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Autor: Huang, Jingyin / Jankiewicz, Kasia / Przytycki, Piotr
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Cocompactly cubulated 2-dimensional Artin groups

Jingyin Huang, Kasia Jankiewicz and Piotr Przytycki*

Abstract. We give a necessary and sufficient condition for a 2-dimensional or a three-generator Artin group A to be (virtually) cocompactly cubulated, in terms of the defining graph of A.

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1. Introduction

We say that a group is (*cocompactly*) *cubulated* if it acts properly (and compactly) by combinatorial automorphisms on a CAT(0) cube complex. We say that a group is *virtually cocompactly cubulated*, if it has a finite index subgroup that is cocompactly cubulated. Such groups either fail to have Kazhdan's property (T) or are finite [32], are bi-automatic [40], satisfy the Tits Alternative [39] and, if cocompactly cubulated, they satisfy rank-rigidity [12]. For more background on CAT(0) cube complexes, see the survey article of Sageev [38].

The *Artin group* with generators s_i and exponents $m_{ij} = m_{ji} \geq 2$, where $i \neq j$, is presented by relations $\underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}$. Here $\underbrace{s_i s_j s_i \cdots}_{m_{ij}}$ denotes the first half of the word $(s_i s_j)^{m_{ij}}$. The *defining graph* of an Artin group has vertices corresponding to s_i and edges labeled m_{ij} between s_i and s_j whenever $m_{ij} < \infty$.

Artin groups that are *right-angled* (i.e. the ones with $m_{ij} \in \{2, \infty\}$) are cocompactly cubulated, and they play a prominent role in theory of special cube complexes of Haglund and Wise [23]. However, much less is known about other Artin groups, in particular about braid groups. In [45] Wise suggested an approach to cubulating Artin groups using cubical small cancellation. However, we failed to execute this approach: we were not able to establish the B(6) condition.

In this article we consider Artin groups that have three generators, or are *2-dimensional*, that is, their corresponding Coxeter groups have finite special

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subgroups of maximal rank 2 (or, equivalently, 2-dimensional Davis complex). We characterise when such a group is virtually cocompactly cubulated. This happens only for very rare defining graphs. An *interior* edge of a graph is an edge that is not a leaf.

Theorem 1.1. *Let A be a 2-dimensional Artin group. Then the following are equivalent.*

(i) *A is cocompactly cubulated,*

(ii) *A is virtually cocompactly cubulated,*

(iii) *each connected component of the defining graph of A is either*

- *a vertex, or an edge, or else*
- *all its interior edges are labeled by 2 and all its leaves are labelled by even numbers.*

Moreover, if A is an arbitrary Artin group, then (iii) implies (i).

Theorem 1.2. *Let A be a three-generator Artin group. Then the following are equivalent.*

(i) *A is cocompactly cubulated,*

(ii) *A is virtually cocompactly cubulated,*

(iii) *the defining graph of A is as in Theorem 1.1(iii) or has two edges labelled by 2.*

1.1. Remarks. From Theorem 1.2 it follows that the 4-strand braid group is not virtually cocompactly cubulated.

Note that, independently, Thomas Haettel [19] has obtained a full classification of cocompactly cubulated Artin groups. His methods do not apply yet to finite index subgroups of Artin groups, but we intend to work together and prove that an Artin group is virtually cocompactly cubulated only if it is cocompactly cubulated.

The equivalence of (i) and (ii) has no counterpart for Coxeter groups, where the group $\widetilde{A}_2$ generated by reflections in the sides of an equilateral triangle in $\mathbb{R}^2$ is virtually cocompactly cubulated, but not cocompactly cubulated.

There are Artin groups that do not satisfy the equivalent conditions from Theorem 1.1, but are cubulated. Namely, it follows from [9, 24] that if the defining graph of A is a tree, then A is the fundamental group of a link complement that is a graph manifold with boundary. Hence by the work of Liu [30] or Przytycki and Wise [36] the Artin group A is cubulated.

Artin groups of *large type*, that is, with all $m_{ij} \geq 3$ are 2-dimensional. For many of them Brady and McCammond constructed 2-dimensional CAT(0) complexes with proper and cocompact action [6]. However, these complexes are built of triangles, not squares.

1.2. Some historical background. Sageev invented a way of *cubulating* groups (i.e. showing that they are cubulated) using codimension 1-subgroups [37], which was later also explained in the language of *walls* in the Cayley complex of the group [17, 33]. Here we give a brief account on some cubulation results, for a more complete one see [25].

Using the technology of walls, Niblo and Reeves cubulated Coxeter groups [32], then Williams [43] and Caprace and Mühlherr [11] analysed when this cubulation is cocompact. It is not known if all Coxeter groups are virtually cocompactly cubulated. Wise cocompactly cubulated small cancellation groups [44], and Ollivier and Wise cocompactly cubulated random groups at density $< \frac{1}{6}$ [34].

Furthermore, using the surfaces of Kahn and Markovic, Bergeron and Wise cocompactly cubulated the fundamental groups of closed hyperbolic 3-manifolds [3, 28], and later Wise cocompactly cubulated the fundamental groups of compact hyperbolic 3-manifolds with boundary [45]. Hagen and Wise cocompactly cubulated hyperbolic free-by-cyclic groups [21].

Groups that are not (relatively) hyperbolic are harder to cubulate cocompactly. Przytycki and Wise cubulated the fundamental groups of all compact 3-dimensional manifolds that are not graph manifolds, as well as graph manifolds with boundary [35, 36]. In [30] Liu gave a criterion for a graph manifold fundamental group to be virtually cubulated *specially* (meaning that the quotient of the action admits a local isometry into the Salvetti complex of a right-angled Artin group), but we do not know if this is equivalent to just being cubulated. Hagen and Przytycki gave a criterion for a graph manifold fundamental group to be cocompactly cubulated [20]. In general, it is difficult to find obstructions for groups to be cubulated. Another result of this type is Wise's characterization of tubular groups that are cocompactly cubulated [46].

1.3. Proof outline for (i) ⇒ (iii) in Theorem 1.1. Given a 2-dimensional Artin group acting properly and cocompactly on a CAT(0) cube complex, we show that its two-generator special subgroups are convex cocompact. More precisely, each of them acts cocompactly on a convex subcomplex which naturally decomposes as a product of a vertical factor and a horizontal factor. Geometrically, the intersection of two such subgroups is either vertical or horizontal. However, if Theorem 1.1(iii) is not satisfied, then this intersection is neither vertical nor horizontal by algebraic considerations.

One of the ingredients of the proof is Theorem 3.8, which asserts that a top rank product of hyperbolic groups acting on a CAT(0) cube complex is always convex cocompact.

1.4. Organization. In Section 2 we give some background on CAT(0) spaces and CAT(0) cube complexes. Section 3 is devoted to the proof of Theorem 3.8. In Section 4 we give some background on Artin groups and discuss some algebraic properties of two-generator Artin groups. Finally, in Section 5 we prove Theorem 1.1 and in Section 6 we prove Theorem 1.2.

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2. Preliminaries

A group is a CAT(0) *group* if it acts properly and cocompactly on a CAT(0) space. We assume the reader is familiar with the basics of CAT(0) spaces and groups. For background, see [7]. In this section we collect some less classical results.

2.1. Asymptotic rank.

The following definition was introduced in [29].

Definition 2.1. Let X be a CAT(κ) space. For $x \in X$ we denote by $\Sigma_x X$ the CAT(1) space that is the completion of the space of directions at x [7, Definition II.3.18]. The *geometric dimension* of X, denoted $\mathrm{GeomDim}(X)$ is defined inductively as follows.

- $\mathrm{GeomDim}(X) = 0$ if X is discrete,
- $\mathrm{GeomDim}(X) \leq n$ if $\mathrm{GeomDim}(\Sigma_x X) \leq n-1$ for any $x \in X$.

Definition 2.2. Let X be a CAT(0) space. Then its *asymptotic rank*, denoted by $\mathrm{asrk}(X)$, is the supremum of the geometric dimension of the asymptotic cones of X.

Theorem 2.3. *Let X and Y be* CAT(0) *spaces. Then*

(1) $\mathrm{asrk}(X \times Y) \geq \mathrm{asrk}(X) + \mathrm{asrk}(Y)$,

(2) *if* $\mathrm{asrk}(X) \leq 1$, *then X is hyperbolic.*

The first assertion follows from Theorem A of [29] and the second assertion follows from Corollary 1.3 of [42].

