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Stable ergodicity and accessibility for certain partially hyperbolic diffeomorphisms with bidimensional center leaves

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Abstract. We consider classes of partially hyperbolic diffeomorphism $f : M \rightarrow M$ with splitting $TM = E^s \oplus E^c \oplus E^u$ and $\dim E^c = 2$. These classes include for instance (perturbations of) the product of Anosov and conservative surface diffeomorphisms, skew products of surface diffeomorphisms over Anosov, partially hyperbolic symplectomorphisms on manifolds of dimension four with bidimensional center foliation whose center leaves are all compact. We prove that accessibility holds in these classes for C^1 open and C^r dense subsets and moreover they are stably ergodic.

Mathematics Subject Classification (2010). 37D30; 37A25, 37C40.

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1. Introduction

Ergodicity plays a fundamental role in Dynamics (and in Probability and Physics) since L. Boltzmann stated the “ergodic hypothesis” which says (roughly speaking) that in an evolution law time average and space average are equal. More precisely, we say that a dynamical system $f : M \rightarrow M$ preserving a finite measure m is ergodic (with respect to m) if any invariant set has zero measure or its complement has zero measure.

E. Hopf [19] proved the ergodicity of the geodesic flow on surfaces of negative curvature. This was extended by Anosov to the geodesic flow on compact manifolds with negative curvature in a cornerstone paper in dynamics [3]. He also proved that conservative (today called) Anosov $C^{1+\alpha}$ diffeomorphisms are ergodic. And, since Anosov diffeomorphisms are open, the above implies that conservative Anosov systems are stably ergodic. We say that a C^r diffeomorphism $f : M \rightarrow M$ preserving a measure m is C^r *stably ergodic* if any sufficiently small C^r perturbation of f preserving m is ergodic.

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In a seminal work, Grayson, Pugh, and Shub [18] proved that the time one map of the geodesic flow of a hyperbolic surface is C^2 stably ergodic. Afterwards, Ch. Pugh and M. Shub recovered (in some sense) Smale's program in the sixties about stability and genericity by restricting to partially hyperbolic diffeomorphisms on manifolds preserving the Lebesgue measure and replacing structural stability by stable ergodicity. They conjectured that among C^2 partially hyperbolic diffeomorphism preserving the Lebesgue measure m , stable ergodicity holds in an open and dense set. They proved important results in this direction and they proposed a program as well (see [25,26], and [27]). The main conjecture is:

Conjecture 1.1 ([27]). *On any compact manifold, ergodicity holds for an open and dense set of C^2 volume preserving partially hyperbolic diffeomorphisms.*

This conjecture splits into two conjectures where accessibility (see Definition 1.6) plays a key role:

Conjecture 1.2 ([27]). *Accessibility holds for an open and dense set of C^2 partially hyperbolic diffeomorphism, volume preserving or not.*

Conjecture 1.3 ([27]). *A partially hyperbolic C^2 volume preserving diffeomorphism with the essential accessibility property is ergodic.*

They also proved [27] a result in the direction of the third conjecture: *A partially hyperbolic C^2 volume preserving diffeomorphism, dynamically coherent, center bunched, and with the essential accessibility property is ergodic.* Since then, a lot of research on the field has been done. See the surveys [7,31,37], and [12] for an account on this progress during the last decades.

In [10], K. Burns and A. Wilkinson improved a lot Pugh–Shub result in two directions: dynamically coherence is not needed and the center bunching condition is much milder than originally stated.

The key fact thus to obtain ergodicity is *accessibility*. In [14] it is proved that accessibility holds for a C^1 open and dense subset of C^r partially hyperbolic diffeomorphism, volume preserving or not. When the center bundle has dimension one, it is proved in [32] that accessibility holds for a C^1 open and C^r dense subset of C^r partially hyperbolic volume preserving diffeomorphism (later extended to the non-volume preserving case in [5]). This in particular implies the main conjecture in its full generality when the center dimension is one.

There has been in the last years a great advance to the main conjecture in the C^1 topology. In fact in [30] it is proved that stably ergodicity is C^1 dense when the center dimension is two. And recently, an outstanding result has been obtained by A. Avila, S. Crovisier, and A. Wilkinson [1]: stable ergodicity is C^1 dense in any case (without any assumption on the dimension of the center bundle).

These results depends heavily on perturbation techniques available in the C^1 topology and not known on higher topologies. The C^r denseness of stable ergodicity, $r \geq 2$, is a complete different problem. Little is known in this case

when the center bundle has dimension greater than one. In [8], the authors prove C^r density of stable ergodicity for group extensions over Anosov diffeomorphisms. A remarkable result has been obtained by F. Rodriguez Hertz [29] for certain automorphisms of the torus \mathbb{T}^d . Also, in [35] are given two examples that can be C^r , $r \geq 2$, approximated by stable ergodic ones. And very recently Z. Zhang [41] obtained C^r density of stable ergodicity for volume preserving diffeomorphisms satisfying some pinching condition and a certain type of dominated splitting on the center. A. Avila and M. Viana have announced C^1 openness and C^r density for certain skew product of surfaces diffeomorphisms over Anosov and our work might have some overlap with theirs although our methods are different.

Our aim in this paper is to contribute to the C^r denseness of stable ergodicity, in particular when the center dimension is two. We prove that for large classes of C^r partially hyperbolic volume preserving diffeomorphisms with two dimensional center bundle, stable ergodicity holds in C^r dense subsets. Precise statements are given in Section 1.2. However, just to give a flavor of them let us state a particular case (see Theorems 4A and 4B).

Theorem 1. *Ergodicity holds in C^1 open and C^r dense subset in the following settings:*

- *Skew products of conservative surfaces diffeomorphisms over conservative Anosov diffeomorphisms.*
- *Partially hyperbolic symplectomorphisms on (M, ω) where $\dim M = 4$ having a bidimensional center foliation whose leaves are all compact.*

The main tool we use to prove the ergodicity is accessibility. Thus, we have to prove that accessibility holds in a C^1 open and C^r dense subset in the setting we are working with. The main idea is to use results on conservative surface dynamics to show that generically one gets accessibility. Indeed, when the center dimension is two and we look to the accessibility class inside a (periodic) compact center leaf we have three possibilities: it has zero, one or two topological dimensions. We prove that generically (see Theorem 2) zero dimensional accessibility classes do not exist. We will use to the full extent results on conservative surface dynamics to prove that also generically one-dimensional accessibility classes do not exist and therefore the accessibility classes are open on the center leaf and so there is just one accessibility class.

1.1. Setting. Let $f : M \rightarrow M$ be a diffeomorphism where M is a compact riemannian manifold without boundary. We say that f is *partially hyperbolic* if the tangent bundle splits into three subbundles $TM = E^s \oplus E^c \oplus E^u$ invariant under the tangent map Df and such that:

- There exists $0 < \lambda < 1$ such that

$$\|Df|_{E^s}\| < \lambda \quad \text{and} \quad \|Df|_{E^u}^{-1}\| < \lambda.$$

- For every $x \in M$ we have

$$\frac{\|Df|_{E_x^s}\|}{m\{Df|_{E_x^c}\}} < 1 \quad \text{and} \quad \frac{\|Df|_{E_x^c}\|}{m\{Df|_{E_x^u}\}} < 1,$$

where $m\{A\}$ is the co-norm of A , i.e., $m\{A\} = \|A^{-1}\|^{-1}$.

By continuity of Df and the compactness of M , there is a positive constant $\eta < 1$ such that the inequalities in the last item hold for η instead of 1. In other words, E^s is uniformly contracted, E^u is uniformly expanded and the behaviour of E^c is between both.

It is well known that the subbundles E^s and E^u uniquely integrate to two foliations $\mathcal{F}^s = \mathcal{F}_f^s$ and $\mathcal{F}^u = \mathcal{F}_f^u$ called the stable and unstable foliation respectively. We denote by $\mathcal{F}^\sigma(x)$ ($\sigma = s, u$) the leaf of the foliation through the point x .

On the other hand it is not always true that the center subbundle E^c is integrable. We say that the partially hyperbolic diffeomorphism f is *dynamically coherent* if the bundles $E^s \oplus E^c$ and $E^c \oplus E^u$ integrate to invariant foliations \mathcal{F}^{cs} and \mathcal{F}^{cu} called the center stable and center unstable foliations respectively. In particular E^c integrates to a (normally hyperbolic) invariant foliation \mathcal{F}^c . Moreover, \mathcal{F}^c and \mathcal{F}^s subfoliates \mathcal{F}^{cs} and \mathcal{F}^c and \mathcal{F}^u subfoliates \mathcal{F}^{cu} , see [9].

We say the center foliation is *r-normally hyperbolic* ($r \geq 1$) if the following holds:

$$\frac{\|Df|_{E_x^s}\|}{m\{Df|_{E_x^c}\}} < 1 \quad \text{and} \quad \frac{\|Df|_{E_x^c}\|^r}{m\{Df|_{E_x^u}\}} < 1.$$

If f is of class C^r and the center foliation is r -normally hyperbolic then the leaves of \mathcal{F}^c are of C^r class (see [20]).

Partially hyperbolic diffeomorphisms are C^1 open. In order to assure that dynamical coherence also holds for C^1 systems nearby we have to require *plaque expansiveness*. This is technical and we will not define it here, we refer to [20] (however, if \mathcal{F}^c is a C^1 foliation or all leaves of \mathcal{F}^c are compact then the center foliation is plaque expansive). The results on [20] (see Theorem 7.4) assure that a normally hyperbolic and plaque expansive foliation \mathcal{F}_f^c of a diffeomorphism f is structurally stable, that is, there exist a neighborhood of $\mathcal{U}(f)$ and a homeomorphism $h : M \rightarrow M$ such that for $g \in \mathcal{U}(f)$ there exists a (normally hyperbolic) foliation \mathcal{F}_g^c such that $h(\mathcal{F}_f^c(x)) = \mathcal{F}_g^c(h(x))$ and $h(\mathcal{F}_f^c(f(x))) = \mathcal{F}_g^c(g(h(x)))$. This result implies that *partially hyperbolic diffeomorphisms, dynamically coherent with center foliation plaque expansive are C^1 open*.

We also say that a partially hyperbolic diffeomorphism f is *center bunched* if:

$$\|Df|_{E_x^s}\| \frac{\|Df|_{E_x^c}\|}{m\{Df|_{E_x^c}\}} < 1 \quad \text{and} \quad \frac{\|Df|_{E_x^c}\|}{m\{Df|_{E_x^c}\}} \frac{1}{m\{Df|_{E_x^u}\}} < 1.$$

This bunching condition is as in [10] where they improve substantially the one stated by Pugh–Shub originally. Notice that the bunching condition is also C^1 open.

We say that a partially hyperbolic diffeomorphism $f : M \rightarrow M$ dynamically coherent has *Global Product Structure* (GPS for short) if there is a covering $\pi : \tilde{M} \rightarrow M$ and a lift $\tilde{f} : \tilde{M} \rightarrow \tilde{M}$ of f such that when we lift the invariant foliations (stable, unstable, center-stable, and center-unstable) to \tilde{M} we have for any \tilde{x}, \tilde{y} in \tilde{M} :

$$\#\{\tilde{\mathcal{F}}^{cs}(\tilde{x}) \cap \tilde{\mathcal{F}}^u(\tilde{y})\} = 1 \quad \text{and} \quad \#\{\tilde{\mathcal{F}}^{cu}(\tilde{x}) \cap \tilde{\mathcal{F}}^s(\tilde{y})\} = 1.$$

The GPS also implies that, in the covering \tilde{M} every stable leaf intersects in one point every center leaf inside a center stable leaf, i.e. if $\tilde{y}, \tilde{z} \in \tilde{\mathcal{F}}^{cs}(\tilde{x})$ then $\#\{\tilde{\mathcal{F}}^s(\tilde{y}) \cap \tilde{\mathcal{F}}^c(\tilde{z})\} = 1$. The same in center-unstable leaves in the covering. In particular, if $y, z \in \mathcal{F}^{cs}(x)$, then $\mathcal{F}^s(y) \cap \mathcal{F}^c(z) \neq \emptyset$, although this intersection might not be unique.

One of the main property of the GPS is that it allow us to define a global projection (in \tilde{M}) onto a given center stable manifold along the holonomy of the unstable foliation. We denote by \mathcal{GPS} the C^1 interior of the set of partially hyperbolic diffeomorphisms dynamically coherent having GPS.

Definition 1.4. Let M be a compact riemannian manifold without boundary and let $r \geq 1$. We denote by $\mathcal{E}^r = \mathcal{E}^r(M)$ the set of C^r diffeomorphisms $f : M \rightarrow M$ (with the C^r topology) such that

- f is partially hyperbolic;
- f is dynamically coherent;
- the center foliation is r -normally hyperbolic and plaque expansive;
- f is center bunched;
- $f \in \mathcal{GPS}$; and
- the set of center leaves that are compact and f -periodic are dense in M .

We remark that \mathcal{E}^r is C^1 open (and hence C^r open as well).

Examples. Here we give some examples of diffeomorphism in \mathcal{E}^r . We restrict ourselves where the center dimension is two.

(1) *Perturbation of product of diffeomorphisms:* Let $g : S \rightarrow S$ be a C^r diffeomorphism of a compact surface and let $f : N \rightarrow N$ be a transitive Anosov diffeomorphism. If the contraction and expansion of f are strong enough we get that $f \times g \in \mathcal{E}^r(M)$ where $M = N \times S$. Notice that the center foliation consists of compact manifolds homeomorphic to S . In particular, the space of center leaves is homeomorphic to N and the dynamics of the center leaves is conjugated to the Anosov diffeomorphism $f : N \rightarrow N$ and hence (lifting to the covering $\tilde{M} = \tilde{N} \times S$ where \tilde{N} is the universal covering of N) $f \times g$ has GPS (and indeed belongs to \mathcal{GPS}). In case $g = \text{id}$ then automatically $f \times g \in \mathcal{E}^r$ for any $r \geq 1$.

(2) *Skew products over Anosov*: Let $f : N \rightarrow N$ be a C^r (transitive) Anosov diffeomorphism and consider S a compact surface. Let $\mathcal{U} \subset \text{Diff}^r(S)$ be the open set such that if $h \in \mathcal{U}$ then $f \times h$ is partially hyperbolic with center fiber $\{x\} \times S$, center bunched, and r -normally hyperbolic. Let $g : N \rightarrow \mathcal{U}$ be a continuous map. For $x \in N$ let's denote by g_x the diffeomorphism $g(x) : S \rightarrow S$. For such a map g consider the skew product $F = f \times_{sp} g : N \times S \rightarrow N \times S$ by

$$F(x, y) = (f(x), g_x(y)).$$

Notice that the center foliation is $\{\{x\} \times S, x \in N\}$ and the dynamics of the center leaves is the one of the Anosov $f : N \rightarrow N$. It follows that F has GPS (and any C^r perturbation has GPS as well) and the periodic center leaves are dense. Thus, we have that $F \in \mathcal{E}^r(N \times S)$. We may consider thus perturbations of F in \mathcal{E}^r and also perturbations in the skew product setting. For this, let $\mathcal{G} = \{g : N \rightarrow \mathcal{U} : \text{continuous}\}$, where $g, \tilde{g} \in \mathcal{G}$ are close if g_x, \tilde{g}_x are C^r close for all $x \in N$. We denote by \mathcal{E}_{sp}^r the set of skew products $f \times_{sp} g$ with $g \in \mathcal{G}$.

If ω is an area (symplectic) form on S we denote by $\mathcal{E}_{sp,\omega}^r$ the set of skew products as above where g_x preserves ω for all $x \in N$. Notice that in general $F \in \mathcal{E}_{sp,\omega}^r$ does not preserve volume. If the base map $f : N \rightarrow N$ preserves a volume form η then $F \in \mathcal{E}_{sp,\omega}^r$ preserves the volume form in $N \times S$ given by $\eta \times \omega$.

(3) Let $f : M \rightarrow M$ be partially hyperbolic diffeomorphism of a four dimensional manifold, dynamically coherent whose center leaves are all compact and bidimensional. Results of Gogolev [17] (see also Carrasco [11] and Bohnet [6]) implies that center foliation is uniformly compact (that is, the leaves have finite holonomy), and that fibers over an Anosov diffeomorphism on the torus \mathbb{T}^2 . In particular it has GPS and the periodic center leaves are dense.

(4) *(Perturbation of) the product of the time t of an Anosov suspension and a rotation*: Consider $f : N \rightarrow N$ the time t map of the suspension of a transitive Anosov diffeomorphism and let $R : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a rotation. Let $f \times R : N \times \mathbb{S}^1 \rightarrow N \times \mathbb{S}^1$. It is not difficult to see that belongs to \mathcal{E}^r for any r as long as f is C^r .

(5) *(Perturbation of) the product of time maps of Anosov suspensions*: Let f, g be time maps of the suspensions of a transitive Anosov diffeomorphisms. Then $f \times g$ belongs to \mathcal{E}^r .

Remark 1.5. We considered time maps of Anosov suspensions so that there is a lift with GPS. There are time-1 map of Anosov flows without GPS, for instance, time-1 map of the geodesic flow in a surface of negative curvature.

1.2. Statements of Results. We denote by $\mathcal{E}_m^r(M)$ the set of diffeomorphisms in \mathcal{E}^r preserving a volume form m on M , and by \mathcal{E}_ω^r the ones in \mathcal{E}^r preserving a symplectic form ω on M . And recall that $\mathcal{E}_{sp}^r(M)$, $\mathcal{E}_{sp,\omega}^r(M)$ are the skew products over Anosov diffeomorphism on $M = N \times S$ where S is a compact surface and ω is an area form on S .

Our results mainly concerns accessibility, so let us introduce the concept.

Definition 1.6. Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism in \mathcal{E}^r . A *su-path* is a continuous curve $\alpha : [0, 1] \rightarrow M$ such that there exists a partition $0 = t_0 < t_1 < \dots < t_n = 1$ such that $\alpha([t_i, t_{i+1}])$ is contained either in a leaf of \mathcal{F}^s or in a leaf of \mathcal{F}^u . The relation $x \sim y$ if there exists a *su-path* from x to y is an equivalence relation on M .

For a point $x \in M$ the accessibility class $AC(x)$ of x is:

$$AC(x) = \{y \in M : \text{there is a } su\text{-path from } x \text{ to } y\}.$$

We say that f is *accessible* if $AC(x) = M$ for some x (and hence for all $x \in M$). On the other hand, we say the accessibility class $AC(x)$ is *trivial* if $AC(x) \cap \mathcal{F}^c(x)$ is totally disconnected.

Our first result concerns trivial accessibility classes (with no restriction on the dimension of the center leaves):

Theorem 2. *Let $r \geq 2$ and let \mathcal{E} denote $\mathcal{E}^r, \mathcal{E}_m^r, \mathcal{E}_\omega^r, \mathcal{E}_{sp}^r$ or $\mathcal{E}_{sp,\omega}^r$. Then, the set \mathcal{R}_0 of diffeomorphisms in \mathcal{E} having no trivial accessibility classes is C^1 open and C^r dense.*

The next result gives a condition to assure accessibility when the center leaves have dimension two. Recall that a *sink* is an attracting periodic orbit. Call a diffeomorphism $g : S \rightarrow S$ *sinkless* if it has no sinks. Define a subset \mathcal{E}_A^r of \mathcal{E}^r by $f \in \mathcal{E}_A^r$ if there is a periodic center leaf $L = f^k(L)$ for which f^k restricted to L is both Axiom A and sinkless. Note that \mathcal{E}_A^r is a C^1 open subset of \mathcal{E}^r .

Theorem 3A. *Assume that $r \geq 2$ and $\dim E^c = 2$. Then, there exists $\widehat{\mathcal{R}} \subset \mathcal{E}_A^r$ which is C^1 open and C^r dense in \mathcal{E}_A^r such that any $f \in \widehat{\mathcal{R}}$ is accessible.*

In the conservative setting define $\mathcal{E}_{A,m}^r := \mathcal{E}_A^r \cap \mathcal{E}_m^r$ and note that this is a C^1 open subset of \mathcal{E}_m^r .

