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Degree three cohomological invariants of reductive groups

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Abstract. We study the degree 3 cohomological invariants with coefficients in $\mathbb{Q}/\mathbb{Z}(2)$ of a split reductive group over an arbitrary field. As an application, we compute the group of reductive indecomposable degree 3 invariants of all split simple algebraic groups.

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1. Introduction

Let G be a linear algebraic group over a field F. Consider a functor

$$G\text{-torsors} : \mathsf{Fields}_F \longrightarrow \mathsf{Sets},$$

where Fields_F is the category of field extensions of F, taking a field K to the set of isomorphism classes of G-torsors over Spec K. Let

$$\Phi : \mathsf{Fields}_F \longrightarrow \mathsf{Abelian\ Groups}$$

be another functor. According to [8], a Φ-*invariant* of G is a morphism of functors

$$I : G\text{-torsors} \longrightarrow \Phi,$$

viewed as functors to Sets. We write $\mathrm{Inv}(G, \Phi)$ for the group of Φ-invariants of G.

An invariant $I \in \mathrm{Inv}(G, \Phi)$ is called *normalized* if $I(E) = 0$ for every trivial G-torsor E. The normalized invariants form a subgroup $\mathrm{Inv}(G, \Phi)_{\mathrm{norm}}$ of $\mathrm{Inv}(G, H)$ and

$$\mathrm{Inv}(G, \Phi) \simeq \Phi(F) \oplus \mathrm{Inv}(G, \Phi)_{\mathrm{norm}}.$$

We will be considering the cohomology functors Φ taking a field K/F to the Galois cohomology $H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ (see Section 3.1) and write $\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the group of *cohomological invariants of G of degree n with coefficients in* $\mathbb{Q}/\mathbb{Z}(j)$.

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If G is connected, then $\mathrm{Inv}^1\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{norm}} = 0$ by [13, Proposition 31.15]. The degree 2 cohomological invariants with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ (equivalently, the invariants with values in the Brauer group Br) of a reductive group were determined in [2, Theorem 2.4]: the group $\mathrm{Inv}^2\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{norm}}$ is isomorphic to the character group of the kernel C of the universal cover of G.

The group of degree 3 invariants $\mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)$ was determined by Rost in the case when G is simply connected (see [8, Part II]). If G is a split simply connected group, the group $\mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}}$ is isomorphic to $S^2(T^*)^W / \mathrm{Dec}(G)$, where T^* is the character group of a split maximal torus $T \subset G$, W is the Weyl group of G and $\mathrm{Dec}(G)$ is the subgroup of the "obvious" elements in $S^2(T^*)^W$ (see §5).

The case of an arbitrary semisimple group G was considered in [20]. If G is split, there is an exact sequence

$$0 \longrightarrow C^* \otimes F^\times \longrightarrow \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \longrightarrow S^2(T^*)^W / \mathrm{Dec}(G) \longrightarrow 0,$$

where C is the kernel of the universal cover of G. If G is simply connected, the group C is trivial and we get Rost's result. The image of $C^* \otimes F^\times$ in $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is the subgroup of *decomposable* invariants. These invariants are obtained from the degree 2 invariants by the cup-product with an element in $F^\times$.

The group of degree 3 invariants of algebraic tori was computed in [2] (this group is trivial for split tori).

In the present paper we consider the case of a split reductive group G, i.e., we generalize [20] in the split case. Our main result is the following theorem (see Theorem 5.1). Note that the exact sequence is the same as in the split semisimple case.

Theorem. *Let G be a split reductive group, $T \subset G$ a split maximal torus, W the Weyl group and C the kernel of the universal cover of the commutator subgroup of G. Then there is an exact sequence*

$$0 \longrightarrow C^* \otimes F^\times \longrightarrow \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \longrightarrow S^2(T^*)^W / \mathrm{Dec}(G) \longrightarrow 0.$$

Note that neither of the approaches of [20] nor [2] could be used in the reductive case. Instead, we employ another method, relating the étale motivic cohomology of the classifying spaces of G and of its Borel subgroup B. The difficult part of the proof is the exactness at the term $S^2(T^*)^W / \mathrm{Dec}(G)$, i.e., to show that every W-invariant quadratic form (on the dual of T^*) gives rise to an invariant of G of degree 3.

We give an application in Section 7. We compute the subgroup of *reductive* invariants of all split (almost) simple groups. The reason we are interested in the reductive invariants is that the group of unramified invariants (an important birational invariant of the classifying space of the group, see [19]) is contained in the group of reductive invariants [19, (10.1)]. In some cases, the group of reductive invariants is trivial (for example, case A_{n-1}, see Section 7), and this allows one to conclude that the group of unramified invariants is also trivial.

We don't impose any characteristic restriction on the base field F, and we take care of the p-part of the group of invariants when $p = \mathrm{char}(F) > 0$ even if F is imperfect. One should be careful since certain étale motivic cohomology groups of algebraic varieties over an imperfect field are not homotopy invariant. The proofs of some statements (e.g., Theorem 3.7) can be simplified if we invert $p = \mathrm{char}(F) > 0$. In this case all the functors considered in the paper are homotopy invariant.

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2. Preliminary results

2.1. K-cohomology and Rost's spectral sequence. Let X be a smooth algebraic variety over F. For any $i \geq 0$, let $X^{(i)}$ be the set of points in X of codimension i. Write $K_d(L)$ for the Milnor K-group of a field L and define the *K-cohomology groups* $A^i(X, K_d)$ as the homology group of the complex (see [22])

$$\coprod_{x \in X^{(i-1)}} K_{d-i+1}\big(F(x)\big) \xrightarrow{\partial} \coprod_{x \in X^{(i)}} K_{d-i}\big(F(x)\big) \xrightarrow{\partial} \coprod_{x \in X^{(i+1)}} K_{d-i-1}\big(F(x)\big).$$

In particular, $A^i(X, K_i) = \mathrm{CH}^i(X)$ is the *Chow group* of classes of algebraic cycles on X of codimension i and $A^0(X, K_1) = F[X]^\times$ is the group of invertible regular functions on X.

Let $f : X \longrightarrow Y$ be a flat morphism. For every point $y \in Y$ write X_y for the fiber $X \times_Y \operatorname{Spec} F(y)$ over y. There is Rost's spectral sequence [22, §8]

$$E_1^{p,q} = \coprod_{y \in Y^p} A^q(X_y, K_{n-p}) \Rightarrow A^{p+q}(X, K_n) \tag{2.1}$$

for every n.

2.2. K-cohomology of G, G/T and G/B. We will be using the following notation in the paper. Let

G be a (connected) split reductive group over a field F,

$T \subset G$ a split maximal torus,

$T^* := \mathrm{Hom}(T, \mathbb{G}_m)$ the character group of T,

W the Weyl group of G,

$B \subset G$ a Borel subgroup containing T (we have $B^* = T^*$),

H the commutator subgroup of G,

$Q = G/H$ a split torus,

$\pi : \widetilde{H} \longrightarrow H$ the simply connected cover of G,

$C = \operatorname{Ker}(\pi)$, thus C is a finite diagonalizable group (not necessarily smooth),

Λ_w the weight lattice of $\widetilde{H}$ (the character group of a maximal split torus of $\widetilde{H}$).

The kernel of the natural homomorphism $\kappa : T^* \longrightarrow \Lambda_w$ is isomorphic to Q^* and its cokernel is isomorphic to C^*.

The smooth projective variety G/B is the *flag variety* for the simply connected group $\widetilde{H}$. By [8, Part 2, §6], there is natural isomorphism

$$\Lambda_w \xrightarrow{\sim} \mathrm{CH}^1(G/B). \tag{2.2}$$

This isomorphism extends to a ring homomorphism $S^*(\Lambda_w) \longrightarrow \mathrm{CH}^*(G/B)$, where S^* stands for the symmetric ring.

Let $E \longrightarrow Y$ be a G-torsor and J the pull-back $E \times_Y \operatorname{Spec}(K)$ for a point $y : \operatorname{Spec}(K) \longrightarrow Y$, so J is a G-torsor over K. Write $f : E/B \longrightarrow Y$ for the morphism induced by the torsor $E \longrightarrow Y$. The fiber of f over y is the (smooth projective) flag variety J/B over K for the group $\operatorname{Aut}_G(J)$ over K, which is a twisted form of G_K.

In the following proposition we collect known results on K-cohomology.