Definition 2.4. If G is a CAT(0) group acting properly and cocompactly on a CAT(0) space X, then the *asymptotic rank* of G is the asymptotic rank of X. By [29, Theorem C] this is the maximal n for which there is a quasi-isometric embedding $\mathbb{R}^n \to X$. Hence it does not depend on the choice of the CAT(0) space X.

Lemma 2.5. *Suppose that G is a* CAT(0) *group, and that G acts properly and cocompactly on a contractible n-dimensional cell complex X (not necessarily* CAT(0)*). Then the asymptotic rank of G is $\leq n$.*

Proof. Choose any G-equivariant length metric on X. We will prove that there does not exist a quasi-isometric embedding $f : \mathbb{R}^k \to X$ for $k > n$. Otherwise, since X is contractible and admits a cocompact action of G, we can assume that f is a continuous quasi-isometry: such f can be defined by induction on consecutive skeleta of the standard cubical subdivision of $\mathbb{R}^k$.

Let $Y \subseteq X$ be the smallest subcomplex containing $f(\mathbb{R}^k)$. Then $f : \mathbb{R}^k \to Y$ is a quasi-isometry. Let $g : Y \to \mathbb{R}^k$ be a quasi-isometry inverse to f, we can again assume that g is continuous. For any $x \in \mathbb{R}^k$ the distance $d(g \circ f(x), x)$ is uniformly bounded and consequently there is a proper geodesic homotopy between $g \circ f$ and the identity map.

Recall that for a topological space X we can consider *locally finite chains* in X, which are formal sums $\Sigma_{\lambda \in \Lambda} a_\lambda \sigma_\lambda$ where a_λ are integers, σ_λ are singular simplices, and any compact set in X intersects the images of only finitely many σ_λ with $a_\lambda \neq 0$. This gives rise to *locally finite homology* of X, denoted by $H_*^{\mathrm{lf}}(X)$. Moreover, proper maps induce homomorphisms on locally finite homology. See [5, Section 2.2] for more discussion.

Since there is a proper geodesic homotopy between $g \circ f$ and the identity map, $g \circ f$ induces the identity on $H_*^{\mathrm{lf}}(\mathbb{R}^k)$, and consequently $f_*\colon H_k^{\mathrm{lf}}(\mathbb{R}^k) \to H_k^{\mathrm{lf}}(Y)$ is injective. This leads to a contradiction, since $H_k^{\mathrm{lf}}(\mathbb{R}^k)$ contains the fundamental class $[\mathbb{R}^k]$ which is a nontrivial element, while $H_k^{\mathrm{lf}}(Y) = 0$ since $\dim(Y) < k$. □

2.2. Gate and parallel set. All CAT(0) cube complexes in our article are finite-dimensional. Throughout this paper the only metric that we consider on a CAT(0) cube complex X is the CAT(0) metric d. The *convex hull* of a subspace $Y \subseteq X$ is the smallest convex subspace containing Y, and is not necessarily a subcomplex, while the *combinatorial convex hull* of Y is the smallest convex subcomplex of X containing Y. For a complete convex subspace $Y \subseteq X$ we denote by $\pi_Y\colon X \to Y$ the closest point projection onto Y.

The following lemma was proved in slightly different contexts by various authors [1,2,4,27]:

Lemma 2.6 ([27, Lemma 2.10]). *Let X be a* CAT(0) *cube complex of dimension n, and let Y_1, Y_2 be convex subcomplexes. Let $\Delta = d(Y_1, Y_2)$, $V_1 = \{y \in Y_1 \mid d(y, Y_2) = \Delta\}$ and $V_2 = \{y \in Y_2 \mid d(y, Y_1) = \Delta\}$. Then:*

(1) *V_1 and V_2 are nonempty convex subcomplexes.*

(2) *π_{Y_1} maps V_2 isometrically onto V_1 and π_{Y_2} maps V_1 isometrically onto V_2. Moreover, the convex hull of $V_1 \cup V_2$ is isometric to $V_1 \times [0, \Delta]$.*

(3) *for every $\epsilon > 0$ there exists $\delta = \delta(\Delta, n, \epsilon) > 0$ such that if $y_1 \in Y_1$, $y_2 \in Y_2$ and $d(y_1, V_1) \geq \epsilon$, $d(y_2, V_2) \geq \epsilon$, then*

$$d(y_1, Y_2) \geq \Delta + \delta d(y_1, V_1), \quad d(y_2, Y_1) \geq \Delta + \delta d(y_2, V_2).$$

We call $V_1 \subseteq Y_1$ the *gate with respect to Y_2*, and $V_2 \subseteq Y_2$ the *gate with respect to Y_1*. We write $\mathcal{G}(Y_1, Y_2) = (V_1, V_2)$. We say that Y_1, Y_2 are *parallel* if $\mathcal{G}(Y_1, Y_2) = (Y_1, Y_2)$.

Lemma 2.7 ([26, Lemma 2.9]). *Let X be a* CAT(0) *cube complex, and let* $(V_1, V_2) = \mathcal{G}(Y_1, Y_2)$ *for some convex subcomplexes* $Y_1, Y_2 \subseteq X$. *Let e be an edge in V_1 and let h be the hyperplane dual to e. Then* $h \cap V_2 \neq \emptyset$.

Lemma 2.8 ([12, Lemma 2.5]). *A decomposition of a* CAT(0) *cube complex as a product of* CAT(0) *cube complexes corresponds to a partition* $\mathcal{H}_1 \sqcup \mathcal{H}_2$ *of the collection of hyperplanes of X such that every hyperplane in $\mathcal{H}_1$ intersects every hyperplane in $\mathcal{H}_2$.*

The following lemma was also proved in [2, Lemma 2.4].

Lemma 2.9. *Let X be a* CAT(0) *cube complex and let* $Y \subseteq X$ *be a convex subcomplex. Let* $\{Y_\lambda\}_{\lambda\in\Lambda}$ *be the collection of all convex subcomplexes that are parallel to Y. Then the combinatorial convex hull P_Y of* $\bigcup_{\lambda\in\Lambda} Y_\lambda$ *admits a natural product decomposition* $P_Y = Y \times Y^\perp$.

P_Y is called the *combinatorial parallel set* of Y.

Proof. Let $\mathcal{H}$ be the collection of hyperplanes in X that separate some points in $\bigcup_{\lambda\in\Lambda} Y_\lambda$ and let $h \in \mathcal{H}$. We claim that either h intersects all Y_λ or it is disjoint from all Y_λ. Indeed, we have $\mathcal{G}(Y, Y_\lambda) = (Y, Y_\lambda)$ for all $\lambda \in \Lambda$. It follows from Lemma 2.7 that if h intersects some Y_λ, then it intersects Y, and hence it intersects all Y_λ.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the collections of hyperplanes satisfying the first assertion and the second assertion in the claim, respectively. For any $h \in \mathcal{H}_2$, there exist $\lambda, \lambda' \in \Lambda$ such that h separates Y_λ from $Y_{\lambda'}$. Thus h intersects every hyperplane in $\mathcal{H}_1$. Note that $\mathcal{H}$ is the collection of hyperplanes that intersect P_Y and $\mathcal{H}_1$ is the collection of hyperplanes that intersect Y. Thus by Lemma 2.8, P_Y admits a product decomposition $P_Y = Y \times Y^\perp$. □

3. Cocompact cores

The main goal of this section is to prove Theorem 3.8 on existence of cocompact cores for top rank products of hyperbolic groups. The first step towards it is to study flats in a CAT(0) cube complex, which we do in Section 3.1. A hurried reader can proceed directly to Section 3.2 and use [47, Theorem 2.6] instead. However, our Theorem 3.4 is of independent interest.

3.1. Combinatorial convex hull of a flat. Throughout this paper a *flat* is a CAT(0) flat, i.e. an isometrically embedded copy of $\mathbb{R}^n$, not necessarily combinatorial. A *half-flat* is an isometrically embedded copy of $\mathbb{R}^{n-1} \times [0, \infty)$.