Theorem 3B. *Assume that $r \geq 2$ and $\dim E^c = 2$. Then, there exists $\widehat{\mathcal{R}} \subset \mathcal{E}_{A,m}^r$ which is C^1 open and C^r dense in $\mathcal{E}_{A,m}^r$ such that any $f \in \widehat{\mathcal{R}}$ is accessible. In particular any $f \in \widehat{\mathcal{R}}$ is stably ergodic.*

Above, we considered a sinkless diffeomorphism $f|_L^k$. By replacing f with its inverse, analogous results also hold when $f|_L^k$ is both Axiom A and “sourceless”.

Finally, the next two results says that accessibility holds generically in the case of skew products or when a symplectic form is preserved (they imply Theorem 1):

Theorem 4A. *Assume that $r \geq 2$ and $\dim E^c = 2$ and consider the space $\mathcal{E}_{sp,\omega}^r$ of skew-products preserving an area form ω . Then, there exists $\mathcal{R} \subset \mathcal{E}_{sp,\omega}^r$ which is C^1 open and C^r dense such that all diffeomorphism in \mathcal{R} are accessible. In particular, if the base map preserve a volume form η , any $f \in \mathcal{R}$ is stably ergodic with respect to the volume induced by $\eta \times \omega$.*

Theorem 4B. *Assume that $r \geq 2$ and $\dim E^c = 2$ and consider the subspace \mathcal{E}_ω^r of \mathcal{E}^r formed by those preserving a symplectic form ω . Then, there exists $\mathcal{R} \subset \mathcal{E}_\omega^r$ which is C^1 open and C^r dense in \mathcal{E}_ω^r such that all diffeomorphism in \mathcal{R} are accessible. In particular they are stable ergodic with respect to the volume induced by the symplectic form ω .*

Organization of the paper. In Section 2 we give general facts concerning accessibility classes and some results regarding its structure when the center has dimension two. In Section 3 we prove some perturbation results in order to obtain later some generic results on the accessibility classes. Section 4 is devoted to prove Theorem 2. The accessibility classes of periodic points are studied in Section 5, where it is proved that generically, when the center subbundle has dimension two, the accessibility classes of hyperbolic periodic points or elliptic (when we restrict to an invariant center leaf) are open. Theorem 3A and Theorem 3B are proved in Section 6. Finally, in Section 7 we prove Theorem 4A and Theorem 4B.

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2. Basic facts on accessibility

In this section we establish some basic results on accessibility. We assume that $f : M \rightarrow M$ belongs to \mathcal{E}^r , $r \geq 2$, although some remarks hold in general.

When $y \in \mathcal{F}^s(x)$ denote by $\Pi_f^s(\mathcal{F}^c(x), \mathcal{F}^c(y))$ the (local) holonomy map from a neighborhood of x in $\mathcal{F}^c(x)$ to a neighborhood of y in $\mathcal{F}^c(y)$ along the stable leaves (inside $\mathcal{F}^{cs}(x) = \mathcal{F}^{cs}(y)$). This map is well defined since the leaves of \mathcal{F}^s are simple connected. By [28] this holonomy map is of class C^1 , i.e. the holonomy map inside center stable leaves along stable leaves is C^1 . The same holds for center unstable leaves and holonomy along unstable leaves and so, for $\Pi_f^u(\mathcal{F}^c(x), \mathcal{F}^c(y))$.

Recall that a diffeomorphism f in \mathcal{E}^r has Global Product Structure, that is, there exists a covering map $\pi : \tilde{M} \rightarrow M$ such that denoting by $\tilde{\mathcal{F}}^*$, $*$ = s, u, cs, cu, c the lift of the stable, unstable, center stable, center unstable, and center leaves respectively, then for every $\tilde{x}, \tilde{y} \in \tilde{M}$ we have:

$$\#\{\tilde{\mathcal{F}}^{cs}(\tilde{x}) \cap \tilde{\mathcal{F}}^u(\tilde{y})\} = 1 \quad \text{and} \quad \#\{\tilde{\mathcal{F}}^{cu}(\tilde{x}) \cap \tilde{\mathcal{F}}^s(\tilde{y})\} = 1.$$

This allows to define a continuous map

$$\Pi_{\tilde{x}}^u : \tilde{M} \rightarrow \tilde{\mathcal{F}}^{cs}(\tilde{x}), \tag{2.1}$$

defined by the holonomy map along unstable leaves:

$$\Pi_{\tilde{x}}^u(\tilde{z}) = \tilde{\mathcal{F}}^u(\tilde{z}) \cap \tilde{\mathcal{F}}^{cs}(\tilde{x}).$$

If we restrict the map $\Pi_{\tilde{x}}^u$ to $\tilde{\mathcal{F}}^{cu}(\tilde{x})$, we have that $\Pi_{\tilde{x}}^u(\tilde{\mathcal{F}}^{cu}(\tilde{x})) = \tilde{\mathcal{F}}^c(\tilde{x})$. In an analogous way we define $\Pi_{\tilde{x}}^s$ the holonomy along stable leaves. And also we define $\Pi_{\tilde{x}}^{su} : \tilde{M} \rightarrow \tilde{\mathcal{F}}^c(\tilde{x})$ by

$$\Pi_{\tilde{x}}^{su} = \Pi_{\tilde{x}}^s \circ \Pi_{\tilde{x}}^u. \tag{2.2}$$

Note that, in general, $\Pi_{\tilde{x}}^{us} \neq \Pi_{\tilde{x}}^{su}$.

Recall that we have defined the accessibility class of $x \in M$ as

$$AC(x) = \{y \in M : \text{there is a } su\text{-path from } x \text{ to } y\}.$$

We define the *center accessibility class* of x as $C(x) = AC(x) \cap \mathcal{F}^c(x)$.

The same definitions for \tilde{M} : for $\tilde{x} \in \tilde{M}$ its *accessibility class* is

$$\widetilde{AC}(\tilde{x}) = \{\tilde{y} \in \tilde{M} : \text{there is a } su\text{-path from } \tilde{x} \text{ to } \tilde{y}\}$$

and $\widetilde{C}(\tilde{x}) = \widetilde{AC}(\tilde{x}) \cap \tilde{\mathcal{F}}^c(\tilde{x})$.

Let us observe that if

$$\tilde{x}, \tilde{z} \in \tilde{M} \text{ with } \tilde{z} \in \widetilde{AC}(\tilde{x}) \implies \Pi_{\tilde{x}}^{su}(\tilde{z}) \in \widetilde{C}(\tilde{x}). \tag{2.3}$$

Lemma 2.1. *Let $\tilde{x} \in \tilde{M}$ and set $x = \pi(\tilde{x})$. Then*

- $\pi(\widetilde{AC}(\tilde{x})) = AC(x)$.
- $\pi(\widetilde{C}(\tilde{x})) \subset C(x)$.

Proof. Note that the projection of a su -path in \tilde{M} is a su -path in M . Then, $\pi(\widetilde{AC}(\tilde{x})) \subset AC(x)$. Reciprocally, the lift of a su -path in M is a su -path in \tilde{M} . The second part also follows easily:

$$\begin{aligned} \pi(\widetilde{C}(\tilde{x})) &= \pi(\widetilde{AC}(\tilde{x}) \cap \tilde{\mathcal{F}}^c(\tilde{x})) \subset \pi(\widetilde{AC}(\tilde{x})) \cap \pi(\tilde{\mathcal{F}}^c(\tilde{x})) \\ &= AC(x) \cap \mathcal{F}^c(x) = C(x). \end{aligned}$$

The proof is complete. □

Lemma 2.2. *The following are equivalent:*

- (a) $AC(x)$ is an open subset.
- (b) $AC(x)$ has non-empty interior.
- (c) $C(x)$ is an open subset of $\mathcal{F}^c(x)$
- (d) $C(x)$ has non-empty interior (in $\mathcal{F}^c(x)$).

Proof. Notice that, by continuity of \mathcal{F}^s , if U is an open set in M then the saturation by stable leaves, i.e. $\cup_{x \in U} \mathcal{F}^s(x)$ is also open in M and the same for the saturation by unstable leaves. Moreover, from the local product structure due the partially hyperbolic structure, given an open set V in a center leaf, its saturation by stable and unstable leaf is also an open set in M . From these simple facts the lemma follows. \square

The same is true for the lift and also equivalent to the above:

Lemma 2.3. *The following are equivalent:*

- (a) $\widetilde{AC}(\tilde{x})$ is an open subset.
- (b) $\widetilde{AC}(\tilde{x})$ has non-empty interior.
- (c) $\widetilde{C}(\tilde{x})$ is an open subset of $\widetilde{\mathcal{F}}^c(x)$.
- (d) $\widetilde{C}(\tilde{x})$ has non-empty interior (in $\widetilde{\mathcal{F}}^c(x)$).
- (e) $C(x)$ is an open subset of $\mathcal{F}^c(x)$.

Proof. The proof of the equivalence among the first four items is analogous as the previous lemma. Let's see the equivalence with the last one. If $\widetilde{AC}(\tilde{x})$ is open we get, by Lemma 2.1 and since π is a covering map, that $AC(x) = \pi(\widetilde{AC}(\tilde{x}))$ is open and hence $C(x)$ is open by lemma above. On the other hand, if $C(x)$ is open in $\mathcal{F}^c(x)$ then $AC(x)$ is also open in M . It follows that $\pi^{-1}(AC(x))$ is open in \widetilde{M} . Since $\pi^{-1}(AC(x)) = \cup_{\beta} \beta(\widetilde{AC}(\tilde{x}))$ where β runs over all covering transformations. It follows, for some β covering transformation, that $\beta(\widetilde{AC}(\tilde{x}))$ has non-empty interior and so $\widetilde{AC}(\tilde{x})$ has non-empty interior. \square

Lemma 2.4. *For any $z \in M$ and any center leaf $\mathcal{F}^c(x)$ we have that $AC(z) \cap \mathcal{F}^c(x) \neq \emptyset$. In particular, f is accessible if and only if for some x it holds that $C(x) = \mathcal{F}^c(x)$.*

Proof. The Global Product Structure implies in particular that for any z and x we have that:

$$\mathcal{F}^u(z) \cap \mathcal{F}^{cs}(x) \neq \emptyset \quad \text{and} \quad \mathcal{F}^{cs}(x) = \bigcup_{y \in \mathcal{F}^c(x)} \mathcal{F}^s(y)$$

which yields $AC(z) \cap \mathcal{F}^c(x) \neq \emptyset$. The second part follows immediately. \square

Recall that for $y \in \mathcal{F}^s(x)$ the (local) holonomy map $\Pi^s(\mathcal{F}^c(x), \mathcal{F}^c(y))$ from a neighborhood U_x^c of x in $\mathcal{F}^c(x)$ to a neighborhood U_y^c of y in $\mathcal{F}^c(y)$ is a C^1 diffeomorphism. If $\gamma : [0, 1] \rightarrow \mathcal{F}^s(x)$ is a path joining x and y then there exists a continuous map $\Gamma^s : U_x^c \times [0, 1] \rightarrow \mathcal{F}^{cs}(x)$ such that:

- $\Gamma^s(z, t) \in \mathcal{F}^s(z) \forall t \in [0, 1]$. In particular $\Gamma^s(z, t) \in AC(z)$.
- $\Gamma^s(z, 0) = z$ for all $z \in U_x^c$.

- $\Gamma^s(z, 1) = \Pi^s(\mathcal{F}^c(x), \mathcal{F}^c(y))(z) \in U_y^c$.
- $\Gamma_0^s : U_x^c \rightarrow U_y^c$ defined by $\Gamma_0^s(z) = \Gamma(z, 1) = \Pi^s(\mathcal{F}^c(x), \mathcal{F}^c(y))(z)$ is a C^1 diffeomorphism.

The same holds when $y \in \mathcal{F}^u(x)$ considering $\Pi^u(\mathcal{F}^c(x), \mathcal{F}^c(y))$ and also for the lift $\tilde{f} : \tilde{M} \rightarrow \tilde{M}$. In particular, if $y \in AC(x)$ and γ is su -path joining x to y , by finite composition of maps as above we get a map (see Figure 1)

$$\Gamma : U_x^c \times [0, 1] \rightarrow M \tag{2.4}$$

such that

- $\Gamma(z, t) \in AC(z)$ for all $t \in [0, 1]$.
- $\Gamma(z, 0) = z$ for all $z \in U_x^c$.
- $\Gamma(z, 1) \in U_y^c$.
- $\Gamma_0 : U_x^c \rightarrow U_y^c$ defined by $\Gamma_0(z) = \Gamma(z, 1)$ is a C^1 diffeomorphism.

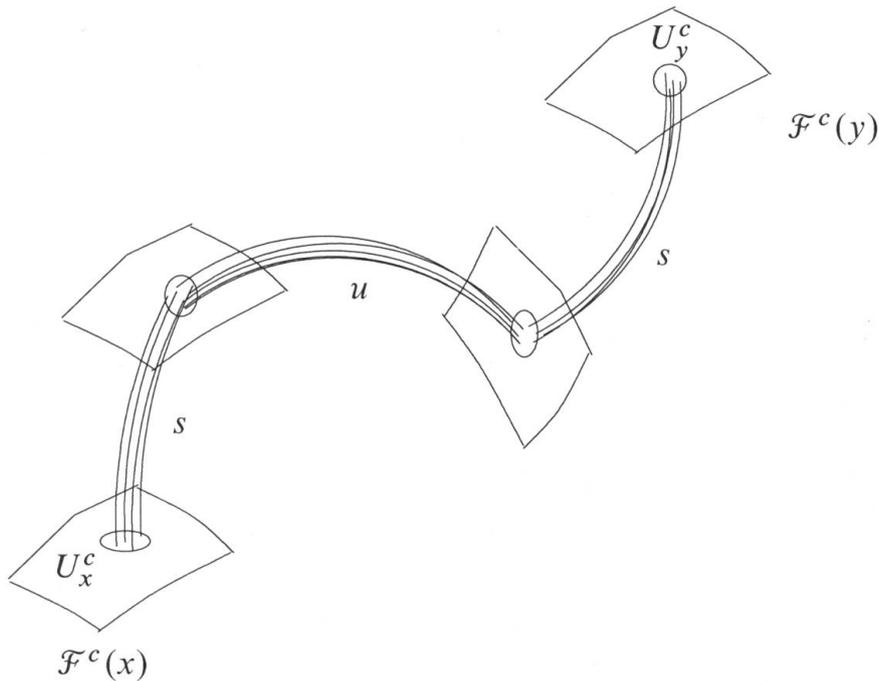


Figure 1. The map Γ .

In the same way, if $\tilde{y} \in \widetilde{AC}(\tilde{x})$ we have a map

$$\tilde{\Gamma} : U_{\tilde{x}}^c \times [0, 1] \rightarrow \tilde{M} \tag{2.5}$$

with the properties above. The above has important consequences. First, we recall the definition of a homogeneous subset.

Definition 2.5. Let X be a riemannian manifold. A subset $Z \subset X$ is said C^r -homogeneous if for every pair of points $x, y \in Z$ there are neighborhoods $U_x, U_y \subset X$ of x and y , respectively, and a C^r -diffeomorphism $\phi : U_x \rightarrow U_y$ such that $\phi(U_x \cap Z) = U_y \cap Z$ and $\phi(x) = y$.

Thus, it follows straightforward from the definitions of Γ and $\tilde{\Gamma}$ that:

Lemma 2.6. $C(x)$ and $\tilde{C}(\tilde{x})$ are C^1 -homogeneous.

Moreover, we have the following:

Lemma 2.7. Let $\tilde{x} \in \tilde{M}$ and $\tilde{y} \in \tilde{C}(\tilde{x})$. Then there are neighborhoods $U_{\tilde{x}}^c$ and $U_{\tilde{y}}^c$ in $\tilde{\mathcal{F}}^c(x)$ and a continuous map $\tilde{\Gamma} : U_{\tilde{x}}^c \times [0, 1] \rightarrow \tilde{\mathcal{F}}^c(\tilde{x})$ such that

- (1) $\tilde{\gamma}(\tilde{z}, 0) = \tilde{z}$ for all $\tilde{z} \in U_{\tilde{x}}^c$;
- (2) $\tilde{\gamma}(\tilde{z}, 1) \in U_{\tilde{y}}^c$ for all $\tilde{z} \in U_{\tilde{x}}^c$;
- (3) $\tilde{\gamma}(\tilde{x}, 1) = \tilde{y}$; and
- (4) $\tilde{\gamma}(\tilde{z}, t) \in \tilde{C}(\tilde{z})$, for all $\tilde{z} \in U_{\tilde{x}}^c$ and $t \in [0, 1]$.

Proof. Just take $\tilde{\gamma} = \Pi_{\tilde{x}}^{su} \circ \tilde{\Gamma}$ where $\Pi_{\tilde{x}}^{su}$ and $\tilde{\Gamma}$ are the maps defined in (2.2) and (2.5), respectively. See also [29]). \square

Remark 2.8. Notice that projecting by $\pi : \tilde{M} \rightarrow M$ we have a similar result: given x in M and $\tilde{x} \in \tilde{M}$ such that $\pi(\tilde{x}) = x$ and $y \in \pi(\tilde{C}(\tilde{x}))$ then there are neighborhoods U_x^c and U_y^c in $\mathcal{F}^c(x)$ and a continuous map $\gamma : U_x^c \times [0, 1] \rightarrow \mathcal{F}^c(x)$ such that

- (1) $\gamma(z, 0) = z$ for all $z \in U_x^c$;
- (2) $\gamma(z, 1) \in U_y^c$ for all $z \in U_x^c$;
- (3) $\gamma(x, 1) = y$; and
- (4) $\gamma(z, t) \in C(z)$, for all $z \in U_x^c$ and $t \in [0, 1]$.

Corollary 2.9. $\tilde{C}(x)$ is connected and arc-connected.

Recall that an accessibility class $AC(x)$ is trivial if $AC(x) \cap \mathcal{F}^c(x)$ is totally disconnected. Notice that by the map Γ defined in (2.4) this does not depend on x , just on the accessibility class.

Lemma 2.10. The accessibility class $AC(x)$ is trivial if and only if $\tilde{C}(\tilde{x}) = \{\tilde{x}\}$ where $\pi(\tilde{x}) = x$.

Proof. If $\tilde{C}(\tilde{x}) \neq \{\tilde{x}\}$ then, since it is connected and arc-connected we have that $\pi(\tilde{C}(\tilde{x})) \subset C(x)$ contains a non trivial connected set and so $C(x)$ is not totally disconnected. On the other hand, if $\tilde{C}(\tilde{x}) = \{\tilde{x}\}$ then $\tilde{AC}(\tilde{x}) \cap \tilde{\mathcal{F}}^c(\tilde{y})$ consists of a single point for any \tilde{y} and in particular for those \tilde{y} with $\pi(\tilde{y}) = x$. Therefore $C(x)$ is at most countable and so totally disconnected. \square

Corollary 2.11. *The set $\{x \in M : AC(x) \text{ is nontrivial}\}$ is open in M .*

Proof. Let $x \in M$ be such that $AC(x)$ is nontrivial and let \tilde{x} such that $\pi(\tilde{x}) = x$. Then $\tilde{C}(\tilde{x}) \neq \{\tilde{x}\}$. Now, from Lemma 2.7 we get for any $\tilde{z} \in \tilde{\mathcal{F}}^c(\tilde{x})$ close enough to \tilde{x} that $\tilde{C}(\tilde{z}) \neq \{\tilde{z}\}$ and the lemma follows. \square

The above says that if we have a nontrivial accessibility class then nearby the classes are nontrivial. We will prove that the same holds when we perturb f as well. We need the following lemma.