Proposition 2.1. *Let G be a split reductive group, $T \subset G$ a split maximal torus and B a Borel subgroup containing T. Let $E \longrightarrow Y$ be a G-torsor with Y a smooth variety. Then*

(1) *The pull-back homomorphism $A^*(E/B, K_*) \longrightarrow A^*(E/T, K_*)$ induced by the natural morphism $E/T \longrightarrow E/B$ is a ring isomorphism.*

(2) *For every smooth variety Z over F, the external product map yields isomorphisms*

$$A^*(Z, K_*) \otimes \mathrm{CH}^*(G/T) \xrightarrow{\sim} A^*(Z \times (G/T), K_*),$$
$$A^*(Z, K_*) \otimes \mathrm{CH}^*(G/B) \xrightarrow{\sim} A^*(Z \times (G/B), K_*).$$

(3) *There are natural isomorphisms $\Lambda_w \xrightarrow{\sim} \mathrm{CH}^1(G/T) \simeq \mathrm{CH}^1(G/B)$.*

(4) *The kernel of the surjective homomorphism*

$$S^2(\Lambda_w) \longrightarrow \mathrm{CH}^2(G/T) \simeq \mathrm{CH}^2(G/B)$$

is equal to the group of W-invariant elements $S^2(\Lambda_w)^W$ in $S^2(\Lambda_w)$. Therefore, $\mathrm{CH}^2(G/T) \simeq \mathrm{CH}^2(G/B) \simeq S^2(\Lambda_w)/S^2(\Lambda_w)^W$.

Proof. (1) The fibers of $E \longrightarrow E/B$ over a field K are B-torsors and hence are split and isomorphic to B_K since B is a *special* group (all B-torsors over fields are trivial). It follows that the fibers of the natural morphism $E/T \longrightarrow E/B$

over K are isomorphic to $(B/T)_K$ and hence are affine spaces over K. By the Homotopy Invariance Property of K-cohomology [6, Theorem 52.13], the pull-back homomorphism is an isomorphism.

(2) It follows from (1) that for the rest of the proof we may consider only the variety G/B. Since G/B is cellular, the statement follows from [7, Proposition 3.7, Lemma 3.8].

(3) follows from (2.2).

(4) was proved in [8, Part 2, Theorem 6.7 and Corollary 6.12]. □

Remark 2.2. There is a natural W-action on G/T. By functoriality of the Chow groups, the groups $\mathrm{CH}^i(G/T)$ and hence $\mathrm{CH}^i(G/B)$ are naturally W-modules. Moreover, the maps in (2.2), (3) and (4) in Proposition 2.1 are homomorphisms of W-modules.

2.3. *K*-cohomology of varieties associated to a torsor. Let $E \longrightarrow Y$ be a G-torsor over a smooth variety Y over F. Set $X = E/T$ or $X = E/B$ and let $f : X \longrightarrow Y$ be the natural morphism. Note that in the case $X = E/B$, the fiber X_y of f over a point $y \in Y$ is a projective homogeneous variety over the field $F(y)$.

Proposition 2.3. *Let $E \longrightarrow Y$ be a G-torsor with Y a smooth variety, $X = E/T$ or $X = E/B$ and $f : X \longrightarrow Y$ the induced morphism.*

(1) *The natural homomorphism*

$$A^0(Y, K_2) \longrightarrow A^0(X, K_2)$$

is an isomorphism.

(2) *There is a natural complex*

$$0 \longrightarrow A^1(Y, K_2) \longrightarrow A^1(X, K_2) \xrightarrow{\alpha} \Lambda_w \otimes F[Y]^\times \longrightarrow 0.$$

The complex is acyclic if the torsor E is trivial.

Proof. By Proposition 2.1(1), it suffices to consider the case $X = E/B$.

(1) Rost's spectral sequence (2.1) for the morphism f yields an exact sequence

$$0 \longrightarrow A^0(X, K_2) \longrightarrow \coprod_{y \in Y^{(0)}} A^0(X_y, K_2) \longrightarrow \coprod_{y \in Y^{(1)}} A^0(X_y, K_1).$$

The fiber X_y is a projective homogeneous G-variety over the field $F(y)$. Therefore, the natural homomorphism $K_i(F(y)) \longrightarrow A^0(X_y, K_i)$ is an isomorphism if $i \le 2$ by [23, Corollary 5.6]. It follows that $A^0(X, K_2) \simeq A^0(Y, K_2)$.

(2) Rost's spectral sequence yields a complex

$$A^1(Y, K_2) \longrightarrow A^1(X, K_2) \longrightarrow \coprod_{y \in Y^{(0)}} A^1(X_y, K_2) \xrightarrow{\partial} \coprod_{y \in Y^{(1)}} A^1(X_y, K_1).$$

As X_y is a projective homogeneous G-variety over $F(y)$, by [16, §3], the natural map

$$A^1(X_y, K_i) \longrightarrow A^1((X_y)_{\mathrm{sep}}, K_i) = \Lambda_w \otimes K_{i-1}(F(y)_{\mathrm{sep}})$$

is injective and it identifies the group $A^1(X_y, K_i)$ with a subgroup of

$$\mathrm{CH}^1(G/B) \otimes K_{i-1}(F(y)) = \Lambda_w \otimes K_{i-1}(F(y))$$

for $i \le 2$. It follows that there is a natural map from $\mathrm{Ker}(\partial)$ to

$$A^0(Y, \Lambda_w \otimes K_1) = \Lambda_w \otimes A^0(Y, K_1) = \Lambda_w \otimes F[Y]^\times.$$

This defines the map α.

If E is a trivial torsor, we have $X \simeq Y \times (G/B)$. By Proposition 2.1,

$$A^1(X, K_2) \simeq A^1(Y, K_2) \oplus (\Lambda_w \otimes F[Y]^\times).$$

Note that the projection of $A^1(X, K_2)$ onto $\Lambda_w \otimes F[Y]^\times$ coincides with the map α. □

2.4. *K*-cohomology of classifying spaces. Let G be an algebraic group. In [24], Totaro defined the Chow ring $\mathrm{CH}^*(\mathrm{B}G)$ of the "classifying space" of G and more generally, Guillot in [10] defined the ring $A^*(\mathrm{B}G, K_*)$ as follows. Fix an integer $i \ge 0$ and choose a generically free representation V of G such that there is a G-equivariant open subset $U \subset V$ with the property $\mathrm{codim}_V(V \setminus U) \ge i + 1$ and a *versal* G-torsor $f : U \longrightarrow U/G$ (see [24, Remark 1.4] or [5, Lemma 9]). Then set

$$A^i(\mathrm{B}G, K_*) := A^i(U/G, K_*).$$

This is independent of the choice of U since the K-cohomology groups are homotopy invariant.

Let $T \subset B$ be a split maximal torus and a Borel subgroup respectively in a split reductive group G. As in the proof of Proposition 2.1(1), the fibers of the natural morphisms $U/T \longrightarrow U/B$ are affine spaces. By the homotopy invariance property,

$$A^i(\mathrm{B}B, K_j) \xrightarrow{\sim} A^i(\mathrm{B}T, K_j).$$

The next statement follows from the Künneth formula [7, Prop. 3.7] (see [2, Example A.5]).

Proposition 2.4. *Let $T \subset B$ be a split maximal torus and a Borel subgroup in G, respectively. Then*

$$A^i(\mathrm{B}B, K_j) \simeq A^i(\mathrm{B}T, K_j) \simeq S^i(T^*) \otimes K_{j-i}(F).$$

More generally if G acts on a variety X over F, one can define the *equivariant K-cohomology groups* $A^i_G(X, K_j)$ (see [10]). In particular, if $X = \mathrm{Spec}(F)$ (with trivial G-action), we have $A^i_G(X, K_j) \simeq A^i(\mathrm{B}G, K_j)$ and if $X \longrightarrow Y$ is a G-torsor, then $A^i_G(X, K_j) \simeq A^i(Y, K_j)$.

The structure morphism $G \longrightarrow \mathrm{Spec}\, F$ yields then a homomorphism

$$A^i(\mathrm{B}B, K_j) = A^i_B(\mathrm{Spec}\, F, K_j) \longrightarrow A^i_B(G, K_j) = A^i(G/B, K_j).$$

The following statement is a consequence of [7, §3] and (2.2).

Lemma 2.5. *The composition*

$$\begin{aligned} T^* \otimes K_{j-1}(F) \simeq A^1(\mathrm{B}B, K_j) \longrightarrow A^1(G/B, K_j) &\simeq \mathrm{CH}^1(G/B) \otimes K_{j-1}(F) \\ &\simeq \Lambda_w \otimes K_{j-1}(F) \end{aligned}$$

coincides with $\kappa \otimes 1$.

3. The motivic cohomology of weight ≤ 2

3.1. The complexes $\mathbb{Q}/\mathbb{Z}(j)$. Let X be a smooth variety over F. For every $j \in \mathbb{Z}$, the complex $\mathbb{Q}/\mathbb{Z}(j)$ is defined in the derived category $D^+\,\mathrm{Sh}_{\text{ét}}(X)$ of étale sheaves of abelian groups on X as the direct sum of two complexes. The first complex is given by the locally constant étale sheaf (placed in degree 0) the colimit over n prime to $\mathrm{char}(F)$ of the Galois modules $\mu_n^{\otimes j}$, where μ_n is the Galois module of n^{th} roots of unity. The second complex is nontrivial only in the case $p = \mathrm{char}(F) > 0$ and it is defined as

$$\operatorname*{colim}_n W_n\Omega^j_{\log}[-j]$$

if $j \geq 0$, with $W_n\Omega^j_{\log}$ the sheaf of *logarithmic de Rham–Witt differentials* (see [11, I.5.7], [12]). The second complex is defined to be zero if $j < 0$.