Lemma 3.1. *Let X be a* CAT(0) *cube complex and let* $F \subseteq X$ *be a flat. Let h be a hyperplane in X intersecting F, and let h^+ and h^- be the halfspaces of h. Then* *either $F \subseteq h$, or $h \cap F$ is a codimension-1 flat in F. In the latter case, both $h^+ \cap F$ and $h^- \cap F$ are half-flats.*

Proof. The carrier N_h of h, which is its neighbourhood, has the form $N_h = h \times [0,1]$. Thus if $F \not\subseteq h$, then $h \cap F$ is a codimension-1 submanifold of F. Moreover, the intersections $h \cap F$, $h^+ \cap F$, and $h^- \cap F$ are convex, thus the lemma follows. □

Lemma 3.2. *Let h be a hyperplane in a* CAT(0) *cube complex X. Suppose that l is a geodesic ray in X starting in h. If $l \not\subseteq h$, then there exists another hyperplane h' in X intersecting l and disjoint from h.*

Proof. Let N_h be the carrier of h. Let B be the first cube outside N_h whose interior is intersected by l. We claim that there is a hyperplane h' intersecting B and disjoint from h. Indeed, pick a vertex $v \in N_h \cap B$ and let e be an edge of B containing v. If the hyperplane dual to e intersects h, then $e \subset N_h$. If this holds for any e, then $B \subset N_h$ by the convexity of N_h, which yields a contradiction. This justifies the claim.

By the claim, there a hyperplane h' intersecting B and disjoint from h. It remains to prove that l intersects h'. Otherwise, since l intersects the interior of the carrier $N_{h'}$, we have that l is contained in $N_{h'}$. Since l starts at h, we have that h intersects $N_{h'}$ and hence it also intersects h', which is a contradiction. □

We will also use a consequence of a result of Haglund [22, Theorem 2.28].

Theorem 3.3. *Let X be a hyperbolic* CAT(0) *cube complex. Then any quasi-isometrically embedded subspace of X is at finite Hausdorff distance from its combinatorial convex hull.*

In the following theorem we generalise our results from [20, Section 3]. Here d_{Haus} denotes the Hausdorff distance.

Theorem 3.4. *Let X be a* CAT(0) *cube complex of asymptotic rank n and let $F \subseteq X$ be an n-flat. Let Y be the combinatorial convex hull of F. Then $d_{\mathrm{Haus}}(F, Y) < \infty$.*

Proof. If F is contained in the carrier $N_h = h \times [0,1]$ of a hyperplane h, then we can replace X by h and F by its projection to h. The combinatorial convex hull Y of F equals $Y' \times [0,1]$, $Y' \times \{0\}$, or $Y' \times \{1\}$, where Y' is the combinatorial convex hull of the projection of F to h. Henceforth we can and will assume that F is not contained in the carrier of any hyperplane.

Let $\mathcal{H}$ be the collection of hyperplanes intersecting F. We define a *pencil of hyperplanes* to be an infinite collection of mutually disjoint hyperplanes $\{h_i\}_{i=-\infty}^{\infty}$ such that for each i, $\{h_j\}_{j=-\infty}^{i-1}$ and $\{h_j\}_{j=i+1}^{\infty}$ are in different halfspaces of h_i. It follows from Lemma 3.1 that every pencil of hyperplanes in $\mathcal{H}$ intersects F in a collection of parallel family of codimension-1 flats. A collection of pencils of hyperplanes in $\mathcal{H}$ is *independent* if their corresponding normal vectors are linearly independent in $F = \mathbb{R}^n$.

Let $\{P_i\}_{i=1}^m$ be a maximal collection of pairwise independent pencils in $\mathcal{H}$. We claim that $m = n$ and that $\{P_i\}$ is independent. Suppose first $m > n$. Note that if two pencils $P, P' \subseteq \mathcal{H}$ are independent, then every hyperplane in P intersects every hyperplane in P'. This gives rise to a quasi-isometric embedding of $\mathbb{R}^m$ into X, contradicting the bound on the asymptotic rank of X. If $m < n$ or if $m = n$ but $\{P_i\}$ is dependent, then there is a geodesic line l in F parallel to $h \cap F$ for all hyperplanes h in all P_i. Using Lemma 3.2, we can then produce a new pencil P formed of some hyperplanes intersecting l. Since P is independent from each P_i, this contradicts the maximality of m. This justifies the claim that $m = n$ and $\{P_i\}$ is independent.

For $1 \le i \le n$, choose $h_i \in P_i$ and let $F_i = h_i \cap F$. We will prove that for any hyperplane $h \in \mathcal{H}$, there exists F_i such that $h \cap F$ is parallel (possibly equal) to F_i. Otherwise, choose a geodesic line l in F transverse to $h \cap F$. By Lemma 3.2, h is contained in a pencil P_h of hyperplanes intersecting l. Note that P_h is independent from each P_i, contradicting the maximality of m.

Let $\mathcal{H}_i \subseteq \mathcal{H}$ be the collection of hyperplanes whose intersection with F is parallel to F_i. The above discussion implies $\mathcal{H} = \bigsqcup_{i=1}^n \mathcal{H}_i$. Moreover, for $i \neq j$, every hyperplane in $\mathcal{H}_i$ intersects every hyperplane in $\mathcal{H}_j$. Let Y be the combinatorial convex hull of F. Since we assumed that F is not contained in the carrier of any hyperplane, the hyperplanes in Y are exactly the intersections with Y of the hyperplanes in $\mathcal{H}$. Two hyperplanes of Y intersect if and only if the corresponding hyperplanes in $\mathcal{H}$ intersect. Hence by Lemma 2.8, we have a product decomposition $Y = Y_1 \times \cdots \times Y_n$.

Let $\pi_i : Y \to Y_i$ be the coordinate projections. Let $l_i = \bigcap_{j \neq i} F_j$, which is a geodesic line in F. Note that for $j \neq i$ we have $l_i \subseteq F_j \subseteq h_j$ and hence the projection $\pi_j(l_i)$ is a single point. Thus the restriction of π_i to l_i is an isometric embedding. It follows that $F = \pi_1(l_1) \times \cdots \times \pi_1(l_n)$. Moreover, since $\pi_i(l_i) = \pi_i(F)$, each Y_i is the combinatorial convex hull of $\pi_i(l_i)$, since otherwise we could pass to a smaller convex subcomplex containing F.

Since each of Y_i contains a line and their product has asymptotic rank $\le n$, by Theorem 2.3(1) each Y_i has asymptotic rank 1. By Theorem 2.3(2) each Y_i is hyperbolic. Thus by Theorem 3.3, we have $d_{\mathrm{Haus}}(\pi_i(l_i), Y_i) < \infty$, and consequently $d_{\mathrm{Haus}}(F, Y) < \infty$. □

While we will not need it in the remaining part of the paper, from the proof above we can deduce the following interesting result which concerns flats that are not necessarily of top rank.

Corollary 3.5. *Let X be a* CAT(0) *cube complex and let $F \subseteq X$ be a flat. Let $Y \subseteq X$ be the combinatorial convex hull of F. Then Y has a natural decomposition $Y = Y_1 \times \cdots \times Y_n \times K$ such that:*

(1) *$n \ge \dim(F)$ and K is a cube.*

(2) *each Y_i contains an isometrically embedded copy of $\mathbb{R}$ that is the projection of a geodesic line in F.*

(3) *no Y_i contains a facing triple of hyperplanes, that is, a collection of three disjoint hyperplanes such that none of them separates the other two.*

Roughly speaking, (3) means that Y_i do not "branch".

3.2. Product of hyperbolic groups.

Definition 3.6. Let X be a CAT(0) cube complex. A group $H \leq Aut(X)$ is *convex cocompact* if there is a convex subcomplex $Y \subseteq X$ that is H-*cocompact*, meaning that H preserves Y and acts on it cocompactly.

Lemma 3.7. *Let X be a* CAT(0) *cube complex and let $H \leq Aut(X)$ be convex cocompact. Then there exists a minimal H-invariant convex subcomplex. Moreover, any minimal H-invariant convex subcomplex is H-cocompact and any two minimal H-invariant convex subcomplexes are parallel.*

Proof. Let $Y \subseteq X$ be an H-cocompact convex subcomplex. Let $\mathcal{P}$ be the poset of H-invariant convex subcomplexes in Y. For the first assertion, by the Kuratowski–Zorn Lemma, it suffices to show that every descending chain of elements $\{Y_\lambda\}_\lambda \subseteq \mathcal{P}$ has a lower bound, or equivalently that their intersection is nonempty. Let $K \subseteq Y$ be compact such that $HK = Y$. Then each $K \cap Y_\lambda$ is nonempty, and by compactness of K so is their intersection.

