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An explicit cycle map for the motivic cohomology of real varieties

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Abstract. We provide a direct construction of a cycle map in the level of representing complexes from the motivic cohomology of real (or complex) varieties to the appropriate ordinary cohomology theory. For complex varieties, this is simply integral Betti cohomology, whereas for real varieties the recipient theory is the bigraded $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant cohomology [19]. Using the finite analytic correspondences from [7] we provide a sheaf-theoretic approach to ordinary equivariant $RO(G)$-graded cohomology for any finite group G. In particular, this gives a complex of sheaves $\mathbb{Z}(p)_\omega$ on a suitable equivariant site of real analytic manifolds-with-corner whose construction closely parallels that of the Voevodsky's motivic complexes $\mathbb{Z}(p)_{\mathcal{M}}$. Our cycle map is induced by the change of sites functor that assigns to a real variety X its analytic space $X(\mathbb{C})$ together with the complex conjugation involution.

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1. Introduction

The motivic cohomology of a smooth algebraic variety X over a perfect field F, as defined by V. Voevodsky in [26], is the hypercohomology of certain *motivic complexes* $\mathbb{Z}(p)_{\mathcal{M}|X}$ of Zariski sheaves on X.

In order to construct the motivic complexes, Voevodsky introduces the *category of finite correspondences*, whose objects are the smooth schemes over F, and the morphisms between X and Y are certain algebraic cycles in $X \times Y$ whose components are finite over X. We give a succinct description of this category and the construction of the motivic complexes in Section 5. For a thorough account of this theory we refer the reader to [21].

The aim of this article is to provide a direct construction of a cycle map from the motivic cohomology of real (or complex) varieties to the appropriate

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ordinary cohomology theory. In the case of complex varieties, this is simply Betti (singular) cohomology with $\mathbb{Z}$-coefficients, whereas in the case of real varieties the natural counterpart is the bigraded $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant cohomology [8,9,19]. A conceptual explanation from the point of view of $\mathbb{A}^1$-homotopy theory for the naturality of this equivariant cohomology theory as the target of the cycle map is found in [6]. This article provides an alternative and direct explanation from a sheaf-theoretic point of view.

Following an approach parallel to Voevodsky's we introduced in [7] the category of *finite analytic correspondences*, whose objects are real analytic manifolds-with-corner and whose morphisms are described in terms of certain *subanalytic chains* in the product of two such manifolds. The benefit of using real analytic manifolds and subanalytic currents lies in the existence of a suitable intersection and slicing theory, developed in this context in [13] and [14]. These constructions are recalled in Section 2 below.

In Section 3.2 we recall the definition and basic properties of $RO(G)$-graded ordinary equivariant cohomology, introduced in [19]. This is a cohomology theory indexed by the ring of orthogonal representations of the group G, which plays a similar role in equivariant topology to the one played by singular cohomology in the non-equivariant context. In particular, these theories coincide when the group is trivial. Denoting by $\mathbf{1}$ the trivial irreducible representation, one gets a natural inclusion $\mathbb{Z} \equiv \mathbb{Z} \cdot \mathbf{1} \subset RO(G)$, and the groups that are indexed by trivial representations coincide with the classical Bredon cohomology, introduced in [3] and [4].

We are mainly interested in the case where $G = \mathfrak{S} := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Here one has $RO(\mathfrak{S}) = \mathbb{Z} \cdot \mathbf{1} \oplus \mathbb{Z} \cdot \xi$, where ξ is the sign representation of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. In this case we adopt the *motivic notation* $H^{n,p}_{\mathrm{Br}}(M; \underline{\mathbb{Z}}) := H_G^{(n-p)\mathbf{1}+p\xi}(M; \underline{\mathbb{Z}})$ and denote the complex $\mathbb{Z}(V)^{\omega}_{\mathrm{G}}$ associated to $V = p \cdot \xi$ simply by $\mathbb{Z}(p)_{\omega}$. This notation matches Voevodsky's notation for motivic cohomology and these equivariant cohomology groups give additional information about the 2-torsion component of the motivic cohomology of real varieties. The bigraded equivariant ring structure for smooth proper curves is computed in [9] and for smooth projective quadrics in [8]. The results are quite close to calculations of motivic cohomology in [22] and [27].

For a fixed finite group G our approach is the following. Given an orthogonal representation V of G, we use finite analytic correspondences to construct in Section 3.2 a complex of abelian sheaves $\mathbb{Z}(V)^{\omega}_{\mathrm{G}}$ on a suitable equivariant site of real analytic manifolds-with-corner.

Our first main result is the following.

Theorem 3.9. *Let X be an oriented real analytic G-manifold. Then, for any finite dimensional G-representation V, with $\dim V = v$, one has natural isomorphisms*

$$\mathbb{H}^n\big(X_{\mathrm{eq}},\ \mathbb{Z}(V)^{\omega}_{\mathrm{G}}|_X\big) \ \cong\ H_G^{V+n-v}(X; \underline{\mathbb{Z}}),$$

between the hypercohomology of X with coefficients in $\mathbb{Z}(V)^{\omega}_{\mathrm{G}}|_X$ and the G-equivariant cohomology of X with $\underline{\mathbb{Z}}$ coefficients, in the direction of V.

In Section 5, we take advantage of the fact that the construction of the complexes $\mathbb{Z}(V)^{\omega}_{\mathrm{G}}$ closely resembles Voevodsky's constructions to provide a very natural description of the cycle map at the level of representing complexes. In a nutshell, given a finite correspondence $\Gamma \in \mathsf{Cor}(X, Y)$, where X and Y are smooth real varieties, the corresponding complex algebraic cycle $\Gamma(\mathbb{C})$ — with the analytic topology — gives a real analytic current $\Gamma(\mathbb{C})$ which becomes a finite analytic correspondence in $\mathcal{I}^{\omega}_{\mathrm{fin}}(X(\mathbb{C}), Y(\mathbb{C}))^{\mathfrak{S}}$. This natural construction yields our next main result.

Theorem 5.7. *Given a smooth real variety X, one has a map of complexes of Zariski sheaves $\mathsf{c}_X : \mathbb{Z}(p)_{\mathcal{M}|X} \longrightarrow R\pi_*\mathbb{Z}(p)_{\mathrm{Br}|X(\mathbb{C})}$ induced by $\pi : X(\mathbb{C})_{\mathrm{eq}} \to X_{\mathrm{Zar}}$ (5.2) and natural in X. This map induces natural bigraded ring homomorphisms*

$$\mathsf{c}_X : H^*_{\mathcal{M}}(X, \mathbb{Z}(\bullet)) \longrightarrow H^{*,\bullet}_{\mathrm{Br}}(X; \underline{\mathbb{Z}}),$$

from motivic cohomology to ordinary $RO(\mathfrak{S})$-graded equivariant cohomology.

We must point out that all the constructions done here, in the case of the trivial group $G = \{e\}$ yield a complex of sheaves $\mathbb{Z}(p)_B$ calculating singular cohomology with coefficients in $\mathbb{Z}(p) = (2\pi\mathbf{i})^p\mathbb{Z} \subset \mathbb{C}$. The same construction provides a cycle map from the motivic cohomology to the singular cohomology of complex varieties, satisfying the properties described in the theorem above.

Using this approach to equivariant ordinary cohomology, we present two basic examples. In Section 3.4 we directly construct the well-known isomorphism

$$H^{p,p}_{\mathrm{Br}}(*; \underline{\mathbb{Z}}) \cong \underbrace{\mathbb{Z}^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}^\times}_{p\text{-times}} \cong \mathbb{Z}^\times,$$

for all $p \geq 1$. Later, in Section 5.3 we use this example to provide a natural ring homomorphism $\rho : K^M_*(F) :\longrightarrow \oplus_{p\geq 0} H^{p,p}_{\mathrm{Br}}(F; \underline{\mathbb{Z}}) := \oplus_{p\geq 0} H^{p,p}_{\mathrm{Br}}(X(\mathbb{C}); \underline{\mathbb{Z}})$, from the Milnor K-theory ring of a number field F to the "diagonal" part of ordinary equivariant cohomology of $X(\mathbb{C})$, where $X := \mathrm{Spec}\,(F \otimes_{\mathbb{Q}} \mathbb{R})$. This map descends to an isomorphism between $K^M_*(F)/2K^M_{\geq 1}(F)$ and $\oplus_{p\geq 0} H^{p,p}_{\mathrm{Br}}(F; \underline{\mathbb{Z}})$. It is worth mentioning that the Milnor K-theory ring of a field F is precisely the diagonal part of the motivic cohomology of $\mathrm{Spec}(F)$, that is, $K^M_*(F) \cong \oplus_{p\geq 0} H^p_{\mathcal{M}}(\mathrm{Spec}(F), \mathbb{Z}(p))$.

In order to prove the main results, particularly Theorem 3.9, we need to endeavor in sheaf-theoretic considerations of independent interest, similar to the discussions in [21] related to complexes of sheaves with *homotopy-invariant* cohomology presheaves. The main steps to prove Theorem 3.9 are summarized as follows.

(i) The sheaf $a\mathcal{P}$ associated to a homotopy-invariant presheaf $\mathcal{P}$ is also homotopy-invariant.

(ii) The previous step along with standard spectral sequences arguments imply that for all $r \in \mathbb{Z}$ the presheaf $X \mapsto \mathbb{H}^r(X_{\mathrm{eq}}; a\mathcal{P}^*_{|X})$ is homotopy-invariant, when $\mathcal{P}^*$ is a complex of presheaves with homotopy-invariant cohomology presheaves.

(iii) We show that Čech hypercohomology on G-Man_ω coincides with sheaf hypercohomology, and prove a Grothendieck-like theorem establishing an isomorphism between the Čech hypercohomology of an equivariant good cover and the usual hypercohomology groups.

In order to keep the exposition as self-contained as possible, we provide the proofs of these technical results in an appendix, where we briefly discuss G-Man_ω as a site with *enough points* and the resulting canonical Godement resolutions of abelian sheaves on G-Man_ω.

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2. Finite analytic correspondences

In this section we review the main properties of the category of *finite analytic correspondences* $\mathsf{Man}_\omega^{\mathsf{fin}}$ introduced in [7]. The constructions take place in the category Man_ω of oriented *real-analytic* manifolds and *real-analytic* maps.

2.1. The category of finite analytic correspondences. Let $\mathcal{I}_k^{\mathrm{loc}}(M)$ be the group of k-dimensional locally integral currents on an m-dimensional oriented smooth manifold M [11, 4.1.24] and, for $k \geq 1$, let $\partial : \mathcal{I}_k^{\mathrm{loc}}(M) \to \mathcal{I}_{k-1}^{\mathrm{loc}}(M)$ denote the boundary map, adjoint to the exterior derivative of differential forms. For $k = 0$, a (locally) integral 0-current is simply a (locally) finite sum of point masses.

If X is an oriented *real analytic* manifold, a k-dimensional locally integral current T in X is called a *k-dimensional subanalytic chain* if $\mathrm{spt}(T)$ is contained in some k-dimensional subanalytic set and $\mathrm{spt}(\partial T)$ is contained in some $(k-1)$-dimensional subanalytic set in case $k \geq 1$.

It follows that T is a locally finite sum of chains corresponding to integration over certain k-dimensional oriented subanalytic strata of some subanalytic stratification of X and, for $k \geq 1$, ∂T similarly comes from $(k-1)$-dimensional strata of this stratification; see [14, p. 64].

Notation 2.1. If X is a real analytic manifold, we denote by $\mathcal{I}_k^{\omega}(X) \subset \mathcal{I}_k^{\mathrm{loc}}(X)$ the group of k-dimensional subanalytic chains on X.

Definition 2.2. A *finite analytic correspondence* T between oriented real analytic manifolds X and Y, of dimensions m and n, respectively, is a current $T \in \mathcal{I}_m^{\omega}(X \times Y)$ satisfying the following conditions:

(FAC.1) T is a closed current (i.e. $\partial T = 0$).

(FAC.2) If $\pi_X : X \times Y \to X$ denotes the natural projection, its restriction $\pi_{X\,|\,\mathrm{spt}(T)}$ to the support of T is a proper map.

(FAC.3) There is $d > 0$ so that for each $x \in X$, one has $\#\{\pi_X^{-1}(x) \cap \operatorname{spt}(T)\} \leq d$. Denote the abelian group of such correspondences between X and Y by $\mathcal{I}^{\omega}_{\mathsf{fin}}(X, Y)$.

Example 2.3. (i) Let $f : X \to Y$ be a real analytic map and let $\Gamma_f \subset X \times Y$ denote its graph, oriented so that the projection on X is an orientation-preserving diffeomorphism. The current $[\![\Gamma_f]\!] \in \mathcal{I}^{\mathrm{loc}}_m(X \times Y)$ represented by integration over Γ_f is a finite analytic correspondence.

(ii) Let $X(\mathbb{C})$ and $Y(\mathbb{C})$ be the complex analytic spaces associated to smooth complex algebraic varieties X and Y, oriented by a choice of $\sqrt{-1}$, and let $\Gamma \subset X \times Y$ be a finite correspondence from X to Y in the sense of [21], and assume that X is irreducible. It follows from [2, Exposé XII, Props. 2.4 and 3.2] that $\Gamma(\mathbb{C}) \subset X(\mathbb{C}) \times Y(\mathbb{C})$ is a closed irreducible analytic subvariety which is finite and surjective over $X(\mathbb{C})$. In particular, $\Gamma(\mathbb{C})$ represents an element $[\![\Gamma(\mathbb{C})]\!]$ in $\mathcal{I}^{\omega}_{\mathsf{fin}}(X(\mathbb{C}), Y(\mathbb{C}))$. See [16, Thm. 3.1.1].

We need a slightly extended version of the notions introduced above to include *oriented real analytic manifolds-with-corner* [15]. Given two such manifolds X and Y, we let $\mathcal{I}^{\omega}_{\mathsf{fin}}(X, Y)$ be the group consisting of those $T \in \mathcal{I}^{\mathrm{loc}}_v(X \times Y)$ for which one can find embeddings $X \subset \widehat{X}$ and $Y \subset \widehat{Y}$ as closed submanifolds-with-corner of oriented real-analytic manifolds $\widehat{X}$ and $\widehat{Y}$ satisfying $\dim X = \dim \widehat{X}$ and $\dim Y = \dim \widehat{Y}$, together with $\widehat{T} \in \mathcal{I}^{\omega}_{\mathsf{fin}}(\widehat{X}, \widehat{Y})$ whose restriction $\widehat{T} \llcorner (X \times Y)$ to $X \times Y$ (see [11, 4.1.7]) is equal to T. Note that if $T \in \mathcal{I}^{\omega}_{\mathsf{fin}}(X, Y)$ with X and Y manifolds-with-corner, then T is not necessarily closed. From now on, the objects of Man_ω will include all oriented real analytic manifolds-with-corner, and the morphisms are analytic maps.

It is useful to think of the elements in $\mathcal{I}^{\omega}_{\mathsf{fin}}(X, Y)$ as multivalued maps from X to Y. Actually, one can associate to a finite analytic current $T \in \mathcal{I}^{\omega}_{\mathsf{fin}}(X, Y)$ a continuous map from X into the group of integral 0-currents in Y, using the slicing techniques introduced in [10]. In general terms, given a smooth map $f : M \to N$ between smooth manifolds and a current T of dimension k on M, the slicing of currents — under appropriate conditions — produces for almost all $y \in N$ a current $\langle T, f, y\rangle$ of dimension $k - \dim(N)$ on M called the *slice* of T over y.

Proposition 2.4 ([7, Prop. 2.5]). *Let $X, Y \in \mathsf{Man}_\omega$ have dimensions m and n, respectively. Denote by $\mathcal{I}_0(Y)$ the group of integral 0-currents in Y with the flat norm topology. Given $T \in \mathcal{I}^{\omega}_{\mathsf{fin}}(X, Y)$ the following holds:*

(i) *The slice $\langle T, \pi_X, x\rangle$ exists for all $x \in X$ and is a 0-dimensional integral current in $X \times Y$.*

(ii) *The function $f_T : X \to \mathcal{I}_0(Y)$ sending $x \in X$ to $\pi_{Y\#}(\langle T, \pi_X, x\rangle)$ is continuous (where $\pi_{Y\#}$ denotes the* push-forward *of currents, i.e. adjoint to the pull-back of forms).*

Let X, Y, and Z be oriented real analytic manifolds-with-corner of dimensions m, n, and k, and use $[\![X]\!]$ to denote the current defined by integration on X

(with the given orientation). Here, for sets or chains in the product space $X \times Y \times Z$, it will be convenient to abuse notation by identifying a corresponding chain or set under the standard identifications of $X \times Y \times Z$ with the $(X \times Y) \times Z$ or with $X \times (Y \times Z)$. For example, if T is a correspondence from Y to Z, we will use, in $X \times Y \times Z$, the notation $[\![X]\!] \times T$ as an abbreviation for the chain $\iota_\#([\![X]\!] \times T)$, where $\iota(x,(y,z)) = (x,y,z)$ for $(x,(y,z)) \in X \times (Y \times Z)$.

Proposition 2.5. *For any analytic correspondences $S \in \mathcal{J}^{\omega}_{\mathrm{fin}}(X,Y)$ and $T \in \mathcal{J}^{\omega}_{\mathrm{fin}}(Y,Z)$, the following statements hold:*

(1) *The intersection current $(S \times [\![Z]\!]) \cap ([\![X]\!] \times T)$ exists in $X \times Y \times X$.*

(2) *Let $p = p_{XZ} : X \times Y \times Z \to X \times Z$ be the projection . Then the restriction of p to the support of $(S \times [\![Z]\!]) \cap ([\![X]\!] \times T)$ is proper.*

(3) *The current $T \circ S := p_\#[(S \times [\![Z]\!]) \cap ([\![X]\!] \times T)]$ lies in $\mathcal{J}^{\omega}_{\mathrm{fin}}(X,Z)$, and is called the composition of T and S.*

Proof. First we need an easy remark about dimensions of subanalytic sets.

If $f : M \to N$ is an analytic map and A is a nonempty subanalytic subset of M such that $A \cap f^{-1}\{y\}$ is finite for all $y \in f(A)$, then $\dim A = \dim f(A)$.

In fact, A admits a locally finite partition $\mathcal{M}$ into analytic strata S so that $f_{|S}$ is an immersion and $\dim S = \dim f(S)$. Hence, $\dim A = \max_{S \in \mathcal{M}} \dim S = \max_{S \in \mathcal{M}} \dim f(S) = \dim f(A)$.

