows that infinite free homotopy classes are extremely common. In fact for $\mathbb{R}$-covered Anosov flows the nonexistence of infinite free homotopy classes is extremely rare.

An essential tool in this article will be the JSJ decomposition of a 3-manifold. In our setting it roughly states that any manifold supporting an Anosov flow has a decomposition by embedded tori into pieces that are either Seifert fibered or hyperbolic (see detailed description in Section 2.2).

We also construct examples of contact Anosov flows (so, in particular, $\mathbb{R}$-covered, see [7]) on manifolds admitting all possible types of JSJ decompositions, by doing Foulon–Hasselblatt [28] surgery on geodesic flows. By the result above, all these flows have some (in fact, infinitely many) infinite free homotopy classes.

Theorem B above is only true for $\mathbb{R}$-covered Anosov flows. In some sense it is not very surprising that this result does not hold for non-transitive Anosov flows. But it turns out that it does not even hold when the flow is transitive:

Theorem C. *There exists (a large family of) non-algebraic transitive Anosov flows such that every periodic orbit is freely homotopic to at most finitely many others.*

In fact, we will prove that all the examples of pseudo-Anosov flows constructed in [10], called totally periodic, are such that every periodic orbit is freely homotopic to at most finitely many others, and many of these examples are Anosov and transitive.

One should also point out that, up until now, all the known examples of Anosov flows with all their free homotopy classes finite were on graph-manifolds, i.e. manifolds so that their JSJ decomposition has only Seifert-fibered pieces. However, we also construct examples of transitive Anosov flows on manifolds containing an atoroidal piece and such that all free homotopy classes are finite.

The first step in order to obtain Theorem B is to use a JSJ decomposition of the manifold which is well adapted to the flow. A *modified JSJ decomposition* is one such that every torus of the decomposition is weakly embedded and *quasi-transverse* to the flow, i.e. transverse except possibly for a finite number of periodic orbits of the flow where the flow is tangent to the entire orbit. Thanks to works of Barbot [5] (for the $\mathbb{R}$-covered case) and Barbot and Fenley [9] (for the general case), any Anosov flow admits a modified JSJ decomposition (see Section 2.2). Using modified JSJ decompositions, we can prove that in general certain types of pieces in the JSJ decomposition *cannot* admit an infinite free homotopy class:

Theorem D. *Suppose that $\mathcal{FH}(\alpha)$ is an* infinite *free homotopy class of a periodic orbit of an Anosov flow on M. Then no orbit of $\mathcal{FH}(\alpha)$ can cross a Seifert-fibered piece on which the flow is periodic, except, possibly, when the piece is a twisted I-bundle over the Klein bottle. In addition any Seifert fibered piece of the modified JSJ decomposition can only contain a bounded number of orbits of $\mathcal{FH}(\alpha)$.*

Note that there are infinitely many examples of non $\mathbb{R}$-covered Anosov flows that admit some infinite free homotopy classes of orbits, but it seems difficult to come up with a topological criterion that would detect such a feature.

Given that infinite free homotopy classes are very common, we now move towards a proof of Theorem A. This is the second part of this article.

To prove Theorem A we will show that the period of orbits in a free homotopy class grow *at least* at a certain rate. The proof of this will be heavily dependent on the geometric type of the manifold or of the pieces of the JSJ decomposition of the manifold and how the geometric type of the pieces relates with the flow lines.

To obtain an upper bound on the number of orbits inside a free homotopy class with period less than a given real number, we need two preliminary key lemmas.

The first result (Proposition 2.26) shows that a free homotopy class can be split into a *uniformly bounded* number of special orbits plus a *uniformly bounded* number of what we call a *string of orbits*. Strings of orbits are particular subsets of free homotopy classes: roughly speaking they are such that they do not involve non separated stable/unstable leaves when lifted to the universal cover. These strings of lozenges come naturally equipped with an indexation by $\mathbb{N}$. The indexation is given by a *chain of lozenges* when lifted to the universal cover (see definition in the next section). The index is given by placement of the lift of the periodic orbit as a corner of a lozenge in this infinite chain of lozenges.

The second result (Lemma 2.28) says that orbits inside a string of orbits, when lifted to the universal cover, are at least linearly far apart with respect to the indexation.

These two results, while not technically difficult given what is already known, are what allows us to bring in geometry into the picture. Using (either directly or indirectly) some hyperbolic properties of the metric inside a piece of the JSJ decomposition, we can obtain a lower bound on the growth of the period of orbits inside a string of orbits. Using the fact that orbits in a string are also *at most* linearly far apart (Lemma 2.29), we also get an upper bound.

Theorem E. *Let $\{\alpha_i\}_{i\in\mathbb{N}}$ be an infinite string of orbits, with the indexation chosen so that α_0 is one of the orbits with the smallest period in the conjugacy class. Then the growth of the period is at least:*

(1) *Exponential in i if the manifold is hyperbolic;*

(2) *Quadratic in i if the $\{\alpha_i\}_{i\in\mathbb{N}}$ intersect an atoroidal piece;*

(3) *Linear in i if $\{\alpha_i\}_{i\in\mathbb{N}}$ goes through two consecutive Seifert-fibered pieces.*

Moreover, the growth of the period is at most exponential in i, independently of the topology of M.

Notice that for the lower bound, there is a very strong dependence on the geometric type of the manifold or the piece of the JSJ decomposition.

This result can then be translated in terms of a counting result inside free homotopy classes thanks to Proposition 2.26:

Theorem F. *Let ϕ^t be an Anosov flow on a 3-manifold M, and $\mathcal{FH}(\alpha_0)$ be the free homotopy class of a closed orbit α_0 of ϕ^t. For any periodic orbit α, let $T(\alpha)$ be the period of α.*

(1) *If M is hyperbolic, then there exists a uniform constant $A_1 > 0$ and a constant C_1 depending on $\mathcal{FH}(\alpha_0)$ (or equivalently on α_0) such that, for t big enough,*

$$\sharp\{\alpha \in \mathcal{FH}(\alpha_0) \mid T(\alpha) < t\} \leq A_1 \log(t) + C_1.$$

(2) *If the JSJ decomposition of M is such that no decomposition torus bounds a Seifert-fibered piece on both sides (so in particular, if all the pieces are atoroidal), then there exists a constant C_1 depending on $\mathcal{FH}(\alpha_0)$ such that, for t big enough,*

$$\sharp\{\alpha \in \mathcal{FH}(\alpha_0) \mid T(\alpha) < t\} \leq C_1\sqrt{t}.$$

(3) *Otherwise, there exist constants $A_1 > 0$ and $B_1 \geq 0$, such that, for t big enough,*

$$\sharp\{\alpha \in \mathcal{FH}(\alpha_0) \mid T(\alpha) < t\} \leq A_1 t + B_1.$$

Furthermore, if M is a graph manifold, then A_1 and B_1 can be chosen independently of $\mathcal{FH}(\alpha_0)$.

So, in any case, the growth of the number of orbits inside a free homotopy class is at most linear in the period — but a priori with constants depending on the particular free homotopy class.

Moreover, independently of the topology of M, the growth of the number of orbits inside an infinite free homotopy class is at least logarithmic in the period. More precisely, there exists a uniform constant $A_2 > 0$ and a constant C_2 depending on $\mathcal{FH}(\alpha_0)$ such that, if $\mathcal{FH}(\alpha_0)$ is infinite, then for any t

$$\sharp\{\alpha \in \mathcal{FH}(\alpha_0) \mid T(\alpha) < t\} \geq \frac{1}{A_2}\log(t) - C_2.$$

Theorem F is not yet enough to get Theorem A. Indeed, we need to show that we have a *uniform* upper bound for the rate of growth of the number of orbits inside a free homotopy class with respect to the period. That is, we need the constants in the previous theorem to be independent of the chosen free homotopy class. We manage to do that, at the cost of getting worse rates of growth:

Theorem G. *Let ϕ^t be an Anosov flow on a 3-manifold M. There exist constants $A_1, A_2, A_3 > 0$ and $t_0 > 0$, depending only on the flow and M, such that for any periodic orbit α_0*

(1) *If M is a graph manifold, then for $t > t_0$,*

$$\sharp\left\{\alpha \in \mathcal{F}\mathcal{H}(\alpha_0) \mid T(\alpha) < t\right\} \leq A_1 t.$$

(2) *If M is hyperbolic, then for $t > t_0$,*

$$\sharp\left\{\alpha \in \mathcal{F}\mathcal{H}(\alpha_0) \mid T(\alpha) < t\right\} \leq A_2 \sqrt{t} \log(t)$$

(3) *Otherwise, for $t > t_0$,*

$$\sharp\left\{\alpha \in \mathcal{F}\mathcal{H}(\alpha_0) \mid T(\alpha) < t\right\} \leq A_3 \sqrt{t} e^{\frac{\sqrt{t}}{2} \log(t)}$$

So, independently of the topology of M, we always have, for $t > t_0$,

$$\sharp\left\{\alpha \in \mathcal{F}\mathcal{H}(\alpha_0) \mid T(\alpha) < t\right\} \leq A_3 \sqrt{t} e^{\frac{\sqrt{t}}{2} \log(t)}.$$

The first two results of Theorem A are an almost direct corollary of Theorem G, thanks to the counting results of Bowen [14] and Margulis [41]. To prove the equidistribution result, i.e. that the measure of maximal entropy can be obtained by a limit of sums of measures supported on the shortest orbit in a free homotopy class, we also use a result of Kifer [39] (following an idea of Babillot and Ledrappier [2]).

We stress that the last three theorems establish a deep connection between counting orbits in infinite free homotopy classes and the JSJ decomposition of the manifold. In addition, they establish a connection with the particular geometry in the pieces of the JSJ decomposition. It is worth noting that this is the first instance establishing such a connection. This relationship does not appear in the aforementioned counting results in homology classes and general counting of orbits of Anosov flows. It follows that the results of this paper establish an important connection between counting results and entropy on the one hand and the topology and geometry of the 3-manifold on the other hand.

Finally, we also deduce from Theorem E the following result about quasigeodesicity of $\mathbb{R}$-covered Anosov flows, which generalizes a result of the second author in [19].

Theorem H. *Let ϕ^t be a $\mathbb{R}$-covered Anosov flow on a closed 3-manifold M. If M admits an atoroidal piece in its JSJ decomposition, i.e. if M is not a graph-manifold, then ϕ^t is* not *quasigeodesic.*

We also conjecture that all the $\mathbb{R}$-covered Anosov flows on graph-manifolds are quasigeodesic flows.

Remark 1.1. Theorems E, F, and G all give coarse results where we only prove the existence of constants. So, in particular, these results are unaffected by any time-change of the flow, as a time-change would only modify the constants by a multiplicative and/or an additive factor.

Moreover, we can also choose any metric on M, and instead of mesuring the growth rate of the period, we can measure the growth rate of the lengths. Since the ratio between the length and the period is uniformly (for a fixed metric and flow) bounded away from zero and infinity, Theorems E, F, and G apply verbatim when we consider lengths instead. The only cost, once again is that the constants will be modified.

Therefore, in this article we will deal from now on with length instead of period, since it is more natural in order to use geometry. Furthermore, since none of the above listed results are changed by reparametrization of the flow, or modification of the metric, we can, and always will, either reparametrize the Anosov flow in such a way that it moves points at unit speed for a given choice of a metric, or choose the metric such that the orbits of the Anosov flow are unit speed.

The topological entropy of a flow, however, is modified by reparametrizations, so one might think that Theorem A would be affected by our choice. This is not the case, as the proof of Theorem A (see Section 7.2) relies only on the *existence* of uniform constants giving a striclty less than exponential growth rate, not on the actual values of the constants.

1.2. Structure of the paper. In Section 2, we cover the background material needed for this article about Anosov flows and their topology. We also prove some new results describing free homotopy classes that are essential for the rest of the article. In particular, we prove the key results Proposition 2.26 and Lemma 2.28.

In Section 3 we describe the Foulon–Hasselblatt surgery and use it to construct a number of contact Anosov flows on manifolds with all possible types of JSJ decompositions.

In Section 4, we study what the existence or nonexistence of infinite free homotopy classes implies for the topology of the manifold. In particular, we prove Theorems B, C and D (Theorem 4.1, Corollaries 4.6 and 4.4 respectively).

Section 5 describes how one can use recent work of Béguin, Bonatti, and Yu [12] to build *non* $\mathbb{R}$-covered Anosov flows on manifolds with all possible types of JSJ decompositions and both finite and infinite free homotopy classes.

Theorem E (Theorems 6.1 and 6.2) is then proved in Section 6, as well as more precise results giving some explicit control of the constants. The proof of that result is split over three subsections (6.1, 6.2, and 6.3), one for each topological type.

In Section 7, we derive the consequences of Theorem E. That is, we first derive Theorem F (Theorem 7.1) and Theorem G (Theorem 7.3). We then explain how to use the latter to finally prove Theorem A (Theorem 7.4, Corollaries 7.5 and 7.6).

Finally, in Section 8, we obtain Theorem H (Theorem 8.1) as yet another consequence of the work done in Section 6.

2. Background and preliminary results

2.1. Generalities on Anosov flows. An Anosov flow is defined as follows.

Definition 2.1. Let M be a compact manifold and $\phi^t\colon M \to M$ a C^1 flow on M. The flow ϕ^t is called Anosov if there exists a splitting of the tangent bundle $TM = \mathbb{R}\cdot X \oplus E^{ss} \oplus E^{uu}$ preserved by $D\phi^t$ and two constants $a, b > 0$ such that:

(1) X is the generating vector field of ϕ^t;

(2) For any $v \in E^{ss}$ and $t > 0$,

$$\|D\phi^t(v)\| \leq be^{-at}\|v\|;$$

(3) For any $v \in E^{uu}$ and $t > 0$,

$$\|D\phi^{-t}(v)\| \leq be^{-at}\|v\|.$$

In the above, $\|\cdot\|$ is any Riemannian (or Finsler) metric on M.

Clearly this definition makes sense for M of any dimension and E^{ss}, E^{uu} of any positive dimension. The results of this article deal with M of dimension 3, so we restrict to this dimension from now on.

The subbundle E^{ss} (resp. E^{uu}) is called the *strong stable distribution* (resp. *strong unstable distribution*). It is a classical result of Anosov [1] that E^{ss}, E^{uu}, $\mathbb{R}\cdot X \oplus E^{ss}$ and $\mathbb{R}\cdot X \oplus E^{uu}$ are integrable and are continuous. We denote by $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$, $\mathcal{F}^s$ and $\mathcal{F}^u$ the respective foliations and we call them the strong stable, strong unstable, stable and unstable foliations.

All of these foliations, as well as the flow, lift to the universal cover $\widetilde{M}$ of M, and we denote the lifts by $\tilde{\phi}^t$, $\widetilde{\mathcal{F}}^s$, $\widetilde{\mathcal{F}}^u$, $\widetilde{\mathcal{F}}^{ss}$ and $\widetilde{\mathcal{F}}^{uu}$. We then define the orbit and leaf spaces of the flow in the following way:

- The *orbit space* of ϕ^t is the quotient space of $\widetilde{M}$ by the relation "being on the same orbit of $\tilde{\phi}^t$". We denote it by $\mathcal{O}$.
- The *stable* (resp. *unstable*) *leaf space* of ϕ^t is the quotient of $\widetilde{M}$ by the relation "being on the same leaf of $\widetilde{\mathcal{F}}^s$ (resp. $\widetilde{\mathcal{F}}^u$)". We denote them by $\mathcal{L}^s$ and $\mathcal{L}^u$ respectively.

The stable and unstable foliations project to two transverse 1-dimensional foliations of the orbit space $\mathcal{O}$. We will keep the same notations for the foliations on $\mathcal{O}$ or on $\widetilde{M}$ and hope it will not be the source of any confusion.

The orbit space $\mathcal{O}$ is always homeomorphic to $\mathbb{R}^2$ [3,19], but in general the leaf spaces are not Hausdorff. So the leaf spaces are examples of simply connected non-Hausdorff 1-manifolds. Therefore we make the following:

Definition 2.2. An Anosov flow is called $\mathbb{R}$-covered if its stable leaf space $\mathcal{L}^s$, or equivalently, its unstable leaf space $\mathcal{L}^u$ is homeomorphic to $\mathbb{R}$.

The proof that one leaf space being Hausdorff implies that the other is can be found in [3,19].

A very important fact about $\mathbb{R}$-covered flows is the following:

Theorem 2.3 (Barbot [3]). *Let ϕ^t be a $\mathbb{R}$-covered Anosov flow on a closed 3-manifold M.*

- *either no leaf of $\widetilde{\mathcal{F}}^s$ intersect every leaf of $\widetilde{\mathcal{F}}^u$ (and vice-versa),*
- *or ϕ^t is orbit equivalent to a suspension of an Anosov diffeomorphism*

Suppose that ϕ^t is $\mathbb{R}$-covered but not orbit equivalent to a suspension. We say in that case that ϕ^t is *skewed*. The previous result implies that the structure of the orbit space and the stable and unstable foliations are particularly nice: Start with a leaf $\lambda^s \in \mathcal{L}^s$. Then the set

$$I^u(\lambda^s) := \{\lambda^u \in \mathcal{L}^u \mid \lambda^u \cap \lambda^s \neq \emptyset\}$$

is an open, non-empty, connected and bounded set in $\mathcal{L}^u \simeq \mathbb{R}$. Hence it admits an upper and lower bound. Let $\eta^s(\lambda^s) \in \mathcal{L}^u$ be the upper bound and $\eta^{-u}(\lambda^s) \in \mathcal{L}^u$ be the lower bound. Similarly, for any $\lambda^u \in \mathcal{L}^u$, define $\eta^u(\lambda^u)$ and $\eta^{-s}(\lambda^u)$ as, respectively, the upper and lower bounds in $\mathcal{L}^s$ of the set of stable leafs that intersects λ^u. We have the following result (see Figure 1):

Proposition 2.4 (Fenley [19], Barbot [3,7]). *Let ϕ^t be a skewed $\mathbb{R}$-covered Anosov flow in a 3-manifold M, where $\mathcal{F}^s$ is transversely orientable. Then, the functions $\eta^s\colon \mathcal{L}^s \to \mathcal{L}^u$ and $\eta^u\colon \mathcal{L}^u \to \mathcal{L}^s$ are Hölder-homeomorphisms and $\pi_1(M)$-equivariant. We have $(\eta^u)^{-1} = \eta^{-u}$, and $(\eta^s)^{-1} = \eta^{-s}$. Furthermore, $\eta^u \circ \eta^s$ and $\eta^s \circ \eta^u$ are strictly increasing homeomorphisms and we can define $\eta\colon \mathcal{O} \to \mathcal{O}$ by*

$$\eta(o) := \eta^u(\widetilde{\mathcal{F}}^u(o)) \cap \eta^s(\widetilde{\mathcal{F}}^s(o)).$$

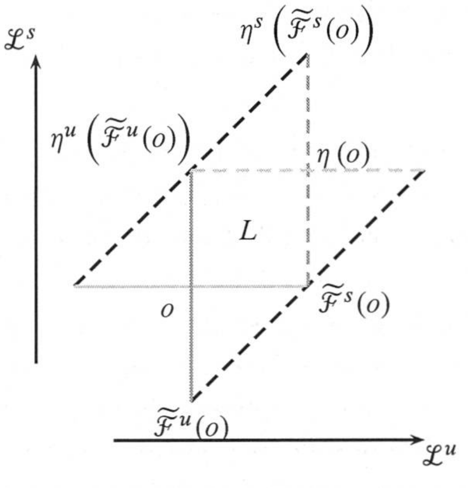


Figure 1. The orbit space in the $\mathbb{R}$-covered case.

If $\mathcal{F}^s$ is not transversely orientable the homeomorphisms η^s, η^u are twisted $\pi_1(M)$-equivariant.

When Anosov flows are non $\mathbb{R}$-covered flows, we can make the following definition:

Definition 2.5. A leaf of $\widetilde{\mathcal{F}}^s$ or $\widetilde{\mathcal{F}}^u$ is called a non-separated leaf if it is non-separated in its respective leaf space ($\mathcal{L}^s$ or $\mathcal{L}^u$) from a distinct leaf. A leaf of $\mathcal{F}^s$ or $\mathcal{F}^u$ is called a *branching* leaf if it is the projection of a non-separated leaf in $\widetilde{\mathcal{F}}^s$ or $\widetilde{\mathcal{F}}^u$.

Non $\mathbb{R}$-covered Anosov flows are generally more complicated than $\mathbb{R}$-covered ones, but we do have the following nice result, that will be quite useful for us:

Theorem 2.6 (Fenley [22, Theorem F]). *For any Anosov flow on a* 3*-manifold, there are only a finite number of branching leaves.*

2.2. Modified JSJ decompositions. A 3-manifold M is irreducible if every embedded sphere bounds a ball [35]. A fundamental result of Jaco–Shalen and Johannson states that 3-manifolds are decomposed into simple pieces. A 3-manifold N is *Seifert fibered* if it has a foliation by circles [18,35]. A 3-manifold N is *atoroidal* if every π_1-injective map from the torus $f : T^2 \to N$ is homotopic into the boundary of N. A Seifert manifold usually has many π_1-injective tori that are not homotopic to the boundary, so these two types of manifolds are opposites. The Jaco–Shalen–Johannson decomposition theorem also called the JSJ, or torus decomposition states the following:

Theorem 2.7. *Let M be a compact, irreducible, orientable* 3*-manifold. Then there is a finite collection $\{T_j\}$ of π_1-injective, embedded tori which cut M into pieces $\{P_i\}$ such that the closure of each component P_i of $M - \cup T_j$ is either Seifert fibered or atoroidal. In addition except for a very small and completely specified class of simple manifolds, the decomposition (in other words the T_j or the P_i up to isotopy) is unique if the collection $\{T_j\}$ is minimal.*

Any manifold supporting an Anosov flow is irreducible [48]. However it may not be orientable, but we will be able to lift to an orientable double cover as explained later.

Definition 2.8. (Birkhoff annulus) Let ϕ^t be an Anosov flow in M^3. A Birkhoff annulus is an a priori only immersed annulus A, such that the interior of A is transverse to the flow and the boundary of A is a union of orbits of the flow (possibly the same orbit).

A π_1-injective, a priori only immersed torus T in M is said to be *quasi-transverse* to the flow ϕ^t if T is a finite union of Birkhoff annuli. An embedded π_1-injective torus is always homotopic to one that is either transverse or to a quasi-transverse torus that is *weakly embedded* [9, Theorem 6.10]. Weakly embedded means that the torus is embedded outside the tangent orbits. Unless the flow ϕ^t is orbit equivalent to a suspension then the torus is always homotopic to a quasi-transverse torus. Moreover, almost always this quasi-transverse torus is unique up to homotopy along the orbits of ϕ^t and unique up to isotopy in the complement of the tangent orbits. In particular the tangent orbits are completely determined by the isotopy class of the torus. There is a special case when there is more than one [9, Lemma 5.5], in which case the torus is associated with a scalloped region (see Definition 2.19 in Section 2.3). In this case up to flow homotopy there are exactly two Birkhoff tori homotopic to T. This is related to the study of $\mathbb{Z}^2$-invariant chains of lozenges, as will be explained further on in this article. In this case the torus is homotopic to two essentially distinct Birkhoff tori, in particular the boundary orbits are not the same for the two tori. In addition the torus is then also isotopic to another torus that is transverse to the flow. So, in summary, the following result describes what we call the modified JSJ decomposition.

Theorem 2.9 ([9], Sections 5 and 6). *Let ϕ^t be an Anosov flow in M orientable, which is not orbit equivalent to a suspension Anosov flow. Let $\{T'_j\}$ be a collection of disjoint, embedded tori given by the JSJ decomposition theorem. Then each torus T'_j is homotopic to a weakly embedded quasi-transverse torus T_j. In case T_j is not unique up to flow homotopy then T'_j is also isotopic to a transverse torus, which will then be denoted by T_j.*

Moreover, the collection $\{T_j\}$ is also weakly embedded, *that is, embedded outside the union of the orbits tangent to the tori T_j that are quasi-transverse to the flow.*

With these choices the tori T_j are unique up to flow homotopy and unique up to flow isotopy outside the tangent orbits. The closure of the complementary components P_i of $\cup T_j$ are called the pieces of the modified JSJ decomposition.

Furthermore, if P_i is not a manifold, then there are arbitrarily small neighborhoods of P_i that are representatives of the corresponding piece P'_j of the torus decomposition of M.

The fact that P_i may not always be a submanifold is due to the possible collapsing of tangent orbits in the union of the tori T_j. For example it could be that two distinct "boundary" components T_j and T_k of P_i have a common tangent orbit γ (and this is quite common as can be seen in [9]). Then, along γ, the piece P_i is not a manifold with boundary, since two "sheets" of the boundary of P_i intersect at γ.

In addition, notice that to ensure the flow uniqueness of the T_j, we need to choose the transverse tori in the case that there are two essentially distinct quasi-transverse tori homotopic to a given T'_j.

Let us now describe how the flow intersects a piece P_i: An orbit intersecting ∂P_i intersects it either in the tangential or transverse part of ∂P_i. If it is tangent then it is *entirely* contained in T_j for some j and so entirely contained in ∂P_i. Otherwise it either enters or exits P_i. Hence the fact that P_i may not be a manifold only affects the orbits that are entirely contained in ∂P_i.

Throughout the article we will use modified JSJ decompositions.

Setup. All the counting questions we consider in this article are left unmodified by passing to finite covers, modulo changing some of the constants. This will be carefully explained later on in the article. Therefore we will always implicitly take a finite cover where M is orientable if necessary, and we will mostly consider modified JSJ decompositions.

Convention 1. We invariably think of γ in $\pi_1(M)$ as both a covering translation of $\widetilde{M}$ and as a homotopy class of curves in M.

Definition 2.10 (intersecting a piece, crossing a piece). Let P be a piece of the torus decomposition of M and let P' be an associated piece of a modified JSJ decomposition. We say that a periodic orbit α intersects P if α is either a tangent orbit in P' or if it intersects $\partial P'$ transversely. In the second case we in addition say that α crosses the piece P. We may also refer to this as α intersects or crosses P', the associated piece of the modified JSJ decomposition.

Notice that P is defined up to isotopy and P' is defined up to homotopy along flow lines and isotopy outside the tangent orbits. Therefore α intersects P or crosses P independently of the particular modified JSJ representative P' and depends only on the isotopy class of P.

Definition 2.11 (periodic piece, free piece). With respect to an Anosov flow, a Seifert fibered piece S of the torus decomposition of M can have one of two possible behaviors:

- Either there exists a Seifert fibration of S and up to powers there exists a periodic orbit in M which is freely homotopic to a regular fiber of S in this Seifert fibration; in which case the piece is called *periodic*;
- Or no periodic orbit is freely homotopic to a regular fiber (even up to powers) of any Seifert fibration of S; and the piece S is then called *free*.

Note that, S is periodic if and only if there is a Seifert fibration of S such that if $h \in \pi_1(S)$ represents a regular fiber of S, then h does not act freely on at least one of the leaf spaces of stable/unstable leaves in $\widetilde{M}$.

The most classical example of a free piece is the geodesic flow of a negatively curved surface. More generally, all the flows that we will construct in Section 3 are free on each of their Seifert-fibered pieces thanks to Barbot's result in [5]. But Anosov flows with periodic pieces are far from uncommon either. A lot of examples were constructed and studied in [9,10] by Barbot and the second author.