For the second and third assertion, let $Y_{\min} \subseteq Y$ be a minimal element of $\mathcal{P}$ and let Y' be any other minimal H-invariant convex subcomplex. Let $(V, V') = \mathcal{G}(Y_{\min}, Y')$. Then both V and V' are H-invariant. By Lemma 2.6(1) both V and V' are convex subcomplexes, hence from minimality of $Y_{\min}$ and Y' we have $V = Y_{\min}$ and $V' = Y'$. Moreover, by Lemma 2.6(2) we have that Y' is H-equivariantly isometric to $Y_{\min}$ and thus it is H-cocompact. □

Theorem 3.8. *Let X be a locally finite* CAT(0) *cube complex of asymptotic rank n. Let $H \leq Aut(X)$ be a subgroup satisfying*

(1) *$H = H_1 \times \cdots \times H_n$, where each H_i is an infinite hyperbolic group, and*

(2) *for some (hence any) point $x \in X$ the orbit map $h \to h \cdot x$ from H to X is a quasi-isometric embedding.*

Then H is convex cocompact. More precisely, if among H_i exactly $\{H_i\}_{i=1}^m$ are not virtually $\mathbb{Z}$, then there is a convex subcomplex $Y \subseteq X$ with a cubical product decomposition $Y = Y_0 \times \prod_{i=1}^m Y_i$ such that

(i) *Y is H-cocompact, and the action $H \curvearrowright Y$ respects the product decomposition, and*

(ii) *the induced action of $\prod_{i=m+1}^n H_i$ on Y_0 is proper and cocompact, in particular Y_0 is quasi-isometric to $\mathbb{R}^{n-m}$, and*

(iii) *for any pair* $i \neq j$ *with* $1 \leq j \leq m$ *and* $1 \leq i \leq n$*, the induced action* $H_i \curvearrowright Y_j$ *is* almost trivial*, i.e. by isometries at uniformly bounded distance from the identity.*

In the proof we need the notion of coarse intersection. Let X be a metric space and let $N_R(Y)$ be the R-neighbourhood of a subspace $Y \subseteq X$. A subspace $V \subseteq X$ is the *coarse intersection* of Y_1 and Y_2 if V is at finite Hausdorff distance from $N_R(Y_1) \cap N_R(Y_2)$ for all sufficiently large R. For example, in Lemma 2.6, in view of its part (3), the gates V_1, V_2 are the coarse intersections of Y_1 and Y_2. However, in general the coarse intersection of two subsets might not exist.

Lemma 3.9 ([31, Lemma 2.2]). *Let X be a finitely generated group with word metric. Then the intersection of a pair of subgroups is their coarse intersection.*

See [31, Chapter 2] for more discussion on coarse intersection.

Proof of Theorem 3.8. We first prove that H is convex cocompact, which we do by the induction on m. Consider first the case $m = 0$. Recall that all CAT(0) cube complexes in the article were assumed to be finite-dimensional. Thus by [8], H acts on X be semi-simple isometries. By the Flat Torus Theorem [7, Chapter II.7], H acts cocompactly on an n-flat $F \subseteq X$. By Theorem 3.4, the combinatorial convex hull Y of F is at finite Hausdorff distance from F. Since X is locally finite, Y is H-cocompact, as desired.

Suppose now that $m \geq 1$. Let $H' = \prod_{i \neq m} H_i$. We first prove that the group H' is convex cocompact. Choose a subgroup $Z \leq H_m$ isomorphic to $\mathbb{Z}$ and choose $h \in H_m$ such that the coarse intersection of hZ and Z is bounded. Let $G = H' \times Z \subset H$. By induction assumption, there exists a G-cocompact convex subcomplex $U \subset X$. Let $V \subset U$ be the gate with respect to $h \cdot U$. Note that both U and $h \cdot U$ are H'-invariant, so V is H'-invariant. By Lemma 2.6(3), V is the coarse intersection of U and $h \cdot U$. Hence, by Lemma 3.9 applied to G and hGh^{-1}, the action $H' \curvearrowright V$ is cocompact.

By Lemma 3.7, there exists a minimal H'-cocompact convex subcomplex, for which we keep the notation V. Then for any $h \in H_m$, the translate $h \cdot V$ is minimal H'-invariant, hence parallel to V by Lemma 3.7. Let $P_V = V \times V^\perp$ be the combinatorial parallel set of V (see Lemma 2.9). We have that P_V is H-invariant. Moreover, since V is H'-invariant, there are induced actions $H \curvearrowright V^\perp$ and $H_m \curvearrowright V^\perp$.

Choose a point $v \in V$. Let $\psi : H_m \to V^\perp$ be the composition of the orbit map $h \to h \cdot v$ with the coordinate projection. We claim that ψ is a quasi-isometric embedding. This follows from assumption (2) and the estimates below, where $\sim$ means equality up to a uniform multiplicative and additive constant. Namely, for any $h_1, h_2 \in H_m$ we have:

$$d_{H_m}(h_1, h_2) \sim d_H(h_1 H', h_2 H') \sim d_X(h_1 \cdot V, h_2 \cdot V) = d_{V^\perp}(\psi(h_1), \psi(h_2))$$

By Theorem 2.3, since V contains an isometrically embedded copy of $\mathbb{R}^{n-1}$, the asymptotic rank of $V^\perp$ is ≤ 1, and hence $V^\perp$ is hyperbolic. Let $V_m \subseteq V^\perp$

be the combinatorial convex hull of $\psi(H_m)$. Then $d_{\mathrm{Haus}}(V_m, \psi(H_m)) < \infty$ by Theorem 3.3. Moreover, V_m is H-invariant under the action $H \curvearrowright V^{\perp}$ since $\psi(H_m)$ is invariant under H. Thus H acts cocompactly on the convex subcomplex $V \times V_m \subseteq P_V$. Notice that since $H' \curvearrowright \psi(H_m)$ is trivial, the action $H' \curvearrowright V_m$ is almost trivial.

By now we already know that H is convex cocompact. As for properties (i)–(iii), if $m = 1$, then it suffices to take $Y_0 = V$ and $Y_1 = V_1$. If $m \geq 2$, to obtain the required decomposition, we consider $X' = V \times V_m$, $H'' = \prod_{i \neq (m-1)} H_i$ and we repeat the previous argument. This gives rise to an H-cocompact convex subcomplex $V' \times V_{m-1} \subseteq V \times V_m$, where V' is a minimal H''-cocompact convex subcomplex. Since V_m is contained in some R-neighbourhood of a V', the intersection $V_{m-1} \cap V_m$ is compact. Moreover, V' and V_{m-1} admit cubical product decompositions

$$V' = (V' \cap V) \times (V' \cap V_m) \quad \text{and} \quad V_{m-1} = (V_{m-1} \cap V) \times (V_{m-1} \cap V_m),$$

thus

$$V' \times V_{m-1} = (V' \cap V) \times (V' \cap V_m) \times (V_{m-1} \cap V) \times (V_{m-1} \cap V_m).$$

The H-action respects the above decomposition. Moreover, the induced action $H' \curvearrowright (V' \cap V_m)$ is almost trivial and the induced action $H'' \curvearrowright (V_{m-1} \cap V)$ is almost trivial. If $m = 2$, then we take $Y_1 = V_1 \cap V$, $Y_2 = V' \cap V_2$, and $Y_0 = (V \cap V') \cup (V_1 \cap V_2)$. If $m \geq 3$, then we let $X'' = V' \times V_{m-1}$, $H''' = \prod_{i \neq (m-2)} H_i$ and we repeat the previous process to obtain further product decomposition. We run this process m times, obtaining the required decomposition as the result of the last step. In each step, we possibly get nontrivial compact factors similar to $V_{m-1} \cap V_m$. We absorb all these compact factors into the factor Y_0 (we can also discard them). □

4. Artin groups

4.1. Background on Artin groups. Let A be an Artin group with defining graph Γ, and generators S. Let W be the Coxeter group defined by Γ. For any $T \subseteq S$ let W_T (respectively A_T) be the *special subgroup* of W (respectively A) generated by T. The special subgroup W_T is naturally isomorphic to the Coxeter group defined by the subgraph Γ_T induced on T [10]. Similarly, by [41] the special subgroup A_T of A is naturally isomorphic to the Artin group defined by Γ_T.