For the finite correspondences S from X to Y and T from Y to Z, we see that, for each $x \in X$, there are only finitely many $y \in Y$ with $(x,y) \in \operatorname{spt}(T)$ and, for each $y \in Y$, only finitely many $z \in Z$ with $(y,z) \in \operatorname{spt}(S)$. Thus, for each $x \in X$, there are only finitely many points $(x,y,z) \in (\operatorname{spt}(X) \times Z) \cap (X \times \operatorname{spt}(T))$. We may now apply the above remark to the projection onto X to see that

$$\dim\big[\big(\operatorname{spt}(X) \times Z\big) \cap \big(X \times \operatorname{spt}(T)\big)\big] = \dim X = m.$$

Note that $S \times [\![Z]\!]$ is an $(m+k)$-dimensional subanalytic chain with $\operatorname{spt}(S \times [\![Z]\!]) = \operatorname{spt}(S) \times Z$ and $\partial(S \times [\![Z]\!]) = (\partial S) \times [\![Z]\!] + (-1)^m S \times \partial [\![Z]\!] = 0 + 0$ and that $[\![X]\!] \times T$ is an $(m+n)$-dimensional subanalytic chain with $\operatorname{spt}([\![X]\!] \times T) = X \times \operatorname{spt}(T)$ and $\partial([\![X]\!] \times T) = 0$.

Thus the two chains $S \times [\![Z]\!]$ and $[\![X]\!] \times T$ have supports intersecting in the correct dimension $(m+k) + (m+n) - (m+n+k) = m$, which establishes the existence of the intersection current in conclusion (1); see [14, §4.5].

For the properness in conclusion (2), assume that K is a compact subset of $X \times Z$ and denote by $q_X : X \times Z \to X$, $q_Y : Y \times Z \to Y$, $\rho_X : X \times Y \to X$, and $\rho_Y : X \times Y \to Y$ the evident projections. It follows that $q_X(K)$ is a compact subset of X and, since $S \in \mathcal{I}^{\omega}_{\mathrm{fin}}(X, Y)$ is a finite correspondence the intersection $A := \rho_X^{-1}(q_X(K)) \cap \mathrm{spt}(S)$ is a compact subset of $X \times Y$. Therefore, $\rho_Y(A)$ is a compact subset of Y and the previous argument shows that $B := q_Y^{-1}(\rho_Y(A)) \cap \mathrm{spt}T$ is a compact subset of $Y \times Z$. It is clear that $\mathrm{spt}(S \times [\![Z]\!]) \cap \mathrm{spt}([\![X]\!] \times T) \cap p^{-1}(K)$ is a closed subset of $(A \times Z) \cap (q_X(K) \times B) \subset q_X(K) \times B$ and the latter is compact.

For conclusion (3) we now readily see that the push-forward current $p_{\#}[(S \times [\![Z]\!]) \cap([\![X]\!] \times T)]$ is a subanalytic chain with zero boundary because $\partial p_{\#} = p_{\#}\partial$. It is a finite analytic correspondence from X to Z because, as we saw above, the projection of its support onto X is a finite to one map. □

Proposition 2.6. *With manifolds X, Y, Z and currents $S \in \mathcal{I}^{\omega}_{\mathrm{fin}}(X, Y)$ and $T \in \mathcal{I}^{\omega}_{\mathrm{fin}}(Y, Z)$ as in Proposition 2.5, suppose that W is an ℓ-dimensional oriented real analytic manifold-with-corner, and $R \in \mathcal{I}^{\omega}_{\mathrm{fin}}(W, X)$. Then*

$$T \circ (S \circ R) = (T \circ S) \circ R.$$

Proof. We use the formula [13, Th. 5.8(11)] to "lift" and apply the associativity of the intersection product of chains that intersect suitably, as in [13, Th. 5.8(7)].

Specifically for the projection p_{WXZ} of $W \times X \times Y \times Z$ onto $W \times X \times Z$ and any subanalytic cycle Q in $W \times X \times Y \times Z$ that intersects $R \times [\![Y]\!] \times [\![Z]\!]$ suitably, $p_{WXZ_{\#}}Q$ intersects $R \times [\![Z]\!]$ suitably, and [13, Th. 5.8(11)] gives the formula

$$(R \times [\![Z]\!]) \cap p_{WXZ_{\#}}Q = p_{WXZ_{\#}}\big[(R \times [\![Y]\!] \times [\![Z]\!]) \cap Q\big].$$

Using the projections $p_{WX} : W \times X \times Z \to W \times X$ and $p_{XZ} : X \times Y \times Z \to X \times Z$ and this formula, we now derive that

$$\begin{aligned}
&(T \circ S) \circ R \\
&\quad = p_{WX_{\#}}\big[(R \times [\![Z]\!]) \cap ([\![W]\!] \times (T \circ S))\big] \\
&\quad = p_{WX_{\#}}\big[(R \times [\![Z]\!]) \cap ([\![W]\!] \times p_{XZ_{\#}}\big[(S \times [\![Z]\!]) \cap ([\![X]\!] \times T)\big])\big] \\
&\quad = p_{WX_{\#}}\big[(R \times [\![Z]\!]) \cap p_{WXZ_{\#}}([\![W]\!] \times \big[(S \times [\![Z]\!]) \cap ([\![X]\!] \times T)\big])\big] \\
&\quad = p_{WX_{\#}}\big[(R \times [\![Z]\!]) \cap p_{WXZ_{\#}}(([\![W]\!] \times S \times [\![Z]\!]) \cap ([\![W]\!] \times [\![X]\!] \times T))\big] \\
&\quad = p_{WX_{\#}}p_{WXZ_{\#}}\big[(R \times [\![Y]\!] \times [\![Z]\!]) \cap (([\![W]\!] \times S \times [\![Z]\!]) \cap ([\![W]\!] \times [\![X]\!] \times T))\big] \\
&\quad = \tilde{p}_{WX_{\#}}\big[(R \times [\![Y]\!] \times [\![Z]\!]) \cap (([\![W]\!] \times S \times [\![Z]\!]) \cap ([\![W]\!] \times [\![X]\!] \times T))\big], \qquad (*)
\end{aligned}$$

where $\tilde{p}_{WX_{\#}}$ is the projection $W \times X \times Y \times Z$ to $W \times X$.

By a similar argument we derive the formula

$$\begin{aligned}
&T \circ (S \circ R) \\
&\quad = \tilde{p}_{WZ_{\#}}\big[((R \times [\![Y]\!] \times [\![Z]\!]) \cap ([\![W]\!] \times S \times [\![Z]\!])) \cap ([\![W]\!] \times [\![X]\!] \times T)\big]. \qquad (**)
\end{aligned}$$

Arguing as in the proof of Proposition 2.5, we see that for each $w \in W$, there are only finitely many points $(w, x, y, z) \in W \times X \times Y \times Z$ so that $(w, x) \in \operatorname{spt}(R)$, $(x, y) \in \operatorname{spt}(S)$, and $(y, z) \in \operatorname{spt}(T)$. Thus,

$$\dim\big[\big(\operatorname{spt}(R) \times Y \times Z\big) \cap \big(W \times \operatorname{spt}(S) \times Z\big) \cap \big(W \times X \times \operatorname{spt}(T)\big)\big] = \ell,$$

which is the correct intersection dimension

$$(\ell + n + k) + (\ell + m + k) + (\ell + m + n) - 2(\ell + m + n + k).$$

Thus the *three* chains $R \times [\![Y]\!] \times [\![Z]\!]$, $[\![W]\!] \times S \times [\![Z]\!]$, and $[\![W]\!] \times [\![X]\!] \times T$ intersect suitably, and the associative law [13, Th. 5.8(7)] implies that $(*) = (**)$, and the proposition follows. $\square$

Definition 2.7. Let $\mathsf{Man}_\omega^{\mathsf{fin}}$ be the category with oriented real analytic manifolds-with-corner as objects and $\mathcal{I}^{\omega}_{\mathsf{fin}}(X, Y)$ as the morphisms between X and Y.

It follows from Propositions 2.5 and 2.6 that one has a faithful embedding

$$\mathsf{j}_{\mathsf{fin}} : \mathsf{Man}_\omega \hookrightarrow \mathsf{Man}_\omega^{\mathsf{fin}}, \tag{2.1}$$

which is the identity on objects and sends an analytic map $f : X \to Y$ to the current defined by its graph $[\![\Gamma_f]\!] \in \mathcal{I}^{\omega}_{\mathsf{fin}}(X, Y)$.

2.2. The category of *equivariant analytic correspondences*.

We now work in the equivariant category G-Man_ω whose objects are oriented real analytic manifolds-with-corner with a finite group G acting by analytic automorphisms (not necessarily orientation-preserving), and whose morphisms are the equivariant analytic maps. There is an induced action of G on $\mathcal{I}^{\omega}_{\mathsf{fin}}(X, Y)$ which, as in the non-equivariant case, leads to the definition of the category of *equivariant finite analytic correspondences* G-$\mathsf{Man}_\omega^{\mathsf{fin}}$ having analytic G-manifolds-with-corner as objects and $\mathcal{I}^{\omega}_{\mathsf{fin}}(X, Y)^G$ as morphisms between X and Y [7, Definition 4.2].

Remark 2.8. It is easy to check that the assignment $\mathcal{I}^{\omega}_{\mathsf{fin}}(X, Y) \to \operatorname{Map}(X, \mathcal{I}_0(Y))$ described in Proposition 2.4(ii) is equivariant. In particular, if $T \in \mathcal{I}^{\omega}_{\mathsf{fin}}(X, Y)^G$ then f_T is an equivariant map, that is $f_T \in \operatorname{Map}(X, \mathcal{I}_0(Y))^G$.

2.3. Homology of finite analytic chains.

The *topological simplex*

$$\blacktriangle^n = \Big\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \,\Big|\, \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0, i = 0, \dots, n\Big\} \tag{2.2}$$

is an oriented analytic manifold-with-corner that we endow with the *trivial* G-action, for any finite group G. This gives a canonical cosimplicial object $\blacktriangle^\bullet = \{\blacktriangle^n \mid n \geq 0\}$ in the categories Man_ω, $\mathsf{Man}_\omega^{\mathsf{fin}}$, G-Man_ω and G-$\mathsf{Man}_\omega^{\mathsf{fin}}$.

Any $X \in \mathsf{Man}_{\omega}^{\mathsf{fin}}$ yields a simplicial object $\mathcal{I}_{\mathsf{fin}}^{\omega}(\blacktriangle^{\bullet}, X)$ whose resulting chain complex (with differentials given by the alternating sum of the simplicial face maps) is denoted $\mathcal{I}_{\mathsf{fin}}^{\omega}(\blacktriangle^{*}, X)$. It follows from Proposition 2.4 that one obtains a map of complexes $s : \mathcal{I}_{\mathsf{fin}}^{\omega}(\blacktriangle^{*}, X) \to \mathsf{Sing}_{*}(\mathcal{I}_0(X))$, where $\mathsf{Sing}_{*}(\mathcal{I}_0(X)) = \mathrm{Hom}_{\mathrm{Top}}(\blacktriangle^{*}, \mathcal{I}_0(X))$. This map will be used to compare the resulting homology theories. Furthermore, the equivariance of the slicing map implies that in the equivariant context s is a map of complexes of G-modules. In particular, for each subgroup $H \subset G$, it maps $\mathcal{I}_{\mathsf{fin}}^{\omega}(\blacktriangle^{*}, X)^H$ into $\mathsf{Sing}_{*}(\mathcal{I}_0(X)^H)$.

Theorem 2.9. *Let X be a compact oriented analytic manifold. The map*

$$s : \mathcal{I}_{\mathsf{fin}}^{\omega}(\blacktriangle^{*}, X) \to \mathsf{Sing}_{*}(\mathcal{I}_0(X))$$

is a quasi-isomorphism, i.e. it gives an isomorphism in homology. More generally, if G is a finite group acting on X by analytic automorphisms and acting trivially on $\blacktriangle^{*}$, then for each subgroup $K \subset G$ the map $s : \mathcal{I}_{\mathsf{fin}}^{\omega}(\blacktriangle^{*}, X)^K \to \mathsf{Sing}_{*}(\mathcal{I}_0(X)^K)$ is a quasi-isomorphism.

Proof. The proof follows from the same arguments found in [7, Th. 3.1]. □

Remark 2.10. For each subgroup $K \subset G$, standard arguments yield an isomorphism between the homology groups $H_{\bullet}(\mathsf{Sing}_{*}(\mathcal{I}_0(X)^K))$ and the homotopy groups $\pi_{\bullet}(\mathcal{I}_0(X)^K)$. The equivariant version of the Dold–Thom theorem in [18] and Theorem 2.9 yield a natural isomorphism between $H_{\bullet}(\mathcal{I}_{\mathsf{fin}}^{\omega}(\blacktriangle^{*}, X)^G)$ and the G-equivariant Bredon homology $H_{\bullet}^{G}(X; \underline{\mathbb{Z}})$ with coefficients in the Mackey functor $\underline{\mathbb{Z}}$.

3. Equivariant analytic presheaves with transfer and ordinary cohomology

For a fixed finite group G we introduce the notion of G-*analytic presheaves with transfers* on G-Man_{ω}. When G is the trivial group this specializes to the *analytic presheaves with transfers*. Our constructions run in parallel with the development of motivic cohomology in [21]. In particular, we define complexes of sheaves on G-Man_{ω} that are the topological counterpart to Voevodsky's motivic complexes.

3.1. Equivariant analytic presheaves with transfers.

Definition 3.1 ([7]). An *equivariant analytic presheaf with transfers* is a contravariant functor $\mathcal{F} : (G\text{-}\mathsf{Man}_{\omega}^{\mathsf{fin}})^{\mathrm{op}} \longrightarrow \mathsf{Ab}$. We denote by G-PST^{ω} the category of equivariant analytic presheaves with transfers and natural transformations.

Example 3.2. Any object $X \in G\text{-}\mathsf{Man}_\omega$, represents an equivariant analytic presheaf with transfers $\mathbb{Z}_{\mathsf{tr}}^{\omega}X \in G\text{-}\mathsf{PST}^{\omega}$ given by

$$\begin{aligned} \mathbb{Z}_{\mathsf{tr}}^{\omega}X : (G\text{-}\mathsf{Man}_\omega^{\mathsf{fin}})^{\mathrm{op}} &\longrightarrow \mathsf{Ab} \\ U &\longmapsto \mathcal{I}_{\mathsf{fin}}^{\omega}(U,X)^{\mathfrak{S}}. \end{aligned}$$

Given a *pointed* object (X,x) in $G\text{-}\mathsf{Man}_\omega$, consising of $X \in G\text{-}\mathsf{Man}_\omega$ and $x \in X^G$, define $\mathbb{Z}_{\mathsf{tr}}^{\omega}(X,x) \in G\text{-}\mathsf{PST}^{\omega}$ as the cokernel of the map $\mathbb{Z} = \mathbb{Z}_{\mathsf{tr}}^{\omega}x \to \mathbb{Z}_{\mathsf{tr}}^{\omega}X$. Since the map $x \to X$ splits, there is a natural splitting $\mathbb{Z}_{\mathsf{tr}}^{\omega}X \cong \mathbb{Z} \oplus \mathbb{Z}_{\mathsf{tr}}^{\omega}(X,x)$.

Example 3.3. Consider $X \in G\text{-}\mathsf{Man}_\omega$. The G-topological group $\mathcal{I}_0(X)$ described in Proposition 2.4 naturally represents an abelian presheaf $\mathbb{Z}_{\mathsf{top}}^{\mathsf{G}}X$ on $G\text{-}\mathsf{Top}$ defined by $U \mapsto G\text{-}\mathsf{Top}(U,\mathcal{I}_0(X))$. If $x \in X^G$ we set $\mathcal{I}_0(X,x) = \mathcal{I}_0X/\mathcal{I}_0(x)$ and $\mathbb{Z}_{\mathsf{top}}^{\mathsf{G}}(X,x)$: $U \mapsto G\text{-}\mathsf{Top}(U,\mathcal{I}_0(X,x))$. In [7, Lem. 4.6] it is shown that these presheaves on $G\text{-}\mathsf{Top}$ extend to presheaves with transfers $\mathbb{Z}_{\mathsf{tr}}^{\mathsf{G_{top}}}X$ and $\mathbb{Z}_{\mathsf{tr}}^{\mathsf{G_{top}}}(X,x)$ on $G\text{-}\mathsf{PST}^{\omega}$.

Given an arbitrary equivariant presheaf with transfers $\mathcal{F} \in G\text{-}\mathsf{PST}^{\omega}$, let

$$\blacktriangle_n\mathcal{F} \in G\text{-}\mathsf{PST}^{\omega} \tag{3.1}$$

denote the functor that sends $U \in G\text{-}\mathsf{Man}_\omega^{\mathsf{fin}}$ to $\blacktriangle_n\mathcal{F}(U) := \mathcal{F}(U \times \blacktriangle^n)$. Using the functoriality of $\mathcal{F}$ one can easily verify that the collection $\blacktriangle_\bullet\mathcal{F} := \{\blacktriangle_n\mathcal{F} \mid n \geq 0\}$ becomes a *simplicial* equivariant analytic presheaf with transfers, and we denote by $\blacktriangle_*\mathcal{F}$ the associated chain complex whose differentials d_n are the alternating sum of the face maps. Denote by $\blacktriangle^*\mathcal{F}$ the complex of $G\text{-}\mathsf{PST}^{\omega}$'s (negatively graded) associated to $\blacktriangle_\bullet\mathcal{F}$. In other words, $\blacktriangle^n\mathcal{F} = \blacktriangle_{-n}\mathcal{F}$ with the differential

$$d^n : \blacktriangle^n\mathcal{F} \to \blacktriangle^{n+1}\mathcal{F}, \text{ defined by } d^n(\alpha) = (-1)^n d_{-n}. \tag{3.2}$$

Definition 3.4. Endow the category $G\text{-}\mathsf{Man}_\omega$ with the Grothendieck topology generated by the pre-topology where an *equivariant covering family* of $U \in G\text{-}\mathsf{Man}_\omega$ is a collection $\{f_i : U_i \to U\}_{i\in I}$ of open embeddings in $G\text{-}\mathsf{Man}_\omega$ satisfying $U = \bigcup_{i\in I} f_i(U_i)$. Denote by $\mathsf{Cov}(U)$ the set of coverings of U and recall that the following holds:

(T1) For $\{U_i \to U\}$ in $\mathsf{Cov}(U)$, and a morphism $V \to U$ in $G\text{-}\mathsf{Man}_\omega$, all fiber products $U_i \times_U V$ exist and $\{U_i \times_U V \to V\}$ is in $\mathsf{Cov}(V)$.

