

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 91 (2016)
Heft: 2

Artikel: Bounded cohomology with coefficients in uniformly convex Banach spaces
Autor: Bestvina, Mladen / Bromberg, Ken / Fujiwara, Koji
DOI: <https://doi.org/10.5169/seals-630557>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 20.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Bounded cohomology with coefficients in uniformly convex Banach spaces

Mladen Bestvina*, Ken Bromberg* and Koji Fujiwara**

Abstract. We show that for acylindrically hyperbolic groups Γ (with no nontrivial finite normal subgroups) and arbitrary unitary representation ρ of Γ in a (nonzero) uniformly convex Banach space the vector space $H_b^2(\Gamma; \rho)$ is infinite dimensional. The result was known for the regular representations on $\ell^p(\Gamma)$ with $1 < p < \infty$ by a different argument. But our result is new even for a non-abelian free group in this great generality for representations, and also the case for acylindrically hyperbolic groups follows as an application.

Mathematics Subject Classification (2010). 20F65; 46B99.

Keywords. Uniformly convex Banach space, second bounded cohomology, acylindrically hyperbolic groups.

1. Introduction

1.1. Quasi-cocycle and quasi-action. Let G be a group and E a normed vector space (usually complete, either over \mathbb{R} or over \mathbb{C}). The linear or rotational part of an isometric G -action on E determines a representation $\rho : G \rightarrow O(E)$ where $O(E)$ is the group of norm-preserving linear isomorphisms $E \rightarrow E$. We will refer to ρ as a *unitary representation*. We will usually write $\rho(g)x$ as $g(x)$ or gx .

The translational part of the G -action is a *cocycle* (with respect to ρ). Namely the translational part is a function $F : G \rightarrow E$ that satisfies

$$F(gg') = F(g) + gF(g') \quad (1.1)$$

for all $g, g' \in G$. Going in the other direction, if ρ is a unitary representation and F a cocycle then the map $g \mapsto (x \mapsto \rho(g)x + F(g))$ determines an (affine) isometric G -action on E . Note that $F(g^{-1}) = -g^{-1}F(g)$. $\rho(g)$ is sometimes called the linear part of the action.

*The first two authors gratefully acknowledge the support by the National Science Foundation.

**The third author is supported in part by Grant-in-Aid for Scientific Research (No. 23244005, 15H05739).

For an isometric quasi-action of G on E the linear part will still be a unitary representation. However, the translational part F will become a *quasi-cocycle* and will only satisfy (1.1) up to a uniformly bounded error so that

$$\Delta(F) := \sup_{g, g' \in G} |F(gg') - F(g) - gF(g')| < \infty. \quad (1.2)$$

The quantity $\Delta(F)$ is the *defect* of the quasi-cocycle.

A basic question is if there are quasi-actions that are not boundedly close to an actual action. Such a quasi-action is *essential*. Since quasi-actions determine unitary representations a more refined question is if there are essential quasi-actions for a given unitary representation.

The above discussion is perhaps more familiar in its algebraic form where it can be rephrased in terms of bounded cohomology. A quasi-cocycle F can be viewed as 1-cochain in the group cohomology twisted by the representation ρ . Condition (1.2), is equivalent to the coboundary δF being a bounded 2-cocycle and will therefore determine a cohomology class in $H_b^2(G; \rho)$, the second bounded cohomology group. Now this cocycle will clearly be trivial in the regular second cohomology group $H^2(G; \rho)$ as it is the coboundary of a 1-cochain. If the cochain F is a bounded distance from a cocycle then δF will also be trivial in $H_b^2(G; \rho)$ so we are interested in the kernel of the map

$$H_b^2(G; \rho) \rightarrow H^2(G; \rho)$$

from bounded cohomology to regular cohomology. In particular this kernel is the vector space $QC(G; \rho)$ of all quasi-cocycles modulo the subspace generated by bounded functions and cocycles. We denote this quotient space $\widetilde{QC}(G; \rho)$. This is the vector space of *essential* quasi-cocycles and it is the main object of study of this paper.

