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On the p-converse of the Kolyvagin–Gross–Zagier theorem

Rodolfo Venerucci

Abstract. Let $A/\mathbf{Q}$ be an elliptic curve having split multiplicative reduction at an odd prime p. Under some mild technical assumptions, we prove the statement:

$$\mathrm{rank}_{\mathbf{Z}} A(\mathbf{Q}) = 1 \text{ and } \#\big(\text{Ш}(A/\mathbf{Q})_{p^\infty}\big) < \infty \implies \mathrm{ord}_{s=1} L(A/\mathbf{Q}, s) = 1,$$

thus providing a 'p-converse' to a celebrated theorem of Kolyvagin–Gross–Zagier.

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Contents

1. Introduction

Let A be an elliptic curve defined over $\mathbf{Q}$, let $L(A/\mathbf{Q}, s)$ be its Hasse–Weil L-function, and let $Ш(A/\mathbf{Q})$ be its Tate–Shafarevich group. The (weak form of the) conjecture of Birch and Swinnerton-Dyer predicts that $Ш(A/\mathbf{Q})$ is finite, and that the order of vanishing $\mathrm{ord}_{s=1}L(A/\mathbf{Q}, s)$ of $L(A/\mathbf{Q}, s)$ at $s = 1$ equals the rank of the Mordell–Weil group $A(\mathbf{Q})$. The main result to date in support of this conjecture comes combining the fundamental work of Kolyvagin [17] and Gross–Zagier [13] (KGZ theorem for short):

$$r_{\mathrm{an}} := \mathrm{ord}_{s=1}L(A/\mathbf{Q}, s) \leq 1 \implies \mathrm{rank}_{\mathbf{Z}}A(\mathbf{Q}) = r_{\mathrm{an}} \text{ and } \#\big(Ш(A/\mathbf{Q})\big) < \infty.$$

Let p be a rational prime, let $r_{\mathrm{alg}} \in \{0, 1\}$, and let $Ш(A/\mathbf{Q})_{p^\infty}$ be the p-primary part of $Ш(A/\mathbf{Q})$. By the *p-converse of the KGZ theorem in rank r_{alg}* we mean the conjectural statement

$$\mathrm{rank}_{\mathbf{Z}}A(\mathbf{Q}) = r_{\mathrm{alg}} \text{ and } \#\big(Ш(A/\mathbf{Q})_{p^\infty}\big) < \infty \overset{?}{\implies} \mathrm{ord}_{s=1}L(A/\mathbf{Q}, s) = r_{\mathrm{alg}}.$$

Thanks to the fundamental work of Bertolini–Darmon, Skinner–Urban and their schools, we have now (at least conceptually) all the necessary tools to attack the p-converse of the KGZ theorem. Notably, assume that p is a prime of *good ordinary reduction* for $A/\mathbf{Q}$. In this case the p-converse of the KGZ theorem in rank 0 follows by [31]. In the preprint [30], Skinner combines Wan's Ph.D. Thesis [38] — which proves, following the ideas and the strategy used in [31], one divisibility in the Iwasawa main conjecture for Rankin–Selberg p-adic L-functions — with the main results of [6] and Brooks's Ph.D. Thesis [9] — extending the results of [6] — to prove many cases of the p-converse of the KGZ theorem in rank 1. In the preprint [39], W. Zhang also proves (among other things) many cases of the p-converse of the KGZ theorem in rank 1 for good ordinary primes, combining the results of [31] with the results and ideas presented in Bertolini–Darmon's proof of (one divisibility in) the anticyclotomic main conjecture [4]. The same strategy also appears in Berti's forthcoming Ph.D. Thesis [1] (see also [2]).

The aim of this note is to prove the p-converse of the KGZ theorem in rank 1 for a prime p of *split multiplicative reduction* for $A/\mathbf{Q}$. Our strategy is different from both the one of [30] and the one of [39], and is based on the (two-variable) Iwasawa theory for the Hida deformation of the p-adic Tate module of $A/\mathbf{Q}$. Together with the results of the author's Ph.D. Thesis [35], and then Nekovář's theory of Selmer

Complexes [20] (on which the results of [35] rely), the key ingredients in our approach are represented by the main results of [5] and [31] (see the outline of the proof given below for more details).[1]

The main result. Let $A/\mathbf{Q}$ be an elliptic curve having *split* multiplicative reduction at an *odd* rational prime p. Let N_A be the conductor of $A/\mathbf{Q}$, let $j_A \in \mathbf{Q}$ be its j-invariant, and let $\overline{\rho}_{A,p} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_p)$ be (the isomorphism class of) the representation of $G_{\mathbf{Q}}$ on the p-torsion submodule $A[p]$ of $A(\overline{\mathbf{Q}})$.

Theorem A. *Let $A/\mathbf{Q}$ and $p \neq 2$ be as above. Assume in addition that the following properties hold:*

1. *$\overline{\rho}_{A,p}$ is irreducible;*
2. *there exists a prime $q \| N_A$, $q \neq p$ such that $p \nmid \mathrm{ord}_q(j_A)$;*
3. *$\mathrm{rank}_{\mathbf{Z}} A(\mathbf{Q}) = 1$ and $Ш(A/\mathbf{Q})_{p^\infty}$ is finite.*

Then the Hasse–Weil L-function $L(A/\mathbf{Q}, s)$ of $A/\mathbf{Q}$ has a simple zero at $s = 1$.

Combined with the KGZ theorem recalled above, this implies:

Theorem B. *Let $A/\mathbf{Q}$ be an elliptic curve having split multiplicative reduction at an odd rational prime p. Assume that $\overline{\rho}_{A,p}$ is irreducible, and that there exists a prime $q \| N_A$, $q \neq p$ such that $p \nmid \mathrm{ord}_q(j_A)$. Then*

$$\mathrm{ord}_{s=1} L(A/\mathbf{Q}, s) = 1 \iff \mathrm{rank}_{\mathbf{Z}} A(\mathbf{Q}) = 1 \text{ and } \#\big(Ш(A/\mathbf{Q})_{p^\infty}\big) < \infty.$$

If this is the case, the whole Tate–Shafarevich group $Ш(A/\mathbf{Q})$ is finite.

Outline of the proof. Let $A/\mathbf{Q}$ be an elliptic curve having split multiplicative reduction at a prime $p \neq 2$, and let $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N_A), \mathbf{Z})^{\mathrm{new}}$ be the weight-two newform attached to A by the modularity theorem of Wiles, Taylor–Wiles *et. al.* Then $N_A = Np$, with $p \nmid N$ and $a_p = a_p(A) = +1$. Assume that $\overline{\rho}_{A,p}$ is irreducible.

Let $\mathbf{f} = \sum_{n=1}^{\infty} \mathbf{a}_n q^n \in \mathbb{I}[\![q]\!]$ be the *Hida family* passing through f. Here $\mathbb{I}$ is a normal local domain, finite and flat over Hida's weight algebra $\Lambda := \mathcal{O}_L[\![\Gamma]\!]$ with $\mathcal{O}_L$-coefficients, where $\Gamma := 1 + p\mathbf{Z}_p$ and $\mathcal{O}_L$ is the ring of integers of a (sufficiently large) finite extension $L/\mathbf{Q}_p$ (cf. Section 2.1). There is a natural injective morphism (Mellin transform) $\mathtt{M} : \mathbb{I} \hookrightarrow \mathscr{A}(U)$, where $U \subset \mathbf{Z}_p$ is a suitable

[1] After this note was written, C. Skinner communicated to the author that, together with W. Zhang, he extended the methods of [39] to obtain (among other results) the p-converse of the KGZ theorem in cases where p is a prime of multiplicative reduction [32]. While there is an overlap between the main result of this note and the result of Skinner–Zhang, neither subsumes the other (cf. the end of this section). Moreover, as remarked above, the methods of proof are substantially different.

p-adic neighbourhood of 2, and $\mathscr{A}(U) \subset L[\![k-2]\!]$ denotes the sub-ring of formal power series in $k-2$ which converge in U (see Section 4.1). Write

$$f_\infty := \sum_{n=1}^{\infty} a_n(k) \cdot q^n \in \mathscr{A}(U)[\![q]\!],$$

with $a_n(k) \in \mathscr{A}(U)$ defined as the image of $\mathbf{a}_n \in \mathbb{I}$ under $\mathtt{M}$. For every *classical point* $\kappa \in U^{\mathrm{cl}} := U \cap \mathbf{Z}^{\geq 2}$, the weight-$\kappa$-specialization $f_\kappa := \sum_{n=1}^{\infty} a_n(\kappa) q^n$ is the q-expansion of a normalised Hecke eigenform of weight κ and level $\Gamma_1(Np)$; moreover $f_2 = f$. For every quadratic character χ of conductor coprime with Np, a construction of Mazur–Kitagawa and Greenberg–Stevens [5, Section 1] attaches to f_∞ and χ a two-variable p-adic analytic L-function $L_p(f_\infty, \chi, k, s)$ on $U \times \mathbf{Z}_p$, interpolating the special complex L-values $L(f_\kappa, \chi, j)$, where $\kappa \in U^{\mathrm{cl}}$, $1 \leq j \leq \kappa - 1$ and $L(f_\kappa, \chi, s)$ is the Hecke L-function of f_κ twisted by χ. (Here s is the *cyclotomic variable*, and k is the *weight-variable*.) Define the *central critical p-adic L-function of* (f_∞, χ):

$$L_p^{\mathrm{cc}}(f_\infty, \chi, k) := L_p(f_\infty, \chi, k, k/2) \in \mathscr{A}(U)$$

as the restriction of the Mazur–Kitagawa p-adic L-function to the *central critical line* $s = k/2$ in the (k,s)-plane.

On the algebraic side, Hida theory attaches to $\mathbf{f}$ a *central critical deformation* $\mathbb{T}_{\mathbf{f}}$ of the p-adic Tate module of $A/\mathbf{Q}$. $\mathbb{T}_{\mathbf{f}}$ is a free rank-two $\mathbb{I}$-module, equipped with a continuous, $\mathbb{I}$-linear action of $G_{\mathbf{Q}}$, satisfying the following interpolation property: let $\kappa \in U^{\mathrm{cl}}$ be a classical point such that $\kappa \equiv 2 \pmod{2(p-1)}$, and let $\mathrm{ev}_\kappa : \mathbb{I} \hookrightarrow \mathscr{A}(U) \to L$ be the morphism induced by evaluation at κ on $\mathscr{A}(U)$. Then the base change $\mathbb{T}_{\mathbf{f}} \otimes_{\mathbb{I},\mathrm{ev}_\kappa} L$ is isomorphic to the central critical twist $V_{f_\kappa}(1-\kappa/2)$ of the contragredient V_{f_κ} of the p-adic Deligne representation of f_κ. Moreover, $\mathbb{T}_{\mathbf{f}}$ is *nearly-ordinary* at p. More precisely, let v be a prime of $\overline{\mathbf{Q}}$ dividing p, associated with an embedding $i_v : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, and denote by $i_v^* : G_{\mathbf{Q}_p} \cong G_v \subset G_{\mathbf{Q}}$ the corresponding decomposition group at v. Then there is a short exact sequence of $\mathbb{I}[G_v]$-modules

$$0 \to \mathbb{T}_{\mathbf{f},v}^+ \to \mathbb{T}_{\mathbf{f}} \to \mathbb{T}_{\mathbf{f},v}^- \to 0,$$

with $\mathbb{T}_{\mathbf{f},v}^\pm$ free of rank one over $\mathbb{I}$. For every number field $F/\mathbf{Q}$, define the (strict) Greenberg Selmer group

$$\mathrm{Sel}_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/F) := \ker\left(H^1(G_{F,S}, \mathbb{T}_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}^*) \longrightarrow \prod_{v|p} H^1(F_v, \mathbb{T}_{\mathbf{f},v}^- \otimes_{\mathbb{I}} \mathbb{I}^*)\right).$$

Here S is a finite set of primes of F containing every prime divisor of $N_A \mathrm{disc}(F)$, $G_{F,S}$ is the Galois group of the maximal algebraic extension of F which is unramified outside $S\cup\{\infty\}$, $\mathbb{I}^* := \mathrm{Hom}_{\mathrm{cont}}(\mathbb{I}, \mathbf{Q}_p/\mathbf{Z}_p)$ is the Pontrjagin dual of $\mathbb{I}$, and the product runs over all the primes v of F which divide p [2]. Write

$$X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/F) := \mathrm{Hom}_{\mathbf{Z}_p}\Big(\mathrm{Sel}_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/F), \mathbf{Q}_p/\mathbf{Z}_p\Big)$$

for the Pontrjagin dual of $\mathrm{Sel}_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/F)$. It is a finitely generated $\mathbb{I}$-module. We now explain the main steps entering in the proof of Theorem A.

Step I: Skinner–Urban's divisibility. Let $K/\mathbf{Q}$ be an imaginary quadratic field in which *p splits*. Assume that the discriminant of $K/\mathbf{Q}$ is coprime to N_A, and write $N_A = N^+N^-$, where N^+ (resp., N^-) is divided precisely by the prime divisors of N_A which are split (resp., inert) in K. Assume the following *generalised Heegner hypothesis* and *ramification hypothesis*:

- N^- is a square-free product of an *odd* number of primes.
- $\overline{\rho}_{A,p}$ is ramified at all prime divisors of N^-.

Under some additional technical hypotheses on the data $(A, K, p, \dots)$ (cf. Hypotheses 1, 2 and 3 below), the main result of [31], together with some auxiliary computations, allows us to deduce the following inequality:

$$\mathrm{ord}_{k=2} L_p^{\mathrm{cc}}(f_\infty/K, k) \leq \mathrm{length}_{\mathfrak{p}_f}\Big(X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K)\Big) + 2. \tag{1}$$

Here $L_p^{\mathrm{cc}}(f_\infty/K,k) := L_p^{\mathrm{cc}}(f_\infty, \chi_{\mathrm{triv}}, k)\cdot L_p^{\mathrm{cc}}(f_\infty, \epsilon_K, k)$, where χ_{triv} is the trivial character and ϵ_K is the quadratic character attached to K. $\mathfrak{p}_f := \ker\big(\mathrm{ev}_2 : \mathbb{I} \hookrightarrow \mathscr{A}(U) \to L\big)$ is the kernel of the morphism induced by evaluation at $k=2$ on $\mathscr{A}(U)$; it is a height-one prime ideal of $\mathbb{I}$, so that the localisation $\mathbb{I}_{\mathfrak{p}_f}$ is a discrete valuation ring. Finally, $\mathrm{length}_{\mathfrak{p}_f}(M)$ denotes the length over $\mathbb{I}_{\mathfrak{p}_f}$ of the localisation $M_{\mathfrak{p}_f}$, for every finite $\mathbb{I}$-module M.

Remark. The main result of Skinner and Urban [31] mentioned above, which proves one divisibility in a three variable main conjecture for GL_2, is a result *over* K, for $K/\mathbf{Q}$ as above, and not over $\mathbf{Q}$. This is why we need to consider a base-change to such a $K/\mathbf{Q}$ in our approach to Theorem A.

Remark. By assumption, $A/\mathbf{Q}$ has split multiplicative reduction at p, and as well-known this implies that $L_p(f_\infty, \chi_{\mathrm{triv}}, k, s)$ has a trivial zero at $(k,s) = (2,1)$ in the sense of [19]. Moreover, the hypothesis $\epsilon_K(p) = +1$ (i.e. p splits in K) implies that $L_p(f_\infty, \epsilon_K, k, s)$ also has such an exceptional zero at $(k,s) = (2,1)$ (see, e.g. [5, Section 1] or [25, Section 5]). This is the reason behind the appearance of the addend 2 in the R.H.S. of (1).

[2] $\mathrm{Sel}_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/F)$ depends on the choice of the set S, even if this dependence is irrelevant for the purposes of this introduction.

Remark. The generalised Heegner hypothesis gives $\epsilon_K(-N_A) = -\epsilon_K(N^-) = +1$. This implies that the Hecke L-series $L(f,s) = L(A/\mathbf{Q},s)$ and $L(f,\epsilon_K,s) = L(A^K/\mathbf{Q},s)$ (where $A^K/\mathbf{Q}$ is the quadratic twist of A by K) have the *same* sign in their functional equations at $s = 1$. The Birch and Swinnerton-Dyer conjecture then predicts that the ranks of $A(\mathbf{Q})$ and $A^K(\mathbf{Q}) \cong A(K)^-$ have the same parity. In particular $\mathrm{rank}_{\mathbf{Z}} A(K)$, and then $\mathrm{ord}_{k=2} L_p^{\mathrm{cc}}(f_\infty/K,k)$ should be *even*.

Step II: Bertolini–Darmon's exceptional-zero formula. Let $K/\mathbf{Q}$ be as in Step I. Assume moreover

- $\mathrm{sign}(A/\mathbf{Q}) = -1$

where $\mathrm{sign}(A/\mathbf{Q}) \in \{\pm 1\}$ denotes the sign in the functional equation satisfied by the Hasse–Weil L-function $L(A/\mathbf{Q},s)$. As remarked above, this implies that $\mathrm{sign}(A^K/\mathbf{Q}) = -1$ too. The analysis carried out in [5, 12, 26] tells us that, for both $\chi = \chi_{\mathrm{triv}}$ and $\chi = \epsilon_K$:

$$\mathrm{ord}_{k=2} L_p^{\mathrm{cc}}(f_\infty,\chi,k) \geq 2; \tag{2}$$

this is once again a manifestation of the presence of an exceptional zero at $(k,s) = (2,1)$ for the Mazur–Kitagawa p-adic L-function $L_p(f_\infty,\chi,k,s)$. Much more deeper, Bertolini and Darmon proved in [5] the formula

$$\frac{d^2}{dk^2} L_p^{\mathrm{cc}}(f_\infty,\chi,k)_{k=2} \doteq \log_A^2(\mathbf{P}_\chi),$$

where $\doteq$ denotes equality up to a non-zero factor, $\log_A : A(\mathbf{Q}_p) \to \mathbf{Q}_p$ is the formal group logarithm, and $\mathbf{P}_\chi \in A(K)^\chi$ is a Heegner point. This formula implies that

$$\mathrm{ord}_{k=2} L_p^{\mathrm{cc}}(f_\infty,\chi,k) = 2 \iff \mathrm{ord}_{s=1} L(A^\chi/\mathbf{Q},s) = 1, \tag{3}$$

i.e. if and only if the Hasse–Weil L-function of the χ-twist $A^\chi/\mathbf{Q}$ has a simple zero at $s = 1$. (Here of course $A^\chi = A$ is $\chi = \chi_{\mathrm{triv}}$ and $A^\chi = A^K$ if $\chi = \epsilon_K$. Recall that by assumption $L(A^\chi/\mathbf{Q},s)$ vanishes at $s = 1$.)

Step III: bounding the characteristic ideal. Let χ denote either the trivial character or a quadratic character of conductor coprime with Np, and write $K_\chi := \mathbf{Q}$ or $K_\chi/\mathbf{Q}$ for the quadratic field attached to χ accordingly. Making use of Nekovář's theory of Selmer Complexes (especially of Nekovář's generalised Cassels–Tate pairings) [20], we are able to relate the structure of the $\mathbb{I}_{\mathfrak{p}_f}$-module $X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K_\chi)^\chi_{\mathfrak{p}_f} := X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K_\chi)^\chi \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}$ to the properties of a suitable *Nekovář's half-twisted weight pairing* (see Section 6.2)

$$\langle -,- \rangle_{V_f,\pi}^{\mathrm{Nek},\chi} : A^\dagger(K_\chi)^\chi \times A^\dagger(K_\chi)^\chi \longrightarrow \mathbf{Q}_p,$$

playing here the rôle of the canonical cyclotomic p-adic height pairing of Schneider, Mazur–Tate *et. al.* in cyclotomic Iwasawa theory. Here, for every $\mathbf{Z}[\mathrm{Gal}(K_\chi/\mathbf{Q})]$-module M, we write M^χ for the submodule of M on which $\mathrm{Gal}(K_\chi/\mathbf{Q})$ acts via χ, and

$A^{\dagger}(K_{\chi})$ is the *extended Mordell–Weil group* of A/K_{χ} introduced in [19]. $\langle -,-\rangle^{\mathrm{Nek},\chi}_{V_f,\pi}$ is a bilinear and *skew-symmetric* form on $A^{\dagger}(K_{\chi})^{\chi}$ (see Section 6). Assume that the following conditions are satisfied:

- $\chi(p)=1$, i.e. p splits in K_{χ};
- $\mathrm{rank}_{\mathbf{Z}} A(K_{\chi})^{\chi}=1$ and $Ш(A/K_{\chi})^{\chi}_{p^{\infty}}$ is finite.

Then $A^{\dagger}(K_{\chi})^{\chi}\otimes\mathbf{Q}_p=\mathbf{Q}_p\cdot q_{\chi}\oplus\mathbf{Q}_p\cdot P_{\chi}$ is a 2-dimensional $\mathbf{Q}_p$-vector space generated by a non-zero point $P_{\chi}\in A(K_{\chi})^{\chi}\otimes\mathbf{Q}$ and a certain *Tate's period* $q_{\chi}\in A^{\dagger}(K_{\chi})^{\chi}$ (which does *not* come from a K_{χ}-rational point of A). In the author's Ph.D. Thesis [35] we proved that

$$\langle q_{\chi},P_{\chi}\rangle^{\mathrm{Nek},\chi}_{V_f,\pi}\doteq\log_A(P_{\chi}) \tag{4}$$

(where $\doteq$ denotes again equality up to a non-zero multiplicative factor), which implies that $\langle -,-\rangle^{\mathrm{Nek},\chi}_{V_f,\pi}$ is non-degenerate on $A^{\dagger}(K_{\chi})^{\chi}$. Together with the results of Nekovář mentioned above, this allows us to deduce that

$$X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_{\chi})^{\chi}_{\mathfrak{p}_f}\cong\mathbb{I}_{\mathfrak{p}_f}/\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f}. \tag{5}$$

Remark. Let $V_f:=\mathrm{Ta}_p(A)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$ be the p-adic Tate module of $A/\mathbf{Q}$, and let $H^1_f(K_{\chi},V_f)$ be the Bloch–Kato Selmer group of V_f over K_{χ}. The pairing $\langle -,-\rangle^{\mathrm{Nek},\chi}_{V_f,\pi}$ is naturally defined on Nekovář's *extended Selmer group* $\widetilde{H}^1_f(K_{\chi},V_f)^{\chi}$, which is an extension of $H^1_f(K_{\chi},V_f)^{\chi}$ by the $\mathbf{Q}_p$-module generated by q_{χ}. Indeed it is the non-degeneracy of $\langle -,-\rangle^{\mathrm{Nek},\chi}_{V_f,\pi}$ on $\widetilde{H}^1_f(K_{\chi},V_f)^{\chi}$ to be directly related to the structure of the $\mathbb{I}_{\mathfrak{p}_f}$-module $X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_{\chi})^{\chi}_{\mathfrak{p}_f}$. On the other hand, $\widetilde{H}^1_f(K_{\chi},V_f)^{\chi}$ contains $A^{\dagger}(K_{\chi})^{\chi}\otimes\mathbf{Q}_p$, and equals it precisely if the p-primary part of $Ш(A/K_{\chi})^{\chi}$ is finite. This explains why we need the finiteness of $Ш(A/K_{\chi})^{\chi}_{p^{\infty}}$ in order to deduce (5).

Remark. The length of $X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_{\chi})^{\chi}_{\mathfrak{p}_f}$ over $\mathbb{I}_{\mathfrak{p}_f}$ can be interpreted as the order of vanishing at $k=2$ of an algebraic p-adic L-function $\mathbb{L}^{\mathrm{cc}}_p(f_{\infty},\chi,k)\in\mathscr{A}(U)$, defined as the Mellin transform of the characteristic ideal of $X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_{\chi})^{\chi}$ (at least assuming that $\mathbb{I}$ is regular). The results of Nekovář briefly mentioned above can be used to prove an analogue in our setting of the algebraic p-adic Birch and Swinnerton-Dyer formulae of Schneider [23] and Perrin-Riou [22], which relates the leading coefficient of $\mathbb{L}^{\mathrm{cc}}_p(f_{\infty},\chi,k)$ at $k=2$ to the determinant of $\langle -,-\rangle^{\mathrm{Nek},\chi}_{V_f,\pi}$, computed on $A^{\dagger}(K_{\chi})^{\chi}/\mathrm{torsion}$.

Remark. Formula (4) is crucial here. Indeed, as remarked above, it allows us to deduce the non-degeneracy of the weight-pairing $\langle -,-\rangle^{\mathrm{Nek},\chi}_{V_f,\pi}$. The analogue of this result in cyclotomic Iwasawa theory (i.e. Schneider conjecture in rank-one) seems out of reach at present.

Remark. The preceding results, and (4) in particular, should be considered as an algebraic counterpart of Bertolini–Darmon's exceptional zero formula (cf. Step II). This point of view is developed in [36] (see also Part I of the author's Ph.D. thesis [35]), and leads to the formulation of two-variable analogues of the Birch and Swinnerton-Dyer conjecture for the Mazur–Kitagawa p-adic L-function $L_p(f_\infty, \chi, k, s)$. Formula (4) — to be considered part of Nekovář's theory — and Bertolini–Darmon's exceptional zero formula, also represent crucial ingredients in the proof, given in [37], of the Mazur–Tate–Teitelbaum exceptional zero conjecture in rank one.