(T2) Given $\{U_i \to U\}$ in $\mathsf{Cov}(U)$, and a family $\{V_{ij} \to U_i\} \in \mathsf{Cov}(U_i)$, for all $i \in I$, the family $\{V_{ij} \to U\}$ obtained by composition also belongs to $\mathsf{Cov}(U)$.

(T3) If $\varphi : U' \to U$ is an isomorphism in $G\text{-}\mathsf{Man}_\omega$, then $\{\varphi : U' \to U\}$ is in $\mathsf{Cov}(U)$.

Let $(G\text{-}\mathsf{Man}_\omega)_{\mathrm{eq}}$ denote the *equivariant analytic site*, consisting of the category $G\text{-}\mathsf{Man}_\omega$ endowed with this topology.

(a) Given $X \in G\text{-}\mathsf{Man}_\omega$, denote by X_{eq} the small equivariant site of X where the objects are the G-invariant open subsets of X and the coverings are as above.

(b) Given a real linear representation V of G of dimension v, let its *representation sphere* S^V be the one-point compactification $V \cup \{\infty\}$. Using Examples 3.2 and 3.3, define two complexes in $G\text{-}\mathsf{PST}^\omega$:

$$\mathbb{Z}(V)_{\mathsf{G}}^{\mathsf{top}} := \left(\blacktriangle^* \mathbb{Z}_{\mathsf{tr}}^{\mathsf{G_{top}}}(S^V, \infty)\right)[-v] \quad \text{and} \quad \mathbb{Z}(V)_{\mathsf{G}}^{\omega} := \left(\blacktriangle^* \mathbb{Z}_{\mathsf{tr}}^{\omega}(S^V, \infty)\right)[-v].$$

By definition,

$$\mathbb{Z}(V)_{\mathsf{G}}^{\omega, j}(U) := \left\{\blacktriangle^* \mathbb{Z}_{\mathsf{tr}}^{\omega}(S^V, \infty)\right\}^{j-v}(U) = \mathbb{Z}_{\mathsf{tr}}^{\omega}(S^V, \infty)(U \times \blacktriangle^{v-j})$$

and the differential

$$d_\omega^j : \mathbb{Z}(V)_{\mathsf{G}}^{\omega, j}(U) \to \mathbb{Z}(V)_{\mathsf{G}}^{\omega, j+1}(U)$$

is $d_\omega^j(\alpha) = (-1)^{-v}(-1)^{v-j} d_{v-j} = (-1)^j d_{v-j}$ by definition of shifted complexes and (3.2), where

$$d_{v-j} : \mathbb{Z}_{\mathsf{tr}}^{\omega}(S^V, \infty)(U \times \blacktriangle^{v-j}) \to \mathbb{Z}_{\mathsf{tr}}^{\omega}(S^V, \infty)(U \times \blacktriangle^{v-j-1})$$

is the simplicial differential. These complexes should respectively be seen as *topological* and *differential-geometric* G-equivariant analogues of Voevodsky's vth motivic complex in the category of smooth schemes over $\mathbb{C}$.

(c) For X in $G\text{-}\mathsf{Man}_\omega$, denote by $\mathbb{Z}(V)_{\mathsf{G}}^{\mathsf{top}}|_X$ and $\mathbb{Z}(V)_{\mathsf{G}}^{\omega}|_X$ the complexes of abelian sheaves on X_{eq} obtained as the sheafification of $\mathbb{Z}(V)_{\mathsf{G}}^{\mathsf{top}}$ and $\mathbb{Z}(V)_{\mathsf{G}}^{\omega}$, respectively, restricted to the small equivariant site of X.

Notation 3.5. Given a site $\mathcal{C}$, denote by $\widehat{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ the categories of presheaves and sheaves of *Sets* on $\mathcal{C}$, respectively. Similarly, denote by $\widehat{\mathcal{C}}_R$ and $\widetilde{\mathcal{C}}_R$ the categories of presheaves and sheaves of *R-modules* on $\mathcal{C}$, for a given ring R.

Definition 3.6. An equivariant analytic presheaf with transfer $\mathcal{F}$ is *homotopy invariant* if for each $U \in G\text{-}\mathsf{Man}_\omega$ the projection $\pi : U \times I \to U$, where $I = [0,1]$ is the unit interval induces an isomorphism $\pi^* : \mathcal{F}(U) \longrightarrow \mathcal{F}(U \times I)$.

Proposition 3.7. *If $\mathcal{F}$ is an equivariant analytic presheaf with transfers then the complex $\blacktriangle^* \mathcal{F}$ has homotopy-invariant cohomology presheaves. In other words, given $n \geq 0$ let $\mathcal{H}^{-n}(\blacktriangle^* \mathcal{F})$ be the G-PST^ω*

$$\mathcal{H}^{-n}(\blacktriangle^* \mathcal{F}) : U \longmapsto \frac{\ker d : \mathcal{F}(U \times \blacktriangle^n) \to \mathcal{F}(U \times \blacktriangle^{n-1})}{\operatorname{Im} d : \mathcal{F}(U \times \blacktriangle^{n+1}) \to \mathcal{F}(U \times \blacktriangle^n)}.$$

Then $\mathcal{H}^{-n}(\blacktriangle^* \mathcal{F})$ is homotopy invariant, for all $n \geq 0$.

Proof. This follows exactly as in [21, Lect. 2]: define $\theta_i : \Delta^{n+1} \to \Delta^n \times \mathbb{A}^1$ by sending the vertex v_j to $v_j \times \{0\}$, for $j \leq i$, and to $v_{j-1} \times \{1\}$ otherwise. Then $s_n = \sum_i \mathcal{F}(1_U \times \theta_i)$ is a chain homotopy from i_1^* to i_0^*, where $i_\alpha : X \hookrightarrow X \times \mathbb{A}^1$ is the inclusion $x \mapsto (x, \alpha)$. By [21, Lemma 2.16] it follows that $\mathcal{H}^{-n}(\blacktriangle^* \mathcal{F})$ is homotopy invariant. $\square$

The notions of *homotopy-invariant presheaves with transfer* and complexes of presheaves with transfer having homotopy-invariant cohomology sheaves play an important role in the development of motivic cohomology. As we shall see in the following self-contained discussion, this is an equally relevant notion in our context.

As in [21, Lect. 2], one can introduce arbitrary colimits and limits (objectwise) in $G\text{-}\mathsf{PST}^\omega$. For example, define the *smash product* of pointed real analytic G-manifolds $(X_1, x_1), \ldots, (X_n, x_n)$ by

$$\begin{aligned}&\mathbb{Z}_{\mathrm{tr}}^{\omega}(X_1 \wedge \cdots \wedge X_n)\\ &:= \mathrm{Coker}\left\{\bigoplus_{j=1}^{n} \mathbb{Z}_{\mathrm{tr}}^{\omega}(X_1 \times \cdots \times \{x_j\} \times \cdots \times X_n) \to \mathbb{Z}_{\mathrm{tr}}^{\omega}(X_1 \times \cdots \times X_n)\right\}. \qquad (3.3)\end{aligned}$$

In particular, given $p \geq 1$, define $\mathbb{Z}_{\mathrm{tr}}^{\omega}(\bigwedge^p X) := \mathbb{Z}_{\mathrm{tr}}^{\omega}(X \wedge \cdots \wedge X)$.

Remark 3.8. In some cases the topological space $X_1 \wedge \cdots \wedge X_n$ admits a real analytic G-manifold structure, such as the case of representation spheres in Definition 3.4. To avoid confusion we do not use $X_1 \wedge \cdots \wedge X_n$ to denote a real analytic G-manifold.

3.2. Ordinary equivariant cohomology and $\mathbb{Z}(V)_G^{\omega}$. The paper [19] introduces a generalization of Bredon's G-equivariant cohomology groups, called *ordinary $RO(G)$-graded equivariant cohomology*. As explained in [19] the appropriate coefficients for this theory are Mackey functors over G. For our purposes the Mackey functor that plays the role of the integers in singular cohomology is the Mackey functor constant with value $\mathbb{Z}$, denoted $\underline{\mathbb{Z}}$ (see for example [7]). For a based G-space X the reduced ordinary RO(G)-graded cohomology with $\underline{\mathbb{Z}}$ coefficients assigns to each orthogonal G-representation V an abelian group $\widetilde{H}_G^V(X; \underline{\mathbb{Z}})$ together with suspension isomorphisms in the direction of arbitrary representations:

$$\sigma^W : \widetilde{H}_G^V(X; \underline{\mathbb{Z}}) \to \widetilde{H}_G^{V+W}(S^W \wedge X; \underline{\mathbb{Z}}),$$

where $S^W = W \cup \{\infty\}$ is the representation sphere of W. As usual, there is an unreduced theory defined by $H_G^V(X; \underline{\mathbb{Z}}) := \widetilde{H}_G^V(X_+; \underline{\mathbb{Z}})$, with $X_+ := X \cup \{+\}$.

The functors $X \mapsto \widetilde{H}_G^V(X; \underline{\mathbb{Z}})$ are contravariant in X and satisfy expected properties such as invariance under equivariant homotopy equivalences, existence of long exact sequences for pairs and sending wedges to products. They also satisfy natural compatibility relations involving the suspension isomorphisms, the morphisms of representations and the direct sum operation (see [20]). Using these properties it is possible to extend the cohomology theory on virtual representations by setting $\widetilde{H}_G^{V-W}(X;\underline{\mathbb{Z}}) := \widetilde{H}_G^V(S^W \wedge X;\underline{\mathbb{Z}})$, yielding an RO(G)-graded theory with a multiplication pairing that sends $\widetilde{H}_G^{\alpha}(X;\underline{\mathbb{Z}}) \otimes \widetilde{H}_G^{\beta}(X;\underline{\mathbb{Z}})$ into $\widetilde{H}_G^{\alpha+\beta}(X;\underline{\mathbb{Z}})$, where $\alpha+\beta$ denotes addition in the representation ring.

The main result in this section is a first indication that $\mathbb{Z}(V)^{\omega}_{\mathsf{G}}$ is indeed the analogue of the vth motivic complex in the category of real analytic G-manifolds.

Theorem 3.9. *Let X be an oriented real analytic G-manifold. Then, for a finite dimensional G-representation V one has natural isomorphisms*

$$\mathbb{H}^n\big(X_{\mathrm{eq}},\, \mathbb{Z}(V)^{\omega}_{\mathsf{G}}|_X\big) \cong H_G^{V+n-v}(X;\underline{\mathbb{Z}}),$$

between the hypercohomology of X with coefficients in $\mathbb{Z}(V)^{\omega}_{\mathsf{G}}|_X$ and the G-equivariant cohomology of X with $\underline{\mathbb{Z}}$ coefficients, in the direction of V.

Proof. In Theorem 3.11 below we show that $\mathbb{Z}(V)^{\mathsf{top}}_{\mathsf{G}}$ computes Bredon cohomology in the direction of V, i.e. that the hypercohomology of the complex $\mathbb{Z}(V)^{\mathsf{top}}_{\mathsf{G}}|_X$ is naturaly isomorphic to the Bredon cohomology groups of X in the direction of the representation V, with the appropriate index shift. The theorem follows as a consequence of Theorem 3.11 and the following lemma. □

Lemma 3.10. *Let V be a finite dimensional G-representation. The slicing maps $s : G\text{-}\mathsf{Man}^{\mathsf{fin}}_{\omega}(U, S^V) \to G\text{-}\mathsf{Top}(U, \mathcal{I}_0(S^V))$ described in Proposition 2.4 induce a quasi-isomorphism $S_X : \mathbb{Z}(V)^{\omega}_{\mathsf{G}|X} \to \mathbb{Z}(V)^{\mathsf{top}}_{\mathsf{G}|X}$, for all $X \in G\text{-}\mathsf{Man}_{\omega}$.*

Proof. Let $x_0 \in X$. Consider a neighbourhood basis $\{U_n\}_{n\in\mathbb{N}}$ of the orbit $G\cdot x_0$ such that $U_n \cong G\times_{G_{x_0}} D_n$, where each D_n is a G_{x_0}-equivariantly contractible analytic open set, as described in Appendix A.1. The homotopy invariance property in Proposition 3.7 applied to the cohomology presheaves of $\mathbb{Z}(V)^{\omega}_{\mathsf{G}|X}$ and $\mathbb{Z}(V)^{\mathsf{top}}_{\mathsf{G}}{}_{|X}$ in these neighbourhoods shows that their stalks $\big(\mathcal{H}^*(\mathbb{Z}(V)^{\omega}_{\mathsf{G}|X})\big)_{G\cdot x_0}$ and $\big(\mathcal{H}^*(\mathbb{Z}(V)^{\mathsf{top}}_{\mathsf{G}}{}_{|X})\big)_{G\cdot x_0}$ are given, respectively, by the homology of the complexes

$$\mathbb{Z}^{\omega}_{\mathsf{tr}}(S^V\!,\infty)(\blacktriangle^*\times G/G_{x_0}) \quad\text{and}\quad \mathbb{Z}^{\mathsf{G_{top}}}_{\mathsf{tr}}(S^V\!,\infty)(\blacktriangle^*\times G/G_{x_0}),$$

shifted by v; see Example 3.3. It is easy to check the commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{I}^{\omega}_{\mathsf{fin}}(\blacktriangle^*\times G/G_{x_0}, S^V)^G & \xrightarrow{S_{\blacktriangle^*\times G/G_{x_0}}} & G\text{-}\mathsf{Top}(\blacktriangle^*\times G/G_{x_0}, \mathcal{I}_0(S^V)) \\
\cong\big\downarrow & & \big\downarrow\cong \\
\mathcal{I}^{\omega}_{\mathsf{fin}}(\blacktriangle^*, S^V)^{G_{x_0}} & \xrightarrow{S_{\blacktriangle^*}} & G_{x_0}\text{-}\mathsf{Top}(\blacktriangle^*, \mathcal{I}_0(S^V)),
\end{array}$$

where the vertical maps are given by intersection with $\blacktriangle^*\times eG_{x_0}\times S^V \subset \blacktriangle^*\times G/G_{x_0}\times S^V$ on the left and by the usual adjunction on the right. The result now follows from Theorem 2.9 and Remark 2.10. □

The key ingredient to prove Theorem 3.9 is the following.

Theorem 3.11. *Let V be an orthogonal representation of G with* $\dim V = v$ *and let X be an oriented analytic manifold. Then there is a natural isomorphism*

$$\mathbb{H}^n(X_{\mathrm{eq}}; \mathbb{Z}(V)^{\mathsf{top}}_{\mathsf{G}|X}) \cong H_G^{V+n-v}(X; \underline{\mathbb{Z}}).$$

First, we need to introduce basic terminology.

Definition 3.12. Let X be an analytic manifold and $\mathcal{U} = \{U_\sigma\}_{\sigma\in I} \in \mathsf{Cov}(X)$, and let $\mathcal{P}$ be an arbitrary abelian presheaf on G-Man_ω. Denote by $\mathcal{P}_{|X}$ (resp., $a\mathcal{P}_{|X}$) the associated presheaf (resp., sheaf) on X_{eq}.

(a) The *Čech nerve* of $\mathcal{U}$ is a *simplicial G-space* over X denoted $\check{\mathsf{N}}(\mathcal{U})_\bullet \to X$. It is obtained from the successive fibered product of $f : \mathbf{U} \to X$, where $\mathbf{U} := \coprod_{\sigma\in I} U_\sigma$. In other words,

$$\check{\mathsf{N}}(\mathcal{U})_k := \underbrace{\mathbf{U}\times_X \cdots \times_X \mathbf{U}}_{(k+1)\text{-times}} = \coprod_{\check{\sigma}\in I^{n+1}} U_{\check{\sigma}},$$

where $U_{\check{\sigma}}$ denotes $U_{\sigma_0}\cap\cdots\cap U_{\sigma_n}$. The realization of $\check{\mathsf{N}}(\mathcal{U})_\bullet$ is a G-space $|\check{\mathsf{N}}(\mathcal{U})_\bullet|$, which is G-homotoy equivalent to X if I is countable [23].

(b) The *nerve* of $\mathcal{U}$ is the *simplicial set* $N(\mathcal{U})_\bullet$ defined by

$$N_k = \{(\sigma_0,\dots,\sigma_n) \in I^{n+1} \mid U_{\sigma_0}\cap\cdots\cap U_{\sigma_n} \neq \varnothing\}.$$

Hence we can write $\check{\mathsf{N}}(\mathcal{U})_k = \coprod_{\check{\sigma}\in N(\mathcal{U})_k} U_{\check{\sigma}}$.

(c) Let $\pi_k : \check{\mathsf{N}}(\mathcal{U})_k \to X$ be the projection and denote $\check{\mathcal{C}}^k_{\mathcal{P}} := \pi_{k*}\pi_k^*(\mathcal{P}_{|X})$. This gives the usual cosimplicial presheaf $\check{\mathcal{C}}^\bullet_{\mathcal{U}}(\mathcal{P})$ on X_{eq} with associated Čech complex of presheaves $\check{\mathcal{C}}^*_{\mathcal{U}}(\mathcal{P})$.

(d) Given a complex of abelian presheaves $\mathcal{P}^*$ on G-Man_ω the naturality of the Čech construction gives a double complex $\check{\mathcal{C}}^\star_{X,\mathcal{U}}(\mathcal{P}^*)$ and we have

$$\Gamma(X, \check{\mathcal{C}}^p_{X,\mathcal{U}}(\mathcal{P}^q)) = \prod_{\check{\sigma}_p\in N(\mathcal{U})_p} \mathcal{P}^q(U_{\check{\sigma}_p}).$$

The Čech hypercohomology $\check{\mathbb{H}}^\bullet(\mathcal{U}; \mathcal{P}^*)$ of the cover $\mathcal{U}$ with coefficients in the complex of presheaves $\mathcal{P}^*$ is defined as the cohomology of the complex of abelian groups $\mathrm{Tot}\big(\Gamma\big(X, \check{\mathcal{C}}^*_{X,\mathcal{U}}(\mathcal{P}^*)\big)\big)$.