Lemma 4.1 ([16, Theorem 1.1]). *Special subgroups of Artin groups are convex with respect to the word metric defined by standard generators.*

A subset $T \subseteq S$ is *spherical* if the special subgroup W_T is finite. The *dimension* of the Artin group A is the maximal cardinality of a spherical subset of S.

The following is a consequence of [15] and [14, Corollary 1.4.2].

Theorem 4.2. *Let A be an Artin group of dimension n. Suppose that*

(A) *$n \leq 2$, or*

(B) *every clique T in Γ is spherical.*

Then there is a finite n-dimensional cell complex that is a $K(A, 1)$.

4.2. Two-generator Artin groups. We start with the description of most two-generator Artin groups as virtually $F_k \times \mathbb{Z}$, where F_k is the free group with k generators.

Lemma 4.3. *Let A be an Artin group with defining graph Γ a single edge labelled by $n > 2$. Then*

(1) *A has a finite index subgroup of form $F_k \times \mathbb{Z}$ with $k \geq 2$, and*

(2) *no power of one of the two standard generators lies in the $\mathbb{Z}$ factor.*

Proof. By [6] (or by our proof of Theorem 5.1) A acts freely and cocompactly on a product of a tree and a line, where a central element acts as a translation in the line factor. By [7, Theorem II.6.12] A virtually decomposes as $A' \times \mathbb{Z}$. The induced action of A' on the tree factor has finite vertex stabilisers so by Bass–Serre theory A' is a graph of finite groups, in particular A' is virtually free, justifying (1). Part (2) follows from the fact that standard generators act hyperbolically on the tree factor. □

Throughout this section by $\bar{x}$ we denote the inverse of x. By x^z we denote the conjugate $\bar{z}xz$.

Let $A_n = \langle a, b \mid \underbrace{aba\ldots}_{n} = \underbrace{bab\ldots}_{n}\rangle$. Denote $\Delta = \underbrace{aba\ldots}_{n} = \underbrace{bab\ldots}_{n}$. Let A'_n be the kernel of the homomorphism sending each generator to the generator of $\mathbb{Z}/2$ i.e. the subgroup consisting of all words of even length. The group A'_n is generated by the elements: $r = ab, s = a\bar{b}, t = \bar{a}b$. If ϕ is a word in an alphabet Λ, and $x \in \Lambda$, then we denote by $\mathrm{Exp}_x(\phi)$ the sum of all the exponents at x in ϕ.

By direct computation we immediately establish the following:

Lemma 4.4. *If n is odd, then the conjugation by Δ is an order two automorphism sending $s \mapsto \bar{s}, t \mapsto \bar{t}, r \mapsto q$, where $q = ba = \bar{s}r\bar{t}$. In particular, Δ^2 is a central element.*

If n is even, then Δ is a central element.

Let z be the element Δ^2 for n odd and the element Δ for n even.

Lemma 4.5. *If n is odd, then we have*

$$b^n = \phi(s, t, r)\Delta,$$

where $\mathrm{Exp}_r(\phi) = 0$.

Proof. Consider the following word ϕ expressed as a product of terms indexed by decreasing i:

$$\phi(s,t,r) = \bar{s} \prod_{i=\frac{n-3}{2}}^{0} \bar{t}^{r^i}$$

Since r^i appear in the expression defining ϕ only as elements that we conjugate by, we have $\mathrm{Exp}_r(\phi) = 0$.

To verify that $b^n = \phi\Delta$, note that

$$\phi = \bar{s} \prod_{i=\frac{n-3}{2}}^{0} \bar{r}^i \bar{t} r^i = \bar{s}(\bar{r}^{\frac{n-3}{2}} \bar{t} r^{\frac{n-3}{2}})(\bar{r}^{\frac{n-3}{2}-1} \bar{t} r^{\frac{n-3}{2}-1}) \cdots (\bar{r}\bar{t}r)\bar{t} = \bar{s}\bar{r}^{\frac{n-1}{2}}(r\bar{t})^{\frac{n-1}{2}}.$$

Since $\bar{s}\bar{r}^{\frac{n-1}{2}} = b\bar{a}(\bar{b}\bar{a})^{\frac{n-1}{2}} = b\bar{\Delta}$ and $r\bar{t}\Delta = \Delta qt = \Delta b^2$, we have

$$\phi(s,t,r)\Delta = \bar{s}\bar{r}^{\frac{n-1}{2}}\Delta b^{n-1} = b\bar{\Delta}\Delta b^{n-1} = b^n. \qquad \square$$

Corollary 4.6. *If n is odd, we have*

$$b^{2n}\bar{z} \in [A'_n, A'_n].$$

Proof. We have

$$b^{2n} = \phi(s,t,r)\Delta\phi(s,t,r)\Delta = \phi(s,t,r)\phi(\bar{s},\bar{t},q)z.$$

Denote the word $\phi(s,t,r)\phi(\bar{s},\bar{t},q)$ by $\psi(s,t,r,q)$. By Lemma 4.5, we have $\mathrm{Exp}_r(\psi) = \mathrm{Exp}_q(\psi) = 0$. We also have $\mathrm{Exp}_s(\psi) = \mathrm{Exp}_t(\psi) = 0$ since the total exponents of s and t in $\phi(s,t,r)$ are equal to the total exponents of $\bar{s}$ and $\bar{t}$ in $\phi(\bar{s},\bar{t},q)$, respectively. Thus $\psi \in [A'_n, A'_n]$. $\square$

4.3. Surface lemma. The following lemma will allow us to utilise the preceding result when discussing finite index subgroups of A_n.

Lemma 4.7. *Let G be a finitely generated group and let $z \in G$ be central. Let H be a finite index normal subgroup of G, and let $h \in H \cap z[G,G]$. Then for any homomorphism $\rho : H \to \mathbb{Z}$ such that $\rho(\langle z\rangle \cap H) \neq \{0\}$, there exist a positive integer m and $g \in G$ with $\rho((h^m)^g) \neq 0$.*

Proof. Let X be a presentation complex for G. Let S be an oriented surface with connected ∂S and basepoint $s \in \partial S$, mapping to X, such that on the level of fundamental groups $\partial S \mapsto h\bar{z}$. Let $\widehat{X}$ be the finite cover of X corresponding to H and let $\widehat{S}$ be a finite cover of S such that $\widehat{S} \to S \to X$ lifts to $\widehat{S} \to \widehat{X}$. Choose a system Σ of nonintersecting arcs that join the basepoint of $\widehat{S}$ to the other preimages of s, one for each of the boundary components of $\widehat{S}$. Consider the surface S' obtained from $\widehat{S}$ by cutting along the arcs of Σ, and the mapping $S' \to \widehat{X}$ that factors through $\widehat{S}$. Then, as the boundary of a surface, $\partial S'$ is mapped to an element $f \in H = \pi_1(\widehat{X})$ contained in $[H, H]$. The arcs of Σ map to paths in $\widehat{X}$ that project to closed paths in X corresponding to some $g_i \in G$. Thus we have $f = \prod_{i=1}^{q} (h^{m_i})^{g_i} \bar{z}^M$, where $m_i \geq 1$ with $M = \sum m_i$.

Since H is normal, each $(h^{m_i})^{g_i}$ lies in H. We have

$$\rho\left(\prod_{i=1}^{q} (h^{m_i})^{g_i}\right) = \rho(z^M) \neq 0.$$

That means that there is at least one element $(h^{m_i})^{g_i}$ such that $\rho((h^{m_i})^{g_i}) \neq 0$. □

Corollary 4.8. *Let n be odd and let H be a finite index normal subgroup of A'_n. Then for any homomorphism $\rho : H \to \mathbb{Z}$ such that $\rho(\langle z\rangle \cap H) \neq \{0\}$, there exist a positive integer m and $g \in A'_n$ such that $b^m \in H$ and $\rho((b^m)^g) \neq 0$.*

Proof. Let k be large enough so that $b^{2nk} \in H$. By Corollary 4.6, we can apply Lemma 4.7 with $G = A'_n$, $h = b^{2nk}$, and z^k in the role of z. □

Corollary 4.9. *Let n be even and let H be a finite index normal subgroup of A_n. Then for any homomorphism $\rho : H \to \mathbb{Z}$ such that $\rho(\langle z\rangle \cap H) \neq \{0\}$, there exist a positive integer m and $g \in A_n$ such that at least one of $(a^m)^g$ and $(b^m)^g$ lies in H and is not mapped to 0 under ρ.*

Proof. Let $k = \frac{n}{2}k'$ be a nonzero integer such that $a^k, b^k \in H$. Since $z^{k'} = (ab)^k$, we have

$$a^k b^k \in z^{k'}[A_n, A_n].$$

By Lemma 4.7, we have $m > 0$ and $g \in A_n$ such that $\rho\big(((a^k b^k)^m)^g\big) \neq 0$. Let $f = (a^k)^g$ and $h = (b^k)^g$. We have $(fh)^m \in f^m h^m [H, H]$. Thus $\rho(f^m h^m) \neq 0$ and so at least one of $f^m = (a^{km})^g$ and $h^m = (b^{km})^g$ is not mapped to 0 under ρ. □

5. The main theorem

In this section we prove Theorem 1.1. The implication (i) ⇒ (ii) is obvious.