We write $H^m\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the étale cohomology of a scheme X with values in $\mathbb{Q}/\mathbb{Z}(j)$. Then

$$H^m\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)\{p\} = \operatorname*{colim}_n H^m\big(X, \mu_{p^n}^{\otimes j}\big)$$

if $p \neq \mathrm{char}\, F$ and

$$H^m\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)\{p\} = \operatorname*{colim}_n H^{m-j}\big(X, W_n\Omega^j_{\log}\big)$$

if $p = \operatorname{char} F > 0$. In the latter case, we have (e.g., see [2])

$$H^m\big(F, \mathbb{Q}/\mathbb{Z}(j)\big)\{p\} = \begin{cases} K_j^M(F) \otimes (\mathbb{Q}_p/\mathbb{Z}_p), & \text{if } m = j; \\ H^2\big(F, K_j^M(F_{\mathrm{sep}})\big)\{p\}, & \text{if } m = j+1; \\ 0, & \text{otherwise,} \end{cases} \tag{3.1}$$

where K_j^M are Milnor's K-groups.

We write $\mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))$ for the Zariski sheaf on X associated to the presheaf $Z \mapsto H^n_{et}(Z, \mathbb{Q}/\mathbb{Z}(j))$.

3.2. Unramified cohomology. For a scheme X and a closed subscheme $Z \subset X$ we write $H^*_Z\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the étale cohomology group of X with support in Z and values in $\mathbb{Q}/\mathbb{Z}(j)$ [21, Ch. III, §1]. For a point $x \in X^{(1)}$ set

$$H^*_x\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) = \operatorname*{colim}_{x \in U} H^*_{\overline{\{x\}} \cap U}\big(U, \mathbb{Q}/\mathbb{Z}(j)\big),$$

where the colimit is taken over all open subsets $U \subset X$ containing x. If X is a variety, write

$$\partial_x : H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^{n+1}_x\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$$

for the boundary homomorphisms arising from the *coniveau spectral sequence* [4, 1.2].

It follows from [4, §6, Examples 7.3(1), 7.4(3)] that the cohomology groups $H^n(X, \mathbb{Q}/\mathbb{Z}(j))$ satisfy the *purity property* (see [3, §2]) and the sequence

$$0 \longrightarrow H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) \longrightarrow H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) \xrightarrow{\partial} \coprod_{x \in X^{(1)}} H^{n+1}_x\big(X, \mathbb{Q}/\mathbb{Z}(j)\big),$$

where $\partial = \coprod \partial_x$, is exact for every smooth irreducible variety X. For every irreducible smooth projective variety X, we have

$$H^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) = H^n_{nr}(F(X), \mathbb{Q}/\mathbb{Z}(j)),$$

the group of elements unramified with respect to all discrete valuations of the field $F(X)$ over F (see [3, Proposition 2.1.8]). This group is a birational invariant of a smooth projective variety.

The following two statements are consequences of a more general theorem [4, Theorem 8.6.1]. We give shorter proofs here of our special cases.

Proposition 3.1. *Let X be a smooth irreducible projective rational variety. Then the natural homomorphism*

$$H^n(F, \mathbb{Q}/\mathbb{Z}(j)) \longrightarrow H^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) = H^n_{nr}(F(X), \mathbb{Q}/\mathbb{Z}(j))$$

is an isomorphism.

Proof. The statement is well known (see [3, Theorem 4.1.5]) if one deletes the p-primary component from $\mathbb{Q}/\mathbb{Z}(j)$ in the case $\mathrm{char}(F) = p > 0$.

In general, we argue by induction on $\dim(X)$. Since the group $H^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$ is a birational invariant, we may assume that $X = \mathbb{P}^{n-1} \times \mathbb{P}^1$. Take an element

$$\alpha \in H^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) = H^n_{nr}(F(X), \mathbb{Q}/\mathbb{Z}(j)) \subset H^n(F(X), \mathbb{Q}/\mathbb{Z}(j)).$$

Pulling back with respect to the morphism $\mathbb{P}^{n-1}_{F(\mathbb{P}^1)} \longrightarrow X$, we have

$$\alpha \in H^0_{\mathrm{Zar}}(\mathbb{P}^{n-1}_{F(\mathbb{P}^1)}, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) = H^n(F(\mathbb{P}^1), \mathbb{Q}/\mathbb{Z}(j))$$

by the induction hypothesis. By [2, Lemma A.6], applied to the projection $X \longrightarrow \mathbb{P}^1$, $\alpha \in H^0_{\mathrm{Zar}}(\mathbb{P}^1, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$. The rest of the proof is as in [2, Proposition 5.1]. The coniveau spectral sequence for the projective line $\mathbb{P}^1$ (see [2, Appendix A]) yields a surjective homomorphism

$$H^n(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)) \longrightarrow H^n_{nr}\big(F(\mathbb{P}^1), \mathbb{Q}/\mathbb{Z}(j)\big).$$

By the projective bundle theorem (classical for the p-primary component if $p \neq \mathrm{char}(F)$ and [9, Th. 2.1.11] if $p = \mathrm{char}(F) > 0$), we have

$$H^n(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)) = H^n(F, \mathbb{Q}/\mathbb{Z}(j)) \oplus H^{n-2}(F, \mathbb{Q}/\mathbb{Z}(j-1))t,$$

where t is a generator of $H^2(\mathbb{P}^1, \mathbb{Z}(1)) = \mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$. As t vanishes over the generic point of $\mathbb{P}^1$, the result follows. □

The following statement is a generalization of Proposition 3.1.

Corollary 3.2. *For any smooth irreducible variety Z and a smooth irreducible projective rational variety P, the pull-back homomorphism*

$$H^0_{\mathrm{Zar}}(Z, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) \longrightarrow H^0_{\mathrm{Zar}}(P \times Z, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$$

is an isomorphism.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
H^0_{\mathrm{Zar}}(Z, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) \arrow[r] \arrow[d, hook] & H^0_{\mathrm{Zar}}(P \times Z, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) \arrow[d, hook] \arrow[dr, hook] & \\
H^n(F(Z), \mathbb{Q}/\mathbb{Z}(j)) \arrow[r, "i"] \arrow[d] & H^0_{\mathrm{Zar}}(P_{F(Z)}, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) \arrow[r, hook] & H^n(F(P \times Z), \mathbb{Q}/\mathbb{Z}(j)) \arrow[d] \\
\coprod_{z \in Z^{(1)}} H^{n+1}_x(X, \mathbb{Q}/\mathbb{Z}(j)) \arrow[rr, hook] & & \coprod_{z \in Z^{(1)}} H^{n+1}_{P \times y}(P \times Z, \mathbb{Q}/\mathbb{Z}(j))
\end{tikzcd}$$

with the exact left column. The map i is an isomorphism by Proposition 3.1. The bottom map is injective by [2, Lemma A7]. By diagram chase, the top homomorphism in the diagram is an isomorphism. □

Lemma 3.3. *The étale sheaf on a smooth variety Y associated to the presheaf*

$$Z \mapsto H^0_{\mathrm{Zar}}(Z, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$$

is trivial if $n \neq 0$ and $n \neq j$.

Proof. Let $y \in Y$ and let $O^{\mathrm{sh}}_{Y,y}$ be the strict henselization of Y at y. Then the stalk at $O^{\mathrm{sh}}_{Y,y}$ of the sheaf in the statement is equal to

$$\mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))(O^{\mathrm{sh}}_{Y,y}) = H^n_{\text{ét}}(O^{\mathrm{sh}}_{Y,y}, \mathbb{Q}/\mathbb{Z}(j)).$$

If p is a prime integer different from $\operatorname{char}(F)$, then the p-component of this group is trivial since $n \neq 0$ and the complex $\mathbb{Q}_p/\mathbb{Z}_p(j)$ is given by a sheaf placed in degree 0. If $p = \operatorname{char}(F)$, the complex $\mathbb{Q}_p/\mathbb{Z}_p(j)$ is given by the de Rham–Witt sheaf shifted by $-j$, hence

$$H^n_{\text{ét}}(O^{\mathrm{sh}}_{Y,y}, \mathbb{Q}/\mathbb{Z}(j))\{p\} = 0$$

since $n \neq j$. □

Let G be a split reductive group and $B \subset G$ be a split Borel subgroup.

Proposition 3.4. *Let $E \longrightarrow Y$ be a G-torsor and $f : X = E/B \longrightarrow Y$ the natural morphism. Then the étale sheaf associated to the presheaf*

$$Z \mapsto H^0_{\mathrm{Zar}}(f^{-1}(Z), \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$$

on Y is trivial if $n > j \geq 0$.