The added technicality in the statement about some Seifert fibration is that in some exceptional cases there is more than one Seifert fibration in S. This happens non trivially for example if S is a twisted I-bundle over the torus or the Klein bottle.

We will later (in particular in Section 6) need the following two lemmas that describe the connection between free homotopy classes and a modified JSJ decomposition. Throughout this article, we use the following convention for our definition of freely homotopic orbit:

Convention 2. We say that two orbits α and β of a flow on M are *freely homotopic* if they are homotopic as non-oriented curves in M. In other words, if $g \in \pi_1(M)$ is a representative of the orbit α, then α and β are freely homotopic if and only if β is represented by $g^{\pm 1}$.

Up to powers this is also equivalent to saying that there exist lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ of α and β to $\widetilde{M}$ such that g stabilizes $\widetilde{\alpha}$ and $\widetilde{\beta}$.

Lemma 2.12. *Let ϕ^t be an Anosov flow on an orientable 3-manifold M. Let $M = \cup_j N_j$ be a modified JSJ decomposition. Let α_0 be a periodic orbit and $\mathcal{F}\mathcal{H}(\alpha_0)$ its free homotopy class. Suppose that some orbit $\beta \in \mathcal{F}\mathcal{H}(\alpha_0)$ crosses a piece $N = N_k$. Then all the orbits $\alpha \in \mathcal{F}\mathcal{H}(\alpha_0)$ also cross N_k.*

In addition, if there exists a connected component β_1 of $\beta \cap N$ between two boundary tori T_1 and T_2 (where we also allow $T_1 = T_2$), then, for any $\alpha \in \mathcal{F}\mathcal{H}(\alpha_0)$, there exists a connected component α_1 of $\alpha \cap N$ between T_1 and T_2 that is in the same free homotopy class as β_1 modulo boundary.

Furthermore, the free homotopy between two segments of orbits can always be realized inside the pieces of the decomposition that the orbits crosses.

Proof. Given the modified JSJ decomposition $M = \cup_j N_j$, we construct the graph G dual to it in the following way:

- The vertices v_j corresponds to the interior of N_j.
- Two vertices v_i, v_j are joint by an edge if N_i and N_j share a common torus boundary.

This graph G lifts to a tree $\widetilde{G}$ that is dual to the lift of the JSJ decomposition of M to its universal cover $\widetilde{M}$. The vertices of $\widetilde{G}$ are copies of the universal cover of some N_j, and the edges corresponds to lifts of the decomposition tori. This graph is used a lot in 3-manifold theory [35].

The piece N_k crossed by β is fixed throughout the proof.

Since $\mathcal{F}\mathcal{H}(\alpha_0)$ represents a free homotopy class, there exist a lift $\widetilde{\mathcal{F}\mathcal{H}}(\alpha_0)$ of $\mathcal{F}\mathcal{H}(\alpha_0)$ to the universal cover such that the collection of orbits $\widetilde{\alpha} \in \widetilde{\mathcal{F}\mathcal{H}}(\alpha_0)$ is exactly the set of orbits of $\widetilde{\phi}^t$ that are individually left invariant by the same element $\gamma \in \pi_1(M)$. We call such a lift a *coherent lift* of $\mathcal{F}\mathcal{H}(\alpha_0)$. We claim that the action of γ on $\widetilde{G}$ is of one of the following types:

(i) Either γ acts freely on the tree $\widetilde{G}$ and γ acts as a translation on an unique axis.

(ii) Or γ acts freely on the set of vertices of $\widetilde{G}$ but fixes an edge, and moreover this fixed edge is unique.

(iii) Or γ has fixed vertices in $\widetilde{G}$, but does not leave invariant three consecutive edges forming a linear subtree of $\widetilde{G}$.

The general theory of group actions on trees (see for instance [49]) states that there are three possibilities: (1) γ acts freely on $\widetilde{G}$ and so γ has a unique axis where it acts as a translation; (2) γ does not fix any vertex of $\widetilde{G}$, but leaves invariant an edge, so γ^2 fixes at least two points (this is what is called an "inversion of an edge") — clearly the invariant edge is unique; (3) γ has a fixed vertex. Given this, the only thing that needs to be justified in the above classification is case (iii). We have to show that γ cannot fix 3 edges of $\widetilde{G}$ forming a linear subtree. Suppose by way of contradiction that this is not true and let $\widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3$ denote the lifts of the tori corresponding to the edges in question. Let $\widetilde{N}_a, \widetilde{N}_b$ be the lifts of the pieces of the JSJ decomposition corresponding to the vertices between $\widetilde{T}_1, \widetilde{T}_2$ and $\widetilde{T}_2, \widetilde{T}_3$ respectively. Projecting to M this means that γ has a representative in the torus T_1 (projection of $\widetilde{T}_1$ to M) and also T_2. This implies that in N_a there is a cylinder or annulus from T_1 to T_2. Notice that T_1 may be the same torus as T_2, but in this case, the annulus cannot be homotoped into T_1 or $\widetilde{T}_1$ would be equal to $\widetilde{T}_2$. In other words there is an essential annulus in N_a. If the piece N_a is atoroidal then it is acylindrical. This is because N_a is in fact hyperbolic and has boundary made up of tori, hence it is acylindrical (see for instance [53]), so this cannot happen. If N_a is Seifert this can only happen if γ is up to powers a representative of the Seifert fiber in N_a. In the same way γ fixes $\widetilde{T}_3$ so N_b is a Seifert piece and γ up to powers represents the Seifert fiber in N_b. Then, up to powers, γ represents the regular fiber in both N_a and N_b. This is disallowed by the minimality requirement in the JSJ decomposition. This proves that γ cannot leave invariant 3 edges forming a linear subtree of $\widetilde{G}$ and yields possibility (iii) above.

In order to prove the lemma, we need to understand a bit better the projection of the class $\mathcal{FH}(\alpha_0)$ onto the graph. A lift $\widetilde{\alpha}$ of an orbit $\alpha \in \mathcal{FH}(\alpha_0)$ to the universal cover projects to a path on $\widetilde{G}$. The path is continuous, but the projection is not: when $\widetilde{\alpha}$ crosses one of the lifts of the T_j the projection immediately goes from a point, to an edge (at the intersection point with T_j), to a point. Suppose that α intersects a torus T of the modified JSJ decomposition.

If it is contained in T then the projection of $\widetilde{\alpha}$ is an edge in $\widetilde{G}$.

If the intersection is transverse and $\widetilde{T}, \widetilde{T}'$ are two lifts intersected by $\widetilde{\alpha}$ then they are distinct. This is because either $\widetilde{T}$ is transverse to the flow, in which case this is obvious or $\widetilde{T}$ is the lift of a quasi-transverse torus. In that case this fact is proved in [10]. The idea of the proof is just that if $\widetilde{T}$ and $\widetilde{T}'$ were the same, then one could build a closed transversal to say the stable foliation by following $\widetilde{\alpha}$, closing it up along $\widetilde{T}$ and making that closed path transverse to the stable foliation by pushing it along the strong unstable foliation. Therefore, a lift $\widetilde{\alpha}$ of α cannot hit twice the same lift of T, and hence the projection to $\widetilde{G}$ intersects an edge at most once. It follows that this projection of $\widetilde{\alpha}$ has to be an infinite path.

So the projection to $\widetilde{G}$ of an orbit $\widetilde{\alpha}$ is either a vertex, or an edge for the special case of the periodic orbits tangent to a decomposition torus, or is an infinite path in $\widetilde{G}$.

We can now establish part of the lemma. Recall that γ is the element of $\pi_1(M)$ that fixes all the orbits $\widetilde{\alpha}$ in the coherent lift $\widetilde{\mathcal{FH}}(\alpha_0)$.

If γ fixes only one vertex $\widetilde{N_i}$ (corresponding to case (ii) above), then the orbits in $\widetilde{\mathcal{F}\mathcal{H}}(\alpha_0)$ are all included in $\widetilde{N_i}$. This case is not possible as the orbit $\beta \in \mathcal{F}\mathcal{H}(\alpha_0)$ crosses the piece N_k, which implies that the projection of $\widetilde{\beta}$ to $\widetilde{G}$ is an infinite path as seen above. Similarly case (iii) cannot happen either, again by the same reason.

It follows that the the action of γ on $\widetilde{G}$ is free (of type (i) above). So all the $\widetilde{\alpha_i}$ project to the axis of γ and the lift of the decomposition tori cuts each orbits in $\widetilde{\mathcal{F}\mathcal{H}}(\alpha_0)$ into freely homotopic connected pieces. This is because $\widetilde{\alpha_i}$ cannot intersect a lift of one of $\{T_j\}$ more than once, so the projection to $\widetilde{G}$ is *exactly* the axis of γ acting on $\widetilde{G}$ and intersects a lift of a piece N_j in a connected arc. Hence all the $\alpha \in \mathcal{F}\mathcal{H}(\alpha_0)$ cross the piece N_k.

Moreover, if β_1 is a segment of $\beta \cap N_k$ between two boundary tori T_1 and T_2, then a lift $\widetilde{\beta_1}$ will connect a lift $\widetilde{T_1}$ of T_1 to a lift $\widetilde{T_2}$ of T_2. Call $\widetilde{N_k}$ the lift of N_k containing $\widetilde{\beta}_1$. By the argument above, any $\alpha \in \mathcal{F}\mathcal{H}(\alpha_0)$ has a coherent lift $\widetilde{\alpha}$ that contains a segment intersecting $\widetilde{T_1}$ and $\widetilde{T_2}$. Call that segment $\widetilde{\alpha}_1$. Notice that this segment $\widetilde{\alpha}_1$ is uniquely determined, because $\widetilde{\alpha}$ intersects $\widetilde{T_1}$ only once.

We define α_1 to be the projection of $\widetilde{\alpha}_1$ on M. All we have left to do now is to show that α_1 is freely homotopic to β_1 relative to the boundary with a free homotopy that stays inside the piece N_k.

The lift $\widetilde{N_k}$ is simply connected, because the tori in the torus decomposition are π_1-injective. Since $\widetilde{T_1}, \widetilde{T_2}$ are path connected we can connect the points of $\widetilde{\alpha}_1, \widetilde{\beta_1}$ in $\widetilde{T_1}$ with an arc a_1 and similarly with an arc a_2 in $\widetilde{T_2}$. This produce a closed loop $a_1 \cup \widetilde{\alpha_1} \cup a_2 \cup \widetilde{\beta_1}$ (we are not paying attention to orientation along the arcs here). Since $\widetilde{N_k}$ is simply connected, this loop is null homotopic in $\widetilde{N_k}$ and projects to a closed loop in N which is null homotopic in N. Notice that the arcs a_1 and a_2 are well defined up to homotopy with endpoints fixed. Hence β_1 and α_1 are freely homotopic in N_k relative to the boundary.

If we keep doing this for all the other components of $\beta - \cup\{T_j\}$, so that at each step the arcs a_1 are chosen to be equal to a previously chosen arc on the same lift $\widetilde{T_1}$, then this produces a free homotopy from any segment of β to a corresponding segment of α as claimed in the lemma.

This finishes the proof of Lemma 2.12. □

Lemma 2.13. *Suppose that α and β are contained in a piece N of the torus decomposition and that they are in the same free homotopy class. Let H be a free homotopy between them. Then we can choose a free homotopy from α to β entirely contained in N, unless, possibly, the image of H intersects a periodic Seifert piece.*

Proof. Fix a piece N of the decomposition, two orbits α, β in N and a free homotopy H (in M) between them. Suppose that H is not already contained in N. As the tori in the torus decomposition are two sided, we can choose the free homotopy to be in general position with respect to the boundary tori of the modified JSJ decomposition.

Let c be the intersection between the image in M of the free homotopy H and the boundary tori of N. All the connected components of c are closed paths on one of the boundary tori. We first deal with the components of c that are homotopically trivial on the tori. We can modify the homotopy H in the following manner: For each such connected component c_i of c, starting with the innermost (since it bounds a disk in the torus), we consider the disc on the torus that c_i bounds. The homotopy H (or more precisely the connected component of H outside of N that bounds c_i) together with that disc forms a sphere. Since M is aspherical (because the universal cover of a 3-manifold supporting an Anosov flow is always $\mathbb{R}^3$ [54]), the sphere that we obtained bounds a ball. We can hence modify H by replacing its part inside of c_i by a disc on the torus and then modifying it slightly to eliminate this intersection with the union of the tori in the modified JSJ decomposition. Doing this process on all the homotopically trivial connected components of c eliminates all such intersections.

If that process removed all the connected components of c, then the homotopy is in N and we are done. Otherwise, we can assume that any remaining component of c is homotopically non trivial in the particular torus. Consider a sub-annulus, outside of N, of the free homotopy between consecutive such intersections, call it A. This annulus A has image in a piece V of the modified JSJ decomposition. If this annulus is homotopic into the boundary of V we modify H so that the annulus A is replaced by its homotopic image inside the boundary of V.

If A is not homotopic into the boundary, then it is an essential annulus in V. As seen before this implies that V is Seifert (not atoroidal) and the core of the annulus is freely homotopic to a regular fiber, up to powers. Since this core is also freely homotopic to a periodic orbit of ϕ^t this implies that V is a periodic Seifert piece. Any further sub-annulus such that no boundary is either α or β has to be homotopic into that boundary of V as proved in the previous lemma.

Recall that V and N are distinct pieces of the JSJ decomposition. So if the free homotopy H cannot be modified to be contained in N it follows that

- There is a homotopy (still denoted by H) made up of 3 annuli: A_1, A, A_2, where:
- A_1 is a free homotopy in N from α to a curve γ_1 in a boundary torus T_1 of N,
- A is a free homotopy contained in the periodic Seifert piece V and
- A_2 is an annulus in N from a curve γ_2 contained in a boundary component T_2 of N to β.

Notice that both γ_1 and γ_2 are isotopic to regular fibers in their respective tori. So if $T_1 = T_2$ then the homotopy H can be modified to be entirely contained in N. Therefore we can assume that $T_1 \neq T_2$, and this finishes the proof of the lemma. □

2.3. Lozenges and separation constant. A *half leaf* of a stable leaf L of $\widetilde{\mathcal{F}}^s$ is a component of the complement of an orbit in L. Similarly if L is an unstable leaf.

Definition 2.14. A *lozenge* L in $\mathcal{O}$ is an open subset of $\mathcal{O}$ such that (see Figure 2): There exist two points $\alpha, \beta \in \mathcal{O}$ and four half leaves $A \subset \widetilde{\mathcal{F}}^s(\alpha)$, $B \subset \widetilde{\mathcal{F}}^u(\alpha)$, $C \subset \widetilde{\mathcal{F}}^s(\beta)$ and $D \subset \widetilde{\mathcal{F}}^u(\beta)$ satisfying:

- For any $\lambda^s \in \mathcal{L}^s$, $\lambda^s \cap B \neq \emptyset$ if and only if $\lambda^s \cap D \neq \emptyset$,
- For any $\lambda^u \in \mathcal{L}^u$, $\lambda^u \cap A \neq \emptyset$ if and only if $\lambda^u \cap C \neq \emptyset$,
- The half-leaf A does not intersect D and B does not intersect C.

Then,

$$L := \{p \in \mathcal{O} \mid \widetilde{\mathcal{F}}^s(p) \cap B \neq \emptyset,\ \widetilde{\mathcal{F}}^u(p) \cap A \neq \emptyset\}.$$

The points α and β are called the *corners* of L and A, B, C and D are called the *sides*.

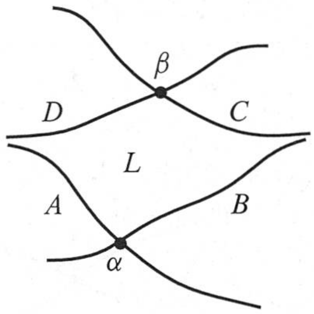


Figure 2. A lozenge with corners α, β and sides A, B, C, D.

In [4], Barbot proves that a lozenge with a periodic corner (i.e. such that its corner orbits project to periodic orbits in M) corresponds to a *Birkhoff annulus*. A Birkhoff annulus is an annulus which is transverse to the flow, except for its two boundary components which are orbits of the flow. Conversely, any Birkhoff annulus gives rise to lozenges, however in general it may be a finite union of lozenges. When the Birkhoff annulus is associated with a single lozenge, we say that the Birkhoff annulus projects to this lozenge. More specifically, Barbot showed that starting with any periodic lozenge, one can construct a Birkhoff annulus that projects to it (see [4]). A *Birkhoff torus* is a torus obtained as a finite union of Birkhoff annuli.

Definition 2.15. A *chain of lozenges* is a connected union of lozenges such that two consecutive lozenges always share a corner or a side.

A *string of lozenges* is a connected union of lozenges that only share corners.

Lozenges sharing sides are particular:

Lemma 2.16. *If two lozenges share a side, then two of their other sides are on non-separated leaves.*

(a) A string of lozenges. (b) A chain of lozenges.

Figure 3. Chain and string of lozenges.

Proof. The two leaves abutting to the shared side are not separated since any leaf in the neighborhood of one of them is in the neighborhood of the other. See Figure 4. □

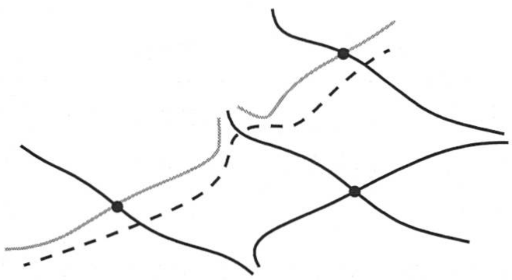


Figure 4. Two lozenges sharing a side. The red (gray) leaves are not separated.

It follows from Theorem 2.6 that only a finite number of lozenges, up to deck transformations, can share a side.

A periodic orbit can in general be the corner of anything from 0 to 4 lozenges, but translating the previous lemma to corners gives:

Lemma 2.17. *Suppose that $\widetilde{\alpha}$ is the corner of* 3 *or* 4 *lozenges, then the opposite corners are on non-separated leaves.*

So up to deck transformations, there are only a finite number of orbits that can be the corner of more than 2 lozenges.

Another fact can also limit the number of lozenges abutting to a particular orbit:

Lemma 2.18. *Suppose that $\widetilde{\alpha}$ is an orbit inside a lozenge L. Then $\widetilde{\alpha}$ is the corner of at most two lozenges.*

Proof. If $\widetilde{\alpha}$ is an orbit inside a lozenge L, then at least two of the quadrants that the stable and unstable leaves of $\widetilde{\alpha}$ define cannot be part of a lozenge as can be seen in Figure 5: The quadrant containing the red (dark gray) leaves cannot be part of a lozenge, since otherwise two stable leaves (and two unstable leaves) would intersect. The other two quadrants can however define lozenges, as can be seen with the blue (mid gray) leaves in Figure 5. □

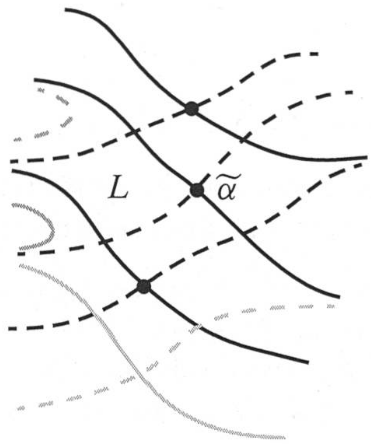


Figure 5. An orbit in a lozenge cannot be the corner of more than two lozenges

Definition 2.19. A *scalloped* chain of lozenges is an bi-infinite chain of lozenges all of which intersect either a common stable leaf or a common unstable leaf.

A *scalloped region* is a scalloped chain of lozenges together with the sides in between two consecutive lozenges.

If α and β are two freely homotopic orbits, then we say that $\widetilde{\alpha}$ and $\widetilde{\beta}$ are *coherent lifts* if there exists $g \in \pi_1(M)$ that fixes both $\widetilde{\alpha}$ and $\widetilde{\beta}$. A property of freely homotopic orbits that will be essential for us is the following

Proposition 2.20 (Fenley [22]). *If α and β are two freely homotopic orbits then any coherent lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ are corners of a chain of lozenges.*

This proposition is the reason why we choose to forget the orientation when talking about free homotopy classes (Convention 2): It is easy to see that if $\widetilde{\alpha}$ and $\widetilde{\beta}$ are the corners of a lozenge, then, up to powers, $\alpha = \pi(\widetilde{\alpha})$ is represented by $g \in \pi_1(M)$ and $\beta = \pi(\widetilde{\beta})$ is represented by g^{-1}. So, forgetting orientation allows us to have a full chain of lozenges associated to a free homotopy class instead of just half of the corners. Moreover, the difference between the number of orbits in a free homotopy class and in a oriented free homotopy class is by a factor of 2. So it would not change any of the counting results in Section 7, but would only change the constants.

In fact the same is true if one lifts the flow to a finite cover of M, modulo changing the constants involved. In particular if needed we can lift to a cover where both $\mathcal{F}^s$ and $\mathcal{F}^u$ are transversely orientable (in which case M is orientable as well). Then given orbits α, β of ϕ^t and integers n, m, not both zero, such that α^n freely homotopic to β^m, it follows that α is either freely homotopic to β or to β^{-1} as *oriented* curves.

We now introduce some terminology that will be needed later on.

Definition 2.21 (minimum distance, Hausdorff distance). Let A, B be two disjoint closed sets in a metric space Z.

The minimum distance between A and B is the infimum of $d(a, b)$ where a is in A and b is in B.

The Hausdorff distance between A and B is the infimum of $r > 0$ such that A is contained in the r neighborhood of B and vice versa. This infimum could be infinite.

We will use later that the minimum and hence the Hausdorff distance (for a given metric on M) between two corners of a lozenge is bounded below.

Lemma 2.22. *There exists $A > 0$, depending only on the flow, such that, if the minimum distance between two stable leaves $\lambda_1, \lambda_2 \in \widetilde{\mathcal{F}}^s$ is less than A, then there exists an unstable leaf $l^u \in \widetilde{\mathcal{F}}^u$ intersecting both λ_1 and λ_2. The same statement stays true with the same A when switching the roles of stable and unstable.*

Moreover, if the minimum distance between two orbits α and β of the lifted flow $\tilde{\phi}^t$ is less than A, then the stable leaf through α intersects the unstable through β and vice-versa.

This lemma is a simple consequence of the product structure of the foliations and the compactness of the manifold.

A far less obvious fact that we will also need later on is that the Hausdorff distance between two corners of a lozenge is also bounded from above (see [27, Corollary 5.3]):

Proposition 2.23 (Fenley [27]). *Let α, β be two freely homotopic orbits such that they admits lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ that are the corners of the same lozenge. Then, there exists $B > 0$ depending only on the flow and on the manifold such that there exists an homotopy H from α to β that moves each point by a distance at most B.*

2.4. From free homotopy class to strings of lozenges. In order to obtain our counting results in Section 7 about free homotopy classes, we will consider some subsets of free homotopy classes which are easier to work with.

We fix some terminology first. Let ϕ^t be an Anosov flow on a 3-manifold M, and α a closed orbit of ϕ^t. Let $\mathcal{F}\mathcal{H}(\alpha)$ be the free homotopy class of α. Recall that we defined (in the proof of Lemma 2.12) a *coherent lift* of $\mathcal{F}\mathcal{H}(\alpha)$ in the following way: Let g be an element of the fundamental group that represents α (so any other element of the conjugacy class of g would also represent α). A coherent lift of $\mathcal{F}\mathcal{H}(\alpha)$ is the set of all the lifts of orbits in $\mathcal{F}\mathcal{H}(\alpha)$ that are invariant under g. Notice that there may be distinct orbits in a coherent lift of $\mathcal{F}\mathcal{H}(\alpha)$ that project to the same orbit in $\mathcal{F}\mathcal{H}(\alpha)$.

By previous results of the second author [22], recalled in Proposition 2.20, a coherent lift of $\mathcal{F}\mathcal{H}(\alpha)$ to the universal cover is composed of the corners of a chain of lozenges.

Definition 2.24. We say that $\{\alpha_i\}_{i\in I}$ is a *string of orbits* in $\mathcal{F}\mathcal{H}(\alpha)$, if it satisfies the following conditions:

- All the α_i are distinct and contained in $\mathcal{F}\mathcal{H}(\alpha)$;
- For a coherent lift of $\mathcal{F}\mathcal{H}(\alpha)$, the orbits $\{\alpha_i\}_{i\in I}$ are the projections of the corners of a string of lozenges $\{\widetilde{\alpha_i}\}$ (see Definition 2.15 above);
- Each $\widetilde{\alpha_i}$ is the corner of at most two lozenges in $\widetilde{M}$.
- Here I is an interval in $\mathbb{Z}$, which could be finite, isomorphic to $\mathbb{N}$ or $\mathbb{Z}$ itself.

There are several slightly different types of string of orbits:

- A string of orbits $\{\alpha_i\}$ is *infinite* if it is indexed by $i \in \mathbb{N}$. We call it *bi-infinite* if it is indexed by $\mathbb{Z}$.
- A string of orbits $\{\alpha_i\}$ is *finite and periodic* if it is finite but the collection $\{\alpha_i\}$ is the projection of corners of an *infinite* string of lozenges. In other words the collection $\{\widetilde{\alpha_i}\}_{i\in\mathbb{Z}}$ is infinite, but there is an element $h \in \pi_1(M)$ and a integer $k > 0$ such that $h \cdot \widetilde{\alpha_i} = \widetilde{\alpha}_{i+k}$. Note that all the orbits in a periodic string are non-trivially freely homotopic to themselves (up to powers).
- A string of orbits $\{\alpha_i\}$ is *finite and non-periodic* otherwise. In other words the string $\{\alpha_i\}$ is *finite*, and it is not the projection of an infinite string $\{\widetilde{\alpha_i}\}, i \in \mathbb{N}$.

Example 2.25. Suppose that ϕ^t is $\mathbb{R}$-covered and that $\mathcal{F}^s$ is transversely orientable. Let α be a periodic orbit. Choose $\widetilde{\alpha}$ a lift of α and, set $\alpha_i = \pi\left(\eta^i(\widetilde{\alpha})\right)$ (where η is the map on the orbit space defined in Proposition 2.4). Then $\{\alpha_i\}$ is either a finite periodic string of orbits or a bi-infinite string of orbits. In addition the free homotopy class of α is exactly the collection $\{\alpha_i\}$.

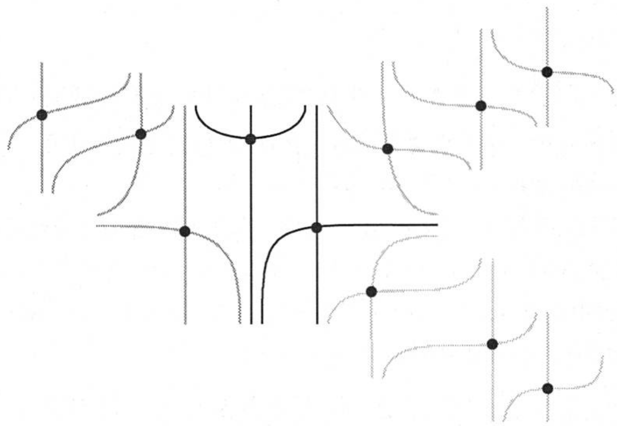


Figure 6. Three different strings of lozenges inside a chain of lozenges

This means that if the flow ϕ^t is $\mathbb{R}$-covered, then the free homotopy classes are quite simple. In the general case however, things tend to be more complicated (see Figure 6). Fortunately, we have the following result.