For the trivial representation on \mathbb{R} a cocycle is just a homomorphism to \mathbb{R} and a quasi-cocycle is usually called a quasi-morphism. When $G = F_2$, the free group on two generators, Brooks [7] gave a combinatorial construction of an infinite dimensional family of essential quasi-morphisms.

1.2. Uniformly convex Banach space and main result. Following the work of Brooks, there is a long history of generalizations of this construction to other groups. Initially, the work focused on the trivial representation. See [4, 5, 12]. This was followed by generalizations to the same groups G but with coefficients in the regular representation $\ell^p(G)$, $1 \leq p < \infty$. See [14, 16].

In this paper we will extend this work to unitary representations in *uniformly convex* Banach spaces. Note that this essentially includes the previous cases since $\ell^p(G)$ is uniformly convex when $1 < p < \infty$.

If one is a bit more careful about how the counting is done then Brooks construction of quasi-morphisms can also be used to produce quasi-cocycles. In

Brooks' original work (i.e., for trivial representations) it is easy to see that the quasi-morphisms are essential. Here we will have to work harder to get the following result.

Theorem 1.1 (Theorem 3.9). *Let ρ be a unitary representation of F_2 on a uniformly convex Banach space $E \neq 0$. Then $\dim \widetilde{QC}(F_2; \rho) = \infty$.*

To show $\widetilde{QC}(F_2; \rho)$ is non-trivial is already hard. We will argue that for a certain Brooks' quasi-cocycle H into a Banach space E , there exists a sequence of elements in F_2 on which H is unbounded. For that we use that E is uniformly convex in an essential way (Lemma 3.4). We also show those quasi-cocycles are not at bounded distance from any cocycle using that E is reflexive (using Lemma 3.6). Those two steps are the novel part of the paper. It seems that the uniform convexity is nearly a necessary assumption for the conclusion. See the examples at the end of this section.

Recently Osin [20] (see also [11]) has identified the class of *acylindrically hyperbolic groups* and this seems to be the most general context where the Brooks' construction can be applied. Osin has shown that acylindrically hyperbolic groups contain *hyperbolically embedded* copies of F_2 and then applying work of Hull–Osin [17] we have the the following corollary to Theorem 3.9. See Section 4 for the proof.

Corollary 1.2. *Let ρ be a unitary representation of an acylindrically hyperbolic group G on a uniformly convex Banach space $E \neq 0$ and assume that the maximal finite normal subgroup has a non-zero fixed vector. Then $\dim \widetilde{QC}(G; \rho) = \infty$.*

A wide variety of groups are acylindrically hyperbolic. In particular our results apply to the following examples. To apply our result, in all examples assume G has no nontrivial finite normal subgroups, or more generally that for the maximal finite normal subgroup N (see [11]) we have that $\rho(N)$ fixes a nonzero vector in E .

Examples 1.3 (Acylindrically hyperbolic groups).

- G is non-elementary word hyperbolic,
- G admits a non-elementary isometric action on a connected δ -hyperbolic space such that at least one element is hyperbolic and WPD,
- $G = \text{Mod}(S)$, the mapping class group of a compact surface which is not virtually abelian,
- $G = \text{Out}(F_n)$ for $n \geq 2$,
- G admits a non-elementary isometric action on a $CAT(0)$ space and at least one element is WPD and acts as a rank 1 isometry.

Remark 1.4. Recall that a Banach space is *superreflexive* if it admits an equivalent uniformly convex norm. It is observed in [1, Proposition 2.3] that if $\rho : G \rightarrow E$ is a unitary representation with E superreflexive, then there is an equivalent uniformly convex norm with respect to which ρ is still unitary. Thus in Corollary 1.2 we may replace “uniformly convex” with “superreflexive”.

Remark 1.5. There is also a more direct approach to going from Theorem 3.9 to our the main theorem. The key point is that any group G covered in the the main theorem acts on a quasi-tree such that there is a free group $F \subset G$ that acts properly and co-compactly on a tree isometrically embedded in the quasi-tree. This is done using the *projection complex* of [2]. Using this one can apply the Brooks' construction to produce quasi-cocycles that when restricted to the free group are exactly the quasi-cocycles of Theorem 3.9. We carry this out in a separate paper [3].