(e) The cover $\mathcal{U}$ is called an *equivariant good cover* if all intersections $U_{\sigma_0}\cap\cdots\cap U_{\sigma_k}$ are equivariantly analytically diffeomorphic to a G-manifold of the form $G\times_H D$ where H is a subgroup of G and D is the intersection of an open disc in an orthogonal representation of H with an H-invariant "corner" of the type $\mathbb{R}^r_+\times\mathbb{R}^{n-r}$. The existence of such covers for an analytic G-manifold is given in [7, Cor. A.7] and the same arguments apply to manifolds-with-corner.

Notation 3.13. Given a double complex (C, d, δ), with $d : C^{p,q} \to C^{p,q+1}$ and $\delta : C^{p,q} \to C^{p+1,q}$, there are two natural spectral sequences converging to the total cohomology $H^*(C)$. We refer to the spectral sequence whose E_1-term is H^d and E_2-term is $H^\delta H^d$ as the *first sequence*. By the *second sequence* we mean the spectral sequence with $E_1 = H^\delta$ and $E_2 = H^d H^\delta$.

The following "steps" to prove Theorem 3.11 have an interest of their own. The first one is the topological counterpart of [21, Thm. 22.1]. Its motivic version is more subtle and plays a fundamental role in the development of motivic cohomology.

Proposition 3.14. *Let $\mathcal{P}$ be a homotopy invariant abelian presheaf on $G\text{-}\mathsf{Man}_\omega$. Then the associated sheaf $a\mathcal{P}$ is homotopy invariant.*

Proof. See Appendix A.3. □

A consequence of this result is the next step.

Theorem 3.15. *Let $\mathcal{P}^*$ be a complex of abelian presheaves on $G\text{-}\mathsf{Man}_\omega$ with homotopy-invariant cohomology presheaves. Then the presheaves*

$$X \mapsto \mathbb{H}^r(X_{\mathrm{eq}}; a\mathcal{P}^*_{|X})$$

are homotopy-invariant.

Proof. See Appendix A.3. □

We now come to the final step.

Proposition 3.16. *Let $\mathcal{U}$ be an equivariant good cover of X and let $\mathcal{P}^*$ be a complex with homotopy invariant cohomology presheaves on $G\text{-}\mathsf{Man}_\omega$. Then the Čech hypercohomology $\check{\mathbb{H}}^\bullet(\mathcal{U}; \mathcal{P}^*)$ of the cover with coefficients on the complex of presheaves $\mathcal{P}^*$, computes the hypercohomology $\mathbb{H}^\bullet(X; a\mathcal{P}^*_{|X})$ of X_{eq} with coefficients in the complex of sheaves $a\mathcal{P}^*_{|X}$.*

Proof. See Appendix A.3. □

Using the steps above we can prove the desired result.

Proof of Theorem 3.11. The proof follows the steps in [28] where a closely related complex is used to represent ordinary equivariant cohomology.

Consider the invariant $\mathfrak{h}^*$ defined on pairs of G-spaces by

$$\mathfrak{h}^r(X, A) = H_G^{V+r-v}(X, A; \underline{\mathbb{Z}})$$

It satisfies the usual axioms for a generalized equivariant cohomology theory: long exact sequence for pairs, G-homotopy invariance and excision. We will construct a suitable filtration on X and follow the arguments in [23] to obtain a spectral sequence converging to $\mathfrak{h}^*(X)$. To obtain the desired filtration we replace X by the realization of the Čech nerve $\check{\mathsf{N}}(\mathcal{U})_\bullet$, for an equivariant good $\mathcal{U}$ of X (see Definition 3.12), and consider its skeletal filtration

$$\mathrm{Sk}_0\,|\check{\mathsf{N}}(\mathcal{U})_\bullet| \subset \mathrm{Sk}_1\,|\check{\mathsf{N}}(\mathcal{U})_\bullet| \subset \cdots \subset \mathrm{Sk}_n\,|\check{\mathsf{N}}(\mathcal{U})_\bullet| \subset \cdots \subset |\check{\mathsf{N}}(\mathcal{U})_\bullet|,$$

where

$$\mathrm{Sk}_n\,|\check{\mathsf{N}}(\mathcal{U})_\bullet| = \mathrm{coeq}\left(\coprod_{\substack{[l]\to[k]\\ k,l\le n}} \blacktriangle^l \times \check{\mathsf{N}}(\mathcal{U})_k \rightrightarrows \coprod_{k\le n} \blacktriangle^k \times \check{\mathsf{N}}(\mathcal{U})_k \right).$$

In this case, [23, Prop. 5.1] yields a spectral sequence converging to $\mathfrak{h}^*(|\check{\mathsf{N}}(\mathcal{U})_\bullet|) \cong \mathfrak{h}^*(X)$ with E_1 term

$$\begin{aligned} E_1^{p,q} &= \mathfrak{h}^{p+q}\big(\mathrm{Sk}_p\,|\check{\mathsf{N}}(\mathcal{U})_\bullet|, \mathrm{Sk}_{p-1}\,|\check{\mathsf{N}}(\mathcal{U})_\bullet|\big) \\ &\cong \mathfrak{h}^{p+q}\big(\blacktriangle^p \times \check{\mathsf{N}}(\mathcal{U})_p, \blacktriangle^p \times \check{\mathsf{N}}(\mathcal{U})_p^d \cup \partial\blacktriangle^p \times \check{\mathsf{N}}(\mathcal{U})_p\big) \\ &\cong \mathfrak{h}^q\big(\check{\mathsf{N}}(\mathcal{U})_p, \check{\mathsf{N}}(\mathcal{U})_p^d\big) \cong \mathfrak{h}^q\big(\check{\mathsf{N}}(\mathcal{U})_p^{nd}\big) = H_G^{V+p-v}\big(\textstyle\coprod_{\check{\sigma}\in N(\mathcal{U})_p^{nd}} U_{\check{\sigma}}; \underline{\mathbb{Z}}\big) \\ &= \prod_{\check{\sigma}\in N(\mathcal{U})_p^{nd}} H_G^{V+p-v}(G/J_{\check{\sigma}}; \underline{\mathbb{Z}}), \end{aligned}$$

where the superscripts d and nd represent the *degenerate* and non-degenerate parts of the corresponding simplicial object and we use the hypothesis that $\mathcal{U}$ is good to write $U_{\check{\sigma}} \cong G \times_{J_{\check{\sigma}}} D$, with D being $J_{\check{\sigma}}$-equivariantly contractible.

The E_2 term is computed as in [23] by considering the cochain complex associated to the simplicial space $\check{\mathsf{N}}(\mathcal{U})_\bullet$ and $\mathfrak{h}$:

$$K_q^p := \mathfrak{h}^q\big(\check{\mathsf{N}}(\mathcal{U})_p\big) = H_G^{V+q-v}\big(\textstyle\coprod_{\check{\sigma}\in N(\mathcal{U})_p} U_{\check{\sigma}}; \underline{\mathbb{Z}}\big) \cong \displaystyle\prod_{\check{\sigma}\in N(\mathcal{U})_p} H_G^{V+p-v}(G/J_{\check{\sigma}}; \underline{\mathbb{Z}}).$$

Its differential is defined as $\sum_i (-1)^i \mathfrak{h}^q(d_i)$, where d_i denotes the ith face map of $\check{\mathsf{N}}(\mathcal{U})_\bullet$. There is an obvious map $E_1^{p,q} \to K_q^p$, which by [23, Prop. 5.1] is compatible with the differentials and induces an isomorphism $E_2^{p,q} \cong H^p(K_q)$.

Now, by Proposition 3.16 the Čech hypercohomology $\check{\mathbb{H}}^\bullet(\mathcal{U}; \mathbb{Z}(V)^{\mathsf{top}}_{\mathsf{G}|X})$ computes $\mathbb{H}^*(X; \mathbb{Z}(V)^{\mathsf{top}}_{\mathsf{G}|X})$. The first spectral sequence gives

$$\begin{aligned} {}'E_1^{p,q}(\mathcal{U}, \mathbb{Z}(V)^{\mathsf{top}}_{\mathsf{G}|X}) &:= \prod_{\check{\sigma}\in N(\mathcal{U})_p} H^q\big(\Gamma\big(U_{\check{\sigma}_p}; \mathbb{Z}(V)^{\mathsf{top}}_{\mathsf{G}|X}\big)\big) \\ &= \prod_{\check{\sigma}\in N(\mathcal{U})_p} \mathcal{H}^q\big(\mathbb{Z}(V)^{\mathsf{top}}_{\mathsf{G}|X}\big)\big(U_{\check{\sigma}_p}\big) \\ &\cong \prod_{\check{\sigma}\in N(\mathcal{U})_p} H_G^{V+p-v}(G/J_{\check{\sigma}}; \underline{\mathbb{Z}}) \cong K_q^p, \end{aligned}$$

where we have used the homotopy invariance of $\mathcal{H}^q(\mathbb{Z}(V)^{\mathrm{top}}_{\mathrm{G}\,|X})$, Proposition 2.9 and Spanier–Whitehead duality:

$$\begin{aligned}\mathcal{H}^q\big(\mathbb{Z}(V)^{\mathrm{top}}_{\mathrm{G}|X}\big)\big(U_{\check{\sigma}}\big) &\cong \mathcal{H}^q\big(\mathbb{Z}(V)^{\mathrm{top}}_{\mathrm{G}|X}\big)\big(G/J_{\check{\sigma}}\big) \cong \widetilde{H}^G_{-q+v}\big(S^V \wedge G/J_{\check{\sigma}+}\big)\\ &\cong H^G_{-V-q+v}\big(G/J_{\sigma_p}\big) \cong H_G^{V+q-v}\big(G/J_{\check{\sigma}}\big).\end{aligned}$$

It is important to note that, under this sequence of natural isomorphisms, the map $\mathcal{H}^q(\mathbb{Z}(V)^{\mathrm{top}}_{\mathrm{G}|X})(U_{\check{\sigma}}) \to \mathcal{H}^q(\mathbb{Z}(V)^{\mathrm{top}}_{\mathrm{G}|X})(U_{\check{\tau}})$ induced by an inclusion $U_{\check{\tau}} \hookrightarrow U_{\check{\sigma}}$ is just the map induced on cohomology by the corresponding map of orbits $G/J_{\check{\tau}} \to G/J_{\check{\sigma}}$. We conclude that the map $E_1^{p,q} \to K_q^p$ yields a map of spectral sequences $E_1^{p,q} \to {}'E_1^{p,q}$ that induces an isomorphism of the E_2-terms. □

Corollary 3.17. *Let V, W be real G-representations of dimensions v and w. Then the complexes $\mathbb{Z}(V \oplus W)^{\omega}_{\mathrm{G}}$ and $\blacktriangle^{\bullet}\mathbb{Z}^{\omega}_{\mathrm{tr}}(S^V \wedge S^W)[-(v+w)]$ are quasi-isomorphic.*

Proof. Given the pointed G-spaces S^V and S^W one can define the abelian sheaf $\mathbb{Z}^{\mathrm{G_{top}}}_{\mathrm{tr}}(S^V \wedge S^W)$ on G-Top exactly as in (3.3):

$$U \longmapsto \frac{\mathbb{Z}^{\mathrm{G_{top}}}_{\mathrm{tr}}\big(S^V \times S^W\big)(U)}{\mathrm{Im}\left\{\big(\mathbb{Z}^{\mathrm{G_{top}}}_{\mathrm{tr}}\big(S^V \times \infty\big) \oplus \mathbb{Z}^{\mathrm{G_{top}}}_{\mathrm{tr}}\big(\{\infty\} \times S^W\big)\big)(U) \to \mathbb{Z}^{\omega}_{\mathrm{tr}}\left(S^V \times S^W\right)(U)\right\}}. \tag{3.4}$$

Note that, even though the quotient $S^V \times S^W / S^V \vee S^W = S^{V\oplus W}$ exists in G-Top, this is not the same as abelian sheaf represented by $S^{V \oplus W}$. But the equivariant Dold–Thom theorem proved in [18] implies that the projection $S^V \times S^W \to S^{V\oplus W}$ induces a quasi-isomorphism $\blacktriangle^{\bullet}\mathbb{Z}^{\mathrm{G_{top}}}_{\mathrm{tr}}(S^V \wedge S^W) \to \blacktriangle^{\bullet}\mathbb{Z}^{\mathrm{G_{top}}}_{\mathrm{tr}}(S^{V\oplus W})$.

To conclude the proof one uses the fact that the map $\blacktriangle^{\bullet}\mathbb{Z}^{\omega}_{\mathrm{tr}}(S^V \wedge S^W) \to \blacktriangle^{\bullet}\mathbb{Z}^{\mathrm{G_{top}}}_{\mathrm{tr}}(S^V \wedge S^W)$, induced by slicing is a quasi-isomorphism by Lemma 3.10. □

Remark 3.18. Given a subgroup $H \leq G$, one has a *forgetful* functor $\varphi : G\text{-}\mathsf{Man}_{\omega} \to H\text{-}\mathsf{Man}_{\omega}$ that determines a morphism of sites $\varphi^{\mathrm{op}} : H\text{-}\mathsf{Man}_{\omega} \to G\text{-}\mathsf{Man}_{\omega}$ inducing a morphism of topoi $\hat{\varphi} = (\varphi^*, \varphi_*) : H\text{-}\mathsf{Man}^{\sim}_{\omega} \to G\text{-}\mathsf{Man}^{\sim}_{\omega}$. It is easy to see that the left adjoint $\varphi^* : G\text{-}\mathsf{Man}^{\sim}_{\omega} \to H\text{-}\mathsf{Man}^{\sim}_{\omega}$ sends $\mathbb{Z}(V)^{\omega}_{\mathrm{G}}$ to $\mathbb{Z}(\mathrm{Res}^G_H(V))^{\omega}_{\mathrm{H}}$. In particular, if $X \in G\text{-}\mathsf{Man}_{\omega}$, one obtains the natural *change of group functor*

$$\varphi^* : H_G^{V+n-v}(X; \underline{\mathbb{Z}}) \to H_H^{\widehat{V}+n-v}(X; \underline{\mathbb{Z}}),$$

where $\widehat{V}$ denotes $\mathrm{Res}^G_H(V)$. In the case where $H = \{e\}$ is the trivial subgroup, this gives the forgetful map into singular cohomology

$$\varphi^* : H_G^{V+n-v}(X; \underline{\mathbb{Z}}) \to H^n_{\mathrm{sing}}(X; \mathbb{Z}).$$

3.3. The case of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Now we specialize to the Galois group $\mathfrak{G} := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and explore its conjugation action on complex points of real varieties.

Let $\xi, \mathbf{1} \cong \mathbb{R}$ denote, respectively, the alternating and trivial one-dimensional representations of $\mathfrak{G}$. Any orthogonal representation of $\mathfrak{G}$ is isomorphic to $\mathbb{R}^{n,q} := \mathbf{1}^{n-q} \oplus \xi^q$, for some $n \geq q \geq 0$, and we denote by $S^{n,q} \in \mathfrak{G}\text{-}\mathsf{Man}_{\omega}^{\mathsf{fin}}$ its one-point compactification. From now on, we use the *motivic notation*

$$H_{\mathrm{Br}}^{n,q}(X;\underline{\mathbb{Z}}) := H_{\mathfrak{G}}^{\mathbb{R}^{n-q,q}}(X;\underline{\mathbb{Z}}). \tag{3.5}$$

Note that $S^1 \subset \mathbb{C}$ endowed with the complex conjugation action is isomorphic to $S^{1,1}$ (with ∞ mapped to $\mathbf{1}$) and the inclusion $S^{1,1} \subset \mathbb{C}^\times$ is an equivariant homotopy equivalence. This can be used to find a convenient expression of the smash product $S^{p,p} := S^{1,1} \wedge \cdots \wedge S^{1,1}$ in the appropriate derived category of sheaves.

Definition 3.19. Define $\mathbb{Z}(p)_\omega := \blacktriangle^* \mathbb{Z}_{\mathsf{tr}}^{\omega}(\bigwedge^p \mathbb{C}^\times)[-p]$.

Lemma 3.20. *The complex $\mathbb{Z}(p)_\omega$ is quasi-isomorphic to*

$$\mathbb{Z}(\mathbb{R}^{p,p})_{\mathfrak{G}}^{\omega} = \blacktriangle^* \mathbb{Z}_{\mathsf{tr}}^{\omega}(S^{p,p})[-p].$$

Proof. As in the proof of Corollary 3.17 we note that slicing induces a map

$$\blacktriangle^* \mathbb{Z}_{\mathsf{tr}}^{\omega}(\textstyle\bigwedge^p \mathbb{C}^\times) \to \blacktriangle^* \mathbb{Z}_{\mathsf{tr}}^{\mathfrak{G}_{\mathsf{top}}}(\textstyle\bigwedge^p \mathbb{C}^\times)$$

which by Lemma 3.10 is a quasi-isomorphism. Since $(S^{p,p}, \infty)$ is a strong deformation retract of $\bigwedge^p(\mathbb{C}^\times, \mathbf{1})$, the result follows. $\square$

3.4. A basic example. We use the complex $\mathbb{Z}(p)_\omega$ to directly compute the cohomology groups $H_{\mathrm{Br}}^{p,p}(*;\underline{\mathbb{Z}})$, where $* = \{pt\}$. For brevity, given a pointed object (X, x_o) in $\mathfrak{G}\text{-}\mathsf{Man}_{\omega}^{\mathsf{fin}}$ we denote

$$Z^p(X) := \mathbb{Z}_{\mathsf{tr}}^{\omega}(\textstyle\bigwedge^p X)(*) = \dfrac{\mathcal{I}_0(X \times \cdots \times X)^{\mathfrak{G}}}{\sum_{j=1}^{p} \mathcal{I}_0(X \times \cdots \times \{x_o\} \times \cdots \times X)^{\mathfrak{G}}}. \tag{3.6}$$

It is clear that the assignment $(X, x_o) \mapsto Z^p(X)$ is functorial on (X, x_o).