Lemma 2.12. *Let $f \in \mathcal{E}^r$. Consider $x \in M$ and let $y \in AC(x)$. Then, there exist U_x, U_y neighborhood of x and y respectively and a C^1 neighborhood of f in \mathcal{E}^r such that for any $g \in \mathcal{U}(f)$ and any $z \in U_x$ it holds that $AC(z) \cap U_y \neq \emptyset$.*

Proof. Fix a su -path from x to y and fix also a small neighborhood U_y . As in the construction of the map Γ , there exists neighborhoods U_x^c and $U_y^c \subset U_y$ such that any $z \in U_x^c$ can be joined by a su -path to a point in U_y^c . By the continuity of the stable and unstable manifolds in compact sets (with respect to f) and the continuous variation of the center foliation, there exists a C^1 neighborhood $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}(f)$ there exists a neighborhood $U_{x,g}^c$ in $\mathcal{F}^c(x, g)$ so that any $z \in U_{x,g}^c$ can be joined by a su -path to a point in a neighborhood $U_{y,g}^c \subset \mathcal{F}^c(y, g)$. We may consider that $U_{y,g}^c$ is contained in U_y for any $g \in \mathcal{U}(f)$. Moreover, we can find a neighborhood U_x so that for any $g \in \mathcal{U}(f)$ and any $z \in U_x$ there is a su -path joining z with a point in $U_{x,g}^c$. \square

As a consequence, we have that a non-trivial accessibility class can not be destroyed by perturbations.

Corollary 2.13. *Let $f \in \mathcal{E}^r$ and let $x \in M$ be such that $AC(x)$ is nontrivial. Then, there exist a neighborhood U_x of x and a neighborhood $\mathcal{U}(f)$ in \mathcal{E}^r (which can be considered C^1 open as well) such that for any $g \in \mathcal{U}(f)$ and $z \in U_x$ the accessibility class $AC(z, g)$ is nontrivial.*

Proof. Let \tilde{x} be such that $\pi(\tilde{x}) = x$. Since $AC(x)$ is nontrivial, then $\tilde{C}(\tilde{x}) \neq \{\tilde{x}\}$. Let $\tilde{y} \in \tilde{C}(\tilde{x})$, $\tilde{y} \neq \tilde{x}$. We have a su -path (in \tilde{M}) joining \tilde{x} to \tilde{y} . By Lemma 2.12 applied to the lift, there are disjoint open sets $U_{\tilde{x}}$ and $U_{\tilde{y}}$ of \tilde{x} and \tilde{y} in \tilde{M} and an open set $\mathcal{U}(f)$ such that for any g in $\mathcal{U}(f)$ we have that any point in $U_{\tilde{x}}$ can be joined by su -path of \tilde{g} with a point in $U_{\tilde{y}}$. And moreover, if we consider $\Pi_{\tilde{g}}^{su} : \tilde{M} \rightarrow \mathcal{F}^c(\tilde{x}, \tilde{g})$, then $\Pi_{\tilde{g}}^{su}(U_{\tilde{x}})$ and $\Pi_{\tilde{g}}^{su}(U_{\tilde{y}})$ are open sets (in $\mathcal{F}^c(\tilde{x}, \tilde{g})$) and disjoint. The result thus follows, since for any $\tilde{z} \in \Pi_{\tilde{g}}^{su}(U_{\tilde{x}})$ there is point in $\Pi_{\tilde{g}}^{su}(U_{\tilde{y}})$ that belongs to $\tilde{C}(\tilde{z}, g)$. \square

Lemma 2.14. *Let $x \in M$ and let $\tilde{x} \in \tilde{M}$ with $\pi(\tilde{x}) = x$. Let $\tilde{\mathcal{F}}_1^c$ and $\tilde{\mathcal{F}}_2^c$ be such that $\pi(\tilde{\mathcal{F}}_i^c) = \mathcal{F}^c(x)$ and let $\tilde{\mathcal{C}}_i = \tilde{AC}(\tilde{x}) \cap \tilde{\mathcal{F}}_i^c$, $i = 1, 2$. Then either $\pi(\mathcal{C}_1)$ and $\pi(\mathcal{C}_2)$ are equal or disjoint.*

Proof. Assume that $\pi(\tilde{\mathcal{C}}_1) \cap \pi(\tilde{\mathcal{C}}_2) \neq \emptyset$. Let z be in this intersection and let $\tilde{z}_i \in \tilde{\mathcal{C}}_i$ be such that $\pi(\tilde{z}_i) = z$. It follows that $\tilde{\mathcal{C}}_i = \tilde{C}(\tilde{z}_i)$. Let β be a covering map, $\beta(\tilde{z}_1) = \tilde{z}_2$. Since β sends su -path in \tilde{M} to su -path we conclude that $\beta(\tilde{\mathcal{C}}_1) \subset \tilde{\mathcal{C}}_2$ and $\beta^{-1}(\tilde{\mathcal{C}}_2) \subset \tilde{\mathcal{C}}_1$. Hence $\pi(\tilde{\mathcal{C}}_1) = \pi(\tilde{\mathcal{C}}_2)$. \square

Corollary 2.15. *Let $x \in M$ and $\tilde{x} \in \tilde{M}$ such that $\pi(\tilde{x}) = x$. If $C(x)$ is open then $\pi(\tilde{C}(\tilde{x}))$ is the connected component of $C(x)$ that contains x .*

Proof. It is a direct consequence of Lemmas 2.1, 2.3, and 2.14. \square

We now investigate the structure of accessibility classes when the center bundle has dimension two, that is $f \in \mathcal{E}^r$ and $\dim E^c = 2$. The following important result is essentially contained in [29].

Theorem 2.16. *Let $f \in \mathcal{E}^r$ and assume that $\dim E^c = 2$. Let $\tilde{x} \in \tilde{M}$. Then one and only one of the following holds:*

- (1) $\tilde{C}(\tilde{x})$ is open;
- (2) $\tilde{C}(\tilde{x}) = \{\tilde{x}\}$;
- (3) $\tilde{C}(\tilde{x})$ is a C^1 one dimensional manifold without boundary.

Proof. The same proof in [29, Proposition 5.2] yields that $\tilde{C}(\tilde{x})$ is either open, consists just of \tilde{x} or it is a topological one dimensional manifold. Now, in case $\tilde{C}(\tilde{x})$ is a topological one dimensional manifold, by the C^1 homogeneity of $\tilde{C}(\tilde{x})$ and the result in [33] which says that a locally compact and C^1 homogeneous subset of a riemannian manifold is a C^1 submanifold, one get that in fact $\tilde{C}(\tilde{x})$ is of class C^1 (and without boundary). \square

Let us denote by $C_0(x)$ the arc-connected component of $C(x)$ that contains x . We remark that when $C(x)$ is open, then $C_0(x)$ is just the connected component of $C(x)$ that contains x .

Corollary 2.17. *Let $f \in \mathcal{E}^r$ and assume that $\dim E^c = 2$. Let $x \in M$ and $\tilde{x} \in \tilde{M}$ with $\pi(\tilde{x}) = x$. Then $\pi(\tilde{C}(\tilde{x})) = C_0(x)$.*

Proof. When $\tilde{C}(\tilde{x})$ is open or trivial then the result follows immediately. Thus, we just have to check it when $\tilde{C}(\tilde{x})$ is a one dimensional submanifold (without boundary).

From Lemma 2.1 and Corollary 2.9 we have $\pi(\tilde{C}(\tilde{x})) \subset C_0(x)$. On the other hand, let β be an arc in $C_0(x)$ starting at x and assume that it is not contained in $\pi(\tilde{C}(\tilde{x}))$. Let $t_0 = \sup\{t : \beta([0, t]) \subset \pi(\tilde{C}(\tilde{x}))\}$. Let $y = \beta(t_0)$ (that belongs to $\pi(\tilde{C}(\tilde{x}))$). Since $\tilde{C}(\tilde{x})$ has no boundary, we have an arc α inside $\pi(\tilde{C}(\tilde{x}))$ having y in its interior, say joining x_1 with x_2 . Now, applying Lemma 2.7 (or Remark 2.8) we have a continuous map $\gamma : U_{x_1}^c \times [0, 1] \rightarrow \mathcal{F}^c(x)$ such that $\gamma(x_1, t) = \alpha(t)$ and that $\gamma(z, t) \in C(z)$ for any t and $z \in U_{x_1}$. Then we conclude that there is an open set $U \subset U_{x_1}$ such that $\gamma(U \times [0, 1]) \cap \beta \neq \emptyset$ (see Figure 2). This implies that $C(x)$ is open, a contradiction. Therefore, $C_0(x) \subset \pi(\tilde{C}(\tilde{x}))$. The proof is finished. \square

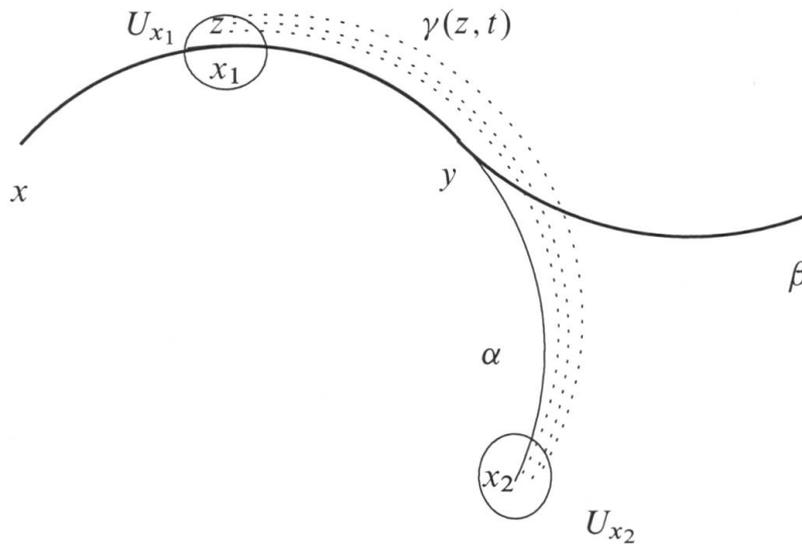


Figure 2. Proof of Corollary 2.17.

Next, we want to show that the set of one dimensional accessibility classes form a lamination. Let's recall the definition of a lamination (we state this for the case of lamination of subsets of a surface for simplicity).

Definition 2.18. Let S be a surface and let $K \subset S$ be closed. We say that K admits a C^1 lamination if K has a partition into C^1 one dimensional manifolds (called leaves of the lamination) such that every point in K has a neighborhood U homeomorphic to $(-1, 1) \times (-1, 1)$ (called charts of the lamination) such that $K \cap U$ correspond to $F \times (-1, 1)$, where F is a closed set in $(-1, 1)$ and every $\{f\} \times (-1, 1)$, for $f \in F$, is inside a leaf of K , and tangent spaces of the leaves vary continuously.

Fix a center leaf $\tilde{\mathcal{F}}^c$. Let \tilde{K} be the union of accessibility classes in $\tilde{\mathcal{F}}^c$ which are C^1 one-dimensional submanifolds. Then \tilde{K} is partitioned by the accessibility classes $\tilde{C}(\tilde{x})$, $\tilde{x} \in \tilde{K}$. We want to prove that this partition form a C^1 lamination.

First we prove that the accessibility classes vary continuously in the C^1 topology:

Proposition 2.19. For $\tilde{x} \in \tilde{K}$, the curves $\tilde{C}(\tilde{x})$ vary continuously in the C^1 topology.

Before proving the proposition we need an elementary lemma from calculus (whose proof is left to the reader). Let us introduce some notations. Let $\gamma : [0, 1] \rightarrow S$ be an C^1 arc of a surface S and consider an ε tubular neighborhood N_γ . This tubular neighborhood is diffeomorphic to $[0, 1] \times [-\varepsilon, \varepsilon]$. Given a point in N_γ we identify with coordinates (t, s) , $t \in [0, 1]$ and $s \in [-\varepsilon, \varepsilon]$.

We call the *left side* of N_γ the boundary $\{0\} \times [-\varepsilon, \varepsilon]$ and *right side* the boundary $\{1\} \times [-\varepsilon, \varepsilon]$. We denote by $\xi : N_\gamma \rightarrow \gamma$ the orthogonal projection, i.e. in local coordinates $\xi(t, s) = t$.

Lemma 2.20. With the notations above, given $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta)$ such that if β is a C^1 curve in N_γ from the left to the right side (and do not intersect γ) then there is some $(t, s) = \beta(\tilde{t})$ such that the angle $\angle(\dot{\beta}(\tilde{t}), \dot{\gamma}(t)) < \delta$.

Now we are ready to give a proof of Proposition 2.19. Since we will be working in a neighborhood of $\widetilde{\mathcal{F}}^c$ we may assume that we are in trivialization chart of the tangent bundle $T\widetilde{\mathcal{F}}^c$, and so we may compare angles and norms of vectors in different tangent spaces.

Proof of Proposition 2.19. We need to prove that if $\tilde{x} \in \widetilde{K}$ and $\tilde{x}_n \in \widetilde{K}$ converges to \tilde{x} then $T_{\tilde{x}_n}\widetilde{\mathcal{C}}(\tilde{x}_n)$ converges to $T_{\tilde{x}}\widetilde{\mathcal{C}}(\tilde{x})$. Assume that it is not true. Then there exists some sequence $\tilde{x}_n \in \widetilde{K}$ converging to some point $\tilde{x} \in \widetilde{K}$ and some $\eta > 0$ such that the angle $\angle(T_{\tilde{x}_n}\widetilde{\mathcal{C}}(\tilde{x}_n), T_{\tilde{x}}\widetilde{\mathcal{C}}(\tilde{x})) > \eta$, for all n .

Let $\tilde{y} \in \widetilde{\mathcal{C}}(\tilde{x})$, $\tilde{y} \neq \tilde{x}$ and let Υ^{su} a su -path joining \tilde{y} to \tilde{x} . We may consider an arc of su -paths, i.e. for each $t \in [0, 1]$ a su -path Υ_t^{su} that vary continuously joining \tilde{x} with some point $\Upsilon_t^{su}(1) \in \widetilde{\mathcal{C}}(\tilde{x})$ such that Υ_0^{su} is the trivial su -path and $\Upsilon_1^{su} = \Upsilon^{su}$. We may assume that the path $\Upsilon_t^{su}(1)$ is the arc joining \tilde{x} and \tilde{y} in $\widetilde{\mathcal{C}}(\tilde{x})$ denoted by $[\tilde{x}, \tilde{y}]$. We will consider tubular neighborhood N of the arc $[\tilde{x}, \tilde{y}]$.

The path Υ_t^{su} allows us to consider (see the last item of properties of the map Γ in (2.4) and equivalent for (2.5)) a map $\phi_t : B(\tilde{x}, r_t) \rightarrow \widetilde{\mathcal{F}}^c$ which is a C^1 diffeomorphism onto its image that contains $\Upsilon_t^{su}(1)$. We may choose $r_t = r$ independent of t . The family ϕ_t varies continuously in the C^1 topology due to the local holonomy is C^1 inside center stable and center unstable leaves, the center leaves vary continuously in the C^1 topology and the path Υ_t^{su} varies continuously with t .

Given $\theta > 0$ there exists $\delta_0 > 0$ and $\rho_0 > 0$ such that for any t if $\text{dist}(\tilde{z}, \tilde{x}) < \rho_0$ and $\angle(T_{\tilde{x}}\widetilde{\mathcal{C}}(\tilde{x}), w) > \theta$ then the angle

$$\angle(d(\phi_t)_{\tilde{x}}(T_{\tilde{x}}\widetilde{\mathcal{C}}(\tilde{x})), d(\phi_t)_{\tilde{z}}(w)) = \angle(T_{\phi_t(\tilde{x})}\widetilde{\mathcal{C}}(\tilde{x}), d(\phi_t)_{\tilde{z}}(w)) > \delta_0. \tag{2.6}$$

On the other hand, given $\delta_1 > 0$ there exists $\varepsilon_1 > 0$ such that if

$$\text{dist}(\xi(\phi_t(\tilde{z})), \phi_t(\tilde{x})) < \varepsilon_1 \text{ then } \angle(T_{\xi(\phi_t(\tilde{z}))}\widetilde{\mathcal{C}}(\tilde{x}), T_{\phi_t(\tilde{x})}\widetilde{\mathcal{C}}(\tilde{x})) < \delta_1. \tag{2.7}$$

Notice also that there exists $\rho > 0$ such that for any t if

$$\text{dist}(\tilde{x}, \tilde{z}) < \rho \text{ then } \text{dist}(\xi(\phi_t(\tilde{z})), \phi_t(\tilde{x})) < \varepsilon_1. \tag{2.8}$$

Consider \hat{x} and \hat{y} in $\widetilde{\mathcal{C}}(\tilde{x})$ between \tilde{x} and \tilde{y} . Denote by $\gamma = [\hat{x}, \hat{y}]$ the arc in $\widetilde{\mathcal{C}}(\tilde{x})$ joining \hat{x} and \hat{y} .

Let $\theta = \eta$ and take $\delta_0 = \delta_0(\theta)$ from (2.6). Choose $\delta_1 > 0$ such that $\delta_0 - \delta_1 = \delta > 0$ and let ε_1 from (2.7). Choose $\rho < \rho_0$ such that (2.8) holds.

For this δ choose an ε tubular neighborhood $N_{[\hat{x}, \hat{y}]}$ as in Lemma 2.20 and such that $N_{[\hat{x}, \hat{y}]} \subset \cup_{t \in [0, 1]} \phi_t(B(\tilde{x}, \rho))$.

Now, if \tilde{x}_n is close enough to \tilde{x} then $\beta_n(t) = \phi_t(\tilde{x}_n)$, $t \in [0, 1]$ is a curve that crosses the $N_{[\hat{x}, \hat{y}]}$ from the left to the right side. On the other hand, if t is such that $\beta_n(t) \in N_{[\hat{x}, \hat{y}]}$ then $\text{span}(\dot{\beta}_n) = d(\phi_t(\tilde{x}_n))(T_{\tilde{x}_n} \tilde{\mathcal{C}}(\tilde{x}_n))$ and so

$$\begin{aligned} \angle(\dot{\beta}_n(t), T_{\xi(\beta_n(t))} \tilde{\mathcal{C}}(\tilde{x})) &= \angle(d\phi_t(\tilde{x}_n)T_{\tilde{x}_n} \tilde{\mathcal{C}}(\tilde{x}_n), T_{\xi(\phi_t(\tilde{x}_n))} \tilde{\mathcal{C}}(\tilde{x}_n)) \\ &> \angle(d\phi_t(\tilde{x}_n)T_{\tilde{x}_n} \tilde{\mathcal{C}}(\tilde{x}_n), d\phi_t(\tilde{x})T_{\tilde{x}} \tilde{\mathcal{C}}(\tilde{x})) \\ &\quad - \angle(T_{\xi(\phi_t(\tilde{x}_n))} \tilde{\mathcal{C}}(\tilde{x}_n), d\phi_t(\tilde{x})T_{\tilde{x}} \tilde{\mathcal{C}}(\tilde{x})) \\ &> \delta_0 - \delta_1 = \delta, \end{aligned}$$

which is a contradiction with Lemma 2.20. □

Corollary 2.21. *Let $\tilde{\mathcal{F}}^c$ be a center leaf in \tilde{M} and assume that there is no trivial accessibility classes. Then the set \tilde{K} of non-open accessibility classes admits a C^1 lamination whose leaves are accessibility classes. The same holds for the set of non open accessibility classes K in \mathcal{F}^c whose leaves are connected components of accessibility classes $C_0(x)$ for $x \in K$.*

Proof. Since there is no trivial accessibility class the set \tilde{K} is closed. From Proposition 2.19 and using transversal sections it is not difficult to construct a chart for each $\tilde{x} \in \tilde{K}$. □

3. Perturbation lemmas

In this section we prove our main perturbation techniques that allow us to prove our theorems. These are Lemmas 3.5 and 3.7. Before we state and prove these lemmas we need some elementary results, the first one says that some perturbations of the identity can be thought as translations in terms of local coordinates, no matter if we are in the conservative world, symplectic world, etc.

Let M be a manifold of dimension d and let $S \subset M$ be an embedded submanifold of M of dimension k . Let $z \in S$ and let U be a neighborhood of z in M and let V be the connected component of $U \cap S$ containing z . We say that we have *local canonical coordinates* in V if we have a coordinate chart (or parametrization) $\varphi : U \rightarrow \mathbb{R}^d$ and $\varphi(V) = V_0 \subset \mathbb{R}^k$ with $\varphi(z) = 0$.