1.3. Known examples with certain Banach spaces. Here are some known vanishing/non-vanishing examples in the literature.

- $E = \mathbb{R}$ and ρ is trivial. In this case $H_b^2(G; \rho)$ is the usual bounded cohomology and quasi-cocycles are quasi-morphisms. As we said this case was known for various kinds of groups.
- $E = \ell^p(G)$ and ρ is the regular representation, see [13, 15]. When $1 < p < \infty$, $\ell^p(G)$ is uniformly convex and our theorem applies. When $p = 1$ or $p = \infty$ then $\ell^p(G)$ is not uniformly, or even strictly, convex. However, for $p = 1$ summation determines a ρ -invariant functional and one can produce a family of quasi-cocycles that when composed with the invariant functional are an infinite dimensional family of non-trivial quasi-morphisms in $\widetilde{QH}(G)$ implying that $\dim \widetilde{QC}(G; \ell^1(G)) = \infty$.

On the other hand,

- When $p = \infty$ given any quasi-cocycle one can explicitly find a cocycle a bounded distance away so $\widetilde{QC}(G; \ell^\infty(G)) = 0$ for any group G .
- If G is countable and exact (e.g., F_2), then $H_b^2(G; \ell_0^\infty(G)) = 0$. In particular, $\widetilde{QC}(G; \ell_0^\infty(G)) = 0$ (Example 3.10). Here $\ell_0^\infty(G)$ is the subspace of $\ell^\infty(G)$ consisting of sequences which are asymptotically 0.

There are also examples where G is not acylindrically hyperbolic but where $\widetilde{QC}(G; \rho)$ is known to be non-zero for certain actions of G on ℓ^p spaces.

- If G has a non-elementary action on a $CAT(0)$ cube complex then $\widetilde{QC}(G; \rho) \neq 0$ where ρ is the representation of G on the space of ℓ^p -functions ($1 \leq p < \infty$) on a certain space where G naturally acts [8]. Note that this class of groups is closed under products so it contains groups that aren't acylindrically hyperbolic.

There are other examples where essentially nothing is known.

- $E = \ell_0^1(G) \subset \ell^1(G)$ is the space of ℓ^1 -functions on G that sum to zero and ρ is the regular representation. Unlike with $\ell^1(G)$, $\ell_0^1(G)$ has no ρ -invariant functionals.
- $E = \mathcal{B}(\ell^2(G))$ the space of bounded linear maps of $\ell^2(G)$ to itself. This example was suggested to us by N. Monod as the non-commutative analogue to $\ell^\infty(G)$.

Acknowledgements. We thank the referee for carefully reading the paper.

2. Quasi-cocycles from trees

Fix $F_2 = \langle a, b \rangle$ and choose a word $w \in F_2$. For simplicity we will assume that w is cyclically reduced. Let E be a normed vector space and $\rho : G \rightarrow O(E)$ a linear representation. Also choose a nonzero $e \in E$. We now set up some notation that will be convenient for what we will do later.

Let $[g, h]$ be an oriented segment in the Cayley graph for F_2 with generators a and b . Then we write $[g, h] \overset{\circ}{\subset} [g', h']$ if $[g, h]$ is a subsegment of $[g', h']$ and the orientations of the two segments agree. We then define

$$w_+(g) = \{h \in G \mid [h, hw] \overset{\circ}{\subset} [1, g]\}$$

and
$$w_-(g) = \{h \in H \mid [h, hw] \overset{\circ}{\subset} [g, 1]\}.$$

Now define a function $H = H_{w,e} : F_2 \rightarrow E$ by

$$H(g) = \sum_{h \in w_+(g)} h(e) - \sum_{h \in w_-(g)} h(e)$$

In other words, to a translate $h \cdot w$ we assign $h(e)$ when traversed in the positive direction, and $-h(e)$ when traversed in negative direction. Note that it follows that $H(g^{-1}) = -g^{-1}H(g)$.