5.1. Implication (iii) ⇒ (i).

Theorem 5.1. *Let A be an Artin group with each connected component of the defining graph:*

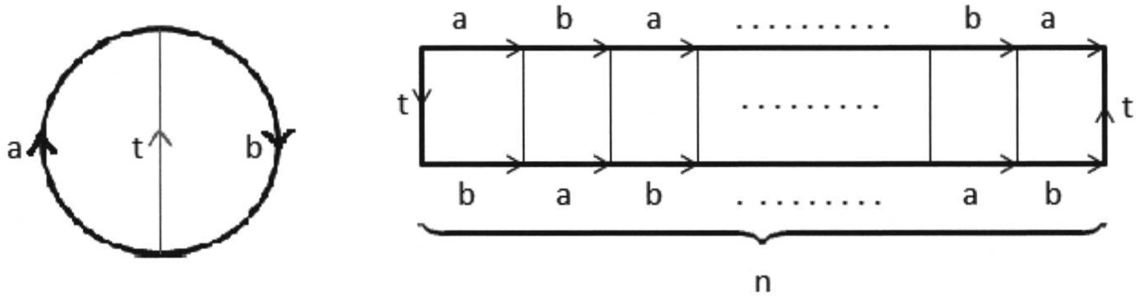
- *a vertex, or an edge, or else*
- *all interior edges labeled by 2 and all leaves labelled by even numbers.*

Then A is the fundamental group of a nonpositively curved cube complex.

Proof. We assume without loss of generality that Γ is connected, since if Γ has more connected components, then A is the fundamental group of the wedge of the complexes obtained for its connected components.

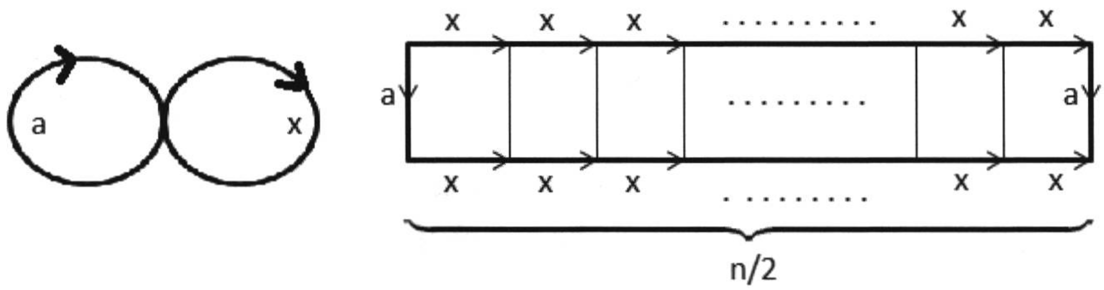
If Γ is a single vertex, then A is the fundamental group of a circle.

If Γ is a single edge labelled by an odd n, then let K_n be the cube complex described in the figure below.



On the left side we see part of the 1-skeleton of K_n consisting of three edges labelled by a, b, t, and the right side indicates how to attach a rectangle (subdivided into n squares) along its boundary path $\underbrace{ab\ldots a}_{n}\,\bar{t}\,\underbrace{\bar{b}\bar{a}\ldots\bar{b}}_{n}\,\bar{t}$. It is easy to check that the link of each of the two vertices in K_n is isomorphic to the spherical join of two points with n points, hence K_n is nonpositively curved. By collapsing the t-edge we obtain the presentation complex for the standard presentation of A, so $\pi_1(K_n) = A$. We learned this construction from Daniel Wise.

If Γ is a single edge labelled by an even n, let $x = ab$. The group A is then presented as $\langle a, x \mid ax^{n/2} = x^{n/2}a\rangle$. Let $K_{n,a}$ be the cube complex described in the figure below.



One can check that the link of the unique vertex in $K_{n,a}$ is isomorphic to the spherical join of two points with n points, hence $K_{n,a}$ is nonpositively curved. It is clear that $\pi_1(K_{n,a}) = A$.

Similarly if we let $y = ba$, then A can be presented as $\langle b, y \mid by^{n/2} = y^{n/2}b\rangle$. We define $K_{n,b}$ in a similar way. Note that the a-circle in $K_{n,a}$ is a locally convex subcomplex, so is the b-circle in $K_{n,b}$.

If Γ contains more than one edge, then let $\Gamma' \subseteq \Gamma$ be the nonempty subgraph induced on all the vertices that have at least two neighbours. Thus the edges of Γ' are precisely the interior edges and by the hypothesis they are labelled by 2. Hence $A_{\Gamma'}$ is a right-angled Artin group. The *Salvetti complex* $S(\Gamma')$ is the nonpositively curved cube complex obtained from the presentation complex of $A_{\Gamma'}$ by adding the missing cubes of higher dimension (see [13]). Let $\{(s_i, t_i)\}_{i=1}^k$ be the collection of leaves of Γ with $s_i \in \Gamma'$. Let n_i be the label of the edge (s_i, t_i), which is even. Let K be the amalgamation of $\{K_{n_i, s_i}\}_{i=1}^k$ and $S(\Gamma')$ along the s_i-circles. Then $\pi_1(K) = A$ and it follows from [7, Proposition II.11.6] that K is nonpositively curved. □

5.2. Implication (ii) ⇒ (iii).

Theorem 5.2. *Let A be a 2-dimensional Artin group. If A is virtually cocompactly cubulated, then each connected component of the defining graph of A is either*

- *a vertex, or an edge, or else*
- *all its interior edges are labeled by 2 and all its leaves are labelled by even numbers.*

Proof. Suppose that there exists a finite index subgroup $\hat{A} \leq A$ that acts properly and cocompactly by combinatorial automorphisms on a CAT(0) cube complex X. Without loss of generality, we assume that $\hat{A}$ is normal in A. It suffices to prove:

(1) no edge of Γ has an odd label, unless it is an entire connected component, and

(2) no interior edge of Γ has an even label ≥ 4.

Let us first prove (1). Suppose to the contrary that Γ has an edge (a, b) with odd label and another edge (b, c). Let A_{ab} be the special subgroup generated by a and b. By A'_{ab} we denote its index-two subgroup that is the kernel of the homomorphism to $\mathbb{Z}/2$ sending both a and b to 1. Let $\hat{A}_{ab} = F_k \times \mathbb{Z}$ be a finite index subgroup of $A'_{ab} \cap \hat{A}$ guaranteed by Lemma 4.3(1). We can also assume that $\hat{A}_{ab}$ is normal in A'_{ab}. Similarly, let A_{bc} be the special subgroup generated by b and c, and let $\hat{A}_{bc} = F_l \times \mathbb{Z}$ be a finite index subgroup of $A_{bc} \cap \hat{A}$. Note that the edge (b, c) might be labelled by 2 and then $l = 1$.

Since $\hat{A}$ is a CAT(0) group, we can speak of its asymptotic rank. By Theorem 4.2(A), there exists a finite 2-dimensional cell complex that is a $K(A, 1)$. Thus by Lemma 2.5, the asymptotic rank of $\hat{A}$ is ≤ 2 and so is the asymptotic rank of X. The subgroup A_{ab} is convex with respect to the standard generators of A by Lemma 4.1 and so $\hat{A}_{ab}$ is quasi-isometrically embedded in $\hat{A}$. We can thus apply Theorem 3.8 to find a convex subcomplex Y_{ab} that is $\hat{A}_{ab}$-cocompact. Moreover, there is a cubical product decomposition $Y_{ab} = V_{ab} \times H_{ab}$ such that the action of $\hat{A}_{ab}$ respects this decomposition, the vertical factor V_{ab} is quasi-isometric to $\mathbb{R}$, and the $\mathbb{Z}$ factor Z of $\hat{A}_{ab}$ acts almost trivially on H_{ab}.