Proposition 2.26. *Let α be a closed orbit of an Anosov flow on a 3-manifold. The free homotopy class $\mathcal{F}\mathcal{H}(\alpha)$ can be decomposed in the following way:*

- *A finite part $\mathcal{F}\mathcal{H}_{\text{finite}}(\alpha)$,*
- *A finite number of disjoint strings of closed orbits (that could be infinite, finite and periodic or just finite).*

Moreover, there exists a uniform bound (i.e. depending only on the manifold and the flow) on the number of elements in $\mathcal{F}\mathcal{H}_{\text{finite}}(\alpha)$. And there exists a uniform bound on the number of different strings that a free homotopy class can contain.

In fact, the statement about the uniform bounds can be made even stronger, as we will see in the proof: Except for a finite number of free homotopy classes, each free homotopy class is either a finite, infinite, or bi-infinite string of orbits. We also want to emphasize that we do not claim that there exists a uniform bound on the number of orbits inside a *finite* free homotopy class (see after Theorem 7.3 for a discussion of that point), but just a bound on the parts of a free homotopy class that are not strings of orbits.

The very important consequence of this proposition for this article is the following: counting orbits inside a free homotopy class is the same, up to a change of constants, as counting orbits in an infinite string.

Proof. Let $\widetilde{\mathcal{F}\mathcal{H}}(\alpha)$ be a coherent lift of the free homotopy class and $g \in \pi_1(M)$ be a generator of the stabilizer of all the lifted orbits.

By Proposition 2.20 the elements of $\widetilde{\mathcal{F}\mathcal{H}}(\alpha)$ are all corners of a chain of lozenges. Moreover, an orbit is a corner of three or more lozenges if and only if the adjacent corners are on branching leaves (see Lemma 2.17).

From $\widetilde{\mathcal{F}\mathcal{H}}(\alpha)$ we construct a graph (V, E) in the following way:

- The vertices are the orbits;
- Two vertices are joined by an edge if they are the two corners of a lozenge.

Note that, even though we will not be using that fact here, the graph defined here is in fact a tree ([9, Proposition 2.12] or [22]).

The stabilizer G of $\widetilde{FH}(\alpha)$ contains g and acts on the graph (V, E). With the assumption that M is orientable and $\mathcal{F}^s$ is transversely orientable, then an element $h \in G$ has a fixed point if and only if $h = g^n$ for some n. We define the quotient graph (V', E') by applying the following rules:

- The vertices v_1, v_2 are identified if there exists $h \in \pi_1(M)$ such that $v_1 = h \cdot v_2$;
- Two edges are identified if their corresponding lozenges are sent one onto the other by an element of G.

Note that in the new graph (V', E'), some edges might go from a vertex to itself.

It is now easy to see that the graph (V', E') has at most a finite number of vertices of degree strictly greater than 2. Indeed, each vertex in this graph of degree > 2 is associated to an orbit which is a corner of at least 3 lozenges. Hence by Lemma 2.17, its neighboring vertices have to be on non-separated leaves, but there are only a finite number of non-separated leaves up to deck transformation (see Theorem 2.6).

Notice that (V', E') is connected. So removing all the vertices of degree > 2 from (V', E') gives a finite number of infinite connected components plus a finite number of finite connected components. Let S be one of these connected components. The only way that S can fail to project to a string of orbits is if some of the lozenges representing the edges in S share sides. But two lozenges share sides if and only if the two opposite corners are on non-separated leaves (see Lemma 2.16). So removing all the corners on non-separated leaves and their adjacent corners leaves only strings of orbits.

So we define $\mathcal{F}\mathcal{H}_{\text{finite}}(\alpha)$ as the set of all the orbits on non-separated leaves plus their adjacent orbit, i.e. the orbits that comes from corners adjacent to the one on non-separated leaves. Clearly, by construction, $\mathcal{F}\mathcal{H}(\alpha) \smallsetminus \mathcal{F}\mathcal{H}_{\text{finite}}(\alpha)$ consists of a finite number of strings of orbits.

The uniform bounds come from the fact that there are a finite number of branching leaves in M. Hence, there are only a finite number of free homotopy classes that are not just a finite non-periodic, finite periodic, infinite, or bi-infinite string of orbits. The fact that we have uniform bounds on the number of different strings is therefore immediate. $\square$

A particularly useful property for us is that strings of orbits that are finite and periodic are actually fairly special, in the sense that they are forced to stay in some topologically limited part of the manifold M:

Proposition 2.27. *Let $\{\alpha_i\}$ be a finite periodic string of orbits. Then $\{\alpha_i\}$ is a complete free homotopy class. In addition they are either entirely contained in a Seifert piece of the modified JSJ decomposition, or are the orbits on one of the quasi-transverse decomposition tori.*

Proof. Let $\widetilde{\alpha_i}$ be a coherent lift of $\{\alpha_i\}$. Let $g \in \pi_1(M)$ be a generator of the stabilizer of all the $\widetilde{\alpha_i}$ and $h \in \pi_1(M)$ such that $h \cdot \widetilde{\alpha_i} = \widetilde{\alpha}_{i+k}$. First, applying h^n, $n \in \mathbb{Z}$, to α_0 shows that the indexation i needs to be bi-infinite, and since all the $\widetilde{\alpha_i}$ are, by definition, assumed to be the corners of at most two lozenges, the part $\mathcal{F}\mathcal{H}_{\text{finite}}(\alpha_0)$ has to be empty and $\{\alpha_i\} = \mathcal{F}\mathcal{H}(\alpha_0)$, which finishes the first part.

Now, since α_0 is freely homotopic to itself, there exists a π_1-injective immersed torus that contains α_0. Using Gabai's version of the Torus theorem [30], we see that this immersed torus is either embedded or the manifold is (a special case of) Seifert-fibered. If it is the second case, we are done, and if the torus is embedded, then it can be isotoped inside a Seifert piece or to one of the modified JSJ decomposition tori (see Section 2.2 or [9]), which finishes the proof. $\square$

We defined strings of orbits in no small part in order to have the following lemma, that we will use time and time again in Section 6. But before stating it we introduce the following convention that we will use for the remainder of this article since it simplifies notations for us:

Convention 3. If $\{\alpha_i\}$ is a finite periodic, non-periodic, infinite, or bi-infinite string of orbits, we choose the indexation so that α_0 is one of the shortest orbits in the string and split the string in two so that i is always taken to be non-negative.

Notice that there are only finitely orbits in the string in $\mathcal{F}\mathcal{H}(\alpha_0)$ that can be the shortest in the string. From now on, a string of orbits will always refer to the result of applying the convention above to a finite or infinite string of orbits.

Lemma 2.28. *There exists $A > 0$, depending only on the flow, such that, if $\{\alpha_i\}$ is a string of orbits and $\{\widetilde{\alpha_i}\}$ is a coherent lift, then, for all i,*

$$d(\widetilde{\alpha}_0, \widetilde{\alpha}_i) \geq Ai.$$

Here d is the minimum distance between $\widetilde{\alpha}_0$ and $\widetilde{\alpha}_i$. In this result, we use Convention 3 so that we can write i instead of $|i|$. Notice that in this and in the following result we do not need to assume that M is orientable or any hypothesis on $\mathcal{F}^s, \mathcal{F}^u$.

Proof. This is just a consequence of Lemma 2.22. Let $\widetilde{\alpha_i}$ be a coherent lift of the α_i. There exists a uniform constant $A > 0$ such that, since the stable leaf of $\widetilde{\alpha_1}$ does not intersect the unstable leaf of $\widetilde{\alpha}_0$, $d(\widetilde{\alpha}_1, \widetilde{\alpha}_0) \geq A$. Moreover, we can choose A such that the minimum distance between the stable leaves of $\widetilde{\alpha}_i$ and $\widetilde{\alpha}_{i+2}$ is at least $2A$, because no unstable leaf intersects both the stable leaf of $\widetilde{\alpha}_i$ and $\widetilde{\alpha}_{i+2}$.

So, using the facts that $\widetilde{M} \simeq \mathbb{R}^3$, that each leaf of the lifted flow is homeomorphic to $\mathbb{R}^2$ and that the stable leaf of $\widetilde{\alpha_i}$ separates $\widetilde{M}$ in two pieces, one containing $\widetilde{\alpha}_{i-1}$ and the other $\widetilde{\alpha}_{i+1}$, we immediately obtain

$$d(\widetilde{\alpha}_0, \widetilde{\alpha}_i) \geq Ai. \qquad \square$$

And, using Proposition 2.23 instead of Lemma 2.22, we get an upper bound:

Lemma 2.29. *There exists $B > 0$, depending only on the flow, such that, if $\{\alpha_i\}$ is a string of orbits, then, for all i, there exists an homotopy H_i between α_0 and α_i that moves points a distance at most Bi.*

3. Examples of $\mathbb{R}$-covered Anosov flows on toroidal manifolds

Obviously, examples of $\mathbb{R}$-covered Anosov flows include suspensions of Anosov diffeomorphism and geodesic flows of negatively curved surfaces. But there are many more examples. For instance, the second author [19] constructed examples of $\mathbb{R}$-covered Anosov flows on hyperbolic manifolds. On the other hand, Barbot proved in [5] that the examples constructed by Handel and Thurston in [34] are $\mathbb{R}$-covered

Anosov flows on graph manifolds, i.e. manifolds such that all their pieces in their JSJ decomposition are Seifert-fibered.

But there also exist $\mathbb{R}$-covered Anosov flows on manifolds admitting all sorts of torus decomposition, i.e. with any number of Seifert fibered pieces and atoroidal pieces, including examples with only atoroidal pieces. For instance, the second author constructed in [26] examples of $\mathbb{R}$-covered Anosov flow on manifolds with some Seifert and some atoroidal pieces. We give here a slightly different construction and note that it can also yield manifolds with only atoroidal pieces.

Our construction will be based on the Foulon–Hasselblatt surgery described in [28]. One of the great advantages of that surgery is that it yields a contact Anosov flow, i.e. an Anosov flow such that its generating vector field is the Reeb field of a contact form. This is helpful in our setting because Barbot showed in [7] that contact Anosov flows are $\mathbb{R}$-covered.

Note that the results in [28] essentially imply the existence of $\mathbb{R}$-covered Anosov flows on manifolds with various torus decompositions, but this was not explicitly stated there.

The Foulon–Hasselblatt surgery is a Dehn surgery done on a tubular neighborhood of an *E-transverse Legendrian knot*. A *Legendrian knot* in a contact manifold is a closed curve tangent to the contact structure. By definition, such a curve is always transverse to the flow. It is called *E-transverse* if it is also transverse to the strong stable and strong unstable subbundles.

E-transverse Legendrian knots are very common. For instance, if ϕ^t is the geodesic flow of a negatively curved surface Σ, then one can take a closed geodesic $(c(t), \dot{c}(t)) \subset T^1\Sigma$ and rotate the tangent vectors by $\pi/2$. The curve $(c(t), \dot{c}(t) + \pi/2)$ is then a E-transverse Legendrian knot.

We can now paraphrase the Foulon–Hasselblatt construction in one theorem, restricting to the case of a geodesic flow (see Theorem 4.2 in [28]). Note that, in [28], there is a missing assumption on the E-transverse Legendrian knot to ensure that the post-surgery flow is still Anosov, but this assumption is automatically verified for the knots obtained by $\pi/2$-rotation of a geodesic.

Theorem 3.1 (Foulon, Hasselblatt [28]). *Let ϕ^t be an Anosov geodesic flow on a unit tangent bundle of a surface $M = T^1\Sigma$. Suppose that γ is a simple closed curve in M, obtained by rotating the vector direction of a geodesic by $\pi/2$. Then, for any small tubular neighborhood U of γ, half of the Dehn surgeries on U yields a manifold N that supports a contact Anosov flow ψ^t.*

Moreover, the orbits of ϕ^t that never enter the surgery locus U are still orbits of the new flow ψ^t and the contact form of ϕ^t and ψ^t are the same on $M \smallsetminus U = N \smallsetminus U$.

In particular, the Foulon–Hasselblatt surgery can be performed, either simultaneously or recursively, on a finite number of disjoint, simple, E-transverse Legendrian knots. Indeed, an E-transverse Legendrian knot that does not enter the surgery locus is still E-transverse and Legendrian for the surgered flow.

The reason we can do only half of the Dehn surgeries is that a certain positivity condition needs to be satisfied in order for the proof that the surgered flow is Anosov to work (see the proof of Theorem 4.3 in [28] or Sections 2.3 and 2.4 in [8])

We can now explain how to build an $\mathbb{R}$-covered Anosov flow such that its torus decomposition consists of one Seifert-fibered piece and one atoroidal one.

Let Σ_3 be a genus 3 surface equipped with a hyperbolic metric, and φ_0^t its geodesic flow. Let c_1 be a geodesic on Σ_3 that splits Σ_3 into two subsurfaces Σ_1, of genus 1, and Σ_2, of genus 2. Let c_2 be a geodesic that fills Σ_2 and does not intersect c_1. Here, "fills" means that any closed geodesic in Σ_2, except for c_1, intersects c_2. Now let γ_1 and γ_2 be the E-transverse Legendrian knots in $T^1\Sigma_3$ obtained by rotating the direction vector of the geodesics $(c_1, \dot{c}_1)$ and $(c_2, \dot{c}_2)$ by $\pi/2$.

Claim 3.2. *For infinitely many Foulon–Hasselblatt surgeries on γ_1 and γ_2, the resulting manifold M has a torus decomposition with one Seifert-fibered piece and one atoroidal piece.*

Proof. This is a folkloric result. For convenience of the reader we provide a proof and refer to other references for more details.

Recall that a *Dehn surgery* is the following process in a 3-manifold N: remove a solid torus neighborhood V of a simple closed curve and glue back by a homeomorphism of the boundary torus Z of V. The torus Z has a basis of its fundamental group given by a *meridian* and a *longitude*. The meridian is the unique curve up to isotopy that is not null homotopic in Z but bounds a solid disc in V. The longitude is a choice of a not null homotopic simple closed curve that has geometric intersection number one with the meridian. There are countably infinitely many homotopically distinct choices of a longitude. The surgery is determined topologically by the *new meridian*. This is the curve up to isotopy in the corresponding component of $N - \mathrm{int}(V)$ which is glued to the meridian in $Z = \partial V$.

Let N be the manifold obtained after a Foulon–Hasselblatt surgery on γ_1. The unit tangent bundle of c_1 is a separating torus T in M splitting it into two Seifert-fibered spaces N_1 and N_2 homeomorphic respectively to $T^1\Sigma_1$ and $T^1\Sigma_2$. The Dehn surgery is done in a torus neighborhood U of γ_1, where γ_1 is contained in the torus T. The boundary of U is also a torus W, which has a well defined meridian. The torus W is isotopic to one W_1 around a flow line α_1 correponding to c_1 (the unit tangent vectors to c_1 in one direction). In W_1 there is a natural longitude, which is a component of the intersection of the stable leaf of α_1 with the boundary of the torus. This then defines uniquely a longitude in W. The surgery done by Foulon–Hasselblatt is such that the new meridian in W is $(1, n)$, where $(1, 0)$ was the old meridian and $(0, 1)$ the longitude, see [28, Theorem 6.2] and [34]. By classical 3-manifold topology techniques, it is not hard to see that doing this Dehn surgery on M is the same as cutting M open along T and regluing with a Dehn twist: the direction correponding to α_1 is sent to itself, but the fiber of the Seifert fibration is taken to a curve homotopic to itself plus n iterations of α_1. As such the Seifert fibrations in each of the two pieces of N do not extend beyond T. It follows that N is a graph manifold with torus decompostion $T^1\Sigma_1$, $T^1\Sigma_2$.

Now we consider the second Dehn surgery, the one around γ_2. If U_{γ_2} is a tubular neighborhood of γ_2, then $N_2 \smallsetminus U_{\gamma_2} \simeq T^1\Sigma_2 \smallsetminus U_{\gamma_2}$ is hyperbolic (see explicit proofs in [28, Appendix B] or [26]). The proofs are done by considering a π_1-injective torus in $T^1\Sigma_2 \smallsetminus U_{\gamma_2}$ and showing that it has to be peripheral, that is, homotopic to the boundary. The important point is that γ_2 fills Σ_2, which implies the property of π_1-injective tori. Hence by the hyperbolic Dehn surgery theorem of Thurston all but a finite number of Dehn surgeries on U_2 will yield a hyperbolic manifold [52]. Therefore, for infinitely many Foulon–Hasselblatt surgeries on γ_2 in N, the surgered manifold will have a torus decomposition consisting of one atoroidal piece (coming from $T^1\Sigma_2$) and a Seifert-fibered piece homeomorphic to N_1. □

To build a contact flow on a manifold with two atoroidal pieces, we can start with Σ_4 a surface of genus 4, choose c_1, c_2 and c_3 three non-intersecting geodesics such that: c_1 splits Σ_4 in two surfaces of genus 2, and c_2 and c_3 each fills one of the split surfaces. Doing Foulon–Hasselblatt surgery on the Legendrian knots obtained from c_1, c_2 and c_3 will almost always give a contact Anosov flow on a manifold with two atoroidal pieces.

It should be clear from that construction how one can build a contact Anosov flow on a manifold with any sort of JSJ decomposition. So in summary, we have:

Theorem 3.3. *There exist contact Anosov flows (so, in particular, $\mathbb{R}$-covered Anosov flows) on manifolds with their torus decomposition consisting of any number of Seifert-fibered pieces and any number of atoroidal pieces (including only atoroidal pieces or only Seifert pieces).*

4. Classifying flows via their free homotopy classes

In this section, we first prove Theorem B.

Theorem 4.1. *Let ϕ^t be a $\mathbb{R}$-covered Anosov flow on a closed 3-manifold M. Suppose that every periodic orbit of ϕ^t is freely homotopic to* at most *a finite number of other periodic orbits. Then either ϕ^t is orbit equivalent to a suspension or ϕ^t is orbit equivalent to a finite cover of the geodesic flow of a negatively curved surface.*

Theorem 4.1 is a consequence of the following:

Theorem 4.2. *Let ϕ^t be a $\mathbb{R}$-covered Anosov flow on a closed, orientable 3-manifold M and suppose that ϕ^t is not orbit equivalent to a suspension. Suppose that $\mathcal{F}^s$ is transversely orientable. Let α be a periodic orbit of ϕ^t. Then α has only finitely many periodic orbits in its free homotopy class, if and only if α is either isotopic into one of the tori of the JSJ decomposition, or isotopic to a curve contained in a Seifert-fibered piece of the JSJ decomposition.*

Proof. Since ϕ^t is $\mathbb{R}$-covered and not orbit equivalent to a suspension, then ϕ^t has the skewed type as explained in Section 2. There are no branching leaves and hence any chain of lozenges is in fact a string of lozenges. Since ϕ^t is skewed each lift $\widetilde{\alpha}$ of α generates an infinite string of lozenges $\mathcal{C}$ in $\widetilde{M}$. Since this is a string of lozenges then a closed orbit β is in $\mathcal{F}\mathcal{H}(\alpha)$ if and only if there is a lift $\widetilde{\beta}$ that is a corner of $\mathcal{C}$. Hence $\mathcal{F}\mathcal{H}(\alpha)$ is finite if and only if the string of orbits obtained by projecting the corners of $\mathcal{C}$ to M is finite, that is, $\mathcal{F}\mathcal{H}(\alpha)$ is finite periodic. So in particular if $\mathcal{F}\mathcal{H}(\alpha)$ is finite, then Proposition 2.27 implies the result.

Let us now deal with the other direction. Suppose that up to isotopy α is on one of the tori or entirely inside a Seifert piece of the JSJ decomposition.

If α is on one of the boundary tori then as an element of $\pi_1(M)$, α is in a $\mathbb{Z}^2$ subgroup of $\pi_1(M)$. If α is contained in a Seifert piece of the JSJ decomposition, then in $\pi_1(M)$, α^2 commutes with an element representing a regular fiber of the Seifert fibration in the piece. In either case α^2 is an element of a subgroup $G \sim \mathbb{Z}^2$ of $\pi_1(M)$. Let $g \in G$ associated with α^2, and $\widetilde{\alpha}$ a lift of α to $\widetilde{M}$ left invariant by g. Let $f \in G$ not leaving $\widetilde{\alpha}$ invariant. Then

$$g(f(\widetilde{\alpha})) = f(g(\widetilde{\alpha})) = f(\widetilde{\alpha}),$$

so $\widetilde{\alpha}$ and $f(\widetilde{\alpha})$ are distinct orbits of $\widetilde{\phi}^t$ that are invariant under g non trivial in $\pi_1(M)$. This implies that $\widetilde{\alpha}$ and $f(\widetilde{\alpha})$ are connected by a chain of lozenges $\mathcal{C}_0$. This chain is a part of a bi-infinite chain $\mathcal{C}$ that is invariant by g. The transformation f acts as a translation in the corners of $\mathcal{C}$, which shows that these corners project to only finitely many closed orbits of ϕ^t in M. Therefore the string of orbits associated to $\mathcal{C}$ is finite. On the other hand, using again that the flow is $\mathbb{R}$-covered, we have that any $\beta \in \mathcal{F}\mathcal{H}(\alpha)$ has a coherent lift $\widetilde{\beta}$ to $\widetilde{M}$ such that $\widetilde{\beta}$ is a corner of this bi-infinite chain $\mathcal{C}$.

This ends the proof of Theorem 4.2. □

Now we prove Theorem 4.1.

Proof of Theorem 4.1. If a finite lift of ϕ^t is a suspension then ϕ^t itself is a suspension [23]. So we assume from now on that M is orientable and both stable and unstable foliations are transversely orientable.

Suppose that every periodic orbit of ϕ^t is freely homotopic to at most a finite number of other periodic orbits and also that ϕ^t is not orbit equivalent to a suspension. We want to show that the flow is, up to finite covers, orbit equivalent to a geodesic flow. All we have to do is to prove that the manifold M is Seifert-fibered, as a previous result of Barbot [5] (see also Ghys [31], or [9] for the generalization to pseudo-Anosov flows) yields the orbit equivalence.

Suppose that M is not Seifert-fibered. If M was hyperbolic then [19, Theorem 4.4] shows that every free homotopy class is infinite, contrary to the hypothesis. It follows that M has at least one torus in its torus decomposition. As ϕ^t is $\mathbb{R}$-covered, it is transitive (see [3]), so there exists a periodic orbit α that is neither contained in one piece of the modified JSJ decomposition nor in one of the tori of the decomposition. To build such a periodic orbit, we can start from a dense orbit and pick a long orbit segment returning inside one of the interiors of the Birkhoff annuli in a Birkhoff torus of the torus decomposition. Using the Anosov closing lemma this orbit is shadowed by a periodic orbit with the same properties. By Theorem 4.2, α has to have an infinite free homotopy class, which gives us a contradiction. □

The second author's construction in [26] gave the first explicit examples of Anosov flows such that some orbits have infinite free homotopy classes and some have finite free homotopy classes. Gathering the results of Barbot [3,5], Fenley [19] and Theorem 4.1, we can now be a bit more precise. Let us say that a flow has a *homogeneous free homotopy type* if either all the closed orbits have infinite free homotopy class or they all have finite free homotopy class.

Theorem 4.2 immediately implies the following:

Corollary 4.3. *Let ϕ^t be an $\mathbb{R}$-covered Anosov flow on M such that $\mathcal{F}^s$ is transversely orientable. Then ϕ^t has a homogeneous free homotopy type if and only if one of the following happens:*

- *M is hyperbolic (and then every closed orbit has an infinite free homotopy class);*
- *M is Seifert-fibered (and then ϕ^t is orbit equivalent to a finite cover of a geodesic flow and there exist k such that all the closed orbits have exactly k orbits in their free homotopy class);*
- *The flow is orbit equivalent to a suspension of an Anosov diffeomorphism (and then every closed orbit has a free homotopy class that is a singleton).*

If $\mathcal{F}^s$ is not transversely orientable the results holds in a double cover of M. In M itself there will be some free homotopy classes that are singletons and in the first two cases, other free homotopy classes that are not singletons.

4.1. Restrictions on infinite free homotopy classes. In this section, we prove Theorem D and then use it to show that Theorem 4.1 is "sharp", in the sense that the assumption that the flow is $\mathbb{R}$-covered cannot be dropped.

It turns out that every periodic piece except for one special case is an obstruction to having an infinite free homotopy class crossing it:

Theorem 4.4. *Suppose that $\mathcal{F}\mathcal{H}(\alpha)$ is an* infinite *free homotopy class of a periodic orbit of an Anosov flow on an orientable manifold M. Then only finitely many orbits of $\mathcal{F}\mathcal{H}(\alpha)$ can be contained in a Seifert-fibered piece of the modified JSJ decomposition. No orbit of $\mathcal{F}\mathcal{H}(\alpha)$ can cross a* periodic *Seifert-fibered piece unless that piece is a twisted I-bundle over the Klein bottle.*

Moreover, there exists a bound C, depending only on the flow and the topology of the manifold, such that if $\mathcal{F}\mathcal{H}(\alpha)$ is a free homotopy class that stays entirely inside *a Seifert piece of the modified JSJ decomposition, or crosses a periodic piece that is not a twisted I-bundle over a Klein bottle, then the number of orbits in $\mathcal{F}\mathcal{H}(\alpha)$ is less than C.*

In order to prove this theorem, we will use the following result. Recall that a Birkhoff annulus is an annulus transverse to the flow except for its boundary components that are periodic orbits (see Section 2.3).

Theorem 4.5 (Barbot, Fenley [9], Theorem B and Section 7). *Suppose that M is orientable. Let P be a periodic Seifert piece of the modified JSJ decomposition of M. There exists a two dimensional complex Z in M, called the spine of P, consisting of a* finite *union of Birkhoff annuli with boundary periodic orbits that up to powers are freely homotopic to the regular fiber of P.*

The submanifold Z is a model for the core of P in the sense that a small neighborhood $N(Z)$ of Z is a representative for the piece P. Moreover, the only periodic orbits inside $N(Z)$ are the boundary periodic orbits in Z, and all the orbits that intersect the piece P are either on one of the periodic orbits, or intersect Z in a segment entering and exiting Z transversely to the boundary.

Finally, let g in $\pi_1(M)$ associated with a periodic orbit in Z and let $\mathcal{C}'_Z$ be the tree of lozenges with corners the fixed points of powers of g and the lozenges that connect these. Let $\mathcal{C}_Z$ be the subtree of $\mathcal{C}'_Z$ that contains all the axes of the elements $f \in \pi_1(P)$ such that f acts freely on $\mathcal{C}'_Z$. This tree $\mathcal{C}_Z$ of lozenges projects to Z in the appropriate sense. Then every corner in $\mathcal{C}_Z$ is the corner of at least two lozenges. In addition unless P is a twisted I-bundle over the Klein bottle, the tree $\mathcal{C}_Z$ is not a linear tree, and there exists n such that any string of lozenges inside the chain $\mathcal{C}_Z$ contains at most n lozenges.

We explain the last statement. The set Z is a *finite* union of Birkhoff annuli, suppose there are m such annuli. If there is a string of lozenges of length more than m, it forces $\mathcal{C}_Z$ to be a linear tree. Under the hypothesis of M orientable, it was shown in [9] that this implies that P is a twisted I-bundle over the Klein bottle.