Proposition 2.1. *The function H constructed above is a quasi-cocycle.*

Proof. This is the standard Brooks argument. Consider the tripod spanned by $1, g, gf$. Call the central point p . We will see that contributions of copies of w in the tripod that do not cross p cancel out leaving only a bounded number of terms.

If $h \cdot w \overset{\circ}{\subset} [1, p]$ then $h(e)$ enters with positive sign in $H(g)$ and in $H(gf)$, so it cancels in the expression $H(gf) - H(g)$. Likewise, if $h \cdot w \overset{\circ}{\subset} [p, 1]$ then $-h(e)$ enters both $H(g)$ and $H(gf)$, so it again cancels.

If $h \cdot w \overset{\circ}{\subset} [p, g]$ then $h(e)$ is a summand in $H(g)$. Since $h \cdot w \overset{\circ}{\subset} [gf, g]$ we also have $g^{-1}h \cdot w \overset{\circ}{\subset} [f, 1]$, so $-g^{-1}h(e)$ is a summand in $H(f)$, and thus we have cancellation in $-H(g) - gH(f)$. There is similar cancellation if $h \cdot w \overset{\circ}{\subset} [g, p]$.

If $h \cdot w \overset{\circ}{\subset} [p, gf]$ or $[gf, p]$ then similarly to the previous paragraph there is cancellation in $H(gf) - gH(f)$.

After the above cancellations in the expression $H(gf) - H(g) - gH(f)$ the only terms left are of the form $\pm h(e)$ where $h(w)$ is contained in the tripod and contains p in its interior. The number of such terms is clearly (generously) bounded by $6|w|$ so we deduce that $\Delta(H) \leq 6|w||e|$. \square

Remark 2.2. Note that if $h \cdot w$ does not overlap w for any $1 \neq h \in F_2$, then $\Delta(H) \leq 6\|e\|$. More generally, for a given w , write $w = u^n v$ as a word such that $|v| < |u|$ and $n > 0$ is maximal. Then, $\Delta(H) \leq 6(n+1)\|e\|$.

Example 2.3. Suppose $w = ab$. Then $H(a^n) = H(b^n) = 0$, while $H((ab)^n) = (1 + ab + (ab)^2 + \cdots + (ab)^{n-1})e \in E$. If the operator $1 - ab : E \rightarrow E$ has a continuous inverse (i.e. if $1 \in \mathbb{C}$ is not in the spectrum of ab) then H is uniformly bounded on the powers of ab since $(1 - ab)H((ab)^n) = e - (ab)^n(e)$ has bounded norm. For example, this happens even for $E = \mathbb{R}^2$ when $\rho(ab)$ is a (proper) rotation.

On the other hand, for the representation $\ell^p(F_2)$ with $1 \leq p < \infty$ and with $e \in \ell^p(F_2)$ defined by $e(1) = 1$, $e(g) = 0$ for $g \neq 1$, the quasi-cocycle H is unbounded on the powers of ab .

3. Nontriviality of quasi-cocycles

In Brooks' original construction of quasi-morphisms $F_2 = \langle a, b \rangle \rightarrow \mathbb{R}$ it is easy to see that the quasi-morphisms are nontrivial. Choosing w to be a reduced word not of the form a^m or b^m it is clear that $H(w^n)$ will be unbounded while $H(a^n)$ and $H(b^n)$ will be zero. By this last fact if G is a homomorphism that is boundedly close to H then G must be bounded on powers of a and b and therefore $G(a) = G(b) = 0$. Since any homomorphism is determined by its behavior on the generators we have $G \equiv 0$ and the nontriviality of H follows.

When the Brooks construction is extended to quasi-cocycles it is no longer clear that the quasi-cocycle is nontrivial. In particular if $H = H_{w,e}$ it may be that $H(w^n)$ is bounded. See Examples 2.3 and 3.5. In fact if 1 is not in the spectrum of $\rho(w)$ then $H(w^n)$ will be bounded for all choices of vectors e . Even if 1 is in the spectrum, when e is chosen arbitrarily $H(w^n)$ may be bounded. To show that the Brooks quasi-cocycles are unbounded we will need to restrict to the class of *uniformly convex* Banach spaces and to look at a wider class of words than powers of w .