Step IV: conclusion of the proof. Assume that the hypotheses of Theorem A are satisfied. Thanks to Nekovář's proof of the parity conjecture [20], $\mathrm{sign}(A/\mathbf{Q}) = -1$. By the main result of [7] and hypothesis 2 in Theorem A, we are then able to find a quadratic imaginary field $K/\mathbf{Q}$ which satisfies the hypotheses needed in Steps I and II, with $N^- = q$, and such that $L(A^K/\mathbf{Q}, s)$ has a simple zero at $s = 1$, i.e.

$$\mathrm{ord}_{s=1} L(A^K/\mathbf{Q}, s) = 1. \tag{6}$$

An application of the KGZ theorem gives

$$\mathrm{rank}_{\mathbf{Z}} A^K(\mathbf{Q}) = 1; \quad \#\Big(\text{Ш}(A^K/\mathbf{Q})_{p^\infty}\Big) < \infty.$$

Together with hypothesis 3 in Theorem A, this implies that the hypotheses needed in Step III are satisfied by both the trivial character $\chi = \chi_{\mathrm{triv}}$ and $\chi = \epsilon_K$. Then

$$4 \overset{(2)}{\leq} \mathrm{ord}_{k=2} L_p^{\mathrm{cc}}(f_\infty/K, k) \overset{(1)}{\leq} \mathrm{length}_{\mathfrak{p}_f}\Big(X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K)\Big) + 2 \overset{(5)}{=} 4,$$

i.e. $\mathrm{ord}_{k=2} L_p^{\mathrm{cc}}(f_\infty/K, k) = 4$. Applying now Bertolini–Darmon's result (3) yields

$$\mathrm{ord}_{s=1} L(A/K, s) = 2,$$

where $L(A/K, s) = L(A/\mathbf{Q}, s) \cdot L(A^K/\mathbf{Q}, s)$ is the Hasse–Weil L-function of A/K. Together with (6), this implies that $L(A/\mathbf{Q}, s)$ has a simple zero at $s = 1$, as was to be shown.

Recent related results. In the recent preprint [32], Skinner and Zhang prove (among other results) a theorem similar to our Theorem A. More precisely, Theorem 1.1 of *loc. cit.* proves instances of the p-converse of the KGZ theorem in rank one, for an elliptic curve with multiplicative reduction at a prime $p \geq 5$. On the one hand, their result does not require the p-primary part of the Tate–Shafarevich group to be finite, but only that the p-primary Selmer group of the elliptic curve has $\mathbf{Z}_p$-corank one. On the other hand, together with the assumptions 1 and 2 of Theorem A, the authors assume extra hypotheses in their statement. For example, they assume that the mod-p Galois representation $\overline{\rho}_{A,p}$ is not finite at p, that the Mazur–Tate–Teitelbaum

L-invariant $\mathscr{L}_p(A/\mathbf{Q}) := \frac{\log_p(q_A)}{\mathrm{ord}_p(q_A)}$ has p-adic valuation 1 (where $q_A \in p\mathbf{Z}_p$ is the Tate period of $A/\mathbf{Q}_p$), and require additional 'p-indivisibility conditions' for the Tamagawa factors of $A/\mathbf{Q}$. (We refer to *loc. cit.* for a precise list of the assumptions.) Finally, it is worth noting that our approach here (cf. preceding Section) is essentially different from that of [32], where the authors extend the results and methods of [39] to the multiplicative setting.

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2. Hida Theory

Fix for the rest of this note an elliptic curve $A/\mathbf{Q}$ having *split* multiplicative reduction at an odd rational prime p. Let N_A be the conductor of $A/\mathbf{Q}$, so that $N_A = Np$, with $p \nmid N$, and let

$$f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(Np), \mathbf{Z})^{\mathrm{new}}$$

be the weight-two newform attached to $A/\mathbf{Q}$ by modularity. Fix a finite extension $L/\mathbf{Q}_p$, with ring of integers $\mathcal{O}_L$ and maximal ideal $\mathfrak{m}_L$, and an embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, under which we identify $\overline{\mathbf{Q}}$ with a subfield of $\overline{\mathbf{Q}}_p$. This also fixes a decomposition group $i_p^* : G_{\mathbf{Q}_p} \hookrightarrow G_{\mathbf{Q}}$ at p (where $G_F := \mathrm{Gal}(\overline{F}/F)$ for every field F).

2.1. The Hida family $\mathbb{I}$. Let $\Gamma := 1 + p\mathbf{Z}_p$, let $\mathbf{Z}_{N,p}^{\times} := \Gamma \times (\mathbf{Z}/pN\mathbf{Z})^{\times}$, and let

$$\mathcal{O}_L[\![\mathbf{Z}_{N,p}^{\times}]\!][T_n : n \in \mathbf{N}] \twoheadrightarrow h^o(N, \mathcal{O}_L)$$

be Hida's universal p-ordinary Hecke algebra with $\mathcal{O}_L$-coefficients. Writing $\Lambda := \mathcal{O}_L[\![\Gamma]\!]$, $h^o(N, \mathcal{O}_L)$ is a finite, flat Λ-algebra [14]. Letting $\mathscr{L} := \mathrm{Frac}(\Lambda)$, there is a decomposition $h^o(N, \mathcal{O}_L) \otimes_\Lambda \mathscr{L} = \prod_j \mathscr{K}_j$ as a finite product of finite field extensions $\mathscr{K}_j/\mathscr{L}$. Let $\mathscr{K} = \mathscr{K}_{j_o}$ be the *primitive component* of $h^o(N, \mathcal{O}_L) \otimes_\Lambda \mathscr{L}$ to which the p-ordinary newform f *belongs* [14, Section 1], and let $\mathbb{I}$ be the integral closure of Λ in the finite extension $\mathscr{K}/\mathscr{L}$. For every $n \in \mathbf{N}$, write $\mathbf{a}_n \in \mathbb{I}$ for the image in $\mathbb{I}$ of the nth Hecke operator T_n. By [14, Corollary 1.5], there exists a unique morphism of $\mathcal{O}_L$-algebras

$$\phi_f : \mathbb{I} \longrightarrow \mathcal{O}_L,$$

such that $\phi_f(\mathbf{a}_n) = a_n$ for every $n \in \mathbf{N}$; moreover, ϕ_f maps the image of $\mathbf{Z}_{N,p}^{\times}$ in $\mathbb{I}$ to 1 (as f has weight two and trivial neben type). $\mathbb{I}$ is a normal local domain, finite and flat over Hida's weight algebra Λ. The domain $\mathbb{I}$ is called the *(branch of the) Hida family passing through* f. This terminology is justified as follows.

An *arithmetic point* on $\mathbb{I}$ is a continuous morphism of $\mathcal{O}_L$-algebras $\psi : \mathbb{I} \to \overline{\mathbf{Q}}_p$, whose restriction to Γ (with respect to the structural morphism $\Lambda \to \mathbb{I}$) is of the form $\psi|_\Gamma(\gamma) = \gamma^{k_\psi - 2} \cdot \chi_\psi(\gamma)$, for an integer $k_\psi \geq 2$ and a finite order character χ_ψ on Γ. We call k_ψ and χ_ψ the *weight* and *(wild) character* of ψ respectively. Write $\mathcal{X}^{\mathrm{arith}}(\mathbb{I})$ for the set of arithmetic points on $\mathbb{I}$. Note that $\phi_f \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})$ is an arithmetic point of weight 2 and trivial character. Let

$$\mathbf{f} = \sum_{n=1}^{\infty} \mathbf{a}_n \cdot q^n \in \mathbb{I}[\![q]\!].$$

Then for every $\psi \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})$, the *specialisation of* $\mathbf{f}$ *at* ψ:

$$f_\psi := \sum_{n=1}^{\infty} \psi(\mathbf{a}_n) \cdot q^n \in S_{k_\psi}(\Gamma_0(Np^{c_\psi+1}), \xi_\psi)$$

is a p-stabilised ordinary newform of tame level N, weight k_ψ and character $\xi_\psi := \chi_\psi \cdot \omega^{2-k_\psi}$. Here $c_\psi \geq 0$ is the smallest positive integer such that $\Gamma^{p^{c_\psi}} \subset \ker(\chi_\psi)$, and $\omega : \mathbf{Z}/(p-1)\mathbf{Z} \cong \mathbf{F}_p^\times \to \mathbf{Z}_p^\times$ is the Teichmüller character. Moreover, we recover f as the ϕ_f-specialisation of $\mathbf{f}$, i.e.

$$f_{\phi_f} := \sum_{n=1}^{\infty} \phi_f(\mathbf{a}_n) q^n = f.$$

Let $\psi \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})$ be an arithmetic point. Denote by $K_\psi := \mathrm{Frac}(\psi(\mathbb{I})) \subset \overline{\mathbf{Q}}_p$ the fraction field of $\psi(\mathbb{I})$, by $\mathfrak{m}_\psi$ its maximal ideal, and by $\mathbf{F}_\psi = \psi(\mathbb{I})/\mathfrak{m}_\psi$ its residue field. Let $\rho_\psi : G_{\mathbf{Q}} \to \mathrm{GL}_2(K_\psi)$ be the contragredient of the Deligne representation associated with f_ψ, and denote by $\overline{\rho}_\psi : G_{\mathbf{Q}} \to \mathrm{Gal}(\mathbf{F}_\psi)$ the semi-simplification of the reduction of ρ_ψ modulo $\mathfrak{m}_\psi$. Then $\overline{\rho}_\psi$ is unramified at every prime $\ell \nmid Np$, and $\mathrm{Trace}(\overline{\rho}_\psi(\mathrm{Frob}_\ell)) = \psi(\mathbf{a}_\ell) \pmod{\mathfrak{m}_\psi}$ for every prime $\ell \nmid Np$, where $\mathrm{Frob}_\ell \in G_{\mathbf{Q}}$ is an arithmetic Frobenius at ℓ. Enlarging L if necessary, one can assume $\mathbf{F}_\psi = \mathbf{F} := \mathcal{O}_L/\mathfrak{m}_L$. Then the representation $\overline{\rho}_\psi$ does not depend, up to isomorphism, on the arithmetic point ψ. Denote by $\overline{\rho}_{\mathbf{f}}$ this isomorphism class, and assume throughout this note the following.

Hypothesis 1 (irr). *$\overline{\rho}_{\mathbf{f}}$ is (absolutely) irreducible.*

Under this assumption, it is known that $\mathbb{H}_{\mathbf{f}} := \big(h^o(N, \mathcal{O}_L) \otimes_\Lambda \mathbb{I}\big) \cap \big(\mathscr{K} \times 0\big)$ is a free $\mathbb{I}$-module of rank one (where we use the decomposition $h^o(N, \mathcal{O}_L) \otimes_\Lambda \mathscr{L} = \mathscr{K} \times \prod_{j \neq j_o} \mathscr{K}_j$ mentioned above).

Remark 2.1. Taking $\psi = \phi_f$ in the discussion above, we deduce that $\overline{\rho}_{\mathbf{f}}$ is isomorphic to the $\mathbf{F}$-base change of the mod-p Galois representation $\overline{\rho}_{A,p}$ attached to the p-torsion submodule $A[p]$ of $A(\overline{\mathbf{Q}})$. (Indeed, Hypothesis 1 is equivalent to require that $\overline{\rho}_{A,p}$ is absolutely irreducible.) Since A has split multiplicative reduction at p, Tate's theory gives us an isomorphism (see [34] or Chapter V of [29])

$$\overline{\rho}_{\mathbf{f}}|_{G_{\mathbf{Q}_p}} \cong \begin{pmatrix} \omega_{\mathrm{cy}} & * \\ 0 & 1 \end{pmatrix},$$

where $\overline{\rho}_{\mathbf{f}}|_{G_{\mathbf{Q}_p}}$ is the restriction of $\overline{\rho}_{\mathbf{f}}$ to $G_{\mathbf{Q}_p}$ and $\omega_{\mathrm{cy}} : G_{\mathbf{Q}_p} \twoheadrightarrow \mathrm{Gal}(\mathbf{Q}_p(\mu_p)/\mathbf{Q}_p) \cong \mathbf{F}_p^\times$ is the mod-p cyclotomic character. As $p \neq 2$, this implies that $\overline{\rho}_{\mathbf{f}}$ is *p-distinguished*, i.e. that condition $\mathbf{(dist)_f}$ in [31] is satisfied.

2.2. Hida's representations $T_{\mathbf{f}}$ and $\mathbb{T}_{\mathbf{f}}$. Let $T_{\mathbf{f}} = (T_{\mathbf{f}}, T_{\mathbf{f}}^+)$ be Hida's p-ordinary $\mathbb{I}$-adic representation attached to $\mathbf{f}$ (see, e.g. [14], [31]). Thanks to our Hypothesis 1, $T_{\mathbf{f}}$ is a free $\mathbb{I}$-module of rank two, equipped with a continuous action of $G_{\mathbf{Q}}$ which is unramified at every prime $\ell \nmid Np$, and such that

$$\det(1 - \mathrm{Frob}_\ell \cdot X|T_{\mathbf{f}}) = 1 - \mathbf{a}_\ell \cdot X + \ell[\ell] \cdot X^2 \tag{7}$$

for every $\ell \nmid Np$. Here $\mathrm{Frob}_\ell = \mathrm{frob}_\ell^{-1}$ is an arithmetic Frobenius at ℓ and $[\cdot] : \mathbf{Z}_{N,p}^\times \subset \mathcal{O}_L[\![\mathbf{Z}_{N,p}^\times]\!] \to \mathbb{I}$ is the structural morphism. Write

$$\chi_{\mathrm{cy},N} : G_{\mathbf{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbf{Q}(\mu_{Np^\infty})/\mathbf{Q}) \cong \mathbf{Z}_{N,p}^\times = \Gamma \times (\mathbf{Z}/Np\mathbf{Z})^\times, \quad \chi_{\mathrm{cy}} : G_{\mathbf{Q}} \twoheadrightarrow \mathbf{Z}_p^\times$$

for the p-adic cyclotomic character (i.e. the composition of $\chi_{\mathrm{cy},N}$ with projection to $\mathbf{Z}_p^\times = \Gamma \times (\mathbf{Z}/p\mathbf{Z})^\times$) and $\kappa_{\mathrm{cy}} : G_{\mathbf{Q}_p} \to \Gamma$ for the composition of χ_{cy} with projection to principal units. Then $[\chi_{\mathrm{cy}}] = [\kappa_{\mathrm{cy}}] = [\chi_{\mathrm{cy},N}]$ as $\mathbb{I}^\times$-valued characters on $G_{\mathbf{Q}}$ (since f has trivial neben type). In particular the determinant representation of $T_{\mathbf{f}}$ is given by

$$\det_{\mathbb{I}} T_{\mathbf{f}} \cong \mathbb{I}\big(\chi_{\mathrm{cy}} \cdot [\kappa_{\mathrm{cy}}]\big). \tag{8}$$

$T_{\mathbf{f}}^+$ is an $\mathbb{I}$-direct summand of $T_{\mathbf{f}}$ of rank one, which is invariant under the action of the decomposition group $G_{\mathbf{Q}_p} \hookrightarrow G_{\mathbf{Q}}$ determined by i_p. Moreover, $T_{\mathbf{f}}^- := T_{\mathbf{f}}/T_{\mathbf{f}}^+$ is an unramified $G_{\mathbf{Q}_p}$-module, and the Frobenius $\mathrm{Frob}_p \in G_{\mathbf{Q}_p}/I_{\mathbf{Q}_p}$ acts on it via multiplication by the p-th Fourier coefficient $\mathbf{a}_p \in \mathbb{I}^\times$ of $\mathbf{f}$. In other words

$$T_{\mathbf{f}}^+ \cong \mathbb{I}\Big(\mathbf{a}_p^{*-1} \cdot \chi_{\mathrm{cy}} \cdot [\kappa_{\mathrm{cy}}]\Big); \quad T_{\mathbf{f}}^- \cong \mathbb{I}\big(\mathbf{a}_p^*\big) \tag{9}$$

as $\mathbb{I}[G_{\mathbf{Q}_p}]$-modules, where $\mathbf{a}_p^* : G_{\mathbf{Q}_p} \twoheadrightarrow G_{\mathbf{Q}_p}/I_{\mathbf{Q}_p} \to \mathbb{I}^\times$ is the unramified character sending Frob_p to $\mathbf{a}_p$, and we write again $\kappa_{\mathrm{cy}} : G_{\mathbf{Q}_p} \twoheadrightarrow \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p) \cong \mathbf{Z}_p^\times \twoheadrightarrow \Gamma$ for the p-adic cyclotomic character on $G_{\mathbf{Q}_p}$.

Given an arithmetic point $\psi \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})$, let V_ψ be the contragredient of the p-adic Deligne representation attached to the eigenform f_ψ: it is a two-dimensional vector space over $K_\psi = \mathrm{Frac}(\mathbb{I}/\ker(\psi))$, equipped with a continuous K_ψ-linear action of $G_{\mathbf{Q}}$ which is unramified at every prime $\ell \nmid Np$, and such that the trace of Frob_ℓ acting on V_ψ equals the ℓth Fourier coefficient $\psi(\mathbf{a}_\ell) = a_\ell(f_\psi)$ of f_ψ, for every

$\ell \nmid Np$. As proved by Ribet, V_ψ is an absolutely irreducible $G_{\mathbf{Q}}$-representation, so that the Chebotarev density theorem, together with the Eichler–Shimura relations (7) tell us that there exists an isomorphism of $K_\psi[G_{\mathbf{Q}}]$-modules

$$T_{\mathbf{f}} \otimes_{\mathbb{I},\psi} K_\psi \cong V_\psi. \tag{10}$$

In other words, $T_{\mathbf{f}}$ interpolates the contragredients of the Deligne representations of the classical specialisations of the Hida family $\mathbf{f}$. (Note: $T_{\mathbf{f}}$ is the *contragredient* of the representation denoted by the same symbol in [31].)

Together with the representations $T_{\mathbf{f}}$, we are particularly interested in a certain self-dual twist $\mathbb{T}_{\mathbf{f}}$ of it, defined as follows. Define the *critical character*

$$[\chi_{\mathrm{cy}}]^{1/2} = [\kappa_{\mathrm{cy}}]^{1/2} : G_{\mathbf{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \cong \mathbf{Z}_p^\times \twoheadrightarrow \Gamma \xrightarrow{\sqrt{\cdot}} \Gamma \xrightarrow{[\cdot]} \mathbb{I}^\times,$$

where the isomorphism is given by the p-adic cyclotomic character χ_{cy}. (As $p \neq 2$ by assumption, $\Gamma = 1 + p\mathbf{Z}_p$ is uniquely 2-divisible, e.g. by Hensel's Lemma, so that $\sqrt{\cdot} : \Gamma \cong \Gamma$ is defined.) Let

$$\mathbb{T}_{\mathbf{f}} := T_{\mathbf{f}} \otimes_{\mathbb{I}} [\chi_{\mathrm{cy}}]^{-1/2} \in {}_{\mathbb{I}[G_{\mathbf{Q}}]}\mathrm{Mod}; \;\; \mathbb{T}_{\mathbf{f}}^{\pm} := T_{\mathbf{f}}^{\pm} \otimes_{\mathbb{I}} [\chi_{\mathrm{cy}}]^{-1/2} \in {}_{\mathbb{I}[G_{\mathbf{Q}_p}]}\mathrm{Mod},$$

where we write for simplicity $[\chi_{\mathrm{cy}}]^{-1/2}$ for the inverse of $[\chi_{\mathrm{cy}}]^{1/2}$. By (8), $\mathbb{T}_{\mathbf{f}}$ satisfies the crucial property:

$$\det_{\mathbb{I}} \mathbb{T}_{\mathbf{f}} \cong \mathbb{I}(1),$$

i.e. the determinant representation of $\mathbb{T}_{\mathbf{f}}$ is given by the p-adic cyclotomic character. As explained in [21], this implies that there exists a skew-symmetric morphism of $\mathbb{I}[G_{\mathbf{Q}}]$-modules

$$\pi : \mathbb{T}_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{T}_{\mathbf{f}} \longrightarrow \mathbb{I}(1),$$

inducing by adjunction isomorphisms of $\mathbb{I}[G_{\mathbf{Q}}]$- and $\mathbb{I}[G_{\mathbf{Q}_p}]$-modules respectively:

$$\mathrm{adj}(\pi) : \mathbb{T}_{\mathbf{f}} \cong \mathrm{Hom}_{\mathbb{I}}(\mathbb{T}_{\mathbf{f}}, \mathbb{I}(1)); \;\; \mathrm{adj}(\pi) : \mathbb{T}_{\mathbf{f}}^{\pm} \cong \mathrm{Hom}_{\mathbb{I}}(\mathbb{T}_{\mathbf{f}}^{\mp}, \mathbb{I}(1)).$$

Let $\mathcal{X}^{\mathrm{arith}}(\mathbb{I})'$ be the set of arithmetic points ψ with trivial character and weight $k_\psi \equiv 2 \pmod{2(p-1)}$. Given $\psi \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})'$, we have $\psi \circ [\chi_{\mathrm{cy}}]^{-1/2}(\mathrm{Frob}_\ell) = \ell^{1-k_\psi/2}$ for every $\ell \nmid Np$. Equation (10) then gives: for every arithmetic point $\psi \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})'$, there exists an isomorphism of $K_\psi[G_{\mathbf{Q}}]$-modules

$$\mathbb{T}_{\mathbf{f}} \otimes_{\mathbb{I},\psi} K_\psi \cong V_\psi(1 - k_\psi/2).$$

In particular, $\mathbb{T}_{\mathbf{f}}$ interpolates the family of self-dual, critical twists $V_\psi(1-k_\psi/2)$, for $\psi \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})'$.

Let v be a prime of $\overline{\mathbf{Q}}$ dividing p, associated with an embedding $i_v : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Write $i_v^* : G_{\mathbf{Q}_p} \hookrightarrow G_{\mathbf{Q}}$ for the embedding determined by i_v, and $G_v := i_v^*(G_{\mathbf{Q}_p})$ for the corresponding decomposition group at v. Let $M_{\mathbf{f}}$ denote either $T_{\mathbf{f}}$ or $\mathbb{T}_{\mathbf{f}}$. Set

$M_{\mathbf{f},v}^{\pm} := M_{\mathbf{f}}^{\pm} \in {}_{\mathbb{I}[G_{\mathbf{Q}_p}]}\mathrm{Mod}$, which we consider as $\mathbb{I}[G_v]$-modules via i_v^*. Then there is a short exact sequence of $\mathbb{I}[G_v]$-modules

$$0 \to M_{\mathbf{f},v}^{+} \overset{i_v^+}{\to} M_{\mathbf{f}} \overset{p_v^-}{\to} M_{\mathbf{f},v}^{-} \to 0, \tag{11}$$

where i_v^+ and p_v^- are defined as follows. Fix $\alpha_v \in G_{\mathbf{Q}_p}$ and $\beta_v \in G_{\mathbf{Q}}$ such that $i_v = \alpha_v \circ i_p \circ \beta_v$. Then one sets $i_v^+ := \beta_v^{-1} \circ i^+ \circ \alpha_v^{-1}$ and $p_v^- := \alpha_v \circ p^- \circ \beta_v$, where $i^+ : M_{\mathbf{f}}^+ \subset M_{\mathbf{f}}$ and $p^- : M_{\mathbf{f}} \twoheadrightarrow M_{\mathbf{f}}^-$ denote the inclusion and projection respectively.

3. The theorem of Skinner–Urban

The aim of this section is to state the main result of [31] in our setting. In order to do that, we recall Skinner–Urban's construction of a three-variable p-adic L-function attached to $\mathbf{f}$ and a suitable quadratic imaginary field, and we introduce the Greenberg-style Selmer groups attached to the Hida family $\mathbf{f}$.

3.1. Cyclotomic p-adic L-functions. For every $\psi \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})$, write $\mathcal{O}_\psi := \psi(\mathbb{I})$. Let $\mathbf{Q}_\infty/\mathbf{Q}$ be the $\mathbf{Z}_p$-extension of $\mathbf{Q}$, let $G_\infty := \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$, and write $\Lambda_\psi^{\mathrm{cy}} := \mathcal{O}_\psi[\![G_\infty]\!]$ for the cyclotomic Iwasawa algebra over $\mathcal{O}_\psi$. Let $\psi \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})$, let ϵ be a quadratic Dirichlet character of conductor C_ϵ coprime with Np, and let S be a finite set of rational primes. We say that an Iwasawa function $\mathcal{L}_\epsilon^S(f_\psi) \in \Lambda_\psi^{\mathrm{cy}}$ is an S-primitive *(cyclotomic) p-adic L-function of* $f_\psi \otimes \epsilon$ if it satisfies the following interpolation property. For every finite order character $\chi \in G_\infty \to \overline{\mathbf{Q}}_p^*$ of conductor p^{c_χ} and every integer $1 \le j \le k_\psi - 1$:

$$\chi_{\mathrm{cy}}^{j-1}\chi\Big(\mathcal{L}_\epsilon^S(f_\psi)\Big) = \psi(\mathbf{a}_p)^{-c_\chi} \cdot \left(1 - \frac{\omega^{1-j}\epsilon\chi(p)\cdot p^{j-1}}{\psi(\mathbf{a}_p)}\right) \times \frac{(p^{c_\chi}C_\epsilon)^j\,(j-1)!\cdot L^{S\setminus\{p\}}(f_\psi, \omega^{j-1}\chi^{-1}\epsilon, j)}{(-2\pi i)^{j-1}G\left(\omega^{j-1}\chi^{-1}\epsilon\right)\cdot\Omega_{f_\psi}^{\mathrm{sgn}(\epsilon)\cdot(-1)^{j-1}}} \in \mathcal{O}_\psi, \tag{12}$$

where the notations are as follows. $L(f_\psi, \mu, s) = L^{\emptyset}(f_\psi, \mu, s)$ denotes the analytic continuation of the complex Hecke L-series $L(f_\psi, \mu, s) := \sum_{n=1}^{\infty}\mu(n)\frac{\psi(\mathbf{a}_n)}{n^s} = \prod_\ell E_\ell(f_\psi \otimes \mu, \ell^{-s})^{-1}$ of f_ψ twisted by μ; for every finite set Σ of rational primes, $L^{\Sigma}(f_\psi, \mu, s) := \prod_{\ell\in\Sigma} E_\ell(f_\psi \otimes \mu, \ell^{-s})\cdot L(f_\psi, \mu, s)$. $G(\mu)$ denotes the Gauss sum of the character μ. Finally, $\Omega_{f_\psi}^{\pm}$ are canonical periods of f_ψ, as defined, e.g. in [31]. We recall that $\Omega_{f_\psi}^{\pm}$ is an element of $\mathbf{C}^\times$, defined only up to multiplication by a p-adic unit in $\mathcal{O}_\psi$, and such that the quotient appearing in the second line of the equation above lies in the number field $\mathbf{Q}(\psi(\mathbf{a}_n) : n \in \mathbf{N})$ generated by the Fourier coefficients of f_ψ. Together with the Weierstraß preparation theorem, this implies that $\mathcal{L}_\epsilon^S(f_\psi)$, if it exists, is unique up to multiplication a unit in $\mathcal{O}_\psi^\times$. For a proof of the existence, see [19, Chapter I].