Proof of Theorem 4.4. Let $\mathcal{F}\mathcal{H}(\alpha)$ be a finite or infinite free homotopy class. Let P be a Seifert-fibered piece of the modified JSJ decomposition of M. We suppose that an orbit of $\mathcal{F}\mathcal{H}(\alpha)$ intersects P. We split the proof in two cases, depending on whether P is periodic or free.

First case. Suppose that P is periodic. We want to show that there exists a uniform bound on the number of orbits in $\mathcal{F}\mathcal{H}(\alpha)$ that can be contained in P. Let Z be the spine of P defined in Theorem 4.5. We denote by $\{\gamma_j\}_{j=1,\dots,k}$ the set of periodic orbits in Z.

If $\beta \in \mathcal{F}\mathcal{H}(\alpha)$ is contained in P, then β is one of the γ_j. This proves the first statement in the case that P is periodic.

Suppose now that β crosses P and that P is not a twisted I-bundle over the Klein bottle. Then the geometric intersection number of β with one of the boundary tori of P is non zero. Therefore the same is true for any $\gamma \in \mathcal{F}\mathcal{H}(\alpha)$ (by Lemma 2.12).

Thanks to Proposition 2.26, we can pick a string of orbits $\{\alpha_i\}$ inside $\mathcal{F}\mathcal{H}(\alpha)$, and, thanks to the uniform control given by Proposition 2.26 on the number of such strings and the number of orbits of $\mathcal{F}\mathcal{H}_{\text{finite}}(\alpha)$, finding a uniform bound for the number of orbits in the string $\{\alpha_i\}$ gives a uniform bound for the number of orbits in $\mathcal{F}\mathcal{H}(\alpha)$.

As explained previously we can assume that the α_i intersect P but are not contained in it. Since the α_i are periodic, they cannot be on the stable or unstable leaves of the $\{\gamma_j\}_{j=1,\dots,k}$. Hence, again by Theorem 4.5, each α_i intersects Z transversely. Let $\mathcal{C}_Z$ be a chain of lozenges in $\mathcal{O}$ that projects to Z given by Theorem 4.5 and $\widetilde{\alpha_i}$ be a coherent lift of the string α_i. Since α_0 intersects Z transversely, we can furthermore choose the lift $\widetilde{\alpha_i}$ such that $\widetilde{\alpha}_0$ (seen in $\mathcal{O}$) is inside one of the lozenges in $\mathcal{C}_Z$. We call that lozenge L_0 and its corners c_0 and c_1. Since $\{\widetilde{\alpha_i}\}$ are the corners of a string of lozenges, up to renaming c_0 and c_1, then c_1 has to be in the lozenge between $\widetilde{\alpha}_0$ and $\widetilde{\alpha}_1$. In particular, according to Lemma 2.18, c_1 can be the corner of at most two lozenges. So, thanks once again to Theorem 4.5, c_1 is the corner of exactly two lozenges. We call L_1 the second lozenge. The orbit α_1 is in L_1, hence we can iterate the argument above to get a third lozenge L_2 such that $L_0 \cup L_1 \cup L_2$ is a string of lozenges. Since P is not a twisted I-bundle over the Klein bottle, then Theorem 4.5 implies that the number of elements in a string of lozenges inside $\mathcal{C}_Z$ is bounded above. Therefore the number of orbits inside the string $\{\widetilde{\alpha_i}\}$ is bounded above by a uniform constant.

This finishes the proof when P is periodic.

Second case. Suppose that P is free. We want to show that there exists a uniform bound on the number of orbits in $\mathcal{F}\mathcal{H}(\alpha)$ that stay inside P.

As explained before we only need to worry about strings of orbits in $\mathcal{F}\mathcal{H}(\alpha)$. Let again $\{\alpha_i\}$ be a string of orbits in $\mathcal{F}\mathcal{H}(\alpha)$ contained in P and let $\{\widetilde{\alpha_i}\}$ be coherent lifts to the universal cover. Let $g \in \pi_1(M)$ be the common stabilizer of the $\widetilde{\alpha_i}$ and let $h \in \pi_1(M)$ be the representative of the fiber of P. Since P is a free Seifert piece, we have $hgh^{-1} = g^{\pm 1}$. Hence, $gh \cdot \widetilde{\alpha}_0 = hg^{\pm 1} \cdot \widetilde{\alpha}_0 = h \cdot \widetilde{\alpha}_0$, so g stabilizes $h \cdot \widetilde{\alpha}_0$. And, by iteration, g stabilizes $h^n \cdot \widetilde{\alpha}_0$ for any $n \in \mathbb{Z}$. Hence all the $h^n \cdot \widetilde{\alpha}_0$ are linked by a chain of lozenges. Moreover, since $\widetilde{\alpha}_0$ cannot be the corner of more than two lozenges, it follows that, for all $n \geq 0$ (or all $n \leq 0$), $h^n \cdot \widetilde{\alpha}_0 \in \{\widetilde{\alpha_i}\}$. In particular, there exists $k \in \mathbb{N}$ such that $\widetilde{\alpha}_k = h \cdot \widetilde{\alpha}_0$ (or $\widetilde{\alpha}_k = h^{-1} \cdot \widetilde{\alpha}_0$), hence the number of orbits in the string $\{\alpha_i\}$ is less than k.

All there is to show now is that k does not depend on the string $\{\alpha_i\}$, but only on h (hence only on P), which will finish the proof. Let $\tau_{\max}(h)$ be the maximum translation length of h inside P, i.e.

$$\tau_{\max}(h) := \sup_{x \in \widetilde{P}} d(x, h \cdot x),$$

where $\widetilde{P}$ is a lift of P in $\widetilde{M}$. Since P is compact, the supremum above is in fact attained, and hence finite.

By Lemma 2.28, there exists $A > 0$ uniform such that $d(\widetilde{\alpha_0}, \widetilde{\alpha_i}) > Ai$, so

$$Ak < d(\widetilde{\alpha_0}, \widetilde{\alpha_k}) = d(\widetilde{\alpha_0}, h \cdot \widetilde{\alpha_0}) \leq \tau_{\max}(h).$$

Hence $k \leq \tau_{\max}(h)/A$, so is bounded above by a uniform constant. This finishes the proof of Theorem 4.4. □

It is easy to see that this result is also true for the more general case of pseudo-Anosov flows.

Now, using the examples of totally periodic Anosov flows constructed in [9] and Theorem 4.4, we can show that Theorem 4.1 is not true for non $\mathbb{R}$-covered flows. Pseudo-Anosov flows are a generalization of Anosov flows where one allows finitely many periodic, singular p-prong orbits where $p \geq 3$ and one only assumes the existence of continuous (weak) stable/unstable foliations and not the strong stable/unstable foliations, see [25,27]. Note that by results of Inaba and Matsumoto [36] and Paternain [43] (see also [15]), for flows on 3-manifolds, being pseudo-Anosov is equivalent to being expansive.

A (pseudo-)Anosov flow on M is *totally periodic* if M is a graph manifold such that all its Seifert pieces are periodic. A consequence of Theorem 4.4 is the following:

Corollary 4.6. *Suppose that ϕ^t is a totally periodic (pseudo-)Anosov flow such that no piece of the JSJ decomposition is a twisted I-bundle over the Klein bottle. Then every periodic orbit is freely homotopic to at most a finite number of other periodic orbits (and there exists a uniform bound on the number of freely homotopic orbits).*

Proof. If necessary we can lift to a double cover such that M is orientable. The fact that this result is true also for pseudo-Anosov flows is just because Theorem 4.5 holds for pseudo-Anosov flow, hence so does Theorem 4.4. In the case of totally periodic pseudo-Anosov flows it follows that one can choose the neighborhoods $N(Z)$ of the spines Z to have boundary transverse to the flow. As explained in [9, section 7] this implies that the tree of lozenges $\mathcal{C}'_Z$ is equal to the "pruned" chain $\mathcal{C}_Z$. In particular if a periodic orbit is freely homotopic into the Seifert piece P, then it is one of the vertical orbits in Z. This proves the result for the vertical orbits in some Seifert piece. Any other orbit crosses a piece. Then again Theorem 4.4, implies the finiteness of the corresponding free homotopy class, since we assumed that no piece is a twisted I-bundle over the Klein bottle. All the bounds are global. □

Note also that Theorem 4.1 for non $\mathbb{R}$-covered flows cannot be true, even if we ask for transitivity.

Corollary 4.7. *There exist (many) non-algebraic transitive Anosov flows such that every periodic orbit is freely homotopic to at most finitely many others (and there exists a uniform bound on the number of freely homotopic orbits).*

Proof. The construction of totally periodic (pseudo)-Anosov flows described in [9, Section 8] can be done in such a way that the resulting flow is transitive, and no piece of the JSJ decomposition is a twisted I-bundle over the Klein bottle. This produces the desired examples. □

As explained in [9], the generalized Bonatti–Langevin examples studied by Barbot [6] are a particular case of totally periodic transitive Anosov flows. They are therefore examples of Anosov flows satisfying the above corollary. But if we consider only the original Bonatti–Langevin example [13], we have something even stronger:

Proposition 4.8. *Every orbit of the Bonatti–Langevin Anosov flow is alone in its free homotopy class.*

Proof. In the Bonatti–Langevin example, every orbit but one intersects a transverse torus T. If β is an orbit intersecting T then $\mathcal{F}\mathcal{H}(\beta) = \{\beta\}$ because if β is freely homotopic to some orbit γ, there is α periodic orbit such that β is freely homotopic to α^{-1}, with orientations induced by the flow and perhaps up to powers. But since T is a transverse torus to ϕ^t this cannot happen.

On the other hand if α is the single orbit not intersecting T, then α is periodic and by the above $\mathcal{F}\mathcal{H}(\alpha) = \{\alpha\}$. This proves the result. □

5. More examples of infinite free homotopy classes

Here we produce a variety of non $\mathbb{R}$-covered examples with infinite free homotopy classes. The starting point is the geodesic flow Φ_0 in the unit tangent bundle M_0 of a closed, orientable, hyperbolic surface S. In [26] the second author constructed the following examples. Let γ be a closed geodesic in S and S_1 the subsurface of S that it fills, and we assume that S_1 is not all of S. Let S_2 be the closure of $S - S_1$. For simplicity we also assume that γ is not simple, so S_1 is not an annulus either. Let α be a closed orbit of Φ_0 that projects to γ in S. Do Fried Dehn surgery along the orbit α to produce a manifold M_1 and a surgered Anosov flow Φ_1. The orbits of Φ_1 are in one to one correspondence with the orbits of Φ_0. Under a positivity condition on the Dehn surgery (satisfied by infinitely many Dehn surgery coefficients) the resulting Anosov flow Φ_1 is still $\mathbb{R}$-covered. The following was proved in [26]:

- Let β be a closed orbit of Φ_1. It corresponds to an orbit β_0 of Φ_0 and that in turn corresponds to a geodesic δ in S.

- If δ is *not* isotopic into S_2 then the free homotopy class of β with respect to the surgered flow Φ_1 is infinite.
- If δ is isotopic into S_2 then the free homotopy class of β with respect to the surgered flow Φ_1 has exactly two elements.

Theorem 5.1. *There is an infinite family of Anosov flows satisfying the following property: each flow is intransitive (hence not $\mathbb{R}$-covered) and has infinitely many orbits such that each one has infinite free homotopy class. It also has orbits with finite free homotopy classes.*

Proof. Start with the geodesic flow Φ_0 and do Fried Dehn surgery as above to obtain the Anosov flow Φ_1. Now consider a geodesic τ in S that is homotopic into S_2 and is not peripheral in S_2. Peripheral means that the curve is homotopic to the boundary. Let ω be a periodic orbit of the flow Φ_1 that corresponds to an orbit of Φ_0 that projects to τ in S. Do a blow up of this orbit, using a derived from Anosov operation. The resulting flow Φ_2 has an expanding orbit. This operation does not affect the periodic orbits β of Φ_1 which correspond to geodesics δ in S contained in S_1 and the free homotopies between the periodic orbits in $\mathcal{F}\mathcal{H}(\beta)$. If the geodesic δ in S_1 corresponding to β is not peripheral then $\mathcal{F}\mathcal{H}(\beta)$ is infinite (with respect to the flow Φ_1), and remains infinite when seeing β as an orbit of Φ_2. Now remove a solid torus neighborhood of the expanding orbit to produce a semi-flow in a manifold with torus boundary and the flow incoming along boundary. Glue a copy of this with a reversed flow such that it is exiting along the boundary, as was done by Franks and Williams in [29]. This can be done to produce a flow that is Anosov, as was carefully proved by Bonatti, Beguin and Yu in [12]. The resulting flow Φ_3 still has the orbits "β" as above and each of these orbits has an infinite free homotopy class, as do infinitely many other orbits of the flow Φ_3. By the construction the flow Φ_3 is not transitive and hence not $\mathbb{R}$-covered. It also has diversified homotopic behavior: if the orbit of Φ_1 corresponds to a peripheral curve in S_1 then the blow up operation does not affect this orbit and one can easily show that the corresponding orbit of Φ_3 has a free homotopy class with exactly two elements.

This finishes the proof of the theorem. □

We also obtain the following result:

Theorem 5.2. *There is an infinite family of transitive Anosov flows that are not $\mathbb{R}$-covered and that have infinitely many orbits, each of which has an infinite free homotopy class.*

Proof. This a modification of the construction in the previous theorem. We use the geodesic τ in S that is homotopic into S_2 and not peripheral in S_2. For simplicity assume that τ is simple. Suppose now that S_2 has high enough genus so there is another simple geodesic τ' homotopic into S_2, not peripheral and disjoint from τ. Let ω' be a periodic orbit of Φ_1 corresponding to the geodesic τ'. Besides doing the blow up of ω, we also do the blow up of ω' now to produce an attracting orbit. Remove neighborhoods of ω and ω' and glue another copy with a reversed flow. The resulting flow is denoted by Φ_4. Exactly as explained for the flow Φ_3 in the previous theorem, the flow Φ_4 has infinitely many periodic orbits with infinite free homotopy classes and also has periodic orbits with finite free homotopy classes. On the other hand since we did the blow up with both a repelling and an attracting orbit, Béguin, Bonatti and Yu [12] proved that the resulting flow is transitive. As it has a transverse torus and is not a suspension, it is not $\mathbb{R}$-covered.

This finishes the proof of the theorem. □

Now it is very easy to see that this can be iterated and blow up finitely many orbits to obtain more complicated flows with the same properties as in these two theorems.

Finally we prove the following:

Theorem 5.3. *There is an infinite family of Anosov flows each of which satisfies the following: the flow ϕ^t is transitive, and not $\mathbb{R}$-covered. The underlying manifold is not hyperbolic but has atoroidal pieces in its torus decomposition. Every free homotopy class of periodic orbits of ϕ has at most 4 elements, and every free homotopy class but two is a singleton.*

Proof. Let ϕ_0 be a suspension Anosov flow on a manifold M_0 and γ_1, γ_2 two periodic orbits of ϕ_0 that have stable and unstable leaves that are annuli. Do a blow up of both of them, turning one into a repelling orbit α_1 and the other an attracting orbit α_2. Remove neighborhoods of α_1, α_2 to produce a manifold M_1 with boundary a union of two tori T_1, T_2 and a semiflow in M_1 that is entering T_1 and exiting T_2. Glue M_1 to a homeomorphic manifold M_2 with a reversed flow. The torus T_1 in M_0 bounds a solid torus and therefore has a well defined meridian up to isotopy, that is, a curve in T_1 that bounds a disk in the solid torus. Because the stable and unstable leaves of γ_1 are annuli, there is also a well defined longitude in T_1 that is a component of the intersection of the local stable leaf of γ_1 with the torus T_1. Similarly the same happens in T_2.

The resulting flow is ϕ in the manifold $M = M_1 \cup M_2$. By results of Béguin, Bonatti and Yu [12] the gluing can be done in such a way that the resulting flow is Anosov and transitive. In addition ϕ admits two transverse tori T_1, T_2 which are not isotopic to each other. It follows that ϕ cannot be orbit equivalent to a suspension Anosov flow. In addition ϕ also cannot be $\mathbb{R}$-covered with skewed type — this is because it admits a torus transverse to the flow. It follows that ϕ is not $\mathbb{R}$-covered.

We will show that for any periodic orbit β of ϕ then the free homotopy class of β has at most 4 elements. In fact we show that $FH(\beta) = \{\beta\}$ except for one free homotopy class. We will also show that $M = M_1 \cup M_2$ is the torus decomposition of M and that M_1, M_2 are atoroidal.

We first show that for any periodic orbit β of ϕ, then $FH(\beta)$ has at most 4 elements. Suppose that there is an orbit freely homotopic to β and distinct from β. Then there is an orbit that is freely homotopic to the inverse of β as oriented curves [20]. Suppose first that β intersects T_1 or T_2. Since T_1 and T_2 are transverse to ϕ this implies that β cannot be freely homotopic to any other periodic orbit of ϕ. This can be done by looking at the algebraic intersection number with T_1 and T_2. Suppose now that β is contained in say M_1 and let A be a possibly immersed annulus that realizes a free homotopy from β to the inverse of a periodic orbit δ. Put this free homotopy in general position with $\partial M_1 = T_1 \cup T_2$. Using the fact that M is irreducible, and cut and paste techniques, we may assume that either A is contained in M_1 or there is a subannulus A_1 of A contained in M_1, such that A_1 is a free homotopy from γ to a closed curve ϵ in ∂M_1. Suppose first that A is entirely contained in M_1. In this case $\epsilon = \delta$ and the free homotopy can be blown back down to a free homotopy between orbits of ϕ_0. This can only happen if they are the same orbit of ϕ_0 as ϕ_0 is a suspension. In particular this implies that β is isotopic in M_1 to a longitude of either T_1 or T_2. In either case the curve β is peripheral in M_1. In particular if one blows back down to M_0 this produces a free homotopy between an orbit of ϕ_0 and one of the blow up orbits γ_1 or γ_2. Again since ϕ_0 is a suspension we obtain that the orbit blown down from β is either γ_1 or γ_2. Again this implies that β is isotopic in M_1 to a longitude of either T_1 or T_2. The same happens from the side of M_2 and therefore this can only happen if the longitudes were glued to each other. Notice that there are two possible such orbits β in M_1: these are the two closed orbits obtained by blowing either γ_1 or γ_2 into 3 periodic orbits and then removing the original orbits γ_1 or γ_2 when removing the solid tori. Therefore this implies that the free homotopy class of β has at most 4 elements. This can only happen for the periodic orbits obtained by blowing up γ_1 or γ_2. So there are two free homotopy classes with four elements. Every other free homotopy class is a singleton. This proves the statement about free homotopy classes.

Let us now prove the statement about the JSJ decomposition of M. We will show that M_1 (and consequently M_2) is atoroidal. Let T be an incompressible torus in M_1. Since T_1, T_2 are incompressible in M, then T is also an incompressible torus in M. As ϕ is not orbit equivalent to a suspension, T can be homotoped into a Birkhoff torus. In other words, T can be a realized as a free homotopy from an orbit to itself. But we just proved above that the only free homotopies are between the blow up orbits from ϕ_0. This shows that T is homotopic and hence isotopic into either T_1 or T_2. This shows that M_1 is atoroidal. This finishes the proof of the theorem. $\square$

6. Growth of period of orbits in strings of closed orbits

We now start the second part of this article, where we study orbits inside a free homotopy class. This section contains the bulk of the work of the second part of the article. Here we prove Theorem E, i.e. that inside an infinite string of orbits, the period grows at least linearly and at most exponentially. We fix a Riemannian metric g on M. We denote by d the distance in M for that particular metric.

Recall (see Remark 1.1) that we may, and will, always choose our metrics and Anosov flows in such a way that the orbits are unit speed, since this does not change the results of Theorems E, F, and G.

We will use the following notations. For any curve c in M, we write $l(c)$ for the length of the curve. In addition if c is a path in the universal cover $\widetilde{M}$, which is the a lift of a closed curve α in M, by $l(c)$ we always mean the length of the corresponding curve α. Recall also that the Hausdorff distance between two sets S_1, S_2 is defined in the following way

$$d_{\mathrm{Haus}}(S_1, S_2) := \max\Big\{ \sup_{x\in S_1} \inf_{y\in S_2} d(x,y), \sup_{y\in S_2} \inf_{x\in S_1} d(x,y)\Big\},$$

and when talking about the distance between two sets, we mean the minimal distance, i.e.

$$d(S_1, S_2) := \inf\big\{d(x,y) \mid x \in S_1, y \in S_2\big\}.$$

We start by stating the result for the upper bound on the length growth, which is the easiest result.

Theorem 6.1. *Let ϕ^t be an Anosov flow in M^3. Let $\{\alpha_i\}_{i\in I}$ be a string of orbits indexed such that α_0 is the shortest. Then the length growth is at most exponential in i. More precisely, there exists $C_1, C_2 > 0$, depending only on the flow and the manifold such that, for all $i \in I$*

$$l(\alpha_i) \leq C_1 l(\alpha_0) e^{C_2 i}.$$

Proof. Let $\widetilde{\alpha_i}$ be a coherent lift of the string α_i and γ the element of $\pi_1(M)$ fixing all of the $\widetilde{\alpha_i}$.

According to Lemma 2.29 there exists $B > 0$ depending only on the flow and the manifold such that there exists an homotopy $H_i(s,t)$, $0 \leq s \leq 1$, $0 \leq t \leq 1$, from $\alpha_0 = H_i(0,\cdot)$ to $\alpha_i = H_i(1,\cdot)$ that moves points a distance at most Bi (that is, for any t, the length of $H_i(\cdot,t)$ is bounded above by Bi).

Let $\widetilde{H_i}$ be a lift of H_i from $\widetilde{\alpha_0}$ to $\widetilde{\alpha_i}$. Let $x = \widetilde{H_i}(0,0) \in \widetilde{\alpha_0}$ and $y = \widetilde{H_i}(1,0) \in \widetilde{\alpha_i}$ be fixed. Let c_0 be the part of $\widetilde{\alpha_0}$ from x to $\gamma\cdot x$ and c_i the part of $\widetilde{\alpha_i}$ from y to $\gamma \cdot y$. Then $\widetilde{H_i}$ is a free homotopy from c_0 to c_i that moves points by at most Bi.

Hence, c_i is included in $N(c_0, Bi)$, the tubular neighborhood of c_0 of radius Bi in $\widetilde{M}$.

We are going to show that the length of c_i (that is, the length of α_i) cannot get too big, because it stays in a part of $\widetilde{M}$ that has a bounded volume (depending on i)

There exists constants $C_1, C_2 > 0$, depending only on the metric on M (in fact just on a lower bound for the curvature), such that the volume of balls in $\widetilde{M}$ of radius r is bounded above by $C_1 e^{C_2 r}$.

Hence,

$$\mathrm{Vol}\big(N(c_0, Bi)\big) \leq l(c_0) C_1 e^{C_2 Bi},$$

where $l(c_0)$ is the length of c_0.

Thanks to Anosov's closing lemma, there exists $\varepsilon > 0$ depending only on the flow, such that, for any orbit $\widetilde{\alpha}$ in $\widetilde{M}$, the tubular neighborhood $N(\widetilde{\alpha}, \varepsilon)$ of $\widetilde{\alpha}$ of radius ε is an *embedded* solid tube in $\widetilde{M}$. Indeed, otherwise the Anosov closing lemma would imply the existence of a closed orbit in $\widetilde{M}$, which is impossible.

Hence, $N(c_i, \varepsilon)$, the tubular neighborhood of c_i of radius ε is embedded in, up to replacing B by $B + \varepsilon$, $N(c_0, Bi)$. So $\mathrm{Vol}\,(N(c_i, \varepsilon)) \leq \mathrm{Vol}\,(N(c_0, Bi))$.

Now, thanks to classical Riemannian comparison theorems, the volume of the embedded tubular neighborhood $N(c_i, \varepsilon)$, will be, for ε small enough, controlled above and below by a term of the form $l(c_i)K(\varepsilon)$, where $K(\varepsilon)$ is either the maximum (for the control above) or the minimum (for the control below) of the area of an embedded disk of radius ε. These maximum and minimum can, in turn, be controlled in terms of bounds of the curvature of the metric, and the area of such a disk in the corresponding model space. In particular, there exists $\varepsilon' > 0$, depending only on ε and the (bounds on the curvature of the) metric on M, such that

$$l(c_i)\varepsilon' \leq \mathrm{Vol}\big(N(c_i, \varepsilon)\big) \leq \mathrm{Vol}\big(N(c_0, Bi)\big) \leq l(c_0)C_1 e^{C_2 Bi}.$$

So up to renaming the constants C_1 and C_2, we get, as claimed,

$$l(\alpha_i) = l(c_i) \leq l(c_0)C_1 e^{C_2 Bi} = l(\alpha_0)C_1 e^{C_2 Bi}. \qquad \square$$

Now we state the result for the lower bound on the growth of period inside a string of lozenges. The proof is much more involved, as t depends in a delicate way on the topology and geometry of M or its pieces. Establishing that result will take the next three subsections.

Theorem 6.2. *Let ϕ^t be an Anosov flow in M^3. Let $\{\alpha_i\}$ be a string of orbits of ϕ^t, with the indexation chosen so that α_0 is the shortest orbit. Then the length growth is at least:*

(1) *Exponential in i if the manifold M is hyperbolic;*

(2) *Quadratic in i if the $\{\alpha_i\}_{i\in\mathbb{N}}$ intersects an atoroidal piece of the JSJ decomposition of M;*

(3) *Linear in i if $\{\alpha_i\}_{i\in\mathbb{N}}$ goes through two consecutive Seifert-fibered pieces of the JSJ decomposition of M.*

Remark 6.3. In some sense the theorem has content only when the string is infinite, since with big enough constants this is trivial for any finite string. We will however see in the next subsections that we can get explicit bounds on the length of α_i depending only on the length of a shortest orbit in the string.

For M Seifert fibered every free homotopy class is finite and uniformly bounded in cardinality. So in the proof we may assume that M is not Seifert fibered. Therefore, using the geometrization theorem, then either (i) M is hyperbolic; or (ii) M is not hyperbolic, but has an atoroidal piece in its JSJ decomposition; or (iii) M is a graph manifold and has at least one torus in its torus decomposition. In the third case of the above theorem, the two consecutive Seifert pieces may be the same piece, but in that case it is assumed that the Seifert fibration does not extend across the gluing torus.

Also up to a double cover we may assume that M is orientable. This does not affect possibilities (1)–(3), up to changing the constants involved.

Remark 6.4. By Lemma 2.12 if an orbit α_j in the string *crosses* a torus T of the JSJ decomposition then every orbit in its free homotopy class also crosses T. The remaining case is that distinct orbits in the string $\{\alpha_i\}$ may be contained in distinct pieces of the modified JSJ decomposition. Again by Lemma 2.12 as we move through the string (say increasing i) the orbits can only change the pieces they are contained in at most two times. So in any case we may choose a substring still denoted by $\{\alpha_i\}$ such that every orbit in this string is contained in the same piece of the modified JSJ decomposition.

6.1. Hyperbolic case. We first start with the hyperbolic case, which is both the easiest and the one for which the period growth is the fastest.