It is a consequence of Darboux Theorem that if ω is a symplectic form in M such that $\omega|_S$ is symplectic and $k = 2j$, $d = 2l$ and we write coordinates in \mathbb{R}^k as $(x_1, \dots, x_j, y_1, \dots, y_j)$ and in \mathbb{R}^d as $(x_1, \dots, x_l, y_1, \dots, y_l)$ then we may assume that the local chart verifies $\varphi^*(\sum_{i=1}^d dx_i \wedge dy_i) = \omega|_U$ and $\varphi^*(\sum_{i=1}^j dx_i \wedge dy_i) = \omega|_V$, see e.g. [24] and [36].

And in case m is a volume form it is well known that we can choose local coordinates in \mathbb{R}^d as (x_1, \dots, x_d) , we may assume also that $\varphi^*(dx_1 \wedge \dots \wedge dx_d) = m$, i.e. in local coordinates the volume form is the standard volume form in \mathbb{R}^d (see e.g. [24]).

The next lemma says that we can glue an arbitrary small translation near a point with the identity outside a neighborhood in the conservative and symplectic setting. Sophisticated versions of this problem can be found in [13] and [2] (pasting lemma).

Lemma 3.1. *Let M be a C^r manifold and let $S \subset M$ be a C^r submanifold of M , $r \geq 1$. Let $z \in S$ and let U be a given neighborhood of z in M such that $V := S \cap U$ has local canonical coordinates. Then, there exist V' , $z \in V' \subset V \subset U$ and $\varepsilon_0 > 0$ such that for any $0 \leq \varepsilon \leq \varepsilon_0$ there exists $\delta > 0$ such that for any $w \in \mathbb{R}^k$, $\|w\| < \delta$ there exists a diffeomorphism $h : M \rightarrow M$ satisfying:*

- (1) h is ε - C^r close to identity;
- (2) $h \equiv \text{id}$ on U^c ;
- (3) h preserves V , i.e. $h(V) = V$;
- (4) $h|_{V'}$ in local coordinates is given by

$$y \mapsto h(y) = y + w.$$

If ω is a symplectic form in M and $\omega|_S$ is symplectic then h can be taken to be a symplectomorphism. If m is a volume form in M , we can take h to preserve the volume form m .

Proof. In the general case (i.e. neither conservative nor symplectic) the solution is easy, just take the time t map (with t small enough) of the flow generated by a vector field X (in the local coordinates) such that $X(x) = v$ for $v \in \mathbb{R}^k$, $\|v\| = 1$ and it is identically zero outside a neighborhood of the origin. The same idea works in the conservative setting taking X to be divergence free vector field and in the symplectic setting taking a Hamiltonian vector field.

Let's consider the conservative setting. Recall that we have local coordinates, i.e. a chart $\varphi : U \rightarrow \mathbb{R}^d$ and $\varphi(V) = V_0 \subset \mathbb{R}^k$ with $\varphi(z) = 0$ and $\varphi^*(dx_1 \wedge \cdots \wedge dx_d) = m$. Let $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a bump function such that it is equal to 1 in a neighborhood of 0 and it is identically zero outside a neighborhood of 0 as well (with closure contained in $\varphi(U)$). Let $w \in \mathbb{R}^k$, $\|w\| = 1$. By a linear change of coordinates (preserving \mathbb{R}^k) we may assume that $w = e_1$. Consider the function $\chi : \mathbb{R}^d \rightarrow \mathbb{R}$ given by

$$\chi(x) = \chi(x_1, x_2, \dots, x_d) = \psi(x)x_2.$$

Then, taking $X(x) = (\frac{\partial \chi}{\partial x_2}, -\frac{\partial \chi}{\partial x_1}, 0, \dots, 0)$ we have that X is divergence free. Now, taking the time t map of the flow generated by X for t small we get the lemma.

Let's consider the symplectic case. We may assume without loss of generality that U is contained in a tubular neighborhood of S and $U = U' \times D$, D fibers of the tubular neighborhood. Choose open balls $V_1 \subset V'_1 \subset U_1 \subset U'_1 \subset U$ centered in z and a C^∞ bump function $\psi : M \rightarrow \mathbb{R}$ so that $\psi|_{U_1^c} \equiv 0$ and $\psi|_{V_1} \equiv 1$.

For simplicity we will assume that S is bidimensional. Let $u = (u_1, u_2)$ be a unit vector in \mathbb{R}^2 and let $H_0^u : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $H_0^u(x, y) = yu_1 - xu_2$. Notice that $X_{H_0^u} = u$ in \mathbb{R}^2 .

Let $\tilde{H}_0^u : \mathbb{R}^2 \rightarrow \mathbb{R}$ be $\tilde{H}_0^u = \psi \cdot H_0^u$ and let $H_1^u : U \rightarrow \mathbb{R}$ be $H_1^u(y) = H_0^u(\varphi(\pi(y)))$, where π is the projection along the fibers of the tubular neighborhood, and let $H^u : M \rightarrow \mathbb{R}$ be such that

- $H^u \equiv 0$ in U^c , and
- $H^u = \psi(y)H_1^u(y)$ if $y \in U$.

Notice that H^u is C^∞ and the C^r norm is bounded by a constant K that does not depend on u .

Let $y \in S \cap V_1$. We claim that $X_{H^u}(y) = X_{H^u}^S(y)$, where X_H^S is the hamiltonian field of $H|_S$. Indeed, $X_{H^u}(y)$ is defined as $\omega(X_{H^u}(y), \cdot) = -dH_y^u$ and $X_{H^u}^S(y)$ as $\omega|_S(X_{H^u}^S(y), \cdot) = -d(H^u|_S)_y$. For $y \in V_1 \cap S$, $H^u(y) = H^u(\pi(y))$ and so $dH^u = dH^u|_S \circ d\pi$ and hence $dH_y^u|(T_y S)^\perp = 0$. Thus, for any $v \in (T_y S)^\perp$, $\omega(X_{H^u}(y), v) = 0$, then $X_{H^u}(y) \in T_y S$ and so, since for any $w \in T_y S$, we have

$$\omega(X_{H^u}(y), w) = \omega|_S(X_{H^u}^S(y), w).$$

We conclude that

$$X_{H^u}(y) = X_{H^u}^S(y).$$

This proves the claim. Finally, taking the time t map of the corresponding hamiltonian flow, for t small enough, we conclude the lemma. \square

For $f \in \mathcal{E}^r$ we denote the *stable manifold of size ε of a center leaf \mathcal{F}_1^c* by $W_\varepsilon^s(\mathcal{F}_1^c) := \bigcup_{z \in \mathcal{F}_1^c} (W_\varepsilon^s(z))$.

Remark 3.2. If \mathcal{F}_1^c is a compact periodic center leaf and $w \in W_\varepsilon^s(\mathcal{F}_1^c)$ then there exists ε_0 such that $w \in W_{\varepsilon_0}^s(\mathcal{F}_1^c)$ but $w \notin \overline{f^n(W_{\varepsilon_0}^s(\mathcal{F}_1^c))}$, for all $n \geq 1$.

Lemma 3.3. *Let $f \in \mathcal{E}^r$ and let \mathcal{F}_1^c be a compact periodic center leaf of f . Let $x \in \mathcal{F}_1^c$ and let B be a small neighborhood of x in M . Then there exist $p_1, p_2, w_1, w_2, z_1, z_2 \in B$, $\varepsilon_0, \varepsilon_1, \varepsilon_2 > 0$, and U_1, U_2 disjoint neighborhoods of w_1, w_2 in M such that, for $i = 1, 2$,*

- (1) $\mathcal{F}^c(p_i), i = 1, 2$, are periodic compact center leaves;
- (2) $w_i \in W_{\varepsilon_0}^s(x) \cap W_{\varepsilon_i}^u(p_i)$;
- (3) $z_i \in W_{\varepsilon_0}^u(\mathcal{F}_1^c) \cap W_{\varepsilon_i}^s(p_i)$;
- (4) $U_i \cap f^n(W_{\varepsilon_0}^s(\mathcal{F}_1^c)) = \emptyset$, for all $n \geq 1$;
- (5) $U_i \cap f^{-n}(W_{\varepsilon_i}^u(\mathcal{F}^c(p_i))) = \emptyset$, for all $n \geq 1$;
- (6) $U_i \cap f^n(W_{\varepsilon_i}^s(\mathcal{F}^c(p_j))) = \emptyset$, for all $n \geq 0$ and $i, j = 1, 2$; and
- (7) $U_i \cap f^{-n}(W_{\varepsilon_0}^u(\mathcal{F}_1^c)) = \emptyset$, for all $n \geq 0$.

Proof. Let $B_0, x \in B_0 \subset B$ be a foliated chart of the center foliation:

$$\varphi : B_0 \rightarrow D^{m-k} \times D^k,$$

where D^{m-k} is a disk in \mathbb{R}^{m-k} and D^k is a disk in \mathbb{R}^k , and the center foliation in B_0 through φ is $\{y\} \times D^k$ and $\varphi(x) = (0, 0)$.

Let P_x be the plaque of x in B_0 . We may assume that $\mathcal{F}_1^c \cap B_0 = P_x$. In B_0 , we identify point in the same plaque, $B_0/\sim \cong D^{m-k}$. Let $P : B_0 \rightarrow B_0/\sim$ the projection map.

For $z \in B_0$, denote by $\hat{z} := P(z)$ the plaque of z and denote by $W_{B_0}^{c*}(z)$, $* = s, u$, the connected component of $W_\varepsilon^*(F^c(z)) \cap B_0$ that contains z , and by $W_{B_0}^*(\hat{z}) := P(W^{c*}(z))$ (B_0 is small compared to ε).

In a neighborhood W of \hat{x} we have local product structure. Since periodic compact center leaves are dense, we may choose $\hat{p}_1 \in W$ such that \hat{p}_1 is contained in a compact periodic center leaf. (This compact center leaf may intersects B_0 in other plaques than \hat{p}_1 , but if it does, intersects finitely many plaques in B_0).

Let $\hat{w}_1 = W_{B_0}^s(\hat{x}) \cap W_{B_0}^u(\hat{p}_1)$ and $w_1 = W_\varepsilon^s(x) \cap P^{-1}(\hat{w}_1)$, $p_1 = W_\varepsilon^u(w_1) \cap P^{-1}(\hat{p}_1)$, $\hat{z}_1 = W_{B_0}^u(\hat{x}) \cap W_{B_0}^s(\hat{p}_1)$, and $z_1 = W_\varepsilon^s(p_1) \cap P^{-1}(\hat{z}_1)$.

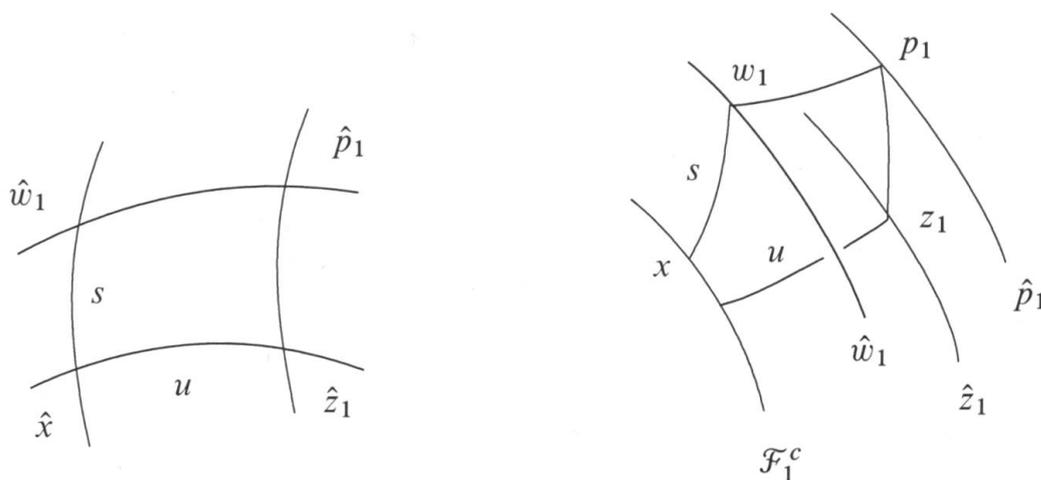


Figure 3. 4-legged path.

Now, we can find $\varepsilon_0, \varepsilon_1 > 0$ such that

- $w_1 \in W_{\varepsilon_0}^s(\mathcal{F}_1^c)$ but $w_1 \notin \overline{f^n(W_{\varepsilon_0}^s(\mathcal{F}_1^c))}$, $n \geq 1$, and
- $w_1 \in W_{\varepsilon_1}^u(\mathcal{F}^c(p_1))$ but $w_1 \notin \overline{f^{-n}(W_{\varepsilon_1}^u(\mathcal{F}^c(p_1)))}$, $n \geq 1$.

Now, we may take \hat{p}_2 close to \hat{p}_1 and contained in a compact periodic leaf and such that, if we set $\hat{w}_2 = W_{B_0}^s(\hat{x}) \cap W_{B_0}^u(\hat{p}_2)$, $w_2 = W_\varepsilon^s(x) \cap P^{-1}(\hat{w}_2)$, $p_2 = W_\varepsilon^u(w_2) \cap P^{-1}(\hat{p}_2)$, $\hat{z}_2 = W_{B_0}^u(\hat{x}) \cap W_{B_0}^s(\hat{p}_2)$, and $z_2 = W_\varepsilon^s(p_2) \cap P^{-1}(\hat{z}_2)$ then, $w_2 \in W_{\varepsilon_0}^s(\mathcal{F}_1^c)$ but $w_2 \notin \overline{f^n(W_{\varepsilon_0}^s(\mathcal{F}_1^c))}$, for $n \geq 1$, if \hat{p}_2 is close enough to \hat{p}_1 . Now, we may choose $\varepsilon_2 > 0$ such that $w_2 \in W_{\varepsilon_2}^u(\mathcal{F}^c(p_2))$ but $w_2 \notin \overline{f^{-n}(W_{\varepsilon_2}^u(\mathcal{F}^c(p_2)))}$, $n \geq 1$.

Observe that trivially it holds that,

$$w_i \notin \overline{f^n(W_\varepsilon^s(\mathcal{F}^c(p_j)))}, \quad n \geq 0, i, j = 1, 2; \text{ and}$$

$$w_i \notin \overline{f^{-n}(W_\varepsilon^u(\mathcal{F}_1^c))}, \quad n \geq 0, i = 1, 2.$$

From this it is easy to find the neighborhoods U_1, U_2 . The proof is complete. \square

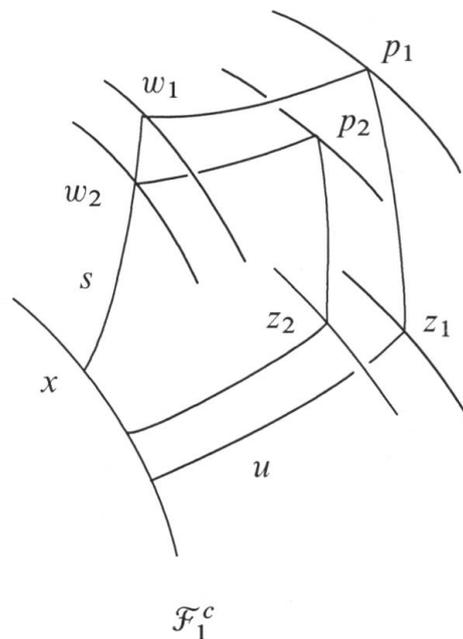


Figure 4. Lemma 3.3.

Given $f \in \mathcal{E}^r$ and $w \in W_\varepsilon^s(x)$ recall that we denote $\Pi_f^s(\mathcal{F}^c(x), \mathcal{F}^c(w))$ the (local) holonomy map along the stable foliation in $\mathcal{F}^{cs}(x)$ from a neighborhood of x in $\mathcal{F}^c(x)$ onto a neighborhood of w in $\mathcal{F}^c(w)$. For the sake of simplicity we set $\hat{x} = \mathcal{F}^c(x), \hat{w} = \mathcal{F}^c(w)$ and $\Pi_f^s(\hat{x}, \hat{w}) = \Pi_f^s(\mathcal{F}^c(x), \mathcal{F}^c(w))$. Moreover, if $w \in \mathcal{F}^{cs}(x)$ (although perhaps $w \notin W^s(x)$) we still set $\Pi_f^s(\hat{x}, \hat{w})$ the local holonomy map from a neighborhood of x in $\hat{x} = \mathcal{F}^c(x)$ onto a neighborhood of a point in $W^s(x) \cap \mathcal{F}^c(w)$ of $\hat{w} = \mathcal{F}^c(w)$ along the stable foliation where the point in $W^s(x) \cap \mathcal{F}^c(w)$ should be clear from the context. Analogous remark and notation holds for Π_f^u .

Remark 3.4. In the setting of Lemma 3.3 notice that $\Pi_f^u(\hat{z}_i, \hat{x})(z_i)$ belongs to $C_0(x)$. Moreover, if $h : M \rightarrow M$ is a diffeomorphism close to the identity such that $h \equiv \text{id}$

in $(U_1 \cup U_2)^c$ and preserves V_i the connected component of $U_i \cup \mathcal{F}^c(w_i)$ that contains w_i then if we define $g = f \circ h^{-1}$ we have:

$$\begin{aligned} \mathcal{F}^c(*, f) &= \mathcal{F}^c(*, g) \quad \text{where } * = x, p_1, p_2; \\ V_i &\subset \mathcal{F}^c(w_i, g); \\ \mathcal{F}_{\text{loc}}^c(z_i, f) &= \mathcal{F}_{\text{loc}}^c(z_i, g); \\ \left. \begin{aligned} \Pi_g^s(\hat{w}_i, \hat{x}) &= \Pi_f^s(\hat{w}_i, \hat{w}_i) \circ h; \\ \Pi_g^u(\hat{w}_i, \hat{p}_i) &= \Pi_f^u(\hat{w}_i, \hat{p}_i); \\ \Pi_g^s(\hat{z}_i, \hat{p}_i) &= \Pi_f^s(\hat{z}_i, \hat{p}_i); \text{ and} \\ \Pi_g^u(\hat{x}, \hat{p}_i) &= \Pi_f^u(\hat{x}, \hat{p}_i). \end{aligned} \right\} \end{aligned} \tag{3.1}$$

Denote $\text{supp}(f \neq g) = \{y : f(y) \neq g(y)\}$. The next lemma says that we can destroy trivial accessibility class.

Lemma 3.5. *Let \mathcal{E} be as in Theorem 2 and let $f \in \mathcal{E}$ and let $\mathcal{F}^c(x)$ be a periodic compact center leaf of f . Assume that $C_0(x) = \{x\}$. Then, there exists neighborhood V_x of x in $\mathcal{F}^c(x)$ and $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ there exists $g \in \mathcal{E}$, with $\text{dist}_{C^r}(f, g) < \varepsilon$ such that:*

- (1) $\text{supp}(f \neq g)$ is disjoint from the f -orbit of $\mathcal{F}^c(x)$,
- (2) for any $y \in V_x$ we have $C_0(y, g) \neq \{y\}$.

Proof. Let $x, w_1, w_2, p_1, p_2, z_1, z_2$ and U_1, U_2 be as in Lemma 3.3. As before, we write $\hat{w}_i = \mathcal{F}^c(w_i, f)$, $\hat{z}_i = \mathcal{F}^c(z_i, f)$, and $\hat{p}_i = \mathcal{F}^c(p_i, f)$.

Since $C_0(x, f) = \{x\}$, we have $\Pi_f^u(\hat{z}_i, \hat{x})(z_i) = x$ and

$$\Pi_f^s(\hat{w}_i, \hat{x}) \circ \Pi_f^u(\hat{p}_i, \hat{w}_i) \circ \Pi_f^s(\hat{z}_i, \hat{p}_i) \circ \Pi_f^u(\hat{x}, \hat{z}_i)(x) = x, \quad i = 1, 2.$$

Let W_0 be a small neighborhood of x in $\mathcal{F}^c(x)$ and let $W_1 \subset W_0$ such that $W_1 \subset \Pi_f^s(\hat{w}_i, \hat{x}) \circ \Pi_f^u(\hat{p}_i, \hat{w}_i) \circ \Pi_f^s(\hat{z}_i, \hat{p}_i) \circ \Pi_f^u(\hat{x}, \hat{z}_i)(W_0)$.

