A classification of torsors over Laurent polynomial rings

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Abstract. Let R_n be the ring of Laurent polynomials in n variables over a field k of characteristic zero and let K_n be its fraction field. Given a linear algebraic k-group G, we show that a K_n-torsor under G which is unramified with respect to $X = \mathrm{Spec}(R_n)$ extends to a unique toral R_n-torsor under G. This result, in turn, allows us to classify all G-torsors over R_n.

1. Introduction

Torsors are the algebraic analogues of the principal homogeneous spaces that one encounters in the theory of Lie groups. While the latter are (under natural assumptions) locally trivial (in the usual topology), this is not the case for the former: a torsor $E \to X$ under a group scheme $\mathfrak{G}$ over X is not necessarily trivial (i.e. isomorphic to $\mathfrak{G}$ with $\mathfrak{G}$ acting on itself by right multiplication) when restricted to any non-empty Zariski open subset U of X. The reason for this is that the Zariski topology is too coarse. The path out of this serious obstacle was initiated by J.-P. Serre (with what is now called the finite étale topology) and then implemented in full generality (together with an accompanying descent theory) by A. Grothendieck. The idea is to have certain morphisms $U \to X$ (e.g. étale or flat and of finite presentation) replace open immersions as trivializing local data.

Torsors have played an important role in number theory (Brauer groups, Tate–Shafarevich group, Manin obstructions) and in the Langlands program (Ngo's proof of the Fundamental Lemma). Somehow surprisingly, torsors have been used over the last decade to solve difficult problems in infinite dimensional Lie theory (see [16] for an extensive list of references. See also [5] and [19]).

The way that torsors arise in this context is the following. The infinite dimensional object $\mathcal{L}$ under consideration (for example, the centreless core of an extended affine Lie algebra [EALA] or a Lie superconformal algebra) has an invariant called the centroid (essentially the linear endomorphisms of the object that commute with their multiplication). These centroids are Laurent polynomial rings $k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ in finitely many variables over a base field k of characteristic 0. We will denote this ring by R_n, or simply by R if no confusion is possible.

The object $\mathcal{L}$ is naturally an R-module and it inherits the algebraic structure of $\mathcal{L}$ (for example $\mathcal{L}$ is a Lie algebra over R). It is when $\mathcal{L}$ is viewed as an object over R that torsors enter into the picture. One can for example classify up to R-isomorphism the objects under consideration using non-abelian étale cohomology. Of course at the end of the day one wants to understand the problem under consideration (say a classification) over k and not R. Thankfully there is a beautiful theory, known as the "centroid trick", that allows this passage.

The above discussion motivates why one is interested in the classification of torsors over R under a reductive R-group scheme $\mathfrak{G}$. We believe that the understanding of such torsors is of its own interest. This is the purpose of the present work. There is an important class of torsors under $\mathfrak{G}$ called *loop torsors* that appear naturally in infinite dimensional Lie theory. (Loop torsors are defined over an arbitrary base in [16]. They have already caught the attention of researchers in other areas. See for example [23].) It is shown in [16] that if E is a torsor over R under $\mathfrak{G}$, then E being a loop torsor is equivalent to E being *toral*, i.e. that the twisted R-group scheme $^E\mathfrak{G}$ admits a maximal torus. This is a remarkable property of the ring R: loop and toral torsors coincide.

Toral torsors under reductive group schemes were completely classified in our paper [5] with the use of Bruhat–Tits theory of buildings (see the Acyclicity Theorem 3.4). We will use toral torsors in what follows (the reader is asked to keep in mind that these are precisely the torsors that arise in infinite dimensional Lie theory). Given a reductive group scheme $\mathfrak{G}$ over R_n we want to classify/describe all the isomorphism classes of R_n-torsors under $\mathfrak{G}$. Since by definition $\mathfrak{G}$ is smooth they are in natural one-to-one correspondence with elements of the pointed set $H^1_{\text{ét}}(R_n, \mathfrak{G})$. We have of course by definition a natural inclusion

$$H^1_{\text{ét,toral}}(R_n, \mathfrak{G}) \subseteq H^1_{\text{ét}}(R_n, \mathfrak{G})$$

where $H^1_{\text{ét,toral}}(R_n, \mathfrak{G})$ is the subset consisting of (isomorphism) classes of toral $\mathfrak{G}$-torsors. From now on by default our topology will be étale; in particular we will denote $H^1_{\text{ét}}$ by H^1 and $H^1_{\text{ét,toral}}$ by H^1_{toral}. One of our main results is the following.

1.1. Theorem. *Under the above notation there is a natural bijection*

$$H^1(R_n, \mathfrak{G}) \longleftrightarrow \bigsqcup_{[E] \in H^1_{\text{toral}}(R_n, \mathfrak{G})} H^1_{\text{Zar}}(R_n, {}^E\mathfrak{G}).$$

The content of the above equality could be put in words as follows. Given a torsor E' over R_n under $\mathfrak{G}$ there exists a *unique* toral torsor E such that E' is locally isomorphic (in the Zariski topology) to E.

The proof of this result is achieved by a careful analysis (of independent interest) of the ramification of the torsors under consideration. We denote by $K_n = k(t_1, \dots, t_n)$ the fraction field of R_n and set $F_n = k((t_1)) \dots ((t_n))$. The precise statement of our other main result is the following.

1.2. Theorem. *Let $\mathfrak{G}$ be an affine R_n-group scheme. Assume that either*

(i) *$\mathfrak{G}$ is reductive and admits a maximal R_n-torus (equivalently $\mathfrak{G}$ is "loop reductive" [16, Cor. 6.3]); or*

(ii) *there exists a linear (smooth, not necessary connected) algebraic group $\mathbf{G}$ over k and a loop torsor E under $\mathbf{G} \times_k R_n$ such that $\mathfrak{G} = {}^E(\mathbf{G} \times_k R_n)$.*

Then we have natural bijections

$$H^1_{\mathrm{toral}}(R_n, \mathfrak{G}) \xrightarrow{\sim} H^1(K_n, \mathfrak{G})_{R_n\text{-unr}} \xrightarrow{\sim} H^1(F_n, \mathfrak{G})$$

where $H^1(K_n, \mathfrak{G})_{R_n\text{-unr}}$ stands for the subset of the Galois cohomology set $H^1(K_n, \mathfrak{G})$ consisting of (isomorphism) classes of $\mathfrak{G}$-torsors over K_n extending everywhere in codimension one (see §3.10).

We need to explain briefly the assumptions. In both cases (i) and (ii) we consider loop (= toral) group schemes because they play a central role in the classification of R_n-torsors (see Theorem 1.1). Also, even though reductive group schemes are the main interest in this paper, in case (ii) we include group schemes over R_n which are not necessary "connected". Far from being a trivial generalization, this case is absolutely essential for applications to infinite dimensional Lie theory. Indeed one is forced to understand twisted forms of R_n-Lie algebras $\mathfrak{g} \otimes_k R_n$, where $\mathfrak{g}$ is a split finite dimensional simple Lie algebra over k. This leads to torsors under the group $\mathbf{G} \times_k R_n$ where $\mathbf{G}$ is the linear algebraic k-group $\mathrm{Aut}(\mathfrak{g})$. Many interesting infinite dimensional Lie objects over k, including extended affine Lie algebras (a particular case of them are the celebrated affine Kac–Moody Lie algebras) and Lie superconformal algebras, follow under the above considerations.

Note that the special case $\mathbf{G} = \mathbf{PGL}_d$ (i.e. R_n-Azumaya algebras) was already quite understood by Brauer group techniques when the base field is algebraically closed [15, §4.4]. Note also that Theorem 1.2 refines our acyclicity theorem, i.e. the bijection $H^1_{\mathrm{toral}}(R_n, \mathfrak{G}) \xrightarrow{\sim} H^1(F_n, \mathfrak{G})$ ([5, Th. 14.1] in case (i), resp. [16, Th. 8.1] in case (ii)).