Proposition 6.5. *Let $\{\alpha_i\}$ be a string of orbits of an Anosov flow on M. If M is hyperbolic, then there exist constants $A, B > 0$, independent of the homotopy class and D_{α_0} depending on α_0 such that*

$$l(\alpha_i) \geq Be^{-D_{\alpha_0}}e^{Ai}.$$

In order to prove this proposition, we recall the following classical lemma of hyperbolic geometry (see for instance [40, Proposition 3.9.11])

Lemma 6.6. *Let $c(t)$, $t \in \mathbb{R}$ be a geodesic of $\mathbb{H}^n$. Let $c_1(t)$, $a \leq t \leq b$, be a curve. Let p, resp. q, be the orthogonal projection of $c_1(a)$, resp. $c_1(b)$, onto c. Suppose that $d(c_1(a), p) = d(c_1(b), q) \geq k$ and that $d(c_1(t), c) \geq k$, for all $a \leq t \leq b$. Then*

$$l(c_1) \geq d(p,q)\cosh k.$$

Proof of Proposition 6.5. We first fix a hyperbolic metric on M. As previously mentioned, by reparametrization, we can now assume that the flow is unit speed for that particular metric. Let $\{\widetilde{\alpha_i}\}$ be a coherent lift of the $\{\alpha_i\}$ and g be a generator of the stabilizer in $\pi_1(M)$ of all $\widetilde{\alpha_i}$. Since g preserves all of the $\widetilde{\alpha_i}$, these curves have the same endpoints on the boundary at infinity $\partial_\infty \mathbb{H}^3$. Let c_g be the axis of g acting on $\mathbb{H}^3$, or equivalently the geodesic with the same two endpoints as the $\widetilde{\alpha_i}$. Since c_g and $\widetilde{\alpha}_0$ have the same endpoints on the boundary at infinity, they are a bounded Hausdorff distance from each other. We denote by D_{α_0} that distance, that is, $D_{\alpha_0} = d_{\mathrm{Haus}}(c_g, \widetilde{\alpha}_0)$.

By Lemma 2.28, there exists a constant $A > 0$, depending only on the flow, such that the minimal distance between $\widetilde{\alpha_i}$ and $\widetilde{\alpha}_0$ is at least Ai. Therefore, the distance between $\widetilde{\alpha_i}$ and c_γ is bounded below by $Ai - D_{\alpha_0}$. Let x be a point on c_g. Define x_i

as the point on $\widetilde{\alpha_i}$ such that the orthogonal projection of x_i onto c_g is x, and (in case there is more than one such point), we take x_i to be a point closest to c_g (but any other choice works as well).

Now $l(\alpha_i)$ is equal to the length of the part of the curve $\widetilde{\alpha_i}$ between x_i and $g \cdot x_i$. Therefore, by Lemma 6.6, we get that

$$l(\alpha_i) \geq d(x, g \cdot x) \cosh(Ai - D_{\alpha_0}) \geq \frac{l(c_g)}{2} e^{Ai} e^{-D_{\alpha_0}}.$$

Replacing $l(c_g)$ by the length of the smallest geodesic in M, we obtain the existence of a universal constant $B > 0$ such that, for all i,

$$l(\alpha_i) \geq B e^{-D_{\alpha_0}} e^{Ai}. \qquad \square$$

Remark 6.7. In order to later obtain counting results with uniform control, we need to give an explicit control of D_{α_0} in terms of $l(\alpha_0)$. The concern here is the following. If D_{α_0} is very big, this means that the curve $\widetilde{\alpha}_0$ has pieces at least D_{α_0} away from c_g and possibly all of $\widetilde{\alpha}_0$ is at least D_{α_0} from c_g. By the lemma this implies that α_0 may have a huge length. Therefore the exponential growth of $l(\alpha_i)$ *with respect to* $l(\alpha_0)$ takes much longer to kick in in terms of i and hence this growth is not uniform amongst strings of orbits. Notice that for example in the case of $\mathbb{R}$-covered Anosov flows in hyperbolic 3-manifolds with $\mathcal{F}^s$ transversely oriented, *every* periodic orbit generates an infinite string of orbits. Therefore there may be infinitely many different strings of orbits (in fact, our bound will prove that there *must be* infinitely many, see Theorem 7.1). To get uniform control we will split α_i into pieces depending on how big D_{α_0} is and also *how much of* $\widetilde{\alpha}_i$ is near c_g or far from c_g. The downside is that, to get this uniform control, we obtain a worse bound of the growth of $l(\alpha_i)$ than in the proposition above.

Lemma 6.8. *Let $\{\alpha_i\}$ be a string of orbits as above. Let a be the length of the shortest geodesic in M. If $l(\alpha_0) < t$, with $t >$ $\max(4, ae/2)$, then, for all i,*

$$l(\alpha_i) \geq B e^{-\sqrt{t}\log(2t/a)} e^{Ai}.$$

Proof. Notice that we only have to worry about the case that D_{α_0} is big, for otherwise the result is immediate given the previous lemma. Let $\widetilde{\alpha}_0$ and c_g be as in the preceding proof. Recall that D_{α_0} is the Hausdorff distance between $\widetilde{\alpha}_0$ and c_g.

First, suppose that $d(\widetilde{\alpha}_0, c_g) > D_{\alpha_0}/\sqrt{t}$. Then, by Lemma 6.6, we have

$$l(\alpha_0) \geq \frac{l(c_g)}{2} e^{D_{\alpha_0}/\sqrt{t}}.$$

So, if a is the length of the shortest geodesic in M, we get

$$e^{-D_{\alpha_0}} \geq \left(\frac{l(c_g)}{2l(\alpha_0)}\right)^{\sqrt{t}} \geq \left(\frac{a}{2t}\right)^{\sqrt{t}}.$$

Hence, by Proposition 6.5, for all i,

$$l(\alpha_i) \geq Be^{-D_{\alpha_0}}e^{Ai} \geq Be^{-\sqrt{t}\log(2t/a)}e^{Ai}.$$

And the lemma is proved in that case.

Now suppose that $d(\widetilde{\alpha_0}, c_g) \leq D_{\alpha_0}/\sqrt{t}$. We then cut $\widetilde{\alpha_0}$ in two (not necessarily connected) pieces (see Figure 7): Let β_0 be the set of points of $\widetilde{\alpha_0}$ that are at most $D_{\alpha_0}/\sqrt{t}$ from c_g, and let γ_0 be the closure of $\widetilde{\alpha_0} \smallsetminus \beta_0$, i.e. γ_0 is the piece of $\widetilde{\alpha_0}$ such that $d(\gamma_0, c_g) \geq D_{\alpha_0}/\sqrt{t}$. By our assumption, β_0 is not empty. And since D_{α_0} is the Hausdorff distance between $\widetilde{\alpha_0}$ and c_g, γ_0 cannot be empty either (because $t > 1$, so $D_{\alpha_0}/\sqrt{t} < D_{\alpha_0}$).

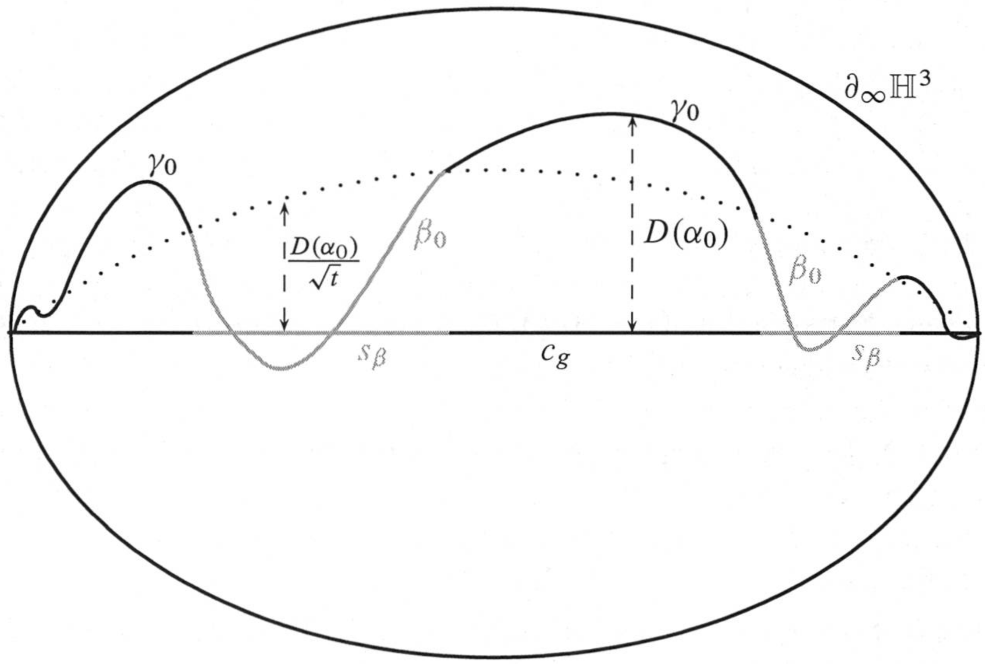


Figure 7. The splitting of the orbit $\widetilde{\alpha_0}$ and the geodesic c_g.

We fix a fundamental domain Ω of $\widetilde{\alpha_0}$ under the action of g. Let s_β be the orthogonal projection of $(\beta_0 \cap \Omega)$ onto c_g and let s_γ be the orthogonal projection of $(\gamma_0 \cap \Omega)$ onto c_g. We write d_β for the length of s_β and d_γ for the length of s_γ. Clearly, $d_\beta + d_\gamma \geq l(c_g)$, so either $d_\beta \geq l(c_g)/2$ or $d_\gamma \geq l(c_g)/2$.

First case. Suppose that $d_\beta \geq l(c_g)/2$. In this case we can redo the proof of Proposition 6.5 for the parts of $\widetilde{\alpha_i}$ that are far enough from c_g: Each curve $\widetilde{\alpha_i}$ has the same endpoints as c_g, hence the orthogonal projection to c_g is surjective. For each i let β_i be the inverse image of s_β of this orthogonal projection.

Since $d(\beta_i, \beta_0) > Ai$ and $d_{\text{Haus}}(\beta_0, c_g) \leq D_{\alpha_0}/\sqrt{t}$, we obtain that $d(\beta_i, c_g) > Ai - D_{\alpha_0}/\sqrt{t}$. Moreover, by construction, the length of the orthogonal projection of β_i onto c_g is at least d_β. Then applying Lemma 6.6 again, we get that

$$l(\alpha_i) \geq l(\beta_i) \geq \frac{d_\beta}{2} e^{Ai - D_{\alpha_0}/\sqrt{t}} \geq \frac{l(c_g)}{4} e^{Ai} e^{-D_{\alpha_0}/\sqrt{t}}.$$

Now, since $D_{\alpha_0} = d_{\mathrm{Haus}}(\widetilde{\alpha}_0, c_g)$, then γ_0 contains a curve that has to go from the annulus of radius $D_{\alpha_0}/\sqrt{t}$ around c_g to the annulus of radius D_{α_0} around c_g. So,

$$D_{\alpha_0}\left(1 - \frac{1}{\sqrt{t}}\right) \leq \frac{l(\gamma_0)}{2} \leq \frac{l(\alpha_0)}{2} < \frac{t}{2}.$$

Taking $t > 4$, we get that $D_{\alpha_0} < t$. Using this and the previous inequality, we get

$$l(\alpha_i) \geq \frac{l(c_g)}{4} e^{Ai} e^{-D_{\alpha_0}/\sqrt{t}} \geq \frac{l(c_g)}{4} e^{Ai} e^{-\sqrt{t}}.$$

So for some universal constant $B > 0$, we get

$$l(\alpha_i) \geq B e^{Ai} e^{-\sqrt{t}},$$

hence the lemma follows for $t > ae^1/2$.

Second case. Suppose that $d_\beta < l(c_g)/2$. It follows that $d_\gamma \geq l(c_g)/2$. Applying Lemma 6.6 once again, we get

$$l(\alpha_0) \geq l(\gamma_0) \geq \frac{l(c_g)}{2} e^{D_{\alpha_0}/\sqrt{t}},$$

So,

$$e^{-D_{\alpha_0}} \geq \left(\frac{l(c_g)}{2l(\alpha_0)}\right)^{\sqrt{t}} \geq \left(\frac{a}{2t}\right)^{\sqrt{t}}.$$

And finally,

$$l(\alpha_i) \geq B e^{-D_{\alpha_0}} e^{Ai} \geq B e^{-\sqrt{t}\log(2t/a)} e^{Ai}.$$

This finishes the proof of the lemma. □

Remark 6.9. The choice of the function $D_{\alpha_0}/\sqrt{t}$ as the transition function from being near c_g to being far from c_g is to some extent arbitrary. Possibly different choices of the transition function could lead to a better inequality in Lemma 6.8. However, it is not clear how to make a better choice, or if it is even possible with that proof. Indeed, if one takes a bigger transition function, say $D_{\alpha_0}/2$, then we get a better bound (in $1/t$) for the part of $\widetilde{\alpha}_0$ that is far from c_g, but a far worse (in fact exponential) bound for the part that is close. Whereas if one takes a smaller transition function, say D_{α_0}/t, then the situation is reversed. In particular, none of these other choices would be good enough to obtain Theorem 7.4, i.e. the answer to Question 1, even though the constants would still be uniform. It is very natural to try bounds of the form D_{α_0}/t^k. The first bound above corresponds essentially to $k = 0$ and the second to $k = 1$. Neither works for Theorem 7.4, and we are lead to $0 < k < 1$. With the transition function D_{α_0}/t^k, if one follows the proof of the proposition from near and far from the geodesic c_g the following happens: One gets a bound in terms of $\exp(-t^k \log t)$ and another in terms of $\exp(-t^{1-k} \log t)$. So clearly the optimal bound *for these types of transition functions* occurs when $k = 1/2$.

6.2. One atoroidal piece.

Proposition 6.10. *Suppose that the orbits $\{\alpha_i\}$ are all entirely contained in an atoroidal piece N of the modified torus decomposition of M or they all cross into this piece N. Then there exists $B > 0$ depending only on M and the flow, and $D_{\alpha_0} > 0$ depending on α_0 such that*

$$l(\alpha_i) \geq B i^2 e^{-D_{\alpha_0}}.$$

In order to prove the above proposition, we need a result on neutered manifolds.

Definition 6.11. A compact manifold N is a *neutered hyperbolic manifold* if $N = V \smallsetminus H$ where V is a complete hyperbolic manifold of finite volume and H is the interior of a disjoint union of horoball neighborhoods centered at the cusps.

The *neutered metric* on N, denoted by d_n, is the path metric obtained from the hyperbolic Riemannian metric in N. We also lift the Riemannian metric to $\widetilde{N}$ and again denote by d_n the path metric in $\widetilde{N}$ (for this lifted Riemannian metric).

The hyperbolic metric on V induces another metric on N, that we denote by d_h. We also write d_h for the metric on $\widetilde{N}$ induced by the hyperbolic metric on $\widetilde{V} = \mathbb{H}^3$. Here we think of $\widetilde{N}$ as a subset of $\widetilde{V} \subset \mathbb{H}^3$.

Note that we may always choose the horoball neighborhoods such that they are spaced at least one unit from each other and we will always assume that in the following. Let $\pi\colon \widetilde{V} \to V$ be the universal cover.

Lemma 6.12. *In $\widetilde{N}$ the following holds: $d_h \leq d_n \leq 2\sinh(d_h/2)$.*

Proof. The first inequality is trivial, so we only prove the second. Let x_1, x_2 be points in $\widetilde{N}$. Let c be the *hyperbolic* geodesic arc from x_1 to x_2. The hyperbolic distance from x to y is exactly the hyperbolic length of c. As long as c is disjoint from $\pi^{-1}(H)$ then c is contained in $\widetilde{N}$ and the hyperbolic length along c is the same as the neutered length. So we suppose this is not the case and let β be the closure of a component of $c \cap \pi^{-1}(H)$. In the upper half space model we can assume that the removed horoball containing β is associated with infinity. We may furthermore assume that the horosphere bounding that horoball is the set of points where $z = 1$. By rotations and translations we can assume that β is actually in $\mathbf{H}^2$ and connects the points $a_0 = \frac{x}{2} + i$ and $a_1 = \frac{-x}{2} + i$ in the upper half plane. Let ρ be the hyperbolic length of β, which is the same as the hyperbolic distance from a_0 to a_1. So ρ is given by

$$\sinh \frac{\rho}{2} = \frac{x}{2}.$$

Notice that x is exactly the length of a segment in the boundary of the horoball, and that is also the neutered length of this segment. Hence any segment β of length ρ can be replaced by a segment in $\widetilde{N}$ of neutered length $x = 2\sinh\frac{\rho}{2}$. The inequality of the lemma follows. $\square$

Proof of Proposition 6.10. We are going to prove that the corresponding parts of the orbits α_i that are inside the atoroidal piece N grow quadratically with the index i. So, if the orbits $\{\alpha_i\}$ are not entirely contained in N, we consider the curves δ_i obtained in the following manner. First, we fix a generator of the fundamental group of each of the decomposition tori (so this is independent of the orbits α_i). Then, by Lemma 2.12, for each i, there exists α_i^N a connected component of $\alpha_i \cap N$ such that each α_i^N are freely homotopic to each other relative to the boundary of N. Let T_1, T_2 be the boundary tori of N that the curves α_i^N intersects.

(1) If $T_1 = T_2$, then we close each α_i^N along a geodesic segment on the torus between its two endpoints, making sure that we choose each geodesic segments in a coherent way, i.e. making sure that the closed paths δ_i are still pairwise freely homotopic to each other.

(2) If $T_1 \neq T_2$, then we close up α_i^N by adding loops l_i^1 and l_i^2, starting at the endpoints of α_i^N, and in the free homotopy class of the fixed generator chosen above of, respectively, T_1 and T_2. Moreover, we choose l_i^1 and l_i^2 to be of minimal length in their homotopy class, so that their length is bounded above by a constant depending *only* on the flow and the manifold.

The path δ_i is obtained by concatenation of l_i^1, α_i^N, l_i^2, and $-\alpha_i^N$.

If the orbits $\{\alpha_i\}$ are contained in N, then we write $\delta_i = \alpha_i$. For convenience, we note that the important features of the δ_i are:

- For all i, $\delta_i \subset N$,
- The curves δ_i are freely homotopic in N,
- The length of $\delta_i \smallsetminus \alpha_i^N$, that is, the length of the pieces of the curve which are not part of an orbit of the flow, are bounded independently of i and independently of the family $\{\alpha_i\}$. Indeed, setting D'

$$D' = \max_{\epsilon} \sup_{c} \inf \big\{ l(d) \mid d \text{ is homotopic to } c \text{ with fixed base point } c(0) \big\},$$

  where ϵ runs over the chosen generators of each of the decomposition tori, and $c\colon [0,1] \to N$ runs over all the curves in the free homotopy class of ϵ, then $l(\delta_i \smallsetminus \alpha_i^N) \leq 2D'$.

We choose a metric on M such that the atoroidal piece N is a neutered hyperbolic manifold. Let d_n, and d_h be the neutered and hyperbolic metrics in N, as in the definition of neutered manifolds.

Let $\widetilde{\delta_i}$ be coherent lifts of the δ_i to the universal cover $\widetilde{N}$ of N. Recall that, Lemma 2.28 gives a uniform $A > 0$, such that the distance in $\widetilde{M}$ between $\widetilde{\alpha_i}$ and $\widetilde{\alpha_0}$ is greater than Ai (and hence the same inequality is true for the d_n distance for the parts of $\widetilde{\alpha_i}$ and $\widetilde{\alpha_0}$ that stays in $\widetilde{N}$). So, the minimal separation for the neutered distance d_n in $\widetilde{N} \subset \widetilde{M}$ between $\widetilde{\delta_i}$ and $\widetilde{\delta_0}$ is at least $Ai - 4D'$, since $l(\delta_i \smallsetminus \alpha_i^N) \leq 2D'$. For convenience, we write $d_n^i := d_n(\widetilde{\delta_0}, \widetilde{\delta_i})$, and, setting $D = 4D'$, we have that $d_n^i > Ai - D$.

Since $\widetilde{N}$ is contained in $\widetilde{V} = \mathbb{H}^3$, we can use the boundary at infinity of $\mathbb{H}^3$: Let g be the element of $\pi_1(N)$ that leaves invariant every $\widetilde{\delta_i}$. Since $N = V \smallsetminus H$, we can see $\pi_1(N)$ as a group of isometries of $\mathbb{H}^3$. More precisely each element of $\pi_1(N)$ seen as a covering translation of $\widetilde{N}$ is the restriction of a hyperbolic isometry of $\mathbb{H}^3$ to $\widetilde{N} \subset \mathbb{H}^3$. So we call c_g the hyperbolic geodesic in $\mathbb{H}^3$ representing g. The geodesic c_g is in general not contained in $\widetilde{N}$, but c_g has the same endpoints as the δ_i on the boundary at infinity $\partial_\infty \mathbb{H}^3$.

Let $d_h^{i,0}$ be the minimum hyperbolic distance between points in $\widetilde{\delta_i}$ and c_g. Let $l_h(c_g)$ be the hyperbolic length of c_g/g, or in other words, $l_h(c_g)$ is the translation length of the hyperbolic element g. Let $l_n(\delta_i)$ be the neutered length of δ_i (which is the same as its length for the Riemannian metric on M), and $l_h(\delta_i)$ be its hyperbolic length. Note that, since δ_i stays in N, its neutered and hyperbolic length are the same. Thanks to Lemma 6.6, we then have

$$l_n(\delta_i) = l_h(\delta_i) \geq l_h(c_g) \cosh d_h^{i,0} \geq l_h(c_g) \frac{e^{d_h^{i,0}}}{2}.$$

In addition if d_n^i is the minimum neutered distance from $\widetilde{\delta_i}$ to $\widetilde{\delta_0}$ and d_h^i is the corresponding minimum hyperbolic distance then by Lemma 6.12, $d_n^i \leq 2\sinh(d_h^i/2)$. Hence

$$e^{d_h^i/2} \geq d_n^i$$

Let $D_{\alpha_0} := d_{\mathrm{Haus},H}(\widetilde{\delta_0}, c_g)$ be the Hausdorff distance for the hyperbolic metric between $\widetilde{\delta_0}$ and c_g. Then

$$d_h^i = d_h(\widetilde{\delta_0}, \widetilde{\delta_i}) \leq d_h(\widetilde{\delta_i}, c_g) + D_{\alpha_0} = d_h^{i,0} + D_{\alpha_0},$$

so

$$\begin{aligned} l_n(\delta_i) \geq l_h(c_g) \frac{e^{d_h^{i,0}}}{2} &\geq \frac{l_h(c_g)}{2} e^{d_h^i} e^{-D_{\alpha_0}} \\ &\geq \frac{l_h(c_g)}{2} e^{-D_{\alpha_0}} (d_n^i)^2 \geq \frac{l_h(c_g)}{2} e^{-D_{\alpha_0}} (Ai - D)^2. \end{aligned}$$

Replacing $l_h(c_g)$ by the minimal translation length of the hyperbolic isometries in $\pi_1(N)$, we can find a constant $B > 0$ depending only on the manifold and the flow such that

$$l_n(\delta_i) \geq Be^{-D_{\alpha_0}}i^2.$$

Since $l_n(\delta_i \setminus \alpha_i^N)$ has length bounded by a uniform constant, we have that for some uniform constant $C > 0$, $l(\alpha_i) \geq C l_n(\delta_i)$. So, replacing the constant B above by B/C, we obtain

$$l(\alpha_i) \geq Be^{-D_{\alpha_0}}i^2. \qquad \square$$

This finishes the proof of Proposition 6.10.

We will later need to control that the constant D_{α_0} obtained in the Proposition 6.10 does not get too big with $l(\alpha_0)$. The following lemma deals with that and its proof is essentially the same as the Lemma 6.8

Lemma 6.13. *Let $\{\alpha_i\}$ and β be as in Proposition 6.10. There exists a uniform constant $C > 0$ such that, if $l(\alpha_0) < t$, with $t > \max(4, Ce^{1/C})$, then, for all i,*

$$l(\alpha_i) \geq Be^{-\sqrt{t}\log(t/C)}i^2.$$

Proof. We use the same notations as in the proof of Proposition 6.10. In particular, h indices refer to distance computed in the hyperbolic metric, while n indices refer to the neutered metric. Length without any index refers to the length in M. Also recall that the curves $\{\delta_i\}$ are contained in N.

First, suppose that $d_h(\widetilde{\delta_0}, c_g) > D_{\alpha_0}/\sqrt{t}$. Then, applying once more Lemma 6.6, we have

$$l_n(\delta_0) = l_h(\delta_0) \geq l_h(c_g)e^{D_{\alpha_0}/\sqrt{t}},$$

where the first equality comes from the fact that δ_0 is entirely in N, hence its neutered and hyperbolic length are equal. Recall that by construction of δ_i, there exists a uniform constant $C > 0$ such that $l(\alpha_i) \geq Cl(\delta_i)$. Setting a to be the smallest translation length of hyperbolic elements in $\pi_1(N)$, we get

$$e^{-D_{\alpha_0}} \geq \left(\frac{l_h(c_g)}{l_n(\delta_0)}\right)^{\sqrt{t}} \geq \left(\frac{Ca}{t}\right)^{\sqrt{t}}.$$

Hence, by Proposition 6.10, for all i,

$$l(\alpha_i) \geq Bi^2e^{-D_{\alpha_0}} \geq Bi^2e^{-\sqrt{t}\log(t/Ca)}$$

And the lemma is proved in that case, up to changing C to aC.

Now suppose that $d_h(\widetilde{\delta_0}, c_g) \leq D_{\alpha_0}/\sqrt{t}$. Then, as in the proof of Lemma 6.8, we let β_0 be the piece of $\widetilde{\delta_0}$ such that its Hausdorff hyperbolic distance is at most $D_{\alpha_0}/\sqrt{t}$, and let γ_0 be the closure of $\widetilde{\delta_0} \setminus \beta_0$, i.e. γ_0 is the piece of $\widetilde{\alpha_0}$ such that

$d_h(\gamma_0, c_g) \geq D_{\alpha_0}/\sqrt{t}$ (see Figure 8). By our assumption, β_0 is not empty. And since D_{α_0} is the Hausdorff distance between $\widetilde{\delta}_0$ and c_g, γ_0 cannot be empty either (because we can assume $t > 1$, so $D_{\alpha_0}/\sqrt{t} < D_{\alpha_0}$).