We may assume that U_i are small so that if V_i is the connected component of $U_i \cap \hat{w}_i$ that contains w_i then $\Pi_f^s(\hat{w}_i, \hat{x})(V_i) \subset W_1$.

Let $\varepsilon > 0$ be given (small) and let V_i' as in Lemma 3.1 for the submanifolds \hat{w}_i corresponding to V_i and let $V_x \subset \Pi_f^s(\hat{w}_i, \hat{x})(V_i')$.

Let $l_i : V_i \rightarrow \hat{w}_i$ be defined by

$$l_i = \Pi_f^u(\hat{p}_i, \hat{w}_i) \circ \Pi_f^s(\hat{z}_i, \hat{p}_i) \circ \Pi_f^u(\hat{x}, \hat{z}_i) \circ \Pi_f^s(\hat{w}_i, \hat{x}).$$

Note that l_i is a C^1 map and

$$\Pi_f^s(\hat{w}_i, \hat{x}) \circ \Pi_f^u(\hat{p}_i, \hat{w}_i) \circ \Pi_f^s(\hat{z}_i, \hat{p}_i) \circ \Pi_f^u(\hat{x}, \hat{z}_i) = \Pi_f^s(\hat{w}_i, \hat{x}) \circ l_i \circ (\Pi_f^s(\hat{w}_i, \hat{x}))^{-1}.$$

Let's look l_1 in the local canonical coordinates and let v in \mathbb{R}^k , $\|v\| < \delta$ be such that $-v$ is a regular value of $l_1 - \text{id}$. For this v , choose $h_1 : M \rightarrow M$ as in Lemma 3.1 (in the appropriate setting for \mathcal{E}) and so $h_1 \circ l_1$ has finitely many fixed points in $\overline{V_1'}$. Indeed, if q is a fixed point in $\overline{V_1'}$ of $h_1 \circ l_1$ then $l_1(q) - q = -v$ and since $-v$ is a regular value of $l_1 - \text{id}$ it is an isolated fixed point.

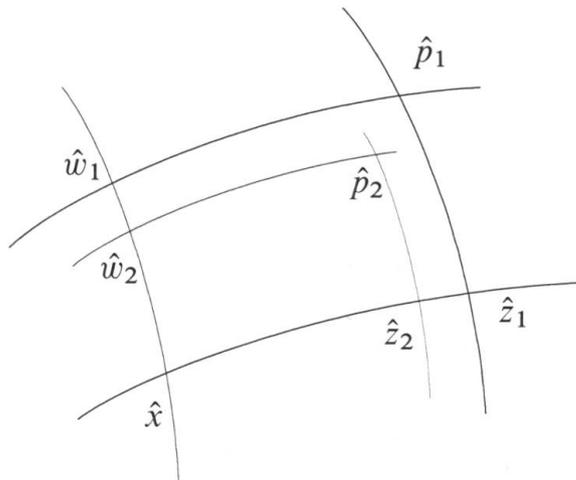


Figure 5. Perturbing trivial accessibility classes.

Let q_1, \dots, q_ℓ be the projection of these fixed points in V_2 , i.e. $\{q_1, \dots, q_\ell\} = \Pi_f^s(\hat{w}_1, \hat{w}_2)(\text{Fix}(h_1 \circ l_1 | \overline{V_1'})) \cap V_2$. Choose $h_2 : M \rightarrow M$ as in Lemma 3.1 (corresponding to U_2, V_2) such that no q_i is fixed by $h_2 \circ l_2$.

Let $g : M \rightarrow M$ be $g = f \circ h^{-1}$ where $h : M \rightarrow M$ is defined by

$$h(z) = \begin{cases} h_1(z) & \text{if } z \in U_1, \\ h_2(z) & \text{if } z \in U_2, \\ z & \text{otherwise.} \end{cases}$$

It is not difficult to see that (see Remark 3.4):

$$\begin{aligned} \Pi_g^s(\hat{w}_i, \hat{x}) \circ \Pi_g^u(\hat{p}_i, \hat{w}_i) \circ \Pi_g^s(\hat{z}_i, \hat{p}_i) \circ \Pi_g^u(\hat{x}, \hat{z}_i) \\ = \Pi_f^s(\hat{w}_i, \hat{x}) \circ h_i \circ l_i \circ (\Pi_f^s(\hat{w}_i, \hat{x}))^{-1}. \end{aligned}$$

Now, the maps

$$\Pi_g^s(\hat{w}_1, \hat{x}) \circ \Pi_g^u(\hat{p}_1, \hat{w}_1) \circ \Pi_g^s(\hat{z}_1, \hat{p}_1) \circ \Pi_g^u(\hat{x}, \hat{z}_1)$$

and

$$\Pi_g^s(\hat{w}_2, \hat{x}) \circ \Pi_g^u(\hat{p}_2, \hat{w}_2) \circ \Pi_g^s(\hat{z}_2, \hat{p}_2) \circ \Pi_g^u(\hat{x}, \hat{z}_2)$$

have no common fixed point. Thus, for $y \in V_x$ we have that either for $i = 1$ or 2 that

$$\Pi_g^s(\hat{w}_i, \hat{x}) \circ \Pi_g^u(\hat{p}_i, \hat{w}_i) \circ \Pi_g^s(\hat{z}_i, \hat{p}_i) \circ \Pi_g^u(\hat{x}, \hat{z}_i)(y) \neq y.$$

This completes the proof. □

We need the following elementary result. For completeness, we give a proof in the appendix. It says roughly that two nondecreasing functions of the interval with arbitrarily small translations have no fixed points in common (this is very simple when the maps are C^1 by transversality).

Proposition 3.6. *Let $\ell_1 : [-a, a] \rightarrow \mathbb{R}$ and $\ell_2 : [-b, b] \rightarrow \mathbb{R}$ be two non-decreasing functions and let $\phi : [-b, b] \rightarrow [-a, a]$ be also a non-decreasing function. Then for any $\varepsilon > 0$ there exist $s, t, |s|, |t| \leq \varepsilon$, such that:*

$$\phi(\{x \in [-b, b] : \ell_2(x) + t = x\}) \cap \{x \in [-a, a] : \ell_1(x) + s = x\} = \emptyset.$$

Proof. See Appendix A. □

We now present the last lemma of this section and it will play a key role in the proof of our main results (Theorems 4A and 4B).

Lemma 3.7. *Let \mathcal{E} be either \mathcal{E}_ω^r or $\mathcal{E}_{sp,\omega}^r$ (i.e. as in Theorems 4A and 4B). Consider $f \in \mathcal{E}$ and let \mathcal{F}_1^c be a periodic compact center leaf. Let $x \in \mathcal{F}_1^c$ and assume that $C_0(x)$ is a C^1 simple closed curve \mathcal{C} . Let U be a neighborhood of \mathcal{C} homeomorphic to an annulus and assume that a family Γ of disjoint essential simple closed curves contained in U is given, with $\mathcal{C} \in \Gamma$. Then, there exist a neighborhood V of \mathcal{C} homeomorphic to an annulus such that for any $\varepsilon > 0$ there exist $g \in \mathcal{E}$ such that*

- (1) $\text{dist}_{C^r}(f, g) < \varepsilon$,
- (2) $\text{supp}(f \neq g)$ is disjoint from the f -orbit of \mathcal{F}_1^c , and
- (3) no curve of Γ contained in V is the accessibility class of a point in V , i.e. for any $y \in V$, $C_0(y, g) \neq \gamma$ for any $\gamma \in \Gamma$.

Proof. Let x be as in statement of the lemma and let $x, w_1, w_2, p_1, p_2, z_1, z_2$ and U_1, U_2 be as in Lemma 3.3. Again, as before, let's write $\hat{w}_i = \mathcal{F}^c(w_i, f)$, $\hat{z}_i = \mathcal{F}^c(z_i, f)$, and $\hat{p}_i = \mathcal{F}^c(p_i, f)$. Let V_i be the connected component of $U_i \cap \hat{w}_i$ that contains w_i and let V'_i be as in Lemma 3.1. Let $W \subset \Pi_f^s(\hat{w}_i, \hat{x})(V'_i)$ be open and containing $x, i = 1, 2$, and let $\mathcal{C}_w = W \cap \mathcal{C}$ (we may assume that \mathcal{C}_w is an arc). Let $\mathcal{C}_i = \Pi_f^s(\hat{x}, \hat{w}_i)(\mathcal{C}_w), i = 1, 2$ (see Figure 6).

In the local canonical coordinates in V_i , let S_i be straight segments transversal to \mathcal{C}_i at w_i , and let $I_i = \Pi_f^s(\hat{w}_i, \hat{x})(S_i \cap V'_i)$. These arcs are transversal to \mathcal{C} at x . We take V a compact neighborhood of \mathcal{C} homeomorphic to a closed annulus, such that both I_i crosses V and intersects \mathcal{C} in just one point. We may suppose that if $\gamma \in \Gamma, \gamma \cap V \neq \emptyset$ then $\gamma \subset V$. Moreover, we redefine I_i to be the connected component of $I_i \cap V$ that contains x and let $S'_i = \Pi_f^s(\hat{x}, \hat{w}_i)(I_i) \subset S_i \cap V'_i$.

Let $\hat{J}_i = \Pi_f^u(\hat{z}_i, \hat{x}) \circ \Pi_f^s(\hat{p}_i, \hat{z}_i) \circ \Pi_f^u(\hat{w}_i, \hat{p}_i)(S_i)$ and we may also assume that \hat{J}_i crosses V . Notice that \hat{J}_i are transversal to \mathcal{C} . Let J_i be a connected component of $\hat{J}_i \cap V$ that crosses V (and we may assume that J_i intersects \mathcal{C} in just

one point). We will define functions $P_i : I_i \rightarrow J_i$, $\varphi : I_1 \rightarrow I_2$, and $\psi : J_1 \rightarrow J_2$ as follows. We will just define $P_1 : I_1 \rightarrow J_1$, the others are completely similar.

We order the arcs I_1, I_2, J_1, J_2 so that all of them crosses \mathcal{C} in “positive” direction.

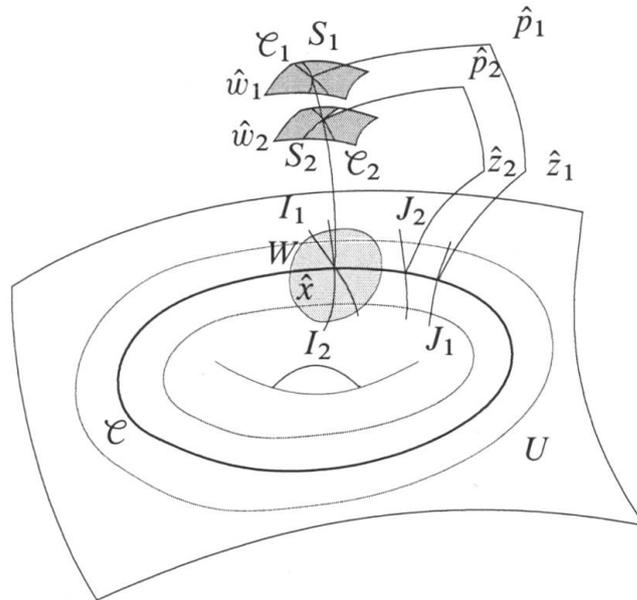


Figure 6. Perturbing closed 1-dimensional accessibility classes.

Let $\gamma \subset V$ be a curve in the family Γ . Let x_γ be the closest point of $\gamma \cap J_1$ (in the order of J_1) to $J_1 \cap \mathcal{C}$, and let y_γ be the closest point of $I_1 \cap \gamma$ (in the order of I_1 to $x = I_1 \cap \gamma$). See Figure 7.

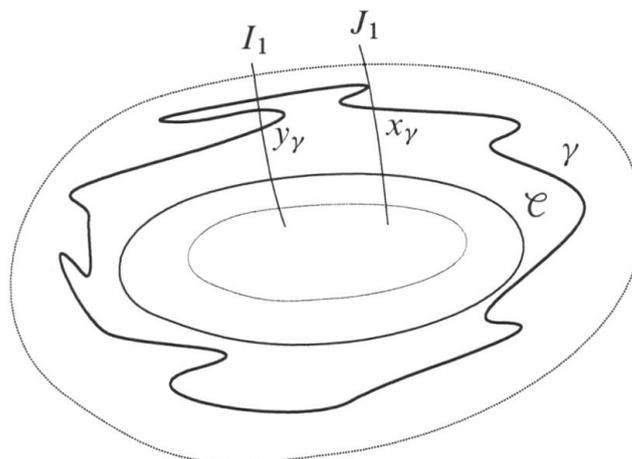


Figure 7. The map P_1 .

Define $P_1(y_\gamma) := x_\gamma$. This is a function from

$$\{y_\gamma : \gamma \in \Gamma\} \subset I_1 \text{ to } \{x_\gamma : \gamma \in \Gamma\} \subset J_1.$$

This function is non-decreasing since for $\gamma, \eta \in \Gamma$, $\gamma \neq \eta$, if $y_\gamma <_{I_1} y_\eta$ then $x_\gamma <_{J_1} x_\eta$ (since each curve of Γ separates V in exactly two components and the curves in Γ are disjoint).

Since the function is non-decreasing it may be extended to a non-decreasing function $P_1 : I_1 \rightarrow J_1$. In the same way we define $P_2 : I_2 \rightarrow J_2$, $\varphi : I_1 \rightarrow I_2$, and $\psi : J_1 \rightarrow J_2$.

Notice that for $\gamma \in \Gamma$ and x_γ as before we have

$$\psi \circ P_1(y_\gamma) = P_2 \circ \varphi(y_\gamma).$$

Let $l_i : S'_i \rightarrow S_i$ defined by

$$l_i = \Pi_f^u(\hat{p}_i, \hat{w}_i) \circ \Pi_f^s(\hat{z}_i, \hat{p}_i) \circ \Pi_f^u(\hat{x}, \hat{z}_i) \circ P_i \circ \Pi_f^s(\hat{w}_i, \hat{x})|_{S'_i}.$$

We claim that l_i is non-decreasing. Indeed, it is equivalent to prove that

$$\tilde{l}_i = \Pi_f^s(\hat{w}_i, \hat{x}) \circ \Pi_f^u(\hat{p}_i, \hat{w}_i) \circ \Pi_f^s(\hat{z}_i, \hat{p}_i) \circ \Pi_f^u(\hat{x}, \hat{z}_i) \circ P_i$$

is non-decreasing, which is equivalent to show that

$$\hat{l}_i = \Pi_f^s(\hat{w}_i, \hat{x}) \circ \Pi_f^u(\hat{p}_i, \hat{w}_i) \circ \Pi_f^s(\hat{z}_i, \hat{p}_i) \circ \Pi_f^u(\hat{x}, \hat{z}_i)$$

from J_i to I_i preserves orientation (since it is a diffeomorphism). Set $x_i = J_i \cap \mathcal{C}$, we know that $\hat{l}_i(x_i) = x$. If \hat{l}_i reverse orientation, then for $y >_{J_i} x_i$ we have that $\hat{l}_i(y) <_{I_i} x$. Since \mathcal{C} is essential in U we get that the accessibility class of $C_0(y)$ must intersect $\mathcal{C} = C_0(x)$ and hence \mathcal{C} is not an accessibility class, a contradiction.

Now, let $\varepsilon > 0$ be given and let δ be as in Lemma 3.1. For $|s| < \delta$ and $|t| < \delta$ we choose h_1 and h_2 as in Lemma 3.1 so that in V'_1 we have $h_1(y) = y + v_1$, v_1 in the direction of S_1 , $\|v_1\| = |s|$ and in V'_2 we have $h_2(y) = y + v_2$, v_2 in the direction of S_2 , $\|v_2\| = |t|$. So, S_1 is invariant by h_1 and S_2 is invariant by h_2 , and parametrizing S_1 and S_2 , these maps have the form $h_{1/S_1}(y) = y + s$ and $h_{2/S_2}(y) = y + t$.

Now define $g = h \circ f$ where

$$h = \begin{cases} h_1(x) & \text{if } x \in U_1, \\ h_2(x) & \text{if } x \in U_2, \\ x & \text{otherwise.} \end{cases}$$

Notice that $\Pi_g^u(\hat{p}_i, \hat{w}_i) = h_i \circ \Pi_f^u(\hat{p}_i, \hat{w}_i)$, $\Pi_g^s(\hat{z}_i, \hat{p}_i) = \Pi_f^s(\hat{z}_i, \hat{p}_i)$, $\Pi_g^u(\hat{x}, \hat{z}_i) = \Pi_f^u(\hat{x}, \hat{z}_i)$, and $\Pi_g^s(\hat{w}_i, \hat{x}) = \Pi_f^s(\hat{w}_i, \hat{x})$.

Now, by Proposition 3.6, we may choose s, t so that if q is a fixed point of

$$h_1 \circ l_1 = \Pi_g^u(\hat{p}_1, \hat{w}_1) \circ \Pi_g^s(\hat{z}_1, \hat{p}_1) \circ \Pi_g^u(\hat{x}, \hat{z}_1) \circ P_1 \circ \Pi_g^s(\hat{w}_1, \hat{x})|_{S'_1}$$

then $[\Pi_f^s(\hat{x}, \hat{w}_1) \circ \varphi \circ \Pi_f^s(\hat{w}_2, \hat{x})]^{-1}(q)$ does not contain any fixed point of $h_2 \circ l_2$.

Thus, by conjugacy with $\Pi_g^s(\hat{x}, \hat{w}_1)$ we have that if q is a fixed point of

$$\hat{l}_1 := \Pi_g^s(\hat{w}_1, \hat{x}) \circ \Pi_g^u(\hat{p}_1, \hat{w}_1) \circ \Pi_g^s(\hat{z}_1, \hat{p}_1) \circ \Pi_g^u(\hat{x}, \hat{z}_1) \circ P_1$$

then $\varphi^{-1}(q)$ does not contain any fixed point of

$$\hat{l}_2 := \Pi_g^s(\hat{w}_2, \hat{x}) \circ \Pi_g^u(\hat{p}_2, \hat{w}_2) \circ \Pi_g^s(\hat{z}_2, \hat{p}_2) \circ \Pi_g^u(\hat{x}, \hat{z}_2) \circ P_2.$$

Now, let $\gamma \in \Gamma, \gamma \subset V$ and let y_γ, x_γ as before. Then $\hat{l}_1(y_\gamma) \in C_0(x_\gamma, g) = C_0(P_1(y_\gamma), g)$. So, if y_γ is not fixed by \hat{l}_1 , we have two possibilities: either

- (1) $\hat{l}_1(y_\gamma) <_{I_1} y_\gamma$, or
- (2) $\hat{l}_1(y_\gamma) >_{I_1} y_\gamma$.

In case (1), $\hat{l}_1(y_\gamma)$ cannot belong to γ by the definition of y_γ and so $C_0(x_\gamma, g)$ is not contained in γ .

In case (2), we conclude that the point

$$z := \Pi_g^u(\hat{z}_1, \hat{x}) \circ \Pi_g^s(\hat{p}_1, \hat{z}_1) \circ \Pi_g^u(\hat{w}_1, \hat{p}_1) \circ \Pi_g^s(\hat{x}, \hat{w}_1)(y_\gamma)$$

satisfies $z <_{J_1} x_\gamma$ and so does not belong to γ which implies that $C_0(y_\gamma, g)$ is not contained in γ .