3.2. Skinner–Urban three variable p-adic L-functions. Let $K/\mathbf{Q}$ be a quadratic *imaginary* field of (absolute) discriminant D_K, let $q_K \nmid 6p$ be a rational prime which splits in K, and let S be a finite set of finite primes of K. We assume that the following hypothesis is satisfied.

Hypothesis 2. *The data (K, p, L, q_K, S) satisfy the following assumptions:*

- *D_K is coprime with $6Np$.*
- *p splits in K.*
- *$L/\mathbf{Q}_p$ contains the finite extension $\mathbf{Q}_p\left(D_K^{1/2}, (-1)^{1/2}, 1^{1/Np}\right)/\mathbf{Q}_p$.*
- *S consists of all the primes of K which divide $q_K D_K Np$.*

Let $\mathcal{K}/K$ be the $\mathbf{Z}_p^2$-extension of K. Then $\mathcal{K} = K_\infty \cdot K_\infty^-$, where K_∞ (resp., K_∞^-) is the cyclotomic (resp., anticyclotomic) $\mathbf{Z}_p$-extension of K. Denote by $G_\infty := \mathrm{Gal}(K_\infty/K) \cong \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ and $D_\infty := \mathrm{Gal}(K_\infty^-/K)$ the Galois groups of K_∞/K and K_∞^-/K respectively, so that $\mathrm{Gal}(\mathcal{K}/K) \cong G_\infty \times D_\infty$, and let $\mathbb{I}_\infty := \mathbb{I}[\![G_\infty]\!]$. Section 12 of [31] constructs an element

$$\mathcal{L}_K^S(\mathbf{f}) \in \mathbb{I}[\![G_\infty \times D_\infty]\!] = \mathbb{I}_\infty[\![D_\infty]\!],$$

satisfying the following property: given $\psi \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})$, write $\psi^{\mathrm{cy}} : \mathbb{I}[\![G_\infty \times D_\infty]\!] \to \Lambda_\psi^{\mathrm{cy}} = \psi(\mathbb{I})[\![G_\infty]\!]$ for the morphism of $\mathcal{O}_L[\![G_\infty]\!]$-algebras whose restriction to $\mathbb{I}$ is ψ, and s.t. $\psi^{\mathrm{cy}}(D_\infty) = 1$. Moreover, fix canonical periods $\Omega_\psi^\pm := \Omega_{f_\psi}^\pm$ for f_ψ. Then, for every $\psi \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})$, there exists $\lambda_\psi \in \mathcal{O}_\psi^\times$ such that

$$\psi^{\mathrm{cy}}\left(\mathcal{L}_K^S(\mathbf{f})\right) = \lambda_\psi \cdot \mathcal{L}^S(f_\psi) \cdot \mathcal{L}_{\epsilon_K}^S(f_\psi), \tag{13}$$

where $\mathcal{L}^S(f_\psi) := \mathcal{L}_1^S(f_\psi)$ (resp., $\mathcal{L}_{\epsilon_K}^S(f_\psi)$) is an S-primitive cyclotomic p-adic L-function of f_ψ (resp., of $f_\psi \otimes \epsilon_K$), computed with respect to the periods $\Omega_\psi^\pm$. Here $\epsilon_K : (\mathbf{Z}/D_K\mathbf{Z})^\times \to \overline{\mathbf{Q}}_p^*$ is the primitive quadratic character attached to $K/\mathbf{Q}$, and we write for simplicity $\mathcal{L}_*^S(f_\psi) := \mathcal{L}_*^{S_o}(f_\psi)$, where $S_o := \{\ell \text{ prime} : \ell | q_K D_K Np\}$ is the set of rational primes lying below the primes in S. More precisely, such a p-adic L-function $\mathcal{L}_K^S(\mathbf{f}) = \mathcal{L}_K^S(\mathbf{f}; 1_\mathbf{f})$ is attached to every generator $1_\mathbf{f}$ of the free rank-one $\mathbb{I}$-module $\mathbb{H}_\mathbf{f}$ (mentioned at the end of Section 2.1), and it is a well defined element of $\mathbb{I}_\infty[\![D_\infty]\!]$ only up to multiplication by a unit in $\mathbb{I}$. We refer to [31, Theorems 12.6 and 12.7 and Proposition 12.8] for the proofs of these facts, and for the interpolation property characterizing $\mathcal{L}_K^S(\mathbf{f})$.

Remark 3.1. Recall that Hypothesis 1 (denoted $(\mathbf{irred})_{\mathbf{f}}$ in [31]) is in order, i.e. that the residual representation $\overline{\rho}_{\mathbf{f}}$ is assumed to be (absolutely) irreducible. As explained in Remark 2.1, we also know that $\overline{\rho}_{\mathbf{f}}$ is p-distinguished, i.e. that condition $(\mathbf{dist})_{\mathbf{f}}$ in [31] is satisfied. These two hypotheses are used by Skinner and Urban in their construction of $\mathcal{L}_K^S(\mathbf{f})$ (cf. Section 3.4.5 and Theorems 12.6 and 12.7 of [31]).

3.3. Greenberg Selmer groups. Let $F/\mathbf{Q}$ be a number field, and let $\mathcal{F}/F$ be a $\mathbf{Z}_p$-power extension of F, i.e. $\mathrm{Gal}(\mathcal{F}/F) \cong \mathbf{Z}_p^r$ for some $r \geq 0$. Write $\mathbb{I}_{\mathcal{F}} := \mathbb{I}[\![\mathrm{Gal}(\mathcal{F}/K)]\!]$ and

$$T_{\mathbf{f}}(\mathcal{F}) := T_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{F}}(\varepsilon_{\mathcal{F}}^{-1}) \in {}_{\mathbb{I}_{\mathcal{F}}[G_F]}\mathrm{Mod},$$

where $\varepsilon_{\mathcal{F}} : G_F \twoheadrightarrow \mathrm{Gal}(\mathcal{F}/F) \subset \mathbb{I}_{\mathcal{F}}^{\times}$ is the tautological representation. Let v be a prime of F dividing p, associated with an embedding $i_v : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}_p}$, and let $i_v^* : G_{F_v} \hookrightarrow G_F$ denote the corresponding decomposition group at v. Define

$$T_{\mathbf{f}}(\mathcal{F})_v^{\pm} := T_{\mathbf{f},v}^{\pm} \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{F}}(\varepsilon_{\mathcal{F},v}^{-1}) \in {}_{\mathbb{I}_{\mathcal{F}}[G_{F_v}]}\mathrm{Mod},$$

where $\varepsilon_{\mathcal{F},v} := \varepsilon_{\mathcal{F}} \circ i_v^* : G_{F_v} \to \mathbb{I}_{\mathcal{F}}^{\times}$. The exact sequence (11) then induces a short exact sequence of $\mathbb{I}_{\mathcal{F}}[G_{F_v}]$-modules

$$0 \to T_{\mathbf{f}}(\mathcal{F})_v^{+} \xrightarrow{i_v^+} T_{\mathbf{f}}(\mathcal{F}) \xrightarrow{p_v^-} T_{\mathbf{f}}(\mathcal{F})_v^{-} \to 0. \tag{14}$$

Let S be a finite set of primes of F, containing all the prime divisors of NpD_F (where $D_F := \mathrm{disc}(F/\mathbf{Q})$ is the discriminant of $F/\mathbf{Q}$), and let $G_{F,S} := \mathrm{Gal}(F_S/F)$ be the Galois group of the maximal algebraic extension F_S/F which is unramified at every finite prime $v \notin S$ of F. As $\mathcal{F}/F$ (being a $\mathbf{Z}_p$-power extension) is unramified outside p, $T_{\mathbf{f}}(\mathcal{F})$ is unramified at every finite prime $v \notin S$ of F, i.e. $T_{\mathbf{f}}(\mathcal{F})$ is a $\mathbb{I}_{\mathcal{F}}[G_{F,S}]$-module. Let $\mathfrak{a} \in \mathrm{Spec}(\mathbb{I}_{\mathcal{F}})$, and write $\mathbb{I}_{\mathcal{F}}^* := \mathrm{Hom}_{\mathrm{cont}}(\mathbb{I}_{\mathcal{F}}, \mathbf{Q}_p/\mathbf{Z}_p)$ for the Pontrjagin dual of $\mathbb{I}_{\mathcal{F}}$, so that $\mathbb{I}_{\mathcal{F}}^*[\mathfrak{a}]$ is the Pontrjagin dual of $\mathbb{I}_{\mathcal{F}}/\mathfrak{a}$. Define the (discrete) *non-strict Greenberg Selmer group*:

$$\begin{aligned} &\mathrm{Sel}_{\mathcal{F}}^S(\mathbf{f}, \mathfrak{a}) \\ &:= \ker\left(H^1\big(G_{F,S}, T_{\mathbf{f}}(\mathcal{F}) \otimes_{\mathbb{I}_{\mathcal{F}}} \mathbb{I}_{\mathcal{F}}^*[\mathfrak{a}]\big) \longrightarrow \prod_{v\mid p} H^1\big(I_v, T_{\mathbf{f}}(\mathcal{F})_v^{-} \otimes_{\mathbb{I}_{\mathcal{F}}} \mathbb{I}_{\mathcal{F}}^*[\mathfrak{a}]\big)\right) \end{aligned} \tag{15}$$

where $I_v = I_{F_v} \subset G_{F_v}$ is the inertia subgroup and the arrow is defined by $\prod_{v\mid p} p_{v*}^{-} \circ \mathrm{res}_v$, p_{v*}^{-} being the morphism induced in cohomology by

$$p_v^{-} : T_{\mathbf{f}}(\mathcal{F}) \twoheadrightarrow T_{\mathbf{f}}(\mathcal{F})_v^{-}.$$

It is a cofinitely generated $\mathbb{I}_{\mathcal{F}}/\mathfrak{a}$-module, i.e. its Pontrjagin dual

$$X_{\mathcal{F}}^{S}(\mathbf{f},\mathfrak{a}) := \mathrm{Hom}_{\mathbb{I}_{\mathcal{F}}}\left(\mathrm{Sel}_{\mathcal{F}}^{S}(\mathbf{f},\mathfrak{a}),\mathbb{I}_{\mathcal{F}}^{*}[\mathfrak{a}]\right) \cong \mathrm{Hom}_{\mathbf{Z}_p}\left(\mathrm{Sel}_{\mathcal{F}}^{S}(\mathbf{f},\mathfrak{a}),\mathbf{Q}_p/\mathbf{Z}_p\right)$$

is a finitely-generated $\mathbb{I}_{\mathcal{F}}/\mathfrak{a}$-module. If $\mathfrak{a}=0$, write more simply

$$\mathrm{Sel}_{\mathcal{F}}^{S}(\mathbf{f}) := \mathrm{Sel}_{\mathcal{F}}^{S}(\mathbf{f},0);\;\; X_{\mathcal{F}}^{S}(\mathbf{f}) := X_{\mathcal{F}}^{S}(\mathbf{f},0).$$

By construction there are natural morphisms of $\mathbb{I}_{\mathcal{F}}/\mathfrak{a}$-modules

$$\mathrm{Sel}_{\mathcal{F}}^{S}(\mathbf{f},\mathfrak{a}) \to \mathrm{Sel}_{\mathcal{F}}^{S}(\mathbf{f})[\mathfrak{a}];\;\; X_{\mathcal{F}}^{S}(\mathbf{f})\otimes_{\mathbb{I}_{\mathcal{F}}}\mathbb{I}_{\mathcal{F}}/\mathfrak{a} \to X_{\mathcal{F}}^{S}(\mathbf{f},\mathfrak{a}). \tag{16}$$

Since $\mathbb{I}$ is a normal domain, so is $\mathbb{I}_{\mathcal{F}} \cong \mathbb{I}[\![X_1,\dots,X_r]\!]$ (with $\mathrm{Gal}(\mathcal{F}/F)\cong\mathbf{Z}_p^r$). Write $\mathrm{Ch}_{\mathcal{F}}^{S}(\mathbf{f})\subset\mathbb{I}_{\mathcal{F}}$ for the characteristic ideal of the $\mathbb{I}_{\mathcal{F}}$-module $X_{\mathcal{F}}^{S}(\mathbf{f})$ (cf. Section 3 of [31]):

$$\mathrm{Ch}_{\mathcal{F}}^{S}(\mathbf{f}) := \Big\{x\in\mathbb{I}_{\mathcal{F}} : \mathrm{ord}_{\mathfrak{a}}(x)\geq \mathrm{length}_{\mathfrak{a}}\left(X_{\mathcal{F}}^{S}(\mathbf{f})\right), \\ \text{for every } \mathfrak{a}\in\mathrm{Spec}(\mathbb{I}_{\mathcal{F}}) \text{ s.t. height}(\mathfrak{a})=1\Big\}.$$

Here $\mathrm{ord}_{\mathfrak{a}}:\mathrm{Frac}(\mathbb{I}_{\mathcal{F}})\to\mathbf{Q}\cup\{\infty\}$ is the (normalised) discrete valuation attached to the height-one prime $\mathfrak{a}$, and $\mathrm{length}_{\mathfrak{a}}:({}_{\mathbb{I}_{\mathcal{F}}}\mathrm{Mod})_{\mathrm{ft}}\to\mathbf{Z}\cup\{\infty\}$ is defined by sending a finite $\mathbb{I}_{\mathcal{F}}$-module M to the length over $(\mathbb{I}_{\mathcal{F}})_{\mathfrak{a}}$ of the localization $M_{\mathfrak{a}}$ of M at $\mathfrak{a}$.

Remark 3.2. Assume that $\mathcal{F}/F$ contains the cyclotomic $\mathbf{Z}_p$-extension $F_\infty\subset F(\mu_{p^\infty})$ of F. Thanks to the work of Kato [15], we know that $X_{\mathcal{F}}^{S}(\mathbf{f})$ is a *torsion* $\mathbb{I}_{\mathcal{F}}$-module (see also Section 3 of [31]), so that $\mathrm{Ch}_{\mathcal{F}}^{S}(\mathbf{f})$ is a *non-zero* divisorial ideal (which is principal if $\mathbb{I}$ is a unique factorization domain).

3.4. The main result of [31]. Let (K,p,L,q_K,S) be as in Section 3.2, and assume (as in *loc. cit.*) that this data satisfies Hypothesis 2. In particular, $K/\mathbf{Q}$ is an *imaginary* quadratic field in which *p splits*. Let $\mathcal{K}=K_\infty\cdot K_\infty^-$ be the $\mathbf{Z}_p^2$-extension of K, and let $\mathcal{L}_K^S(\mathbf{f})\in\mathbb{I}_{\mathcal{K}}=\mathbb{I}[\![\mathrm{Gal}(\mathcal{K}/K)]\!]$ be Skinner–Urban's three variable p-adic L-function. Together with Hypotheses 1 and 2, we have to consider:

Hypothesis 3 (ram). *Decompose $N=N^+N^-$, where $N^+=N_K^+$ (resp., $N^-=N_K^-$) is divided precisely by the prime divisors of $N=N_A/p$ which are split (resp., inert) in K. Then:*

- *N^- is square-free, and has an* odd *number of prime divisors.*
- *The residual representation $\overline{\rho}_{\mathbf{f}}$ is ramified at every prime $\ell\|N^-$.*

The following fundamental and deep result is Theorem 3.26 of [31].

Theorem 3.3 (Skinner–Urban [31]). *Assume that Hypotheses* 1, 2 *and* 3 *hold. Then*

$$\mathrm{Ch}_{\mathcal{K}}^{S}(\mathbf{f})\subseteq\left(\mathcal{L}_K^S(\mathbf{f})\right).$$

4. Restricting to the central critical line

The aim of this section is to specialise Skinner–Urban's result to the *(cyclotomic) central critical line* in the *weight-cyclotomic* space. More precisely, we use Theorem 3.3 to compare the order of vanishing of a certain *central-critical p-adic L-function* of the weight variable with the structure of a certain *central-critical Selmer group* attached to Hida's half-twisted representation $\mathbb{T}_{\mathbf{f}}$.

In this section, the notations and hypotheses of Section 3.4 are in force. In particular, we assume that Hypotheses 1, 2 and 3 are satisfied.

4.1. The (localised) Hida family. Let $\phi_f \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})$ be the arithmetic point of weight 2 and trivial character introduced in Section 2.1, with associated p-stabilised weight-two newform $f \in S_2(\Gamma_0(Np), \mathbf{Z})^{\mathrm{new}}$. Write $\mathfrak{p}_f := \ker(\phi_f) \in \mathrm{Spec}(\mathbb{I})$. By [14, Corollary 1.4], the localisation $\mathbb{I}_{\mathfrak{p}_f}$ is a discrete valuation ring, unramified over the localisation of $\Lambda = \mathcal{O}_L[\![\Gamma]\!]$ at the prime $\widetilde{\mathfrak{p}} = \mathfrak{p}_f \cap \Lambda$. Fix a topological generator $\gamma_{\mathrm{wt}} \in \Gamma = 1 + p\mathbf{Z}_p$, and write $\varpi_{\mathrm{wt}} := \gamma_{\mathrm{wt}} - 1$. Then ϖ_{wt} is a generator of the prime $\widetilde{\mathfrak{p}}$, so that

$$\mathfrak{p}_f \cdot \mathbb{I}_{\mathfrak{p}_f} = \varpi_{\mathrm{wt}} \cdot \mathbb{I}_{\mathfrak{p}_f}, \tag{17}$$

i.e. $\varpi_{\mathrm{wt}} \in \Lambda$ is a uniformiser of the discrete valuation ring $\mathbb{I}_{\mathfrak{p}_f}$.

Let $W \subset \mathbf{Z}_p$ be a non-empty open neighbourhood of 2. Denote by $\mathscr{A}(W) \subset \overline{\mathbf{Q}}_p[\![k-2]\!]$ the subring of formal power series in $k-2$ which converge for every $k \in W$. As explained in [12] (see also [21]), there exist an open neighbourhood $U = U_f \subset \mathbf{Z}_p$ of 2, and a natural morphism (the *Mellin transform centred at* ϕ_f)

$$\mathtt{M} : \mathbb{I} \longrightarrow \mathscr{A}(U),$$

characterised by the following properties: for every $x \in \mathbb{I}$ write $\mathtt{M}_x(k) := \mathtt{M}(x)(k) \in \mathscr{A}(U)$. Then: (i) for every $x \in \mathbb{I}$, $\mathtt{M}_x(2) = \phi_f(x)$ and (ii) for every $\gamma \in \Gamma \subset \mathbb{I}^\times$, $\mathtt{M}_{[\gamma]}(k) = \gamma^{k-2} := \exp_p\big((k-2) \cdot \log_p(\gamma)\big) \in \mathscr{A}(\mathbf{Z}_p)$ ($[\cdot] : \Lambda \to \mathbb{I}$ being the structural morphism). For every positive integer n, write $a_n(k) := \mathtt{M}(\mathbf{a}_n) \in \mathscr{A}(U)$ for the image of the n-th Hecke operator $\mathbf{a}_n \in \mathbb{I}$ under $\mathtt{M}$, and consider the formal q-expansion with coefficients in $\mathscr{A}(U)$:

$$f_\infty := \sum_{n=1}^{\infty} a_n(k) q^n \in \mathscr{A}(U)[\![q]\!].$$

This is the 'portion' of the Hida family $\mathbf{f}$ we are mostly interested in. More precisely, let

$$U^{\mathrm{cl}} := \big\{k \in U \cap \mathbf{Z} : k \geq 2; k \equiv 2 \pmod{2(p-1)}\big\}$$

be the subset of *classical points*, which is a dense subset of U. For every classical point $\kappa \in U^{\mathrm{cl}}$, the composition $\phi_\kappa : \mathbb{I} \xrightarrow{\mathtt{M}} \mathscr{A}(U) \xrightarrow{\mathrm{ev}_\kappa} \overline{\mathbf{Q}}_p$ (where ev_κ is evaluation at κ) is an arithmetic point of weight κ and trivial character, and the *weight-κ specialisation*

$f_\kappa := f_{\phi_\kappa} = \sum_{n=1}^{\infty} a_n(\kappa)q^n \in S_\kappa(\Gamma_0(Np))$ is a p-ordinary normalised eigenform of weight κ and level $\Gamma_0(Np)$. By construction: $f = f_2$. Moreover, N divides the conductor of f_κ for every $\kappa \in U^{\mathrm{cl}}$ (and f_κ is old at p for $\kappa > 2$, i.e. f_κ is the p-stabilisation of a newform of level $\Gamma_0(N)$ when $\kappa > 2$ [14]).

4.2. The central critical p-adic L-function. Let

$$\mathscr{A}(U \times \mathbf{Z}_p \times \mathbf{Z}_p) \subset \overline{\mathbf{Q}}_p[\![k-2, s-1, r-1]\!]$$

be the subring of formal power series converging for every $(k, s, r) \in U \times \mathbf{Z}_p \times \mathbf{Z}_p$. Let $\chi_{\mathrm{cy}} : G_\infty \cong 1 + p\mathbf{Z}_p$ be the p-adic cyclotomic character, and fix an isomorphism $\chi_{\mathrm{acy}} : D_\infty \cong 1 + p\mathbf{Z}_p$. We can uniquely extend the Mellin transform $\mathtt{M}$ to a morphism of rings

$$\widetilde{\mathtt{M}} : \mathbb{I}[\![G_\infty \times D_\infty]\!] \longrightarrow \mathscr{A}(U \times \mathbf{Z}_p \times \mathbf{Z}_p),$$

by mapping every $\sigma \in D_\infty$ (resp., $\sigma \in G_\infty$) to the analytic function on $\mathbf{Z}_p$ represented by the power series $\widetilde{\mathtt{M}}(\sigma) := \chi_{\mathrm{acy}}(\sigma)^{r-1} = \exp_p\left((r-1) \cdot \log_p\left(\chi_{\mathrm{acy}}(\sigma)\right)\right)$ (resp., $\widetilde{\mathtt{M}}(\sigma) := \chi_{\mathrm{cy}}(\sigma)^{s-1}$). We then define the *$S$-primitive analytic three-variable p-adic L-function of f_∞/K*:

$$L_p^S(f_\infty/K, k, s, r) := \widetilde{\mathtt{M}}\big(\mathcal{L}_K^S(\mathbf{f})\big) \in \mathscr{A}\left(U \times \mathbf{Z}_p \times \mathbf{Z}_p\right).$$

In the rest of this note, the *(cyclotomic) central critical line*

$$\ell^{\mathrm{cc}} := \{(k, s, r) \in U \times \mathbf{Z}_p \times \mathbf{Z}_p : r = 1;\ s = k/2\}$$

will play a key role. Let $\mathfrak{l}$ be a prime of K contained in S, which does not divide p. Let $\ell \neq p$ be the rational prime lying below it: $\mathfrak{l} \cap \mathbf{Z} = \ell\mathbf{Z}$. Define the *central critical ℓ-Euler factor of f_∞/K* as

$$E_\ell(f_\infty/K, k) := \left(1 - \frac{a_\ell(k)}{\langle\ell\rangle^{k/2}\,\omega(\ell)} + \frac{\mathbf{1}_N(\ell)}{\ell}\right) \cdot \left(1 - \frac{\epsilon_K(\ell)a_\ell(k)}{\langle\ell\rangle^{k/2}\,\omega(\ell)} + \frac{\mathbf{1}_{ND_K}(\ell)}{\ell}\right) \in \mathscr{A}(\mathbf{Z}_p),$$

where $\langle\ell\rangle := \omega(\ell)^{-1}\ell \in 1 + p\mathbf{Z}_p$ is the projection of ℓ to principal units and $\mathbf{1}_M$ denotes the trivial Dirichlet character modulo M, for every $M \in \mathbf{N}$. Then

$$E_\ell(f_\infty/K, \kappa) = E_\ell(f_\kappa, \ell^{-\kappa/2}) \cdot E_\ell(f_\kappa \otimes \epsilon_K, \ell^{-\kappa/2})$$

for every classical point $\kappa \in U^{\mathrm{cl}}$, where $E_\ell(*, X)$ is the ℓ-th Euler factor of the eigenform $*$, so that the Hecke L-series of $*$ is given by the product $L(*, s) =$

$\prod_{q \text{ prime}} E_q(*, q^{-s})^{-1}$ (cf. Section 3.1). Define the *central critical S-Euler factors* of f_∞/K by

$$E_S(f_\infty/K, k) := \prod_{\ell | q_K N D_K} E_\ell(f_\infty/K, k),$$

where the product runs over the rational primes lying below a prime $\mathfrak{l} \nmid p$ of S (cf. Hypothesis 2). One has $E_\ell(f_\infty/K, 2) \neq 0$ for every $\ell | N D_K q_K$, so that, up to shrinking the p-adic disc U if necessary, one can assume that $E_S(f_\infty/K, k) \in \mathscr{A}(U)^\times$. Define finally the *central critical p-adic L-function* of f_∞/K:

$$L_p^{\mathrm{cc}}(f_\infty/K, k) := E_S(f_\infty/K, k)^{-1} \cdot L_p^S(f_\infty/K, k, k/2, 1) \in \mathscr{A}(U). \tag{18}$$

Note that, while the definition of $L_p^S(f_\infty/K, k, s, r)$ depends on the choice of the isomorphism $\chi_{\mathrm{acy}} : D_\infty \cong 1 + p\mathbf{Z}_p$, the analytic function $L_p^{\mathrm{cc}}(f_\infty/K, k)$ is independent of this choice.