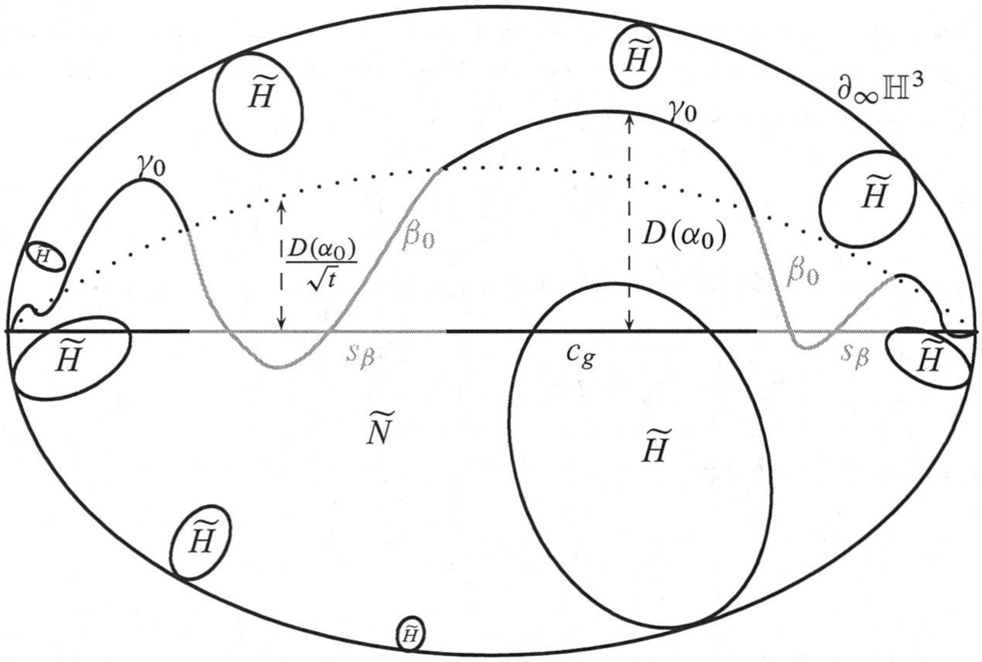


Figure 8. The orbit $\widetilde{\delta}_0$ in $\widetilde{N} = \widetilde{V} \smallsetminus \widetilde{H}$ and the geodesic c_g in $\widetilde{V} = \mathbb{H}^3$.

Let Ω be a fundamental domain of $\widetilde{\delta}$ under the action of g. Let s_β be the orthogonal projection (for the hyperbolic metric) of $(\beta_0 \cap \Omega)$ onto c_g and s_γ the orthogonal projection of $(\gamma_0 \cap \Omega)$ onto c_g (see Figure 8). We write $l_{h,\beta}$ for the hyperbolic length of s_β and $l_{h,\gamma}$ for the hyperbolic length of s_γ. Clearly, $l_{h,\beta} + l_{h,\gamma} \geq l_h(c_g)$, so either $l_{h,\beta} \geq l_h(c_g)/2$ or $l_{h,\gamma} \geq l_h(c_g)/2$.

First case. Suppose that $l_{h,\beta} \geq l_h(c_g)/2$. Here we redo the proof of Proposition 6.10 for the parts of $\widetilde{\delta}_i$ that are far enough from c_g. Let $\pi_{c_g} : \mathbb{H}^3 \to c_g$ be the orthogonal projection and let $\beta_i = \widetilde{\delta}_i \cap (\pi_{c_g})^{-1}(s_\beta)$. Notice that β_i is not necessarily connected.

Let $d_{h,\beta}^{i,0} = d_h(\beta_i, c_g)$. Let $l_n(\beta_i)$ be the neutered length of β_i and $l_h(\beta_i)$ its hyperbolic length. Once again, since we are talking about lengths of curves in N (and not distances between points), $l_n(\beta_i) = l_h(\beta_i)$, but we keep the different subscripts to help remembering which metric we are considering at each time. Thanks to Lemma 6.6, and our assumption that $l_{h,\beta} \geq l_h(c_g)/2$, we have

$$l_n(\delta_i) \geq l_n(\beta_i) = l_h(\beta_i) \geq l_{h,\beta} \cosh d_{h,\beta}^{i,0} \geq \frac{l_h(c_g)}{4} e^{d_{h,\beta}^{i,0}}. \tag{6.1}$$

In addition if $d_{n,\beta}^i$ is the minimum neutered distance from $\widetilde{\beta}_i$ to $\widetilde{\beta}_0$ and $d_{h,\beta}^i$

is the corresponding minimum hyperbolic distance then by Lemma 6.12, $d^i_{n,\beta} \leq 2\sinh(d^i_{h,\beta}/2)$. Hence

$$e^{d^i_{h,\beta}/2} \geq d^i_{n,\beta}.$$

Recall that, by construction of δ_i, we have $d_n(\delta_i, \delta_0) > Ai - D$, where D is a uniform constant (see proof of Proposition 6.10). Therefore, the same inequality holds for $d^i_{n,\beta} = d_n(\beta_i, \beta_0)$, so we have

$$e^{d^i_{h,\beta}/2} \geq Ai - D$$

Now, by construction, β_0 is such that $d_{\mathrm{Haus},H}(\beta_0, c_g) \leq D_{\alpha_0}/\sqrt{t}$, so

$$d^i_{h,\beta} \leq d^{i,0}_{h,\beta} + \frac{D_{\alpha_0}}{\sqrt{t}}.$$

Hence, using (6.1), we get

$$\begin{aligned} l_n(\delta_i) \geq \frac{l_h(c_g)}{4} e^{d^{i,0}_{h,\beta}} &\geq \frac{l_h(c_g)}{4} e^{d^i_{h,\beta}} e^{-D_{\alpha_0}/\sqrt{t}} \\ &\geq \frac{l_h(c_g)}{4} e^{-D_{\alpha_0}/\sqrt{t}} (d^i_{n,\beta})^2 \geq \frac{l_h(c_g)}{4} e^{-D_{\alpha_0}/\sqrt{t}} (Ai - D)^2. \end{aligned}$$

Since $D_{\alpha_0} = d_{\mathrm{Haus},H}(\widetilde{\delta_0}, c_g)$, γ_0 contains a curve that has to go from the annulus of radius $D_{\alpha_0}/\sqrt{t}$ around c_g to the annulus of radius D_{α_0} around c_g, which implies that

$$D_{\alpha_0}\left(1 - \frac{1}{\sqrt{t}}\right) \leq \frac{l_n(\gamma_0)}{2}.$$

Taking once again $C > 0$ to be a uniform constant such that $l(\alpha_i) \geq C l(\delta_i)$, we obtain, for $t > 4$,

$$D_{\alpha_0} \leq l_n(\gamma_0) \leq l_n(\delta_0) \leq \frac{l(\alpha_0)}{C} < \frac{t}{C}.$$

Using this and the previous inequality, we obtain

$$l(\alpha_i) \geq C l_n(\delta_i) \geq C \frac{l_h(c_g)}{4} e^{-D_{\alpha_0}/\sqrt{t}} (Ai - D)^2 \geq B i^2 e^{-\sqrt{t}/C},$$

where $B > 0$ is some universal constant. The lemma follows for $t \geq C e^{1/C}$.

Second case. Suppose now that $l_{h,\gamma} \geq l_h(c_g)/2$. Here we apply Lemma 6.6 once again, and obtain

$$\frac{l(\alpha_0)}{C} \geq l_n(\delta_0) = l_h(\delta_0) \geq l_h(\gamma_0) \geq \frac{l_h(c_g)}{2} e^{D_{\alpha_0}/\sqrt{t}}.$$

So,

$$e^{-D_{\alpha_0}} \geq \left(\frac{C l_h(c_g)}{2l(\alpha_0)}\right)^{\sqrt{t}} \geq \left(\frac{Ca}{2t}\right)^{\sqrt{t}}.$$

Using Proposition 6.10, we get

$$l(\alpha_i) \geq Bi^2 e^{-D_{\alpha_0}} \geq Bi^2 e^{-\sqrt{t}\log(2t/Ca)},$$

which yields the lemma after changing C to $Ca/2$. □

6.3. Two Seifert-fibered pieces.

Proposition 6.14. *Let ϕ^t be an Anosov flow on M. There are uniform constants $A_1, A_2 > 0$ such that the following happens: Let $\{\alpha_i\}$ be a string of orbits. Let S_1 and S_2 be two Seifert pieces glued in M along one of the decomposition tori (so S_1 and S_2 are allowed to be the same with two boundary tori glued together). Suppose that α_i intersects both Seifert pieces S_1 and S_2 consecutively. Then*

$$l(\alpha_i) \geq A_1 i - A_2 - l(\alpha_0).$$

Proof. This proof will split into several different cases, depending on the topological type of the Seifert pieces S_1 and S_2, and on the dynamical type of the flow (i.e. free or periodic) on them.

Suppose first that either S_1 or S_2 is periodic and is not a twisted I-bundle over the Klein bottle. This is the easy case, because by Theorem 4.4, there exists a uniform bound on the number of orbits in $\{\alpha_i\}$. The result follows therefore trivially.

The remaining possibilities are either that the flow is free on both S_1 and S_2 or that one is free and the other is a twisted I-bundle over the Klein bottle, or both are twisted I-bundles over the Klein bottle.

We first show that the last situation cannot happen. If S_1 and S_2 are both twisted I-bundles over the Klein bottle, then S_1 and S_2 have a unique boundary torus T. Hence we have that $M = S_1 \cup S_2$. Since $\pi(T)$ is a subgroup of index 2 in $\pi_1(S_1)$ and in $\pi_1(S_2)$, it follows that $\pi_1(T)$ is a subgroup of index at most 4 in $\pi_1(M)$. This is impossible since a 3-manifold supporting an Anosov flow cannot have a finite index subgroup homeomorphic to $\mathbb{Z}^2$. Indeed, this would contradict the fact that the fundamental group of a 3-manifold supporting an Anosov flow has exponential growth [46].

So we can now suppose that either the flow is free on both S_1 and S_2 or that one is free and the other is a twisted I-bundle over the Klein bottle. This situation is quite complicated and requires a long proof.

Note for future reference that since one of S_1 or S_2 is free then the tori constituting the common boundaries of S_1 and S_2 are quasi-transverse but cannot be transverse. Indeed, let T be such a torus and consider $\mathcal{C}$ a bi-infinite chain of lozenges that is $\pi_1(T)$-invariant. As explained in [9,10] if T is homotopic to a transverse torus, then all consecutive lozenges in $\mathcal{C}$ are adjacent (i.e. $\mathcal{C}$ is a scalloped chain). This contradicts the fact that S_1 (or S_2) is a free piece [10].

Now, if T is an incompressible torus in the boundary of S_1 and S_2, then either there exists a loop on T that is freely homotopic to a periodic orbit of the flow or the flow is a suspension Anosov flow [21]. By hypothesis, the flow is not a suspension, so any boundary tori of S_1 and S_2 must have one generator of their fundamental group freely homotopic to a periodic orbit. Since we can assume that T is a quasi-transverse torus, there is a periodic orbit contained in T. If both S_1 and S_2 are free Seifert pieces, this orbit cannot be freely homotopic to the Seifert fiber direction. This remark will be important for us in the following way: Since S_1 and S_2 are glued along the quasi-transverse torus T, the gluing is a Dehn twist that has to preserve the periodic orbits in T. Therefore the gluing is a Dehn twist around the orbits on T. If $(1,0)$ represents the closed orbits in T then the gluing is given by the matrix $\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right]$.

If both S_1 and S_2 are free, there is a lot of structure of the flow when restricted to these pieces. First of all we choose models for S_1, S_2 that have every boundary a quasi-transverse torus. In [11] the following facts are proved: the stable foliation restricted to S_1 (or S_2) is transverse to the boundary and it is an $\mathbb{R}$-covered foliation. In addition since S_1 and S_2 are free one can choose the Seifert fibration in the respective piece to be transverse to the stable foliation.

As S_1, S_2 are Seifert fibered spaces, let B_1 and B_2 be the bases of respectively S_1 and S_2. In other words B_1, B_2 are the quotients of S_1, S_2 by the respective Seifert fibrations. If S_1 or S_2 is periodic then, at this point, it is a twisted I-bundle over the Klein bottle, and its base is not a hyperbolic orbifold.

First case. Suppose that B_1 and B_2 are hyperbolic orbifolds. So in particular, both S_1 and S_2 are free.

Choose a hyperbolic metric on B_1 and B_2 and lift this to metrics in S_1, S_2 respectively, such that the leaves of the stable foliation are hyperbolic surfaces and local holonomy along the Seifert fibers is a hyperbolic isometry. The Seifert fibrations do not agree along the common boundary of S_1 and S_2. So in S_2 we make an interpolation between the two Riemannian metrics near these boundaries.

Let δ_i^1 and δ_i^2 be connected components of, respectively, $\alpha_i \cap S_1$ and $\alpha_i \cap S_2$ such that $\delta_i = \delta_i^1 \cup \delta_i^2$ is (possibly a subset of) a connected component of $\alpha_i \cap (S_1 \cup S_2)$. We as usual choose all these connected components in such a way that they are freely homotopic to each other inside each piece. Let T be the decomposition torus in between δ_i^1 and δ_i^2, and x_i be the point on T that separates δ_i between δ_i^1 and δ_i^2.

Let $\widetilde{S_1}$ be the universal cover of S_1. Note that $\widetilde{S_1} = \widetilde{B_1} \times \mathbb{R}$, where $\widetilde{B_1}$ is the universal cover of B_1. Note that $\widetilde{B_1} \subset \mathbb{H}^2$. Let $\widetilde{\delta_i^1}$ be a coherent lift of the δ_i^1 and $\widetilde{x_i} = \widetilde{\delta_i^1} \cap \widetilde{T}$ be the endpoint on $\widetilde{T}$.

The horizontal foliation is the stable foliation. The vertical foliation is the Seifert fibration in each piece. For any $x, y \in \widetilde{S_1}$, we write $d^1_{\mathrm{Hor}}(x,y)$ for the distance along the horizontal foliation and $d^1_{\mathrm{Ver}}(x, y)$ for the distance along the vertical foliation. Since the pieces S_1 and S_2 are glued together by a Dehn twist along T, there exists a constant $C_1 > 0$ such that a vertical leaf for the fibration on S_1 is sent to a line of slope C_1 in the coordinates given by the horizontal and vertical foliations on S_2.

Once more using Lemma 2.28, we know that for some uniform $A > 0$, $d(\widetilde{\delta}_0, \widetilde{\delta}_i) \geq Ai$, so

$$Ai \leq d^1_{\mathrm{Hor}}(\widetilde{x}_0, \widetilde{x}_i) + d^1_{\mathrm{Ver}}(\widetilde{x}_0, \widetilde{x}_i).$$

Suppose that

$$d^1_{\mathrm{Hor}}(\widetilde{x}_0, \widetilde{x}_i) \geq \frac{AC_1 i}{2 + C_1},$$

then the result follows from the following claim.

Claim 6.15. *Let $C > 0$. There exists a uniform constant $C_2 \geq 0$ such that, if $d^1_{\mathrm{Hor}}(\widetilde{x}_0, \widetilde{x}_i) \geq Ci$ then $l(\delta^1_i) \geq Ci - l(\delta^1_0) - C_2$.*

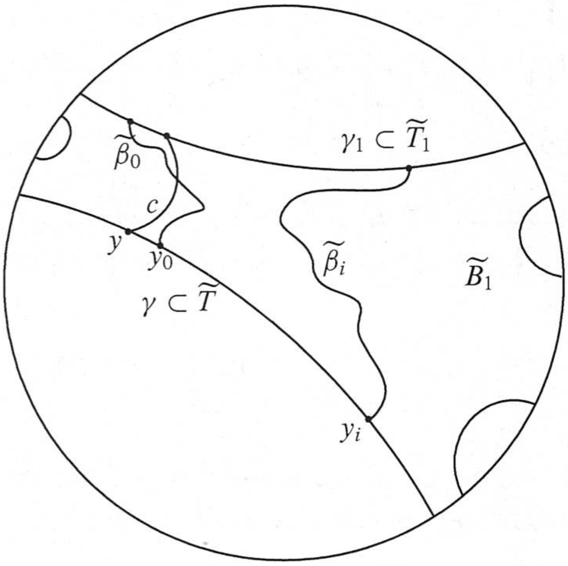


Figure 9. Large horizontal distance between y_0 and y_i.

Proof. We first fix one horizontal leaf $\widetilde{B}_1 = \widetilde{B}_1 \times \{0\}$ inside $\widetilde{S}_1 = \widetilde{B}_1 \times \mathbb{R}$ and write $\widetilde{\beta}_i$ for the projection (through the vertical foliation) of $\widetilde{\delta}_i$ onto $\widetilde{B}_1$. Let y_i be the projection along the Seifert fibration direction of $\widetilde{x}_i$ onto B_1. We write d_{B_1} for the hyperbolic distance on $\widetilde{B}_1 \subset \mathbb{H}^2$. By definition, $d_{B_1}(y_0, y_i) = d^1_{\mathrm{Hor}}(\widetilde{x}_0, \widetilde{x}_i) \geq Ci$.

Let T_1 be the decomposition torus containing the other endpoint of δ^1_i. Let $\widetilde{T}_1$ be the coherent lift of T_1, i.e. the lift such that the other endpoint of $\widetilde{\delta}_i$ is on $\widetilde{T}_1$. Note that T_1 and T might be the same torus, however, $\widetilde{T}_1 \neq \widetilde{T}$ since the δ_i are not homotopically trivial in S_1 relative to the boundary. This last fact comes from the quasi-transversality of T and T_1. Let $\gamma = \widetilde{T} \cap \widetilde{B}_1$ and $\gamma_1 = \widetilde{T}_1 \cap \widetilde{B}_1$. By our choice of metric, γ and γ_1 are geodesics boundaries of $\widetilde{B_1}$ inside $\mathbb{H}^2$. Let c be the geodesic in B_1 realizing the minimal distance between γ and γ_1 and let $y = c \cap \gamma$ the endpoint of c on γ (see Figure 9).

We have that $d_{B_1}(y, y_i) \geq Ci - d_{B_1}(y, y_0)$. Now, since the $\widetilde{\beta_i}$ are curves from γ_1 to γ, that c is the geodesic arc perpendicular to both γ_1 and γ, that the metric on B_1 is hyperbolic, and that $l(c)$ is greater than some constant depending only on the manifold M (so up to scaling the metric, we can suppose that $l(c) \geq 1$), we get that for some uniform constant C_0,

$$l(\widetilde{\beta_i}) \geq d_{B_1}(y, y_i) + d(\gamma, \gamma_1) - C_0 \geq Ci - d_{B_1}(y, y_0) - C_2,$$

where C_2 can be chosen to depend only on the manifold and the JSJ decomposition. Note that we also have

$$l(\widetilde{\beta_0}) \geq d_{B_1}(y, y_0) + d(\gamma, \gamma_1) - C_0,$$

so, $d_{B_1}(y, y_0) \leq l(\widetilde{\beta_0}) + C_0$. So finally, since $l(\delta_i^1) \geq l(\widetilde{\beta_i})$, we obtain

$$l(\delta_i^1) \geq l(\widetilde{\beta_i}) \geq Ci - l(\delta_0^1) - (C_2 + C_0),$$

and the claim is proved. □

So the proposition is proved if $d^1_{\mathrm{Hor}}(\widetilde{x_0}, \widetilde{x_i}) \geq AC_1 i/(2+C_1)$, with a constant A_1 that can be taken to be $A_1 := AC_1/(2 + C_1)$. We now suppose that $d^1_{\mathrm{Hor}}(\widetilde{x_0}, \widetilde{x_i}) < AC_1 i/(2 + C_1)$. Then,

$$d^1_{\mathrm{Ver}}(\widetilde{x_0}, \widetilde{x_i}) \geq Ai - \frac{AC_1 i}{2 + C_1} = \frac{2Ai}{2 + C_1}.$$

Using the fact that S_1 and S_2 are glued together by a Dehn twist on T such that the slope in the horizontal/vertical coordinates is C_1, we get that

$$d^2_{\mathrm{Hor}}(\widetilde{x_0}, \widetilde{x_i}) \geq C_1 d^1_{\mathrm{Ver}}(\widetilde{x_0}, \widetilde{x_i}) - d^1_{\mathrm{Hor}}(\widetilde{x_0}, \widetilde{x_i}).$$

Hence,

$$d^2_{\mathrm{Hor}}(\widetilde{x_0}, \widetilde{x_i}) \geq C_1 \frac{2Ai}{2 + C_1} - \frac{AC_1 i}{2 + C_1} = \frac{AC_1 i}{2 + C_1}.$$

Therefore, we can apply the previous claim in the piece S_2 and thus finish the proof, with the same constant $A_1 = AC_1/(2 + C_1)$ as the first case.

Let us recap what we showed up to now: The proposition is proven if either S_1 or S_2 is a periodic piece that is not a twisted I-bundle over the Klein bottle, or if both S_1 and S_2 are free and have an underlying orbifold admitting an hyperbolic structure. We assume now that this is not the case. Whether S_1, S_2 are periodic or not, let B_1, B_2 be their base spaces.

Second case. Up to renaming S_1 and S_2, we can suppose that either S_1 is periodic and a twisted I-bundle over the Klein bottle or S_1 is free with B_1 not an hyperbolic orbifold.

So whether S_1 is periodic or free, B_1 is not an hyperbolic orbifold. Denoting by χ_O the orbifold Euler characteristic, by χ the topological Euler characteristic, and by n_j the order of the cone points of B_1, we then have (since S_1 is Seifert, and so the singular points of B_1 can only be elliptic points)

$$\chi_O(B_1) = \chi(B_1) - \sum_j \left(1 - \frac{1}{n_j}\right) \geq 0.$$

Since $\chi_O(B_1) \geq 0$, we must have $\chi(B_1) \geq \sum_j \left(1 - \frac{1}{n_j}\right) \geq 0$. Hence the list of topological types for B_1 is: the disk, the sphere, the real projective plane, the annulus, the Möbius band, the Klein bottle and the torus. Now, by assumption, S_1 has at least one boundary torus, so B_1 can only be a disk, an annulus or a Möbius band.

If B_1 is an annulus or a Möbius band, then $\chi(B_1) = 0$, so B_1 cannot have any cone point. Suppose that B_1 is an annulus, then, since M is orientable S_1 has to be orientable, so the Seifert fibration in S_1 has to be orientable. Hence S_1 has to be a torus times an interval, but no Seifert piece of a JSJ decomposition can be $\mathbb{T}^2 \times I$, so we have a contradiction in this case.

Hence, if $\chi(B_1) = 0$, then B_1 has to be a Möbius band. And in that case, since M is orientable, the S^1 fibration has to be non-orientable. So S_1 is a regular neighborhood of a one sided Klein bottle. In particular, S_1 has only one boundary component, and $\pi_1(S_1) = \pi_1(K)$, where K is the Klein bottle. We let N_1 be a manifold which is a regular neighborhood of a one sided Klein bottle. In particular, N_1 is a twisted I-bundle over the Klein bottle.

Before continuing with the case $S_1 = N_1$, let us consider the other case left.

Suppose that B_1 is a disk, then, since $\chi_O(B_1) \geq 0$, B_1 has at most 2 singular fibers. Suppose that B_1 has either 0 or 1 singular fibers. Then S_1 is a solid torus, so its boundary torus T is compressible, which is ruled out since T is a tori of the JSJ decomposition of an irreducible manifold, so in particular T has to be incompressible [35,37].

So B_1 has two singular fibers of order 2 each. In addition, since the disk is orientable and M is assumed to be orientable too, the S^1 fibration has to be orientable. Call N_2 that Seifert manifold. A presentation of the fundamental group of N_2 is the following

$$\pi_1(N_2) = \langle c, d, h \mid [c, h] = [d, h] = 1,\ c^2 = d^2 = h \rangle.$$

In particular, setting $a = c$ and $b = d^{-1}c$, we see that

$$\pi_1(N_2) = \langle a, b \mid a^{-1}ba = b^{-1} \rangle = \pi_1(K).$$

So N_2 has also the fundamental group of the Klein bottle, and N_2 is also a regular neighborhood of a Klein bottle. It follows that the manifolds N_1, N_2 are homeomorphic, but the Seifert fibrations are different. This is one of the few manifolds where this happens.

Therefore, independently of whether $S_1 = N_1$ or $S_1 = N_2$, we always have that S_1 has a *unique* boundary torus, that we call T and $\pi_1(S_1) = \pi_1(K)$. Moreover, B_1 always admit a finite cover which is an annulus and the universal cover of S_1 is $\widetilde{S_1} = \widetilde{B_1} \times \mathbb{R}$. We can replace the Seifert fibration in N_2 such that the $\mathbb{R}$ factor is always the fiber direction and $\widetilde{B_1}$ is homeomorphic to $[0,1] \times \mathbb{R}$.

We now turn our attention towards S_2. If we suppose that B_2, the base orbifold of S_2 is not hyperbolic, then, the argument just above shows that $S_2 = N_1$ or $S_2 = N_2$. In particular, S_2 has only one boundary torus T and $M = S_1 \cup S_2$. As we saw earlier in the proof, this is impossible since otherwise $\pi_1(T)$ would be a subgroup of index at most 4 in $\pi_1(M)$, which would contradict the fact that $\pi_1(M)$ has exponential growth.

Therefore, B_2 is an hyperbolic orbifold.

So in either case we obtain, up to switching S_1 and S_2, that S_1 is a twisted I-bundle over the Klein bottle, and that S_2 is a free Seifert piece with hyperbolic orbifold base.

In order to finish the proof we will apply a trick that will allow us to reduce the proof to what we did in the first case. The manifold S_1 is a twisted I-bundle over the Klein bottle. This can be described as follows. First let T with coordinates (x, y) defined mod 1. Let $j : T \to T$ be the free involution of T given by $j(x, y) = (x + 1/2, 1 - y)$. Now define S_1 to be the quotient of $T \times [0,1]$ by $g(p,t) = (j(p), 1-t)$. Up to isotopy there are two Seifert fibrations in S_1: the first $\mathcal{F}_1$ is given by the curves $y =$ const, $t =$ const, the second $\mathcal{F}_2$ is defined by the curves $x =$ const, $t =$ const. The stable foliation when restricted to S_1 has up to isotopy an annular leaf $y = 0$. The Seifert fibration $\mathcal{F}_1$ cannot be made transverse to the stable foliation in S_1, but $\mathcal{F}_2$ can be made transverse.

Let $f: T \subset S_2 \to T \subset S_1$ be the Dehn twist giving the gluing between S_2 and S_1. Now, if V is a vertical fiber of S_2, it travels through S_1 and comes back to S_2, it becomes the curve $f^{-1} \circ j \circ f(V)$. In particular, since $f(V)$ is a curve of slope $C_1 > 0$ in the horizontal/vertical coordinates in S_1, $j \circ f(V)$ will be of *negative* slope and hence $f^{-1} \circ j \circ f(V)$ will have an even more negative slope.

The trick is the following: S_1 has a double cover $T \times [0,1]$. Since K is one sided in S_1 this produces a double cover M_2 of M made up of $T \times [0,1]$ and two copies of $M - \mathrm{int}(S_1)$ glued along $T \times \{0\}$ and $T \times \{1\}$. In particular the Seifert piece S_2 lifts to two Seifert fibered spaces S_3 and S_4 contained in M_2 and each one is homeomorphic to S_2. In addition since S_1 lifts to $T \times [0,1]$ then $S_3 \cup (T \times [0,1])$ is also a Seifert fibered space. But the Seifert fibration cannot be extended to S_4 because as explained in the paragraph above the Seifert fiber V' in S_3 (a lift of the Seifert fiber V in S_2) moves through $T \times [0,1]$ (corresponding to the curve V moving through S_1) to a curve V'' which is a lift of $f^{-1} \circ j \circ f(V)$) and as explained in the paragraph above this is *not* a curve isotopic to a lift of the fiber V in S_2. This shows that the Seifert fibration in $S_3 \cup (T \times [0,1])$ cannot extend into S_4. Notice that $S_3 \cup (T \times [0,1])$ is homeomorphic to S_3 which is in turn homeomorphic to S_2 and hence has hyperbolic base orbifold. In addition the Seifert fibration in S_3 cannot extend to any other parts of M_2 or else the projection to M would extend the Seifert fibration of S_2 in M. The important conclusion for us is that $S_3 \cup (T \times [0,1])$ and S_4 are Seifert fibered pieces of the JSJ decomposition of M_2. The Anosov flow in M lifts to an Anosov flow in M_2 and the string of orbits also does, with a factor of at most 2 in the periods of the orbits. Now $S_3 \cup (T \times [0,1])$ is free and with hyperbolic base orbifold. The same is true of S_4. The lifted string of orbits crosses through $S_3 \cup (T \times [0,1])$ into S_4. So we reduced the last possibility to the first case and the result follows.