Finally, assume that y_γ is fixed by \hat{l}_1 and let \bar{x}_γ be the closest point of $\gamma \cap I_2$ (in the order of I_2) to x . Then we know that \bar{y}_γ is not fixed by \hat{l}_2 and we apply the previous argument. Thus, no curve $\gamma \in \Gamma$ is an accessibility class. The proof is finished. □

4. Proof of Theorem 2

Let $r \geq 2$ and let \mathcal{E} be as in Theorem 2, i.e. \mathcal{E} is $\mathcal{E}^r, \mathcal{E}_m^r, \mathcal{E}_\omega^r, \mathcal{E}_{sp}^r$ or $\mathcal{E}_{sp,\omega}^r$. We have to prove the set \mathcal{R}_0 of diffeomorphisms in \mathcal{E} having no trivial accessibility classes is C^1 open and C^r dense. This result is a consequence of Lemma 3.5, as follows.

Let's consider $\Gamma_0 : \mathcal{E} \rightarrow \mathcal{C}(M) = \{\text{compact subsets of } M\}$ (endowed with the Hausdorff topology)

$$\Gamma_0(f) = \{x \in M : AC(x) \text{ is trivial}\}. \tag{4.1}$$

We observe that $\Gamma_0(f)$ is indeed a compact set, it follows from Corollary 2.11.

Lemma 4.1. *The map Γ_0 is upper semicontinuous, i.e. given $f \in \mathcal{E}$ and a compact set K such that $\Gamma_0(f) \cap K = \emptyset$ then there exists a neighborhood $\mathcal{U}(f)$ of f in \mathcal{E} (which is also C^1 open) such that $\Gamma_0(g) \cap K = \emptyset$ for all $g \in \mathcal{U}(f)$.*

Proof. Let $y \notin \Gamma_0(f)$. From Corollary 2.13 there exist $U(y)$ and $\mathcal{U}_y(f)$ (which is also C^1 open) such that for any $g \in \mathcal{U}_y(f)$ and $z \in U(y)$ we have that $AC(z, g)$ is non-trivial.

Now, consider the family of U_y with $y \in K$. We may cover K with finitely many of them, say $K \subset \bigcup_{i=1}^n U_{y_i}$.

Let $\mathcal{U}(f) \subset \bigcap_{i=1}^n \mathcal{U}_{y_i}(f)$. If $g \in \mathcal{U}(f)$ and $z \in K$ then $z \in U_{y_i}$ and $g \in \mathcal{U}_{y_i}(f)$ for some i and so $AC(z, g)$ is non-trivial. The proof of the lemma is complete. \square

By taking $K = M$ in the previous lemma, we get:

Corollary 4.2. *If for some $f \in \mathcal{E}$ we have that $\Gamma_0(f) = \emptyset$ then there is a neighborhood $\mathcal{U}(f) \subset \mathcal{E}$ (which is C^1 open) such that for any $g \in \mathcal{U}(f)$ we have $\Gamma_0(g) = \emptyset$.*

Now, we are ready to conclude:

Proof of Theorem 2. Let \mathcal{G}_0 be the set of continuity points of Γ_0 . This is a residual set in \mathcal{E} (since \mathcal{E} with the C^r topology is a Baire space). We claim that if $f \in \mathcal{G}_0$ then $\Gamma_0(f) = \emptyset$. Otherwise, let $x \in \Gamma_0(f)$ and we may assume that x belongs to a periodic compact center leaf (see Lemma 2.4).

Indeed, by the continuity of Γ_0 at f we have that for any neighborhood V of x there exists $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}(f)$ there is $x_g \in V$ such that $AC(x_g, g)$ is trivial. A direct application of Lemma 3.5 yields a contradiction and the claim is proved.

From this and Corollary 4.2 we get that the set

$$\mathcal{R}_0 = \{f \in \mathcal{E} : \Gamma_0(f) = \emptyset\} \tag{4.2}$$

is C^1 open and C^r dense in \mathcal{E} . This set \mathcal{R}_0 is just the set of diffeomorphisms where any accessibility class is nontrivial. \square

5. The accessibility class of periodic points

Through this section, we consider \mathcal{E} to be either $\mathcal{E}^r, \mathcal{E}_m^r, \mathcal{E}_\omega^r, \mathcal{E}_{sp}^r$ or $\mathcal{E}_{sp,\omega}^r$ and with $\dim E^c = 2$.

Let $f \in \mathcal{E}$ and let \mathcal{F}_1^c be a compact periodic center leaf of period k and let $p \in \mathcal{F}_1^c$ be a periodic point of f . Let U be a neighborhood of p in M . We denote by $C_0(p, U, f)$ the *local accessibility class* of p , that is, the set $y \in \mathcal{F}_1^c$ that can be joined to p by su -path contained in the neighborhood U of p .

We will classify the periodic points with respect to its behaviour on the central leaf. In particular we say

- p is *center-hyperbolic of saddle type* if p is hyperbolic of saddle type with respect to $f|_{\mathcal{F}_1^c}$,

- p is *center-attractor* or *center-repeller* if p is attractor or repeller w.r.t. $f|_{\mathcal{F}_1^c}^k$,
- p is *center-elliptic* if p is elliptic w.r.t. $f|_{\mathcal{F}_1^c}^k$.

Assume that p is a center-hyperbolic periodic point of saddle type $f|_{\mathcal{F}_1^c}^k$. We denote by $CW^s(p)$ the *stable manifold of p with respect to $f|_{\mathcal{F}_1^c}^k$* . We write $CE_p^s \subset T_p\mathcal{F}_1^c$ the tangent space to $CW^s(p)$. Analogously, we denote $CW^u(p)$ the *unstable manifold of p with respect to $f|_{\mathcal{F}_1^c}^k$* and $CE_p^u \subset T_p\mathcal{F}_1^c$ the tangent space to $CW^u(p)$.

We say that a periodic point p of period $\tau(p)$ of f is *generic* if:

- p is hyperbolic in the case $\mathcal{E} = \mathcal{E}^r$ or \mathcal{E}_{sp}^r .
- -1 and 1 are not eigenvalues of $Df_p^{\tau(p)}$ in the case $\mathcal{E} = \mathcal{E}_m^r, \mathcal{E}_\omega^r$ or $\mathcal{E}_{sp,\omega}^r$.

Lemma 5.1. *There exists a residual set \mathcal{G}_1 in \mathcal{E} such that if $f \in \mathcal{G}_1$ and p is a center-hyperbolic periodic point of saddle type of f for $f|_{\mathcal{F}_1^c}^k$ then there exist neighborhoods U_c and U of p , U_c in $\mathcal{F}^c(p)$ and U in M , such that $U_c \setminus (CW_{loc}^s(p) \cup CW_{loc}^u(p))$ has four connected component, $U \cap \mathcal{F}^c(p) \subset U_c$, and $C_0(p, U, f)$ is not contained in $CW_{loc}^s(p) \cup CW_{loc}^u(p)$.*

We say that a periodic point p as in the previous lemma satisfies the *Property (L)*.

Proof. Let $H_n = \{f \in \mathcal{E} : \text{all points in } \text{Fix}(f^n) \text{ are generic and every center-hyperbolic periodic point } p \in \text{Fix}(f^n) \text{ of saddle type satisfies Property (L)}\}$.

Claim. H_n is open and dense in \mathcal{E} . In fact, notice that $H_n^0 = \{f \in \mathcal{E} : \text{Fix}(f^n) \text{ generic}\}$ is open and dense in \mathcal{E} . Thus, to prove the claim it is enough to show that H_n is open and dense in H_n^0 . It is immediate that H_n is open in H_n^0 . Let us show that H_n is dense. Let $f \in H_n^0$. We know that there are finitely many center-hyperbolic periodic points in $\text{Fix}(f^n)$. Choose a neighborhood U_c for each one as in Property (L). By similar arguments as Lemma 3.5 it is not difficult to get $g \in H_n$ arbitrarily close to f satisfying Property (L).

Finally, set $\mathcal{G}_1 = \bigcap_{n \geq 0} H_n$ and the lemma is proved. □

Theorem 5.2. *There exists a residual subset \mathcal{R}_* in \mathcal{E} such that if $f \in \mathcal{R}_*$ and p is a periodic point which is neither a center-attractor nor a center-repeller for $f|_{\mathcal{F}_1^c}^k$ on a compact periodic center leaf \mathcal{F}_1^c then $C_0(p, f)$ is open. Moreover, if p is center-hyperbolic of saddle type then there exist an open set V in M (contained in a ball around p) and a neighborhood $\mathcal{U}(f)$ (which is also C^1 open) such that for any $g \in \mathcal{U}(f)$, we have $V \subset AC(p_g, g)$ where p_g is the continuation of p for $g \in \mathcal{U}(f)$.*

Proof. Let $\mathcal{R}_* = \mathcal{R}_0 \cap \mathcal{G}_1$ where \mathcal{R}_0 is as in Theorem 2 (see also (4.2)). Let $f \in \mathcal{R}_*$ and let p be a periodic point of f . Since $f \in \mathcal{R}_*$, $C_0(p) := C_0(p, f)$ is either open or a one dimensional C^1 submanifold.

Assume first that p is a center-elliptic periodic point of $f|_{\mathcal{F}_1^c}$ of period $\tau(p)$. Now, since $C_0(p)$ is invariant under $f^{\tau(p)}$ and there is no invariant direction of $Df|_{\mathcal{F}^c(p)}$ we easily conclude that $C_0(p)$ is open.

Assume now that p is a center-hyperbolic periodic point of $f|_{\mathcal{F}_1^c}$ of saddle type of period $\tau(p)$ and assume, by contradiction, that $C_0(p)$ is not open, that is, $C_0(p)$ is a one dimensional C^1 submanifold. Then p satisfies Property (L). This implies that there exists a non trivial connected set $C \subset C_0(p, U, f) \subset C_0(p)$ and not contained in $CW_{loc}^s(p) \cup CW_{loc}^u(p)$.

On the other hand, $T_p C_0(p)$ must be an invariant direction (by the invariance of $C_0(p)$) by $Df_p^{\tau(p)}$ and so $T_p C_0(p) = CE_p^s$ or CE_p^u . Assume that $T_p C_0(p) = CE_p^s$. Thus, $C_0(p)$ is locally a graph around p (via the exponential map) of a map from $CE_p^s \rightarrow CE_p^u$.

Now, this graph is not contained in $CW_{loc}^s(p)$ (since p satisfies Property (L) and the connected set $C \subset C_0(p, U, f)$). But notice that this graph is locally invariant, by the invariance of $C_0(p)$. This is a contradiction since there is a unique locally invariant graph, namely $CW_{loc}^s(p)$. Analogously, if $T_p C_0(p) = CE_p^u$ we use the same argument for f^{-1} . Thus, we have proved that $C_0(p)$ is open.

As the accessibility class $C_0(p)$ of a center-hyperbolic periodic point of saddle type p is open and that $CW_{loc}^s(p)$ and $CW_{loc}^u(p)$ intersect $C_0(p)$ then, by invariance we get that $CW^s(p)$ and $CW^u(p)$ are contained in $C_0(p)$.

From Property (L), we know that there exists $y_f \in U_c \setminus CW_{loc}^s(p) \cup CW_{loc}^u(p)$ so that $y_f \in C_0(p, U, f)$. Let's order the four connected component clockwise beginning with the one that contains y_f . See Figure 8.

Let $B_c(y_f)$ be a ball centered in y_f contained in $U_c \setminus (CW_{loc}^s(p) \cup CW_{loc}^u(p))$, that is, $B_c(y_f) \subset (I)$.

We know (see Lemma 2.7) that there exists a continuous map $\gamma : B_c(p) \times [0, 1] \rightarrow U_c$ such that

- $\gamma(z, 0) = z,$
- $\gamma(z, 1) \subset B_c(y_f),$ and
- $\gamma(z, t) \subset C_0(z, U, f).$

Let V_c be an open set such that $\overline{V_c} \subset B_c(p) \cap (III)$. Thus, for any $z \in V_c$ we have for some t_0 that $\gamma(z, t_0) \in (CW_{loc}^s(p) \cup CW_{loc}^u(p)) \subset C_0(p)$ and so $V_c \subset C_0(p)$.

Finally, we saturate V_c by local (strong) stable and unstable manifolds to obtain an open set $V \subset M$. This set V satisfies the requirement of the theorem for $\mathcal{U}(f)$ small enough by the continuation of center leaves, strong stable and unstable leaves, the continuation of p , and the continuation of γ (see also Lemma 2.12) and so Property (L) is open. The theorem is proved. □

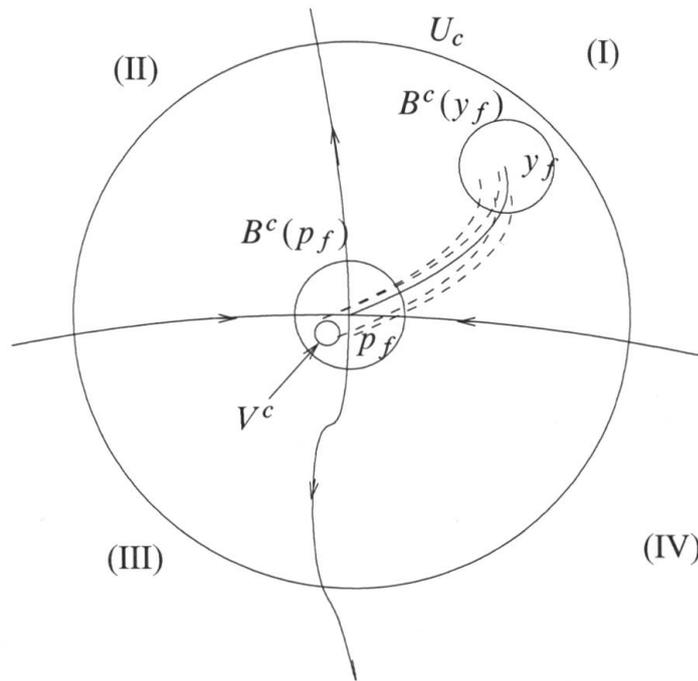


Figure 8. Accessibility class of hyperbolic periodic points.

Let $f \in \mathcal{R}_*$ and let p be a center-hyperbolic periodic point of saddle type (belonging to a periodic compact center leaf). Let $\mathcal{U}(f)$ corresponding to p and f in Theorem 5.2 (we denote by p_g the continuation of p for $g \in \mathcal{U}(f)$).

Let $\Gamma_1 : \mathcal{U}(f) \rightarrow \mathcal{C}(M)$,

$$\Gamma_1(g) = M \setminus AC(p_g, g). \tag{5.1}$$

Notice that Γ_1 is well defined since $AC(p_g, g)$ is open for $g \in \mathcal{U}(f)$.

Proposition 5.3. *The map Γ_1 is upper semicontinuous, i.e. given $g \in \mathcal{U}(f)$ and a compact set K such that $\Gamma_1(g) \cap K = \emptyset$ then there exists a neighborhood $\mathcal{V}(g) \subset \mathcal{U}(f)$ (which is also C^1 open) such that for any $h \in \mathcal{V}(g)$ we have that $\Gamma_1(h) \cap K = \emptyset$.*

Proof. Let V be the fixed open set in M from Theorem 5.2, that is, $V \subset AC(p_g, g)$ for every $g \in \mathcal{U}(f)$. Let $g \in \mathcal{U}(f)$ and K compact with $\Gamma_1(g) \cap K = \emptyset$ be given. Let $y \in K$, then there exists U_y and $\mathcal{U}_y(g)$ such that for any $h \in \mathcal{U}_y(g)$ and any $z \in U_y$ we have that $AC(z, h) \cap V \neq \emptyset$ (see Lemma 2.12), in other words $U_y \subset AC(p_h, h)$ for any $h \in \mathcal{U}_y(g)$.

Now, cover K with finitely many of these open sets U_y , that is, $K \subset \bigcup_{i=1}^n U_{y_i}$. Let $\mathcal{V}(g) = \bigcap_{i=1}^n \mathcal{U}_{y_i}(g)$. Then, for every $h \in \mathcal{V}(g)$ we have that $K \subset AC(p_h, h)$. The proof of proposition is finished. \square

Corollary 5.4. *Assume that for $g \in \mathcal{U}(f)$ we have that $\Gamma_1(g) = \emptyset$, then g is accessible. Moreover, there exists $\mathcal{V}(g)$ (which is also C^1 open) such that any $h \in \mathcal{V}(g)$ is accessible.*

6. Proof of Theorems 3A and 3B

For simplicity, we prove both theorems together since the proof is essentially the same. Let \mathcal{E} be either \mathcal{E}_A^r or $\mathcal{E}_{A,m}^r$ and with $\dim E^c = 2$, that is, the set of \mathcal{E}^r or \mathcal{E}_m^r where $\dim E^c = 2$ and having a compact periodic leaf so that the dynamics in this leaf (in the period) is both Axiom A and sinkless. Let \mathcal{R}_0 be as in Theorem 2 and \mathcal{R}_* as in Theorem 5.2 (both restricted to \mathcal{E}_A^r or $\mathcal{E}_{A,m}^r$). Thus, $\mathcal{R}_0 \cap \mathcal{R}_*$ is residual in \mathcal{E} . Let $f \in \mathcal{R}_0 \cap \mathcal{R}_*$ (although in the proof of Theorem 5.2 we construct $\mathcal{R}_* \subset \mathcal{R}_0$). We will prove that f is accessible. Let's see the properties we know for f :

- Any accessibility class is nontrivial.
- The accessibility classes of center-hyperbolic periodic points of saddle type of f are open.
- If p is a center-hyperbolic periodic point of saddle type, then it satisfies Property (L).
- There exists a compact periodic center leaf \mathcal{F}_1^c such that $f|_{\mathcal{F}_1^c}^k$ is an Axiom-A diffeomorphism and sinkless, where k is the period of \mathcal{F}_1^c .
- If p is center-hyperbolic periodic point of saddle type, then the stable and unstable manifolds in the center leaf $CW^s(p)$ and $CW^u(p)$ are contained in the accessibility class of p .

Let's denote by Λ a basic piece from Smale's spectral decomposition of $f|_{\mathcal{F}_1^c}^k$ which is not a periodic center-repeller. Recall that the stable and unstable manifolds $CW^s(\mathcal{O}(p))$ and $CW^u(\mathcal{O}(p))$ are dense in Λ for any periodic point $p \in \Lambda$. Let $x \in \Lambda$ be any point. We know that $C_0(x)$ is open or a one dimensional C^1 manifold without boundary containing x . In any case, we have that it intersects $CW^s(\mathcal{O}(p))$ or $CW^u(\mathcal{O}(p))$ and therefore $C_0(x) \cap C_0(f^i(p)) \neq \emptyset$ for some i . Therefore, $C_0(x) = C_0(f^i(p))$. This means that Λ is contained in (the open set) $\cup_{q \in \mathcal{O}(p)} C_0(q)$ for $p \in \Lambda$ periodic. By the invariance of $\cup_{q \in \mathcal{O}(p)} C_0(q)$ we also have that $CW^s(\Lambda)$ and $CW^u(\Lambda)$ are contained in $\cup_{q \in \mathcal{O}(p)} C_0(q)$. Let F_0 be the set of periodic center-repellers which is a finite set. Let F_1 be the set of center-hyperbolic periodic points of saddle type. Since there are no periodic center-attractor we have that $\text{Per}(f|_{\mathcal{F}_1^c}) = F_0 \cup F_1$.

Since every point in \mathcal{F}_1^c is contained in the stable manifold (inside the center leaf) of the basic pieces, we conclude that $\mathcal{F}_1^c \setminus F_0 \subset \cup_{p \in F_1} C_0(p)$ and that $C_0(p)$ is open for any periodic point in F_1 . By connectedness we conclude that $\mathcal{F}_1^c \setminus F_0 \subset C_0(p)$ (for any periodic point $p \in F_1$). Since the accessibility classes are non trivial for

every $q \in F_0$, we have that $C_0(q) \cap C_0(p) \neq \emptyset$ and so $C_0(q) = C_0(p)$. Thus $\mathcal{F}_1^c = C_0(p)$ for any $p \in F_1$. That is, the center leaf \mathcal{F}_1^c is just one center accessibility class and by Lemma 2.4 we have that f is accessible, as we claimed.