Proof. As the G-torsor f is trivial locally in the étale topology, we may assume that the torsor E is trivial, i.e., $f^{-1}(Z) \simeq (G/B) \times Z$. It follows from Corollary 3.2 that

$$H^0_{\mathrm{Zar}}(f^{-1}(Z), \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) \simeq H^0_{\mathrm{Zar}}(Z, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))).$$

The statement now follows from Lemma 3.3. □

3.3. The complexes $\mathbb{Z}_X(j)$. Let X be a smooth variety over F. We consider the *motivic complexes* $\mathbb{Z}_X(j)$ of *weight* $j = 0, 1$ and 2 in the category $D^+ \operatorname{Sh}_{\text{ét}}(X)$. The complex $\mathbb{Z}_X(0)$ is $\mathbb{Z}$ (placed in degree 0) and $\mathbb{Z}(1) = \mathbb{G}_m[-1]$. We write $\mathbb{Z}_X(2)$ for the motivic complex $\Gamma_X(2)$ defined in [14] and [15]. This complex is conjecturally quasi-isomorphic to Voevodsky's complex $\mathbb{Z}_X(2)$.

We use the following notation for the *étale motivic cohomology* of weight $j \leq 2$:

$$H^{n,j}(X) := H^n_{et}(X, \mathbb{Z}(j)).$$

By [12, Theorem 1.1], we have the following isomorphisms for the étale motivic cohomology of weight 2:

$$H^{n,2}(X) = \begin{cases} 0, & \text{if } n \leq 0; \\ K_3(F(X))_{\text{ind}}, & \text{if } n = 1; \\ A^0(X, K_2), & \text{if } n = 2; \\ A^1(X, K_2), & \text{if } n = 3, \end{cases} \tag{3.2}$$

where

$$K_3(L)_{\text{ind}} := \operatorname{Coker}\big(K_3(L) \longrightarrow K_3^Q(L)\big)$$

for a field L and $K_3^Q(L)$ is Quillen's K-group of L.

We will be using the following proposition proved in [12, Theorem 1.1].

Proposition 3.5. *There is a natural exact sequence*

$$0 \longrightarrow \mathrm{CH}^2(X) \longrightarrow H^{4,2}(X) \longrightarrow H^0_{\text{Zar}}(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \longrightarrow 0$$

for a smooth variety X.

3.4. Homology of the complex $\mathbb{Z}_f(2)$. Let G be a split reductive algebraic group over F. Choose a maximal split torus T and a Borel subgroup B such that $T \subset B \subset G$. Let $E \longrightarrow Y$ be a G-torsor with Y a smooth variety, $X = E/T$ or $X = E/B$ and $f : X \longrightarrow Y$ the induced morphism.

We write $\mathbb{Z}_f(2)$ for the cone of the natural morphism $\mathbb{Z}_Y(2) \longrightarrow Rf_*(\mathbb{Z}_X(2))$ in the category $D^+ \operatorname{Sh}_{\text{ét}}(Y)$. Thus, we have an exact triangle

$$\mathbb{Z}_Y(2) \longrightarrow Rf_*(\mathbb{Z}_X(2)) \longrightarrow \mathbb{Z}_f(2) \longrightarrow \mathbb{Z}_Y(2)[1]. \tag{3.3}$$

We compute the cohomology sheaves $\mathcal{H}^n(\mathbb{Z}_f(2))$ of the complexes $\mathbb{Z}_f(2)$ for small values of n.

Proposition 3.6. *Let G be a split reductive algebraic group over F, Λ_w the weight lattice of the commutator subgroup of G. Let $E \longrightarrow Y$ be a G-torsor with Y a geometrically irreducible smooth variety, $X = E/T$ or $X = E/B$ and let $f : X \longrightarrow Y$ be the induced morphism. Then*

$$\mathcal{H}^n(\mathbb{Z}_f(2)) = \begin{cases} 0, & \text{if } n \leq 2; \\ \Lambda_w \otimes \mathbb{G}_m, & \text{if } n = 3, \end{cases}$$

and there is an exact sequence of étale sheaves on Y

$$0 \longrightarrow \big[S^2(\Lambda_w)/S^2(\Lambda_w)^W\big] \longrightarrow \mathcal{H}^4(\mathbb{Z}_f(2)) \longrightarrow N \longrightarrow 0,$$

where N is the étale sheaf on Y associated to the presheaf

$$Z \mapsto H^0_{\text{Zar}}(f^{-1}(Z), \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))).$$

The sheaf N is trivial if $X = E/B$.

Proof. The complex $\mathbb{Z}(2)$ is supported in degrees 1 and 2, hence $\mathcal{H}^n(\mathbb{Z}_f(2)) = 0$ if $n < 0$. The triangle (3.3) yields then an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{H}^0(\mathbb{Z}_f(2)) \longrightarrow \mathcal{H}^1(\mathbb{Z}_Y(2)) \xrightarrow{s} R^1 f_*(\mathbb{Z}_X(2)) \longrightarrow \\ \mathcal{H}^1(\mathbb{Z}_f(2)) \longrightarrow \mathcal{H}^2(\mathbb{Z}_Y(2)) \xrightarrow{t} R^2 f_*(\mathbb{Z}_X(2)) \longrightarrow \mathcal{H}^2(\mathbb{Z}_f(2)) \longrightarrow 0 \end{aligned} \tag{3.4}$$

of étale sheaves on Y and the isomorphisms

$$R^n f_*(\mathbb{Z}_X(2)) \simeq \mathcal{H}^n(\mathbb{Z}_f(2)) \quad \text{for } n \geq 3.$$

By [21, Proposition III.1.13], $\mathcal{H}^n(\mathbb{Z}_Y(2))$ and $R^n f_*(\mathbb{Z}_X(2))$ are the étale sheaves on Y associated to the presheaves

$$Z \mapsto H^{n,2}(Z) \quad \text{and} \quad Z \mapsto H^{n,2}(f^{-1}Z),$$

respectively.

It follows from (3.2) that $\mathcal{H}^1(\mathbb{Z}_Y(2))$ and $R^1 f_*(\mathbb{Z}_X(2))$ are the étale sheaves on Y associated to the presheaves

$$Z \mapsto K_3F(Z)_{\text{ind}} \quad \text{and} \quad Z \mapsto K_3F(f^{-1}Z)_{\text{ind}},$$

respectively. To show that the map s in (3.4) is an isomorphism, we may assume that the torsor $E \longrightarrow Y$ is trivial. The variety X is rational, hence the natural homomorphism

$$K_3F(Z)_{\text{ind}} \longrightarrow K_3F(f^{-1}Z)_{\text{ind}}$$

is an isomorphism by [17, Lemma 4.2] since the field extension $F(f^{-1}Z)/F(Z)$ is purely transcendental. Thus, the morphism s in the sequence (3.4) is an isomorphism.

By (3.2), the étale sheaves $\mathcal{H}^2(\mathbb{Z}_Y(2))$ and $R^2 f_*(\mathbb{Z}_X(2))$ on Y are associated to the presheaves

$$Z \mapsto H^{2,2}(Z) = A^0(Z, K_2) \quad \text{and} \quad Z \mapsto H^{2,2}(f^{-1}Z) = A^0(f^{-1}Z, K_2),$$

respectively.

By Proposition 2.3(1), the morphism t in the sequence (3.4) is an isomorphism. The exactness of (3.4) implies that

$$\mathcal{H}^n(\mathbb{Z}_f(2)) = 0, \quad \text{if } n \leq 2.$$

The étale sheaf $\mathcal{H}^3(\mathbb{Z}_f(2)) = R^3 f_*(\mathbb{Z}_X(2))$ on Y is associated to the presheaf

$$Z \mapsto H^{3,2}(f^{-1}Z) = A^1(f^{-1}Z, K_2).$$

It follows from Proposition 2.3(2) that there is a natural isomorphism

$$\mathcal{H}^3(\mathbb{Z}_f(2)) \simeq \Lambda_w \otimes \mathbb{G}_m\,.$$

Now consider the case $n = 4$. The étale sheaf $\mathcal{H}^4(\mathbb{Z}_f(2)) = R^4 f_*(\mathbb{Z}_X(2))$ on Y is associated to the presheaf

$$Z \mapsto H^{4,2}(f^{-1}Z).$$

To study this sheaf, consider another étale sheaf M associated to the presheaf

$$Z \mapsto \mathrm{CH}^2(f^{-1}Z).$$

Let z be a generic point of Z and L an algebraic closure of $F(z)$. The fiber $f^{-1}(z)$ is split over L, i.e., it is isomorphic to $(G/T)_L$ if $X = E/T$ and to $(G/B)_L$ if $X = E/B$. The composition (see Proposition 2.1(4))

$$\mathrm{CH}^2(f^{-1}Z) \longrightarrow \mathrm{CH}^2(f^{-1}(z)) \longrightarrow \mathrm{CH}^2(G/D)_L = S^2(\Lambda_w)/S^2(\Lambda_w)^W$$