4.3. The central critical Selmer group: a control theorem. Fix topological generators $\gamma_+ \in G_\infty$, $\gamma_- \in D_\infty$ and $\gamma_{\mathrm{wt}} \in \Gamma$, and write $\varpi_? := \gamma_? - 1$. We can (and will) assume that $\chi_{\mathrm{cy}}(\gamma_+) = \gamma_{\mathrm{wt}}$, where we write again $\chi_{\mathrm{cy}} : G_\infty \cong 1 + p\mathbf{Z}_p = \Gamma \subset \mathbb{I}^\times$ for the isomorphism induced by the p-adic cyclotomic character. Let

$$\Theta_K^+ : \mathrm{Gal}(\mathcal{K}/K) = G_\infty \times D_\infty \twoheadrightarrow G_\infty \overset{\chi_{\mathrm{cy}}}{\cong} \Gamma \xrightarrow{\sqrt{\cdot}} \Gamma \xrightarrow{[\cdot]} \mathbb{I}^\times$$

be the *cyclotomic central critical Greenberg character*. We can extend uniquely Θ_K^+ to a morphism of $\mathbb{I}$-algebras, denoted again by the same symbol, $\Theta_K^+ : \mathbb{I}_{\mathcal{K}} \to \mathbb{I}$. As easily seen, its kernel $\mathfrak{P}^{\mathrm{cc}}$ is given by

$$\mathfrak{P}^{\mathrm{cc}} := \ker\left(\Theta_K^+ : \mathbb{I}_{\mathcal{K}} \twoheadrightarrow \mathbb{I}\right) = (\varpi_{\mathrm{cc}}, \varpi_-) \cdot \mathbb{I}_{\mathcal{K}}; \quad \varpi_{\mathrm{cc}} := [\gamma_{\mathrm{wt}}] - \gamma_+^2 \in \mathbb{I}_{\mathcal{K}},$$

i.e. $\mathfrak{P}^{\mathrm{cc}}$ is generated by ϖ_- and ϖ_{cc}. In analogy with the definitions above, we define the *(cyclotomic) S-primitive central critical (non-strict) Greenberg Selmer group of* $\mathbf{f}/K$ by

$$\mathrm{Sel}^{S,\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K) := \ker\left(H^1(G_{K,S}, \mathbb{T}_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}^*) \longrightarrow \prod_{v|p} H^1(I_v, \mathbb{T}^-_{\mathbf{f},v} \otimes_{\mathbb{I}} \mathbb{I}^*)\right).$$

Here $\mathbb{T}_{\mathbf{f}} = (\mathbb{T}_{\mathbf{f}}, \mathbb{T}_{\mathbf{f}}^+)$ is Hida's half-twisted representation defined in Section (2.2) and S is as in Section 3.2. Moreover, the arrow refers again to $\prod_{v|p} p^-_{v*} \circ \mathrm{res}_v$, where $p_v^- : \mathbb{T}_{\mathbf{f}} \twoheadrightarrow \mathbb{T}^-_{\mathbf{f},v}$ is the projection introduced in equation (11) [3]. Denote by

[3]We should keep in mind that the cyclotomic variable plays a non trivial role in the definition of Hida's half-twisted representation $\mathbb{T}_{\mathbf{f}}$. This explains the appearance of the subscript $\mathbf{Q}_\infty$ in the notation $\mathrm{Sel}^{S,\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K)$.

$X^{S,\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K)$ the Pontrjagin dual of $\mathrm{Sel}^{S,\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K)$:

$$X^{S,\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K) := \mathrm{Hom}_{\mathbf{Z}_p}\left(\mathrm{Sel}^{S,\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K), \mathbf{Q}_p/\mathbf{Z}_p\right).$$

With these notations, and the ones introduced in Section 3.3, we have the following perfect control theorem.

Proposition 4.1. *There exists a canonical isomorphism of $\mathbb{I}$-modules*

$$X^S_{\mathcal{K}}(\mathbf{f}) \otimes_{\mathbb{I}_{\mathcal{K}}} \mathbb{I}_{\mathcal{K}}/\mathfrak{P}^{\mathrm{cc}} \cong X^{S,\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K).$$

Proof. Let $\mathfrak{a}_1 = (\varpi_-) \in \mathrm{Spec}(\mathbb{I}_{\mathcal{K}})$ and $\mathfrak{a}_2 := (\varpi_{\mathrm{cc}}) \in \mathrm{Spec}(\mathbb{I}_{K_\infty})$. (We remind that $\mathcal{K} = K_\infty \cdot K^-_\infty$ is the $\mathbf{Z}_p^2$-extension of K and K_∞/K is the cyclotomic $\mathbf{Z}_p$-extension.) As $\mathbb{I}_{\mathcal{K}}/\mathfrak{a}_1 \cong \mathbb{I}_{K_\infty}$ and $T_{\mathbf{f}}(\mathcal{K})/\mathfrak{a}_1 \cong T_{\mathbf{f}}(K_\infty)$:

$$T_{\mathbf{f}}(\mathcal{K}) \otimes_{\mathbb{I}_{\mathcal{K}}} \mathbb{I}^*_{\mathcal{K}}[\mathfrak{a}_1] \cong T_{\mathbf{f}}(\mathcal{K})/\mathfrak{a}_1 \otimes_{\mathbb{I}_{\mathcal{K}}/\mathfrak{a}_1} \mathbb{I}^*_{K_\infty} \cong T_{\mathbf{f}}(K_\infty) \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}^*_{K_\infty},$$

and similarly $T_{\mathbf{f}}(\mathcal{K})^-_v \otimes_{\mathbb{I}_{\mathcal{K}}} \mathbb{I}^*_{\mathcal{K}}[\mathfrak{a}_1] \cong T_{\mathbf{f}}(K_\infty)^-_v \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}^*_{K_\infty}$ for every $v|p$. In particular $\mathrm{Sel}^S_{\mathcal{K}}(\mathbf{f}, \mathfrak{a}_1)$ is canonically isomorphic to $\mathrm{Sel}^S_{K_\infty}(\mathbf{f})$. Moreover, by [31, Proposition 3.9], the maps (16) induce isomorphisms

$$\mathrm{Sel}^S_{K_\infty}(\mathbf{f}) \cong \mathrm{Sel}^S_{\mathcal{K}}(\mathbf{f})[\mathfrak{a}_1]; \quad X^S_{\mathcal{K}}(\mathbf{f}) \otimes_{\mathbb{I}_{\mathcal{K}}} \mathbb{I}_{\mathcal{K}}/\mathfrak{a}_1 \cong X^S_{K_\infty}(\mathbf{f}). \tag{19}$$

Similarly, Θ^+_K induces an isomorphism: $\mathbb{I}_{K_\infty}/\mathfrak{a}_2 \cong \mathbb{I}$, an isomorphism of $\mathbb{I}[G_{K,S}]$-modules: $T_{\mathbf{f}}(K_\infty)/\mathfrak{a}_2 \cong \mathbb{T}_{\mathbf{f}}$ and isomorphisms of $\mathbb{I}[G_{K_v}]$-modules: $T_{\mathbf{f}}(K_\infty)^\pm_v/\mathfrak{a}_2 \cong \mathbb{T}^\pm_{\mathbf{f},v}$ for every $v|p$. (Indeed, write $\Theta_K : \mathbb{I}_\infty = \mathbb{I}_{K_\infty} \twoheadrightarrow \mathbb{I}$ for the 'restriction' of Θ^+_K to $\mathbb{I}_\infty$. Then $\Theta_K \circ \varepsilon^{-1}_{K_\infty} = [\chi_{\mathrm{cy}}]^{-1/2}$ on $G_{K,S}$, so that

$$\begin{aligned} T_{\mathbf{f}}(K_\infty)/\mathfrak{a}_2 \cong T_{\mathbf{f}}(K_\infty) \otimes_{\mathbb{I}_\infty, \Theta_K} \mathbb{I} &= T_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_\infty\big(\varepsilon^{-1}_{K_\infty}\big) \otimes_{\mathbb{I}_\infty, \Theta_K} \mathbb{I} \\ &\cong T_{\mathbf{f}} \otimes_{\mathbb{I}} [\chi_{\mathrm{cy}}]^{-1/2} = \mathbb{T}_{\mathbf{f}}. \end{aligned}$$

The same argument justifies the statement for the $\pm$-parts at a prime $v|p$.) As above (i.e. retracing the definitions), this gives a canonical isomorphism of Selmer groups

$$\mathrm{Sel}^{S,\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K) \cong \mathrm{Sel}^S_{K_\infty}(\mathbf{f}, \mathfrak{a}_2). \tag{20}$$

Let us consider the following commutative diagram with (tautological) exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Sel}^S_{K_\infty}(\mathbf{f}, \mathfrak{a}_2) \arrow[r] \arrow[d, "\alpha"] & H^1(G_{K,S}, T_{\mathbf{f}}(K_\infty) \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}^*_{K_\infty}[\mathfrak{a}_2]) \arrow[r] \arrow[d, "\beta"] & \prod_{v|p} H^1(I_v, T_{\mathbf{f}}(K_\infty)^-_v \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}^*_{K_\infty}[\mathfrak{a}_2]) \arrow[d, "\gamma"] \\ 0 \arrow[r] & \left(\mathrm{Sel}^S_{K_\infty}(\mathbf{f})\right)[\mathfrak{a}_2] \arrow[r] & \left(H^1(G_{K,S}, T_{\mathbf{f}}(K_\infty) \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}^*_{K_\infty})\right)[\mathfrak{a}_2] \arrow[r] & \left(\prod_{v|p} H^1(I_v, T_{\mathbf{f}}(K_\infty)^-_v \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}^*_{K_\infty})\right)[\mathfrak{a}_2], \end{tikzcd}$$

where the vertical maps are the natural ones induced by the inclusion $\mathbb{I}^*_{K_\infty}[\mathfrak{a}_2] \subset \mathbb{I}^*_{K_\infty}$ (cf. (16)). We claim that α is an isomorphism of $\mathbb{I}$-modules:

$$\alpha : \mathrm{Sel}^S_{K_\infty}(\mathbf{f}, \mathfrak{a}_2) \cong \mathrm{Sel}^S_{K_\infty}(\mathbf{f})[\mathfrak{a}_2]. \tag{21}$$

The map β sits into a short exact sequence (arising from $0 \to \mathbb{I}^*_{K_\infty}[\mathfrak{a}_2] \to \mathbb{I}^*_{K_\infty} \overset{\varpi_{\mathrm{cc}}}{\to} \mathbb{I}^*_{K_\infty} \to 0$):

$$0 \to H^0(G_{K,S}, T_{\mathbf{f}}(K_\infty) \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}^*_{K_\infty})/\varpi_{\mathrm{cc}} \to H^1(G_{K,S}, T_{\mathbf{f}}(K_\infty) \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}^*_{K_\infty}[\mathfrak{a}_2]) \xrightarrow{\beta} H^1(G_{K,S}, T_{\mathbf{f}}(K_\infty) \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}^*_{K_\infty})[\mathfrak{a}_2] \to 0.$$

Hypotheses 1 and 2 imply that the restriction of $\overline{\rho}_{\mathbf{f}}$ to G_K is irreducible. Then the first H^0 vanishes, and β is an isomorphism. By the Snake Lemma, the morphism α is injective, and its cockerel is a sub-module of $\ker(\gamma)$. To prove the claim (21) it is then sufficient to show that

$$\ker(\gamma) = 0. \tag{22}$$

Looking again at the exact I_v-cohomology sequence arising from $0 \to \mathbb{I}^*_{K_\infty}[\mathfrak{a}_2] \to \mathbb{I}^*_{K_\infty} \overset{\varpi_{\mathrm{cc}}}{\to} \mathbb{I}^*_{K_\infty} \to 0$, we have

$$\ker(\gamma) \cong \prod_{v|p} H^0(I_v, T_{\mathbf{f}}(K_\infty)^-_v \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}^*_{K_\infty}) \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}_{K_\infty}/\varpi_{\mathrm{cc}}. \tag{23}$$

Note that $T_{\mathbf{f}}(K_\infty)^-_v \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}^*_{K_\infty} \cong \mathbb{I}^*_{K_\infty}\big(\mathbf{a}^*_p \cdot \varepsilon^{-1}_{K_\infty}\big)$ (cf. Section 2.2). Since $\mathbb{I}_{K_\infty}/(\gamma_+ - 1)\mathbb{I}_{K_\infty} \cong \mathbb{I}$, one finds

$$H^0(I_v, T_{\mathbf{f}}(K_\infty)^-_v \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}^*_{K_\infty}) = \mathbb{I}^*_{K_\infty}(\mathbf{a}^*_p)[\gamma_+ - 1] = \mathbb{I}^*(\mathbf{a}^*_p)$$

(recall that $\mathbf{a}^*_p$ is the unramified character on $G_{\mathbf{Q}_p}$ sending an arithmetic Frobenius to $\mathbf{a}_p$). Finally, note that $\varpi_{\mathrm{cc}} := [\gamma_{\mathrm{wt}}] - \gamma_+^2$ acts as $\varpi_{\mathrm{wt}} = [\gamma_{\mathrm{wt}}] - 1$ on $\mathbb{I}^* = \mathbb{I}^*_{K_\infty}[\gamma_+ - 1]$, so that $\mathbb{I}^*$ is ϖ_{cc}-divisible, and hence

$$H^0(I_v, T_{\mathbf{f}}(K_\infty)^-_v \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}^*_{K_\infty}) \otimes_{\mathbb{I}_{K_\infty}} \mathbb{I}_{K_\infty}/\varpi_{\mathrm{cc}} = 0$$

for every prime $v|p$ of K. Together with (23), this implies that (22) holds true, and then proves the claim (21). When combined with the isomorphism (20), this gives canonical isomorphisms of $\mathbb{I}$-modules

$$\mathrm{Sel}^{S,\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K) \cong \mathrm{Sel}^S_{K_\infty}(\mathbf{f})[\mathfrak{a}_2]; \quad X^{S,\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K) \cong X^S_{K_\infty}(\mathbf{f})/\mathfrak{a}_2.$$

Since $\mathfrak{P}^{\mathrm{cc}} = (\mathfrak{a}_1, \mathfrak{a}_2) \cdot \mathbb{I}_{\mathcal{K}}$, combined with the second isomorphism in (19), this concludes the proof. □

4.4. Specialising Skinner–Urban to the central critical line. We can finally state the following corollary of the theorem of Skinner–Urban. For every $f(k) \in \mathscr{A}(U)$, write $\mathrm{ord}_{k=2} f(k) \in \mathbf{N} \cup \{\infty\}$ to denote the order of vanishing of $f(k)$ at $k = 2$. Given a finite $\mathbb{I}$-module M, write as usual $\mathrm{length}_{\mathfrak{p}_f}(M)$ for the length of the localisation $M_{\mathfrak{p}_f}$ over the discrete valuation ring $\mathbb{I}_{\mathfrak{p}_f}$.

Corollary 4.2. *Assume that Hypotheses* 1, 2 *and* 3 *are satisfied. Then*

$$\mathrm{ord}_{k=2} L_p^{\mathrm{cc}}(f_\infty/K,k) \leq \mathrm{length}_{\mathfrak{p}_f}\big(X_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}/K)\big).$$

Proof. Combining Skinner–Urban's Theorem 3.3 with Proposition 4.1, we easily deduce that the characteristic ideal of $X_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}/K)$ is contained in the principal ideal generated by the projection $\mathcal{L}_K^S(\mathbf{f}) \bmod \mathfrak{P}^{\mathrm{cc}}$ (cf. the proof of [31, Corollary 3.8]). In other words

$$\Big\{\text{Characteristic ideal of } X_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}/K)\Big\} \subset \Big(\mathcal{L}_K^S(\mathbf{f}) \bmod \mathfrak{P}^{\mathrm{cc}}\Big).$$

In particular, writing $\mathrm{ord}_{\mathfrak{p}_f} : \mathrm{Frac}(\mathbb{I}) \to \mathbf{Z} \cup \{\infty\}$ for the valuation attached to $\mathfrak{p}_f$,

$$\mathrm{ord}_{\mathfrak{p}_f}\big(\mathcal{L}_K^S(\mathbf{f}) \bmod \mathfrak{P}^{\mathrm{cc}}\big) \leq \mathrm{length}_{\mathfrak{p}_f}\Big(X_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}/K)\Big).$$

Write for simplicity $\mathcal{L}_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}/K) := \mathcal{L}_K^S(\mathbf{f}) \bmod \mathfrak{P}^{\mathrm{cc}}$. To conclude the proof it remains to verify that

$$\mathrm{ord}_{\mathfrak{p}_f}\mathcal{L}_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}/K) = \mathrm{ord}_{k=2} L_p^S(f_\infty/K,k,k/2,1). \tag{24}$$

Note that, by the definition of the Mellin transform $\widetilde{\mathtt{M}}$ (and the normalisation $\chi_{\mathrm{cy}}(\gamma_+) = \gamma_{\mathrm{wt}}$) we have

$$\begin{aligned}\widetilde{\mathtt{M}}(\varpi_{\mathrm{cc}})(k,s,r) = \gamma_{\mathrm{wt}}^{k-2} - \gamma_{\mathrm{wt}}^{2(s-1)} &= \gamma_{\mathrm{wt}}^{2(s-1)}\Big(\gamma_{\mathrm{wt}}^{2(k/2-s)} - 1\Big)\\ &\equiv 0 \bmod (s-k/2)\cdot \mathscr{A}(U \times \mathbf{Z}_p \times \mathbf{Z}_p),\end{aligned} \tag{25}$$

and then $\widetilde{\mathtt{M}}(\varpi_{\mathrm{cc}})(k,k/2,1) = 0$. Similarly, writing $\ell_{\mathrm{wt}} := \log_p(\gamma_{\mathrm{wt}})$ and $\ell_- := \log_p(\chi_{\mathrm{acy}}(\gamma_-))$, we have

$$\begin{aligned}\mathtt{M}(\varpi_{\mathrm{wt}})(k) &\equiv \ell_{\mathrm{wt}}\cdot(k-2) \bmod (k-2)^2;\\ \widetilde{\mathtt{M}}(\varpi_-)(k,s,r) &\equiv \ell_-\cdot(r-1) \bmod (r-1)^2.\end{aligned} \tag{26}$$

Assume now that $\mathcal{L}_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}/K) \in \mathfrak{p}_f^m\mathbb{I}_{\mathfrak{p}_f} - \mathfrak{p}_f^{m+1}\mathbb{I}_{\mathfrak{p}_f}$, for some integer $m \geq 0$, so that $\mathrm{ord}_{\mathfrak{p}_f}\mathcal{L}_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}/K) = m$. Since $\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f}$ is a principal ideal generated by ϖ_{wt} (17), equation (26) gives

$$\mathrm{ord}_{k=2}\mathtt{M}\Big(\mathcal{L}_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}/K)\Big)(k) = \mathrm{ord}_{\mathfrak{p}_f}\mathcal{L}_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}/K).$$

On the other hand, we have by construction $\mathcal{L}_K^S(\mathbf{f}) \equiv \mathcal{L}_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}) \bmod \mathfrak{P}^{\mathrm{cc}}$, so that equations (25) and (26) give

$$L_p^S(f_\infty/K,k,k/2,1) := \widetilde{\mathtt{M}}\big(\mathcal{L}_K^S(\mathbf{f})\big)(k,k/2,1) = \mathtt{M}\Big(\mathcal{L}_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}/K)\Big)(k).$$

Combining the preceding two equations, we deduce that (24) holds in this case. Assume finally that $\mathcal{L}_K^S(\mathbf{f}) \in \mathfrak{P}^{\mathrm{cc}}$, i.e. $\mathcal{L}_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}/K) = 0$. (This is the case '$m = \infty$'.) Then $L_p^S(f_\infty/K,k,k/2,1) \equiv 0$ by (25) and (26), so that (24) holds also in this case (giving $\infty = \infty$). □

5. Bertolini–Darmon's exceptional zero formula

Throughout this section, the notations and assumptions are as in Section 4. In particular, we assume that Hypotheses 1–3 are satisfied.

Let $\kappa \in U^{\mathrm{cl}}$ be a classical point in U, let $\phi_k \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})$ be the associated arithmetic point (of weight κ and trivial character), and let $f_\kappa \in S_\kappa(\Gamma_0(Np))$ be the corresponding p-stabilised newform (cf. Section 4.1). Write $\phi_\kappa^\dagger = \phi_\kappa \times \chi_{\mathrm{cy}}^{\kappa/2-1} \times 1 : \mathbb{I}[\![G_\infty \times D_\infty]\!] \to \overline{\mathbf{Q}}_p$ for the morphism of $\mathcal{O}_L$-algebras such that $\phi_\kappa^\dagger(\sigma \times h) = \chi_{\mathrm{cy}}(\sigma)^{\kappa/2-1}$ for every $\sigma \times h \in G_\infty \times D_\infty$, and such that $\phi_\kappa^\dagger(x) = \phi_\kappa(x)$ for every $x \in \mathbb{I}$. Since $\kappa \equiv 2 \bmod 2(p-1)$, $p \neq 2$, and p splits in K (i.e. $\epsilon_K(p) = 1$), equations (12) and (13) yield

$$\phi_\kappa^\dagger\left(\mathcal{L}_K^S(\mathbf{f})\right) = \lambda_\kappa D_K^{\frac{\kappa-2}{2}} \left(1 - \frac{p^{\frac{\kappa}{2}-1}}{a_p(\kappa)}\right)^2 \frac{(\kappa/2-1)! \cdot L^{S\setminus\{p\}}(f_\kappa, \kappa/2)}{(-2\pi i)^{\kappa/2-1}\Omega_{\phi_\kappa}^+} \cdot \frac{G(\epsilon_K)(\kappa/2-1)! \cdot L^{S\setminus\{p\}}(f_\kappa, \epsilon_K, \kappa/2)}{(-2\pi i)^{\kappa/2-1}\Omega_{\phi_\kappa}^-}$$

By the very definition of the central critical p-adic L-function $L_p^{\mathrm{cc}}(f_\infty/K, k)$ we then deduce: for every $\kappa \in U^{\mathrm{cl}}$

$$L_p^{\mathrm{cc}}(f_\infty/K, \kappa) = \lambda_\kappa D_K^{\frac{\kappa-2}{2}} \left(1 - \frac{p^{\frac{\kappa}{2}-1}}{a_p(\kappa)}\right)^2 \frac{(\kappa/2-1)! L(f_\kappa, \kappa/2)}{(-2\pi i)^{\kappa/2-1}\Omega_{\phi_\kappa}^+} \cdot \frac{G(\epsilon_K)(\kappa/2-1)! L(f_\kappa, \epsilon_K, \kappa/2)}{(-2\pi i)^{\kappa/2-1}\Omega_{\phi_\kappa}^-}.$$

Since U^{cl} is a dense subset of U, if we compare this formula with [5, Theorem 1.12], we obtain a factorisation

$$L_p^{\mathrm{cc}}(f_\infty/K, k) = D_K^{\frac{k-2}{2}} L_p(f_\infty, k, k/2) L_p(f_\infty, \epsilon_K, k, k/2). \tag{27}$$

Here, for every quadratic Dirichlet character χ of conductor coprime with Np, $L_p(f_\infty, \chi, k, s) \in \mathscr{A}(U \times \mathbf{Z}_p)$ is a Mazur–Kitagawa two-variable p-adic L-function attached to f_∞ and χ in [5, Section 1] (see also [12, 16, 25]), and we write simply $L_p(f_\infty, k, s) := L_p(f_\infty, \chi_{\mathrm{triv}}, k, s)$ when $\chi = \chi_{\mathrm{triv}}$ is the trivial character. Like $L_p^{\mathrm{cc}}(f_\infty/K, s)$ (once the periods $\Omega_{\phi_\kappa}^\pm$ are fixed for $\kappa \in U^{\mathrm{cl}}$), $L_p(f_\infty, \chi, k, s)$ is characterised by its interpolation property (namely [5, Theorem 1.12]) up to multiplication by a nowhere-vanishing analytic function on U, so the preceding equality has to be interpreted up to multiplication by such a unit in $\mathscr{A}(U)$.

The following *exceptional-zero* formula is the main result (Theorem 5.4) of [5], where it is proved under a technical assumption (namely the existence of a prime $q \| N$) subsequently removed by Mok in [18]. Write $\mathrm{sign}(A/\mathbf{Q}) \in \{\pm 1\}$ for the sign in the functional equation satisfied by the Hecke L-series $L(A/\mathbf{Q}, s) = L(f, s)$.