We will also have to work harder to show that a cocycle G that is bounded on powers of the generators is bounded everywhere. In fact we cannot do this in general but instead will show that in a reflexive Banach space (which includes uniformly convex Banach spaces) either the cocycle is bounded or the original representation, when restricted to a non-abelian subgroup, has an eigenvector. In this latter case it is easy to construct many nontrivial quasi-cocycles.

3.1. Uniformly convex and reflexive Banach spaces. We will use basic facts about Banach spaces. General references are [6, 18]. The following concept was introduced by Clarkson [10].

Definition 3.1. A Banach space E is *uniformly convex* if for every $\epsilon > 0$ there is $\delta > 0$ such that $x, y \in E$, $|x| \leq 1$, $|y| \leq 1$, $|x - y| \geq \epsilon$ implies $|\frac{x+y}{2}| \leq 1 - \delta$.

The original definition in [10] replaces $|x|, |y| \leq 1$ above with equalities, but it is not hard to see that the two are equivalent.

Proposition 3.2. (i) ℓ^p spaces are uniformly convex for $1 < p < \infty$ [10]. ℓ^1 and ℓ^∞ spaces are not uniformly convex and not reflexive.

(ii) A uniformly convex Banach space is reflexive (the Milman–Pettis theorem).

(iii) If E is uniformly convex, then for any $R > 0$ there are $\epsilon > 0$ and $\mu > 0$ so that the following holds. If $|v| \leq R$ and $f : E \rightarrow \mathbb{R}$ is a functional of norm 1 with $f(v) = |v|$ and if e is a vector of norm $\geq 1/2$ with $f(e) \geq -\mu$ then $|v + e| \geq |v| + \epsilon$.

Proof. We only prove (iii). Choose $\delta \in (0, 1)$ so that $|x|, |y| \leq 1, |x - y| \geq \frac{1}{2(R+1)}$ implies $|\frac{x+y}{2}| \leq 1 - \delta$. Then choose $\epsilon, \mu > 0$ so that $\epsilon < \frac{1}{8}$ and $\frac{\frac{1}{8} - \frac{\mu}{2}}{\frac{1}{8} + \epsilon} > 1 - \delta$. Suppose f, v, e satisfy the assumptions but $|v + e| < R + \epsilon$. If $|v| \leq 1/8$ then $|v + e| \geq |e| - |v| \geq 1/4 \geq |v| + 1/8$ and we are done. So assume that $|v| > 1/8$. Then for $x = \frac{v}{|v| + \epsilon}, y = \frac{v+e}{|v| + \epsilon}$ we have $|x|, |y| \leq 1$ and $|x - y| \geq \frac{1}{2(|v| + 1)} \geq \frac{1}{2(R+1)}$, so we must have $|\frac{x+y}{2}| \leq 1 - \delta$. Thus

$$1 - \delta \geq \left| \frac{x + y}{2} \right| = \left| \frac{v + e/2}{|v| + \epsilon} \right| \geq \frac{|v| - \frac{\mu}{2}}{|v| + \epsilon} \geq \frac{\frac{1}{8} - \frac{\mu}{2}}{\frac{1}{8} + \epsilon}$$

since $f(v + e/2) = |v| + f(e)/2 \geq |v| - \frac{\mu}{2}$ and $|f| = 1$. This contradicts the choice of μ, ϵ . \square

Lemma 3.3. Let ρ be a unitary representation of a group F on a reflexive Banach space E . If there is a linear functional f and a vector $e \in E$ such that the F -orbit of e lies in the half space $\{f \geq \mu\}$ with $\mu > 0$ then there is an F -invariant vector $e' \neq 0 \in E$ and an F -invariant functional ϕ with $\phi(e') \geq \mu$. If e is F -invariant, then we can take $e' = e$.

Proof. Let Λ be the convex hull of the F -orbit of e in the weak topology on E . Since E is reflexive, Λ is weakly compact. The convex hull Λ is also F -invariant so by the Ryll–Nardzewski fixed point theorem it will contain an F -invariant vector e' . Since $e' \in \Lambda$, $f(e') \geq \mu$ and therefore $e' \neq 0$.