yields a morphism of M to the constant sheaf $\left[S^2(\Lambda_w)/S^2(\Lambda_w)^W\right]$ over Y. We claim that this morphism is an isomorphism. It suffices to assume that E is trivial over Z, i.e., $f^{-1}Z \simeq Z \times (G/T)$ if $X = E/T$ and $f^{-1}Z \simeq Z \times (G/B)$ if $X = E/B$. By Proposition 2.1(2), we have

$$\mathrm{CH}^2(f^{-1}Z) \simeq \mathrm{CH}^2(Z) \oplus \left(\mathrm{CH}^1(Z) \otimes \mathrm{CH}^1(G/T)\right) \oplus \left(\mathrm{CH}^0(Z) \otimes \mathrm{CH}^2(G/T)\right)$$

if $X = E/T$, and

$$\mathrm{CH}^2(f^{-1}Z) \simeq \mathrm{CH}^2(Z) \oplus \left(\mathrm{CH}^1(Z) \otimes \mathrm{CH}^1(G/B)\right) \oplus \left(\mathrm{CH}^0(Z) \otimes \mathrm{CH}^2(G/B)\right)$$

if $X = E/B$. Note that the projection of $\mathrm{CH}^2(f^{-1}Z)$ onto the last direct summand coincides with the map $M(Z) \longrightarrow \left[S^2(\Lambda_w)/S^2(\Lambda_w)^W\right]$. The sheaves associated to the presheaves $Z \mapsto \mathrm{CH}^i(Z)$ are trivial for $i > 0$. The claim is proved.

By Proposition 3.5, M is a subsheaf of $\mathcal{H}^4(\mathbb{Z}_f(2))$ and the quotient sheaf is the sheaf N associated to the presheaf $Z \mapsto H^0_{\mathrm{Zar}}(f^{-1}(Z), \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(j)))$. By Proposition 3.4, the latter sheaf is trivial if $X = E/B$. $\square$

3.5. Motivic cohomology of varieties associated to a torsor. We study certain étale motivic cohomology of E/T and E/B for a torsor E under a split reductive group G. The group $H^{4,2}$ is *not* homotopy invariant (for the p-component if $p = \mathrm{char}(F)$) and the natural map $H^{4,2}(E/B) \longrightarrow H^{4,2}(E/T)$ is not isomorphism in general. The Weyl group W acts naturally on E/T and hence on $H^{4,2}(E/T)$ (see Remark 2.2). We will show that $H^{4,2}(E/B)$ is isomorphic to a W-submodule of $H^{4,2}(E/T)$.

Theorem 3.7. *Let G be a split reductive group over a field F, B a Borel subgroup, $E \longrightarrow Y$ a G-torsor with Y smooth connected variety and $f : X = E/B \longrightarrow Y$ the induced morphism. Then there are exact sequences of W-modules*

$$\begin{aligned} 0 \longrightarrow A^1(Y, K_2) \longrightarrow A^1(X, K_2) \longrightarrow \Lambda_w \otimes F[Y]^\times \\ \longrightarrow H^{4,2}(Y) \longrightarrow H^{4,2}(X) \longrightarrow H^4(Y, \mathbb{Z}_f(2)) \end{aligned}$$

and

$$0 \longrightarrow \Lambda_w \otimes \mathrm{CH}^1(Y) \longrightarrow H^4(Y, \mathbb{Z}_f(2)) \longrightarrow S^2(\Lambda_w)/S^2(\Lambda_w)^W \longrightarrow \Lambda_w \otimes \mathrm{Br}(Y).$$

The W-action on $H^{4,2}(Y)$ is trivial and $H^{4,2}(X)$ is a W-submodule of $H^{4,2}(E/T)$.

Proof. Let $X' = E/T$ and $g : X' \longrightarrow Y$ the induced morphism. Since $\mathcal{H}^n(\mathbb{Z}_f(2)) = 0 = \mathcal{H}^n(\mathbb{Z}_g(2))$ for $n \leq 2$ and

$$\mathcal{H}^3(\mathbb{Z}_f(2)) = \Lambda_w \otimes \mathbb{G}_m = \mathcal{H}^3(\mathbb{Z}_g(2)) \tag{3.5}$$

by Proposition 3.6, there are exact triangles in $D^+ \operatorname{Sh}_{\text{ét}}(Y)$:

$$\Lambda_w \otimes \mathbb{G}_m[-3] \longrightarrow \tau_{\leq 4}\mathbb{Z}_f(2) \longrightarrow \mathcal{H}^4(\mathbb{Z}_f(2))[-4] \longrightarrow \Lambda_w \otimes \mathbb{G}_m[-2], \tag{3.6}$$

$$\Lambda_w \otimes \mathbb{G}_m[-3] \longrightarrow \tau_{\leq 4}\mathbb{Z}_g(2) \longrightarrow \mathcal{H}^4(\mathbb{Z}_g(2))[-4] \longrightarrow \Lambda_w \otimes \mathbb{G}_m[-2], \tag{3.7}$$

where $\tau_{\leq 4}$ is the truncation functor. It follows that

$$H^3(Y, \tau_{\leq 4}\mathbb{Z}_f(2)) = H^3(Y, \mathbb{Z}_f(2)) = \Lambda_w \otimes F[Y]^\times = H^3(Y, \mathbb{Z}_g(2)).$$

Applying the cohomology functor to the exact triangles 3.6 and 3.7, we get a diagram with the exact rows induced by the morphism $X' \longrightarrow X$:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda_w \otimes \mathrm{CH}^1(Y) \arrow[r] \arrow[d, equal] & H^4(Y, \mathbb{Z}_f(2)) \arrow[r] \arrow[d, "\alpha"] & H^0(Y, \mathcal{H}^4(\mathbb{Z}_f(2))) \arrow[r] \arrow[d, "\beta"] & \Lambda_w \otimes \mathrm{Br}(Y) \arrow[d, equal] \\
0 \arrow[r] & \Lambda_w \otimes \mathrm{CH}^1(Y) \arrow[r] & H^4(Y, \mathbb{Z}_g(2)) \arrow[r] & H^0(Y, \mathcal{H}^4(\mathbb{Z}_g(2))) \arrow[r] & \Lambda_w \otimes \mathrm{Br}(Y),
\end{tikzcd}$$

There is a natural W-action on X' and g is W-equivariant (with W acting trivially on Y). Therefore, W acts on the complex $\mathbb{Z}_g(2)$. It follows that the bottom sequence in the diagram is a sequence of W-module homomorphisms.

By Proposition 3.6,

$$H^0(Y, \mathcal{H}^4(\mathbb{Z}_f(2))) \simeq S^2(\Lambda_w)/S^2(\Lambda_w)^W,$$

β is injective and $H^0(Y, \mathcal{H}^4(\mathbb{Z}_f(2)))$ is isomorphic to the kernel of the W-equivariant homomorphism

$$H^0(Y, \mathcal{H}^4(\mathbb{Z}_g(2))) \longrightarrow N(Y),$$

where N is defined in Proposition 3.6.

By 5-Lemma, α is also injective and $H^4(Y, \mathbb{Z}_f(2))$ is isomorphic to the kernel of the W-equivariant composition

$$H^4(Y, \mathbb{Z}_g(2)) \longrightarrow H^0(Y, \mathcal{H}^4(\mathbb{Z}_h(2))) \longrightarrow N(Y).$$

It follows that the top row in the diagram is a sequence of W-equivariant homomorphisms. This gives the second exact sequence in the statement of the theorem.

Applying the cohomology functor to the exact triangle (3.3) and using (3.2) and (3.5) we get exact sequences

$$\begin{aligned} 0 \longrightarrow A^1(Y,K_2) \longrightarrow A^1(X,K_2) \longrightarrow \Lambda_w \otimes F[Y]^\times \longrightarrow H^{4,2}(Y) \\ \longrightarrow H^{4,2}(X) \longrightarrow H^4(Y,\mathbb{Z}_f(2)) \longrightarrow H^{5,2}(Y), \end{aligned} \tag{3.8}$$

$$\begin{aligned} 0 \longrightarrow A^1(Y,K_2) \longrightarrow A^1(X',K_2) \longrightarrow \Lambda_w \otimes F[Y]^\times \longrightarrow H^{4,2}(Y) \\ \longrightarrow H^{4,2}(X') \longrightarrow H^4(Y,\mathbb{Z}_g(2)) \longrightarrow H^{5,2}(Y). \end{aligned} \tag{3.9}$$

The first exact sequence in the statement of the theorem is (3.8). Comparing the exact sequences (3.8) and (3.9) via the morphism $X' \longrightarrow X$, we get a commutative diagram similar to the one above. We have shown that $H^4(Y,\mathbb{Z}_f(2))$ is a W-submodule of $H^4(Y,\mathbb{Z}_g(2))$. Again, by 5-Lemma we see that $H^{4,2}(X)$ is a W-submodule of $H^{4,2}(X')$. It follows that (3.8) is an exact sequence of W-module homomorphisms. □

4. Cohomology of classifying spaces

4.1. Balanced elements. Let $A^\bullet$ be a cosimplicial abelian group and write $h_*(A^\bullet)$ for the homology groups of the associated complex of abelian groups. If $A^\bullet$ is a constant cosimplicial abelian group (all coface and codegeneracy maps are the identity), we have $h_0(A^\bullet) = A^0$ and $h_i(A^\bullet) = 0$ for all $i > 0$.