Theorem 5.1 (Bertolini–Darmon [5]). *Let χ be a quadratic Dirichlet character of conductor coprime with $N_A = Np$, such that*

$$\chi(-N) = -\mathrm{sign}(A/\mathbf{Q}); \quad \chi(p) = a_p(A) = +1.$$

If χ is non-trivial (resp., $\chi = 1$), let $K_\chi/\mathbf{Q}$ be the quadratic extension attached to χ (resp., let $K_\chi := \mathbf{Q}$). Then

1. *$L_p(f_\infty, \chi, k, k/2)$ vanishes to order at least 2 at $k = 2$.*
2. *There exists a global point $\mathbf{P}_\chi \in A(K_\chi)^\chi$ [4] such that*

$$\frac{d^2}{dk^2} L_p(f_\infty, \chi, k, k/2)_{k=2} \doteq \log_A^2(\mathbf{P}_\chi),$$

where $\log_A : A(\overline{\mathbf{Q}}_p) \to \overline{\mathbf{Q}}_p$ is the formal group logarithm[5], and $\doteq$ denotes equality up to multiplication by a non-zero (explicit) factor in $\mathbf{Q}_p^\times$.

3. *$\mathbf{P}_\chi$ has infinite order if and only if the Hecke L-series $L(f, \chi, s)$ has a simple zero at $s = 1$.*

In the preceding result, χ is allowed to be a generic Dirichelt character of conductor coprime with Np. Applying the theorem to both $\chi = \chi_{\mathrm{triv}}$ and $\chi = \epsilon_K$, we obtain the following corollary.

Corollary 5.2. *Assume that $\mathrm{sign}(A/\mathbf{Q}) = -1$, and that Hypotheses 1, 2 and 3 are satisfied. Denote by $L(A/K, s) := L(f, s) \cdot L(f, \epsilon_K, s)$ the complex Hasse–Weil L-function of A/K. Then $L_p^{\mathrm{cc}}(f_\infty/K, k)$ vanishes to order at least 4 at $k = 2$, and*

$$\mathrm{ord}_{k=2} L_p^{\mathrm{cc}}(f_\infty/K, k) = 4 \iff \mathrm{ord}_{s=1} L(A/K, s) = 2.$$

Proof. Since $\mathrm{sign}(A/\mathbf{Q}) = -1$, the hypotheses of the preceding theorem are satisfied by $\chi = \chi_{\mathrm{triv}}$. Moreover, since p splits in K by Hypothesis 2, $\epsilon_K(p) = +1$, and $\epsilon_K(-N) = -\epsilon(N^-) = +1$ by Hypothesis 3. Then $\chi = \epsilon_K$ also satisfies the hypotheses of the theorem. The corollary then follows by applying the theorem to both $\chi = \chi_{\mathrm{triv}}$ and $\chi = \epsilon_K$, and using the factorisation (27). □

[4] By $A(K_\chi)^\chi$ we mean the subgroup of $A(K_\chi)$ on which $\mathrm{Gal}(K_\chi/\mathbf{Q})$ acts via χ.

[5] Writing $\Phi_{\mathrm{Tate}} : \overline{\mathbf{Q}}_p^\times / q_A^{\mathbf{Z}} \cong A(\overline{\mathbf{Q}}_p)$ for the Tate p-adic uniformization of $A/\mathbf{Q}_p$ (see Section 6.3 below), one can define $\log_A := \log_{q_A} \circ \Phi_{\mathrm{Tate}}^{-1} : A(\overline{\mathbf{Q}}_p) \to \overline{\mathbf{Q}}_p$, where $\log_{q_A}$ is the branch of the p-adic logarithm vanishing at the Tate period $q_A \in p\mathbf{Z}_p$ of $A/\mathbf{Q}_p$.

6. Bounding the characteristic ideal via Nekovář's duality

Recall the arithmetic prime $\phi_f \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})$ defined in Section 4.1, and write as above $\mathfrak{p}_f := \ker(\phi_f)$, which is a height-one prime ideal of $\mathbb{I}$. Let χ be a quadratic Dirichlet character of conductor coprime with Np. If χ is non-trivial (resp., $\chi = 1$), let $K_\chi/\mathbf{Q}$ be the corresponding quadratic extension (resp., let $K_\chi := \mathbf{Q}$), and let D_χ be the discriminant of K_χ. Fix a finite set S of primes of K_χ containing all the prime divisors of NpD_χ, and decomposition groups $G_{K_{\chi,w}} := \mathrm{Gal}(\overline{\mathbf{Q}}_\ell/K_{\chi,w}) \hookrightarrow G_{K_\chi}$ at w, for every $w \in S$ dividing the rational prime ℓ (where $K_{\chi,w}$ denotes the completion of K_χ at w). Define the *strict Greenberg Selmer group* of $\mathbb{T}_{\mathbf{f}}/K_\chi$ (cf. Section 2.2):

$$\mathrm{Sel}^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_\chi) := \ker\left(H^1(G_{K_\chi,S}, \mathbb{T}_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}^*) \longrightarrow \prod_{v|p} H^1(K_{\chi,v}, \mathbb{T}^-_{\mathbf{f},v} \otimes_{\mathbb{I}} \mathbb{I}^*)\right),$$

where $G_{K_\chi,S}$ denotes as usual the Galois group of the maximal algebraic extension of K_χ which is unramified outside $S \cup \{\infty\}$. Let

$$X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_\chi) := \mathrm{Hom}_{\mathbf{Z}_p}\left(\mathrm{Sel}^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_\chi), \mathbf{Q}_p/\mathbf{Z}_p\right) [6].$$

For every $\mathbf{Z}[\mathrm{Gal}(K_\chi/\mathbf{Q})]$-module M, write M^χ for the submodule of M on which $\mathrm{Gal}(K_\chi/\mathbf{Q})$ acts via χ (so that $M^\chi := M$ is χ is trivial, and M^χ is the submodule of M on which the nontrivial automorphism of $\mathrm{Gal}(K_\chi/\mathbf{Q})$ acts as -1 if χ is nontrivial). The aim of this section is to prove the following theorem.

Theorem 6.1. *Let χ be a quadratic Dirichlet character of conductor coprime with Np. Assume that:*

(i) $\chi(p) = 1$, *i.e. p splits in K_χ;*

(ii) $\mathrm{rank}_{\mathbf{Z}} A(K_\chi)^\chi = 1$;

(iii) *the p-primary subgroup $\text{Ш}(A/K_\chi)^\chi_{p^\infty}$ of $\text{Ш}(A/K_\chi)^\chi$ is finite.*

Then the localisation at $\mathfrak{p}_f$ of $X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_\chi)^\chi$ is isomorphic to the residue field of the discrete valuation ring $\mathbb{I}_{\mathfrak{p}_f}$:

$$X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_\chi)^\chi \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f} \cong \mathbb{I}_{\mathfrak{p}_f}/\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f}.$$

6.1. Nekovář's theory. In this section we recall the needed results from Nekovář's theory of Selmer complexes [20]. Unless explicitly specified, all notations and conventions are as in *loc. cit.*

[6] The Selmer groups already defined depend in general on the choice of the set S. On the other hand, we are interested here only in the structure of the localisation of $X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_\chi)$ at $\mathfrak{p}_f$, and such a localisation does not depend, up to canonical isomorphism, on the choice of S.

6.1.1. Nekovář's Selmer complexes. Given a ring R, write $\mathrm{D}(R) := \mathrm{D}({}_R\mathrm{Mod})$ for the derived category of complexes of R-modules, and $\mathrm{D}^b_{\mathrm{ft}}(R) \subset \mathrm{D}(R)$ (resp., $\mathrm{D}^b_{\mathrm{cf}}(R) \subset \mathrm{D}(R)$) for the subcategory of cohomologically bounded complexes, with cohomology of finite (resp., cofinite) type over R.

Recall the self-dual, ordinary $\mathbb{I}$-adic representation $\mathbb{T}_{\mathbf{f}} = (\mathbb{T}_{\mathbf{f}}, \mathbb{T}^{\pm}_{\mathbf{f}})$, defined in Section 2.2. Denote by

$$\mathbb{A}_{\mathbf{f}} := \mathrm{Hom}_{\mathrm{cont}}(\mathbb{T}_{\mathbf{f}}, \mu_{p^\infty}); \;\; \mathbb{A}^{\pm}_{\mathbf{f}} := \mathrm{Hom}_{\mathrm{cont}}(\mathbb{T}^{\mp}_{\mathbf{f}}, \mu_{p^\infty})$$

the Kummer dual p-ordinary representation. Set $T_f := \mathbb{T}_{\mathbf{f}}/\mathfrak{p}_f\mathbb{T}_{\mathbf{f}}$ and $T^{\pm}_f := \mathbb{T}^{\pm}_{\mathbf{f}}/\mathfrak{p}_f\mathbb{T}^{\pm}_{\mathbf{f}}$. Then one has

$$A_f := \mathrm{Hom}_{\mathrm{cont}}(T_f, \mu_{p^\infty}) \cong \mathbb{A}_{\mathbf{f}}[\mathfrak{p}_f]; \;\; A^{\pm}_f := \mathrm{Hom}_{\mathrm{cont}}(T^{\mp}_f, \mu_{p^\infty}) \cong \mathbb{A}^{\pm}_{\mathbf{f}}[\mathfrak{p}_f].$$

Given a multiplicative subset $\mathscr{S}$ of a ring R, and an R-module M, write as usual $\mathscr{S}^{-1}M$ for the localisation of M at $\mathscr{S}$. Fix a multiplicative subset $\mathscr{S}$ of $\mathbb{I}$ or $\mathcal{O}_L$, let

$$X \in \{\mathscr{S}^{-1}\mathbb{T}_{\mathbf{f}}, \mathscr{S}^{-1}T_f, \mathbb{A}_{\mathbf{f}}, A_f\}$$

and let $R_X \in \{\mathscr{S}^{-1}\mathbb{I}, \mathscr{S}^{-1}\mathcal{O}_L, \mathbb{I}, \mathcal{O}_L\}$ be the corresponding 'coefficient ring'. For every prime $v|p$ of K_χ, set $X^+_v := \mathscr{S}^{-1}\mathbb{T}^+_{\mathbf{f}}$ (resp., $\mathscr{S}^{-1}T^+_f$, $\mathbb{A}^+_{\mathbf{f}}$, A^+_f) if $X = \mathscr{S}^{-1}\mathbb{T}_{\mathbf{f}}$ (resp., $\mathscr{S}^{-1}T_f$, $\mathbb{A}_{\mathbf{f}}$, A_f), and $X^-_v := X/X^+_v$. The exact sequence (11) then induces short exact sequences of $R_X[G_{K_\chi,v}]$-modules

$$0 \to X^+_v \overset{i^+_v}{\to} X \overset{p^-_v}{\to} X^-_v \to 0.$$

(Recall that $\mathbb{T}^+_{\mathbf{f},w} := \mathbb{T}^+_{\mathbf{f}}$ for every prime $w|p$ of $\overline{\mathbf{Q}}$, cf. equation (11).)

As in [20, Section 6], define local conditions $\Delta_S(X) = \{\Delta_v(X)\}_{v\in S}$ for X/K_χ as follows[7]. For a prime $v \in S$ dividing p, let $\Delta_v(X)$ be the morphism

$$i^+_v(X) : U^+_v(X) := C^\bullet_{\mathrm{cont}}(K_{\chi,v}, X^+_v) \longrightarrow C^\bullet_{\mathrm{cont}}(K_{\chi,v}, X),$$

i.e. $\Delta_v(X)$ is the Greenberg local condition attached to the $R_X[G_{K_{\chi,v}}]$-submodule $i^+_v : X^+_v \subset X$. For every $S \ni w \nmid p$, we define $\Delta_w(X)$ to be the *full* local condition: $i^+_w(X) : U^+_w(X) := 0 \to C^\bullet_{\mathrm{cont}}(K_{\chi,w}, X)$ (resp., the *empty* local

[7] Let R be a local complete Noetherian ring with finite residue field of characteristic p, and let T be an R-module of finite or cofinite type, equipped with a continuous, linear action of $G_{K_\chi,S}$. For every $w \in S$, fix a decomposition group G_w at w, i.e. $G_w := G_{K_\chi,w} \hookrightarrow G_{K_\chi} \twoheadrightarrow G_{K_\chi,S}$. According to Nekovář's theory of Selmer complexes, a *local condition* at $w \in S$ for T is the choice $\Delta_w(T)$ of a complex of R-modules $U^+_w(T)$, together with a morphism of complexes $i^+_w(T) : U^+_w(T) \to C^\bullet_{\mathrm{cont}}(K_{\chi,w}, T)$. For $G = G_{K_\chi,S}$ or G_w ($w \in S$), $C^\bullet_{\mathrm{cont}}(G, T)$ (also denoted $C^\bullet_{\mathrm{cont}}(K_{\chi,w}, T)$ when $G = G_w$) is the complex of continuous (non-homogeneous) T-valued cochains on G. If $\mathscr{R}$ is a localisation of R, and $\mathscr{T} := T \otimes_R \mathscr{R}$, set $C^\bullet_{\mathrm{cont}}(*, \mathscr{T}) := C^\bullet_{\mathrm{cont}}(*, T) \otimes_R \mathscr{R}$. Then a *local condition* for $\mathscr{T}$ at $w \in S$ is a morphism $i^+_w(T) \otimes \mathscr{R} : U^+_w(T) \otimes_R \mathscr{R} \to C^\bullet_{\mathrm{cont}}(K_{\chi,w}, \mathscr{T})$, obtained as the base change of a local condition $i^+_w(T)$ for T at w.

condition: $i_w^+(X) = \mathrm{id} : U_w^+(X) := C^\bullet_{\mathrm{cont}}(K_{\chi,w}, X) \to C^\bullet_{\mathrm{cont}}(K_{\chi,w}, X))$ in case $X \in \{\mathscr{S}^{-1}\mathbb{T}_{\mathbf{f}}, \mathscr{S}^{-1}T_f\}$ (resp., $X \in \{\mathbb{A}_{\mathbf{f}}, A_f\}$). The associated *Nekovář's Selmer complex* [20] is defined as the complex of R_X-modules

$$\begin{aligned}\widetilde{C}^\bullet_f(K_\chi, X) &= \widetilde{C}^\bullet_f(G_{K_\chi,S}, X; \Delta_S(X))\\ &:= \mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G_{K_\chi,S}, X) \oplus \bigoplus_{v\in S} U_v^+(X) \xrightarrow{\mathrm{res}_S - i_S^+} \bigoplus_{v\in S} C^\bullet_{\mathrm{cont}}(K_{\chi,v}, X)\right)[-1],\end{aligned}$$

where $\mathrm{res}_S = \oplus_{v\in S}\mathrm{res}_v$ and $i_S^+ = \oplus_{v\in S} i_v^+(X)$. It follows by standard results on continuous Galois cohomology groups [20, Section 4] (essentially due to Tate [33]) that $\widetilde{C}^\bullet_f(K_\chi, X)$ is cohomologically bounded, with cohomology of finite (resp., cofinite) type over R_X if X is of finite (resp., cofinite) type over R_X. Let

$$\begin{aligned}\widetilde{\mathbf{R}\Gamma}_f(K_\chi, X) &\in \mathrm{D}^b_{\mathrm{ft},(\mathrm{resp.,\ cf})}(R_X);\\ \widetilde{H}^*_f(K_\chi, X) := H^*\left(\widetilde{\mathbf{R}\Gamma}_f(K_\chi, X)\right) &\in \left({}_{R_X}\mathrm{Mod}\right)_{\mathrm{ft},(\mathrm{resp.,\ cf})}\end{aligned}$$

be the image of $\widetilde{C}^\bullet_f(K_\chi, X)$ in the derived category and its cohomology respectively. If $\mathcal{X} \in \{\mathbb{T}_{\mathbf{f}}, T_f\}$ and $R_{\mathcal{X}} \in \{\mathbb{I}, \mathcal{O}_L\}$ is the corresponding coefficient ring, then

$$\begin{aligned}\widetilde{\mathbf{R}\Gamma}_f(K_\chi, X) &\cong \widetilde{\mathbf{R}\Gamma}_f(K_\chi, \mathcal{X}) \otimes_{R_{\mathcal{X}}} R_X;\\ \widetilde{H}^*_f(K_\chi, X) &\cong \widetilde{H}^*_f(K_\chi, \mathcal{X}) \otimes_{R_{\mathcal{X}}} R_X,\end{aligned}$$

which we consider as equalities in what follows.

Let $X \in \{\mathscr{S}^{-1}\mathbb{T}_{\mathbf{f}}, \mathscr{S}^{-1}T_f\}$ (resp., $X \in \{\mathbb{A}_{\mathbf{f}}, A_f\}$), and let $S \ni w \nmid p$. Define the $R_X[G_{K_\chi,w}]$-module $X_w^- := X$ (resp., $X_w^- := 0$). By the definition of Nekovář's Selmer complexes, there is a long exact cohomology sequence of R_X-modules [20, Section 6]:

$$\begin{aligned}\cdots \to \bigoplus_{w\in S} H^{q-1}(K_{\chi,w}, X_w^-) \to \widetilde{H}^q_f(K_\chi, X)&\\ \to H^q(G_{K_\chi,S}, X) \to \bigoplus_{w\in S} H^q(K_{\chi,w}, X_w^-) \to \cdots.&\end{aligned}$$

In particular this gives an exact sequence of R_X-modules

$$X^{G_{K_\chi,S}} \to \bigoplus_{w\in S} H^0(K_{\chi,w}, X_w^-) \to \widetilde{H}^1_f(K_\chi, X) \to \mathfrak{S}(K_\chi, X) \to 0. \tag{28}$$

Here $\mathfrak{S}(K_\chi, X) = \mathfrak{S}(G_{K_\chi,S}, X)$ is the *(S-primitive, strict) Greenberg Selmer group of X/K_χ*, defined by

$$\mathfrak{S}(K_\chi, X) := \ker\left(H^1(G_{K_\chi,S}, X) \longrightarrow \prod_{w\in S} H^1(K_{\chi,w}, X_w^-)\right).$$

6.1.2. A control theorem. We know that $\mathbb{I}_{\mathfrak{p}_f}$ is a discrete valuation ring, and that its maximal ideal $\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f}$ is generated by $\varpi_{\mathrm{wt}} := \gamma_{\mathrm{wt}} - 1 \in \Lambda$ (see (17)). Write $V_f := T_f \otimes_{\mathcal{O}_L} L$ and $\mathbb{T}_{\mathbf{f},\mathfrak{p}_f} := \mathbb{T}_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}$. By [20, Propositions 3.4.2 and 3.5.10], the arithmetic point $\phi_f \in \mathcal{X}^{\mathrm{arith}}(\mathbb{I})$ induces an exact triangle in $\mathrm{D}^b_{\mathrm{ft}}(\mathbb{I}_{\mathfrak{p}_f})$:

$$\widetilde{\mathbf{R}\Gamma}_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f}) \xrightarrow{\varpi_{\mathrm{wt}}} \widetilde{\mathbf{R}\Gamma}_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f}) \xrightarrow{\phi_{f*}} \widetilde{\mathbf{R}\Gamma}_f(K, V_f),$$

and then an isomorphism in $\mathrm{D}^b_{\mathrm{ft}}(L)$:

$$c_f : \mathbf{L}\phi_f^*\Big(\widetilde{\mathbf{R}\Gamma}_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})\Big) \cong \widetilde{\mathbf{R}\Gamma}_f(K_\chi, V_f), \tag{29}$$

where $\mathbf{L}\phi_f^* : \mathrm{D}^-(\mathbb{I}_{\mathfrak{p}_f}) \to \mathrm{D}(L)$ is the left derived functor of the base-change functor $\phi_f^*(\cdot) := \cdot \otimes_{\mathbb{I},\phi_f} L$. (Note that, since $f = f_2$ has integral Fourier coefficients, the residue field $\mathbb{I}_{\mathfrak{p}_f}/\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f}$ of $\mathbb{I}_{\mathfrak{p}_f}$ equals L.) This induces in cohomology short exact sequences of L-modules

$$0 \to \widetilde{H}^q_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})/\varpi_{\mathrm{wt}} \to \widetilde{H}^q_f(K_\chi, V_f) \xrightarrow{i_{\mathrm{wt}}} \widetilde{H}^{q+1}_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})[\varpi_{\mathrm{wt}}] \to 0. \tag{30}$$

6.1.3. Nekovář's duality I: global cup-products. Let $\mathcal{X} \in \{\mathbb{T}_{\mathbf{f}}, T_f\}$, and let $\mathcal{R} \in \{\mathbb{I}, \mathcal{O}_L\}$ be the corresponding coefficient ring. For $\mathscr{S} \in \{\mathbb{I} - \mathfrak{p}_f, \mathcal{O}_L - \mathfrak{m}_L\}$ (where $\mathfrak{m}_L$ is the maximal ideal of $\mathcal{O}_L$), write $X := \mathscr{S}^{-1}\mathcal{X} \in \{\mathbb{T}_{\mathbf{f},\mathfrak{p}_f}, V_f\}$ and $R_X := \mathscr{S}^{-1}\mathcal{R} \in \{\mathbb{I}_{\mathfrak{p}_f}, L\}$. Let

$$\pi_X : X \otimes_{R_X} X \to R_X(1)$$

be the localization at $\mathscr{S}$ of the perfect duality $\pi : \mathbb{T}_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{T}_{\mathbf{f}} \to \mathbb{I}(1)$ if $\mathcal{X} = \mathbb{T}_{\mathbf{f}}$, or the localisation at $\mathscr{S}$ of its ϕ_f-base change $\pi_f := \phi_f^*(\pi) : T_f \otimes_{\mathcal{O}_L} T_f \to \mathcal{O}_L(1)$ if $\mathcal{X} = T_f$ (see Section 2.2). As a manifestation of Nekovář's wide generalization of Poitou–Tate duality, Section 6 of [20] attaches to π_X a morphism in $\mathrm{D}^b_{\mathrm{ft}}(R_X)$:

$$\cup^{\mathbf{Nek}}_{\boldsymbol{\pi}_X} : \widetilde{\mathbf{R}\Gamma}_f(K_\chi, X) \otimes^{\mathbf{L}}_{R_X} \widetilde{\mathbf{R}\Gamma}_f(K_\chi, X) \longrightarrow \tau_{\geq 3}\mathbf{R}\Gamma_{c,\mathrm{cont}}(K_\chi, R_X(1)) \cong R_X[-3],$$

where $\mathbf{R}\Gamma_{c,\mathrm{cont}}(K_\chi, -)$ denotes the complex of *cochains with compact support* [20, Section 5], and the isomorphism comes (essentially) by the sum of the invariant maps of local class field theory for $v \in S$. The pairings $\cup^{\mathbf{Nek}}_{\boldsymbol{\pi}}$ on $\widetilde{\mathbf{R}\Gamma}_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})$ and $\cup^{\mathbf{Nek}}_{\boldsymbol{\pi}_f}$ on $\widetilde{\mathbf{R}\Gamma}_f(K_\chi, V_f)$ are compatible with respect to the isomorphism $c_f : \mathbf{L}\phi_f^*\big(\widetilde{\mathbf{R}\Gamma}_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})\big) \cong \widetilde{\mathbf{R}\Gamma}_f(K_\chi, V_f)$ in $\mathrm{D}(L)$ described in (29).

The global cup-product pairing $\cup^{\mathbf{Nek}}_{\boldsymbol{\pi}_X}$ gives in cohomology pairings

$$_q\cup^{\mathrm{Nek}}_{\pi_X} : \widetilde{H}^q_f(K_\chi, X) \otimes_{R_X} \widetilde{H}^{3-q}_f(K_\chi, X) \longrightarrow R_X \tag{31}$$

(for every $q \in \mathbf{Z}$).

Writing $\mathscr{R}_X := \mathrm{Frac}(R_X)$, they induce by adjunction isomorphisms

$$\mathrm{adj}\left({}_q\cup^{\mathrm{Nek}}_{\pi_X}\right) : \widetilde{H}^q_f(K_\chi, X) \otimes_{R_X} \mathscr{R}_X \cong \mathrm{Hom}_{\mathscr{R}_X}\left(\widetilde{H}^{3-q}_f(K_\chi, X) \otimes_{R_X} \mathscr{R}_X, \mathscr{R}_X\right), \tag{32}$$

as follows from [20, Proposition 6.7.7], since $\mathbf{R}\Gamma_{\mathrm{cont}}(K_{\chi,w}, X) \cong 0$ is acyclic for every prime $w \nmid p$ of K_χ. (See also [20, Propositions 12.7.13.3 and 12.7.13.4].)

6.1.4. Nekovář's duality II: generalised Pontrjagin duality. Let X denote either $\mathbb{T}_{\mathbf{f}}$ or T_f, let R_X be either $\mathbb{I}$ or $\mathcal{O}_L$ (accordingly), and let $\mathbb{A}_X := \mathrm{Hom}_{\mathrm{cont}}(X, \mu_{p^\infty})$ be the (discrete) Kummer dual of X. Appealing again to Nekovář's generalised Poitou–Tate duality, we have Pontrjagin dualities

$$\widetilde{H}^{3-q}_f(K_\chi, \mathbb{A}_X) \cong \mathrm{Hom}_{\mathrm{cont}}\left(\widetilde{H}^q_f(K_\chi, X), \mathbf{Q}_p/\mathbf{Z}_p\right) =: \widetilde{H}^q_f(K_\chi, X)^*. \tag{33}$$

We refer the reader to [20, Section 6] for the details.