The structure of the paper is as follows. In Section 2, we establish useful generalities about unramified functors. Section 3 discusses unramified non-abelian cohomology. In Section 4, we prove Theorem 1.2. Section 5 is devoted to applications including a disjoint union decomposition for the set $H^1(R_n, \mathfrak{G})$ (Theorem 1.1).

Two important particular cases are considered in detail: the cases of orthogonal groups and projective linear groups. This illustrates that our main result, which may look rather abstract and remote in appearance, can lead to new very concrete classifications/descriptions of familiar objects.

2. Unramified functors

We follow essentially the setting of [6]. Let S be a scheme. If X is an integral S-scheme we denote by $\kappa(X)$ the fraction field of X. Let $\mathcal{F}$ be an S-functor, that is a contravariant functor $X \mapsto \mathcal{F}(X)$ from the category of S-schemes into the category of sets. If X is integral normal one defines the following two subsets of $\mathcal{F}(\kappa(X))$:

$$\mathcal{F}(\kappa(X))_{X\text{-loc}} := \bigcap_{x \in X} \mathrm{Im}\Big(\mathcal{F}(O_{X,x}) \to \mathcal{F}(\kappa(X))\Big)$$

and

$$\mathcal{F}(\kappa(X))_{X\text{-unr}} := \bigcap_{x \in X^{(1)}} \mathrm{Im}\Big(\mathcal{F}(O_{X,x}) \to \mathcal{F}(\kappa(X))\Big).$$

The first subset $\mathcal{F}(\kappa(X))_{X\text{-loc}}$ is called the subset of *local classes* with respect to X and the second one $\mathcal{F}(\kappa(X))_{X\text{-unr}}$ is called the subset of *unramified classes* with respect to X. Obviously, we have the inclusions

$$\mathcal{F}(\kappa(X))_{X\text{-loc}} \subseteq \mathcal{F}(\kappa(X))_{X\text{-unr}} \subseteq \mathcal{F}(\kappa(X)).$$

2.1. Lemma. *Let Y be an integral normal scheme over S. Let $f : Y \to X$ be a dominant morphism of S-schemes. Consider the map $\mathcal{F}(f^*) : \mathcal{F}(\kappa(X)) \to \mathcal{F}(\kappa(Y))$ induced by the comorphism $f^* = f^*_{\kappa(X)} : \kappa(X) \to \kappa(Y)$. Then:*

(1) $\mathcal{F}(f^*)(\mathcal{F}(\kappa(X))_{X\text{-loc}}) \subseteq \mathcal{F}(\kappa(Y))_{Y\text{-loc}}.$

(2) *If f is flat then $\mathcal{F}(f^*)(\mathcal{F}(\kappa(X))_{X\text{-unr}}) \subseteq \mathcal{F}(\kappa(Y))_{Y\text{-unr}}.$*

Proof. (1) The comorphism f^* allows us to view $\kappa(X)$ as a subfield $\kappa(X) \hookrightarrow \kappa(Y)$ of the field $\kappa(Y)$. Let $\gamma \in \mathcal{F}(\kappa(X))_{X\text{-loc}}$. We want to show that its image $\gamma_{\kappa(Y)} := \mathcal{F}(f^*)(\gamma) \in \mathcal{F}(\kappa(Y))$ under the base change is local with respect to Y. Let $y \in Y$ and put $x = f(y)$. The commutative square

$$\kappa(X) \subset \kappa(Y)$$

$$\cup \qquad\quad \cup$$

$$O_{X,x} \subset O_{Y,y}$$

induces a commutative diagram

$$
\begin{array}{ccc}
\mathcal{F}(\kappa(X)) & \longrightarrow & \mathcal{F}(\kappa(Y)) \\
\uparrow & & \uparrow \\
\mathcal{F}(O_{X,x}) & \longrightarrow & \mathcal{F}(O_{Y,y}).
\end{array}
$$

Since $\gamma \in \mathrm{Im}\big(\mathcal{F}(O_{X,x}) \to \mathcal{F}(\kappa(X))\big)$, it follows that $\gamma_{\kappa(Y)}$ is contained in $\mathrm{Im}\big(\mathcal{F}(O_{Y,y}) \to \mathcal{F}(\kappa(Y))\big)$. Thus $\gamma_{\kappa(Y)} \in \mathcal{F}(\kappa(Y))_{Y\text{-loc}}$.

(2) Assume now that $\gamma \in \mathcal{F}(\kappa(X))_{X\text{-unr}}$ and let $y \in Y^{(1)}$. As above, we set $x = f(y)$. Without loss of generality we may assume that

$$X = \mathrm{Spec}(A) = \mathrm{Spec}(O_{X,x})$$

and that

$$Y = \mathrm{Spec}(B) = \mathrm{Spec}(O_{Y,y})$$

where B is a DVR. Let v be the discrete valuation on $\kappa(Y)$ corresponding to the valuation ring B. For brevity we denote by K (resp. L) the fraction field of A (resp. B). If $v(K^{\times}) = 0$, then $K \subset B$ and therefore

$$\gamma_{\kappa(Y)} \in \mathrm{Im}\big(\mathcal{F}(O_{Y,y}) \to \mathcal{F}(\kappa(Y))\big).$$

Assume now that $v(K^{\times}) \neq 0$. Then $\mathfrak{m}_A B \neq B$ so that $A \to B$ is a local morphism. By [11, 0.6.6.2], since B is flat over A the ring B is a faithfully flat A-module. It follows that $A = B \cap K$ (apply [2, §I.3, §5, Prop. 10] with $F = K$ and $F' = A$. An alternative proof can be given by appealing to [12, 2.1.13]). Let $A_v = \{x \in K^{\times} \mid v(x) \geq 0\}$ be the valuation ring of $v_{|K}$. Then $A_v = K \cap B = A$, so that A is a DVR. This implies that $\gamma \in \mathrm{Im}\big(\mathcal{F}(O_{X,x}) \to \mathcal{F}(\kappa(X))\big)$, and the commutative diagram above yields that $\gamma_{\kappa(Y)} \in \mathrm{Im}\big(\mathcal{F}(O_{Y,y}) \to \mathcal{F}(\kappa(Y))\big)$. Thus $\gamma_{\kappa(Y)} \in \mathcal{F}(\kappa(Y))_{Y\text{-unr}}$. $\qquad\square$

We shall discuss next the case of non-abelian cohomology functors, but we remark that this technique and considerations can be applied to various interesting functors such as Brauer groups, Witt groups, unramified Galois cohomology...

3. Non-abelian cohomology

3.1. Some terminology. Let X be a scheme and let $\mathfrak{G}$ be an X-group scheme. The pointed set of non-abelian Čech cohomology on the flat (resp. étale, Zariski) site of X with coefficients in $\mathfrak{G}$, is denoted by $H^1_{\mathrm{fppf}}(X, \mathfrak{G})$ (resp. $H^1_{\mathrm{ét}}(X, \mathfrak{G})$, $H^1_{\mathrm{Zar}}(X, \mathfrak{G})$). These pointed sets measure the isomorphism classes of sheaf torsors over X under $\mathfrak{G}$

with respect to the chosen topology (see [20, Ch. IV §1] and [9] for basic definitions and references). If $X = \operatorname{Spec}(R)$, following customary usage and depending on the context, we also use the notation $H^1_{\mathrm{fppf}}(R, \mathfrak{G})$ instead of $H^1_{\mathrm{fppf}}(X, \mathfrak{G})$. Similarly for the étale and Zariski sites.