Let G be a split reductive group over F. Choose a generically free representation V of G such that there is a G-equivariant open subset $U \subset V$ with the property $\operatorname{codim}_V(V \setminus U) \geq 3$ and a *versal* G-torsor $f : U \longrightarrow U/G$ (see [24, Remark 1.4] or [5, Lemma 9]). Moreover, we may assume that $(U/G)(F) \neq \emptyset$.

Write U^n for the product of n copies of U with the diagonal action of G. Let $H : \mathrm{SmVar}(F) \longrightarrow \mathrm{Ab}$ be a contravariant functor from the category of smooth varieties over F to the category of abelian groups. Then $H(U^\bullet/G)$ is a cosimplicial abelian group. We have the two maps

$$H(p_i) : H(U/G) \longrightarrow H(U^2/G), \quad i = 1,2,$$

where $p_i : U^2/G \longrightarrow U/G$ are the projections. An element $v \in H(U/G)$ is called *balanced* if $H(p_1)(v) = H(p_2)(v)$. We write $H(U/G)_{\mathrm{bal}}$ for the subgroup of balanced elements in $H(U/G)$. In other words, $H(U/G)_{\mathrm{bal}} = h_0\big(H(U^\bullet/G)\big)$ (see [2]).

We write $\overline{H}(X)$ for the cokernel of the homomorphism $H(\operatorname{Spec} F) \longrightarrow H(X)$ for a smooth variety X over F induced by the structure morphism of X.

4.2. Cohomology of the classifying space. By [2, Corollary 3.5], the group

$$H^0_{\mathrm{Zar}}\big(BG, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) := H^0_{\mathrm{Zar}}\big(U/G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)_{\mathrm{bal}}$$

is independent of the choice of U and we have an isomorphism

$$H^0_{\mathrm{Zar}}\big(\mathrm{B}G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) \xrightarrow{\sim} \mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big). \tag{4.1}$$

By [20, §3d] and Proposition 3.5, the group

$$H^{4,2}(BG) := H^{4,2}(U/G)_{\mathrm{bal}} \tag{4.2}$$

is also independent of the choice of U and there is an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(BG) \longrightarrow H^{4,2}(\mathrm{B}G) \longrightarrow \mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big) \longrightarrow 0. \tag{4.3}$$

Note that since the functor $H^{4,2}$ is not homotopy invariant in general, we cannot use the machinery of [24] and [10] to define $H^{4,2}(BG)$.

Theorem 4.1. *Let G be a split reductive group over F, $T \subset G$ a split maximal torus and C the kernel of the universal cover of the commutator subgroup of G. Then there is an exact sequence*

$$0 \longrightarrow C^* \otimes F^\times \longrightarrow \overline{H}^{4,2}(\mathrm{B}G) \longrightarrow S^2(T^*)^W \longrightarrow 0.$$

Proof. Applying Theorem 3.7 to the versal G-torsors $f^n : U^n \longrightarrow U^n/G$ for all n we get an exact sequence

$$\begin{aligned} A^1(U^n/B, K_2) \longrightarrow \Lambda_w \otimes F[U^n/G]^\times \longrightarrow H^{4,2}(U^n/G) \\ \longrightarrow H^{4,2}(U^n/B) \longrightarrow H^4(U^n/G, \mathbb{Z}_{f^n}(2)) \end{aligned} \tag{4.4}$$

of W-modules. The W-action on $H^{4,2}(U^n/G)$ is trivial. By Proposition 2.4,

$$A^1(U^n/B, K_2) = A^1(\mathrm{B}B, K_2) = T^* \otimes F^\times.$$

Note that

$$F^\times \subset F[U^n/G]^\times \subset F[U^n]^\times = F[V^n]^\times = F^\times$$

by the assumption on the codimension of U in V, hence $F[U^n/G]^\times = F^\times$.

Now we would like to determine the first homomorphism in the exact sequence (4.4). Let Z be the generic fiber of $U^n/B \to U^n/G$. The variety Z is a twisted form of G/B over the function field $F(U^n/G)$, so $Z_L \simeq (G/B)_L$ over an algebraic closure L of $F(U^n/G)$.

The pull-back map for the morphism $Z \longrightarrow U^n/B$ yields a composition

$$\begin{aligned} T^* \otimes F^\times \simeq A^1(\mathrm{B}B, K_2) = A^1(U^n/B, K_2) \longrightarrow A^1(Z, K_2) \\ \longrightarrow A^1(Z_L, K_2) \simeq A^1((G/B)_L, K_2) = \Lambda_w \otimes L^\times. \end{aligned}$$

By definition, the first homomorphism in the exact sequence (4.4) is given by this composition. The composition coincides with

$$T^* \otimes F^\times \simeq A^1(\mathrm{B}B, K_2) \longrightarrow A^1((\mathrm{B}B)_L, K_2) \xrightarrow{\iota} A^1((G/B)_L, K_2) = \Lambda_w \otimes L^\times$$

with the map ι induced by the structure morphism $G_L \longrightarrow \operatorname{Spec} L$ (see 2.4).

By Lemma 2.5, this composition is induced by the homomorphism $\kappa : T \longrightarrow \Lambda_w$. It follows that the first homomorphism in the exact sequence (4.4) is isomorphic to $\kappa \otimes 1 : T^* \otimes F^\times \longrightarrow \Lambda_w \otimes F^\times$. Therefore, its cokernel is isomorphic to $C^* \otimes F^\times$ since $C^* = \operatorname{Coker}(T^* \longrightarrow \Lambda_w)$. Thus, (4.4) yields an exact sequence

$$0 \longrightarrow C^* \otimes F^\times \longrightarrow H^{4,2}(U^n/G) \longrightarrow H^{4,2}(U^n/B) \longrightarrow H^4(U^n/G, \mathbb{Z}_f(2)). \quad (4.5)$$

Since B is special (see proof of Proposition 2.1), every invariant of B is constant. The exact sequence (4.3) for the group B and Proposition 2.4 then yield:

$$\overline{H}^{4,2}(\mathrm{B}B) \simeq \mathrm{CH}^2(\mathrm{B}B) \simeq S^2(T^*). \quad (4.6)$$

Taking the balanced elements in the exact sequence (4.5) of cosimplicial groups and recalling the definitions of $H^{4,2}(\mathrm{B}G)$ and $H^{4,2}(\mathrm{B}B)$ given in (4.2), we get a sequence of homomorphisms of W-modules

$$0 \longrightarrow C^* \otimes F^\times \longrightarrow \overline{H}^{4,2}(\mathrm{B}G) \longrightarrow \overline{H}^{4,2}(\mathrm{B}B) \longrightarrow H^4(U/G, \mathbb{Z}_f(2)),$$

where $f = f^1 : U \longrightarrow U/G$. Note that the sequence is exact by [2, Lemma A.2] since the first term in (4.5) is a constant cosimplicial group.

We will use the following simple lemma.

Lemma 4.2. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D$ be an exact sequence of W-modules. Suppose that W acts trivially on A and B. Then the induced sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C^W \longrightarrow D^W$ is exact.*

Recall that W acts trivially on $C^* \otimes F^\times$ and $\overline{H}^{4,2}(\mathrm{B}G)$. Taking into account Lemma 4.2 and using (4.6) we get an exact sequence

$$0 \longrightarrow C^* \otimes F^\times \longrightarrow \overline{H}^{4,2}(\mathrm{B}G) \longrightarrow S^2(T^*)^W \longrightarrow H^4(U/G, \mathbb{Z}_f(2))^W.$$

It suffices to show that the last term in the sequence is trivial. Recall that we write Q for the factor group of G by the commutator subgroup. We have $Q^* = G^* = \mathrm{CH}^1(\mathrm{B}G)$. The second sequence in Theorem 3.7 reads as follows:

$$0 \longrightarrow \Lambda_w \otimes Q^* \longrightarrow H^4(U/G, \mathbb{Z}_f(2)) \longrightarrow S^2(\Lambda_w)/S^2(\Lambda_w)^W,$$

Since $H^1(W, S^2(\Lambda_w)^W) = 0$, we have $[S^2(\Lambda_w)/S^2(\Lambda_w)^W]^W = 0$ and conclude that $H^4(U/G, \mathbb{Z}_f(2))^W = 0$. □

5. Degree 3 invariants

Let G be a split reductive group over F and $T \subset G$ a split maximal torus. We have the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & C^* \otimes F^\times \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & \mathrm{CH}^2(\mathrm{B}G) \arrow[r] \arrow[dr, "\gamma"'] & \overline{H}^{4,2}(\mathrm{B}G) \arrow[r] \arrow[d] & \mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \arrow[r] & 0 \\
 & & S^2(T^*)^W \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

with the exact row and column by (4.3) and Theorem 4.1.