Finally, since $f \in \mathcal{R}_*$ and fixing a center-hyperbolic periodic point of saddle type p of f , in the setting of Section 5, we have a neighborhood $\mathcal{U}(f)$ and a map Γ_1 defined in $\mathcal{U}(f)$ as in (5.1). Due to what we just proved $\Gamma_1(f) = \emptyset$ holds and so by Corollary 5.4 there exists $\mathcal{V}(f)$ such that any $g \in \mathcal{V}(f)$ is accessible. Thus,

$$\widehat{\mathcal{R}} = \bigcup_{f \in \mathcal{R}_0 \cap \mathcal{R}_*} \mathcal{V}(f)$$

is C^1 open and C^r dense in \mathcal{E} and formed by accessible diffeomorphisms. This completes the proof of Theorems 3A and 3B. \square

6.1. Examples. We present here some examples where Theorems 3A and 3B apply.

Example 6.1. This example can be thought as a conservative version of the well known Shub's example on \mathbb{T}^4 [34].

Consider the Lewowicz family (see [22]) of conservative diffeomorphisms on \mathbb{T}^2 :

$$f_c(x, y) = \left(2x - \frac{c}{2\pi} \sin(2\pi x) + y, x - \frac{c}{2\pi} \sin(2\pi x) + y \right), \quad c \in \mathbb{R}.$$

Notice that when $c = 0$ f_c is Anosov and when $1 < c < 5$ the fixed point at $(0, 0)$ is an elliptic fixed point. We just consider for instance $c \in [0, 2]$. From this family it is not difficult to construct a continuous map

$$g : \mathbb{T}^2 \rightarrow \text{Diff}^\infty(\mathbb{T}^2)$$

such that for two points $p, q \in \mathbb{T}^2$ given, we have $g(p) = f_0$ and $g(q) = f_2$. Now, given $r \geq 2$, consider a conservative Anosov diffeomorphism $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ having $p, q \in \mathbb{T}^2$ as fixed points and with enough strong expansion and contraction so the map

$$F : \mathbb{T}^2 \times \mathbb{T}^2 \rightarrow \mathbb{T}^2 \times \mathbb{T}^2 \text{ defined as } F(x, y) = (A(x), g_x(y))$$

belongs to $\mathcal{E}_{A,m}^r(\mathbb{T}^2 \times \mathbb{T}^2)$. The center foliation is thus $\{x\} \times \mathbb{T}^2$ and at $\mathcal{F}^c(p) = \{p\} \times \mathbb{T}^2$ the map F supports an Anosov (and hence is an Axiom-A diffeomorphism without periodic attractors). Our theorem implies that a generic arbitrarily small C^r perturbation (preserving the Lebesgue measure on $\mathbb{T}^2 \times \mathbb{T}^2$) is stably ergodic. The same example can be considered also just in the skew-product setting.

Remark 6.2. Notice that due to the presence of an elliptic point on $\{q\} \times \mathbb{T}^2$ the center bundle E^c does not admit any dominated splitting. By the result in [4] we may find a perturbation of F (and stably ergodic) such that the center Lyapunov exponent is nonzero. This implies that the center foliation (which is two dimensional) is not absolutely continuous.

Theorem 3A admits some generalizations or different versions. We just give some examples and an idea of how to prove stable ergodicity.

Example 6.3. Consider $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ a conservative Anosov diffeomorphism and let $F_0 : \mathbb{T}^2 \times \mathbb{S}^1 \rightarrow \mathbb{T}^2 \times \mathbb{S}^1$ as $F_0 = f \times id$. This is a conservative partially hyperbolic diffeomorphism on \mathbb{T}^3 with one dimensional center and the center foliation is by circles. Let p be a fixed point of f . It is not difficult to construct a (conservative) perturbation F of F_0 such that on the corresponding $\mathcal{F}^c(p, F)$ the dynamics is a north-south Morse–Smale dynamics and F satisfies the same generic conditions as in Theorem 3A. Then, the map $F \times F : \mathbb{T}^6 \rightarrow \mathbb{T}^6$ belongs to $\mathcal{E}_m^r(\mathbb{T}^6)$, the center foliation is by \mathbb{T}^2 and $F|_{\mathcal{F}^c(p,p)}$ is an Axiom-A diffeomorphism (the product of the two Morse–Smale on the circle). Theorem 3A does not apply in this case because $F|_{\mathcal{F}^c(p,p)}$ have a center-attracting and a center-repelling periodic point. Nevertheless, by the similar arguments as in the proof of Theorem 3A one gets that the union of the accessibility classes of the two center-hyperbolic saddles in $\mathcal{F}^c(p, p)$ is open and if it is not the whole center leaf $\mathcal{F}^c(p, p)$ then its complement consist of a closed C^1 curve which is a connection between the attractor and the repeller. Since this curve does not separate the leaf $\mathcal{F}^c(p, p)$ we have that the union of the accessibility classes of the two center-hyperbolic saddles is just one accessibility class $C_0(q)$, for q any periodic saddle, which is open and $AC(q)$ has full measure in \mathbb{T}^6 . This means that F is essentially accessible and hence ergodic. Since the above situation is C^r open we conclude that F is C^r stably ergodic.

The same argument also applies to the next example.

Example 6.4. Consider $f_0 : M \rightarrow M$ to be the time one map of the suspension of a conservative Anosov diffeomorphisms on \mathbb{T}^2 . This is a conservative partially hyperbolic diffeomorphism whose leaves of the center foliation are the orbits of the suspension. Let p be a fixed point of the Anosov and the center leaf $\mathcal{F}^c(p, f_0)$ is a circle where f_0 is the identity. We then find a conservative and generic perturbation f of f_0 such that f restricted to $\mathcal{F}^c(p, f)$ is a Morse–Smale system. The diffeomorphism $f \times f : M \times M \rightarrow M \times M$ belongs to $\mathcal{E}_m^r(M \times M)$ with two dimensional center leaves and in the leaf $\mathcal{F}^c((p, p), f \times f)$ is an Axiom-A on a two torus and in the same situation as the previous example. The same argument yields that $f \times f$ is stably ergodic.

7. Proof of Theorems 4A and 4B

The proof of Theorems 4A and 4B are essentially the same. Indeed, the important thing we need is that if $L = f^k(L)$ is a periodic compact leaf then $f|_L^k$ preserves area. This automatically holds in the case of conservative skew products. We will see that this property also holds if a symplectic form is preserved.

In this section we denote by \mathcal{E} either \mathcal{E}_ω^r or $\mathcal{E}_{sp,\omega}^r$ and with $\dim E^c = 2$. The key fact about preserving a symplectic form ω in \mathcal{E}_ω^r is the following folklore result (see [39, Lemma 2.5]):

Lemma 7.1. *Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism preserving a symplectic form ω . Then $\omega|_{E^c}$ is non degenerate (and so symplectic). In particular if f is dynamically coherent and $\dim E^c = 2$ then ω is an area form in $\mathcal{F}^c(x)$ for any x . Furthermore, if $\mathcal{F}^c(x)$ is k -periodic then $f|_{\mathcal{F}^c(x)}^k$ is a conservative diffeomorphism.*

Throughout this section in order to simplify notation we omit the word center when we refer to the classification of periodic point in a center leaf in Section 5.

The next lemma says that generically we have compact leaves with periodic points.

Lemma 7.2. *There exists a C^1 open and C^r dense set \mathcal{G}_2 in \mathcal{E} such that if $f \in \mathcal{G}_2$ then there exists a periodic compact leaf having a hyperbolic periodic point.*

Proof. Notice that the set in \mathcal{E} having a hyperbolic periodic point on some compact periodic leaf is C^1 open.

Let $f_0 \in \mathcal{E}$ and let \mathcal{F}_1^c be a compact periodic leaf. For simplicity we assume it is fixed. We may assume also that \mathcal{F}_1^c is orientable (otherwise we go to the double covering). If \mathcal{F}_1^c is not the two torus then $f_0|_{\mathcal{F}_1^c}$ has periodic points. Let $f \in \mathcal{E}$ be a Kupka–Smale diffeomorphism and arbitrarily C^r close to f_0 . It follows that there is a hyperbolic periodic point in $\mathcal{F}_1^c(f)$ since once we have elliptic periodic points we have hyperbolic periodic points (see [40]).

If \mathcal{F}_1^c is the two torus and f_0 has no periodic points in \mathcal{F}_1^c , then by composing with a translation in the torus (and extending this perturbation on \mathcal{F}_1^c to M) we may change the mean rotation number (the mean rotation number of the composition of two maps is the sum of the mean rotation number of each one) to get a rational mean rotation number. Using a result by Franks [16] we get a periodic point, which by perturbation we may assume that it is hyperbolic or elliptic. And then we argue as before. \square

Remark 7.3. In this situation we are working with $(\mathcal{E}_\omega^r$ or $\mathcal{E}_{sp,\omega}^r)$, if p is a hyperbolic periodic point of f then for the restriction to the center manifold that contains p we have that p is a hyperbolic periodic point of saddle type (since the restriction to the center manifold preserves area).

Let \mathcal{R}_0 from Theorem 2, and let \mathcal{KS} be the set of Kupka–Smale diffeomorphisms in \mathcal{E} which is a residual set, and let \mathcal{R}_* as in Theorem 5.2. We consider

$$f \in \mathcal{R}_0 \cap \mathcal{R}_* \cap \mathcal{G}_2 \cap \mathcal{KS} \quad (7.1)$$

and let \mathcal{F}_1^c be a compact k -periodic leaf containing a hyperbolic periodic point p , $\mathcal{F}_1^c = \mathcal{F}^c(p)$.

Let $\mathcal{U}(f)$ be as in Theorem 5.2 and let Γ_1 as in (5.1). Let $\mathcal{R}_1(f) \subset \mathcal{U}(f)$ be the residual subset of continuity points of Γ_1 .

We will assume for simplicity that the compact leaf $S = \mathcal{F}^c(p, f)$ is fixed by f . When g varies on $\mathcal{U}(f)$ the compact leaf $\mathcal{F}^c(p_g, g)$ varies continuously (by a homeomorphism on M close to the identity), and thus there is a natural identification between $\mathcal{F}^c(p_g, g)$ with $\mathcal{F}^c(p, f)$ as the surface S . In order to avoid unimportant technicalities we will assume that $\mathcal{F}^c(p_g, g) = S$ for any $g \in \mathcal{U}(f)$. Now consider the following functions $\Gamma_2, \Gamma_3 : \mathcal{U}(f) \rightarrow \mathcal{C}(S)$, where $\mathcal{C}(S)$ is the set of compact subset of S with the Hausdorff topology:

$$\Gamma_2(g) = S \setminus C_0(p_g, g) \quad \text{and} \quad \Gamma_3(g) = \overline{C_0(p_g, g)}, \tag{7.2}$$

where $C_0(p_g, g)$ is the connected component of $C(p_g, g)$ that contains p_g .

Lemma 7.4. *The function Γ_2 is upper semicontinuous and the function Γ_3 is lower semicontinuous. That is, given $g \in \mathcal{U}(f)$, a compact set $K \subset S$ and an open set $U \subset S$ such that $K \cap \Gamma_2(g) = \emptyset$ and $U \cap \Gamma_3(g) \neq \emptyset$ then there exists $\mathcal{U}(g)$ such that $K \cap \Gamma_2(h) = \emptyset$ and $U \cap \Gamma_3(h) \neq \emptyset$ for any $h \in \mathcal{U}(g)$.*

Proof. The proof that Γ_2 is upper semicontinuous is similar as the proof of Proposition 5.3. Let $V' = V \cap S$ where V is as in Theorem 5.2 and let $K \subset S$ a compact set as in statement, that is $K \subset C_0(p_g, g)$. Let $y \in K$. There exists $U_y \subset S$ and $\mathcal{U}_y(g)$ such that for any $z \in U_y$ and $h \in \mathcal{U}_y(g)$ we have that $AC(z, h) \cap V' \neq \emptyset$. On the other hand we can assume that $U_y \subset C_0(p_g, g)$ and this means that the su -path (of g) joining $z \in U_y$ with V' when lifted to the covering \tilde{M} is a path that starts and ends on a same center leaf (which projects to S) and so the same happens for h near g . This implies that $U_y \subset C_0(p_h, h)$ for any $h \in \mathcal{U}_y(g)$. Then, covering K with finitely many sets U_y and taking the corresponding intersection of the $\mathcal{U}_y(g)$ we conclude the statement on Γ_2 .

Let's prove the semicontinuity of Γ_3 . Let $U \subset S$ be an open set such that $U \cap \Gamma_3(g) \neq \emptyset$. In particular $U \cap C_0(p_g, g) \neq \emptyset$. Let y be in this intersection and let U_y and $\mathcal{U}_y(g)$ as before. We may assume that $U_y \subset U$. Then, for any $h \in \mathcal{U}_y(g)$ we have that $U_y \subset C_0(p_h, h)$ and so $U \cap \Gamma_3(h) \neq \emptyset$. □

• Let $\mathcal{R}_2(f)$ and $\mathcal{R}_3(f)$ be the sets of continuity points of Γ_2 and Γ_3 respectively. These are residual subsets of $\mathcal{U}(f)$. We set

$$\mathcal{R}_{\mathcal{U}(f)} = \mathcal{R}_0 \cap \mathcal{E}_2 \cap \mathcal{R}_* \cap \mathcal{R}_1(f) \cap \mathcal{R}_2(f) \cap \mathcal{R}_3(f) \cap \mathcal{K}\mathcal{S} \cap \mathcal{U}(f), \tag{7.3}$$

which is a residual subset of $\mathcal{U}(f)$. The next result implies our Theorems 4A and 4B.

Theorem 7.5. *Let $g \in \mathcal{R}_{\mathcal{U}(f)}$. Then $C_0(p_g, g) = \mathcal{F}^c(p_g, g)$, i.e. g is accessible.*

Indeed, for any f as in (7.1) we consider $\mathcal{R}_{\mathcal{U}(f)}$ as in (7.3) and we set

$$\mathcal{R} = \bigcup_{f \in \mathcal{R}_0 \cap \mathcal{R}_* \cap \mathcal{E}_2 \cap \mathcal{K}\mathcal{S}} \mathcal{R}_{\mathcal{U}(f)}.$$

It follows that \mathcal{R} is residual (and hence C^r dense) in \mathcal{E} and every $g \in \mathcal{R}$ is accessible from Theorem 7.5. On the other hand, Corollary 5.4 implies that the accessible ones are C^1 open.

The rest of the section is thus devoted to prove Theorem 7.5.

Lemma 7.6. *Let $g \in \mathcal{U}(f)$ and let K be a connected component of the boundary $\partial C_0(p_g, g)$. Then, for every $x \in K$ we have that $C_0(x) \subset K$.*

Proof. Let $x \in K$ and let $y \in C_0(x)$ and assume that $y \notin K$. Since $C_0(x)$ is connected, we may assume, without loss of generality, that $y \notin \partial C_0(p_g, g)$. Therefore, since y cannot belong to $C_0(p_g)$, we have that $y \notin \overline{C_0(p_g)}$.

On the other hand we know (by Lemma 2.7 and Remark 2.8) there is a continuous map $\gamma : B_x \rightarrow B_y$, such that for any $z \in B_x$, $\gamma(z) \in C_0(z)$, where B_x and B_y are small neighborhoods x and y , respectively, and we may take $B_y \subset \mathcal{F}^c(p_g, g) \setminus \overline{C_0(p_g)}$. Since $x \in \partial C_0(p_g)$ then we may take $z \in B_x \cap C_0(p_g)$ and hence $\gamma(z) \in C_0(p_g) \cap B_y$, a contradiction. \square

Remark 7.7. Let $g \in \mathcal{R}_{\mathcal{U}(f)}$ and let K be a connected component of $\partial C_0(p_g, g)$. Then K has no periodic point. This is because K has empty interior and we know that for any periodic point q of g , $C_0(q, g)$ is open.

Lemma 7.8. *Let $g \in \mathcal{R}_{\mathcal{U}(f)}$ and let $h \in \mathcal{U}(f)$ such that $h = g$ in $\mathcal{F}^c(p_g, g)$. Then there is no periodic point of h in $\partial C_0(p_h, h)$.*

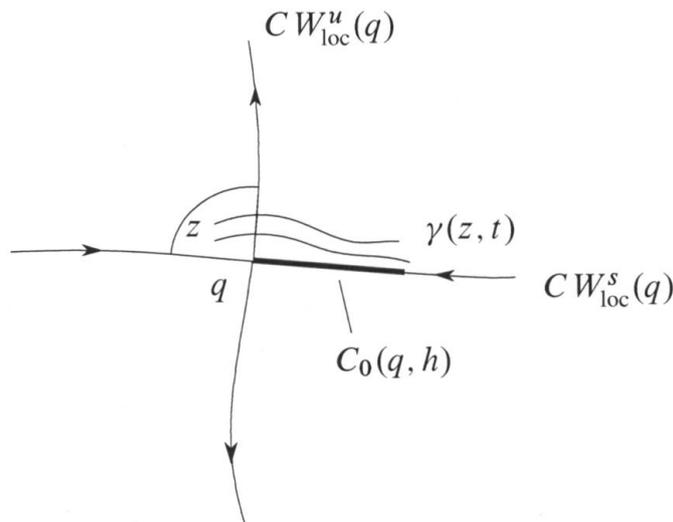


Figure 9.

Proof. Assume, by contradiction, that there exists a periodic point $q \in \partial C_0(p_h, h)$, $q \in K$ a connected component of $\partial C_0(p_h, h)$. Since $h = g$ on $\mathcal{F}^c(p_g, g) = \mathcal{F}^c(p_h, h)$ we have that q is a periodic point of g and hence q is either elliptic or

hyperbolic (for $g/\mathcal{F}^c(p_g, g)$ and thus for $h/\mathcal{F}^c(p_g, g)$). If q is elliptic then we know that $C_0(q, h)$ is open and we get a contradiction.

Assume that q is hyperbolic. If $C_0(q, h)$ is open we are done. If not, we know that $CW^s(q) \subset C_0(q, h)$ or $CW^u(q) \subset C_0(q, h)$. For instance, assume that $CW^s(q) \subset C_0(q, h)$. Since $h = g$ on $\mathcal{F}^c(p_g, g)$ then every periodic point of h in $\mathcal{F}^c(p_g, g)$ is elliptic nondegenerated or hyperbolic and there is no saddle connections (since g is \mathcal{KS}). A theorem of J. Mather in [23, Theorem 5.2] implies that $CW^u(q) \subset \overline{CW^s(q)} \subset \overline{C_0(q, h)} \subset K$. We know that there exists a continuous map $\gamma : B(q) \times [0, 1] \rightarrow \mathcal{F}^c(p_g, g)$ such that $\gamma(q, \cdot) \subset CW^s_{loc}(q, h)$, $\gamma(q, \cdot)$ is not constant and for every $z \in B(q)$, $\gamma(z, t) \in C_0(z, h)$. Therefore, for z belonging to an appropriate component of $B(q) \setminus (CW^s_{loc}(q) \cup CW^u_{loc}(q))$ we have that $\gamma(z, t_z) \in CW^u_{loc}(q)$ for some t_z and so $z \in K$.

This implies that K has nonempty interior, a contradiction. □

Proposition 7.9. *Let $h \in \mathcal{U}(f)$ and let K be a connected component of $\partial C_0(p_h, h)$. Then, the partition of K by connected component of accessibility classes form a C^1 lamination.*

Proof. It is a direct consequence of Corollary 2.21. □

We need a general result about C^1 lamination of subsets of the plane.

Proposition 7.10. *Let $K \subset \mathbb{R}^2$ be a compact and connected set with empty interior and supporting a C^1 lamination. Then*

- (1) $\mathbb{R}^2 \setminus K$ has at least two connected components, and
- (2) if $\mathbb{R}^2 \setminus K$ has exactly two connected component then $K \cong \mathbb{S}^1$.

Proof. If K contains a leaf which is diffeomorphic to a circle then clearly K separates \mathbb{R}^2 . On the other hand, if K does not contains such a leaf then by [15], $\mathbb{R}^2 \setminus K$ has at least four connected components, this proves item 1.