6.1.5. Nekovář's duality III: generalised Cassels–Tate pairings. Section 10 of [20] — which provides a generalisation of a construction of Flach [10] — attaches to $\pi : \mathbb{T}_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{T}_{\mathbf{f}} \to \mathbb{I}(1)$ a *skew-symmetric* pairing

$$\cup^{\mathbf{CT}}_{\boldsymbol{\pi}} : \widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f}})_{\mathrm{tors}} \otimes_{\mathbb{I}} \widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f}})_{\mathrm{tors}} \longrightarrow \mathrm{Frac}(\mathbb{I})/\mathbb{I},$$

where $M_{\mathrm{tors}} = \ker\left(M \xrightarrow{i} M \otimes_{\mathbb{I}} \mathrm{Frac}(\mathbb{I})\right)$ denotes the $\mathbb{I}$-torsion submodule of M. Denote by

$$\cup^{\mathrm{CT}}_{\pi} : \widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})_{\mathrm{tors}} \otimes_{\mathbb{I}_{\mathfrak{p}_f}} \widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})_{\mathrm{tors}} \longrightarrow \mathrm{Frac}(\mathbb{I}_{\mathfrak{p}_f})/\mathbb{I}_{\mathfrak{p}_f} \tag{34}$$

its localization at $\mathfrak{p}_f$, $N_{\mathrm{tors}} := N[\varpi_{\mathrm{wt}}^\infty]$ denoting now the $\mathbb{I}_{\mathfrak{p}_f}$-torsion submodule of N (see (17)). As proved in [20, Proposition 12.7.13.4], $\cup^{\mathrm{CT}}_\pi$ is a *perfect* pairing, i.e. its adjoint

$$\mathrm{adj}\left(\cup^{\mathrm{CT}}_\pi\right) : \widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})_{\mathrm{tors}} \cong \mathrm{Hom}_{\mathbb{I}_{\mathfrak{p}_f}}\left(\widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})_{\mathrm{tors}}, \mathrm{Frac}(\mathbb{I}_{\mathfrak{p}_f})/\mathbb{I}_{\mathfrak{p}_f}\right) \tag{35}$$

is an isomorphism. We call $\cup^{\mathrm{CT}}_\pi$ *Nekovář (localized) Cassels–Tate pairing* on $\mathbb{T}_{\mathbf{f},\mathfrak{p}_f}$. This is the pairing denoted $\cup_{\pi(\mathfrak{p}_f),0,2,2}$ in *loc. cit.* We refer to Sections 2.10.14, 10.2 and 10.4 of [20] for the definition of $\cup^{\mathbf{CT}}_{\boldsymbol{\pi}}$.

6.1.6. Comparison with Bloch–Kato Selmer groups. Recall that $V_f := T_f \otimes_{\mathcal{O}_L} L$, and $V^{\pm}_{f,v} := T^{\pm}_{f,v} \otimes_{\mathcal{O}_L} L$ for $v|p$. Then $V_f \cong \mathbb{T}_{\mathbf{f},\mathfrak{p}_f} \otimes_{\mathbb{I}_{\mathfrak{p}_f},\phi_f} L$ is isomorphic to the ϕ_f-base change of the localisation $\mathbb{T}_{\mathbf{f},\mathfrak{p}_f}$, and similarly $V^{\pm}_{f,v}$ is isomorphic to the ϕ_f-base change of the localisation of $\mathbb{T}^{\pm}_{\mathbf{f},v}$ at $\mathfrak{p}_f$. By (7), combined with the

Chebotarev density theorem and [28, Chapters V and VII], there is an isomorphism of $L[G_{K_\chi,S}]$-modules (cf. Section 2.2)

$$V_f \cong V_p(A) \otimes_{\mathbf{Q}_p} L, \tag{36}$$

where $V_p(A) := \mathrm{Ta}_p(A) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is the p-adic Tate module of $A/\mathbf{Q}$ with $\mathbf{Q}_p$-coefficients. We fix from now on such an isomorphism, and we will use it to identify V_f with $V_p(A) \otimes_{\mathbf{Q}_p} L$.

Consider the classical (or Bloch–Kato [8]) Selmer group attached to $V_p(A)/K_\chi$ via Kummer theory:

$$\mathrm{Sel}_p(A/K_\chi) := \ker\left(H^1(K_{\chi,S}, V_p(A)) \longrightarrow \prod_{v|p} \frac{H^1(K_{\chi,v}, V_p(A))}{A(K_{\chi,v})\widehat{\otimes}\mathbf{Q}_p} \right)$$

(it is easily verified using Tate local duality and [28, Chapter VII] that $H^1(K_{\chi,w}, V_p(A)) = 0$ for $w \nmid p$), sitting in a short exact sequence

$$0 \to A(K_\chi)\widehat{\otimes}\mathbf{Q}_p \to \mathrm{Sel}_p(A/K_\chi) \to V_p\left(Ш(A/K_\chi)\right) \to 0, \tag{37}$$

where $Ш(A/K_\chi)$ is the Tate–Shafarevich group of A/K_χ and

$$V_p(\cdot) := \varprojlim_{n\geq 1}(\cdot)_{p^n} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

is the p-adic Tate module of the abelian group $(\cdot)$ with $\mathbf{Q}_p$-coefficients. R. Greenberg [11] has proved that

$$\mathrm{Sel}_p(A/K_\chi) \otimes_{\mathbf{Q}_p} L = \mathfrak{S}(K_\chi, V_f).$$

Since $a_p = a_p(A) = +1$ (as $A/\mathbf{Q}_p$ has split multiplicative reduction), the $G_{\mathbf{Q}_p}$-representation $V_f = V_p(A)\otimes_{\mathbf{Q}_p} L$ is a Kummer extension of the trivial representation L, i.e. $V_{f,v}^+ \cong L(1)$ and $V_{f,v}^- \cong L$ for every $v|p$ (where L is the trivial representation of $G_{K_{\chi,v}}$ and $L(1) := L \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(1)$ is its Tate twist). As $H^0(G_{K_\chi,S}, V_f) \subset H^0(G_{K_{\chi,w}}, V_f) = 0$ for every $w \nmid p$ (by [28, Chapter VII] and local Tate duality), (28) gives rise to an exact sequence

$$0 \to \bigoplus_{v|p} L \to \widetilde{H}^1_f(K_\chi, V_f) \to \mathrm{Sel}_p(A/K_\chi) \otimes_{\mathbf{Q}_p} L \to 0. \tag{38}$$

(See Section 6.3 below for more details.)

6.1.7. Galois conjugation. Let X be as in Section 6.1.1. Section 8 of [20] defines a natural action of $\mathrm{Gal}(K_\chi/\mathbf{Q})$ on $\widetilde{H}^q_f(K_\chi, X)$, making it a $R_X[\mathrm{Gal}(K_\chi/\mathbf{Q})]$-module. If τ is a nontrivial automorphism on K_χ, we will write $\tau(x)$ or x^τ for its action on $x \in \widetilde{H}^q_f(K_\chi, X)$. To be short, all the relevant constructions we discussed above commute with the action of $\mathrm{Gal}(K_\chi/\mathbf{Q})$. In particular, we mention the following properties.

Nekovář's global cup products ${}_q\cup^{\mathrm{Nek}}_{\pi\chi}$ (defined in (31)) are $\mathrm{Gal}(K_\chi/\mathbf{Q})$-equivariant [20, Section 8].

Nekovář's Pontrjagin duality isomorphisms (33) are $\mathrm{Gal}(K_\chi/\mathbf{Q})$-equivariant [20, Prop. 8.8.9].

The abstract Cassels–Tate pairing $\cup^{\mathrm{CT}}_\pi$ is $\mathrm{Gal}(K_\chi/\mathbf{Q})$-equivariant [20, Section 10.3.2].

The exact sequences (28), (30) and (38) are $\mathrm{Gal}(K_\chi/\mathbf{Q})$-equivariant. (In case $K_\chi/\mathbf{Q}$ is quadratic and p splits in K_χ, the action of the non-trivial element $\tau \in \mathrm{Gal}(K_\chi/K)$ on the first term $\bigoplus_{v|p} L = L \oplus L$ in (38) is given by permutation of the factors: $(q,q')^\tau = (q',q)$ for every $q,q' \in L$.)

6.2. The half-twisted weight pairing. Define *Nekovář's half-twisted weight pairing* by the composition

$$\begin{aligned}\langle -,-\rangle^{\mathrm{Nek}}_{V_f,\pi} : \widetilde{H}^1_f(K_\chi, V_f) \otimes_L \widetilde{H}^1_f(K_\chi, V_f) \\ \xrightarrow{i_{\mathrm{wt}}\otimes i_{\mathrm{wt}}} \widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})[\varpi_{\mathrm{wt}}] \otimes_{\mathbb{I}_{\mathfrak{p}_f}} \widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})[\varpi_{\mathrm{wt}}] \\ \xrightarrow{\cup^{\mathrm{CT}}_\pi} \Big(\mathrm{Frac}(\mathbb{I}_{\mathfrak{p}_f})/\mathbb{I}_{\mathfrak{p}_f}\Big)[\varpi_{\mathrm{wt}}] \overset{\theta_{\mathrm{wt}}}{\cong} \mathbb{I}_{\mathfrak{p}_f}/\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f} \overset{\phi_f}{\cong} L \overset{\times \ell_{\mathrm{wt}}}{\cong} L,\end{aligned}$$

where the notations are as follows. The morphism $i_{\mathrm{wt}} : \widetilde{H}^1_f(K_\chi, V_f) \to \widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})[\varpi_{\mathrm{wt}}]$ is the one appearing in the exact sequence (30) (taking $q=1$). $\cup^{\mathrm{CT}}_\pi$ is Nekovář's Cassels–Tate pairing attached to $\pi : \mathbb{T}_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{T}_{\mathbf{f}} \to \mathbb{I}(1)$, and defined in Section 6.1.5. $\theta_{\mathrm{wt}} : \big(\mathrm{Frac}(\mathbb{I}_{\mathfrak{p}_f})/\mathbb{I}_{\mathfrak{p}_f}\big)[\varpi_{\mathrm{wt}}] \cong \mathbb{I}_{\mathfrak{p}_f}/\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f}$ is defined by $\theta_{\mathrm{wt}}\big(\frac{a}{\varpi_{\mathrm{wt}}} \bmod \mathbb{I}_{\mathfrak{p}_f}\big) := a \bmod \mathfrak{p}_f$, for every $a \in \mathbb{I}_{\mathfrak{p}_f}$. (We remind that $\varpi_{\mathrm{wt}} \in \Lambda$ is a uniformiser of $\mathbb{I}_{\mathfrak{p}_f}$ by (17)). Finally, $\ell_{\mathrm{wt}} := \log_p(\gamma_{\mathrm{wt}})$ (where $\varpi_{\mathrm{wt}} := \gamma_{\mathrm{wt}} - 1$). Note that both the morphisms i_{wt} and θ_{wt} depend on the choice of the uniformiser ϖ_{wt}. Multiplication by ℓ_{wt} serves the purposes of removing the dependence on this choice.

Since $\cup^{\mathrm{CT}}_\pi$ is a skew-symmetric, $\mathrm{Gal}(K_\chi/\mathbf{Q})$-equivariant pairing, and since i_{wt} is a $\mathrm{Gal}(K_\chi/\mathbf{Q})$-equivariant morphism (cf. Section 6.1.7), $\langle -,-\rangle^{\mathrm{Nek}}_{V_f,\pi}$ is a *skew-symmetric,* $\mathrm{Gal}(K_\chi/\mathbf{Q})$*-equivariant pairing*. (Of course, here we consider on L the trivial $\mathrm{Gal}(K_\chi/\mathbf{Q})$-action.)

The aim of this section is to prove the following key proposition, whose proof uses all the power of Nekovář's results mentioned above. Let χ be (as above) a quadratic Dirichlet character of conductor coprime with Np. Write $\langle -,-\rangle^{\mathrm{Nek},\chi}_{V_f,\pi}$ for the restriction of $\langle -,-\rangle^{\mathrm{Nek}}_{V_f,\pi}$ to $\widetilde{H}^1_f(K_\chi, V_f)^\chi \otimes_L \widetilde{H}^1_f(K_\chi, V_f)^\chi$. (Of course, if χ is the trivial character, i.e. if $K_\chi = \mathbf{Q}$, we are defining nothing new.) Given an $\mathbb{I}$-module M, we say that M is *semi-simple at* $\mathfrak{p}_f$ if $M_{\mathfrak{p}_f}$ is a semi-simple $\mathbb{I}_{\mathfrak{p}_f}$-module, and we write $\mathrm{length}_{\mathfrak{p}_f}(M)$ to denote the length of $M_{\mathfrak{p}_f}$ over $\mathbb{I}_{\mathfrak{p}_f}$.

Proposition 6.2. *Let χ be a quadratic Dirichlet character of conductor coprime with Np, and assume that p splits in K_χ. Then the following conditions are equivalent:*

1. $\langle -,- \rangle_{V_f,\pi}^{\mathrm{Nek},\chi}$ *is a non-degenerate L-bilinear form on $\widetilde{H}^1_f(K_\chi, V_f)^\chi$.*
2. $$\mathrm{length}_{\mathfrak{p}_f}\left(\widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f}})^\chi\right) = \dim_L\left(\widetilde{H}^1_f(K_\chi, V_f)^\chi\right).$$
3. $\widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f}})^\chi$ *is a torsion $\mathbb{I}$-module, which is semi-simple at $\mathfrak{p}_f$.*

If these properties are satisfied, then $X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_\chi)^\chi$ is a torsion $\mathbb{I}$-module, which is semi-simple at $\mathfrak{p}_f$, and

$$\mathrm{length}_{\mathfrak{p}_f}\left(X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_\chi)^\chi\right) = \dim_{\mathbf{Q}_p}\left(\mathrm{Sel}_p(A/K_\chi)^\chi\right).$$

The proposition will be an immediate consequence of the following three lemmas (in which we will prove separately the equivalences 1 $\iff$ 3, 3 $\iff$ 2 and the last statement, respectively).

Lemma 6.3. $\langle -,- \rangle_{V_f,\pi}^{\mathrm{Nek},\chi}$ *is non-degenerate if and only if $\widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})^\chi$ is a torsion, semi-simple $\mathbb{I}_{\mathfrak{p}_f}$-module.*

Proof. Taking the χ-component of the exact sequence (30), we see that the restrictions

$$i^\chi_{\mathrm{wt}} = i^{q,\chi}_{\mathrm{wt}} : \widetilde{H}^q_f(K_\chi, V_f)^\chi \longrightarrow \widetilde{H}^{q+1}_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})^\chi[\varpi_{\mathrm{wt}}]$$

of the morphisms $i_{\mathrm{wt}} = i^q_{\mathrm{wt}}$ defined in (30) are surjective. Since $\widetilde{H}^0_f(K_\chi, V_f) \subset H^0(G_{K_{\chi,S}}, V_f) = 0$, this implies in particular that $\widetilde{H}^1_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})^\chi$ is torsion free, and $i^{1,\chi}_{\mathrm{wt}}$ is injective if and only if $\widetilde{H}^1_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})^\chi = 0$. Moreover, since χ is quadratic and ${}_q\cup^{\mathrm{Nek}}_\pi$ is $\mathrm{Gal}(K_\chi/\mathbf{Q})$-equivariant, the duality isomorphism (32) shows that the latter condition is equivalent to the fact that $\widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})^\chi$ is a torsion $\mathbb{I}_{\mathfrak{p}_f}$-module.

Write for simplicity $N := \widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})_{\mathrm{tors}}$ for the $\mathbb{I}_{\mathfrak{p}_f}$-torsion submodule of $\widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})$. Since $\cup^{\mathrm{CT}}_\pi$ is $\mathrm{Gal}(K_\chi/\mathbf{Q})$-equivariant, $p \neq 2$ and χ is quadratic, the isomorphism (35) restricts to an isomorphism

$$\mathrm{adj}\left(\cup^{\mathrm{CT}}_\pi\right) : N^\chi \cong \mathrm{Hom}_{\mathbb{I}_{\mathfrak{p}_f}}(N^\chi, \mathrm{Frac}(\mathbb{I}_{\mathfrak{p}_f})/\mathbb{I}_{\mathfrak{p}_f}).$$

Let $\cup^{\mathrm{CT},\chi}_{\pi,\varpi_{\mathrm{wt}}} : N^\chi[\varpi_{\mathrm{wt}}] \otimes N^\chi[\varpi_{\mathrm{wt}}] \to \left(\mathrm{Frac}(\mathbb{I}_{\mathfrak{p}_f})/\mathbb{I}_{\mathfrak{p}_f}\right)[\varpi_{\mathrm{wt}}]$ denote the restriction of $\cup^{\mathrm{CT}}_\pi$ to the ϖ_{wt}-torsion of N^χ. It follows by the preceding isomorphism that the right (or left) radical of $\cup^{\mathrm{CT},\chi}_{\pi,\varpi_{\mathrm{wt}}}$ equals $\mathcal{N}^\chi := \varpi_{\mathrm{wt}} N^\chi \cap N^\chi[\varpi_{\mathrm{wt}}]$. In other words, $\cup^{\mathrm{CT},\chi}_{\pi,\varpi_{\mathrm{wt}}}$ is non-degenerate if and only if $\mathcal{N}^\chi = 0$. On the other hand, as ϖ_{wt} is a uniformiser for $\mathbb{I}_{\mathfrak{p}_f}$, the structure theorem for finite modules over discrete valuation rings gives an isomorphism of $\mathbb{I}_{\mathfrak{p}_f}$-modules $N^\chi \cong \bigoplus_{j=0}^{\infty}\left(\mathbb{I}_{\mathfrak{p}_f}/(\varpi_{\mathrm{wt}})^j\right)^{e_j}$, for positive integers e_j such that $e_j = 0$ for $j \gg 0$. Then $\mathcal{N}^\chi = 0$ if and only if $e_j = 0$ for every $j > 1$, i.e. if and only if N^χ is semi-simple.

Since $i_{\mathrm{wt}}^\chi = i_{\mathrm{wt}}^{1,\chi}$ is surjective, it follows by the definitions that $\langle -,-\rangle_{V_f,\pi}^{\mathrm{Nek},\chi}$ is non-degenerate (i.e. has trivial right=left radical) if and only if i_{wt}^χ is injective and $\cup_{\pi,\varpi_{\mathrm{wt}}}^{\mathrm{CT},\chi}$ has trivial radical. Together with the preceding discussion, this concludes the proof of the lemma. □

Lemma 6.4. $\mathrm{length}_{\mathfrak{p}_f}\left(\widetilde{H}_f^2(K_\chi,\mathbb{T}_{\mathbf{f}})^\chi\right) \geq \dim_L\left(\widetilde{H}_f^1(K_\chi,V_f)^\chi\right)$, *and equality holds if and only if* $\widetilde{H}_f^2(K_\chi,\mathbb{T}_{\mathbf{f},\mathfrak{p}_f})^\chi$ *is a torsion, semi-simple* $\mathbb{I}_{\mathfrak{p}_f}$*-module.*

Proof. Write for simplicity $\varpi := \varpi_{\mathrm{wt}}$, $M_* := \widetilde{H}_f^*(K_\chi,\mathbb{T}_{\mathbf{f},\mathfrak{p}_f})^\chi$, and $\mathscr{M}_* := \widetilde{H}_f^*(K_\chi,V_f)^\chi$, so that there are short exact sequences of L-modules (30):

$$0 \to M_q/\varpi \to \mathscr{M}_q \to M_{q+1}[\varpi] \to 0.$$

We can assume that M_2 is a torsion $\mathbb{I}_{\mathfrak{p}_f}$-module, hence $M_1 = 0$ by the duality isomorphism (32) (cf. the preceding proof). Then $\mathscr{M}_1 \cong M_2[\varpi]$ and

$$\dim_L \mathscr{M}_1 = \dim_L M_2[\varpi]. \tag{39}$$

The structure theorem for finite, torsion modules over principal ideal domains yields an isomorphism

$$M_2 = \bigoplus_{j=1}^{\infty}\left(\mathbb{I}_{\mathfrak{p}_f}/\varpi^j\right)^{m(j)},$$

where $m : \mathbf{N} \to \mathbf{N}$ is a function such that $m(j) = 0$ for $j \gg 0$. Since $\left(\mathbb{I}_{\mathfrak{p}_f}/\varpi^j\right)[\varpi] \cong \mathbb{I}_{\mathfrak{p}_f}/\varpi$ for $j \geq 1$:

$$\begin{aligned}\mathrm{length}_{\mathfrak{p}_f} M_2 = \sum_{j=0}^{\infty} m(j)\cdot j &= \sum_{j=1}^{\infty} m(j) + \sum_{j=2}^{\infty} m(j)\cdot(j-1)\\ &= \dim_L M_2[\varpi] + \sum_{j=2}^{\infty} m(j)\cdot(j-1).\end{aligned}$$

Together with (39), this gives $\mathrm{length}_{\mathfrak{p}_f} M_2 \geq \dim_L \mathscr{M}_1$, with equality if and only if $m(j) = 0$ for every $j \geq 2$, i.e. if and only if M_2 is a semi-simple $\mathbb{I}_{\mathfrak{p}_f}$-module. □

Lemma 6.5. *Assume that* $\widetilde{H}_f^2(K_\chi,\mathbb{T}_{\mathbf{f},\mathfrak{p}_f})^\chi$ *is a torsion, semi-simple* $\mathbb{I}_{\mathfrak{p}_f}$*-module. Then* $X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K_\chi)^\chi \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}$ *is a torsion, semi-simple* $\mathbb{I}_{\mathfrak{p}_f}$*-module, and*

$$\mathrm{length}_{\mathfrak{p}_f}\left(X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K_\chi)^\chi\right) = \dim_{\mathbf{Q}_p}\left(\mathrm{Sel}_p(A/K_\chi)^\chi\right).$$

Proof. Since $\mathrm{adj}(\pi) : \mathbb{T}_{\mathbf{f}} \cong \mathrm{Hom}_{\mathbb{I}}(\mathbb{T}_{\mathbf{f}}, \mathbb{I}(1))$ and $\mathbb{T}_{\mathbf{f}}$ is a free $\mathbb{I}$-module, there is a canonical isomorphism of $\mathbb{I}[G_{K_\chi,S}]$-modules

$$\mathbb{T}_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}^* \cong \mathrm{Hom}_{\mathbb{I}}(\mathbb{T}_{\mathbf{f}}, \mathbb{I}(1)) \otimes_{\mathbb{I}} \mathrm{Hom}_{\mathrm{cont}}(\mathbb{I}, \mathbf{Q}_p/\mathbf{Z}_p) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathbb{T}_{\mathbf{f}}, \mu_{p^\infty}) =: \mathbb{A}_{\mathbf{f}},$$

the second isomorphism being defined by composition: $\psi \otimes \mu \mapsto \mu \circ \psi$. Similarly, the isomorphism of $\mathbb{I}[G_{\mathbf{Q}_p}]$-modules $\mathrm{adj}(\pi) : \mathbb{T}_{\mathbf{f}}^- \cong \mathrm{Hom}_{\mathbb{I}}(\mathbb{T}_{\mathbf{f}}^+, \mathbb{I}(1))$ gives an isomorphism of $\mathbb{I}[G_{\mathbf{Q}_p}]$-modules $\mathbb{T}_{\mathbf{f}}^- \otimes_{\mathbb{I}} \mathbb{I}^* \cong \mathbb{A}_{\mathbf{f}}^-$. (Recall here that $\mathbb{A}_{\mathbf{f}}$ and $\mathbb{A}_{\mathbf{f}}^-$ are the Kummer duals of $\mathbb{T}_{\mathbf{f}}$ and $\mathbb{T}_{\mathbf{f}}^+$ respectively.) This implies that $\mathrm{Sel}_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K_\chi) = \mathfrak{S}\left(K_\chi, \mathbb{A}_{\mathbf{f}}\right)$. (Note that $\mathbb{A}_{\mathbf{f},w}^- := 0$ for every $S \ni w \nmid p$, so that we impose no condition at $w \nmid p$ in both the definitions of $\mathrm{Sel}_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K_\chi)$ and $\mathfrak{S}(K_\chi, \mathbb{A}_{\mathbf{f}})$.) By (28) one then obtains an exact sequence

$$H^0(G_{K_\chi,S}, \mathbb{A}_{\mathbf{f}}) \to \bigoplus_{v|p} H^0(K_{\chi,v}, \mathbb{A}_{\mathbf{f},v}^-) \to \widetilde{H}^1_f(K_\chi, \mathbb{A}_{\mathbf{f}}) \to \mathrm{Sel}_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K_\chi) \to 0. \tag{40}$$

We claim that the localisation at $\mathfrak{p}_f$ of the Pontrjagin dual of $H^0(G_{K_\chi,S}, \mathbb{A}_{\mathbf{f}})$ vanishes, i.e.

$$H^0(G_{K_\chi,S}, \mathbb{A}_{\mathbf{f}})^*_{\mathfrak{p}_f} := \mathrm{Hom}_{\mathbf{Z}_p}\Big(H^0(G_{K_\chi,S}, \mathbb{A}_{\mathbf{f}}), \mathbf{Q}_p/\mathbf{Z}_p\Big) \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f} = 0. \tag{41}$$

Indeed, let w be a prime of K_χ. By Tate local duality, $H^0(K_{\chi,w}, \mathbb{A}_{\mathbf{f}})$ is the Pontrjagin dual of $H^2(K_{\chi,w}, \mathbb{T}_{\mathbf{f}})$, so that the inclusion $H^0(G_{K_\chi,S}, \mathbb{A}_{\mathbf{f}}) \subset H^0(K_{\chi,w}, \mathbb{A}_{\mathbf{f}})$ induces a surjection $H^2(K_{\chi,w}, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f}) \twoheadrightarrow H^0(G_{K_\chi,S}, \mathbb{A}_{\mathbf{f}})^*_{\mathfrak{p}_f}$ on (localised) Pontrjagin duals. As $\mathbf{R}\Gamma_{\mathrm{cont}}(K_{\chi,w}, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f}) \cong 0 \in \mathrm{D}(\mathbb{I}_{\mathfrak{p}_f})$ is acyclic for every prime $w \nmid p$ (as easily proved, cf. [20, Proposition 12.7.13.3(i)]), the claim (41) follows. Since $\widetilde{H}^1_f(K_\chi, \mathbb{A}_{\mathbf{f}})$ is the Pontrjagin dual of $\widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f}})$ by Nekovář's duality isomorphism (33), applying first $\mathrm{Hom}_{\mathbf{Z}_p}(-, \mathbf{Q}_p/\mathbf{Z}_p)$ and then $- \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}$ to (40), and using (41), yield a short exact sequence of $\mathbb{I}_{\mathfrak{p}_f}$-modules

$$0 \to X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K_\chi) \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f} \to \widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f}) \to \bigoplus_{v|p} H^2(K_{\chi,v}, \mathbb{T}_{\mathbf{f},v}^+ \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}) \to 0, \tag{42}$$

where we used once again local Tate duality to rewrite the Pontrjagin dual of $H^0(K_{\chi,v}, \mathbb{A}_{\mathbf{f},v}^-)$ as $H^2(K_{\chi,v}, \mathbb{T}_{\mathbf{f},v}^+)$. Lemma 6.6 below gives an isomorphism of $\mathbb{I}_{\mathfrak{p}_f}$-modules

$$H^2(K_{\chi,v}, \mathbb{T}_{\mathbf{f},v}^+ \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}) \cong H^2(\mathbf{Q}_p, \mathbb{T}_{\mathbf{f}}^+ \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}) \cong \mathbb{I}_{\mathfrak{p}_f}/\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f},$$

for every $v|p$. Since p splits in (the at most quadratic field) K_χ, taking the χ-component of (42) gives a short exact sequence of $\mathbb{I}_{\mathfrak{p}_f}$-modules

$$0 \to X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K_\chi)^\chi \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f} \to \widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})^\chi \to \mathbb{I}_{\mathfrak{p}_f}/\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f} \to 0.$$

(Note that, if χ is nontrivial, the nontrivial automorphism of $\mathrm{Gal}(K_\chi/\mathbf{Q})$ acts by permuting the factors in the sum

$$H^2(K_{\chi,v_1}, \mathbb{T}^+_{\mathbf{f},v_1} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}) \oplus H^2(K_{\chi,v_2}, \mathbb{T}^+_{\mathbf{f},v_2} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}) =: V \oplus V,$$

where $\{v|p\} = \{v_1, v_2\}$. Then the ϵ-component of $V \oplus V$ is equal to either the subspace $\{(v,v) : v \in V\} \cong V$ if $\epsilon = 1$ or to $\{(v,-v) : v \in V\} \cong V$ if $\epsilon = \chi$.) In particular, $X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_\chi)^\chi$ is a torsion module, which is semi-simple at $\mathfrak{p}_f$ if $\widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})^\chi$ is. Moreover, if $\widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f},\mathfrak{p}_f})^\chi$ is indeed semi-simple, the preceding equation and Lemma 6.4 give

$$\begin{aligned}\mathrm{length}_{\mathfrak{p}_f}\left(X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_\chi)^\chi\right) &= \mathrm{length}_{\mathfrak{p}_f}\left(\widetilde{H}^2_f(K_\chi, \mathbb{T}_{\mathbf{f}})^\chi\right) - 1\\ &= \dim_L\left(\widetilde{H}^1_f(K_\chi, V_f)^\chi\right) - 1.\end{aligned}$$

Since $\dim_L \widetilde{H}^1_f(K_\chi, V_f)^\chi = \dim_{\mathbf{Q}_p} \mathrm{Sel}_p(A/K_\chi)^\chi + 1$ by (38), this concludes the proof of the lemma. □

Lemma 6.6. $H^2(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}) \cong \mathbb{I}_{\mathfrak{p}_f}/\mathfrak{p}_f \mathbb{I}_{\mathfrak{p}_f}$.