Let $\mathbb{Z}[T^*]$ be the group ring of T^* and let

$$c_i : \mathbb{Z}[T^*] \longrightarrow S^i(T^*)$$

be abstract Chern classes (see [20, 3c]). In particular, if $a = \sum_i e^{x_i} \in \mathbb{Z}[T^*]$ (we use the exponential notation for the elements in $\mathbb{Z}[T^*]$), then

$$c_1(a) = \sum_i x_i \in S^1(T^*) = T^* \quad \text{and} \quad c_2(a) = \sum_{i<j} x_i x_j \in S^2(T^*).$$

By [24, Corollary 3.2], the group $\mathrm{CH}^2(\mathrm{B}G)$ is generated by the second Chern classes of representations of G. Since the representation ring of G is equal to $\mathbb{Z}[T^*]^W$, the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{Z}[T^*]^W \arrow[r] \arrow[d, "c_2"] & \mathbb{Z}[T^*] \arrow[d, "c_2"] & \\
\mathrm{CH}^2(\mathrm{B}G) \arrow[r] & \mathrm{CH}^2(\mathrm{B}T) \arrow[r, equal] & S^2(T^*)
\end{tikzcd}$$

implies that the image of γ, denoted $\mathrm{Dec}(G)$, is generated by the image of the restriction $\mathbb{Z}[T^*]^W \longrightarrow S^2(T^*)^W$ of c_2. As

$$c_2(x + y) = c_2(x) + c_2(y) + c_1(x)c_1(y)$$

for all $x, y \in \mathbb{Z}[T^*]^W$, the square of $(T^*)^W$ is in $\mathrm{Dec}(G)$. The group $\mathbb{Z}[T^*]^W$ is generated by the sums $\sum_i e^{x_i}$, where $\{x_i\}$ is the W-orbit of a character in T^*. Since the composition

$$\mathbb{Z}[T^*]^W \longrightarrow S^2(T^*)^W \longrightarrow S^2(T^*)^W/(T^{*W})^2$$

is a group homomorphism, the subgroup $\operatorname{Dec}(G) \subset S^2(T^*)^W$ is generated by elements of the following types:

1) $\sum_{i<j} x_i x_j$, where $\{x_i\}$ is the W-orbit of a character in T^*,

2) xy, where $x, y \in (T^*)^W = Q^*$.

In other words, $\operatorname{Dec}(G)$ is the subgroup of the "obvious" elements in $S^2(T^*)^W$.

By [2, Theorem 2.4], the group of the Brauer invariants of G

$$\operatorname{Inv}^2\big(G, \mathbb{Q}/\mathbb{Z}(1)\big)_{\mathrm{norm}} = \operatorname{Inv}(G, \mathrm{Br})_{\mathrm{norm}}$$

is isomorphic to $\operatorname{Pic}(G) = C^*$. The homomorphism

$$C^* \otimes F^\times \longrightarrow \operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$$

is given by the cup-product, and invariants in the image of this homomorphism are called *decomposable invariants* (see [20]).

Theorem 5.1. *Let G be a split reductive group, $T \subset G$ a split maximal torus and C the kernel of the universal cover of the commutator subgroup of G. Then the sequence*

$$0 \longrightarrow C^* \otimes F^\times \longrightarrow \operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \longrightarrow S^2(T^*)^W / \operatorname{Dec}(G) \longrightarrow 0$$

is exact.

Proof. Everything except the injectivity of the first homomorphisms follows from a diagram chase. Let H be the commutator subgroup of G. The composition

$$C^* \otimes F^\times \longrightarrow \operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\mathrm{res}} \operatorname{Inv}^3\big(H, \mathbb{Q}/\mathbb{Z}(2)\big)$$

is injective and it identifies $C^* \otimes F^\times$ with the decomposable invariants of H by [20, Theorem 4.2]. The injectivity of the first homomorphism in the composition follows. □

Write $\operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$ for the factor group of $\operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ by the subgroup of decomposable invariants. We have a natural isomorphism

$$\operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \simeq S^2(T^*)^W / \operatorname{Dec}(G). \tag{5.1}$$

6. Restriction to the commutator subgroup

Let G be a split reductive group and H its commutator subgroup. We shall study the restriction homomorphism

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2)).$$

Consider the *polar* homomorphism

$$\mathrm{pol} : S^2(\Lambda_w) \longrightarrow \Lambda_w \otimes \Lambda_w, \quad xy \mapsto x \otimes y + y \otimes x.$$

By [18, Proposition 2.2], $\mathrm{pol}(S^2(\Lambda_w)^W)$ is contained in $\Lambda_w \otimes \Lambda_r$, where Λ_r is the root lattice.

By [19, §9], the embedding of Λ_r into Λ_w factors as follows:

$$\Lambda_r \xrightarrow{\sigma} T^* \xrightarrow{\tau} \Lambda_w.$$

Let α be the composition

$$S^2(\Lambda_w)^W \xrightarrow{\mathrm{pol}} (\Lambda_w \otimes \Lambda_r)^W \xrightarrow{1\otimes\sigma} (\Lambda_w \otimes T^*)^W.$$

Let S be a split maximal torus of H contained in T. The character group S^* is the image of τ. We have a commutative diagram

$$\begin{tikzcd}
S^2(S^*)^W \arrow[rrr, "\mathrm{pol}"] \arrow[d, hook] & & & S^* \otimes S^* \arrow[d, hook] \arrow[dr, hook] & \\
S^2(\Lambda_w)^W \arrow[r, "\alpha"] & (\Lambda_w \otimes T^*)^W \arrow[rr, "\beta"] & & \Lambda_w \otimes S^* \arrow[r, hook] & \Lambda_w \otimes \Lambda_w .
\end{tikzcd}$$

Note that the kernel of the homomorphism $\Lambda_w \otimes T^* \longrightarrow \Lambda_w \otimes S^*$ is equal to $\Lambda_w \otimes Q^*$, where $Q = G/H = T/S$. Since $(\Lambda_w \otimes Q^*)^W = 0$, the homomorphism β is injective. Therefore, we have the following commutative key diagram

$$\begin{tikzcd}
 & S^2(S^*)^W \arrow[dl] \arrow[dr] & \\
S^2(\Lambda_w)^W \arrow[dr, "\alpha"'] & (S^* \otimes T^*)^W \arrow[r, hook] \arrow[d, hook] & S^* \otimes S^* \arrow[d, hook] \\
 & (\Lambda_w \otimes T^*)^W \arrow[r, hook, "\beta"] \arrow[d] & \Lambda_w \otimes S^* \arrow[d] \\
C^* \otimes Q^* \arrow[r, hook] & C^* \otimes T^* \arrow[r, two heads] & C^* \otimes S^*
\end{tikzcd}$$

with vertical exact sequences, where $C^* = \Lambda_w/S^*$ is the character group of the kernel C of a universal cover $\widetilde{H} \longrightarrow H$. A diagram chase yields a homomorphism

$$\theta : S^2(S^*)^W \longrightarrow C^* \otimes Q^*. \tag{6.1}$$

Lemma 6.1. *An element $u \in S^2(S^*)^W$ belongs to the image of*

$$S^2(T^*)^W \longrightarrow S^2(S^*)^W$$

if and only if $\operatorname{pol}(u)$ *belongs to the image of*

$$(S^* \otimes T^*)^W \longrightarrow S^* \otimes S^*.$$

Proof. Let X be the kernel of the natural homomorphism $S^2(T^*) \longrightarrow S^2(S^*)$. We have an exact sequence

$$0 \longrightarrow S^2(Q^*) \longrightarrow X \longrightarrow S^* \otimes Q^* \longrightarrow 0$$

and the following commutative diagram with exact rows

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & S^2(T^*) \arrow[r, two heads] \arrow[d] & S^2(S^*) \arrow[d, "\operatorname{pol}"] \\
S^* \otimes Q^* \arrow[r, hook] & S^* \otimes T^* \arrow[r, two heads] & S^* \otimes S^*,
\end{tikzcd}$$

where the middle arrow is the composition of $\operatorname{pol} : S^2(T^*) \longrightarrow T^* \otimes T^*$ with the natural homomorphism $T^* \otimes T^* \longrightarrow S^* \otimes T^*$.

Since W acts trivially on $S^2(Q^*)$, we have $H^1(W, S^2(Q^*)) = 0$. It follows that the right vertical map in the commutative diagram

$$\begin{tikzcd}
S^2(T^*)^W \arrow[r] \arrow[d] & S^2(S^*)^W \arrow[r] \arrow[d, "\operatorname{pol}"] & H^1(W, X) \arrow[d, hook] \\
(S^* \otimes T^*)^W \arrow[r] & (S^* \otimes S^*)^W \arrow[r] & H^1(W, S^* \otimes Q^*)
\end{tikzcd}$$

with exact rows is injective. The result follows by diagram chase. □

The following statement is a consequence of Lemma 6.1 and the key diagram chase.