Consider $\mathrm{Min}(Z) = \mathbb{R} \times V_0 \subseteq V_{ab}$ for the induced action of Z, where $\mathbb{R}$ is an axis of Z. Since Z is contained in the centre of $\hat{A}_{ab}$, we have an induced action of $\hat{A}_{ab}$ on $\mathbb{R} \times V_0$ respecting this decomposition. The factor V_0 is bounded, so V_0

contains a fixed-point of the action of $\hat{A}_{ab}$. Thus $\mathbb{R} \times V_0$ contains an $\hat{A}_{ab}$–invariant line l. Let $\rho : \hat{A}_{ab} \to \mathrm{Isom}(l)$ be the induced map. Note that $\rho(\hat{A}_{ab})$ does not flip the ends of l. Moreover, since V_{ab} is a cube complex, the translation lengths on l are discrete. This gives rise to a homomorphism $\rho : \hat{A}_{ab} \to \mathbb{Z}$ assigning to each element of $\hat{A}_{ab}$ its translation length on l. Note that $\rho(Z) \neq 0$. By Corollary 4.8 applied to $H = \hat{A}_{ab}$, there exists a nonzero integer m and $g \in A'_{ab}$ such that $\rho((b^m)^g) \neq 0$.

By normality of $\hat{A}$, we have $(\hat{A}_{bc})^g \leq \hat{A}$. Let Y_{bc} be a convex $(\hat{A}_{bc})^g$-cocompact subcomplex guaranteed again by Theorem 3.8. By [41] we have $A_{ab} \cap A_{bc} = A_b$, and hence the groups $\langle b^m \rangle^g$ and $\hat{A}_{ab} \cap (\hat{A}_{bc})^g$ have a common finite index subgroup B. Let $Y \subset Y_{ab}$ be the gate with respect to Y_{bc}. Then Y is the coarse intersection of Y_{ab} and Y_{bc} by Lemma 2.6(3). By Lemma 3.9, Y is B-cocompact.

Since Y is a convex subcomplex, it has a product structure $Y = Y_V \times Y_H$ where $Y_V \subseteq V_{ab}$ and $Y_H \subseteq H_{ab}$. We have $\rho(B) \neq 0$, so Y_V is unbounded. Since Y is quasi-isometric to $\mathbb{R}$, the factor Y_H is bounded. Since Z acts almost trivially on H_{ab}, any of its orbits in Y_{ab} is at a finite Hausdorff distance from Y. Hence Z is commensurable with B. Thus there exists an integer $j \neq 0$ such that $(b^g)^j \in Z$, and hence $b^j \in Z$, contradicting Lemma 4.3(2).

Let us now prove (2). Suppose that Γ has edges (a,b), (b,c), and (c',a) (here c and c' are possibly the same), where (a,b) has an even label ≥ 4. Let $\hat{A}_{ab}, \hat{A}_{bc}, \hat{A}_{c'a}$ be finite index subgroups of $A_{ab} \cap \hat{A}$, $A_{bc} \cap \hat{A}$, $A_{c'a} \cap \hat{A}$, respectively, that are isomorphic to a product of a free group and $\mathbb{Z}$. Assume moreover that $\hat{A}_{ab}$ is normal in A_{ab}. Let $Y_{ab} = V_{ab} \times H_{ab}$ be a convex $\hat{A}_{ab}$-cocompact subcomplex, and let $\rho : \hat{A}_{ab} \to \mathbb{Z}$ be defined as before. By Corollary 4.9, there exist a nonzero integer m and $g \in A_{ab}$ such that at least one of $(a^m)^g$ and $(b^m)^g$ lies in $\hat{A}_{ab}$ and is not mapped to 0 under ρ. Without loss of generality we can assume $\rho((b^m)^g) \neq 0$. The rest of the argument is identical as in the proof of (1). □

6. 3-generator Artin groups

This section is devoted to the proof of Theorem 1.2. Let A be the three-generator Artin group with $m_{ab} = 3, m_{bc} = 2$, and $m_{ac} = 3, 4$, or 5, and let W be the Coxeter group with the same defining graph. Consider a longest word in a, b, c which is a minimal length representative of the element it represents in W. This word represents also an element of A, which we call Δ.

Lemma 6.1. (i) *The centre Z of A is generated by Δ^2 for $m_{ac} = 3$ and by Δ for $m_{ac} = 4$ or 5.*

(ii) *The intersections of A_{ab} and A_{bc} with Z are trivial.*

(iii) *In A we have $A_{ab} \times Z \cap A_{bc} \times Z = A_b \times Z$.*

Proof. Assertion (i) follows from [18, Theorem 4.21].

For (ii), let $\Delta_{ab} = aba$. By [18, Proposition 4.17], each element of A_{ab} is represented by $\Delta_{ab}^{-k}\phi(a,b)$, where ϕ is a positive word in a, b, and $k \geq 0$. If we had $\phi(a,b) = \Delta_{ab}^{k}\Delta^{l}$ for some $l > 0, k \geq 0$, then by [18, Theorem 4.14] this equality would also hold in the Artin semigroup, contradicting the fact that Δ is expressed as a positive word involving all a, b, c. The same argument works for A_{bc}.

For (iii) we need to show $A_{ab} \times Z \cap A_{bc} \times Z \subseteq A_b \times Z$. Since b and c commute, it suffices to show that for each $m \neq 0$ we have $c^m \notin A_{ab} \times Z$. If $m_{ac} = 3$, then this follows from a well known fact that A/Z is the mapping class group of the four punctured disc, where A_{ab} fixes a curve around the first three punctures and c is a half-Dehn twist in a curve around the third and the fourth.

If $m_{ac} = 4$ or 5, assume for contradiction that $c^m = gz$, for some $z \in Z$ and $g \in A_{ab}$. Thus $gc^m = g^2z = gzg = c^m g$. Let $g = \Delta_{ab}^{-k}\phi(a,b)$, where ϕ is a positive word in a, b, and $k \geq 0$ is even. Thus $\phi(a,b)c^m\Delta_{ab}^{k} = \Delta_{ab}^{k}c^m\phi(a,b)$.

By [18, Theorem 4.14] this equality also holds in the Artin semigroup. The relation $acac = caca$ or $acaca = cacac$ involves on each side 2 occurences of c separated by an occurence of a. The word $\phi(a,b)c^m\Delta_{ab}^{k}$ does not contain such a subword, and this property is invariant under the replacements $bc = cb, aba = bab$. Thus to pass from $\phi(a,b)c^m\Delta_{ab}^{k}$ to $\Delta_{ab}^{k}c^m\phi(a,b)$ one can only use $bc = cb$, and $aba = bab$, which is the relation defining A_{ab}. Thus there is l such that in A_{ab} we have $\phi(a,b)b^l = \Delta_{ab}^{k}$. Hence $g = b^{-l}$. Thus $c^m = b^{-l}z$, contradicting assertion (ii). □

We also need the following consequence of rank-rigidity [12].

Lemma 6.2. *Let G be a cocompactly cubulated group with centre containing $Z \cong \mathbb{Z}$. Then G has a finite index subgroup $G_0 \times Z$ with G_0 cocompactly cubulated.*

Proof. Suppose that G acts properly and cocompactly by cubical automorphisms on a CAT(0) cube complex X. By [12, Corollary 6.4(iii)], if we replace X with its essential core, and G with a finite-index subgroup, we obtain a cubical product decomposition of X respected by G, such that for each factor there is an element $g \in G$ acting on it as a rank one isometry. Let X_V be a factor on which Z acts freely, and combine all other factors into X_H, so that $X = X_H \times X_V$. Let $g \in G$ act on X_V as a rank one isometry.

Note that the generator z of Z acts on X_V as a rank one isometry. Otherwise an axis of g would not be parallel to an axis of z. Hence g and z would generate $\mathbb{Z}^2$ acting properly on X_V, contradicting the fact that g has rank one. Consider $\mathrm{Min}(Z) = \mathbb{R} \times Y \subseteq X_V$, where $\mathbb{R}$ is an axis of Z. Since Z is contained in the centre of G, we have an induced action of G on $\mathbb{R} \times Y$ respecting this decomposition. Since z has rank one, we have that Y does not contain a geodesic ray, and hence is bounded. Consequently, Y contains a fixed-point of the action of G. Thus X_V contains a G-invariant line l.