Proof. Write $\varpi := \varpi_{\mathrm{wt}}$. Since $\mathbb{T}^+_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}/\varpi \cong V^+_f \cong L(1)$ as $G_{\mathbf{Q}_p}$-modules (see Section 6.1.6), there are short exact sequences of L-modules

$$\begin{aligned}0 \to H^j(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f})/\varpi \to H^j(\mathbf{Q}_p, \mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} L&\\ \to H^{j+1}(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f})[\varpi] \to 0.&\quad (43)\end{aligned}$$

Taking $j = 0$ one finds $H^1(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f})[\varpi] = 0$, i.e. $H^1(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f})$ is a free $\mathbb{I}_{\mathfrak{p}_f}$-module. It is immediately seen by the explicit description of $\mathbb{T}^\pm_{\mathbf{f}}$ given in (9) that $H^0(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}}) = 0$ and $H^0(\mathbf{Q}_p, \mathbb{T}^-_{\mathbf{f}}) = 0$. Since $\mathbb{T}^-_{\mathbf{f}} \cong \mathrm{Hom}_{\mathbb{I}}(\mathbb{T}^+_{\mathbf{f}}, \mathbb{I}(1))$ (under the duality π from Section 2.2), Tate local duality tells us that $H^2(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}})$ is a torsion $\mathbb{I}$-module. Since $\mathbb{T}^+_{\mathbf{f}}$ is free of rank one over $\mathbb{I}$, Tate's formula for the local Euler characteristic now gives $\sum_{k=0}^2 (-1)^k \mathrm{rank}_{\mathbb{I}} H^j(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}}) = -1$. Together with what already proved, this allows us to conclude $H^1(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}) \cong \mathbb{I}_{\mathfrak{p}_f}$. Taking now $j = 1$ and $j = 2$ in (43) we find exact sequences

$$\begin{gathered}0 \to \mathbb{I}_{\mathfrak{p}_f}/\varpi \to H^1(\mathbf{Q}_p, \mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} L \to H^2(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f})[\varpi] \to 0;\\ H^2(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f})/\varpi \cong H^2(\mathbf{Q}_p, \mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} L.\end{gathered}$$

Since $\dim_{\mathbf{Q}_p} H^1(\mathbf{Q}_p, \mathbf{Q}_p(1)) = 2$ and $\dim_{\mathbf{Q}_p} H^2(\mathbf{Q}_p, \mathbf{Q}_p(1)) = 1$, and since $\mathbb{I}_{\mathfrak{p}_f}/\varpi \cong L$, it follows that both the ϖ-torsion $H^2(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f})[\varpi]$ and the ϖ-cotorsion $H^2(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f})/\varpi$ have dimension 1 over $L = \mathbb{I}_{\mathfrak{p}_f}/\varpi$. The structure theorem for finite torsion modules over principal ideal domains then gives $H^2(\mathbf{Q}_p, \mathbb{T}^+_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}) \cong \mathbb{I}_{\mathfrak{p}_f}/\varpi^n$ for some $n \geq 1$. To conclude the proof, it remains to show that $n = 1$, i.e. that $H^2(\mathbf{Q}_p, \mathbb{T}_{\mathbf{f}}^+ \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f})$ is semi-simple, or equivalently that the composition

$$\begin{aligned}\mathcal{H} : H^1(\mathbf{Q}_p, L(1)) \twoheadrightarrow H^2(\mathbf{Q}_p, \mathbb{T}_{\mathbf{f}}^+ \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f})[\varpi] \hookrightarrow H^2(\mathbf{Q}_p, \mathbb{T}_{\mathbf{f}}^+ \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}) \\ \twoheadrightarrow H^2(\mathbf{Q}_p, \mathbb{T}_{\mathbf{f}}^+ \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f})/\varpi \cong H^2(\mathbf{Q}_p, L(1)) \overset{\mathrm{inv}_p}{\cong} L\end{aligned}$$

is non-zero. To do this, identify $H^1(\mathbf{Q}_p, L(1)) \cong \mathbf{Q}_p^\times \widehat{\otimes} L$ via Kummer theory, and let $q \in \mathbf{Q}_p^\times$. We want to compute the image $\mathcal{H}(q) = \mathcal{H}(q \widehat{\otimes} 1) \in L$. Identify $\mathbb{T}_{\mathbf{f}}^+$ with $\mathbb{I}(\mathbf{a}_p^{*-1} \chi_{\mathrm{cy}}[\chi_{\mathrm{cy}}]^{1/2})$ (cf. Section 2.2), and write $c_q : G_{\mathbf{Q}_p} \to L(1)$ for a 1-cocycle representing $q \widehat{\otimes} 1$. Since $\mathbb{I}_{\mathfrak{p}_f}$ is a L-algebra and $\phi_f : \mathbb{I}_{\mathfrak{p}_f} \twoheadrightarrow L$ is a morphism of L-algebras, one can consider $c_q : G_{\mathbf{Q}_p} \to \mathbb{T}_{\mathbf{f}}^+ \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}$ as 1-cochain which lifts c_q under ϕ_f. The differential (in $C^\bullet_{\mathrm{cont}}(\mathbf{Q}_p, \mathbb{T}_{\mathbf{f}}^+ \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f})$) of c_q is then given by

$$\begin{aligned} dc_q(g,h) &= \mathbf{a}_p^*(g)^{-1} \cdot \chi_{\mathrm{cy}}(g) \cdot [\chi_{\mathrm{cy}}(g)]^{1/2} \cdot c_q(h) - c_q(gh) + c_q(g) \\ &= \chi_{\mathrm{cy}}(g) \cdot \Big(\mathbf{a}_p^*(g)^{-1} \cdot [\chi_{\mathrm{cy}}(g)]^{1/2} - 1\Big) \cdot c_q(h), \end{aligned}$$

where we used the cocyle relation (in $C^\bullet_{\mathrm{cont}}(\mathbf{Q}_p, L(1))$) for the second equality. Retracing the definitions given above, the class $\mathcal{H}(q)$ is then the image under inv_p of the class represented by the 2-cocycle

$$\vartheta(g,h) := \chi_{\mathrm{cy}}(g) \cdot c_q(h) \cdot \phi_f \left(\frac{\mathbf{a}_p^*(g)^{-1} \cdot [\chi_{\mathrm{cy}}]^{1/2}(g) - 1}{\varpi} \right) \in L(1). \tag{44}$$

Consider the Tate local cup-product pairing

$$\langle -, - \rangle_{\mathbf{Q}_p}^{\mathrm{Tate}} : H^1(\mathbf{Q}_p, L) \times H^1(\mathbf{Q}_p, L(1)) \to L.$$

Noting that

$$\Phi_{\mathbf{f}} := \phi_f \left(\frac{\mathbf{a}_p^{*-1} \cdot [\chi_{\mathrm{cy}}]^{1/2} - 1}{\varpi} \right) \in \mathrm{Hom}_{\mathrm{cont}}(G_{\mathbf{Q}_p}^{\mathrm{ab}}, L) = H^1(\mathbf{Q}_p, L),$$

the equality (44) can be rewritten as

$$\mathcal{H}(q) = \mathrm{inv}_p\big(\text{class of } \vartheta\big) = \langle \Phi_{\mathbf{f}}, q \rangle_{\mathbf{Q}_p}^{\mathrm{Tate}} \in L. \tag{45}$$

Let $g_0 \in I_{\mathbf{Q}_p}$ be such that $\chi_{\mathrm{cy}}(g_0)^{1/2} = \gamma_{\mathrm{wt}}$ (where $\varpi = [\gamma_{\mathrm{wt}}] - 1$), and let $g \in I_{\mathbf{Q}_p}$. Then $\kappa_{\mathrm{cy}}(g)^{1/2} = \gamma_{\mathrm{wt}}^z$ for some $z \in \mathbf{Z}_p$, satisfying $\frac{1}{2} \log_p \big(\chi_{\mathrm{cy}}(g)\big) = z \cdot \log_p(\gamma_{\mathrm{wt}})$.

(Recall that $\kappa_{\mathrm{cy}} : G_{\mathbf{Q}_p} \twoheadrightarrow 1 + p\mathbf{Z}_p$ is the composition of the p-adic cyclotomic character $\chi_{\mathrm{cy}} : G_{\mathbf{Q}_p} \twoheadrightarrow \mathbf{Z}_p^\times$ with projection to principal units.) Since $\mathbf{a}_p^*(g) = 1$ this implies

$$\begin{aligned} \Phi_{\mathbf{f}}(g) &= \phi_f\left(\frac{\mathbf{a}_p^*(g)^{-1} \cdot [\chi_{\mathrm{cy}}]^{1/2}(g) - 1}{\varpi}\right) \\ &= \phi_f\left(\frac{[\gamma_{\mathrm{wt}}^z] - 1}{\varpi}\right) = z = \frac{1}{2}\frac{\log_p\left(\chi_{\mathrm{cy}}(g)\right)}{\log_p(\gamma_{\mathrm{wt}})}. \end{aligned} \tag{46}$$

Let now $\mathrm{Frob}_p \in \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{un}}/\mathbf{Q}_p) =: G_{\mathbf{Q}_p}^{\mathrm{un}}$ be an arithmetic Frobenius, where $\mathbf{Q}_p^{\mathrm{un}}/\mathbf{Q}_p$ is the maximal unramified extension of $\mathbf{Q}_p$, and $G_{\mathbf{Q}_p}^{\mathrm{un}}$ is viewed as a subgroup of the abelianisation $G_{\mathbf{Q}_p}^{\mathrm{ab}}$ of $G_{\mathbf{Q}_p}$ under the canonical decomposition $G_{\mathbf{Q}_p}^{\mathrm{ab}} \cong \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p) \times G_{\mathbf{Q}_p}^{\mathrm{un}}$. Using the Mellin transform introduced in Section 4.1, and the well-known formula of Greenberg–Stevens [12]: $\frac{d}{dk}a_p(k)_{k=2} = -\frac{1}{2}\mathscr{L}_p(A)$, where $\mathscr{L}_p(A) := \frac{\log_p(q_A)}{\mathrm{ord}_p(q_A)}$ for the Tate period $q_A \in p\mathbf{Z}_p$ of $A/\mathbf{Q}_p$ (see the following section), one easily computes

$$\Phi_{\mathbf{f}}(\mathrm{Frob}_p^n) = \phi_f\left(\frac{\mathbf{a}_p^*(\mathrm{Frob}_p^n)^{-1} - 1}{\varpi}\right) = \frac{1}{2}\mathscr{L}_p(A) \cdot \frac{n}{\log_p(\gamma_{\mathrm{wt}})}. \tag{47}$$

Let $\mathrm{rec}_p : \mathbf{Q}_p^\times \to G_{\mathbf{Q}_p}^{\mathrm{ab}}$ be the reciprocity map of local class field theory [24]. By combining the explicit formula for rec_p given by Lubin–Tate theory with formulae (46) and (47) above yields

$$\begin{aligned} \Phi_{\mathbf{f}}\big(\mathrm{rec}_p(q)\big) &= \phi_f\left(\frac{\mathbf{a}_p^*(\mathrm{rec}_p(q))^{-1} \cdot [\chi_{\mathrm{cy}}]^{1/2}(\mathrm{rec}_p(q)) - 1}{\varpi}\right) \\ &= -\frac{1}{2}\frac{1}{\log_p(\gamma_{\mathrm{wt}})} \cdot \log_{q_A}(q) \end{aligned}$$

for every $q \in \mathbf{Q}_p^\times$, where $\log_{q_A} : \mathbf{Q}_p^\times \to \mathbf{Q}_p$ is the branch of the p-adic logarithm vanishing at the Tate period q_A. Equation (45) and another application of local class field theory then give (cf. [24])

$$\mathcal{H}(q) = \langle \Phi_{\mathbf{f}}, q\rangle_{\mathbf{Q}_p}^{\mathrm{Tate}} = \Phi_{\mathbf{f}}\big(\mathrm{rec}_p(q)\big) \doteq \log_{q_A}(q),$$

where $\doteq$ denotes equality up to a non-zero factor. This clearly proves that $\mathcal{H}$ is non-zero, hence (as explained above) that $H^2(\mathbf{Q}_p, \mathbb{T}_{\mathbf{f}}^+ \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f})$ is a semi-simple $\mathbb{I}_{\mathfrak{p}_f}$-module. This concludes the proof of the lemma. □

6.3. Algebraic exceptional zero formulae. Since $A/\mathbf{Q}_p$ has split multiplicative reduction, it is a *Tate curve* [34], [29, Chapter V], i.e. isomorphic (as a rigid analytic variety) to a Tate curve $\mathbb{G}_m/q_A^{\mathbf{Z}}$ over $\mathbf{Q}_p$, where $q_A \in p\mathbf{Z}_p$ is the so called Tate period of $A/\mathbf{Q}_p$. In particular, there exists a $G_{\mathbf{Q}_p}$-equivariant isomorphism

$$\Phi_{\mathrm{Tate}} : \overline{\mathbf{Q}}_p^{\times}/q_A^{\mathbf{Z}} \cong A(\overline{\mathbf{Q}}_p). \tag{48}$$

Write $K_{\chi,p} := K_\chi \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong \prod_{v|p} K_{\chi,v}$, and write $\iota_v : K_\chi \hookrightarrow K_{\chi,v} \subset \overline{\mathbf{Q}}_p$ for the resulting embedding of K_χ in its completion at v. Following [19] and [3], define the *extended Mordell–Weil group* of A/K_χ:

$$A^\dagger(K_\chi) := \left\{\left(P, (y_v)_{v|p}\right) \in A(K_\chi) \times K_{\chi,p}^\times : \Phi_{\mathrm{Tate}}(y_v) = \iota_v(P),\ \text{for every } v|p\right\}.$$

In concrete terms, an element of $A^\dagger(K_\chi)$ is a K_χ-rational point on A, together with a distinguished lift under Φ_{Tate} for every prime $v|p$. Then $A^\dagger(K_\chi)$ is an extension of the usual Mordell–Weil group $A(K_\chi)$ by a free $\mathbf{Z}$-module of rank $\#\{v|p\}$. In other words there is a short exact sequence

$$0 \to \bigoplus_{v|p} \mathbf{Z} \to A^\dagger(K_\chi) \to A(K_\chi) \to 0, \tag{49}$$

where the first map sends the canonical v-generator to

$$q_v := \left(0, q_A^v\right) \in A^\dagger(K_\chi), \tag{50}$$

$q_A^v \in K_{\chi,p}^\times$ being the element having q_A as v-component and 1 elsewhere. When $K_\chi/\mathbf{Q}$ is quadratic, $A^\dagger(K_\chi)$ has a natural $\mathrm{Gal}(K_\chi/\mathbf{Q})$-action, coming from the diagonal action on $A(K_\chi) \times K_{\chi,p}^\times$ (with $\mathrm{Gal}(K_\chi/\mathbf{Q})$ acting on $K_{\chi,p} := K_\chi \otimes_{\mathbf{Q}} \mathbf{Q}_p$ via its action on the first component). Recall the Kummer map $A(K_\chi)\widehat{\otimes}\mathbf{Q}_p \hookrightarrow \mathrm{Sel}_p(A/K_\chi)$ [28, Chapter X]. The following lemma is proved in [35, Section 4] (see in particular Lemma 4.1 and Lemma 4.3). For every abelian group $\mathcal{A}$, write for simplicity $\mathcal{A} \otimes L := \left(\mathcal{A}\widehat{\otimes}\mathbf{Z}_p\right) \otimes_{\mathbf{Z}_p} L$.

Lemma 6.7. *There exists a unique injective and* $\mathrm{Gal}(K_\chi/\mathbf{Q})$*-equivariant morphism of L-modules*

$$i_A^\dagger : A^\dagger(K_\chi) \otimes L \longrightarrow \widetilde{H}^1_f(K_\chi, V_f),$$

satisfying the following properties:

(i) $i_A^\dagger$ *gives rise to an injective morphism of short exact sequences of* $L[\mathrm{Gal}(K_\chi/\mathbf{Q})]$*-modules:*

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{v|p} L \arrow[r] \arrow[d, equal] & A^\dagger(K_\chi) \otimes L \arrow[r] \arrow[d, hook, "i_A^\dagger"'] & A(K_\chi) \otimes L \arrow[r] \arrow[d, hook, "\text{Kummer}"] & 0 \\
0 \arrow[r] & \bigoplus_{v|p} L \arrow[r] & \widetilde{H}^1_f(K_\chi, V_f) \arrow[r] & \mathrm{Sel}_p(A/K_\chi) \otimes_{\mathbf{Q}_p} L \arrow[r] & 0,
\end{tikzcd}$$

the bottom row being (38).

(ii) *Let* $\mathbb{P} = (P, (y_v)_{v|p}) \in A^\dagger(K_\chi)$ *be such that* $y_v \in \mathcal{O}^\times_{K_\chi,v}$ *for every* $v|p$. *Then the image of* $i^\dagger_A(\mathbb{P})$ *under the natural map* $\widetilde{H}^1_f(K_\chi, V_f) \to \bigoplus_{v|p} H^1(K_{\chi,v}, V^+_{f,v})$ *lies in the finite subspace* $\bigoplus_{v|p} H^1_f(K_{\chi,v}, V^+_{f,v})$[8].

In particular, $i^\dagger_A : A^\dagger(K_\chi) \otimes L \cong \widetilde{H}^1_f(K_\chi, V_f)$ *is an isomorphism provided that* $Ш(A/K_\chi)_{p^\infty}$ *is finite.*

We will consider from now on $A^\dagger(K_\chi)$ (or precisely $A^\dagger(K_\chi)/\text{torsion}$) as a submodule of $\widetilde{H}^1_f(K_\chi, V_f)$ via the injection $i^\dagger_A$. In particular $\langle P, Q\rangle^{\mathrm{Nek}}_{V_f,\pi} := \langle i^\dagger_A(P), i^\dagger_A(Q)\rangle^{\mathrm{Nek}}_{V_f,\pi}$ for every $P, Q \in A^\dagger(K_\chi)$.

For every $\alpha \in \mathbf{Z}_p$, let $\boldsymbol{\alpha} = (\alpha^{1/p}, \alpha^{1/p^2}, \ldots)$ be a (fixed) compatible system of p^n-th roots of α in $\overline{\mathbf{Q}}_p$. Using the Tate parametrisation (and recalling that $q_A \in p\mathbf{Z}_p$ has positive p-adic valuation), we can identify $V_p(A)$ with the $\mathbf{Q}_p$-module generated by $\mathbf{1} \in \mathbf{Z}_p(1)$ and $\boldsymbol{q_A}$. Thanks to our fixed isomorphism (36), the duality $\pi_f := \pi \otimes_{\mathbb{I}_{\mathfrak{p}_f},\phi_f} L$ induces a duality $\pi_f : V_p(A) \otimes_{\mathbf{Q}_p} V_p(A) \to \mathbf{Q}_p(1)$. Denote by $\pi_{f,\mathbf{1}} : V_p(A) \otimes_{\mathbf{Q}_p} V_p(A) \to \mathbf{Q}_p$ the composition of π_f with the isomorphism $\mathbf{Q}_p(1) \cong \mathbf{Q}_p$; $\mathbf{1} \mapsto 1$. We can then state the main result of this section.

Theorem 6.8. *Let* $\big(P, \widetilde{P}\big) \in A^\dagger(K_\chi)$, *with* $\widetilde{P} = \big(\widetilde{P}_v\big)_{v|p} \in K^\times_{\chi,p}$. *Then*

$$\big\langle q_v, \big(P, \widetilde{P}\big)\big\rangle^{\mathrm{Nek}}_{V_f,\pi} = c(\pi) \cdot \log_{q_A}\big(N_{K_{\chi,v}/\mathbf{Q}_p}\big(\widetilde{P}_v\big)\big),$$

where $\log_{q_A} : \overline{\mathbf{Q}}^\times_p \to \overline{\mathbf{Q}}_p$ *is the branch of the p-adic logarithm vanishing at* q_A, $N_{K_{\chi,v}/\mathbf{Q}_p} : K^\times_{\chi,v} \to \mathbf{Q}^\times_p$ *is the norm, and the* non-zero *constant* $c(\pi) \in \mathbf{Q}^\times_p$ *(depending on* π, *but* not *on* $(P, \widetilde{P})$*) is given by* $c(\pi) = \frac{1}{2}\pi_{f,\mathbf{1}}(\mathbf{1} \otimes \boldsymbol{q_A})$.