As described in Remark 2.8, one associates to $T \in \mathcal{I}_{\mathsf{fin}}^{\omega}(\blacktriangle^1, X \times \cdots \times X)^{\mathfrak{G}}$ a map $f_T : \blacktriangle^1 \to \mathcal{I}_0(X \times \cdots \times X)^{\mathfrak{G}}$, which induces a continuous map

$$f_T : \blacktriangle^1 \to Z^p(X) \tag{3.7}$$

for the quotient topology on $Z^p(X)$. Note that if $[T]$ denotes the class of T in

$$\blacktriangle^1 \mathbb{Z}_{\mathsf{tr}}^{\omega}(\textstyle\bigwedge^p X)(*) = \dfrac{\mathcal{I}_{\mathsf{fin}}^{\omega}(\blacktriangle^1, X \times \cdots \times X)^{\mathfrak{G}}}{\sum_{j=1}^{p} \mathcal{I}_{\mathsf{fin}}^{\omega}(\blacktriangle^1, X \times \cdots \times \{x_o\} \times \cdots \times X)^{\mathfrak{G}}}$$

then f_T depends only on $[T]$. Furthermore, the simplicial boundary map

$$d_1 : \blacktriangle^1 \mathbb{Z}_{\mathsf{tr}}^{\omega}(\textstyle\bigwedge^p X)(*) \longrightarrow \mathbb{Z}_{\mathsf{tr}}^{\omega}(\textstyle\bigwedge^p X)(*) = Z^p(X) \tag{3.8}$$

is defined by $d_1[T] = f_T(1) - f_T(0)$, when one identifies $\mathbb{A}^1$ with the unit interval $[0,1]$. Denote the image of d_1 by $B^p(X) \subset Z^p(X)$. In the particular case where $(X, x_o) = (\mathbb{C}^\times, 1)$ one has a short exact sequence

$$0 \to B^p(\mathbb{C}^\times) \to Z^p(\mathbb{C}^\times) \xrightarrow{q} H^{p,p}_{\mathrm{Br}}(*; \underline{\mathbb{Z}}) \to 0, \tag{3.9}$$

since $\mathbb{Z}(p)^{p+1}_\omega = 0$.

Proposition 3.21. *There is a canonical isomorphism*

$$\overline{\rho} : H^{p,p}_{Br}(*; \underline{\mathbb{Z}}) \xrightarrow{\cong} \mathbb{Z}^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}^\times .$$

Proof. Denote an element in the p-fold cartesian product $\mathbb{C}^{\times p} := \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$ by $\mathbf{z} = (z_1, \ldots, z_p)$ and let $\delta_{\mathbf{z}} \in \mathcal{I}_0(\mathbb{C}^{\times p})$ denote the 0-current represented by $\mathbf{z}$.

We start recalling a standard exact sequence of topological groups

$$0 \to \mathfrak{S} * \mathcal{I}_0(\mathbb{C}^{\times p}) \to \mathcal{I}_0(\mathbb{C}^{\times p})^{\mathfrak{S}} \xrightarrow{\pi} \mathcal{I}_0(\mathbb{R}^{\times p}) \otimes \mathbb{Z}/2 \to 0, \tag{3.10}$$

where $\mathfrak{S} * \mathcal{I}_0(\mathbb{C}^{\times p})$ denotes the closed subgroup generated by Galois sums of the form $\delta_z + \delta_{\overline{z}} \in \mathcal{I}_0(\mathbb{C}^{\times p})$. This sequence appears as the top row of the diagram below, which summarizes the arguments that follow.

$$\begin{tikzcd}
\mathfrak{S} * \mathcal{I}_0(\mathbb{C}^{\times p}) \arrow[rr, hook] & & \mathcal{I}_0(\mathbb{C}^{\times p})^{\mathfrak{S}} \arrow[rr, two heads, "\pi"] \arrow[dd, two heads, "\phi_1"] & & \mathcal{I}_0(\mathbb{R}^{\times p}) \otimes \mathbb{Z}/2 \arrow[dd, two heads, "\phi_2"] \\
& \ker \rho \arrow[rr, equal] \arrow[dr, hook] & & B^p(\mathbb{C}^\times) \arrow[dl, hook'] & \\
\mathbb{A}^1 \arrow[rr, dotted, "f_T"] & & Z^p(\mathbb{C}^\times) \arrow[rr, two heads, "\widehat{\pi}"] \arrow[ddll, two heads, "q"'] \arrow[ddrr, two heads, "\rho"] & & Z^p(\mathbb{R}^\times) \otimes \mathbb{Z}/2 \arrow[d, two heads, "\mu_*"] \\
& & & & Z^p(\mathbb{Z}^\times) \otimes \mathbb{Z}/2 \arrow[d, two heads, "u"] \\
H^{p,p}_{\mathrm{Br}}(*; \underline{\mathbb{Z}}) \arrow[rrrr, leftrightarrow, dashed, "\cong"] & & & & \mathbb{Z}^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}^\times
\end{tikzcd} \tag{3.11}$$

In this diagram:

(1) The vertical arrows ϕ_1 and ϕ_2 are the natural quotient homomorphisms (3.6).

(2) The homomorphism μ_* is induced by $\mu : \mathbb{R}^\times \to \mathbb{Z}^\times$, $\mu : x \mapsto \frac{x}{|x|}$.

(3) The vertical arrow u is induced by the universal homomorphism $\mathbb{Z}\{\mathbb{Z}^{\times p}\} = \mathcal{I}_0(\mathbb{Z}^{\times p}) \to \mathbb{Z}^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}^\times$, and all vertical arrows are continuous surjections.

(4) It is clear that π descends to the quotients inducing $\widehat{\pi}$, and we define

$$\rho := u \circ \mu_* \circ \widehat{\pi}. \tag{3.12}$$

(5) The diagonal sequences are short exact sequences, by definition.

It is easy to see that an element $\beta \in \mathcal{I}_0(\mathbb{C}^{\times p})^{\mathfrak{G}}$ can be uniquely written as $\beta = A + R^+ + R^-$, where $A = \sum_\lambda n_\lambda(\delta_{\mathbf{z}^\lambda} + \delta_{\bar{\mathbf{z}}^\lambda})$, $R^+ = \sum_\iota m_\iota \delta_{\mathbf{x}^\iota}$, and $R^- = \sum_J r_J \delta_{\mathbf{y}^J}$, where $\mathbf{z}^\lambda \in \mathbb{C}^{\times p} - \mathbb{R}^{\times p}$, $\mathbf{y}_J \in (\mathbb{R}_{<0})^p$, for all J, and $\mathbf{x}^\iota \in \mathbb{R}^{\times p} - (\mathbb{R}_{<0})^p$ for all ι.

A diagram chase shows that $\rho(\beta) = \big(\sum_J r_J\big)\{-1,\dots,-1\}$, where $\{-1,\dots,-1\}$ is the generator of $\mathbb{Z}^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}^\times$. It follows that

$$\beta = A + R^+ + R^- \in \ker\rho \ \text{ if and only if } \deg R^- := \sum_J r_J \text{ is even.} \tag{3.13}$$

If $\mathbf{z}$ is in $\mathbb{C}^{\times p} - \mathbb{R}^{\times p}$, assume without loss of generality that the first coordinate z_1 is in $\mathbb{C}^\times - \mathbb{R}^\times$, and consider the map $g_{z_1} : \blacktriangle^1 \to \mathbb{C}^\times$ defined by $g_{z_1}(t) = t\, z_1 + (1-t)$. Let $[\![\Gamma_{\mathbf{z}}]\!] \in \mathcal{I}^{\omega}_{\text{fin}}(\blacktriangle^1, \mathbb{C}^{\times p})$ denote the current associated to the graph of

$$g_{\mathbf{z}} : t \in \blacktriangle^1 \longmapsto \big(g_{z_1}(t), z_2, \dots, z_p\big) \in \mathbb{C}^{\times p},$$

as in Example 2.3(i). It follows that $\Gamma^{\text{av}}_{\mathbf{z}} := [\![\Gamma_{\mathbf{z}}]\!] + \sigma_*[\![\Gamma_{\bar{\mathbf{z}}}]\!] \in \mathcal{I}^{\omega}_{\text{fin}}(\blacktriangle^1, \mathbb{C}^{\times p})^{\mathfrak{G}}$, where σ_* is the action on currents induced by complex conjugation, and that

$$d_1(\Gamma^{\text{av}}_{\mathbf{z}}) = \delta_{\mathbf{z}} + \delta_{\bar{\mathbf{z}}} - \delta_{\mathbf{z}'} - \delta_{\bar{\mathbf{z}}'}, \tag{3.14}$$

where $\mathbf{z}' = (1, z_2, \dots, z_p)$.

Now, for $a \in \mathbb{R}^\times$ define $h_a : \blacktriangle^1 \to \mathbb{R}^\times$ by $h_a(t) = ta + (1-t)\frac{a}{|a|}$. Given $\mathbf{a} = (a_1, \dots, a_p) \in \mathbb{R}^{\times p}$ define $h_{\mathbf{a}} : \blacktriangle^1 \to \mathbb{R}^{\times p}$ by $t \mapsto (h_{a_1}(t), \dots, h_{a_p}(t))$, and let $[\![W_{\mathbf{a}}]\!] \in \mathcal{I}^{\omega}_{\text{fin}}(\blacktriangle^1, \mathbb{R}^{\times p})$ denote its graph. It is clear that

$$d_1[\![W_{\mathbf{a}}]\!] = \delta_{\mathbf{a}} - \delta_{\mu(\mathbf{a})}, \quad \text{where } \mu(\mathbf{a}) = \left(\frac{a_1}{|a_1|}, \dots, \frac{a_1}{|a_1|}\right). \tag{3.15}$$

Given $\beta = A + R^+ + R^-$ as above, define $W_\beta = \Gamma_A + W_{R^+} + W_{R^-}$, where $\Gamma_A = \sum_\lambda n_\lambda [\![\Gamma^{\text{av}}_{\mathbf{z}^\lambda}]\!]$, $W_{R^+} = \sum_\iota m_\iota [\![W_{\mathbf{x}^\iota}]\!]$, and $W_{R^-} = \sum_\iota r_J [\![W_{\mathbf{y}^J}]\!]$. It follows from (3.14) and (3.15) that one has

$$\begin{aligned} d_1(W_\beta) &= d_1(W_A) + d_1(W_{R^+}) + d_1(W_{R^-}) \\ &= (A - A') + (R^+ - R') + \Big(R^- - \sum_J r_J\Big)\delta_{\ell_p} \\ &= \beta - (A' + R') - \Big(\sum_J r_J\Big)\delta_{\ell_p}, \end{aligned}$$

where A', R' are sums of elements of the form $\delta_{(\epsilon_1,\dots,\epsilon_p)}$ where $\epsilon_k = 1$, for some $k = 1,\dots,p$, and $\ell_p = (-1,\dots,-1) \in \mathbb{R}^{\times p}$. In other words, the class of β in $Z^p(\mathbb{C}^\times)$ can be written as

$$\beta = \Big(\sum_J r_J\Big)\delta_{\ell_p} + d_1(W_\beta). \tag{3.16}$$

Now, let $\gamma : \mathbb{A}^1 \to \mathbb{C}^\times$ denote an analytic parametrization of the upper half of the unit circle centered at the origin in $\mathbb{C}^\times$, oriented from 1 to -1. Define $\gamma_p : \mathbb{A}^1 \to \mathbb{C}^{\times p}$ by $\gamma_p(t) = (\gamma(t), -1, \dots, -1)$ and let $[\![\Gamma_p]\!] \in \mathcal{I}^\omega_{\mathrm{fin}}(\mathbb{A}^1, \mathbb{C}^{\times p})$ denote its graph. It follows that $\Theta_p := [\![\Gamma_p]\!] + \sigma_*[\![\Gamma_p]\!]$ lies in $\mathcal{I}^\omega_{\mathrm{fin}}(\mathbb{A}^1, \mathbb{C}^{\times p})^{\mathfrak{G}}$ and that

$$d_1(\Theta_p) = 2\delta_{\ell_p} - 2\delta_{(1,-1,\dots,-1)}. \tag{3.17}$$

Hence, the class of δ_{ℓ_p} in $Z^p(\mathbb{C}^\times)$ satisfies

$$d_1(\Theta_p) = 2\delta_{\ell_p}, \tag{3.18}$$

since $\delta_{(1,-1,\dots,-1)} = 0 \in Z^p(\mathbb{R}^\times)$.

By (3.13), if $\beta \in \mathrm{Ker}(\rho_p)$ then $\sum_J r_J = 2k$, for some integer k, and using (3.16) and (3.18) one concludes that $\beta = (\sum_J r_J)\delta_{\ell_p} + d_1(W_\beta) = d_1\left(k\Theta_p + W_\beta\right)$. This shows that

$$\ker\rho \subset B^p(\mathbb{C}^\times). \tag{3.19}$$

Now, by definition, if $\beta \in B^p(\mathbb{C}^\times)$ one can find $T \in \mathcal{I}^\omega_{\mathrm{fin}}(\mathbb{A}^1, \mathbb{C}^{\times p})^{\mathfrak{G}}$ satisfying $\beta = f_T(1) - f_T(0)$. The resulting $f_T : \mathbb{A}^1 \to Z^p(\mathbb{C}^\times)$ is continuous and hence $\rho \circ f_T$ is a constant map, since $\mathbb{A}^1$ is connected. See (3.7) and (3.11). It follows that $\rho(\beta) = \rho(f_T(1) - f_T(0)) = \rho\circ f_T(1) - \rho\circ f_T(0) = \{1,\dots,1\}$, and hence $\beta \in \ker(\rho)$. This shows that $B^p(\mathbb{C}^\times) \subset \ker(\rho)$, which together with (3.19) proves the proposition. $\square$

Remark 3.22. Let $\varepsilon \in H^{1,1}_{\mathrm{Br}}(*;\underline{\mathbb{Z}})$ denote the generator, i.e. $\rho(\varepsilon) = \{-1\}$. Using the description of the cup product in (5.3), one sees that $\varepsilon^p := \varepsilon \cup \cdots \cup \varepsilon \in H^{p,p}_{\mathrm{Br}}(*;\underline{\mathbb{Z}})$ is the generator, i.e. $\rho(\varepsilon^p) = \{-1,\dots,-1\}$. This gives a proof of the well-known fact that the graded ring $\oplus_{p\geq 0} H^{p,p}_{\mathrm{Br}}(*;\underline{\mathbb{Z}})$ is isomorphic to $\mathbb{Z}[\varepsilon] \cong \mathbb{Z}[\mathrm{x}]/\langle 2\mathrm{x}\rangle$.

4. A de Rham realization of $\mathbb{Z}(p)_\omega$

Let $H^*_{\mathrm{sing}}(X;\mathbb{Z}(p)) \cong H^k_{\mathrm{sing}}(X;\mathbb{Z}) \otimes \mathbb{Z}(p)$ denote the singular cohomology groups with coefficients in $\mathbb{Z}(p) = (2\pi\sqrt{-1})^p\mathbb{Z} \subset \mathbb{C}$. We have a commutative diagram

$$\begin{tikzcd} H^{n,p}_{\mathrm{Br}}(X;\underline{\mathbb{Z}}) \arrow[r, "\varphi"] & H^n_{\mathrm{sing}}(X;\mathbb{Z}(p))^{\mathfrak{G}} \arrow[r, hook] \arrow[d, "j'"] & H^n_{\mathrm{sing}}(X;\mathbb{Z}(p)) \arrow[d, "j"] \\ & H^n_{\mathrm{sing}}(X;\mathbb{C})^{\mathfrak{G}} \cong H^n_{\mathrm{DR}}(X)^{\mathfrak{G}} \arrow[r, hook] & H^n_{\mathrm{DR}}(X) \end{tikzcd} \tag{4.1}$$

where the first row gives the "change of groups functor" described in Remark 3.18 for the subgroup $\{e\} \subset \mathfrak{G}$, and vertical arrows are change of coefficients functors for singular cohomology, along with deRham's theorem. The group $H^n_{\text{sing}}(X; \mathbb{Z}(p))^{\mathfrak{G}}$ consists of the invariants of the simultaneous action of $\mathfrak{G}$ on X and $\mathbb{Z}(p)$, and $H^n_{\text{DR}}(X)^{\mathfrak{G}}$ are the invariants induced by the action $\theta \mapsto \overline{\sigma^*\theta}$ on differential forms.

Our goal is to describe a direct realization of the map $H^{n,p}_{\text{Br}}(X; \underline{\mathbb{Z}}) \to H^n_{\text{DR}}(X)^{\mathfrak{G}}$ on the level of complexes of sheaves on manifold X. This description is relevant for a subsequent study of regulator maps for real varieties.

Given $X \in \mathfrak{G}\text{-}\mathsf{Man}_\omega$ with $\dim X = m$, it follows from Definition 3.19 and (3.3) that

$$\mathbb{Z}(p)^j_\omega(X) := \frac{\mathcal{I}^\omega_{\text{fin}}\big(X \times \blacktriangle^{p-j}, (\mathbb{C}^\times)^p\big)^{\mathfrak{G}}}{\operatorname{Im}\big\{ \oplus_{j=1}^p \mathcal{I}^\omega_{\text{fin}}\big(X \times \blacktriangle^{p-j}, \mathbb{C}^\times \times \cdots \times \{1\} \times \cdots \times \mathbb{C}^\times\big)^{\mathfrak{G}} \to \mathcal{I}^\omega_{\text{fin}}\big(X \times \blacktriangle^{p-j}, (\mathbb{C}^\times)^{p^{\mathfrak{G}}}\big)\big\}}.$$

Let

$$\pi_1 : X \times \blacktriangle^{p-j} \times (\mathbb{C}^\times)^p \longrightarrow X,$$
$$\pi_3 : X \times \blacktriangle^{j-p} \times (\mathbb{C}^\times)^p \longrightarrow (\mathbb{C}^\times)^p,$$

and

$$\pi_{12} : X \times \blacktriangle^{p-j} \times (\mathbb{C}^\times)^p \longrightarrow X \times \blacktriangle^{p-j}$$

denote the projections. By definition, an element $\alpha \in \mathbb{Z}(p)^j_\omega(X)$ is represented by a subanalytic current $T \in \mathcal{I}^{\text{loc}}_{m+p-j}(X \times \blacktriangle^{p-j} \times (\mathbb{C}^\times)^p)^{\mathfrak{G}}$ such that, among other properties, $\pi_{12|_{\text{spt}(T)}}$ is proper. In particular, $\pi_{1|_{\text{spt}(T)}}$ is also a proper map.

We now consider the form $\omega_p = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_p}{z_p} \in \Omega^p(\{\mathbb{C}^\times\}^p)$ which has the following properties:

(a) $\overline{\sigma^*\omega_p} = \omega_p$ and $\overline{\sigma^*\overline{\omega}_p} = \overline{\omega}_p$ where σ is the action of $\mathfrak{G}$ on the ambient manifold.

(b) If $\mathbf{T}^p \subset \mathbb{C}^{\times p}$ is the compact torus $\mathbf{T}^p := S^1 \times \cdots \times S^1$ then $\overline{\omega}_{p|\mathbf{T}^p} = (-1)^p\, \omega_{p|\mathbf{T}^p}$.