Let $\rho : G \to \mathrm{Isom}(l)$ be the induced map. Note that $\rho(G)$ does not flip the ends of l. Moreover, since X_V is a cube complex, the translation lengths on l are discrete. Thus the image of ρ can be identified with $\mathbb{Z}$, which contains $\rho(Z)$ as a finite index subgroup. Let $G_0 = \ker(\rho)$. Thus $Z \times G_0$ is a finite index subgroup of G. Moreover, G_0 acts properly by cubical automorphisms on $X_H \subset X$. Since the action of Z on X_V is proper, the action of G_0 on X_H is cocompact. □

We complement Lemma 6.2 with the following:

Lemma 6.3. *Let $G = G_0 \times Z$ be finitely generated, with $Z \cong \mathbb{Z}$. Let $H < G$ be a finite product of finitely generated free groups of rank ≥ 2 that is quasi-isometrically embedded.*

(i) *The map $H \to G/Z$ is a quasi-isometric embedding.*

(ii) *Let G be cocompactly cubulated. If we require that $H \cap Z$ is trivial, then assertion (i) holds also if in the product we allow free groups of rank* 1.

Proof. If H is a free group of rank ≥ 2, then we choose in H a free generating set $S^{\pm}$. In Z we consider the generating set $\{\pm 1\}$ and in G_0 any symmetric generating set. Let $|\cdot|_H, |\cdot|_Z, |\cdot|_{G_0}$ denote the corresponding word-lengths. Let π_{G_0}, π_Z be the coordinate projections from G to G_0, Z, respectively. By assumption, there exists a constant c such that for any $h \in H$, we have $|h|_H \leq c\big(|\pi_{G_0}(h)|_{G_0} + |\pi_Z(h)|_Z\big)$. Viewing h as a reduced word over $S^{\pm}$, choose $s \in S^{\pm}$ such that the word $w = hsh^{-1}s^{-1}$ is reduced. Then $|\pi_Z(w)|_Z = 0$, and applying the above inequality with w in place of h we obtain $2|h|_H + 2 \leq c|\pi_{G_0}(w)|_{G_0} \leq 2c\big(|\pi_{G_0}(h)|_{G_0} + |\pi_{G_0}(s)|_{G_0}\big)$. Consequently $|h|_H \leq c|\pi_{G_0}(h)|_{G_0} + a$ for some uniform constant a, and thus the restriction of π_{G_0} to H is a quasi-isometric embedding, as desired.

Similarly, if H is a product of free groups H_i of rank ≥ 2, then we choose generating sets $S_i^{\pm}$ in H_i. Let $h = \prod h_i$ with $h_i \in H_i$. To get an estimate on $|h|_H$, it suffices to use a product of reduced words $w = \prod h_i s_i h_i^{-1} s_i^{-1}$, with $s_i \in S_i^{\pm}$. This proves assertion (i).

If G is cocompactly cubulated, then by Lemma 6.2, after passing to a finite index subgroup, the quotient G/Z acts properly and cocompactly on a CAT(0) cube complex X. Let $H = \mathbb{Z}^n \times H_0 \leq G$, where H_0 is a finite product of finitely generated free groups of rank ≥ 2. We keep the notation H for the isomorphic image of H in G/Z. Then H preserves $\mathrm{Min}(\mathbb{Z}^n) = \mathbb{R}^n \times Y \subseteq X$ and respects its product structure. We fix $v \in \mathbb{R}^n$ and $y \in Y$. From assertion (i), the orbit map $h_0 \to (h_0 \cdot v, h_0 \cdot y)$ from H_0 to $\mathbb{R}^n \times Y$ is a quasi-isometric embedding. Since the commutator of H_0 acts trivially on the $\mathbb{R}^n$ factor, using the same argument as for assertion (i), we obtain c satisfying $|h_0|_{H_0} \leq c d_Y(y, h_0 \cdot y)$. On the other hand, there is c' such that for $f \in \mathbb{Z}^n$ we have $|f|_{\mathbb{Z}^n} \leq c' d_{\mathbb{R}^n}(v, f \cdot v)$. Let d be the maximum of the displacements $d_{\mathbb{R}^n}(v, s \cdot v)$ over the generators s of H_0. For $fh_0 \in H$ consider the maximum norm $\|fh_0\| = \max\{|f|_{\mathbb{Z}^n}, 2c'd|h_0|_{H_0}\}$. If $|f|_{\mathbb{Z}^n} \geq 2c'd|h_0|_{H_0}$, then

$$c'd_{\mathbb{R}^n}(v, fh_0 \cdot v) \geq |f|_{\mathbb{Z}^n} - c'd|h_0|_{H_0} \geq \frac{1}{2}|f|_{\mathbb{Z}^n} \geq \frac{1}{2}\|fh_0\|.$$

Otherwise, if $|f|_{\mathbb{Z}^n} < 2c'd|h_0|_{H_0}$, then

$$cd_Y(y, fh_0 \cdot y) = cd_Y(y, h_0 \cdot y) \geq |h_0|_{H_0} > \frac{1}{2c'd}\|fh_0\|.$$

This proves assertion (ii). □

Proof of Theorem 1.2. The implication (i) $\Rightarrow$ (ii) is obvious. The implication (iii) $\Rightarrow$ (i) follows from Theorem 5.1 unless the defining graph Γ of A has two edges (a,c), (b,c) with label 2. By Theorem 5.1, A_{ab} is the fundamental group of a nonpositively curved cube complex K. Then $K \times S^1$ is a nonpositively curved cube complex with fundamental group A.

The implication (ii) $\Rightarrow$ (iii) follows from Theorem 5.2 if A is 2-dimensional. Suppose now that A is not 2-dimensional. Then the labels of Γ are $m_{ab} = 3$, $m_{bc} = 2$, and $m_{ac} = 3, 4$, or 5. Let Z be the centre of A described in Lemma 6.1(i).

Suppose that there exists a normal finite index subgroup $\hat{A} \leq A$ that is cocompactly cubulated. Let $\hat{Z} = \hat{A} \cap Z$. By Lemma 6.2, up to replacing $\hat{A}$ with a further finite index subgroup, we have $\hat{A} = \hat{A}_0 \times \hat{Z}$, where $\hat{A}_0$ is cocompactly cubulated. We keep the notation $\hat{A}_0$ for its isomorphic image in the quotient A/Z. Note that $\hat{A}_0 \leq A/Z$ is a normal finite index subgroup.

By Theorem 4.2(B), the Artin group A is the fundamental group of a 3-dimensional cell complex which is a $K(A, 1)$. Thus, by Lemma 2.5, the asymptotic rank of $\hat{A}$ is ≤ 3. Hence the asymptotic rank of $\hat{A}_0$ is ≤ 2.

By Lemma 6.1(ii), the intersections of A_{ab} and A_{bc} with Z are trivial. Thus A_{ab} and A_{bc} embed into A/Z under the quotient map, and we keep the notation A_{ab} and A_{bc} for their images in A/Z. By Lemma 6.1(iii) in A/Z we have $A_{ab} \cap A_{bc} = A_b$.

Let $\hat{A}_{ab} = F_k \times \mathbb{Z}$ be a finite index subgroup of $A'_{ab} \cap \hat{A}_0$ guaranteed by Lemma 4.3(1). We can assume that $\hat{A}_{ab}$ is normal in A'_{ab}. Let $\hat{A}_{bc} = A_{bc} \cap \hat{A}_0 = \mathbb{Z}^2$. By Lemmas 4.1 and 6.3(ii), $\hat{A}_{ab}, \hat{A}_{bc} < A/Z$ are quasi-isometric embeddings.

From this point we argue to reach a contradiction exactly as in part (1) of the proof of Theorem 5.2. □

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J. Huang, Department of Mathematics and Statistics, McGill University,
Montreal, Quebec H3A 0B9, Canada

E-mail: jingyin.huang@mcgill.ca

K. Jankiewicz, Department of Mathematics and Statistics, McGill University,
Montreal, Quebec H3A 0B9, Canada

E-mail: kasia@math.mcgill.ca

P. Przytycki, Department of Mathematics and Statistics, McGill University,
Montreal, Quebec H3A 0B9, Canada

E-mail: piotr.przytycki@mcgill.ca