If $\mathfrak{G}$ is in addition affine over X, by faithfully flat descent all of our sheaf torsors are representable. They are thus *torsors* in the usual sense. For a $\mathfrak{G}$-torsor E we denote by $^E\mathfrak{G}$ the twisted form of $\mathfrak{G}$ by inner automorphisms; it is an affine group scheme over X. Furthermore, if $\mathfrak{G}$ is smooth all torsors are locally trivial for the étale topology. In particular, $H^1_{\mathrm{ét}}(X, \mathfrak{G}) = H^1_{\mathrm{fppf}}(X, \mathfrak{G})$. These assumptions on $\mathfrak{G}$ hold in most of the situations that arise in our work. Also, as we mentioned in the introduction, by default our topology will be étale so that instead of $H^1_{\mathrm{ét}}(X, \mathfrak{G})$ we will write $H^1(X, \mathfrak{G})$.

Given an X-group $\mathfrak{G}$ and a morphism $Y \to X$ of schemes, we let $\mathfrak{G}_Y$ denote the Y-group $\mathfrak{G} \times_X Y$ obtained by base change. For convenience, we will denote $H^1(Y, \mathfrak{G}_Y)$ by $H^1(Y, \mathfrak{G})$.

Assuming $\mathfrak{G}$ is smooth and affine of finite type, a *maximal torus* $\mathfrak{T}$ of $\mathfrak{G}$ is a subgroup X-scheme $\mathfrak{T}$ of $\mathfrak{G}$ such that $\mathfrak{T} \times_X \overline{\kappa(x)}$ is a maximal $\overline{\kappa(x)}$-torus of $\mathfrak{G} \times_X \overline{\kappa(x)}$ for each point $x \in X$ [28, XII.1]. Here $\overline{\kappa(x)}$ denotes an algebraic closure of $\kappa(x)$. A $\mathfrak{G}$-torsor E is *toral* if the twisted group scheme $^E\mathfrak{G}$ admits a maximal X-torus. We denote by $H^1_{\mathrm{fppf,toral}}(X, \mathfrak{G})$ the subset of $H^1_{\mathrm{fppf}}(X, \mathfrak{G})$ consisting of (isomorphism) classes of toral X-torsors under $\mathfrak{G}$.

3.2. Torsion bijection. If E is an X-torsor under $\mathfrak{G}$ (not necessarily toral), according to [17, III.2.6.3.1] there exists a natural bijection

$$\tau_E : H^1_{\mathrm{fppf}}(X, {}^E\mathfrak{G}) \to H^1_{\mathrm{fppf}}(X, \mathfrak{G}),$$

called the torsion bijection, which takes the class of the trivial torsor under $^E\mathfrak{G}$ to the class of E. It is easy to see that its restriction to classes of toral torsors induces a bijection

$$H^1_{\mathrm{fppf,toral}}(X, {}^E\mathfrak{G}) \to H^1_{\mathrm{fppf,toral}}(X, \mathfrak{G}).$$

3.3. Acyclicity theorem. The following theorem is the main tool for proving our main results.

3.4. Theorem. *Let $\mathfrak{G}$ be an affine R_n-group scheme. Assume that either*

(i) *$\mathfrak{G}$ is reductive and admits a maximal R_n-torus (equivalently $\mathfrak{G}$ is "loop reductive" [16, Cor. 6.3]); or*

(ii) *there exists a linear (smooth, not necessary connected) algebraic group $\mathbf{G}$ over k and a loop torsor E under $\mathbf{G} \times_k R_n$ such that $\mathfrak{G} = {}^E(\mathbf{G} \times_k R_n)$.*

Then the natural map

$$H^1_{\mathrm{toral}}(R_n, \mathfrak{G}) \longrightarrow H^1(F_n, \mathfrak{G})$$

is bijective.

Proof. See [5, Th. 14.1] in case (i) and [16, Th. 8.1] in case (ii).   □

3.5. Grothendieck–Serre's conjecture.

3.5. Grothendieck–Serre's conjecture. The following conjecture is due to Grothendieck–Serre ([26, Remarque 3], [18, Remarque 1.1.a]).

3.6. Conjecture. *Let R be a regular local ring with fraction field K. If $\mathfrak{G}$ a reductive group scheme over R then the natural map $H^1(R, \mathfrak{G}) \to H^1(K, \mathfrak{G})$ has trivial kernel.*

If R contains an infinite field k (of any characteristic), the conjecture has been proven by Fedorov–Panin [13, 22]. When $\mathfrak{G} = \mathbf{G} \times_k R$ where $\mathbf{G}$ is a reductive k-group, the so called "constant" case, this was established before by Colliot-Thélène and Ojanguren [7]. For our considerations we need a similar result for group schemes which are not necessary "connected".

3.7. Lemma. *Let R be a regular local ring with fraction field K. Let $\mathfrak{G}$ be an affine smooth group scheme over R which is an extension of a finite twisted constant group scheme $\mathbf{F}$ over R by a reductive group scheme $\mathfrak{G}^0$ over R. Assume that Grothendieck–Serre's conjecture holds for $\mathfrak{G}^0$. Then the map $H^1(R, \mathfrak{G}) \to H^1(K, \mathfrak{G})$ has trivial kernel.*

Proof. We consider the commutative exact diagram of pointed sets

$$
\begin{array}{ccccccc}
\mathbf{F}(R) & \longrightarrow & H^1(R, \mathfrak{G}^0) & \longrightarrow & H^1(R, \mathfrak{G}) & \longrightarrow & H^1(R, \mathbf{F}) \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \lambda} \\
\mathbf{F}(K) & \longrightarrow & H^1(K, \mathfrak{G}^0) & \longrightarrow & H^1(K, \mathfrak{G}) & \longrightarrow & H^1(K, \mathbf{F}).
\end{array}
$$

Since λ is injective, an easy diagram chase shows that $\operatorname{Ker} \eta$ surjects onto $\operatorname{Ker} \nu$. But by hypothesis $\operatorname{Ker} \eta$ vanishes, so the assertion follows.   □

As a corollary of Fedorov–Panin's theorem we get the following facts.

3.8. Corollary. *Let R be a regular local ring containing an infinite field with fraction field K. Let $\mathfrak{G}$ be an affine smooth group scheme over R which is an extension of a finite twisted constant group scheme over R by a reductive group scheme over R. Then the natural map $H^1(R, \mathfrak{G}) \to H^1(K, \mathfrak{G})$ has trivial kernel.*

3.9. Corollary. *Let X be an integral smooth affine variety over an infinite field with function field K. Let $\mathfrak{G}$ be an affine smooth group scheme over X which is an extension of a finite twisted constant group scheme over X by a reductive group scheme over X. Then the sequence of pointed sets*

$$
1 \longrightarrow H^1_{\mathrm{Zar}}(X, \mathfrak{G}) \longrightarrow H^1(X, \mathfrak{G}) \longrightarrow H^1(K, \mathfrak{G})
$$

is exact.

3.10. Rational torsors everywhere locally defined. For a smooth affine group scheme $\mathfrak{G}$ over an integral normal scheme X Colliot-Thélène and Sansuc [8, §6] introduced the following sets:

$$D^{\mathfrak{G}}(X) := \mathrm{Im}\Big(H^1(X, \mathfrak{G}) \to H^1(\kappa(X), \mathfrak{G}) \Big),$$

$$H^1(\kappa(X), \mathfrak{G})_{X\text{-loc}} := \bigcap_{x \in X} D^{\mathfrak{G}}(\mathcal{O}_{X,x}) \subseteq H^1(\kappa(X), \mathfrak{G}),$$

and

$$H^1(\kappa(X), \mathfrak{G})_{X\text{-unr}} := \bigcap_{x \in X^{(1)}} D^{\mathfrak{G}}(\mathcal{O}_{X,x}) \subseteq H^1(\kappa(X), \mathfrak{G}).$$

Clearly, we have the inclusions

$$D^{\mathfrak{G}}(X) \subseteq H^1(\kappa(X), \mathfrak{G})_{X\text{-loc}} \subseteq H^1(\kappa(X), \mathfrak{G})_{X\text{-unr}}.$$

In our terminology introduced in § 2 the two last sets are nothing but the local and unramified subsets with respect to X attached to the functor $\mathcal{F}$ given by $\mathcal{F}(Y) = H^1(Y, \mathfrak{G})$ for each X-scheme Y. Unramified classes have the following geometrical characterization.