(c) If

$$S \in \operatorname{Im}\big\{\oplus_{j=1}^p \mathcal{I}^\omega_{\text{fin}}(X \times \blacktriangle^{p-j}, \{\mathbb{C}^\times \times \cdots \times \{1\} \times \cdots \times \mathbb{C}^\times\}) \to \mathcal{I}^\omega_{\text{fin}}(X \times \blacktriangle^{p-j}, (\mathbb{C}^\times)^p)\big\}$$

then $S \llcorner \pi_3^*\omega_p = 0$ and $S \llcorner \pi_3^*\overline{\omega}_p = 0$.

It follows that if $\alpha = [T]$ as above, then

$$\alpha \llcorner \pi_3^*\overline{\omega}_p := T \llcorner \pi_3^*\overline{\omega}_p \in {}'\mathcal{D}_{m-j}\big(X \times \blacktriangle^{p-j} \times (\mathbb{C}^\times)^p\big)^{\mathfrak{G}}$$

is a well-defined deRham current of dimension $m - j$, which is invariant under the Galois group action, and so is

$$\tau(\alpha) := (-1)^{j(m+p+1)} \pi_{1\#}\big(\alpha \llcorner \pi_3^*\overline{\omega}_p\big) \in {}'\mathcal{D}^j(X)^{\mathfrak{G}}. \tag{4.2}$$

Proposition 4.1. *The assignment* $\alpha \in \mathbb{Z}(p)^j_\omega(X) \longmapsto \tau(\alpha) \in {}'\mathcal{D}^j(X)^{\mathfrak{S}}$ *defines a map of complexes* $\tau : \mathbb{Z}(p)^*_{\omega|X} \longrightarrow {}'\mathcal{D}^{*\mathfrak{S}}_{|X}$. *In the level of hypercohomology, this map realizes the composition* $j' \circ \varphi : H^{n,p}_{Br}(X;\underline{\mathbb{Z}}) \to H^n_{DR}(X)^{\mathfrak{S}}$ *displayed in* (4.1).

Proof. Given $T \in \mathcal{I}^\omega_{\text{fin}}(X \times \blacktriangle^{p-j}, (\mathbb{C}^\times)^p)^{\mathfrak{S}}$, by definition one can find an open subset $\blacktriangle^{p-j} \subset U \subset \Delta^{p-j}(\mathbb{R})$ and a (closed) current $\widehat{T} \in \mathcal{I}^\omega_{\text{fin}}(X \times U, (\mathbb{C}^\times)^p)^{\mathfrak{S}}$ such that $T = \widehat{T} \cap [\![X \times \blacktriangle^{p-j}]\!]$. Hence, using the boundary formula for the intersection [13, Th. 5.8(9)], we obtain

$$\begin{aligned}
\partial T &= \partial\big(\widehat{T} \cap [\![X \times \blacktriangle^{p-j}]\!]\big)\\
&= (-1)^{2p}(\partial\widehat{T}) \cap [\![X \times \blacktriangle^{p-j}]\!] + \widehat{T} \cap \partial[\![X \times \blacktriangle^{p-j}]\!] = \widehat{T} \cap \partial\big([\![X]\!] \times [\![\blacktriangle^{p-j}]\!]\big)\\
&= (-1)^m \widehat{T} \cap \big([\![X]\!] \times \partial[\![\blacktriangle^{p-j}]\!]\big) = (-1)^m \sum_{s=0}^{p-j} \widehat{T} \cap \big([\![X]\!] \times \iota_{s\#}[\![\blacktriangle^{p-j-1}]\!]\big)\\
&= (-1)^m d_{p-j}(T) = (-1)^m(-1)^j d^j_\omega(T) = (-1)^{m+j} d^j_\omega(T),
\end{aligned}$$

where d_{p-j} is the simplicial differential and d^j_ω is the differential in the complex $\mathbb{Z}(p)_\omega$. See Definition 3.4(b).

Therefore,

$$\begin{aligned}
d\tau([T]) &= (-1)^{j+1}\partial\tau([T]) = (-1)^{j+1}(-1)^{j(m+p+1)}\partial\pi_{1\#}(T \llcorner \pi_3^*\overline{\omega}_p)\\
&= (-1)^{j+1}(-1)^{j(m+p+1)}\pi_{1\#}(\partial\{T \llcorner \pi_3^*\overline{\omega}_p\})\\
&= (-1)^{j+1}(-1)^{j(m+p+1)}(-1)^p\pi_{1\#}(\{\partial T\} \llcorner \pi_3^*\overline{\omega}_p)\\
&= (-1)^{j+1}(-1)^{j(m+p+1)}(-1)^p(-1)^{m+j}\pi_{1\#}(\{d^j_\omega(T)\} \llcorner \pi_3^*\overline{\omega}_p)\\
&= (-1)^{(j+1)(m+p+1)}\pi_{1\#}(\{d^j_\omega(T)\} \llcorner \pi_3^*\overline{\omega}_p)\\
&= \tau\big(d^j_\omega[T]\big).
\end{aligned}$$

□

5. The cycle map from motivic cohomology

5.1. Motivic cohomology. We start with a brief introduction to Voevodsky's motivic complexes $\mathbb{Z}(p)_{\mathcal{M}}$ in the category of smooth varieties over a field F, closely following [21]. The similarities between these constructions and the approach taken with analytic currents in the previous sections will become evident.

Definition 5.1. Let $X = \coprod_i X_i$ and Y be smooth algebraic varieties over a field F, with X_i irreducible. For each i define $\mathsf{Cor}(X_i, Y)$ as the free abelian group on the irreducible $W \subset X_i \times Y$ that are finite and surjective onto X_i. Define $\mathsf{Cor}(X, Y) := \oplus_i \mathsf{Cor}(X_i, Y)$. The finiteness and surjectivity condition guarantee that if $W \subset X \times Y$ and $V \subset Y \times Z$ are irreducible finite correspondences, then $W \times Z$ and $X \times V$ intersect properly. One defines $V \circ W$ as $p_{13*}((W \times Z) \cap (X \times V))$, where $(W \times Z) \cap (X \times V)$

is the intersection of algebraic cycles [12], and $p_{13} : X \times Y \times Z \to X \times Z$ is the projection. As shown in [21, Lecture 1], this induces an associative composition pairing

$$\circ : \mathsf{Cor}(X, Y) \times \mathsf{Cor}(Y, Z) \longrightarrow \mathsf{Cor}(X, Z).$$

(a) The additive category Cor_F of *finite correspondences* has smooth varieties over F as objects and $\mathsf{Cor}(X, Y)$ as the morphisms from X to Y, with the empty variety $\emptyset$ as the zero object and disjoint union as coproduct. Compare with Propositions 2.5 and 2.6, and Definition 2.7.

(b) A *presheaf with transfers* is a (contravariant) additive functor $\mathcal{F} : \mathsf{Cor}_F^{\mathrm{op}} \to \mathsf{Ab}$. We denote by $\mathsf{PST}(F)$ the functor category with presheaves with transfers as objects and natural transformations as morphisms. This is an abelian category with enough injectives and projectives; see [21, Lecture 2].

Example 5.2. Let X be a smooth variety over F.

(a) The presheaf with transfers represented by X is denoted by $\mathbb{Z}_{\mathrm{tr}}(X)$. In other words, $\mathbb{Z}_{\mathrm{tr}}(X) : U \mapsto \mathbb{Z}_{\mathrm{tr}}(X)(U) := \mathsf{Cor}(U, X)$.

(b) We use $\mathbb{G}_{\mathrm{m}}$ to denote the pointed multiplicative algebraic group $(\mathbb{G}_{\mathrm{m}}, 1)$ defined by $\mathbb{G}_{\mathrm{m}}(R) = R^{\times}$, where $R^{\times}$ is the group of units of the F-algebra R. In the context of $\mathfrak{G}$-analytic spaces, the realization of $\mathbb{G}_{\mathrm{m}}$ is $\mathbb{G}_{\mathrm{m}}(\mathbb{C}) = \mathbb{C}^{\times}$ with the analytic topology and endowed with the action of $\mathfrak{G}$ given by complex conjugation.

(c) Given pointed varieties $(X_1, x_1), \dots, (X_n, x_n)$ one defines $\mathbb{Z}_{\mathrm{tr}}(X_1 \wedge \cdots \wedge X_n)$ in the same fashion as in the analytic case (3.3). In other words,

$$\begin{aligned} &\mathbb{Z}_{\mathrm{tr}}(X_1 \wedge \cdots \wedge X_n) \\ &\quad = \mathrm{Coker}\Big\{ \bigoplus_{j=1}^{n} \mathbb{Z}_{\mathrm{tr}}(X_1 \times \cdots \times \{x_j\} \times \cdots \times X_n) \to \mathbb{Z}_{\mathrm{tr}}(X_1 \times \cdots \times X_n) \Big\}. \end{aligned}$$

In particular, given $p \geq 1$, define $\mathbb{Z}_{\mathrm{tr}}(\bigwedge^p X) := \mathbb{Z}_{\mathrm{tr}}(X \wedge \cdots \wedge X)$.

The *standard algebraic n-simplex* over F is the affine variety

$$\Delta^n = \mathrm{Spec}\, F[x_0, \dots, x_n] / \langle x_0 + \cdots + x_n - 1 \rangle.$$

In particular, given any extension K of F the K-valued points $\Delta^n(K)$ corresponds to the hyperplane in K^{n+1} given by the equation $x_0 + \cdots + x_n = 1$. The collection of these algebraic simplices forms the standard cosimplicial variety $\Delta^\bullet$.

Given a presheaf with transfers $\mathcal{F}$ define a simplicial presheaf with transfers $\Delta_\bullet \mathcal{F}$ by $\Delta_n \mathcal{F} : U \mapsto \mathcal{F}(U \times \Delta^n)$. Denote the corresponding (chain) complex of presheaves with transfers by $\Delta_* \mathcal{F}$ and let $\Delta^* \mathcal{F}$ denote the associated (negatively graded) complex with $\Delta^n \mathcal{F} = \Delta_{-n} \mathcal{F}$.

Definition 5.3. The *motivic complex or weight* p is the complex of presheaves with transfers defined as $\mathbb{Z}(p)^*_{\mathcal{M}} := \Delta^*\mathbb{Z}_{\mathrm{tr}}(\bigwedge^p \mathbb{G}_{\mathrm{m}})[-p]$. In particular, given a smooth algebraic variety U over F and $-\infty < j \leq p$, one has

$$\begin{aligned}
&\mathbb{Z}(p)^j_{\mathcal{M}}(U)\\
&= \mathbb{Z}_{\mathrm{tr}}\big(\textstyle\bigwedge^p \mathbb{G}_{\mathrm{m}}\big)\big(U \times \Delta^{p-j}\big)\\
&= \frac{\mathsf{Cor}\big(\Delta^{p-j} \times U, \mathbb{G}_{\mathrm{m}}^p\big)}{\mathrm{Im}\,\big(\oplus_{i=1}^p \mathsf{Cor}\big(\Delta^{p-j} \times U, \mathbb{G}_{\mathrm{m}} \times \cdots \times \{1\} \times \cdots \times \mathbb{G}_{\mathrm{m}}\big) \to \mathsf{Cor}\big(\Delta^{p-j} \times U, \mathbb{G}_{\mathrm{m}}^p\big)\big)}.
\end{aligned}$$

Compare with Definitions 3.4 and 3.19.

Remark 5.4. It is shown in [21, Cor. 3.3] that, given any smooth variety X over F the restriction $\mathbb{Z}(p)^*_{\mathcal{M}|X}$ of $\mathbb{Z}(p)^*_{\mathcal{M}}$ to the (small) Zariski site X_{Zar} of X is indeed a complex of sheaves in the Zariski topology of X.

Our approach to ordinary equivariant cohomology expressed in Theorem 3.9 was designed to provide a topological perspective on the following definition.

Definition 5.5 ([21]). Given a smooth variety over the field F, the *motivic cohomology* groups $H^n_{\mathcal{M}}(X, \mathbb{Z}(p))$ of X are defined as the hypercohomology groups $H^n_{\mathcal{M}}(X, \mathbb{Z}(p)) := \mathbb{H}^n(X_{\mathrm{Zar}}; \mathbb{Z}(p)^*_{\mathcal{M}|X})$.

5.2. Real varieties and cycle maps. In this section we present the desired cycle map, utilizing the terminology introduced in Definition 3.4.

Lemma 5.6. *The assignment $X \to (X(\mathbb{C}), \sigma)$ of the analytic space $X(\mathbb{C})$ together with the complex conjugation involution $\sigma : X(\mathbb{C}) \to X(\mathbb{C})$ induces a morphism of sites*

$$\pi : \big(\mathfrak{S}\text{-}\mathsf{Man}_\omega\big)_{eq} \to \big(\mathsf{Sm}_{\mathbb{R}}\big)_{Zar}. \tag{5.1}$$

Given $X \in \mathsf{Sm}_{\mathbb{R}}$, we also denote by

$$\pi : X(\mathbb{C})_{eq} \to X_{Zar} \tag{5.2}$$

the induced morphism between the corresponding small sites.

Proof. Let $u : \mathsf{Sm}_{\mathbb{R}} \to \mathfrak{S}\text{-}\mathsf{Man}_\omega$ denote the functor that sends X to $(X(\mathbb{C}), \sigma)$, and $f : X \to Y$ to $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$.

Let $X = \bigcup_{i \in I} U_i$ be a Zariski open cover of X, which we denote by $\{U_i \to X\}$. It is clear that $\{U(\mathbb{C})_i \to X(\mathbb{C})\}$ is an open cover of $X(\mathbb{C})$ in the analytic topology by $\mathfrak{S}$-invariant open subsets, since each U_i is a real open subvariety of X. In other words, the functor u sends covering families in $\mathsf{Sm}_{\mathbb{R}}$ to covering families in $\mathfrak{S}\text{-}\mathsf{Man}_\omega$.

Given $\{U_i \to U\}$ in $\mathsf{Cov}(\mathsf{Sm}_{\mathbb{R}})$ and a morphism $f : V \to U$ in $\mathsf{Sm}_{\mathbb{R}}$, then $U_i \times_U V = f^{-1}(U_i)$ is a Zariski open subset of V. It follows from [2, Exposé XII, §1.2] that $u(U_i \times_U V) = (U_i \times_U V)(\mathbb{C}) = f^{-1}(U_i)(\mathbb{C}) = f_{\mathbb{C}}^{-1}(U_i(\mathbb{C})) = U_i(\mathbb{C}) \times_{U(\mathbb{C})} V(\mathbb{C}) = u(U_i) \times_{u(U)} u(V)$.

This shows that u induces the desired morphism of sites. See [25, Def. 1.2.2] and [24, Tag 00X0]. $\square$

We now present our final result, where the cycle map from motivic cohomology is realized in the level of complexes.

Theorem 5.7. *Given a smooth real variety X, one has map of complexes of Zariski sheaves $\mathsf{c}_X : \mathbb{Z}(p)_{\mathcal{M}|X} \longrightarrow R\pi_*\mathbb{Z}(p)_{\omega|X(\mathbb{C})}$ induced by $\pi : X(\mathbb{C})_{\mathrm{eq}} \to X_{\mathrm{Zar}}$ (5.2) and natural on X. This map induces natural bigraded ring homomorphisms*

$$\mathsf{c}_X : H^*_{\mathcal{M}}\big(X, \mathbb{Z}(\bullet)\big) \longrightarrow H^{*,\bullet}_{\mathrm{Br}}(X; \underline{\mathbb{Z}}),$$

from motivic cohomology to ordinary $RO(\mathfrak{S})$-graded equivariant cohomology.

Proof. Let $U \in \mathsf{Sm}_{\mathbb{R}}$ be a smooth real variety and let $\Gamma \subset U \times \Delta^{p-j} \times (\mathbb{G}_{\mathrm{m}})^p$ be a finite correspondence representing an element in $\mathbb{Z}(p)^j_{\mathcal{M}}(U)$. It suffices to assume that Γ is irreducible.

Let $\Gamma(\mathbb{C})$ be the complex analytic subvariety of $U(\mathbb{C}) \times \Delta^{p-j}(\mathbb{C}) \times \mathbb{C}^{\times p}$ associated to Γ. As explained in Example 2.3(ii)), $\Gamma(\mathbb{C})$ represents an element $[\Gamma(\mathbb{C})] \in \mathcal{J}^{\omega}_{\mathsf{fin}}(U(\mathbb{C}) \times \Delta^{p-j}(\mathbb{C}), \mathbb{C}^{\times p})$ invariant under the action of the Galois group $\mathfrak{S}$.

The inclusion $\iota : U(\mathbb{C}) \times \blacktriangle^{p-j} \hookrightarrow U(\mathbb{C}) \times \Delta^{p-j}(\mathbb{C})$ induces a pull-back map

$$\iota^* : \mathcal{J}^{\omega}_{\mathsf{fin}}\big(U(\mathbb{C}) \times \Delta^{n-p}(\mathbb{C}), \mathbb{C}^{\times p}\big) \longrightarrow \mathcal{J}^{\omega}_{\mathsf{fin}}\big(U(\mathbb{C}) \times \blacktriangle^{n-p}, \mathbb{C}^{\times p}\big),$$

and we denote $\Gamma_{\blacktriangle} := \iota^*(\Gamma(\mathbb{C}))$. Hence $\Gamma_{\blacktriangle}$ lies in $\mathfrak{S}\text{-}\mathsf{Man}^{\mathsf{fin}}_{\omega}(U(\mathbb{C}) \times \blacktriangle^{p-n}, \mathbb{C}^{\times p})$ and represents an element in $\mathbb{Z}(p)^j_{\mathrm{Br}}(U(\mathbb{C}))$.

Furthermore, using the naturality and compatibility of inclusions of faces

$$\begin{tikzcd} \blacktriangle^{n-p-1} \arrow[r, hook] \arrow[d, hook] & \blacktriangle^{n-p} \arrow[d, hook] \\ \Delta^{n-p-1}(\mathbb{C}) \arrow[r, hook] & \Delta^{n-p}(\mathbb{C}) \end{tikzcd}$$

one shows that the assignment $\Gamma \mapsto \Gamma_{\blacktriangle}$ gives a map of complexes

$$c_U : \mathbb{Z}(p)^*_{\mathcal{M}}(U) \longrightarrow \mathbb{Z}(p)^*_{\mathrm{Br}}\big(U(\mathbb{C})\big) = \pi_*\big(\mathbb{Z}(p)^*_{\mathrm{Br}}\big)(U)$$

natural on U. Taking an injective resolution $0 \to \mathbb{Z}(p)^*_{\mathrm{Br}|X} \to I^*$ gives the desired morphism of complexes.