This finally ends the proof of Proposition 6.14. □

7. Consequences for counting orbits

7.1. Counting orbits in free homotopy classes. First, it is a classical result that the number of periodic orbits of an Anosov flow grows exponentially fast with the period [14,41]. Moreover (when the flow is transitive and not a suspension of an Anosov diffeomorphism) the exponential rate of growth is the topological entropy of the flow. Several authors also studied the growth of periodic orbits when restricted to a given *homology* class (see for instance [2,38,44,50] and references therein). In a fixed homology class, the rate of growth is still exponential, and they obtain some precise expression of the exponential rate.

Thanks to our previous results, we can give bounds for the rate of growth of the number of orbits inside a fixed *free homotopy* class. Let us first explain what we exactly mean by that: The free homotopy class of an orbit is only well determined up to conjugacy, so when talking about a fixed free homotopy class, we fix a conjugacy class in $\pi_1(M)$. We will write $\mathrm{Cl}(h)$ for the conjugacy class of an element $g \in \pi_1(M)$. If α is a closed orbit of ϕ^t, then we write $\mathrm{Cl}(\alpha)$ for the conjugacy class in the fundamental group of M that represents α. So

$$\mathcal{F}\mathcal{H}(\alpha) = \{\beta \text{ closed orbit} \mid \mathrm{Cl}(\beta) = \mathrm{Cl}(\alpha)\}.$$

Theorem 7.1. *Let ϕ^t be an Anosov flow on a 3-manifold M, and h be an element of the fundamental group of M.*

(1) *If M is hyperbolic, then there exists a uniform constant $A_1 > 0$ and a constant $C_{1,h}$ depending on h such that, for t big enough,*

$$\sharp\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h),\ l(\alpha) < t\} \leq \frac{1}{A_1}\log(t) + C_{1,h}.$$

(2) *If the JSJ decomposition of M is such that no decomposition tori bounds a Seifert-fibered piece on both sides (so in particular, if all the pieces are atoroidal), then there exists a constant $C_{1,h}$ depending on h such that, for t big enough,*

$$\sharp\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h),\ l(\alpha) < t\} \leq C_{1,h}\sqrt{t}.$$

(3) *Otherwise, there exist constants $A_1 > 0$ and $B \geq 0$, that do not depend on h if M is a graph manifold, such that, for t big enough,*

$$\sharp\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h),\ l(\alpha) < t\} \leq A_1 t + B.$$

So, in any case, the orbit growth inside a conjugacy class is at most linear in the period.

Moreover, independently of the topology of M, there exists a uniform constant $A_2 > 0$ and a constant $C_{2,h}$ depending on h such that, if the set

$$\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h)\}$$

is infinite, then for any t

$$\sharp\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h),\ l(\alpha) < t\} \geq \frac{1}{A_2}\log(t) - C_{2,h}.$$

Remark 7.2. Note that, when M is hyperbolic, the growth rate of the number of orbits inside a free homotopy class is exactly logarithmic.

Proof. First note that, if $\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h)\}$ is empty or finite, then the first parts of the theorem follows trivially, so we restrict our attention to elements h that yields an infinite free homotopy class of orbits.

In order to prove the result, we also note that, thanks to Proposition 2.26, counting the number of orbits in a free homotopy class is, up to a uniform factor, the same thing as counting orbits inside an infinite string. Hence the above result is just a transcription of Theorem 6.2 and Theorem 6.1, if we bound the worst case scenario in the finitely many strings of orbits in any free homotopy class.

Let $\{\alpha_i\}_{i\in\mathbb{N}}$ be an infinite string of orbits inside $\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h)\}$ and suppose that α_0 is the shortest orbit in the string. We will prove the three different cases of the theorem separately and then prove the lower bound.

(1) If M is hyperbolic, then, according to Proposition 6.5, there exist $A, B > 0$ independent of the homotopy class and D_{α_0} depending on the length of α_0 such that

$$l(\alpha_i) \geq B e^{-D_{\alpha_0}} e^{Ai}.$$

So, if $l(\alpha_i) < t$, then

$$i < \frac{1}{A}\log\left(\frac{t e^{D_{\alpha_0}}}{B}\right) = \frac{1}{A}\log(t) + C_{\alpha},$$

where $C_\alpha = (D_{\alpha_0} - \log(B))/A$ is a constant depending only on the infinite string chosen. Hence,

$$\sharp\{\alpha_i \mid l(\alpha) < t\} \leq \frac{1}{A}\log(t) + C_\alpha.$$

Using the above inequality and Proposition 2.26, we get, up to a renaming of constants that, for t big enough,

$$\sharp\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h),\ l(\alpha) < t\} \leq \frac{1}{A}\log(t) + C_h.$$

So the first case is proven.

(2) Suppose now that M is such that no decomposition tori bounds a Seifert-fibered piece on both sides. Since no infinite free homotopy class can stay entirely in a unique Seifert piece (by Theorem 4.4), the $\{\alpha_i\}_{i\in\mathbb{N}}$ has to go through one of the decomposition tori or be contained in an atoroidal piece of the JSJ decomposition. And since the tori do not bound a Seifert-fibered piece on both sides, the orbits $\{\alpha_i\}$ have to enter an atoroidal piece of the modified JSJ decomposition or be contained in an atoroidal piece of the JSJ decomposition. We can hence apply Proposition 6.10. There exists $B > 0$ depending only on M and the flow, and $D_{\alpha_0} > 0$ depending on α_0 such that

$$l(\alpha_i) \geq B i^2 e^{-D_{\alpha_0}}.$$

So, if $l(\alpha_i) < t$, then

$$i^2 < \frac{t e^{D_{\alpha_0}}}{B},$$

so,

$$\sharp\{\alpha_i \mid l(\alpha_i) < t\} \leq \sqrt{t}\left(\frac{e^{D_{\alpha_0}}}{B}\right)^{1/2}.$$

Which implies, using again Proposition 2.26, that for some constant C_h, depending only on h and for t big enough,

$$\sharp\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h),\ l(\alpha) < t\} \leq C_h\sqrt{t}.$$

(3) Since linear growth is faster than a square root growth, the last case is proven by what we did above as soon as the orbits α_i enters an atoroidal piece. As mentioned above, thanks to Theorem 4.4, an infinite string cannot stay entirely in a unique Seifert piece. So the only case we are left to deal with is when the string crosses a decomposition torus that bounds two Seifert pieces (note that the torus can also bound the same Seifert piece on both side, but this is the same for us). We can then apply Proposition 6.14 and deduce the result in the same manner as above.

Finally, to prove the lower bound on the growth rate, we use Theorem 6.1; There exist uniform constants $C_1, C_2 > 0$ such that

$$l(\alpha_i) \leq C_1 l(\alpha_0) e^{C_2 i}.$$

Hence, for any i such that $i < \big(\log t - \log(C_1 l(\alpha_0))\big)/C_2$, we have $l(\alpha_i) < t$. So

$$\sharp\{\alpha_i \mid l(\alpha) < t\} \geq \frac{\log t}{C_2} - \frac{\log(C_1 l(\alpha_0))}{C_2},$$

and this finishes the proof. □

As we saw, the constants given in the preceding theorem depend on the free homotopy class we start with, but thanks to the Lemmas 6.8 and 6.13, we can also obtain some uniform growth rate for the upper bounds.

Theorem 7.3. *Let ϕ^t be an Anosov flow on a 3-manifold M. There exist uniform constants $A_1, \dots, A_7 > 0$ and t_0 such that, if h is an element of the fundamental group of M, then for $t \geq t_0$,*

(1) *If M is a graph manifold, then*

$$\sharp\big\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h),\ l(\alpha) < t\big\} \leq A_1 t + A_2.$$

(2) *If M is hyperbolic, then*

$$\sharp\big\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h),\ l(\alpha) < t\big\} \leq A_3 \log(t) + A_3\sqrt{t}\log(A_4 t) + A_5$$

(3) *Otherwise,*

$$\sharp\big\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h),\ l(\alpha) < t\big\} \leq A_6\sqrt{t}e^{\frac{\sqrt{t}}{2}\log(t/A_7)}$$

So, independently of the topology of M, we can rename t_0 and A_6, A_7 so that we always have, for $t \geq t_0$,

$$\sharp\big\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h),\ l(\alpha) < t\big\} \leq A_6\sqrt{t}e^{\frac{\sqrt{t}}{2}\log(t/A_7)}.$$

Note that Theorem 7.3 is not trivial even when looking at finite free homotopy classes, as opposed to Theorem 7.1. Indeed, one consequence of Theorem 7.3 is that there exists constants $A_6, A_7 > 0$ uniform, such that if $t_{\max}$ is the longest orbit in a given finite free homotopy class, then this class has cardinality bounded above by $A_6\sqrt{t_{\max}}e^{\sqrt{t_{\max}}\log(t_{\max}/A_7)/2}$.

Proof. The proof of this theorem is almost the same as Theorem 7.1, we just replace the use of Proposition 6.5 by Lemma 6.8 and Proposition 6.10 by Lemma 6.13. The only difference is that, when we want a uniform control, the worse, i.e. fastest, control we get is in the case of manifolds containing one or more atoroidal pieces in their JSJ decomposition. Here are the details.

As before, choose a string of orbits $\{\alpha_i\}$ contained in the free homotopy class associated with h, that is, $\{\alpha \text{ closed orbit} \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h)\}$. Note that, in order to get uniform controls on the constants and on the size of t, we do have to deal with finite strings of orbits also.

Recall that there is a uniform bound on the numbers of strings of orbits in any free homotopy class and a uniform bound on the number of orbits in a free homotopy class outside a string (by Proposition 2.26). Therefore counting the number of orbits of length less than t inside a string implies the result of Theorem 7.3 up to a change of (uniform) constants.

(1) Suppose that M is a graph-manifold. Then, either $\{\alpha_i\}$ stays in a Seifert fibered piece, or it intersects at least two different Seifert-fibered pieces. In the first case, by Theorem 4.4, there exists a uniform bound on the number of orbits in $\{\alpha_i\}$, and the result follows trivially for t big enough (and independently of h in $\pi_1(M)$). In the latter case, we can apply Proposition 6.14 and get that, for some uniform constants $A_1, A_2 > 0$, if $l(\alpha_i) < t$, then $l(\alpha_0) < t$ and

$$i < \left(\frac{A_2}{A_1}\right) + \left(\frac{2}{A_1}\right)t,$$

which implies the result up to renaming A_1, A_2.

(2) If M is hyperbolic, then we can apply Lemma 6.8 and get that, if $l(\alpha_i) < t$ and $t > \max(4, \frac{ae}{2})$ where a is the length of the shortest geodesic in M, then

$$i < \frac{1}{A}\log\left(\frac{t}{B}e^{\sqrt{t}\log(2t/a)}\right),$$

where A, B and a are uniform constants. Then

$$i < \frac{1}{A}\left(\log t - \log B + \sqrt{t}\log\left(\frac{2t}{a}\right)\right) \leq A_3\log t + A_3\sqrt{t}\log(A_4 t) + A_5,$$

where $A_3 = \frac{1}{A}$, $A_4 = \frac{2}{A}$ and $A_5 = |\frac{\log B}{A}|$. This implies the result in the hyperbolic case.

(3) In the general case, we have three possibilities: The first possibility is that the orbits of the string stay in a Seifert piece. Then Theorem 4.4 yields the result. Notice that there is a global bound on the number of orbits. The second option is that the orbits intersect two consecutive Seifert-fibered pieces. In this case we have uniform growth bounded above by a linear function, by Proposition 6.14. The third and final option is that the orbits have to enter (cross) an atoroidal piece. In this final case, we can use Lemma 6.13, and we obtain that for some uniform constants, if $l(\alpha_i) < t$, then

$$i^2 < \frac{t}{B}e^{\sqrt{t}\log(t/C)},$$

therefore

$$i < \frac{\sqrt{t}}{\sqrt{B}}e^{\sqrt{t}\log(t/C)/2}.$$

Then take $A_6 = \frac{1}{\sqrt{B}}$ and $A_7 = C$.

The third function is eventually bigger than the other two so up to changing A_6, A_7 and t_0, we obtain the final statement of the theorem. $\square$

7.2. Counting the conjugacy classes. We can now use Theorem 7.3 to prove Theorem A, and thereby answer, in the case of Anosov flows on a 3-manifold, the question raised by Plante and Thurston in [46]. Now, in this section, we are not allowed anymore to use reparametrizations of the Anosov flow (as they modify the topological entropy). To make that point clear, we will switch our notation back to those of the introduction and talk about the period of orbits instead of the length.

Recall that if $h \in \pi_1(M)$, then we denote by $\alpha_{\mathrm{Cl}(h)}$ a periodic orbit in the conjugacy class $\mathrm{Cl}(h)$ with smallest period if there is a periodic orbit in it. We also write

$$\mathrm{CCl}(t) := \left\{\mathrm{Cl}(h) \mid h \in \pi_1(M),\ T\left(\alpha_{\mathrm{Cl}(h)}\right) < t\right\}$$

for the set of conjugacy class in $\pi_1(M)$ that admit a periodic orbit representative of period less than t. Plante and Thurston [46] asked if the number of elements in $\mathrm{CCl}(t)$ grew exponentially with t. We have

Theorem 7.4. *Let ϕ^t be an Anosov flow on a 3-manifold M.*

Then the number of conjugacy classes in $\pi_1(M)$ grows exponentially fast with the period of the shortest representative. Moreover, the exponential growth rate is equal to the exponential growth rate of the number of periodic orbits.

More precisely, there exists constants $A_6, A_7 > 0$ and $t_0 > 0$, given by Theorem 7.3, such that,

$$\sharp\mathrm{CCl}(t) \leq \sharp\left\{\alpha \text{ closed orbit} \mid T(\alpha) < t\right\},$$

and, for $t \geq t_0$,

$$\sharp\mathrm{CCl}(t) \geq \frac{1}{A_6\sqrt{t}} e^{-\frac{\sqrt{t}}{2}\log(t/A_7)}\, \sharp\left\{\alpha \text{ closed orbit} \mid T(\alpha) < t\right\}.$$

Moreover:

- *If M is hyperbolic, then there exist $A_3, A_4, A_5 > 0$ such that, for all $t \geq t_0$,*

$$\sharp\mathrm{CCl}(t) \geq \frac{1}{A_3\log(t) + A_3\sqrt{t}\log(A_4 t) + A_5}\, \sharp\left\{\alpha \text{ closed orbit} \mid T(\alpha) < t\right\}.$$

- *If M is a graph manifold, then there exist $A_1, A_2 > 0$ such that, for all $t \geq t_0$,*

$$\sharp\mathrm{CCl}(t) \geq \frac{1}{A_1 t + A_2}\, \sharp\left\{\alpha \text{ closed orbit} \mid T(\alpha) < t\right\}.$$

With this result, and Margulis' [41] or Bowen's [14] counting results we obtain the following:

Corollary 7.5. *Let ϕ^t be a* transitive *Anosov flow on a* 3*-manifold M. Then*

$$\lim_{t\to+\infty}\frac{1}{t}\log \sharp\mathrm{CCl}(t) = h_{\mathrm{top}},$$

where h_{top} is the topological entropy of the flow.

Note that, if we use more precise asymptotics of the number of periodic orbits (see for instance [47]), we could deduce a more precise control on $\sharp\mathrm{CCl}(t)$. However, even in the best possible case, i.e. when M is hyperbolic, our results are not quite enough to deduce an actual asymptotic formula for $\sharp\mathrm{CCl}(t)$.

Note finally that Plante and Thurston asked the question about the growth of conjugacy classes in the setting of Anosov flows on a manifold of any dimension, so in a setting much more general than ours. It is possible that parts of our method can be extended directly to higher dimensions for codimension one Anosov flows. However, we are not aware of any previous results on that particular question. Moreover, as previously mentioned, if the Verjovsky conjecture is true, then that question is void for codimension one Anosov flow in higher dimensional manifolds.

Before proving Theorem 7.4, we also state another easy consequence, which is that the shortest orbit representatives of conjugacy classes are equidistributed. For α a periodic orbit of ϕ^t, we can define a probability measure supported on it by setting

$$\delta_\alpha := \frac{1}{T(\alpha)}\mathrm{Leb}_\alpha,$$

where Leb_α is the image of the Lebesgue measure on $[0, T(\alpha)]$ under the map $x \mapsto \phi^t x$, with $x \in \alpha$.

Corollary 7.6. *Let ϕ^t be a* transitive *Anosov flow on a* 3*-manifold M. Then, the Bowen–Margulis measure μ_{BM} of ϕ^t (i.e. measure of maximal entropy) can be obtained as*

$$\mu_{BM} = \lim_{t\to+\infty}\frac{1}{\sharp\mathrm{CCl}(t)}\sum_{\mathrm{Cl}(h)\in\mathrm{CCl}(t)}\delta_{\alpha_{\mathrm{Cl}(h)}}.$$

Proof of Theorem 7.4. The first inequality in the theorem is trivial: If a conjugacy class has its smallest representative of length less than t, then there exists at least an orbit with length less than t, so

$$\sharp\big\{\mathrm{Cl}(h) \mid h \in \pi_1(M),\ T\big(\alpha_{\mathrm{Cl}(h)}\big) < t\big\} \le \sharp\big\{\alpha \mid T(\alpha) < t\big\}.$$

We now prove the second inequality. For any $h \in \Gamma$, we set

$$N(\mathrm{Cl}(h), t) := \sharp\big\{\alpha \text{ closed orbit of } \phi^t \mid \mathrm{Cl}(\alpha) = \mathrm{Cl}(h),\ T(\alpha) < t\big\}.$$

By Theorem 7.3, there exist uniform constants $A_6, A_7 > 0$, such that, for t big enough — that is $t \ge t_0$ (note that this is where we need to know that the t does not depend on the conjugacy class of h for the control given in Theorem 7.3 to work)

$$\sharp\{\alpha \mid T(\alpha) < t\} = \sum_{\mathrm{Cl}(h)\in \mathrm{CCl}(t)} N(\mathrm{Cl}(h), t) \leq A_6\sqrt{t}e^{\frac{\sqrt{t}}{2}\log(t/A_7)}\,\sharp\,\mathrm{CCl}(t).$$

Which gives the second inequality. The other inequalities follow in the same manner.

The exponential growth of the number of closed orbits of an Anosov flow is always positive (even when the flow is not transitive [14]). Therefore $\sharp\mathrm{CCl}(t)$ has exponential growth as well. This is because $\{\alpha \mid T(\alpha) < t\}$ grows at least as fast as e^{bt} for some $b > 0$ and $bt - \frac{\sqrt{t}}{2}\log\left(\frac{t}{A_7}\right) \geq ct$ for some $c > 0$ and for all $t \geq t_1$ for some uniform time t_1. Therefore the number of conjugacy classes also grows exponentially fast with the length of the shortest representative. □

Proof of Corollary 7.5. The second inequality in Theorem 7.4 yields

$$\frac{1}{t}\log\sharp\{\alpha \mid T(\alpha) < t\} \leq \frac{1}{t}\left(\log(A_6\sqrt{t}) + \frac{\sqrt{t}}{2}\log(t/A_7)\right) + \frac{1}{t}\log\,\sharp\,\mathrm{CCl}(t).$$

Passing to the limit (if it exists), and also using the first inequality of Theorem 7.4, gives

$$\lim_{t\to+\infty}\frac{1}{t}\log\sharp\{\alpha \mid T(\alpha) < t\} = \lim_{t\to+\infty}\frac{1}{t}\log\sharp\mathrm{CCl}(t),$$

And, since the flow is transitive, then Bowen's result in [14] (or Margulis [41]) shows that the above limit exists and

$$\lim_{t\to+\infty}\frac{1}{t}\log\sharp\{\alpha \mid T(\alpha) < t\} = h_{\mathrm{top}},$$

so this proves Corollary 7.5. □

To prove Corollary 7.6, one could follow Bowen's original proof [14] that the closed orbits of an Anosov flow are equidistributed. Instead we copy the proof of equidistribution of closed orbits under homological constraints given by Babillot and Ledrappier in [2]. Their proof is based on the following result of Kifer [39].

Theorem 7.7 (Kifer [39]). *If $\mathcal{K}$ is a closed subset of the set of ϕ^t-invariant probability measures (equipped with the weak*-topology), then*

$$\limsup_{t\to+\infty}\frac{1}{t}\log\sharp\{\alpha \mid \delta_\alpha \in \mathcal{K},\ T(\alpha) < t\} \leq \sup_{\mu\in\mathcal{K}} h_\mu,$$

where h_μ is the measure-theoretic entropy of μ.

Proof of Corollary 7.6. Let U be an open neighborhood of μ_{BM} and write U^c for its complementary set. Since μ_{BM} is the measure of maximal entropy and it is unique, there exists $\varepsilon_0 > 0$ such that $\sup_{\mu\in U^c} h_\mu \leq h_{\mathrm{top}} - \varepsilon_0$.

Recall that $h_{top} \geq h_\mu$ for any ϕ^t-invariant probability measure μ. Hence, by Kifer's result, for t big enough

$$\sharp\{\mathrm{Cl}(h) \mid \mathrm{Cl}(h) \in \mathrm{CCl}(t) \text{ and } \delta_{\alpha_{\mathrm{Cl}(h)}} \in U^c\} \leq \sharp\{\alpha \mid \delta_\alpha \in U^c \text{ and } T(\alpha) < t\} \leq e^{t(h_{\mathrm{top}}-\varepsilon_0/2)}.$$

Now, recall from Theorem 7.4, that for t big enough,

$$\sharp\,\mathrm{CCl}(t) \geq \frac{1}{A_6\sqrt{t}} e^{-\frac{\sqrt{t}}{2}\log(t/A_7)} \sharp\{\alpha \mid T(\alpha) < t\},$$

so, since the flow is transitive, for t big enough,

$$\sharp\,\mathrm{CCl}(t) \geq \frac{1}{A_6\sqrt{t}} e^{-\frac{\sqrt{t}}{2}\log(t/A_7)} e^{t(h_{\mathrm{top}}-\varepsilon_0/10)} \geq e^{t(h_{\mathrm{top}}-\varepsilon_0/5)}.$$

These two equations imply that

$$\frac{1}{\sharp\mathrm{CCl}(t)} \sharp\{\mathrm{Cl}(h) \mid \mathrm{Cl}(h) \in \mathrm{CCl}(t) \text{ and } \delta_{\alpha_{\mathrm{Cl}(h)}} \in U^c\} < e^{-3t\varepsilon_0/10},$$

for t big enough. Consider the sum

$$\frac{1}{\sharp\mathrm{CCl}(t)} \sum_{\mathrm{Cl}(h)\in\mathrm{CCl}(t)\cap U} \delta_{\alpha_{\mathrm{CCl}(t)}} + \frac{1}{\sharp\mathrm{CCl}(t)} \sum_{\mathrm{Cl}(h)\in\mathrm{CCl}(t)\cap U^c} \delta_{\alpha_{\mathrm{CCl}(t)}}$$

By the above, the total mass of the second sum tends to zero when $t \to \infty$. Hence any weak*-limit of the sum of the two terms has to be in U since the first sum is in U and the second part will converge to the zero measure.

Since U is arbitrary, this shows that any weak limit of the original total sum has be the Bowen Margulis measure. □

8. Quasigeodesic behavior and $\mathbb{R}$-covered Anosov flows

Let us first recall that a quasigeodesic is a quasi-isometric embedding of the real line or a segment of the real line into an open complete manifold. A quasi-isometry between metric spaces (X,d), (Y,d') is a map $f\colon X \to Y$ such that there are constants $k, c > 0$ so that for any a, b in X,

$$\frac{1}{k} d(a,b) - c \leq d'(f(a), f(b)) \leq k d(a,b) + c.$$

The map f need not be continuous. A flow on a compact manifold M is *quasigeodesic* if the orbits of its lift to the universal cover are quasigeodesics with universal constants, that is, independent of the particular flow line. The metric in the domain is the path metric along the flow lines. The quasigeodesic question for flows is particularly important if the manifold is hyperbolic, because in $\mathbb{H}^3$ a quasigeodesic is a bounded distance from a minimal geodesic, with the bound depending only on the k, c of the associated quasi-isometry [33,53]. As such quasigeodesics are extremely important and useful in the whole theory of hyperbolic 3-manifolds.

Surprisingly enough the quasigeodesic question for flows in closed hyperbolic 3-manifolds is easier to deal with for certain classes of pseudo-Anosov flows, rather than the more restrictive Anosov flows. In particular there is a huge amount of examples of pseudo-Anosov quasigeodesic flows in closed, hyperbolic 3-manifolds. For example, suspensions of pseudo-Anosov diffeomorphisms on surfaces [55]. In addition any transversely oriented, $\mathbb{R}$-covered foliation in a closed hyperbolic 3-manifold admits a transverse quasigeodesic pseudo-Anosov flow [16,24,25]. The quasigeodesic property is then used to study the asymptotic behavior of the leaves of the foliation lifted to the universal cover $\mathbb{H}^3$ [17,25].

As for the "supposedly" much simpler case of Anosov flows, the only examples of Anosov flows that are known to be quasigeodesic are geodesic flows and suspensions of Anosov diffeomorphisms, and in each case the underlying manifold is not hyperbolic. The second author proved 20 years ago, in [19], that $\mathbb{R}$-covered Anosov flows on hyperbolic manifolds cannot be quasigeodesic. We prove here that the only $\mathbb{R}$-covered Anosov flows that could possibly be quasigeodesic are on graph-manifolds.

Theorem 8.1. *Let ϕ^t be an $\mathbb{R}$-covered Anosov flow on a 3-manifold M. If M admits an atoroidal piece in its JSJ decomposition, then ϕ^t is not quasigeodesic.*

It seems likely however that $\mathbb{R}$-covered Anosov flows on graph-manifolds are indeed quasigeodesic, so we make the following:

Conjecture 8.2. *Let ϕ^t be an $\mathbb{R}$-covered Anosov flow on a 3-manifold M. The flow is quasigeodesic if and only if M is a graph-manifold.*

Remark 8.3. Note that many flows, for instance contact (a.k.a. Reeb) flows, are geodesible, i.e. there exists a metric making the flow-lines geodesics. So any contact Anosov flow on a 3-manifold admitting an atoroidal piece is geodesible, but, according to Theorem 8.1, *not* quasigeodesic. This is not a contradiction as being geodesible is a local property, while being quasigeodesic is a global one: it is measured in the large scale when lifted to the universal cover. What we can deduce is that the flow-lines are non-minimizing geodesics, in fact they are unboundedly bad at measuring distances in the universal cover. Another consequence is that a metric making the flow-lines geodesic must have positive curvature.

In order to prove Theorem 8.1, we will first need to prove that there always exist periodic orbits in the interior of an atoroidal piece.

Proposition 8.4. *Let ϕ^t be a $\mathbb{R}$-covered Anosov flow on a 3-manifold M. Let P be a piece of a modified JSJ decomposition of M. Then there exists periodic orbits in the interior of P.*

A different proof than the one we are going to present would show that the above result holds for any transitive Anosov flow, not only the $\mathbb{R}$-covered ones. But since we do not need the more general result, and the proof in the $\mathbb{R}$-covered case is much nicer, we only present this one here.