3.11. Lemma. *Let $\gamma \in H^1(\kappa(X), \mathfrak{G})$. Then $\gamma \in H^1(\kappa(X), \mathfrak{G})_{X\text{-unr}}$ if and only if there exists an open subset U of X and a class $\widetilde{\gamma} \in H^1_{\text{ét}}(U, \mathfrak{G})$ such that*

(i) $\gamma = (\widetilde{\gamma})_{\kappa(X)}$;

(ii) $X^{(1)} \subset U$.

Proof. In one direction the statement is obvious. The other one was treated in [15, Cor. A.8]. $\square$

A special case of Lemma 2.1 is then the following.

3.12. Lemma. *Let $f : Y \to X$ be a dominant morphism of integral and normal S-schemes. Let*

$$\mathcal{F}(f^*) : H^1(\kappa(X), \mathfrak{G}) \to H^1(\kappa(Y), \mathfrak{G}_Y)$$

be the map induced by the comorphism $f^* : \kappa(X) \to \kappa(Y)$. Then.

(1) $\mathcal{F}(f^*)(H^1(\kappa(X), \mathfrak{G})_{X\text{-loc}}) \subseteq H^1(\kappa(Y), \mathfrak{G})_{Y\text{-loc}}$.

(2) *If f is flat then*

$$\mathcal{F}(f^*)(H^1(\kappa(X), \mathfrak{G})_{X\text{-unr}}) \subseteq H^1(\kappa(Y), \mathfrak{G})_{Y\text{-unr}}.$$

3.13. Remark. If X is regular and the "purity conjecture" holds for $\mathfrak{G}$ and local rings of X, i.e. $D^{\mathfrak{G}}(\mathcal{O}_{X,x}) = H^1(\kappa(X), \mathfrak{G})_{\mathcal{O}_{X,x}\text{-unr}}$ for all points $x \in X$, then assertion (1) implies that (2) holds without flatness assumption for f.

We now combine earlier work by Colliot-Thélène–Sansuc and Nisnevich Theorem [21] on Grothendieck–Serre conjecture for reductive groups over DVR.

3.14. Proposition. *Assume that X is a regular integral scheme and that $\mathfrak{G}$ is an extension of a finite twisted constant group scheme by a reductive group scheme $\mathfrak{G}^0$. Then*

(1) *$D^{\mathfrak{G}}$ defines a contravariant functor for the category of regular integral X-schemes to sets.*

(2) *If $X = \mathrm{Spec}(k)$, then $H^1(\ ,\mathfrak{G})_{\mathrm{loc}}$ defines a contravariant functor for the category of smooth integral k-varieties to sets.*

Proof. Nisnevich's theorem [21] states that if A is a DVR and $\mathfrak{G}$ is a reductive group over A then the natural map $H^1(A, \mathfrak{G}) \to H^1(K, \mathfrak{G})$, where K is a fraction field of A, is injective. By Lemma 3.7, it holds more generally for a group $\mathfrak{G}$ which is an extension of a finite twisted constant group by a reductive group. In particular, if $\gamma_1, \gamma_2 \in H^1(A, \mathfrak{G})$ have the same image in $H^1(K, \mathfrak{G})$, they have the same specialization modulo the maximal ideal of A. By [8, 6.6.1], this specialization property holds more generally over an arbitrary valuation ring. Then the assertions follow from [8, 6.6.3] and [6, Proposition 2.1.10]. $\qquad\square$

4. Proof of Theorem 1.2

Let $\mathfrak{G}$ be an affine group scheme over R_n satisfying condition (i) or (ii) in Theorem 1.2. Clearly,

$$\mathrm{Im}\,[H^1(R_n, \mathfrak{G}) \to H^1(K_n, \mathfrak{G})] \subset H^1(K_n, \mathfrak{G})_{R_n\text{-unr}},$$

so that we have the factorization

$$H^1_{\mathrm{toral}}(R_n, \mathfrak{G}) \xrightarrow{\ \phi\ } H^1(K_n, \mathfrak{G})_{R_n\text{-unr}} \xrightarrow{\ \psi\ } H^1(F_n, \mathfrak{G}). \qquad (*)$$

The Acyclicity Theorem 3.4 states that the composite map $\psi \circ \phi$ is bijective; in particular, ϕ is injective and ψ is surjective.

4.1. Lemma. *The following are equivalent.*

(i) *ϕ is bijective.*

(ii) *The map*
$$^E\psi : H^1(K_n, {}^E\mathfrak{G})_{R_n\text{-unr}} \longrightarrow H^1(F_n, {}^E\mathfrak{G})$$
has trivial kernel for all toral R_n-torsors E under $\mathfrak{G}$.

Proof. (i) $\Longrightarrow$ (ii): Assume that ϕ is bijective. The above factorization $(*)$ and the bijectivity of $\psi \circ \phi$ yield that we have bijections

$$H^1_{\mathrm{toral}}(R_n, \mathfrak{G}) \xrightarrow{\ \sim\ } H^1(K_n, \mathfrak{G})_{R_n\text{-unr}} \xrightarrow{\ \sim\ } H^1(F_n, \mathfrak{G}).$$

Let now E be a toral R_n-torsor under $\mathfrak{G}$. Recall that the torsion bijection map

$$\tau_E : H^1(R_n, {}^E\mathfrak{G}) \overset{\sim}{\longrightarrow} H^1(R_n, \mathfrak{G})$$

induces a bijection

$$H^1_{\mathrm{toral}}(R_n, {}^E\mathfrak{G}) \overset{\sim}{\longrightarrow} H^1_{\mathrm{toral}}(R_n, \mathfrak{G})$$

(see § 3.2). The commutative diagram of torsion bijections

$$
\begin{array}{ccccc}
H^1_{\mathrm{toral}}(R_n, \mathfrak{G}) & \overset{\sim}{\longrightarrow} & H^1(K_n, \mathfrak{G})_{R_n\text{-unr}} & \overset{\sim}{\longrightarrow} & H^1(F_n, \mathfrak{G}) \\
\wr \big\uparrow & & \wr \big\uparrow & & \wr \big\uparrow \\
H^1_{\mathrm{toral}}(R_n, {}^E\mathfrak{G}) & \longrightarrow & H^1(K_n, {}^E\mathfrak{G})_{R_n\text{-unr}} & \overset{{}^E\psi}{\longrightarrow} & H^1(F_n, {}^E\mathfrak{G})
\end{array}
$$

shows that ${}^E\psi$ is bijective, so a fortiori has trivial kernel.