Proof. This is Corollary 4.6 of [35]. (In *loc. cit.* $\pi : \mathbb{T}_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{T}_{\mathbf{f}} \to \mathbb{I}(1)$ is normalised in such a way that $\pi_{f,\mathbf{1}}$ takes the value 1 on $\mathbf{1} \otimes \boldsymbol{q_A}$, so that the constant $c(\pi)$ becomes $1/2$.) For a more general statement, see also [36]. □

6.4. Proof of Theorem 6.1. Assume that $\chi(p) = 1$, i.e. that p splits in K_χ. Moreover, assume that

$$\mathrm{rank}_{\mathbf{Z}} A(K_\chi)^\chi = 1; \;\; \#\Big(Ш(A/K_\chi)^\chi_{p^\infty}\Big) < \infty, \tag{51}$$

and let $P_\chi \in A(K_\chi)^\chi$ be a generator of $A(K_\chi)^\chi$ modulo torsion. Fix a lift $P^\dagger_\chi = \big(P_\chi, (\widetilde{P}_{\chi,v})_{v|p}\big) \in A^\dagger(K_\chi)^\chi$ of P_χ under (49), and define a χ-*period*

$$q_\chi \in A^\dagger(K_\chi)^\chi$$

[8]More precisely, by the definition of Nekovář's Selmer complexes, we have a natural surjective morphism of complexes $p^+_f : \widetilde{\mathbf{R\Gamma}}_f(K_\chi, V_f) \twoheadrightarrow \bigoplus_{v|p} \mathbf{R\Gamma}_{\mathrm{cont}}(K_{\chi,v}, V^+_{f,v})$. The map referred to in the lemma is the morphism induced in cohomology by p^+_f. Moreover, we recall that the *finite (of Bloch–Kato) subspace* $H^1_f(K_{\chi,v}, -)$ is defined to be the subspace of $H^1(K_{\chi,v}, -)$ made of crystalline classes, i.e. classes with trivial image in $H^1(K_{\chi,v}, - \otimes B_{\mathrm{cris}})$ [8].

as follows. If χ is the trivial character, i.e. $K_\chi = \mathbf{Q}$, then let

$$q_\chi := (0, q_A) \in A^\dagger(\mathbf{Q}) \subset A(\mathbf{Q}) \times \mathbf{Q}_p^\times.$$

Similarly, if $K_\chi/\mathbf{Q}$ is quadratic, let

$$q_\chi := \big(0, (q_A, q_A^{-1})\big) \in A^\dagger(K_\chi)^\chi \subset A(K_\chi) \times K_{\chi,\mathfrak{p}}^\times \times K_{\chi,\overline{\mathfrak{p}}}^\times,$$

where $p\mathcal{O}_{K_\chi} = \mathfrak{p} \cdot \overline{\mathfrak{p}}$. By the exact sequence of $\mathbf{Z}[\mathrm{Gal}(K_\chi/\mathbf{Q})]$-modules (49), our assumptions, and Lemma 6.7 one has

$$\widetilde{H}^1_f(K_\chi, V_f)^\chi \overset{i_A^\dagger}{\cong} \big(A(K_\chi) \otimes L\big)^\chi = L \cdot q_\chi \oplus L \cdot P_\chi^\dagger. \tag{52}$$

Since $\langle -,- \rangle_{V_f,\pi}^{\mathrm{Nek}}$ is a skew-symmetric bilinear form, $\langle q_\chi, q_\chi \rangle_{V_f,\pi}^{\mathrm{Nek}} = 0$ and $\left\langle P_\chi^\dagger, P_\chi^\dagger \right\rangle_{V_f,\pi}^{\mathrm{Nek}} = 0$. Moreover, in case $K_\chi = \mathbf{Q}$, Theorem 6.8 gives

$$\left\langle q_\chi, P_\chi^\dagger \right\rangle_{V_f,\pi}^{\mathrm{Nek}} \doteq \log_{q_A}(\widetilde{P}_{\chi,p}) = \log_A(P_\chi),$$

where $\log_A := \log_{q_A} \circ \Phi_{\mathrm{Tate}}^{-1} : A(\mathbf{Q}_p) \cong \mathbf{Q}_p$ is the formal group logarithm on $A/\mathbf{Q}_p$, and $\doteq$ denotes equality up to multiplication by a non-zero element of $L^\times$. In case $K_\chi/\mathbf{Q}$ is quadratic, write as above $(p) = \mathfrak{p} \cdot \overline{\mathfrak{p}}$, and $\iota_\mathfrak{p} : K_\chi \subset K_{\chi,\mathfrak{p}} \cong \mathbf{Q}_p$ and $\iota_{\overline{\mathfrak{p}}} : K_\chi \subset K_{\chi,\overline{\mathfrak{p}}} \cong \mathbf{Q}_p$ for the completions of K at $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ respectively. Then $\iota_{\overline{\mathfrak{p}}} = \iota_\mathfrak{p} \circ \tau$, where τ is the non-trivial element of $\mathrm{Gal}(K_\chi/\mathbf{Q})$. Since $P_\chi^\dagger \in A^\dagger(K_\chi)^\chi$, we have $P_\chi^\tau = -P_\chi$ and $\widetilde{P}_{\chi,\mathfrak{p}} = \widetilde{P}_{\chi,\overline{\mathfrak{p}}}^{-1}$. As $q_\chi := q_\mathfrak{p} - q_{\overline{\mathfrak{p}}}$ (by the definitions), another application of Theorem 6.8 allows us to compute

$$\begin{aligned}
\left\langle q_\chi, P_\chi^\dagger \right\rangle_{V_f,\pi}^{\mathrm{Nek}} &= \left\langle q_\mathfrak{p}, P_\chi^\dagger \right\rangle_{V_f,\pi}^{\mathrm{Nek}} - \left\langle q_{\overline{\mathfrak{p}}}, P_\chi^\dagger \right\rangle_{V_f,\pi}^{\mathrm{Nek}} \doteq \log_{q_A}\big(\widetilde{P}_{\chi,\mathfrak{p}}\big) - \log_{q_A}\big(\widetilde{P}_{\chi,\overline{\mathfrak{p}}}\big) \\
&= \log_A(\iota_\mathfrak{p}(P_\chi)) - \log_A(\iota_{\overline{\mathfrak{p}}}(P_\chi)) = \log_A\big(\iota_\mathfrak{p}\big(P_\chi - P_\chi^\tau\big)\big) \\
&= 2 \cdot \log_A\big(P_\chi\big),
\end{aligned}$$

where we write again (with a slight abuse of notation)

$$\log_A : A(K_\chi) \xrightarrow{\iota_\mathfrak{p}} A(\mathbf{Q}_p) \xrightarrow{\log_A} \mathbf{Q}_p.$$

The preceding discussion can be summarised by the following formulae (valid for χ trivial or quadratic):

$$\begin{aligned}
\det \langle -,- \rangle_{V_f,\pi}^{\mathrm{Nek},\chi} &:= \det \begin{pmatrix} \langle q_\chi, q_\chi \rangle_{V_f,\pi}^{\mathrm{Nek}} & \langle q_\chi, P_\chi^\dagger \rangle_{V_f,\pi}^{\mathrm{Nek}} \\ \langle P_\chi^\dagger, q_\chi \rangle_{V_f,\pi}^{\mathrm{Nek}} & \langle P_\chi^\dagger, P_\chi^\dagger \rangle_{V_f,\pi}^{\mathrm{Nek}} \end{pmatrix} \\
&\doteq \det \begin{pmatrix} 0 & \log_A(P_\chi) \\ -\log_A(P_\chi) & 0 \end{pmatrix} \doteq \log_A^2(P_\chi)
\end{aligned}$$

(where we used again the fact that $\langle -,-\rangle^{\mathrm{Nek}}_{V_f,\pi}$ is skew-symmetric to compute $\langle P^{\dagger}_{\chi}, q_{\chi}\rangle^{\mathrm{Nek}}_{V_f,\pi} = -\langle q_{\chi}, P^{\dagger}_{\chi}\rangle^{\mathrm{Nek}}_{V_f,\pi}$, and we wrote as above $\doteq$ to denote equality up to multiplication by a non-zero element in $L^{\times}$). Since $P_{\chi} \in A(K_{\chi})$ is a point of infinite order, and $\log_A$ gives an isomorphism between $A(\mathbf{Q}_p) \otimes \mathbf{Q}_p$ and $\mathbf{Q}_p$, $\log_A(P_{\chi}) \neq 0$, so that

$$\det \langle -,-\rangle^{\mathrm{Nek},\chi}_{V_f,\pi} \neq 0.$$

Recalling that q_{χ} and $P^{\dagger}_{\chi}$ generate $\widetilde{H}^1_f(K_{\chi}, V_f)^{\chi}$ as an L-vector space by (52), this implies that $\langle -,-\rangle^{\mathrm{Nek},\chi}_{V_f,\pi}$ is non-degenerate, and the last statement of Proposition 6.2 finally gives

$$\mathrm{length}_{\mathfrak{p}_f}\left(X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_{\chi})^{\chi}\right) = \dim_{\mathbf{Q}_p}\left(\mathrm{Sel}_p(A/K_{\chi})^{\chi}\right) \overset{(37)\text{ and }(51)}{=} 1.$$

This means that $X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K_{\chi})^{\chi} \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f} \cong \mathbb{I}_{\mathfrak{p}_f}/\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f}$ as $\mathbb{I}_{\mathfrak{p}_f}$-modules, as was to be shown.

7. Proof of the main result

This section is entirely devoted to the proof of Theorem A stated in the introduction.

7.1. An auxiliary imaginary quadratic field. We will need the following crucial lemma, which follows combining the main result of [7], Nekovář's proof of the parity conjecture [20], and the KGZ Theorem.

Lemma 7.1. *Let $N_A = Np$ be the conductor of $A/\mathbf{Q}$ (with $p \nmid N$). Assume that the following properties hold:*

(a) *there exists a prime $q \neq p$ such that $q \| N_A$;*

(b) $\mathrm{rank}_{\mathbf{Z}} A(\mathbf{Q}) = 1$ *and* $Ш(A/\mathbf{Q})_{p^{\infty}}$ *is finite.*

Then there exists an imaginary quadratic field $F/\mathbf{Q}$, of discriminant D_F, satisfying the following properties:

1. *D_F is coprime to $6N_A$;*
2. *q (resp., every prime divisor of N_A/q) is inert (resp., splits) in F;*
3. $\mathrm{ord}_{s=1} L(A^F/\mathbf{Q}, s) = 1$;
4. $\mathrm{rank}_{\mathbf{Z}} A(F) = 2$ *and* $Ш(A/F)_{p^{\infty}}$ *is finite.*

(In 3: $A^F/\mathbf{Q}$ is the ϵ_F-twist of $A/\mathbf{Q}$, ϵ_F being the quadratic character of F.)

Proof. By condition (b) and Nekovář's proof of the parity conjecture [20, Section 12]

$$\mathrm{sign}(A/\mathbf{Q}) = -1$$

(where $\mathrm{sign}(A/\mathbf{Q})$ denotes the sign in the functional equation satisfied by the Hasse–Weil L-series $L(A/\mathbf{Q},s)$). Let χ be a quadratic Dirichlet character of conductor c_χ coprime with $6N_A$ such that:

(α_χ) $\chi(q) = -1$ and $\chi(\ell) = +1$ for every prime divisor ℓ of N_A/q;

(β_χ) $\chi(-1) = +1$,

and let $A^\chi/\mathbf{Q}$ be the χ-twist of $A/\mathbf{Q}$. As $q \| N_A$, we deduce by [27, Theorem 3.66] and the preceding properties

$$\mathrm{sign}(A^\chi/\mathbf{Q}) = \chi(-N_A)\cdot\mathrm{sign}(A/\mathbf{Q}) = -\chi(N_A) = +1.$$

The main result of [7] then guarantees the existence of a quadratic Dirichlet character ψ, of conductor coprime with $6c_\chi N_A$, such that

(α_ψ) $\psi(\ell) = +1$ for every prime divisor ℓ of $6c_\chi N_A$;

(β_ψ) $\psi(-1) = -1$;

(γ_ψ) $\mathrm{ord}_{s=1} L(A^{\chi\psi}/\mathbf{Q},s) = 1$.

Define $F = F_{\chi\psi}$ as the quadratic field attached to $\chi\psi$, so $\chi\psi = \epsilon_F$ and $L(A^{\chi\psi}/\mathbf{Q},s) = L(A^F/\mathbf{Q},s)$ is the Hasse–Weil L-series of the F-twist of $A/\mathbf{Q}$. In particular, property 3 in the statement is satisfied. By the KGZ theorem, it follows by (γ_ψ) that $A(F)^{\epsilon_F}$ has rank one and $Ш(A/F)^{\epsilon_F}$ is finite. Together with (b), this gives

$$\mathrm{rank}_{\mathbf{Z}} A(F) = 2;\quad \#\Big(Ш(A/F)_{p^\infty}\Big) < \infty,$$

i.e. property 4 in the statement. Property 1 is clear by construction. Moreover, by (α_χ), (β_χ), (α_ψ) and (β_ψ) we deduce $\epsilon_F(-1) = -1$, $\epsilon_F(q) = -1$ and $\epsilon_F(\ell) = +1$ for every prime divisor of N_A/q. This means precisely that $F/\mathbf{Q}$ is an imaginary quadratic field satisfying property 2 in the statement, thus concluding the proof. □

7.2. Proof of Theorem A. Assume that $A/\mathbf{Q}$ and $p > 2$ satisfy the hypotheses listed in Theorem A, i.e.

(α) $\overline{\rho}_{A,p}$ is an irreducible $G_{\mathbf{Q}}$-representation;

(β) there exists a prime $q \neq p$ at which A has multiplicative reduction (i.e. $q \| N_A$);

(γ) $p \nmid \mathrm{ord}_q(j_A)$;

(δ) $\mathrm{rank}_{\mathbf{Z}} A(\mathbf{Q}) = 1$ and $Ш(A/\mathbf{Q})_{p^\infty}$ is finite.

Let $K/\mathbf{Q}$ be a quadratic imaginary field such that

(ϵ) D_K is coprime with $6N_A$;

(ζ) q is inert in K;

(η) every prime divisor of N_A/q splits in K;

(θ) $\mathrm{rank}_{\mathbf{Z}} A(K) = 2$ and $Ш(A/K)_{p^\infty}$ is finite;

(ι) $\mathrm{ord}_{s=1} L(A^K/\mathbf{Q}, s) = 1$.

The existence of such a $K/\mathbf{Q}$ has been proved in Lemma 7.1 above. Finally, let $L/\mathbf{Q}_p$ be a finite extension containing $\mathbf{Q}_p\left(D_K^{1/2}, (-1)^{1/2}, 1^{1/Np}\right)/\mathbf{Q}_p$, let $q_K \nmid 6p$ be a rational prime which splits in K, and let S be the set of primes of K consisting of all the prime divisors of $q_K N_A D_K$. Then:

Lemma 7.2. *The data* $(\mathbf{f}, K, p, L, q_K, S)$ *satisfy Hypotheses* 1, 2 *and* 3.

Proof. By construction and properties (ϵ) and (η), Hypothesis 2 is satisfied. Since $\overline{\rho}_{\mathbf{f}}$ is isomorphic (by definition) to the semi-simplification of $\overline{\rho}_{A,p}$, assumption (α) is nothing but a reformulation of Hypothesis 1. To prove that Hypothesis 3 holds true, note that (with the notations of *loc. cit.*) $N^+ = N_A/pq$ and $N^- = q$ by (ζ) and (η) above. Then N^- is a square-free product of an odd number of primes. It thus remains to prove that $\overline{\rho}_{A,p} \cong \overline{\rho}_{\mathbf{f}}$ is ramified at q. By Tate's theory, we know that $A/\overline{\mathbf{Q}}_q$ is isomorphic to the Tate curve $\mathbb{G}_m/t_q^{\mathbf{Z}}$ over the quadratic unramified extension of $\mathbf{Q}_p$, where $t_q \in q\mathbf{Z}_q$ is the Tate period of $A/\mathbf{Q}_q$, satisfying $\mathrm{ord}_q\left(t_q\right) = -\mathrm{ord}_q(j_A)$ [34], [29, Chapter V]. Then

$$A[p] := A(\overline{\mathbf{Q}})[p] \cong \left\{t_q^{\frac{i}{p}} \cdot \zeta_p^j : 0 \leq i, j < p\right\}$$

as $I_{\mathbf{Q}_q}$-modules, where $t_q^{1/p} \in \overline{\mathbf{Q}}_q$ and $\zeta_p \in \overline{\mathbf{Q}}_q$ are fixed primitive pth roots of t_q and 1 respectively. As $\mathbf{Q}_q(\zeta_p)/\mathbf{Q}_q$ is unramified, $\overline{\rho}_{A,p}$ is ramified at q precisely if $\mathbf{Q}_q(t_q^{1/p})/\mathbf{Q}_q$ is ramified. Recalling that $t_q \in q\mathbf{Z}_q$ and $\mathrm{ord}_q(t_q) = -\mathrm{ord}_q(j_A)$, this is the case if and only if $p \nmid \mathrm{ord}_q(j_A)$. Then Hypothesis 3 follows from (γ). $\square$

In order to prove Theorem A, we need one more simple lemma. Omitting S from the notations, recall the dual Selmer groups $X_{\mathbf{Q}_\infty}^{\mathrm{cc}}(\mathbf{f}/K) := X_{\mathbf{Q}_\infty}^{S,\mathrm{cc}}(\mathbf{f}/K)$ and $X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K)$ introduced in Sections 4.3 and 6 respectively.

Lemma 7.3. $\mathrm{length}_{\mathfrak{p}_f}\left(X_{\mathbf{Q}_\infty}^{\mathrm{cc}}(\mathbf{f}/K)\right) \leq \mathrm{length}_{\mathfrak{p}_f}\left(X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K)\right) + 2.$

Proof. As remarked in the proof of Lemma 6.5, the perfect, skew-symmetric duality $\pi : \mathbb{T}_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{T}_{\mathbf{f}} \to \mathbb{I}(1)$ induces a natural isomorphism of $\mathbb{I}[G_{\mathbf{Q}_p}]$-modules: $\mathbb{T}_{\mathbf{f}}^- \otimes_{\mathbb{I}} \mathbb{I}^* \cong \mathrm{Hom}_{\mathrm{cont}}(\mathbb{T}_{\mathbf{f}}^+, \mu_{p^\infty}) =: \mathbb{A}_{\mathbf{f}}^-$. By construction and the inflation-restriction sequence, there is then an exact sequence

$$0 \to \mathrm{Sel}_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K) \to \mathrm{Sel}_{\mathbf{Q}_\infty}^{\mathrm{cc}}(\mathbf{f}/K) \to \bigoplus_{v|p} H^1\left(\mathrm{Frob}_v, \left(\mathbb{A}_{\mathbf{f}}^-\right)^{I_v}\right),$$

where $I_v := I_{K_v}$ is the inertia subgroup of G_{K_v}, $\mathrm{Frob}_v \in G_{K_v}/I_{K_v}$ is the arithmetic Frobenius at v, and we write for simplicity $H^*(\mathrm{Frob}_v, -) := H^*(G_{K_v}/I_{K_v}, -)$.

(Here the reference to the fixed set S is again omitted, so that $\mathrm{Sel}^{\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K) := \mathrm{Sel}^{S,\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K)$.) Taking Pontrjagin duals and then localising at $\mathfrak{p}_f$ gives an exact sequence of $\mathbb{I}_{\mathfrak{p}_f}$-modules:

$$\bigoplus_{v|p} H^1\left(\mathrm{Frob}_v, (\mathbb{A}_{\mathbf{f}}^-)^{I_v}\right)^*_{\mathfrak{p}_f} \to X^{\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K)_{\mathfrak{p}_f} \to X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K)_{\mathfrak{p}_f} \to 0,$$

where $(-)^*_{\mathfrak{p}_f}$ is an abbreviation for $((-)^*)_{\mathfrak{p}_f} = (-)^* \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{p}_f}$. As p splits in K, one deduces

$$\begin{aligned}\mathrm{length}_{\mathfrak{p}_f}&\left(X^{\mathrm{cc}}_{\mathbf{Q}_\infty}(\mathbf{f}/K)\right)\\ &\le \mathrm{length}_{\mathfrak{p}_f}\left(X^{\mathrm{cc}}_{\mathrm{Gr}}(\mathbf{f}/K)\right) + 2\cdot \mathrm{length}_{\mathfrak{p}_f}\left(H^1\left(\mathrm{Frob}_p, (\mathbb{A}_{\mathbf{f}}^-)^{I_p}\right)^*\right), \end{aligned} \tag{53}$$

where $I_p := I_{\mathbf{Q}_p} \subset G_{\mathbf{Q}_p}$ is the inertia subgroup and $\mathrm{Frob}_p \in G_{\mathbf{Q}_p}/I_{\mathbf{Q}_p}$ is the arithmetic Frobenius at p.

By equation (9), $\mathbb{T}_{\mathbf{f}}^+ \cong \mathbb{I}\left((\mathbf{a}_p^*)^{-1}\cdot \chi_{\mathrm{cy}} \cdot [\chi_{\mathrm{cy}}]^{1/2}\right)$ as $G_{\mathbf{Q}_p}$-modules. Then its Kummer dual $\mathbb{A}_{\mathbf{f}}^-$ is isomorphic to $\mathbb{I}^*\left(\mathbf{a}_p^* \cdot [\chi_{\mathrm{cy}}]^{-1/2}\right)$. Let $\gamma \in 1 + p\mathbf{Z}_p$ be a topological generator, let $[\gamma] \in \mathbb{I}$ be its image under the structural morphism $[\cdot] : \Lambda \to \mathbb{I}$, and let $\varpi = [\gamma] - 1 \in \Lambda$. Since $\mathbf{a}_p^*$ is an unramified character and $[\rho] \equiv 1 \bmod \varpi$ for every $\rho \in 1 + p\mathbf{Z}_p$, one has isomorphisms of Frob_p-modules

$$H^0(I_p, \mathbb{A}_{\mathbf{f}}^-) = \mathbb{A}_{\mathbf{f}}^-[\varpi] \cong (\mathbb{I}/\varpi\mathbb{I})^*(\mathbf{a}_p^*). \tag{54}$$

Applying $H^1(\mathrm{Frob}_p, -)$ to (54) then yields

$$H^1\left(\mathrm{Frob}_p, (\mathbb{A}_{\mathbf{f}}^-)^{I_p}\right) = \left(\frac{\mathbb{I}}{\varpi\cdot\mathbb{I}}\right)^* / (\mathbf{a}_p - 1)\left(\frac{\mathbb{I}}{\varpi\cdot\mathbb{I}}\right)^*.$$

Taking the Pontrjagin duals and then localising at $\mathfrak{p}_f$ one deduces

$$\begin{aligned} H^1\left(\mathrm{Frob}_p, (\mathbb{A}_{\mathbf{f}}^-)^{I_p}\right)^*_{\mathfrak{p}_f} &\cong \left(\left(\frac{\mathbb{I}}{\varpi\cdot\mathbb{I}}\right)^{**}[\mathbf{a}_p - 1]\right)_{\mathfrak{p}_f} \\ &\cong \left(\frac{\mathbb{I}_{\mathfrak{p}_f}}{\varpi\cdot\mathbb{I}_{\mathfrak{p}_f}}\right)[\phi_f(\mathbf{a}_p) - 1] = \mathbb{I}_{\mathfrak{p}_f}/\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f}. \end{aligned} \tag{55}$$

Indeed, as remarked in (17), ϖ is a uniformiser of $\mathbb{I}_{\mathfrak{p}_f}$. Moreover, $\mathfrak{p}_f := \ker(\phi_f)$ and $\phi_f(\mathbf{a}_p) = a_p(2) = +1$ (as $A/\mathbf{Q}_p$ is split multiplicative), so that $\mathbf{a}_p - 1$ acts trivially on $\mathbb{I}_{\mathfrak{p}_f}/\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f}$ and (55) follows. In particular, (55) yields

$$\mathrm{length}_{\mathfrak{p}_f}\left(H^1\left(\mathrm{Frob}_p, (\mathbb{A}_{\mathbf{f}}^-)^{I_p}\right)^*\right) = 1.$$

Together with equation (53), this concludes the proof of the lemma. □

We can finally conclude the proof of Theorem A. To be short, we have

$$\begin{aligned} 4 \overset{\text{Cor. 5.2}}{\leq} \operatorname{ord}_{k=2} L_p^{\mathrm{cc}}(f_\infty/K, k) &\overset{\text{Cor. 4.2}}{\leq} \operatorname{length}_{\mathfrak{p}_f}\left(X_{\mathbf{Q}_\infty}^{\mathrm{cc}}(\mathbf{f}/K)\right) \\ &\overset{\text{Lemma 7.3}}{\leq} \operatorname{length}_{\mathfrak{p}_f}\left(X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K)\right) + 2 \overset{\text{Th. 6.1}}{=} 4. \end{aligned} \tag{56}$$

Indeed, hypothesis (δ) gives $\dim_{\mathbf{Q}_p} \mathrm{Sel}_p(A/\mathbf{Q}) = 1$, and then (as in the proof of Lemma 7.1) Nekovář's proof of the parity conjecture guarantees that $\mathrm{sign}(A/\mathbf{Q}) = -1$. Together with Lemma 7.2, this implies that the hypotheses of Corollary 5.2 are satisfied, and then that the first inequality in (56) holds true. Lemma 7.2 also allows us to apply Skinner–Urban's Corollary 4.2, which gives the second inequality in (56). The third inequality in (56) is the content of the preceding lemma. Finally, let χ denote either the trivial character or the quadratic character ϵ_K of K, and let $K_\chi := \mathbf{Q}$ or $K_\chi := K$ accordingly. Then (δ) and (θ) imply that (with the notations of Section 6)

$$\operatorname{rank}_{\mathbf{Z}} A(K_\chi)^\chi = 1; \quad \#\left(\text{Ш}(A/K_\chi)_{p^\infty}^\chi\right) < \infty.$$

Moreover, we know that p splits in K_χ (i.e. in K, by hypothesis (η)). Then the hypotheses (i), (ii) and (iii) of Theorem 6.1 are satisfied by both our χ's, and by applying the theorem twice yields

$$X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K)_{\mathfrak{p}_f} \cong X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/\mathbf{Q})_{\mathfrak{p}_f} \oplus X_{\mathrm{Gr}}^{\mathrm{cc}}(\mathbf{f}/K)_{\mathfrak{p}_f}^{\epsilon_K} \cong \mathbb{I}_{\mathfrak{p}_f}/\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f} \oplus \mathbb{I}_{\mathfrak{p}_f}/\mathfrak{p}_f\mathbb{I}_{\mathfrak{p}_f}\,[9],$$

justifying the last equality in (56).

Equation (56) proves that $\operatorname{ord}_{k=2} L_p^{\mathrm{cc}}(f_\infty/K, k) = 4$. It then follows by Bertolini–Darmon's Corollary 5.2 that the Hasse–Weil L-function of A/K has a double zero at $s = 1$:

$$\operatorname{ord}_{s=1} L(A/K, s) = 2.$$

Since $L(A/K, s) = L(A/\mathbf{Q}, s) \cdot L(A^K/\mathbf{Q}, s)$ is the product of the Hasse–Weil L-functions of $A/\mathbf{Q}$ and its K-twist $A^K/\mathbf{Q}$, and since $L(A^K/\mathbf{Q}, s)$ has a simple zero at $s = 1$ by (ι) above, we finally deduce

$$\operatorname{ord}_{s=1} L(A/\mathbf{Q}, s) = 1.$$

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