Now, if $\mathbb{R}^2 \setminus K$ has exactly two connected components, by the above it follows that K contains a leaf W_0 that is diffeomorphic to a circle, which is unique otherwise the complement has at least three connected components. Arguing by contradiction, assume that there are other leaves of the lamination than W_0 .

Let $W(x)$ be the leaf of lamination through $x \in K$. Orientate the leaf in an arbitrary way. It follows that the α and ω limit set of the leaf must contain W_0 . Otherwise, the result of [15] applies and the complement of K has at least four connected components.

Consider a transversal section J through W_0 . By the above, every point in $K \setminus J$ is in an arc of the lamination having both ends in J . The same arguments in the paper of [15] yields that the lamination could be extended to a foliation with singularities in the sphere having at most one singularity of index 1 and the others have index less than 1/2. It follows that the complement of K has at least 3 connected components, a contradiction. □

We now state a theorem that will be useful in our context.

Theorem 7.11 (Xia [38], Koropecski [21]). *Let S be a compact surface without boundary and let $f : S \rightarrow S$ be a homeomorphism such that $\Omega(f) = S$. Let K be a compact connected invariant set. Then, one of the following holds:*

- (1) K has a periodic point;
- (2) $K = S = \mathbb{T}^2$;
- (3) K is an annular domain, i.e. there exists an open neighborhood U of K homeomorphic to an annulus and $U \setminus K$ has exactly two components (each one homeomorphic to an annulus).

Proposition 7.12. *Let $g \in \mathcal{R}_{\mathcal{U}(f)}$ and let $h \in \mathcal{U}(f)$ such that $h = g$ on $\mathcal{F}^c(p_g, g)$. Then, any connected component K of $\partial C_0(p_h, h)$ is a simple closed C^1 curve invariant for some power of h (and g).*

Proof. Let K be a connected component of $\partial C_0(p_h, h)$. By Proposition 7.9 we know that K admits a C^1 lamination. We have three possibilities:

- (1) $K \subset U$ where U is homeomorphic to a disk and K does not separate U ;
- (2) $K \subset U$ where U is homeomorphic to a disk and K does separate U ;
- (3) none of the above, i.e. in any neighborhood U of K we have non null-homotopic closed curves (in $\mathcal{F}^c(p_h, h)$).

Proposition 7.10 implies that (1) cannot happen. Let's consider situation (2). We consider an open set $U_0 \subset U$, where U_0 is a connected component of the complement of $\overline{C_0(p_h, h)}$ and $\partial U_0 \subset K$. Since $h|_{\mathcal{F}^c(p_h, h)}$ preserves the form $\omega|_{\mathcal{F}^c(p_h, h)}$ we have for some integer m that $h^m(U_0) = U_0$. This implies that $h^m(K) \cap K \neq \emptyset$. From the fact that $C_0(p_h, h)$ is invariant we get $h^m(K) = K$. Since K has no periodic point (from Lemma 7.8) and is not the whole surface, we have from Theorem 7.11 that K is an annular domain and by Proposition 7.10, we have that K is a simple closed curve.

Finally, let's consider situation (3). Notice that there are finitely many components satisfying (3). On the other hand, h maps a connected component K satisfying (3), to another one also satisfying (3). Therefore, for some m we have that $h^m(K) = K$, for any K in (3). Applying Theorem 7.11 and Proposition 7.10, we get the result as before. \square

Lemma 7.13. *Let $g \in \mathcal{R}_{\mathcal{U}(f)}$. Then $\overline{C_0(p_g, g)} = \mathcal{F}^c(p_g, g)$.*

Proof. Assume that this is not the case, and so, there is a connected component \mathcal{C} of $\partial C_0(p_g, g)$ (which is a simple closed curve) and an open annulus U which is a neighborhood of \mathcal{C} , such that one component of $U \setminus \mathcal{C} \subset C_0(p_g, g)$ and the other one is contained in the complement of $\overline{C_0(p_g, g)}$. We know that $g^m(\mathcal{C}) = \mathcal{C}$ for some m .

Consider the family Γ of g^m invariant simple closed and essential C^1 curve in U . Notice that curves in this family are disjoint or coincide. This is because, since g is \mathcal{KS} , these curves cannot have rational rotation number. Now, if $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$, by invariance we have that they intersects along the nonwandering set $\Omega(g^m|\mathcal{C}_1) = \Omega(g^m|\mathcal{C}_2)$. But if one (and hence both) are Denjoy maps there must exists a wandering open set $U \subset \mathcal{F}^c(p_g, g)$, a contradiction since g preserves area on the compact leaf $\mathcal{F}^c(p_g, g)$.

Let $V \subset \bar{V} \subset U$ be an annulus neighborhood of \mathcal{C} as in Lemma 3.7. Since $g \in \mathcal{R}_{\mathcal{U}(f)}$ and in particular g is a continuity point of the maps Γ_2 and Γ_3 (see (7.2)) it is not difficult to see that there exists $\mathcal{V}(g)$ such that if $h \in \mathcal{V}(g)$ and $h = g$ on $\mathcal{F}^c(p_g, g)$ then $\partial C_0(p_h, h)$ must have a connected component in V which must be an h^m -invariant (and so g^m -invariant) essential simple closed C^1 curve. By Lemma 3.7 we get a contradiction. \square

Now we are ready to finish the proof of Theorem 7.5 and hence Theorems 4A and 4B.

End of proof of Theorem 7.5. Let $g \in \mathcal{R}_{\mathcal{U}(f)}$ and we already know that $\overline{C_0(p_g, g)} = \mathcal{F}^c(p_g, g)$. We want to prove that $C_0(p_g, g) = \mathcal{F}^c(p_g, g)$. We argue by contradiction and we assume that $C_0(p_g, g) \neq \mathcal{F}^c(p_g, g)$ and let $\mathcal{C}_i = \mathcal{C}_i(g)$, $i = 1, \dots, \ell$ be the connected components of $\partial C_0(p_g, g)$. We know that every \mathcal{C}_i is a simple closed C^1 curve non null-homotopic invariant for g^{m_i} , for some m_i and let U_i be annulus neighborhoods of \mathcal{C}_i .

Since g is a continuity point of Γ_2 and Γ_3 we get that there exists a neighborhood $\mathcal{V}(g)$ such that if $h \in \mathcal{V}(g)$ then

- $\overline{C_0(p_h, h)} = \mathcal{F}^c(p_h, h)$.
- $\mathcal{F}^c(p_h, h) \setminus C_0(p_h, h) \cap U_i \neq \emptyset, i = 1, \dots, \ell$.

Consider the family of essential simple closed C^1 curves g^m -invariant and contained in U_i and let V_i be as in Lemma 3.7. Since $g \in \mathcal{R}_{\mathcal{U}(f)}$ and for $\mathcal{V}(g)$ as above we have for any $h \in \mathcal{V}(g)$ and such that $h = g$ on $\mathcal{F}^c(p_g, g)$ that $\partial C_0(p_h, h)$ must have a connected components $\mathcal{C}_i(h)$ (which are simple closed curves) contained in every V_i . By Lemma 3.7 this curves cannot be essential in V_i . This implies that $\mathcal{C}_i(h)$ must be null-homotopic. And therefore $\overline{C_0(p_h, h)} \neq \mathcal{F}^c(p_h, h)$, a contradiction. \square

A. Proof of Proposition 3.6

A.1. Bounded variation. Recall that $f : [a, b] \rightarrow \mathbb{R}$ is of *bounded variation* if:

$$\sup \left\{ \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| : a = x_0 < x_1 < \dots < x_n = b \right\} < \infty,$$

and this supremum is denoted by $V(f; [a, b])$.

Lemma A.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ of bounded variation. We have the following:*

- (1) *if $[a_1, b_1] \subset [a, b]$ then $V(f; [a_1, b_1]) \leq V(f; [a, b])$,*
- (2) *if (a_1, b_1) and (a_2, b_2) are disjoint intervals contained in $[a, b]$ then we have that $V(f; [a_1, b_1]) + V(f; [a_2, b_2]) \leq V(f; [a, b])$. The same holds for any finite disjoint collection of intervals (a_i, b_i) 's,*
- (3) *if (a_1, b_1) and (a_2, b_2) are disjoint intervals in $[a, b]$ and $f([a_1, b_1]) \cup f([a_2, b_2]) \supset [c, d]$ then $V(f; [a_1, b_1]) + V(f; [a_2, b_2]) \geq d - c$. A similar statement holds for any finite disjoint collection of (a_i, b_i) 's,*
- (4) *if f is the difference of two non-decreasing maps then f is of bounded variation.*

Proof. We just prove item 3, the others follows immediately from the definition of bounded variation. Let's assume that $c \in f([a_1, b_1])$. If $d \in f([a_1, b_1])$ then we are done. So, assume that $d \notin f([a_1, b_1])$ and so $d \in f([a_2, b_2])$. Let $c^* = \sup(f([a_1, b_1]))$ and $d^* = \inf(f([a_1, b_1]))$ then $c^* \geq d^*$, $c^* \geq c$, and $d^* \leq d$. Then $V(f; [a_1, b_1]) \geq c^* - c$ and $V(f; [a_2, b_2]) \geq d - d^*$. Then,

$$V(f; [a_1, b_1]) + V(f; [a_2, b_2]) \geq (d - d^*) + (c^* - c) \geq d - c.$$

By induction, we prove the statement for finite collections of intervals. \square

A.2. Proof of Proposition 3.6. Let's state it again for simplicity:

Proposition 3.6. *Let $\ell_1 : [-a, a] \rightarrow \mathbb{R}$ and $\ell_2 : [-b, b] \rightarrow \mathbb{R}$ be two non-decreasing maps and let $\phi : [-b, b] \rightarrow [-a, a]$ be also a non-decreasing map. Then for any $\varepsilon > 0$ there exist s, t , $|s|, |t| \leq \varepsilon$, such that:*

$$\phi(\{x \in [-b, b] : \ell_2(x) + t = x\}) \cap \{x \in [-a, a] : \ell_1(x) + s = x\} = \emptyset.$$

For a non-decreasing map $f : [a, b] \rightarrow \mathbb{R}$ and $y \in [a, b]$ we denote by $f_-(y) = \lim_{x \nearrow y} f(x)$ and $f_+(y) = \lim_{x \searrow y} f(x)$. We say that $f : [a, b] \rightarrow \mathbb{R}$ has a *jump* in $z \in (a, b)$ if $f_-(z) \neq f_+(z)$ (i.e. if z is a discontinuity point of f). Moreover, we say that f has a *jump of size ε* if f has a jump in some z such that $|f_-(z) - f_+(z)| \geq \varepsilon$.

Lemma A.2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be non-decreasing. Then, for any $\varepsilon > 0$ there is $\delta > 0$ such that if $0 < y - x < \delta$ then either $f(y) - f(x) < \varepsilon$ or there exists a jump of size $\varepsilon/2$ between x and y .*

Proof. By contradiction, suppose that there is $\varepsilon_0 > 0$ such that for any $\delta > 0$, there exist x_δ, y_δ such that $0 < y_\delta - x_\delta < \delta$ and $f(y_\delta) - f(x_\delta) \geq \varepsilon_0$, and there is no jump of size $\varepsilon_0/2$.

Let (x_n) and (y_n) be two sequences such that $0 < y_n - x_n < 1/n$ and $f(y_n) - f(x_n) \geq \varepsilon_0$, and there is no jump of size $\varepsilon_0/2$ between x_n and y_n , for every n . Let x be an accumulation point of $\{x_n\}$. Since f is non-decreasing, we may assume that x_n approaches x from the left (otherwise $\lim f(y_n) - f(x_n) = f_+(x) - f_+(x) = 0$).

We may assume then that $x_n \nearrow x$. By the same argument, we have that $\{y_n\}$ has to approach x from the right, and we may assume that and so $y_n \searrow x$. Thus, $f_+(x) - f_-(x) \geq \varepsilon_0$, which is a contradiction, since $x_n \leq x \leq y_n$. \square

Let $g_1 : [-a, a] \rightarrow \mathbb{R}$ defined by $g_1 = \ell_1(x) - x$ and $g_2 : [-b, b] \rightarrow \mathbb{R}$ defined by $g_2 = \ell_2(x) - x$. Notice that, g_1, g_2 are of bounded variation and that

$$\{x : \ell_1(x) + s = x\} = g_1^{-1}(s) \quad \text{and} \quad \{x : \ell_2(x) + t = x\} = g_2^{-1}(s).$$

Lemma A.3. *For any $s_1 < s_2$ the following hold:*

- (1) $\overline{\phi^{-1}(g_1^{-1}(s_1))} \cap \overline{\phi^{-1}(g_1^{-1}(s_2))}$ contains at most finitely many points.
- (2) For any y in the above intersection there exists $\delta > 0$ such that either
 - (a) $(y - \delta, y) \cap \phi^{-1}(g_1^{-1}(s_1)) = \emptyset$ and $(y, y + \delta) \cap \phi^{-1}(g_1^{-1}(s_2)) = \emptyset$, or
 - (b) $(y - \delta, y) \cap \phi^{-1}(g_1^{-1}(s_2)) = \emptyset$ and $(y, y + \delta) \cap \phi^{-1}(g_1^{-1}(s_1)) = \emptyset$.

Proof. Let $y \in \overline{\phi^{-1}(g_1^{-1}(s_1))} \cap \overline{\phi^{-1}(g_1^{-1}(s_2))}$. Observe that $\phi^{-1}(g_1^{-1}(s_1)) \cap \phi^{-1}(g_1^{-1}(s_2)) = \emptyset$. We claim that y cannot be accumulated at one side (either right or left) by both sets $\phi^{-1}(g_1^{-1}(s_1))$ and $\phi^{-1}(g_1^{-1}(s_2))$. Otherwise, assume this for the left, let $x_n \nearrow y$, $x_n \in \phi^{-1}(g_1^{-1}(s_1))$ and $z_n \nearrow y$, $z_n \in \phi^{-1}(g_1^{-1}(s_2))$. Then, $\phi(x_n) \nearrow \phi_-(y)$ and $\phi(z_n) \nearrow \phi_-(y)$. Hence, $s_1 = g_1(\phi(x_n)) = \ell_1(\phi(x_n)) - \phi(x_n)$ and so $s_1 = (\ell_1)_-(\phi_-(y)) - \phi_-(y)$. Analogously, $s_2 = g_1(\phi(z_n)) = \ell_1(\phi(z_n)) - \phi(z_n)$ and so $s_2 = (\ell_1)_-(\phi_-(y)) - \phi_-(y)$, a contradiction since $s_1 \neq s_2$. This proves item 2.

To prove item 1, let's assume that for y in the intersection situation (a) holds. Then $s_2 = (\ell_1)_-(\phi_-(y)) - \phi_-(y)$ and $s_1 = (\ell_1)_+(\phi_+(y)) - \phi_+(y)$. So, $(\ell_1)_+(\phi_+(y)) = s_1 + \phi_+(y)$ and $(\ell_1)_+(\phi_-(y)) = s_2 + \phi_-(y)$. Since $\phi(y) \leq \phi_+(y)$ and ℓ_1 is non-decreasing then $(\ell_1)_+(\phi_-(y)) \leq (\ell_1)_+(\phi_+(y))$. Then, $s_1 + \phi_+(y) \geq s_2 + \phi_-(y)$. Therefore,

$$\phi_+(y) - \phi_-(y) \geq s_2 - s_1.$$

This means that the jump of ϕ at y is at least of size $s_2 - s_1$ and there are at most finitely many of them. The proof is complete in this case.

Now, assume that (b) holds and so $s_1 = (\ell_1)_-(\phi_-(y)) - \phi_-(y)$ and $s_2 = (\ell_1)_+(\phi_+(y)) - \phi_+(y)$. Let $\varepsilon = s_2 - s_1$ and let $\delta (< \varepsilon)$ from Lemma A.2 applied to ℓ_1 . Notices that

$$\begin{aligned} (\ell_1)_+(\phi_+(y)) - (\ell_1)_-(\phi_-(y)) &= s_2 + \phi_+(y) - s_1 - \phi_-(y) \\ &= (s_2 - s_1) + \phi_+(y) - \phi_-(y) \geq s_2 - s_1. \end{aligned}$$

If y is a continuity point of ϕ then we have a ℓ_1 -jump of size $s_2 - s_1$ at $\phi(y) = \phi_+(y) = \phi_-(y)$ and there can be just finitely many of them. On the other hand, there can be finitely many y 's such that the ϕ -jump at y is at least δ . So, we just consider

the set of y 's such that $\phi_+(y) - \phi_-(y) < \delta$. By Lemma A.2 there exists a ℓ_1 -jump in $[\phi_-(y), \phi_+(y)]$ of size at least $\varepsilon/2$. For different y 's the intervals $[\phi_-(y), \phi_+(y)]$ are disjoint. Since there are finitely many ℓ_1 -jumps of size at least $\varepsilon/2$, we conclude that there are finitely many y 's in

$$\overline{\phi^{-1}(g_1^{-1}(s_1))} \cap \overline{\phi^{-1}(g_1^{-1}(s_2))}$$

and the lemma is proved. □

Now we can prove Proposition 3.6.

Proof of Proposition 3.6. Recall that $g_1(x) = \ell_1(x) - x$ and $g_2(x) = \ell_2(x) - x$. Assume, by contradiction, that for some $\varepsilon > 0$ we have that for any $s, t, |s|, |t| \leq \varepsilon$ we have

$$\phi(\{x \in [-b, b] : \ell_2(x) + t = x\}) \cap \{x \in [-a, a] : \ell_1(x) + s = x\} \neq \emptyset.$$

We know that g_2 is of bounded variation, set $M = V(g_2; [-b, b])$ and let k be an integer, $k > M/(2\varepsilon)$. Consider a partition of $[-\varepsilon, \varepsilon] -\varepsilon \leq s_1 < s_2 < \dots < s_k \leq \varepsilon$, and let $S_i = \overline{\phi^{-1}(g_1^{-1}(s_i))}$. Notice that from our contradicting assumption that $g_2(S_i) \supset [-\varepsilon, \varepsilon]$. From Lemma A.3 we have that $S_i \cap S_j$ contains at most finitely many points for $i \neq j$. And if $i \neq j \neq l \neq i$ then $S_i \cap S_j \cap S_l = \emptyset$.

Let y_1, \dots, y_m be the set of points that belongs to more than one S_i . For each y_i let δ_i from Lemma A.3 such that $(y_i - \delta_i, y_i)$ intersects just one of the sets $S_j, j = 1, \dots, k$, and the same for $(y_i, y_i + \delta_i)$.

Let

$$\widehat{S}_j = S_j \cap \left[\bigcup_{i=1}^M (y_i - \delta_i, y_i + \delta_i) \right]^c.$$

The sets $\widehat{S}_j, j = 1, \dots, k$ are compact and disjoint. For each j , choose

$$\widehat{U}_j = \bigcup_{l=1}^{m_j} [a_l, b_l]$$

such that $\widehat{S}_j \subset \widehat{U}_j$ and $\widehat{U}_j \cap \widehat{U}_i = \emptyset$ if $j \neq i$.

Let

$$U_j = \widehat{U}_j \cup \left(\bigcup_{i: S_j \cap [y_i - \delta_i, y_i] \neq \emptyset} [y_i - \delta_i, y_i] \right) \cup \left(\bigcup_{i: S_j \cap [y_i, y_i + \delta_i] \neq \emptyset} [y_i, y_i + \delta_i] \right).$$

We can write U_j as a union of finitely many compact and disjoint intervals $I_j(1), \dots, I_j(m_j)$.

Now, we have:

- $g_2(U_j) \supset [-\varepsilon, \varepsilon]$ for any $j = 1, \dots, k$; and
- $\text{int}(U_j) \cap \text{int}(U_l) = \emptyset$ if $j \neq l$.

Therefore, we have

$$\sum_{i=1}^m V(g_2; I_j(i)) \geq 2\varepsilon,$$

and so, from Lemma A.1, we get

$$V(g_2; [-b, b]) \geq \sum_{j=1}^k \sum_{i=1}^m V(g_2; I_j(i)) \geq k2\varepsilon > M \geq V(g_2; [-b, b]),$$

a contradiction. This completes the proof. \square

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