(ii) $\Longrightarrow$ (i): We have noticed above that ϕ is injective, hence it remains to prove it is surjective. Let $[E'] \in H^1(K_n, \mathfrak{G})_{R_n\text{-unr}}$. Since $\psi \circ \phi$ is bijective there exists a class $[E] \in H^1(R_n, \mathfrak{G})$ such that $\psi([E']) = \psi(\phi([E]))$. It follows from the above commutative diagram that under the torsion bijection $H^1(K_n, {}^E\mathfrak{G})_{R_n\text{-unr}} \to H^1(K_n, \mathfrak{G})_{R_n\text{-unr}}$ the class $[E']$ corresponds to an element in $H^1(K_n, {}^E\mathfrak{G})_{R_n\text{-unr}}$ lying in the kernel of ${}^E\psi$. Since by our hypothesis $\mathrm{Ker}({}^E\psi) = 1$, this implies that the class $[E']$ corresponds to the trivial one in $H^1(K_n, {}^E\mathfrak{G})_{R_n\text{-unr}}$ or equivalently $\phi([E]) = [E']$. $\square$

Note that hypotheses (i) and (ii) in Theorem 1.2 are stable with respect to twisting by a toral R_n-torsor under $\mathfrak{G}$. Therefore the above lemma reduces the proof of Theorem 1.2 to showing that for all group schemes $\mathfrak{H}$ over R_n satisfying conditions (i) or (ii) in Theorem 1.2 the natural map

$$\psi : H^1(K_n, \mathfrak{H})_{R_n\text{-unr}} \longrightarrow H^1(F_n, \mathfrak{H})$$

has trivial kernel. To prove this fact we proceed by induction on $n \geq 1$ by allowing the base field k to vary.

$\boldsymbol{n = 1.}$ Since we are in dimension one, by Lemma 3.11 the map

$$H^1(R_1, \mathfrak{H}) \longrightarrow H^1(K_1, \mathfrak{H})_{R_1\text{-unr}}$$

is onto. Therefore, Lemma 4.1 applied to the group scheme $\mathfrak{G} = \mathfrak{H}$ and the trivial torsor $E = 1$ shows that ψ has trivial kernel.

$n \geq 2.$ Consider the following field tower:

$$K_n \subset F_{n-1}(t_n) \subset F_n.$$

Let $\gamma \in \mathrm{Ker}(\psi)$ and let γ' be its image in $H^1(F_{n-1}(t_n), \mathfrak{H})$. Since the morphism of affine schemes $\mathrm{Spec}(F_{n-1}[t_n^{\pm 1}]) \to \mathrm{Spec}(R_n)$ is flat and dominant, by Lemma 3.12 (2) we have

$$\gamma' \in H^1\big(F_{n-1}(t_n), \mathfrak{H}\big)_{F_{n-1}[t_n^{\pm 1}]\text{-unr}}.$$

Since $F_n = F_{n-1}((t_n))$ and $\gamma_{F_n} = 1$, we then conclude that

$$\gamma' \in \mathrm{Ker}\Big(H^1\big(F_{n-1}(t_n), \mathfrak{H}\big)_{F_{n-1}[t_n^{\pm 1}]\text{-unr}} \to H^1\big(F_{n-1}((t_n)), \mathfrak{H}\big)\Big).$$

But according to the case $n = 1$ (applied to the base field F_{n-1}) the last kernel is trivial. Thus $\gamma' = 1$, i.e.

$$\gamma \in \mathrm{Ker}\Big(H^1\big(K_n, \mathfrak{H}\big)_{R_n\text{-unr}} \to H^1(F_{n-1}(t_n), \mathfrak{H})\Big). \qquad (**)$$

Now, we observe that the field $F_{n-1}(t_n) = k((t_1))\ldots((t_{n-1}))(t_n)$ embeds into $k(t_n)((t_1))\ldots((t_{n-1})) = k'((t_1))\ldots((t_{n-1}))$ with $k' = k(t_n)$, so that we have a commutative diagram

$$
\begin{array}{ccc}
H^1\big(K_n, \mathfrak{H}\big)_{R_n\text{-unr}} & \longrightarrow & H^1\big(F_{n-1}(t_n), \mathfrak{H}\big) \\
\cap & & \downarrow \\
H^1\big(k'(t_1, \ldots, t_{n-1}), \mathfrak{H}\big)_{R_{n-1} \otimes_k k'\text{-unr}} & \longrightarrow & H^1\big(k'((t_1))\ldots((t_{n-1})), \mathfrak{H}\big),
\end{array}
$$

where the left vertical inclusion again is due to Lemma 3.12 (2). By the induction hypothesis applied to the base field k', the bottom horizontal map has trivial kernel. Therefore the top horizontal map has trivial kernel as well. By $(**)$, this implies $\gamma = 1$.

5. Applications

5.1. A disjoint union decomposition for R_n-torsors.
We shall use now our Theorem 1.2 and Fedorov–Panin's Theorem to prove Theorem 1.1. In fact we will prove a little bit more general result by allowing $\mathfrak{G}$ to be any group scheme satisfying the conditions of Theorem 1.2.

5.2. Theorem.
Let $\mathfrak{G}$ be as in Theorem 1.2. Then there is a natural bijection

$$\bigsqcup_{[E] \in H^1_{\mathrm{toral}}(R_n, \mathfrak{G})} H^1_{\mathrm{Zar}}(R_n, {}^E\mathfrak{G}) \xrightarrow[\sim]{\Theta} H^1(R_n, \mathfrak{G}).$$

Proof. Recall first that the torsion bijection τ_E (see § 3.2) allows us to embed

$$H^1_{\text{Zar}}(R_n, {}^E\mathfrak{G}) \hookrightarrow H^1(R_n, {}^E\mathfrak{G}) \xrightarrow[\sim]{\tau_E} H^1(R_n, \mathfrak{G}).$$

and this, in turn, induces a natural map

$$\Theta : \bigsqcup_{[E] \in H^1_{\text{toral}}(R_n, \mathfrak{G})} H^1_{\text{Zar}}(R_n, {}^E\mathfrak{G}) \to H^1_{\text{ét}}(R_n, \mathfrak{G}).$$

Surjectivity of Θ. Assume $[\gamma] \in H^1(R_n, \mathfrak{G})$. Since the generic class $\gamma_{K_n} \in H^1(K_n, \mathfrak{G})$ is R_n-unramified by Theorem 1.2 there is a unique toral class $[E] \in H^1(R_n, \mathfrak{G})_{\text{toral}}$ such that $[E]_{K_n} = \gamma_{K_n}$. Consider the following commutative diagram

$$
\begin{array}{ccc}
H^1(R_n, \mathfrak{G}) & \longrightarrow & H^1(K_n, \mathfrak{G}) \\
{\scriptstyle \tau_E} \uparrow {\scriptstyle \wr} & & {\scriptstyle \tau_E} \uparrow {\scriptstyle \wr} \\
H^1_{\text{Zar}}(R_n, {}^E\mathfrak{G}) \overset{\iota}{\hookrightarrow} H^1(R_n, {}^E\mathfrak{G}) & \longrightarrow & H^1(K_n, {}^E\mathfrak{G})
\end{array}
$$

with an exact bottom horizontal line (see Corollary 3.9). It follows from the diagram that that $\tau_E^{-1}(\gamma) \in \text{Im}\,\iota$. Hence there exists a unique class $\eta \in H^1_{\text{Zar}}(R_n, {}^E\mathfrak{G})$ such that $\eta = \tau_E^{-1}(\gamma)$. By construction, $\Theta(\eta) = \gamma$.

Injectivity of Θ. Let E, E' be two toral torsors under $\mathfrak{G}$ and let

$$\eta \in H^1_{\text{Zar}}(R_n, {}^E\mathfrak{G}), \quad \eta' \in H^1_{\text{Zar}}(R_n, {}^{E'}\mathfrak{G})$$

be such that $\Theta(\eta) = \Theta(\eta')$, i.e. $\tau_E(\eta) = \tau_{E'}(\eta') \in H^1(R_n, \mathfrak{G})$. Since η is locally trivial in the Zariski topology we conclude that

$$\tau_E(\eta)_{K_n} = \tau_E(1)_{K_n} = [E]_{K_n}$$

and similarly for E'. It follows that

$$[E]_{K_n} = [E']_{K_n} \in H^1(K_n, \mathfrak{G})_{R_n\text{-unr}}$$

and therefore, by Theorem 1.2, we have $[E] = [E']$. But $\tau_E = \tau_{E'}$ is bijective. Therefore, $\eta = \eta'$. $\qquad\qquad \square$

5.3. Example. Let $\mathbf{G}$ be a reductive k-group with the property that all of its semisimple quotients are isotropic (for example, k-split). By a result of Raghunathan [25, Th. B] one has

$$H^1_{\text{Zar}}(\mathbf{A}^n_k, \mathbf{G}) = 1$$

for all $n \geq 0$. According to [15, Prop. 2.2], any Zariski locally trivial $\mathbf{G}$-torsor over R_n can be extended to a $\mathbf{G}$-torsor over $\mathbf{A}^n_k$. Therefore $H^1_{\text{Zar}}(R_n, \mathbf{G}) = 1$. This implies that the map $H^1(R_n, \mathbf{G}) \to H^1(K_n, \mathbf{G})$ has trivial kernel. In other words, rationally trivial R_n-torsors under $\mathbf{G}$ are trivial.