We stress that the whole point of this proposition is that the periodic orbit is contained in the *interior* of P as opposed to the boundary of P. Since ∂P is made up of Birkhoff tori, there are always periodic orbits contained in ∂P.

Proof. Let T be a quasi-transverse boundary torus of P and A be an open Birkhoff annulus (i.e. transverse) of T such that orbits through A enter the piece P. We are going to show that there exist orbits through A that stay in the interior of the piece P and do not accumulate on the boundary. For any such orbit, there must exist a subsegment of the orbit in the interior of P that comes back close to itself. We can then apply the Anosov closing lemma to it and get a periodic orbit in the interior of P.

Recall that, since the flow is $\mathbb{R}$-covered, each leaf space is homeomorphic to $\mathbb{R}$ and the orbit space is homeomorphic to a diagonal band in $\mathcal{L}^s \times \mathcal{L}^u$ (see Proposition 2.4). We will be using this fact in all the proof.

The annulus A lifts to a lozenge in the orbit space $\mathcal{O}$ that we denote by $\widetilde{A}$. What we mean by that is that if V is a lift of A to the universal cover $\widetilde{M}$ then the *set of orbits* intersected by the interior of V is a lozenge in $\mathcal{O}$. Let α be an orbit intersecting A, α leaves the piece P if and only if it intersects one of the exiting annuli of P. Lifting that to the orbit space, it means that, if $\widetilde{\alpha}$ is a lift of α in $\widetilde{A}$, then α exits P if and only if $\widetilde{\alpha}$ is also inside one of the lozenges in $\mathcal{O}$ that projects to one of the exiting annuli. Moreover, α accumulates on one of the boundary tori of P if and only if it is on the stable leaf of one of the periodic orbits of the boundary tori.

In the same way let $\{\widetilde{B}_i\}_i$ be the (countable) set of lozenges in $\mathcal{O}$ that are all the lifts of the exiting annuli of P. We are going to show that $\widetilde{A} \setminus \cup_i \widetilde{B}_i$ is an uncountable set in $\mathcal{O}$, that is an uncountable set of orbits. In addition the set $\widetilde{A} \setminus \cup_i \overline{\widetilde{B}_i}$, where we remove also the sides of the lozenges, is still uncountable, so any orbit in that set projects to an orbit of ϕ^t that enters P through A and never leaves P, and never accumulates on ∂P.

Since we are interested in the set $\widetilde{A} \setminus \cup_i \widetilde{B}_i$, we can already remove all the $\widetilde{B}_i$ that do not intersect $\widetilde{A}$ from our considerations. So from now on, $\{\widetilde{B}_i\}_{i\in\mathbb{N}}$ is the set of all the lifts of the exiting annuli of P such that $\widetilde{A} \cap \widetilde{B}_i \neq \emptyset$.

The first thing to remark is that $\widetilde{A} \cap \widetilde{B}_i$ is an open set in $\mathcal{O}$ and it cannot contain one of the corners of A or $\widetilde{B}_i$. Indeed, the corners of $\widetilde{A}$ and $\widetilde{B}_i$ are periodic orbits on the boundary tori, so in particular when projected to M these closed orbits do not intersect any of the open Birkhoff annuli contained in the boundary tori of P — entering or exiting. This same remark applies to $\widetilde{B}_i \cap \widetilde{B}_j$ for any $i \neq j$.

Therefore, either $\widetilde{B}_i$ intersects all the stable leaves in $\widetilde{A}$, or it intersects all the unstable ones (see Figure 10). We say that $\widetilde{B}_i$ is *vertical* if it intersects all the stable leaves of $\widetilde{A}$ (as the red (dark gray) lozenge $\widetilde{B}_j$ in Figure 10), and *horizontal* otherwise (as the blue (mid gray) lozenge $\widetilde{B}_i$ in Figure 10).

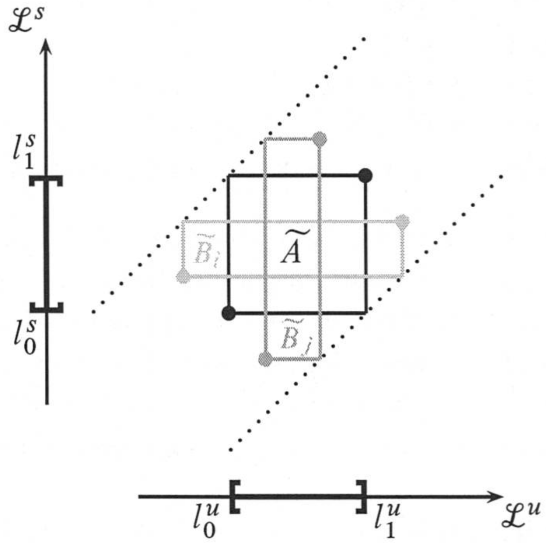


Figure 10. The lozenge $\widetilde{A}$ with a vertical and an horizontal intersecting lozenges.

Since the intersection $\widetilde{B}_i \cap \widetilde{B}_j$ cannot contain any of their corners, then, up to switching i and j, we have:

(1) either $\widetilde{B}_i$ is horizontal and $\widetilde{B}_j$ is vertical, (see Figure 10)

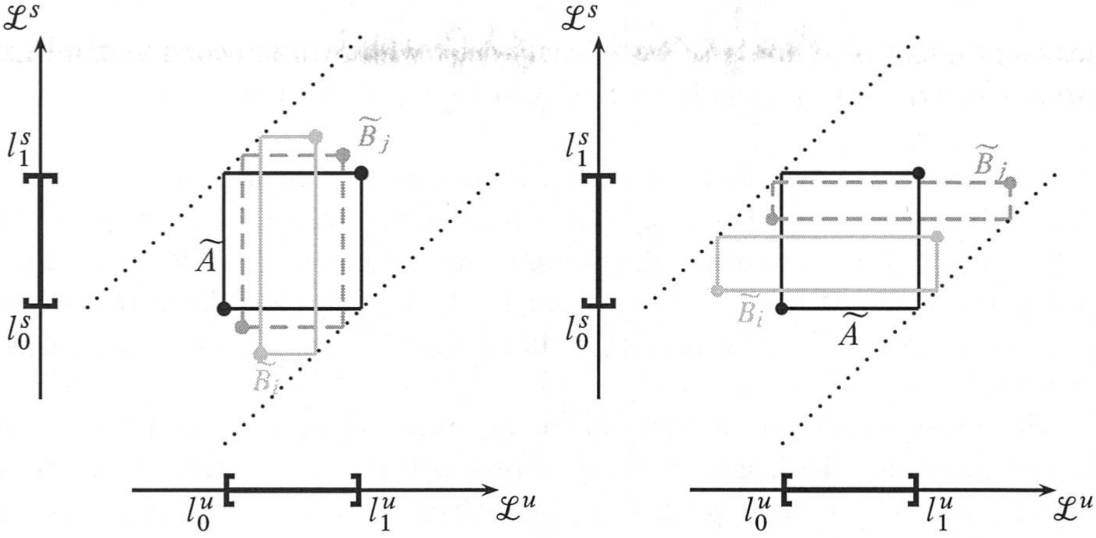
(2) or $\widetilde{B}_i \cap A \subset \widetilde{B}_j \cap A$, in particular $\widetilde{B}_i$ and $\widetilde{B}_j$ are both vertical or both horizontal (see Figure 11a)

(3) or $\widetilde{B}_i$ and $\widetilde{B}_j$ are disjoint, and $\widetilde{B}_i$ and $\widetilde{B}_j$ are both vertical or both horizontal (see Figure 11b).

It turns out that there are no vertical lozenges in $\{\widetilde{B}_i\}$ (vertical lozenges would appear if we were considering A as an *exiting* Birkhoff annulus of some other piece and took the intersection with some entering Birkhoff annulus). However, we do not really need this fact and continue as if there were some, since it saves us some work.

Let l_0^s and l_1^s be the stable sides of $\widetilde{A}$, and let l_0^u and l_1^u be the unstable sides. Note that, if l^s is the stable side of any horizontal lozenge $\widetilde{B}_i$, then $l^s \in [l_0^s, l_1^s] \subset \mathcal{L}^s = \mathbb{R}$ (see Figure 10). And, similarly, if l^u is the unstable side of any vertical lozenge, then $l^u \in [l_0^u, l_1^u]$.

We claim that for any $\widetilde{B}_i$, there are at most finitely many j such that $\widetilde{B}_i \cap A \subset \widetilde{B}_j \cap A$: Suppose that this is not the case. Then we can suppose that $\{\widetilde{B}_j \cap A\}$ is an increasing sequence. Call δ_j^0 and δ_j^1 their corners. The sequence $\{\delta_j^1\}$ stays in a compact part of the orbit space (see Figure 11a). More precisely, $\{\delta_j^1\}$ stays in the compact rectangle delimited by the stable and unstable leaves of δ_0^1 and of the top corner of $\widetilde{A}$. Hence, $\{\delta_j^1\}$ admits a converging subsequence. But this is impossible since the δ_j^1 are lifts of a finite number of periodic orbits.



(a) Case (2) $\widetilde{B}_i \cap A \subset \widetilde{B}_j \cap A$ (b) Case (3) $\widetilde{B}_i$ and $\widetilde{B}_j$ are disjoint

Figure 11. Possible types of intersections of the lozenges $\widetilde{B}_i$ (blue (mid gray)) and $\widetilde{B}_j$ (red (dark gray)) with $\widetilde{A}$ (black).

Therefore from the family $\{\widetilde{B}_i\}_{i\in\mathbb{N}}$, we can extract a subfamily $\{\widetilde{B}_i\}_{i\in I}$ such that the intersection of $\widetilde{B}_i$ with $\widetilde{A}$ is maximal for $i \in I$. So, for any $i \neq j \in I$, either $\widetilde{B}_i$ and $\widetilde{B}_j$ are not of the same type, or they are disjoint. In fact for any $\widetilde{B}_j$ of the original family, the intersection $\widetilde{B}_j \cap \widetilde{A}$ is contained in some $\widetilde{B}_i \cap \widetilde{A}$, where $i \in I$.

For any vertical lozenge $\widetilde{B}_i$, $i \in I$, we set I_i^u to be the closed interval consisting of the closure of the set of unstable leaves of $\widetilde{B}_i$. And for any horizontal lozenge, we set I_i^s to be the closed interval consisting of the closure of the set of its stable leaves. We claim that, if $i \neq j$ in I are two indices such that $\widetilde{B}_i$ and $\widetilde{B}_j$ are both vertical, then $I_i^u \cap I_j^u = \emptyset$. If not, then $\widetilde{B}_i, \widetilde{B}_j$ must share a side. If this is true then they also share a corner orbit as they are lozenges with periodic corners. But, because of the structure of skewed $\mathbb{R}$-covered Anosov flows, there are no adjacent lozenges: a contradiction.

The above implies that each of $[l_0^s, l_1^s] \smallsetminus \cap_{i\in I} I_i^s$ and $[l_0^u, l_1^u] \smallsetminus \cap_{i\in I} I_i^u$ is either a Cantor set or contains an open interval. Hence,

$$\widetilde{A} \smallsetminus \cup_i \widetilde{B}_i = \left([l_0^s, l_1^s] \smallsetminus \cap_{i\in I} I_i^s\right) \times \left([l_0^u, l_1^u] \smallsetminus \cap_{i\in I} I_i^u\right),$$

is uncountable, which finishes the proof.

In fact, the above set is a Cantor set times an interval. Indeed $[l_0^s, l_1^s] \smallsetminus \cap_{i\in I} I_i^s$ cannot contain an open set because the flow is transitive, so for each exiting annulus there is a lift of a dense orbit that intersects A and this exiting annulus. □

Proof of Theorem 8.1. Suppose that the flow is quasigeodesic, but M is not a graph-manifold. Up to taking a double cover we may assume that $\mathcal{F}^s$ is transversely orientable. Then either M is hyperbolic or there exist an atoroidal piece in its JSJ decomposition. The first case was already dealt with by the second author in [19]. More specifically, if ϕ^t is an **R**-covered Anosov flow in M hyperbolic, then ϕ^t is not quasigeodesic.

So we suppose that there exists an atoroidal piece P that is not all of M. By Proposition 8.4, there exists a periodic orbit α in the interior of P. Since the flow is $\mathbb{R}$-covered, P is atoroidal, and α is not on the boundary tori, Theorem 4.2 shows that α has an infinite free homotopy class. Let $\{\alpha_i\}_{i\in\mathbb{Z}}$ be the infinite free homotopy class of α, indexed so that α_0 is the shortest, and let $\widetilde{\alpha_i}$ be coherent lifts to the universal cover.

The idea now is to use what we did in the proof of Proposition 6.10: We showed in that proof that the length of the $\widetilde{\alpha_i}$ grows at least quadratically in the distance between $\widetilde{\alpha_i}$ and a certain geodesic c_g, but since $\widetilde{\alpha_i}$ is a quasigeodesic, its length cannot grow more than linearly in that distance, and we obtain a contradiction. As before, by length of $\widetilde{\alpha}_i$, we actually mean the length of a fundamental domain.

Let us be more precise. We use the same notations as in the proof of Proposition 6.10. In particular, P is equipped with a neutered metric d_n and $\widetilde{P}$ can be seen inside the hyperbolic space $\mathbb{H}^3$. We use an n subscript to refer to the neutered distance and h subscript for the hyperbolic distance.

Let $g \in \pi_1(P)$ be the stabilizer of the $\widetilde{\alpha_i}$. Since P is a neutered manifold, we can think of g as a hyperbolic isometry. Let c_g be the geodesic in $\mathbb{H}^3$ associated to g. Let x be a point on c_g which projects to a point inside P, and let H_x be the hyperbolic hyperplane through x and orthogonal to c_g. Finally, let x_i be the closest point on $\widetilde{\alpha_i} \cap H_x$. Using Lemma 6.6, we get

$$l_n(\widetilde{\alpha_i}) = l_h(\widetilde{\alpha_i}) \geq l_h(c_g)\frac{e^{d_h(x_i,x)}}{2}.$$

And, using Lemma 6.12, we have

$$e^{d_h(x_i,x)/2} \geq d_n(x_i, x).$$

So, we have

$$l_n(\widetilde{\alpha_i}) \geq \frac{l_h(c_g)}{2} d_n(x_i, x)^2. \tag{8.1}$$

Now, since we assumed that ϕ^t is a quasigeodesic flow, the orbits $\widetilde{\alpha_i}$ are quasigeodesics and since they stay in the atoroidal piece P, they are quasigeodesics for the neutered distance. So there exist constants $C_1 \geq 1$ and $C_2 \geq 0$ such that

$$l_n(\widetilde{\alpha_i}) \leq C_1 d_n(x_i, g \cdot x_i) + C_2.$$

And the triangle inequality gives that

$$\begin{aligned} d_n(x_i, g \cdot x_i) &\leq d_n(x_i, x) + d_n(x, g \cdot x) + d_n(g \cdot x, g \cdot x_i) \\ &= 2d_n(x_i, x) + d_n(x, g \cdot x). \end{aligned}$$

So, setting $C_3 = C_1 d_n(x, g \cdot x) + C_2$, we get

$$l_n(\widetilde{\alpha_i}) \leq 2C_1 d_n(x_i, x) + C_3.$$

Together with equation (8.1), this gives, for all i

$$2C_1 d_n(x_i, x) + C_3 \geq \frac{l_h(c_g)}{2} d_n(x_i, x)^2. \tag{8.2}$$

But $d_n(\widetilde{\alpha_i}, \widetilde{\alpha_0}) \geq Ai$ for some uniform $A > 0$, thanks to Lemma 2.28. So $d_n(x_i, x) \geq Ai - d_{\mathrm{Haus}}(\widetilde{\alpha_0}, c_g)$, hence the equation (8.2) cannot hold for big i, and we obtained our contradiction. □

References

[1] D. V. Anosov, Geodesic flows on closed Riemannian manifolds of negative curvature, *Trudy Mat. Inst. Steklov.*, **90** (1967), 209pp. Zbl 0176.19101 MR 224110

[2] M. Babillot and F. Ledrappier, Lalley's theorem on periodic orbits of hyperbolic flows, *Ergodic Theory Dynam. Systems*, **18** (1998), no. 1, 17–39. Zbl 01268726 MR 1609507

[3] T. Barbot, Caractérisation des flots d'Anosov en dimension 3 par leurs feuilletages faibles, *Ergodic Theory Dynam. Systems*, **15** (1995), no. 2, 247–270. Zbl 0826.58025 MR 1332403

[4] T. Barbot, Mise en position optimale de tores par rapport à un flot d'Anosov, *Comment. Math. Helv.*, **70** (1995), no. 1, 113–160. Zbl 0838.58029 MR 1314942

[5] T. Barbot, Flots d'Anosov sur les variétés graphées au sens de Waldhausen, *Ann. Inst. Fourier (Grenoble)*, **46** (1996), no. 5, 1451–1517. Zbl 0861.58028 MR 1427133

[6] T. Barbot, Generalizations of the Bonatti–Langevin example of Anosov flow and their classification up to topological equivalence, *Comm. Anal. Geom.*, **6** (1998), no. 4, 749–798. Zbl 0916.58033 MR 1652255

[7] T. Barbot, Plane affine geometry and Anosov flows, *Ann. Sci. École Norm. Sup. (4)*, **34** (2001), no. 6, 871–889. Zbl 1098.37513 MR 1872423

[8] T. Barbot, *De l'hyperbolique au globalement hyperbolique*, Habilitation à diriger des recherches, Université Claude Bernard de Lyon, 2005.

[9] T. Barbot and S. Fenley, Pseudo–Anosov flows in toroidal manifolds, *Geom. Topol.*, **17** (2013), no. 4, 1877–1954. Zbl 1317.37038 MR 3109861

[10] T. Barbot and S. Fenley, Classification and rigidity of totally periodic pseudo-anosov flows in graph manifolds, *Ergodic Theory Dynam. Systems*, **35** (2015), 1681–1722. Zbl 1352.37081 MR 3377280

[11] T. Barbot and S. Fenley, Free seifert pieces of pseudo-Anosov flows, 2015. arXiv:1512.06341

[12] F. Béguin, C. Bonatti, and B. Yu, Building Anosov flows on 3-manifolds, *Geom. Topol.*, **21** (2017), no. 3, 1837–1930. Zbl 06726513 MR 3650083

[13] C. Bonatti and R. Langevin, Un exemple de flot d'Anosov transitif transverse à un tore et non conjugué à une suspension, *Ergodic Theory Dynam. Systems*, **14** (1994), no. 4, 633–643. Zbl 00700895 MR 1304136

[14] R. Bowen, Periodic orbits for hyperbolic flows, *Amer. J. Math.*, **94** (1972), 1–30. Zbl 0254.58005 MR 298700

[15] M. Brunella, Surfaces of section for expansive flows on three-manifolds, *J. Math. Soc. Japan*, **47** (1995), no. 3, 491–501. Zbl 0828.58024 MR 1331326

[16] D. Calegari, The geometry of **R**-covered foliations, *Geom. Topol.*, **4** (2000), 457–515 (electronic). Zbl 0964.57014 MR 1800151

[17] J. W. Cannon and W. P. Thurston, Group invariant Peano curves, *Geom. Topol.*, **11** (2007), 1315–1355. Zbl 1136.57009 MR 2326947

[18] D. B. A. Epstein, Curves on 2-manifolds and isotopies, *Acta Math.*, **115** (1966), 83–107. Zbl 0136.44605 MR 214087

[19] S. R. Fenley, Anosov flows in 3-manifolds, *Ann. of Math. (2)*, **39** (1994), no. 1, 79–115. Zbl 0796.58039 MR 1259365

[20] S. R. Fenley, Quasigeodesic Anosov flows and homotopic properties of flow lines, *J. Differential Geom.*, **41** (1995), no. 2, 479–514. Zbl 0832.57009 MR 1331975

[21] S. R. Fenley, Incompressible tori transverse to Anosov flows in 3-manifolds, *Ergodic Theory Dynam. Systems*, **17** (1997), no. 1, 105–121. Zbl 0924.57022 MR 1440770

[22] S. R. Fenley, The structure of branching in Anosov flows of 3-manifolds, *Comment. Math. Helv.*, **73** (1998), no. 2, 259–297. Zbl 0999.37008 MR 1611703

[23] S. R. Fenley, Surfaces transverse to pseudo-Anosov flows and virtual fibers in 3-manifolds, *Topology*, **38** (1999), no. 4, 823–859. Zbl 0926.57009 MR 1679801

[24] S. R. Fenley, Foliations, topology and geometry of 3-manifolds: **R**-covered foliations and transverse pseudo-Anosov flows, *Comment. Math. Helv.*, **77** (2002), no. 3, 415–490. Zbl 1010.57008 MR 1933786

[25] S. R. Fenley, Ideal boundaries of pseudo-Anosov flows and uniform convergence groups, with connections and applications to large scale geometry, *Geom. Topol.*, **16** (2012), no. 1, 1–110. Zbl 1279.37026 MR 2872578

[26] S. R. Fenley, Diversified homotopic behavior of closed orbits of some $\mathbb{R}$-covered Anosov flows, *Ergodic Theory Dynam. Systems*, **36** (2016), 767–780. Zbl 1366.37073 MR 3480342

[27] S. R. Fenley, Quasigeodesic pseudo-Anosov flows in hyperbolic 3-manifolds and connections with large scale geometry, *Advances in Math.*, **303** (2016), 192–278. Zbl 1366.37085 MR 3552525

[28] P. Foulon and B. Hasselblatt, Contact Anosov flows on hyperbolic 3-manifolds, *Geom. Topol.*, **17** (2013), no. 2, 1225–1252. Zbl 1277.37057 MR 3070525

[29] J. Franks and B. Williams, Anomalous Anosov flows, in *Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979)*, 158–174, Lecture Notes in Math., 819, Springer, Berlin, 1980. Zbl 0463.58021 MR 591182

[30] D. Gabai, Convergence groups are Fuchsian groups, *Ann. of Math. (2)*, **136** (1992), no. 3, 447–510. Zbl 0785.57004 MR 1189862

[31] É. Ghys, Flots d'Anosov sur les 3-variétés fibrées en cercles, *Ergodic Theory Dynam. Systems*, **4** (1984), no. 1, 67–80. Zbl 0527.58030 MR 758894

[32] É. Ghys, Codimension one Anosov flows and suspensions, in *Dynamical systems (Valparaiso, 1986)*, 59–72, Lecture Notes in Math., 1331, Springer, Berlin, 1988. Zbl 0672.58033 MR 961093

[33] M. Gromov, Hyperbolic groups, in *Essays in group theory*, 75–263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987. Zbl 0634.20015 MR 919829

[34] M. Handel and W. P. Thurston, Anosov flows on new three manifolds, *Invent. Math.*, **59** (1980), no. 2, 95–103. Zbl 0435.58019 MR 577356

[35] J. Hempel, 3-*Manifolds*, Ann. of Math. Studies, 86, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1976. Zbl 0345.57001 MR 415619

[36] T. Inaba and S. Matsumoto, Nonsingular expansive flows on 3-manifolds and foliations with circle prong singularities, *Japan. J. Math. (N.S.)*, **16** (1990), no. 2, 329–340. Zbl 0729.58046 MR 1091166

[37] W. Jaco and P. Shalen, Seifert fibered spaces in 3-manifolds, *Mem. Amer. Math. Soc.*, **21** (1979), no. 220, viii+192pp. Zbl 0415.57005 MR 539411

[38] A. Katsuda and T. Sunada, Closed orbits in homology classes, *Inst. Hautes Études Sci. Publ. Math.*, (1990), no. 71, 5–32. Zbl 0728.58026 MR 1079641

[39] Y. Kifer, Large deviations, averaging and periodic orbits of dynamical systems, *Comm. Math. Phys.*, **162** (1994), no. 1, 33–46. Zbl 0797.58068 MR 1272765

[40] W. Klingenberg, *Riemannian geometry*, second edition, de Gruyter Studies in Mathematics, 1, Walter de Gruyter & Co., Berlin, 1995. Zbl 0911.53022 MR 1330918

[41] G. A. Margulis, *On some aspects of the theory of Anosov systems*, with a survey by Richard Sharp: Periodic orbits of hyperbolic flows, translated from the Russian by Valentina Vladimirovna Szulikowska, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004. Zbl 1140.37010 MR 2035655

[42] W. Parry and M. Pollicott, An analogue of the prime number theorem for closed orbits of Axiom A flows, *Ann. of Math. (2)*, **118** (1983), no. 3, 573–591. Zbl 0537.58038 MR 727704

[43] M. Paternain, Expansive flows and the fundamental group, *Bol. Soc. Brasil. Mat. (N.S.)*, **24** (1993), no. 2, 179–199. Zbl 0796.58040 MR 1254982

[44] R. Phillips and P. Sarnak, Geodesics in homology classes, *Duke Math. J.*, **55** (1987), no. 2, 287–297. Zbl 0642.53050 MR 894581

[45] J. F. Plante, Anosov flows, transversely affine foliations, and a conjecture of Verjovsky, *J. London Math. Soc. (2)*, **23** (1981), no. 2, 359–362. Zbl 0465.58020 MR 609116

[46] J. F. Plante and W. P. Thurston, Anosov flows and the fundamental group, *Topology*, **11** (1972), 147–150. Zbl 0246.58014 MR 295389

[47] M. Pollicott and R. Sharp, Error terms for closed orbits of hyperbolic flows, *Ergodic Theory Dynam. Systems*, **21** (2001), no. 2, 545–562. Zbl 0988.37024 MR 1827118

[48] H. Rosenberg, Foliations by planes, *Topology*, **7** (1968), 131–138. Zbl 0157.30504 MR 228011

[49] J.-P. Serre, *Trees*, translated from the French by John Stillwell, Springer-Verlag, Berlin-New York, 1980. Zbl 0548.20018 MR 607504

[50] R. Sharp, Closed orbits in homology classes for Anosov flows, *Ergodic Theory Dynam. Systems*, **13** (1993), no. 2, 387–408. Zbl 0783.58059 MR 1235480

[51] S. Simić, Codimension one Anosov flows and a conjecture of Verjovsky, *Ergodic Theory Dynam. Systems*, **17** (1997), no. 5, 1211–1231. Zbl 0903.58026 MR 1477039

[52] W. P. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc. (N.S.)*, **6** (1982), no. 3, 357–381. Zbl 0496.57005 MR 648524

[53] W. P. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Silvio Levy (ed.), Princeton Mathematical Series, 35, Princeton University Press, Princeton, NJ, 1997. Zbl 0873.57001 MR 1435975

[54] A. Verjovsky, Codimension one Anosov flows, *Bol. Soc. Mat. Mexicana (2)*, **19** (1974), no. 2, 49–77. Zbl 0323.58014 MR 431281

[55] A. Zeghib, Sur les feuilletages géodésiques continus des variétés hyperboliques, *Invent. Math.*, **114** (1993), no. 1, 193–206. Zbl 0789.57019 MR 1235023

Received July 22, 2015

T. Barthelmé, Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
Current address: Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
E-mail: thomas.barthelme@queensu.ca

S. R. Fenley, Department of Mathematics, Florida State University, Tallahassee, FL 32306, USA
E-mail: fenley@math.fsu.edu