To end the proof we first recall that the functor $U \mapsto U(\mathbb{C})$ is compatible with (fibered) products, as explained in the proof of Lemma 5.6. Now, one can simply repeat the simplicial arguments used in [21, Construction 3.11] to obtain a product $\mathbb{Z}(p)_\omega \otimes \mathbb{Z}(q)_\omega \longrightarrow \mathbb{Z}(p+q)_\omega$ that gives the multiplicative structure in equivariant cohomology. The same argument is used to give the multiplication in motivic cohomology, and this yields a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}(p)^*_{\mathcal{M}} \otimes \mathbb{Z}(q)^*_{\mathcal{M}} \arrow[r] \arrow[d] & \mathbb{Z}(p+q)^*_{\mathcal{M}} \arrow[d] \\
R\pi_*\mathbb{Z}(p)_\omega \otimes^{\mathbb{L}} R\pi_*\mathbb{Z}(q)_\omega \arrow[r] & R\pi_*\mathbb{Z}(p+q)_\omega.
\end{tikzcd} \tag{5.3}$$

The result follows. □

5.3. Example: Number fields and Milnor K-theory. Given a number field F, let $\Gamma_\mathbb{R}$ and $\Gamma_\mathbb{C}$ denote the sets of real and complex embeddings of F, respectively. Since the complex embeddings come in conjugate pairs, one can think of $\Gamma_\mathbb{C}$ as a finite $\mathfrak{S}$-set $\mathfrak{S}$-isomorphic to $\Gamma_\mathbb{C}^+ \times \mathfrak{S}$, where $\Gamma_\mathbb{C}^+$ contains one chosen embedding from each pair of conjugate embeddings in $\Gamma_\mathbb{C}$.

Consider the real variety $X := \operatorname{Spec}(F \otimes_\mathbb{Q} \mathbb{R})$. The space $X(\mathbb{C})$ is isomorphic to $\Gamma_\mathbb{R} \coprod \Gamma_\mathbb{C}$ as $\mathfrak{S}$-spaces. Denoting $H^{n,p}_{\mathrm{Br}}(F;\underline{\mathbb{Z}}) := H^{n,p}_{\mathrm{Br}}(X(\mathbb{C});\underline{\mathbb{Z}})$ one has:

$$\begin{aligned}
H^{n,p}_{\mathrm{Br}}(F;\underline{\mathbb{Z}}) &= H^{n,p}_{\mathrm{Br}}(\Gamma_\mathbb{R};\underline{\mathbb{Z}}) \times H^{n,p}_{\mathrm{Br}}(\Gamma_\mathbb{C};\underline{\mathbb{Z}}) \equiv H^{n,p}_{\mathrm{Br}}(*;\underline{\mathbb{Z}})^{\Gamma_\mathbb{R}} \times H^{n,p}_{\mathrm{Br}}(\mathfrak{S};\mathbb{Z})^{\Gamma_\mathbb{C}^+} \\
&\cong H^{n,p}_{\mathrm{Br}}(*;\underline{\mathbb{Z}})^s \times H^{n,p}_{\mathrm{Br}}(\mathfrak{S};\underline{\mathbb{Z}})^t \\
&\cong H^{n,p}_{\mathrm{Br}}(*;\underline{\mathbb{Z}})^s \times H^{n}_{\mathrm{sing}}(*;\mathbb{Z})^t,
\end{aligned} \tag{5.4}$$

where $\Gamma_\mathbb{R} = \{\varphi_1, \ldots, \varphi_s\}$ and $\Gamma_\mathbb{C}^+ = \{\eta_1, \ldots, \eta_t\}$, and the last isomorphism follows from the general isomorphism $H^V_G(X \times G/K;\underline{\mathbb{Z}}) \cong H^{\mathrm{Res}^G_K(V)}_K(X;\underline{\mathbb{Z}})$ for $RO(G)$-graded ordinary cohomology, along with the fact that ordinary equivariant cohomology is singular cohomology with the group is trivial. In particular,

$$H^{1,1}_{\mathrm{Br}}(*;\underline{\mathbb{Z}}) \equiv (\mathbb{Z}^\times)^{\Gamma_\mathbb{R}} \cong (\mathbb{Z}^\times)^s. \tag{5.5}$$

See Proposition 3.21.

Consider the composition $F^\times \times \Gamma_\mathbb{R} \xrightarrow{\mathrm{ev}} \mathbb{R}^\times \xrightarrow{\mu} \mathbb{Z}^\times$, where ev is the evaluation map $\mathrm{ev}(x,\varphi) = \varphi(x)$ and $\mu(x) = \frac{x}{|x|}$. Taking adjoints one gets a homomorphism

$$\mathbf{c} : F^\times \equiv H^1_\mathcal{M}(\operatorname{Spec} F, \mathbb{Z}(1)) \longrightarrow H^{1,1}_{\mathrm{Br}}(F;\underline{\mathbb{Z}}) \cong (\mathbb{Z}^\times)^{\Gamma_\mathbb{R}} \tag{5.6}$$

which is precisely the cycle map given in Theorem 5.7 Since $\oplus_{p\geq 0} H^{p,p}_{\mathrm{Br}}(F;\underline{\mathbb{Z}})$ is a graded commutative ring this map induces a homomorphism of graded rings

$$\varrho : T(F^\times) \to \bigoplus_{p\geq 0} H^{p,p}_{\mathrm{Br}}(F;\underline{\mathbb{Z}}), \tag{5.7}$$

where $T(F^\times)$ is the tensor algebra of $F^\times$.

Using the cup product one gets

$$\begin{array}{ccccc} F^\times \otimes F^\times & \xrightarrow{\varrho} & H^{1,1}_{\mathrm{Br}}(F;\underline{\mathbb{Z}}) \otimes H^{1,1}_{\mathrm{Br}}(F;\underline{\mathbb{Z}}) & \xrightarrow{\cup} & H^{2,2}_{\mathrm{Br}}(F;\underline{\mathbb{Z}}) \\ & & \| & & \| \\ & & (\mathbb{Z}^\times)^{\Gamma_{\mathbb{R}}} \otimes (\mathbb{Z}^\times)^{\Gamma_{\mathbb{R}}} & \longrightarrow & (\mathbb{Z}^\times \otimes_{\mathbb{Z}} \mathbb{Z}^\times)^{\Gamma_{\mathbb{R}}}, \end{array}$$

where the right vertical equality comes from Proposition 3.21. Now, note that for $1 \neq x \in \mathbb{R}^\times$ either x or $1-x$ is positive and this implies that for $a \in F^\times$ one has $\cup \circ \varrho\{a \otimes (1-a)\} = 0$. It follows that ϱ descends to a homomorphism

$$\bar{\varrho} : K^M_*(F) \equiv \oplus_{p \geq 0} H^p_{\mathcal{M}}\big(\operatorname{Spec} F, \mathbb{Z}(p)\big) \longrightarrow \oplus_{p\geq 0} H^{p,p}_{\mathrm{Br}}(F;\underline{\mathbb{Z}}), \tag{5.8}$$

from the Milnor K-theory ring of F to the "diagonal" subring of the ordinary bigraded equivariant cohomology of F.

Remark 5.8. Bass and Tate have shown that

$$K^M_*(\mathbb{R})/2K^M_{\geq 1}(\mathbb{R}) \cong \mathbb{Z}[\varepsilon] \cong \oplus_{p\geq 0} H^{p,p}_{\mathrm{Br}}(\mathbb{R};\underline{\mathbb{Z}}),$$

and their isomorphism is realized by the cycle map described above.

A. Points, sheaves and hypercohomology on G-Man_ω

A.1. Points on G-Man_ω. Denote by G-$\mathsf{Man}^\wedge_\omega$ and G-$\mathsf{Man}^\sim_\omega$, respectively, the category of presheaves and sheaves of sets on the site G-Man_ω, as in Definition 3.4. Fix a complete G-universe $\mathbf{U}$, which can be an orthogonal representation isomorphic to a countably infinite direct sum of the regular representation of G.

For each subgroup $H \leq G$, each H-subrepresentation $V \subset \operatorname{Res}^G_H(\mathbf{U})$ and $0 \leq k \leq v = \dim V$, consider the collection

$$\Phi_H(V,k) := \{G \times_H D_{1/r}(V,k) \mid r \in \mathbb{N}\}$$

of objects in G-Man_ω, where $D_{1/r}(V,k) \subset V$ is the intersection of an open disk of radius $1/r$ centered around the origin with an H-invariant "corner" isomorphic to $\mathbb{R}^k_+ \times \mathbb{R}^{v-k}$. The collection $\Phi_H(V,k)$ induces a functor

$$p_H(V,k)^* : G\text{-}\mathsf{Man}^\wedge_\omega \to \mathsf{Sets}; \quad \mathcal{P} \mapsto \operatorname*{colim}_{r\in\mathbb{N}} \mathcal{P}(G \times_H D_{1/r}(V,k)).$$

The functor $p_H(V,k)^*$ preserves finite projective limits and arbitrary inductive limits of sheaves. It follows that $p_H(V,k)^*$ admits a right-adjoint $p^H(V,k)_* : \mathsf{Sets} \to G$-$\mathsf{Man}^\sim_\omega$ (see [1, IV, Cor. 1.7]) and the pair of adjoint functors defines a morphism of topoi $p_H(V,k) : \mathsf{Sets} \to G$-$\mathsf{Man}^\sim_\omega$, i.e. a *point* of the topos G-$\mathsf{Man}^\sim_\omega$.

The following fact is easy to verify.

Lemma A.1. *The set* $\Phi := \{p_H(V,k) \mid H \leq G,\ V \subset \mathbf{U}, 0 \leq k \leq n \in \mathbb{N}\}$ *forms a conservative set of points in the topos* $G\text{-}\mathsf{Man}_\omega^{\sim}$.

In this case, the site is said to have *enough points*, in other words, the collection $\{p_H(V,k)^*\}$ detects isomorphisms. The existence of a conservative family of points yields a canonical *Godement resolution* of abelian sheaves or sheaves of modules over a ring R. A succinct account of the construction can be found in [17, IV-§2] and the results we need are summarized in the following statement, where we also use $G\text{-}\mathsf{Man}_\omega^{\sim}$ to denote the category of abelian sheaves on $G\text{-}\mathsf{Man}_\omega$.

Corollary A.2. *There is a canonical functor*

$$\mathbf{G} := G_{\mathbb{Z}}^* : G\text{-}\mathsf{Man}_\omega^{\sim} \longmapsto \mathbf{C}^+(G\text{-}\mathsf{Man}_\omega^{\sim}),$$

where $\mathbf{C}^+(G\text{-}\mathsf{Man}_\omega^{\sim})$ *denotes the category of bounded below complex of abelian sheaves on the site* $G\text{-}\mathsf{Man}_\omega$. *This comes with a natural augmentation functor* $\mathcal{F} \to \mathbf{G}(\mathcal{F})$ *that gives a resolution of* $\mathcal{F}$ *by flabby sheaves. (The* canonical Godement resolution *for a site with enough points.)*

Remark A.3. Given a complex of abelian sheaves $\mathcal{F}^*$ in $G\text{-}\mathsf{Man}_\omega$, we obtain a double complex by applying the Godement resolution to each sheaf in the complex and use the same notation $\mathbf{G}(\mathcal{F}^*)$ to denote the total complex associated to this double complex. The resulting map $\mathcal{F}^* \to \mathbf{G}(\mathcal{F}^*)$ is a quasi-isomorphism of complexes of sheaves and, since $\mathbf{G}(\mathcal{F}^*)$ is a complex of flabby sheaves, one can calculate the hypercohomology of any $X \in G\text{-}\mathsf{Man}_\omega$ with values in $\mathcal{F}^*$ as

$$\mathbb{H}^n(X_{\mathrm{eq}}; \mathcal{F}^*) \cong H^n\big(\Gamma\big(X, \mathbf{G}(\mathcal{F}^*)\big)\big). \tag{A.1}$$

A.2. Čech hypercohomology. Given $X \in G\text{-}\mathsf{Man}_\omega$, the set of coverings of X forms a directed set $\mathsf{Cov}(X)$ under the partial order given by refinements of coverings. The rth Čech hypercohomology of X with coefficients in a complex of abelian presheaves $\mathcal{P}^*$ on $G\text{-}\mathsf{Man}_\omega$ is defined as the colimit

$$\check{\mathbb{H}}^r(X_{\mathrm{eq}}; \mathcal{P}^*) := \operatorname*{colim}_{\mathcal{U} \in \mathsf{Cov}(X)} \check{\mathbb{H}}^r(\mathcal{U}; \mathcal{P}^*), \tag{A.2}$$

where $\check{\mathbb{H}}^r(\mathcal{U}; \mathcal{P}^*)$ is defined as the cohomology of the complex of abelian groups $\mathrm{Tot}\big(\Gamma\big(X, \check{\mathcal{C}}^*_{\mathcal{P}^*|X,\mathcal{U}}\big)\big)$; see Definition 3.12(d).

Proposition A.4. *Let* $\mathcal{P}^*$ *and* X *be as above. The natural map*

$$\psi : \check{\mathbb{H}}^r(X_{\mathrm{eq}}; \mathcal{P}^*) \to \mathbb{H}^r(X_{\mathrm{eq}}; a\mathcal{P}^*_{|X})$$

is an isomorphism.

Proof. The proof is essentially the same as in the non-equivariant case; see [5]. Since manifolds are paracompact, one can formulate a suitable notion of equivariant paracompactness (for coverings in $\mathsf{Cov}(U)$), and the usual arguments apply. □

A.3. On complexes of presheaves with homotopy invariant cohomology. The main purpose of this final section is to provide detailed proofs of the key steps in the proof of Theorem 3.11, namely, Propositions 3.14 and 3.16, and Theorem 3.15.

We first introduce a family of coverings of the interval $I = [0,1] \subset \mathbb{R}$, as follows. For each $n \in \mathbb{N}$ and $i = 1, \dots, n$, define $J_i^n := \big(\frac{i-1}{n} - \varepsilon(n), \frac{i}{n} + \varepsilon(n)\big) \cap I$, where $\varepsilon(n) = \frac{1}{100n^2}$ (or any sufficiently small $\varepsilon(n)$). Define $\mathcal{J}^n := \{J_i^n \mid i = 1, \dots, n\}$ and verify that $\mathcal{J}^{n+1}$ refines $\mathcal{J}^n$ for all $n \in \mathbb{N}$. Using Lebesgue's number lemma, one sees that the directed family $\{\mathcal{J}^n \mid n \in \mathbb{N}\}$ forms a cofinal family of coverings of I.

Given any covering $\mathcal{U} = \{U_\sigma \mid \sigma \in \Lambda\} \in \mathsf{Cov}(X)$, denote

$$\mathcal{J}^n(\mathcal{U}) := \{U_\sigma \times J_i^n \mid \sigma \in \Lambda, \text{ and } i = 1, \dots, n\}, \tag{A.3}$$

and observe that the collection of coverings of the form $\mathcal{J}^n(\mathcal{U})$ forms a cofinal family in $\mathsf{Cov}(X \times I)$. When $\mathcal{U} := \{X\}$, simply denote $\mathcal{J}^n(X) := \mathcal{J}^n(\mathcal{U})$.

Lemma A.5. *Given a homotopy invariant abelian presheaf $\mathcal{P}$ on G-Man_ω and $X \in G$-Man_ω, then for $n > 0$ the Čech cohomology of the cover $\mathcal{J}^n(X)$ is given by*

$$\check{H}^q\big(\mathcal{J}^n(X); \mathcal{P}\big) = \begin{cases} 0, & q > 0, \\ \mathcal{P}(X \times I) \cong \mathcal{P}(X), & q = 0. \end{cases}$$

Proof. This follows from a routine calculation. □

As a consequence, we obtains the following *tube lemma*.

Corollary A.6. *Let $a\mathcal{P}$ be the abelian sheaf on G-Man_ω associated to a separated homotopy invariant abelian presheaf $\mathcal{P}$. Given an element $\hat{\mathbf{s}} \in a\mathcal{P}(X \times I)$ and $x_0 \in X$, one can find an open G-invariant neighborhood U of x_0 and $\bar{\sigma} \in \mathcal{P}(U \times I)$ such that $\iota(\bar{\sigma}) = \rho(\hat{\mathbf{s}})$, where ι and ρ are the natural homomorphisms*

$$\mathcal{P}(U \times I) \xhookrightarrow{\iota} a\mathcal{P}(U \times I) \xleftarrow{\rho} a\mathcal{P}(X \times I).$$

Proof. For each $t \in I$ one can find a G-invariant neighborhood $U_t \times N_t$ of $(x_0, t) \in X \times I$, together with $\mathbf{s}_t \in \mathcal{P}(U_t \times N_t)$ such that, under the induced maps

$$\mathcal{P}(U_t \times N_t) \xhookrightarrow{\iota} a\mathcal{P}(U_t \times N_t) \xleftarrow{\rho_t} a\mathcal{P}(X \times I)$$

one has $\iota(\mathbf{s}_t) = \rho_t(\hat{\mathbf{s}})$. One can find $n > 0$ sufficiently large so that the finite cover $\mathcal{J}^n$ refines the cover $\{N_t\}_{t \in I}$ of the interval. For each $i = 1, \dots, n$ find $t_i \in I$ such that $J_i^n \subset N_{t_i}$ and define $U := \bigcap_{i=1}^n U_{t_i}$. Finally, denote by $\sigma_i \in \mathcal{P}(U \times J_i^n)$ the restriction of $\mathbf{s}_{t_i}$ to $U \times J_i^n$.