We expect that the similar result holds in a more general case.

5.4. Conjecture. *Let $\mathfrak{G}$ be a loop reductive group scheme over R_n. If all semisimple quotients of $\mathfrak{G}$ are isotropic, then $H^1_{\mathrm{Zar}}(R_n, \mathfrak{G}) = 1$.*

5.5. Remark. Note that Artamonov's freeness result [1] as well as Parimala's result [24] for quadratic forms over R_2 are particular special cases of this conjecture.

Theorem 5.2 gives a classification of all $\mathfrak{G}$-torsors that involves first the description of all its toral torsors and then to studying locally trivial in the Zariski topology torsors under its twisted toral forms. In the following two subsections we show how our theorem works in particular cases of orthogonal groups and projective linear groups.

5.6. The case of orthogonal groups. Let q_{split} be a split quadratic form over k of dimension $d \geq 1$ and let $\mathbf{O}(q_{\mathrm{split}})$ be the corresponding split orthogonal group. It is well known that $H^1(R_n, \mathbf{O}(q_{\mathrm{split}}))$ classifies non-singular quadratic R_n-forms of dimension d [9, III.5.2]. This allows us to identify classes of torsors under $\mathfrak{G} = \mathbf{O}(q_{\mathrm{split}, R_n})$ with classes of d-dimensional quadratic forms over the ring R_n.

5.7. Proposition. *For each subset $I \subseteq \{1, \ldots, d\}$, we put $t_I = \prod_{i \in I} t_i \in R_n^\times$ with the convention $t_\emptyset = 1$.*

(1) *Each class in $H^1_{\mathrm{toral}}(R_n, \mathbf{O}(q_{\mathrm{split}}))$ contains a unique R_n-quadratic form of the form*

$$\bigoplus_{I \subseteq \{1, \ldots, d\}} \langle t_I \rangle \otimes q_{I, R_n}$$

where all q_I's with $I \neq \emptyset$ are non-singular anisotropic quadratic forms over k such that

$$d = \bigoplus_{I \subseteq \{1, \ldots, d\}} \dim(q_I).$$

(2) *Let q be a non-singular quadratic R_n-form of dimension d. Then there exists a unique quadratic R_n-form q_{loop} as in (1) such that q is a Zariski R_n-form of q_{loop}. Furthermore, q is isometric to q_{loop} if and only if q is diagonalizable.*

Proof. (1) By Acyclicity Theorem 3.4 it suffices to compute $H^1(F_n, \mathbf{O}(q_{\mathrm{split}}))$ or equivalently isometry classes of d-dimensional quadratic forms over F_n.

Let q be such quadratic form. We want to show that it is as in (1). By the Witt theorem we may assume without loss of generality that q is anisotropic. We proceed by induction on $n \geq 0$. The case $n = 0$ is obvious. Assume that $n \geq 1$. Note that $F_n = F_{n-1}((t_n))$. Springer's decomposition [10, §19], then shows that $q \cong q' \oplus \langle t_n \rangle q''$ where q' and q'' are (unique) anisotropic quadratic forms over F_{n-1}. By induction on n, q' and q'' are of the required form, hence the assertion for q follows. The unicity is clear.

(2) The first assertion follows from Theorem 5.2. If q is isometric to q_{loop}, then q is diagonalizable since so is q_{loop}. Conversely, assume that q is diagonalizable: $q = \langle b_1, \ldots, b_d \rangle$. Since

$$R_n^\times/(R_n^\times)^2 \xrightarrow{\sim} k^\times/(k^\times)^2 \times \langle t_1^{\epsilon_1} \ldots t_n^{\epsilon_n} \mid \epsilon_1, \ldots, \epsilon_n = 0, 1 \rangle.$$

all coefficients of q are of the shape $b_i = a_i\, t_{I_i}$ with $I_i \subseteq \{1, \ldots, d\}$ and $a_i \in k^\times$. Then we can renumber $b_1, \ldots, b_d$ in such a way that q is as in (1) and we are done.   □

5.8. Corollary. *Toral classes in* $H^1(R_n, \mathbf{O}(q_{\text{split}}))$ *correspond to diagonalizable* R_n-*quadratic forms.*

5.9. The projective linear case. Let $\mathfrak{G} = \mathbf{PGL}_d$. The set $H^1(R_n, \mathbf{PGL}_d)$ classifies Azumaya R_n-algebras of degree d [3, Prop. 2.5.3.13]. Recall also that if $\mathcal{A}$ is an Azumaya algebra over R_n of degree d, then there is a one-to-one correspondence between commutative étale R_n-subalgebras of $\mathcal{A}$ of dimension d and maximal R_n-tori of the group scheme $\mathbf{PGL}_1(\mathcal{A})$. (This is a general well-known fact [28, XIV.3.21.(b)]. Indeed, if S is a commutative étale R_n-subalgebra of $\mathcal{A}$ of degree d then $\mathfrak{T} = R_{S/R_n}(\mathbf{G}_m)/\mathbf{G}_m$ is a maximal R_n-torus of $\mathbf{PGL}_1(\mathcal{A})$. Conversely, to a maximal R_n-subtorus $\mathfrak{T}$ of $\mathbf{PGL}_1(\mathcal{A})$ one associates the R_n-subalgebra $\mathcal{A}^{\mathfrak{T}}$ of fixed points under the natural action of $\mathfrak{T}$. This subalgebra $\mathcal{A}^{\mathfrak{T}}$ has the required properties locally and hence globally).

From the above discussion it follows that $H^1_{\text{toral}}(R_n, \mathbf{PGL}_d)$ consists of isomorphism classes of Azumaya R_n-algebras $\mathcal{A}$ of degree d having étale commutative R_n-subalgebras of dimension d.

We pass to description of locally trivial torsors under $\mathbf{PGL}_1(\mathcal{A})$.