Proposition 6.2. *The sequence*

$$S^2(T^*)^W \longrightarrow S^2(S^*)^W \xrightarrow{\theta} C^* \otimes Q^*$$

is exact.

The first homomorphism in the proposition takes $\mathrm{Dec}(G)$ surjectively onto $\mathrm{Dec}(H)$ (see the proof of [19, Lemma 5.2]). It follows that $\theta(\mathrm{Dec}(H)) = 0$. Theorem 5.1, applied to H, gives then a composition

$$\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow S^2(S^*)^W / \mathrm{Dec}(H) \longrightarrow C^* \otimes Q^*.$$

The isomorphisms (5.1) for H and G, the injectivity of the map

$$S^2(T^*)^W / \mathrm{Dec}(G) \longrightarrow S^2(S^*)^W / \mathrm{Dec}(H)$$

(see [19, Lemma 5.2(2)]) and Proposition 6.2 yield the following theorem.

Theorem 6.3. *Let G be a split reductive group, $H \subset G$ the commutator subgroup, $Q = G/H$ and C the kernel of the universal cover $\widetilde{H} \longrightarrow H$. Then the sequence*

$$0 \longrightarrow \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow C^* \otimes Q^*$$

is exact.

Corollary 6.4. *If H is either simply connected or adjoint, then the restriction homomorphism* $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))$ *is an isomorphism.*

Proof. If H is simply connected, then $C^* = 0$. If H is adjoint, $S^* = \Lambda_r$, so the surjection $T^* \longrightarrow S^*$ is split by the map $\Lambda_r \longrightarrow T^*$. It follows that the map $S^2(T^*)^W \longrightarrow S^2(S^*)^W$ is surjective, hence θ is zero in Proposition 6.2. □

7. Reductive invariants

Let H be a split semisimple group. A reductive group G is called a *strict reductive envelope* of H (see [19, §10]), if H is the commutator subgroup of G and the (scheme-theoretic) center of G is a torus. By [19, §10], if G is a strict reductive envelope of H, the restriction map

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \longrightarrow \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$$

is injective (see also Theorem 6.3) and its image $\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{red}}$ is independent of the choice of G. This is the subgroup of *reductive indecomposable* invariants of H.

By [8, Part II, 6.10], the group $S^2(\Lambda_w)^W$ is free abelian with canonical basis q_j indexed by the set of irreducible components of the Dynkin diagram of G.

We write $\alpha_{ij} \in \Lambda_r$ for the simple roots of the j^{th} component of the root system of H and $w_{ij} \in \Lambda_w$ for the corresponding fundamental weights. Let d_{ij} be the square of the length of the co-root $(\alpha_{ij})^\vee$.

Proposition 7.1. *Let $q = \sum_j k_j q_j \in S^2(S^*)^W \subset S^2(\Lambda_w)^W$ with $k_j \in \mathbb{Z}$. Let $I \in \operatorname{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$ be the element corresponding to q under the isomorphism (5.1). Then I is reductive indecomposable if and only if the order of $\overline{w}_{ij}$ in C^* divides $d_{ij}k_j$ for all i and j.*

Proof. By construction, the composition of θ in (6.1) with the injective map $C^* \otimes Q^* \longrightarrow C^* \otimes T^*$ factors into the composition

$$S^2(S^*)^W \longrightarrow S^2(\Lambda_w)^W \xrightarrow{\text{pol}} \Lambda_w \otimes \Lambda_r \longrightarrow C^* \otimes \Lambda_r \longrightarrow C^* \otimes T^*.$$

As G is strict, Λ_r is a direct summand of T^*, hence the last map in the sequence is injective. Therefore, the sequence

$$S^2(T^*)^W \longrightarrow S^2(S^*)^W \xrightarrow{\theta'} C^* \otimes \Lambda_r$$

is exact by Proposition 6.2.

It follows from Theorem 6.3 that I is reductive indecomposable if and only if q belongs to the kernel of θ'. By [19, §10], the polar form of q_j is equal to

$$\sum_i d_{ij} w_{ij} \otimes \alpha_{ij} \in \Lambda_w \otimes \Lambda_r.$$

Since the roots α_{ij} form a $\mathbb{Z}$-basis for Λ_r, q belongs to the kernel of θ' if and only if the order of $\overline{w}_{ij}$ in C^* divides $d_{ij}k_j$ for all i and j. □

Remark 7.2. The implication "⇒" of Proposition 7.1 was earlier proved in [19, Proposition 10.6].

We compute the group $\operatorname{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{red}}$ for a simple group H. If H is ether simply connected or adjoint, then all invariants are reductive indecomposable by Corollary 6.4. In what follows we consider all simple groups H that are neither simply connected nor adjoint (thus, the order of the center of the simply connected cover of H is not a prime integer).

Case A_{n-1}. Let H be a split simple group of type A_{n-1}, i.e., $H = \mathbf{SL}_n/\mu_m$ for some m dividing n. We claim that $\operatorname{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{red}} = 0$. It is sufficient to show that the p-primary component is trivial for every prime p. Let r be the largest power of p dividing m. Note that the group $\mathbf{GL}_n/\mu_m$ is a strict envelope of H and the kernel of the natural homomorphism $\mathbf{GL}_n/\mu_r \longrightarrow \mathbf{GL}_n/\mu_m$ is finite of degree prime to p. It follows from [19, Proposition 7.1] that the p-primary components of the groups of degree 3 invariants of H and $\mathbf{SL}_n/\mu_r$ are isomorphic. Replacing m by r we may assume that m is a p-power.

Let q be the canonical generator of $S^2(S^*)^W$. It is proved in [1, Theorem 4.1] that if $\operatorname{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is nonzero, then $mq \in \operatorname{Dec}(H)$. On the other hand, by

Proposition 7.1, if I is a reductive indecomposable invariant of H corresponding to a multiple kq of q, then k divides the order of the first fundamental weight in $C^* = \mathbb{Z}/m\mathbb{Z}$. The latter is equal to m, i.e., m divides k, hence $kq \in \mathrm{Dec}(H)$ and therefore, the invariant I is trivial.

Remark 7.3. The group $G = \mathbf{GL}_n/\mu_m$ is a strict envelope of $H = \mathbf{SL}_n/\mu_m$. A G-torsor over a field K is a central simple algebra A of degree n over K and exponent dividing m. Thus, every reductive indecomposable invariant of H is an invariant of such algebras. We have shown that every normalized degree 3 invariant of A is decomposable, i.e., it is equal to $[A] \cup (x) \in H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ for some $x \in F^\times$, where $[A]$ is the class of A in $\mathrm{Br}(K) = H^2(K, \mathbb{Q}/\mathbb{Z}(1))$. In other words, central simple algebras of fixed degree and exponent have no nontrivial indecomposable degree 3 invariants.

Case D_n. If H is the special orthogonal group $\mathbf{O}^+_{2n}$, then $\mathrm{Inv}^3_{\mathrm{ind}}(H, \mathbb{Q}/\mathbb{Z}(2)) = 0$ (see the proof of [19, Proposition 8.2]). Finally $H = \mathbf{HSpin}_{2n}$ is the half-spin group when $n \geq 4$ is even. Let q be the canonical generator of $S^2(S^*)^W$. It is shown in [1, Theorem 5.1] that

$$\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} = \begin{cases} 0, & \text{if } n \equiv 2 \text{ modulo } 4 \text{ or } n = 4; \\ 2\mathbb{Z}q/4\mathbb{Z}q, & \text{if } n \equiv 4 \text{ modulo } 8 \text{ and } n.4; \\ \mathbb{Z}q/4\mathbb{Z}q, & \text{if } n \equiv 0 \text{ modulo } 8, \end{cases}$$

where q is the canonical generator of $S^2(\Lambda_w)^W$. The orders of the fundamental weights in $C^* = \mathbb{Z}/2\mathbb{Z}$ are equal to 1 or 2. By Proposition 7.1,

$$\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{red}} = \begin{cases} 0, & \text{if } n \equiv 2 \text{ modulo } 4 \text{ or } n = 4; \\ 2\mathbb{Z}q/4\mathbb{Z}q, & \text{if } n \equiv 0 \text{ modulo } 4 \text{ and } n > 4. \end{cases}$$

Remark 7.4. It is shown in [20, Section 4b] that the group

$$\mathrm{Inv}^3(\overline{H}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} = \mathrm{Inv}^3(\overline{H}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{red}}$$

for the split adjoint simple group $\overline{H}$ of type D_n when n is divisible by 4, is isomorphic to $2\mathbb{Z}q/4\mathbb{Z}q$. Therefore, in this case the pull-back homomorphism

$$\mathrm{Inv}^3(\overline{H}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{red}} \longrightarrow \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{red}}$$

is an isomorphism. In particular, the value of a reductive degree 3 invariant of the half-spin group H at an H-torsor depends only on the corresponding central simple algebra of degree $2n$ with a quadratic pair (see [13]).

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