The collection $\check{\sigma} = (\sigma_i)$ belongs to $\check{C}^0(\mathcal{J}^n(U); \mathcal{P})$. Note that the Čech differential $\check{C}^0(\mathcal{J}^n(U); \mathcal{P}) \xrightarrow{\delta} \check{C}^1(\mathcal{J}^n(U); \mathcal{P})$ sends $\check{\sigma}$ to $\delta\check{\sigma} = (\gamma_{ij})$ where

$$\gamma_{ij} = \sigma_j|_{U \times (J_i^n \cap J_j^n)} - \sigma_i|_{U \times (J_i^n \cap J_j^n)} \in \mathcal{P}\big(U \times \big(J_i^n \cap J_j^n\big)\big).$$

Now,

$$\begin{aligned}\iota(\delta\breve{\sigma}) &= \iota(\sigma_j|_{U\times(J_i^n\cap J_j^n)}) - \iota(\sigma_i|_{U\times(J_i^n\cap J_j^n)})\\ &= \rho(\hat{\mathbf{s}})|_{U\times(J_i^n\cap J_j^n)} - \rho(\hat{\mathbf{s}})|_{U\times(J_i^n\cap J_j^n)} = 0.\end{aligned}$$

Since $\mathcal{P}$ is a separated presheaf, ι is injective and one concludes that $\delta\breve{\sigma} = 0$. Therefore, $\breve{\sigma}$ represents a class $\bar{\sigma}$ in $\check{H}^0(\mathcal{J}^n(U);\mathcal{P}) = \mathcal{P}(U\times I) \cong \mathcal{P}(U)$, according to Lemma A.5, with the property that $\bar{\sigma}_{|U\times J_i^n} = \sigma_i$, for all $i = 1,\dots,n$. As a consequence, one has

$$\iota(\bar{\sigma})_{|U\times J_i^n} = \iota(\sigma_i) = \rho(\hat{\mathbf{s}}) = \hat{\mathbf{s}}_{|U\times J_i^n}.$$

for all i. Since $\mathcal{J}^n(U) = \{U\times J_i^n\}$ is a cover of $U\times I$ and $a\mathcal{P}$ is a sheaf, one concludes that $\iota(\bar{\sigma}) = \rho(\hat{\mathbf{s}})$. □

Lemma A.7. *Let $\mathcal{S}$ be a homotopy invariant abelian sheaf on G-Man_ω. Given $X \in G$-Man_ω, let $\pi : X\times I \to X$ denote the projection. Then for all $q > 0$ one has*

$$R^q\pi_*\left(\mathcal{S}_{|X\times I}\right) = 0.$$

Proof. Given $x \in X$ denote $H := G_x$ and let $\mathsf{N}(x)$ denote the set of equivariant open neighborhoods U of x isomorphic to $G\times_H B$, where B is H-equivariantly analytically contractible to a point. In particular, the orbit of x is an equivariant strong deformation retract of U.

The stalk of $R^q\pi_*\left(\mathcal{S}_{|X\times I}\right)$ at x is given by

$$\begin{aligned}R^q\pi_*(\mathcal{S}_{|X\times I})_x &:= \operatorname*{colim}_{U\in\mathsf{N}(x)} H^q\big(\pi^{-1}U;\mathcal{S}_{|\pi^{-1}(U)}\big) = \operatorname*{colim}_{U\in\mathsf{N}(x)} H^q\big(U\times I;\mathcal{S}_{|U\times I}\big)\\ &= \operatorname*{colim}_{U\in\mathsf{N}(x)} \check{H}^q\big(U\times I;\mathcal{S}_{|U\times I}\big),\end{aligned}$$

where the last isomorphism follows from Proposition A.4.

Given $U \in \mathsf{N}(x)$ and $\mathbf{a} \in \check{H}^q(U\times I;\mathcal{S}_{|U\times I})$, one can find a locally finite covering $\mathcal{V}$ of U, $n \in \mathbb{N}$, and $\hat{\mathbf{a}} \in \check{H}^q(\mathcal{J}^n(\mathcal{V});\mathcal{S})$ that represents $\mathbf{a}$. In other words, the natural map $\rho : \check{H}^q(\mathcal{J}^n(\mathcal{V});\mathcal{S}) \to \check{H}^q(U\times I;\mathcal{S})$ sends $\hat{\mathbf{a}}$ to $\mathbf{a}$.

Let $W \in \mathsf{N}(x)$ be a neighborhood of x that is contained in the intersection of all (finitely many) elements of $\mathcal{V}$ containing x, and consider the covering $\mathcal{W} = \{W\}$ of W. One gets a commutative diagram

$$\begin{tikzcd}
\check{H}^q(\mathcal{J}^n(\mathcal{V});\mathcal{S}) \arrow[r, "\rho"] \arrow[d] & \check{H}^q(U\times I;\mathcal{S}) \cong H^q(U\times I;\mathcal{S}) \arrow[d] \\
\check{H}^q(\mathcal{J}^n(\mathcal{W});\mathcal{S}) \arrow[r, "\rho"'] & \check{H}^q(W\times I;\mathcal{S}) \cong H^q(W\times I;\mathcal{S}).
\end{tikzcd}$$

If $q > 0$, the group on the lower left corner is zero, by Lemma A.5. Hence, for an arbitrary $x \in X$ the stalk of $R^q\pi_*\left(\mathcal{S}_{|X\times I}\right)$ at x is zero and this concludes the proof. □

Corollary A.8. *If $\mathcal{S}$ is a homotopy invariant abelian sheaf on G-Man_ω then for all $n \geq 0$ the presheaf $X \longmapsto H^n(X;\mathcal{S}_{|X})$ is homotopy-invariant.*

Proof. The E_2-term of the Leray spectral sequence associated to the map $\pi : X\times I \to X$ is given by $E_2^{p,q} = H^p(X; R^q\pi_*(\mathcal{S}_{|X\times I})) \Longrightarrow H^{p+q}(X\times I;\mathcal{S}_{|X\times I})$. The previous lemma shows that $E_2^{p,q} = 0$ if $q > 0$. Hence $H^p\left(X\times I;\mathcal{S}_{|X\times I}\right) \cong H^p(X;\pi_*\mathcal{S}_{|X\times I})$. Finally, the homotopy invariance of $\mathcal{S}$ gives $\pi_*\mathcal{S}_{|X\times I} = \mathcal{S}_{|X}$. □

Lemma A.9. *Let $\mathcal{P}^*$ be an abelian presheaf on G-Man_ω and let $H^q_{\mathcal{P}^*}$ denote its qth cohomology presheaf. Then the natural map of complexes $\mathcal{P}^* \to a\mathcal{P}^*$ induces an isomorphism $aH^q_{\mathcal{P}^*} \cong aH^q_{a\mathcal{P}^*}$.*

Proof. This can be directly verified on stalks, using the fact that in the category of abelian groups homology commutes with filtered colimits. □

We now have all the ingredients to prove

Proposition 3.14. *Let $\mathcal{P}$ be a homotopy invariant abelian presheaf on G-Man_ω. Then the associated sheaf $a\mathcal{P}$ is homotopy invariant.*

Proof. Let $\mathcal{P}_0$ denote the subpresheaf of $\mathcal{P}$ defined by

$$\mathcal{P}_0(X) := \operatorname*{colim}_{\mathcal{U}\in\mathsf{Cov}(X)} \ker\left\{\mathcal{P}(X) \to \prod_{\lambda\in\Lambda}\mathcal{P}(U_\lambda)\right\}$$

and let $\mathcal{P}_s := \mathcal{P}/\mathcal{P}_0$ denote the quotient presheaf. This is the separated presheaf associated to $\mathcal{P}$. The following facts are standard:

(i) The quotient map $\mathcal{P} \to \mathcal{P}_s$ induces an isomorphism $a\mathcal{P} \xrightarrow{\cong} a\mathcal{P}_s$ between their respective associated sheaves

(ii) The natural map of presheaves $\mathcal{P}_s \to a\mathcal{P}_s$ is injective.

We next need to show that the presheaf $\mathcal{P}_s$ is homotopy invariant. Given $X\in G$-Man_ω, consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P}_0(X) \arrow[r, "a"] \arrow[d, shift right=1ex, "\pi^*"'] & \mathcal{P}(X) \arrow[r] \arrow[d, shift right=1ex, "\pi^*"', "\cong"] & \mathcal{P}_s(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{P}_0(X\times I) \arrow[r, "b"] \arrow[u, shift right=1ex, "i_0^*"'] & \mathcal{P}(X\times I) \arrow[r] \arrow[u, shift right=1ex, "i_0^*"', "\cong"] & \mathcal{P}_s(X\times I) \arrow[r] & 0.
\end{tikzcd}$$

Since $\pi\circ i_0 = 1_X$, it follows that the maps i_0^* are surjective. The equality $ai_0^* = i_0^*b$ then shows that the leftmost i_0^* is injective, as well. It follows from the five-lemma that the rightmost arrow is an isomorphism and hence $\mathcal{P}_s$ is homotopy invariant.

Using the isomorphism $a\mathcal{P} \cong a\mathcal{P}_s$, it suffices to assume that $\mathcal{P}$ is a separated homotopy invariant presheaf from now on.

Given a G-invariant open subset U of X one gets a commutative diagram

$$\begin{tikzcd} \mathcal{P}(U \times I) \arrow[r, hook, "\imath"] \arrow[d, "i_0^*"', "\cong"] & a\mathcal{P}(U \times I) \arrow[d, "i_0^*"] & a\mathcal{P}(X \times I) \arrow[l, "\rho"'] \arrow[d, "i_0^*"] \\ \mathcal{P}(U) \arrow[r, hook, "\jmath"] & a\mathcal{P}(U) & a\mathcal{P}(X). \arrow[l, "\tau"'] \end{tikzcd}$$

Now, suppose that $\hat{\mathbf{s}} \in \ker\{i_0^* : a\mathcal{P}(X \times I) \to a\mathcal{P}(X)\}$. Given a point $x_0 \in X$ it follows from Corollary A.6 that one can find U as above, along with $\bar{\sigma} \in \mathcal{P}(U \times I)$ satisfying $\imath(\bar{\sigma}) = \rho(\hat{\mathbf{s}})$. Chasing the diagram one obtains

$$\jmath \circ i_0^*(\bar{\sigma}) = i_0^* \circ \imath(\bar{\sigma}) = i_0^*(\rho(\hat{\mathbf{s}})) = \tau \circ i_0^*(\hat{\mathbf{s}}) = 0.$$

Since $\mathcal{P}$ is separated, $\jmath$ is injective and hence $i_0^*(\bar{\sigma}) = 0$. From the homotopy invariance of $\mathcal{P}$ one knows that the leftmost vertical arrow is an isomorphism, thus showing that $\bar{\sigma} = 0$. Hence, $\hat{\mathbf{s}}_{|U\times I} = \rho(\hat{\mathbf{s}}) = \imath(\bar{\sigma}) = 0$.

It follows that one can find a cover $\mathcal{U} = \{U_x\}_{x\in X} \in \mathsf{Cov}(X)$ such that for each $x \in X$ one has $\hat{\mathbf{s}}_{|U_x\times I} = 0$. One concludes that $\hat{\mathbf{s}} = 0$ since $a\mathcal{P}$ is a sheaf.

This shows that i_0^*, the rightmost vertical arrow in the diagram is injective, hence an isomorphism since the identity $\pi \circ i_0 = 1_X$ shows that i_0^* is surjective. □

As a direct consequence we get

Theorem 3.15. *Let $\mathcal{P}^*$ be a complex of abelian presheaves on G-Man_ω with homotopy-invariant cohomology presheaves. Then the presheaves*

$$X \mapsto \mathbb{H}^r\left(X_{\mathrm{eq}}; a\mathcal{P}^*_{|X}\right)$$

are homotopy-invariant.

Proof. Let $\mathcal{H}^q := aH^*_{a\mathcal{P}^*}$ denote the cohomology sheaf associated to the complex of sheaves $a\mathcal{P}^*$. Using Lemma A.9 one concludes that $\mathcal{H}^q \cong aH^q_{\mathcal{P}^*}$, and hence it follows from Proposition 3.14 that $\mathcal{H}^q$ is a homotopy invariant abelian sheaf since, by hypothesis, $H^q_{\mathcal{P}^*}$ is homotopy invariant.

Now, we can use Corollary A.8 to see that for each $X \in G$-Man_ω the corresponding map of hypercohomology spectral sequences

$$\pi^* : E_2^{p,q}(X) := H^p\left(X; \mathcal{H}^q_{|X}\right) \longrightarrow E_2^{p,q}(X \times I) := H^p\left(X \times I; \mathcal{H}^q_{|X\times I}\right) \quad \text{(A.4)}$$

is an isomorphism. Therefore, the projection $\pi : X\times I \to X$ induces an isomorphism between the abutments of these spectral sequences, thus proving the theorem. □

Corollary A.10. *If $i : A \hookrightarrow U$ is a strong deformation retract in the category G-Man_ω and $\mathcal{P}^*$ is a complex of abelian presheaves on G-Man_ω with homotopy invariant cohomology presheaves then one has an isomorphism*

$$i^* : \mathbb{H}^\bullet\big(U_{\mathrm{eq}}; a\mathcal{P}^*_{|U}\big) \longrightarrow \mathbb{H}^\bullet\big(A_{\mathrm{eq}}; a\mathcal{P}^*_{|A}\big).$$

Remark A.11. The key example here are the "neighborhoods" described in A.1, where we have an inclusion $G/H \hookrightarrow G \times_H D$, sending gH to $(gH, 0)$, where 0 is the center of disc containing D.

We conclude with the proof of

Proposition 3.16. *Let $\mathcal{U}$ be an equivariant good cover of X and let $\mathcal{P}^*$ be a complex with homotopy invariant cohomology presheaves on G-Man_ω. Then the Čech hypercohomology $\check{\mathbb{H}}^\bullet(\mathcal{U}; \mathcal{P}^*)$ of the cover with coefficients on the complex of presheaves $\mathcal{P}^*$, computes the hypercohomology $\mathbb{H}^\bullet\big(X; a\mathcal{P}^*_{|X}\big)$ of X_{eq} with coefficients in the complex of sheaves $a\mathcal{P}^*_{|X}$.*

Proof. Let $\rho : a\mathcal{P}^*_{|X} \to \mathbf{G}(a\mathcal{P}^*)_{|X}$ be the Godement quasi-isomorphism described in Remark A.3. The functoriality of this construction guarantees that if $U \subseteq X$ is an open G-invariant subset then

$$\Gamma\big(U, \mathbf{G}(a\mathcal{P}^*)_{|X}\big) = \Gamma\big(U, \mathbf{G}(a\mathcal{P}^*)_{|U}\big). \tag{A.5}$$

Standard arguments with flabby sheaves and Čech functors show that if $\mathcal{F}$ is a flabby abelian sheaf on X_{eq} and $\mathcal{U} \in \mathsf{Cov}(X)$, one has an acyclic complex

$$0 \to \Gamma(X; \mathcal{F}) \to \Gamma\big(X; \check{\mathcal{C}}^*_{\mathcal{F},\mathcal{U}}\big).$$

It follows that the second spectral sequence of the double complex $\Gamma(X, \check{\mathcal{C}}^p_{\mathbf{G}^q(a\mathcal{P}^*_{|X}),\mathcal{U}})$ has the form

$$''E_1^{p,q}(\mathcal{U}, \mathbf{G}(a\mathcal{P}^*_{|X})) = \begin{cases} 0, & p > 0, \\ \Gamma\big(X, \mathbf{G}^q(a\mathcal{P}^*_{|X})\big), & p = 0, \end{cases}$$

and, as a consequence,

$$\check{\mathbb{H}}^r\big(\mathcal{U}; \mathbf{G}(a\mathcal{P}^*_{|X})\big) \cong \mathbb{H}^r\big(X_{\mathrm{eq}}; a\mathcal{P}^*_{|X}\big). \tag{A.6}$$

Now, by naturality, the Godement resolution $\rho : a\mathcal{P}^*_{|X} \to \mathbf{G}(a\mathcal{P}^*_{|X})$ gives maps of first spectral sequences

$$'E_1^{p,q}\big(\mathcal{U}, \mathcal{P}^*_{|X}\big) \longrightarrow {'E_1^{p,q}}\big(\mathcal{U}, a\mathcal{P}^*_{|X}\big) \longrightarrow {'E_1^{p,q}}\big(\mathcal{U}, \mathbf{G}(a\mathcal{P}^*_{|X})\big),$$

as explained in Notation 3.13. This composition is a product of terms of the form

$$H^q\big(\Gamma\big(U_{\check{\sigma}_p}; \mathcal{P}^*\big)\big) \to H^q\big(\Gamma\big(U_{\check{\sigma}_p}; \mathbf{G}(a\mathcal{P}^*_{|X})\big)\big) \cong \mathbb{H}^q\big(U_{\check{\sigma}_p}; a\mathcal{P}^*_{|U_{\check{\sigma}_p}}\big), \tag{A.7}$$

where the isomorphism follows from (A.5) and (A.1).

Using the equivariant isomorphisms $U_{\check{\sigma}_p} \cong G \times_{H_{\check{\sigma}_p}} D^n$, one sees that the natural projection $U_{\check{\sigma}_p} \to \pi_0(U_{\check{\sigma}_p}) \cong G/H_{\check{\sigma}_p}$ is an equivariant (analytic) homotopy equivalence that induces the vertical maps in the following commutative diagram, whose horizontal maps are defined in (A.7):

$$\begin{tikzcd} H^q(\Gamma(U_{\check{\sigma}_p};\mathcal{P}^*)) \arrow[r] & \mathbb{H}^q(U_{\check{\sigma}_p};a\mathcal{P}^*) \\ H^q(\Gamma(G/H_{\check{\sigma}_p};\mathcal{P}^*)) \arrow[r] \arrow[u] & \mathbb{H}^q(G/H_{\check{\sigma}_p};a\mathcal{P}^*). \arrow[u] \end{tikzcd}$$

Since $\mathcal{P}^*$ has homotopy invariant cohomology presheaves, the left vertical arrow is an isomorphism, and Proposition 3.15 shows that the right vertical arrow is also an isomorphism. Finally, the bottom horizontal arrow is an isomorphism since $G/H_{\check{\sigma}_p}$ is zero-dimensional and sheaf cohomology coincides with Čech cohomology, as shown in Proposition A.4. It follows that the top horizontal arrow is an isomorphism, and by (3.13) one sees that the map ${}'E_1^{p,q}(\mathcal{U},\mathcal{P}^*_{|X}) \longrightarrow {}'E_1^{p,q}(\mathcal{U},\mathbf{G}(a\mathcal{P}^*_{|X}))$ gives an isomoprhism of ${}'E_1$-terms, showing that the spectral sequences converge to the same groups. By definition, the former spectral sequence converges to $\check{\mathbb{H}}^\bullet(\mathcal{U};\mathcal{P}^*)$ and the latter converges to $\mathbb{H}^\bullet(X_{\mathrm{eq}};a\mathcal{P}^*_{|X})$, by (A.6). This concludes the proof. □

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