5.10. Lemma. *Let* $\mathcal{A}$ *be an Azumaya algebra over* R_n. *Then the natural map* $H^1_{\text{Zar}}(R_n, \mathbf{GL}_1(\mathcal{A})) \to H^1_{\text{Zar}}(R_n, \mathbf{PGL}_1(\mathcal{A}))$ *is bijective.*

Proof. The exact sequence

$$1 \to \mathbf{G}_m \to \mathbf{GL}_1(\mathcal{A}) \to \mathbf{PGL}_1(\mathcal{A}) \to 1$$

gives rise to a commutative diagram with exact rows (of pointed sets)

$$
\begin{array}{ccccccc}
1 & \longrightarrow & H^1(R_n, \mathbf{GL}_1(\mathcal{A})) & \xrightarrow{\phi} & H^1(R_n, \mathbf{PGL}_1(\mathcal{A})) & \longrightarrow & \mathrm{Br}(R_n) \\
& & \downarrow & & \psi_1 \downarrow & & \psi_2 \downarrow \\
& & H^1(K_n, \mathbf{GL}_1(A)) = 1 & \longrightarrow & H^1(K_n, \mathbf{PGL}_1(\mathcal{A})) & \longrightarrow & \mathrm{Br}(K_n).
\end{array}
$$

Every class in $H^1(R_n, \mathbf{GL}_1(\mathcal{A}))$ is rationally trivial, hence by Fedorov–Panin's result [13, 22] it is locally trivial in the Zariski topology. In other words

$$H^1(R_n, \mathbf{GL}_1(\mathcal{A})) = H^1_{\text{Zar}}(R_n, \mathbf{GL}_1(\mathcal{A})).$$

Clearly

$$\phi(H^1_{\text{Zar}}(R_n, \mathbf{GL}_1(\mathcal{A}))) \subseteq H^1_{\text{Zar}}(R_n, \mathbf{PGL}_1(\mathcal{A})).$$

Conversely, let $\gamma \in H^1_{\text{Zar}}(R_n, \mathbf{PGL}_1(\mathcal{A}))$. Since $\psi_1(\gamma) = 1$ and since ψ_2 is injective, we obtain $\gamma \in \text{Im}(\phi)$.

Finally, it remains to note that the above diagram shows that ϕ has trivial kernel and this is true for all Azumaya algebras over R_n. Then the standard twisting argument enables us to conclude that ϕ is injective. $\qquad\square$

Thus, the disjoint union decomposition of the set of isomorphism classes of torsors under $\mathbf{PGL}_d$ becomes

$$\bigsqcup_{[\mathcal{A}] \in H^1_{\text{toral}}(R_n, \mathbf{PGL}_d)} H^1_{\text{Zar}}(R_n, \mathbf{GL}_1(\mathcal{A})) \xrightarrow[\sim]{\Theta} H^1(R_n, \mathbf{PGL}_d).$$

In general we can not say much about the subset $H^1_{\text{Zar}}(R_n, \mathbf{GL}_1(\mathcal{A}))$. Recall only that it classifies right invertible $\mathcal{A}$-modules (so that the above decomposition is coherent with [15, Prop. 4.8]). More precisely, if $\mathcal{A}$ is a toral Azumaya R_n-algebra and $\mathcal{L}$ is a right invertible $\mathcal{A}$-module, then the class $[\mathcal{L}]$ corresponds to the class of the Azumaya algebra $\text{End}_A(\mathcal{L})$ under the map Θ.

However, when the base field k is algebraically closed, toral Azumaya algebras over R_n are easy to classify explicitly, and also much more information about Zariski trivial torsors is available due to Artamonov's freeness statements [1].

More precisely, let k be an algebraically closed field. Choose a coherent system of primitive roots of unity $(\zeta_n)_{n \geq 1}$ in k. Given integers r, s satisfying $1 \leq r \leq s$, we let $A(x, y)^s_r$ denote the Azumaya algebra of degree s over the Laurent polynomial ring $k[x^{\pm 1}, y^{\pm 1}]$ defined by a presentation

$$X^s = x, \ Y^s = y, \ YX = \zeta_s^r XY.$$

5.11. Lemma. *Let $s_1, \ldots, s_m, r_1, \ldots, r_m$ be positive integers such that $(s_i, r_i) = 1$ for all $i = 1, \ldots, m$. Then the Azumaya algebra*

$$\mathcal{A} = A(t_1, t_2)^{s_1}_{r_1} \otimes A(t_3, t_4)^{s_2}_{r_2} \otimes \cdots \otimes A(t_{2m-1}, t_{2m})^{s_m}_{r_m}$$

is a division algebra.

Proof. Indeed, using the residue method it is easy to see that it is a division algebra even over the field $F_{2m} = k((t_1)) \ldots ((t_{2m}))$. $\qquad\square$

We next recall that the group $\text{GL}_d(\mathbb{Z})$ acts in a natural way on the ring R_n, hence it acts on $H^1(R_n, \mathbf{PGL}_d)$ (for details see [16, 8.4]).

5.12. Theorem. *Assume that k is algebraically closed.*

(1) $H^1_{\mathrm{toral}}(R_n, \mathbf{PGL}_d)$ *consists of* $\mathrm{GL}_d(\mathbb{Z})$*-orbits of classes of* R_n*-algebras of the following shape:*

$$\mathcal{A} = M_{s_0}(R_n) \otimes_{R_n} A(t_1, t_2)^{s_1}_{r_1} \otimes A(t_3, t_4)^{s_2}_{r_2} \otimes \cdots \otimes A(t_{2m-1}, t_{2m})^{s_m}_{r_m}$$

where the integers $m, s_0, s_1, r_1, \ldots, s_m, r_m$ *satisfy the following conditions:*

(i) $0 \le 2m \le n$; $1 \le r_i \le s_i$ *and* $(r_i, s_i) = 1$ *for all* $i = 1, \ldots, m$;

(ii) $s_0 \ge 1$ *and* $s_0 s_1 \ldots s_m = d$.

(2) $\mathcal{A} \otimes_{R_n} F_n$ *is division if and only if* $s_0 = 1$.

(3) *Let* $\mathcal{A}$ *be an Azumaya algebra as in* (1). *If* $s_0 \ge 2$ *we have*

$$H^1_{\mathrm{Zar}}(R_n, \mathbf{GL}_1(\mathcal{A})) = 1.$$

Proof.   (1) If $\mathcal{A} \otimes_{R_n} F_n$ is division, this is [16, Th. 4.7]. Assume now that $\mathcal{A} \otimes_{R_n} F_n$ is not division. By Wedderburn's theorem there exists an integer $s \ge 1$ and a central division F_n-algebra A' such that $\mathcal{A} \otimes_{R_n} F_n \cong M_s(A')$. The acyclicity theorem provides a toral R_n-Azumaya algebra $\mathcal{A}'$ such that $\mathcal{A}' \otimes_{R_n} F_n \cong A'$. Since $M_s(\mathcal{A}')$ and $\mathcal{A}$ are isomorphic over F_n, the acyclicity theorem again shows that $\mathcal{A} \cong M_s(\mathcal{A}')$. It remains to note that $\mathcal{A}'$ being a division algebra over R_n is of the required form by the first case.

(2) The assertion follows from Lemma 5.11.

(3) Write $\mathcal{A} = M_s(\mathcal{A}')$ with $\mathcal{A}'$ is division and $s \ge 2$. Morita equivalence provides a one-to-one correspondence between invertible $\mathcal{A}$-modules and finitely generated projective $\mathcal{A}'$-modules of relative rank s. Artamonov's result [1] states that those $\mathcal{A}'$-modules are free [1, Cor. 3] since $s \ge 2$, so that invertible $\mathcal{A}$-modules are free as well. This implies $H^1_{\mathrm{Zar}}(R_n, \mathbf{GL}_1(\mathcal{A})) = 1$. $\square$

5.13. Remarks.

(a) The third statement shows that for each invertible $\mathcal{A}$-module P the module $P \oplus \mathcal{A}$ is free.

(b) The third statement refines Steinmetz's results in the 2-dimension case ($n = 2$) [29, Th. 4.8] where the case $s \ge 3$ was considered only. Note that [29] provides some other cases for classical groups when all Zariski locally trivial torsors are trivial.

5.14. Applications to R_n-Lie algebras. We next consider the special case $\mathbf{G} = \mathrm{Aut}(\mathfrak{g})$ where $\mathfrak{g}$ is a split simple Lie algebra over k of finite dimension. For such group the set $H^1_{\mathrm{\acute{e}t}}(R_n, \mathbf{G})$ classifies R_n-forms of the Lie algebra $\mathfrak{g} \otimes_k R_n$ and $H^1_{\mathrm{toral}}(R_n, \mathbf{G})$

classifies loop objects, i.e. those which arise from loop cocycles. More precisely, by [16, §6] we have

$$\mathrm{Im}\Big(H^1\big(\pi_1(R_n, 1), \mathbf{G}(k_s)\big) \to H^1(R_n, \mathbf{G})\Big) = H^1_{\mathrm{toral}}(R_n, \mathbf{G}).$$

Theorems 1.2 and 5.2 have the following consequences.

5.15. Corollary.

(1) Let $\widetilde{\mathcal{L}}$ be a K_n-form of the Lie algebra $\mathfrak{g} \otimes_k K_n$. If $\widetilde{\mathcal{L}}$ is unramified, i.e. extends everywhere in codimension one, then $\widetilde{\mathcal{L}}$ is isomorphic to the generic fiber of a unique multiloop Lie algebra $\mathcal{L}$. (Of course $\mathcal{L}$, being a multiloop algebra, is a twisted from of the R_n-Lie algebra $\mathfrak{g} \otimes_k R_n$.)

(2) Let $\mathcal{L}$ be any R_n-form of $\mathfrak{g} \otimes_R R_n$. Then there exists a (unique up to R_n-isomorphism) multiloop Lie algebra $\mathcal{L}_{\mathrm{loop}}$ over R_n such that $\mathcal{L}$ is a Zariski R_n-form of $\mathcal{L}_{\mathrm{loop}}$.

References

[1] V. A. Artamonov, Projective modules over crossed products, *J. Algebra*, **173** (1995), 696–714. Zbl 0828.16031 MR 1327876

[2] N. Bourbaki, *Éléments de mathématique. Algèbre commutative, Ch. 1 à 4*, Reprint of the 1985 original, Springer, 2006. Zbl 1103.13001 MR 782296

[3] B. Calmès and J. Fasel, Groupes classiques, in *Autours des schémas en groupes. Vol. II*, 1–133, Panor. Synthéses, 46, Soc. Math. France, Paris, 2015. MR 3525594

[4] V. Chernousov, P. Gille, and A. Pianzola, Torsors over the punctured affine line, *American Journal of Mathematics*, **134** (2012), 1541–1583. Zbl 1279.14059 MR 2999288

[5] V. Chernousov, P. Gille, and A. Pianzola, Conjugacy theorems for loop reductive group schemes and Lie algebras, *Bulletin of Mathematical Sciences*, **4** (2014), 281–324. Zbl 06398510 MR 3228576

[6] J.-L. Colliot-Thélène, Birational invariants, purity and the Gersten conjecture, in *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, 1–64, Proc. Sympos. Pure Math., 58.1, AMS, 1995. Zbl 0834.14009 MR 1327280

[7] J.-L. Colliot-Thélène and M. Ojanguren, Espaces principaux homogènes localement triviaux, *I.H.É.S. Publ. Math.*, **75** (1992), 97–122. Zbl 0795.14029 MR 1179077

[8] J.-L. Colliot-Thélène and J.-J. Sansuc, Fibrés quadratiques et composantes connexes réelles, *Math. Annalen*, **244** (1979), 105–134. Zbl 0418.14016 MR 0550842

[9] M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*. Avec un appendice Corps de classes local par Michiel Hazewinkel. Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. Zbl 0203.23401 MR 302656

[10] R. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications, 56, American Mathematical Society, Providence, RI, 2008. Zbl 1165.11042 MR 2427530

[11] A. Grothendieck (avec la collaboration de J. Dieudonné), Eléments de Géométrie Algébrique I: Le langage des schémas, *Publications mathématiques de l'I.H.É.S.*, **4** (1960), 1–228. Zbl 03193326 MR 0217083

[12] A. Grothendieck (avec la collaboration de J. Dieudonné), Eléments de Géométrie Algébrique IV: Étude locale des schémas et des morphismes de schémas, *Publications mathématiques de l'I.H.É.S.*, (1964–1967), no. 20, 24, 28, 32, 259pp. Zbl 0212.52801 MR 0173675

[13] R. Fedorov and I. Panin, A proof of Grothendieck–Serre conjecture on principal bundles over regular local rings containing infinite fields, *Publications Mathématiques de l'I.H.É.S*, **122** (2015), 169–193. Zbl 1330.14077 MR 3415067

[14] P. Gille and A. Pianzola, Galois cohomology and forms of algebras over Laurent polynomial rings, *Mathematische Annalen*, **338** (2007), 497–543. Zbl 1131.11070 MR 2302073

[15] P. Gille and A. Pianzola, Isotriviality and étale cohomology of Laurent polynomial rings, *J. Pure Appl. Algebra*, **212** (2008), 780–800. Zbl 1132.14042 MR 2363492

[16] P. Gille and A. Pianzola, Torsors, reductive group schemes and extended affine Lie algebras, *Mem. Amer. Math. Soc.*, **226** (2013), no. 1063, vi+112 pp. Zbl 06304949 MR 3052313

[17] J. Giraud, Cohomologie non abélienne, Die Grundlehren der mathematischen Wissenschaften, 179, Springer-Verlag, 1971. Zbl 0226.14011 MR 0344253

[18] A. Grothendieck, Le groupe de Brauer. II, Théorie cohomologique, in *Dix exposés sur la cohomologie des schémas*, 67–87, Adv. Stud. Pure Math., 3, North-Holland, Amsterdam, 1968. Zbl 0198.25803 MR 0244270

[19] V. Kac, M. Lau, and A. Pianzola, Differential conformal superalgebras and their forms, *Advances in Mathematics*, **222** (2009), 809–861. Zbl 1217.17018 MR 2553371

[20] J. S. Milne, *Étale cohomology*, Princeton University Press, 1980. Zbl 0433.14012 MR 0559531

[21] Y. A. Nisnevich, Espaces homogènes principaux rationnellement triviaux et arithmétique des schémas en groupes réductifs sur les anneaux de Dedekind, *C. R. Acad. Sci. Paris Sér. I Math.*, **299** (1984), 5–8. Zbl 0587.14033 MR 0756297

[22] I. Panin, A. Stavrova, and N. Vavilov, On Grothendieck–Serre's conjecture concerning principal G-bundles over reductive group schemes. I, *Compositio Math.*, **151** (2015), 535–567. Zbl 1317.14102 MR 3320571

[23] G. Pappas and X. Zhu, Local models of Shimura varieties and a conjecture of Kottwitz, *Inventiones Mathematicae*, **194** (2013), 147–254. Zbl 1294.14012 MR 3103258

[24] R. Parimala, Quadratic spaces over Laurent extensions of Dedekind domains, *Trans. Amer. Math. Soc.*, **277** (1983), 569–578. Zbl 0514.13005 MR 0694376

[25] M. S. Raghunathan, Principal bundles on affine space and bundles on the projective line, *Math. Ann.*, **285** (1989), 309–332. Zbl 0661.14016 MR 1016097

[26] J.-P. Serre, Espaces fibrés algébriques, *Séminaire Claude Chevalley*, **3** (1958), Exposé no. 1, 37pp.

[27] Séminaire de Géométrie algébrique de l'I.H.É.S., 1960–1961, Revêtements étales et groupe fondamental, dirigé par A. Grothendieck, Lecture Notes in Math., 224, Springer, 1971. Zbl 0234.14002 MR 0354651

[28] Séminaire de Géométrie algébrique de l'I.H.É.S., 1962–1964, Schémas en groupes, dirigé par M. Demazure et A. Grothendieck, Lecture Notes in Math., 151-153, Springer, 1970. Zbl 0207.51401 Zbl 0209.24201 Zbl 0212.52810 MR 2867621 MR 2867622

[29] W. A. Steinmetz Zikesch, Algèbres de Lie de dimension infinie et théorie de la descente, *Mém. Soc. Math. Fr. (Nouv. Sér)*, **129** (2012), 99pp. Zbl 1319.17014 MR 3059229

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