Since e' is a functional on the reflexive Banach space E^* and the F -orbit of f will be contained in the half space $\{e' \geq \mu\}$ we similarly get a F -invariant vector $\phi \in E^*$ with $e'(\phi) = \phi(e') \geq \mu$. \square

Note that if E contains a nonzero vector that is F -invariant, then the Hahn–Banach theorem supplies a functional that satisfies the conditions of the lemma and so there is also a nonzero F -invariant functional.

3.2. Detecting unboundedness.

Lemma 3.4. *Let ρ be any unitary representation of $F_2 = \langle a, b \rangle$ into a uniformly convex Banach space E . Then one of the following holds:*

- (i) *for every $e \neq 0 \in E$ and any $1 \neq w \in F_2$ not of the form $a^m b^n$ nor $b^m a^n$ the quasi-cocycle $H = H_{w,e}$ is unbounded on F_2 , or*
- (ii) *there is a free subgroup $F \subset F_2$ with $F \cong F_2$, a linear functional g , a vector e and a $\mu > 0$ such that the F -orbit of e is contained in the half-space $\{g \leq -\mu\}$. In particular, there is an F -invariant vector $e' \neq 0$ in the half space.*

Proof. We first make some observations about words in F_2 . Given a word w as in (i) we can find buffer words B and B' of the form $a^\ell b^\ell$ or $b^\ell a^\ell$ and a subgroup $F = \langle a^m, b^m \rangle$ with $m \gg \ell, |w|$ such that if $w' = BwB'$ and $y_1, y_2, \dots, y_n \in F$ then in the reduced word for the element $x = y_1 w' y_2 w' \cdots y_n w'$ there is exactly one copy of w for each w' and no other copies of either w or w^{-1} . Note that the word $y_1 w' y_2 w' \cdots y_n w'$ may not be reduced and in its reduced version there may be cancellations in the w' . However, the buffer words will prevent these cancellations from reaching w . The restrictions on w ensure that w does not appear as a subword of some y_i . In particular, $|H(w')| = |e|$ and $H(xyw') = H(x) + xH(yw') = H(x) + xyH(w')$ for any $y \in F$.

For simplicity, normalize so that $|e| = 1$, so $|H(w')| = 1$. Assume that (ii) doesn't hold, and that H is bounded on F_2 . Let F_w be the set of words of the form

$$y_1 w' y_2 w' \cdots y_n w', (y_i \in F)$$

and let $R = \sup_{x \in F_w} |H(x)| < \infty$. Let $\epsilon, \mu > 0$ be as in Proposition 3.2(iii). Choose an $x \in F_w$ such that $|H(x)| > R - \epsilon$. We will find a $y \in F$ such that $|H(xyw')| > R$ to obtain a contradiction since $xyw' \in F_w$.

Let ϕ be a linear functional of norm 1 such that $\phi(H(x)) = |H(x)|$. Let $\psi = \phi \circ x$. Since (ii) doesn't hold, there exists a $y \in F$ with $\psi(yH(w')) > -\mu$. (We are applying the negation of (ii) not to e but to $H(w')$, which is in the F_2 -orbit of e , but it is easy to see that this follows from the corresponding fact for e by replacing F with a conjugate.) So, $\phi(xyH(w')) > -\mu$. Then by Proposition 3.2(iii), $|H(xyw')| = |H(x) + xyH(w')| \geq |H(x)| + \epsilon > R$, contradiction.

For an F -invariant vector in (ii), see the proof of Lemma 3.3. \square

We give an application of Lemma 3.4.

Example 3.5. Choose an embedding $\rho : F_2 \subset U(2)$ so that every nontrivial element is conjugate to a matrix of the form

$$\begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix}$$

with $t, s, \frac{t}{s}$ all irrational.

(Such representations can be constructed by noting that they form the complement of countably many proper subvarieties in $\text{Hom}(F_2, U(2))$.) Put $E = \mathbb{C}^2$.

Then any $H = H_{w,e}$ with $0 \neq e \in E, 1 \neq w \in F_2$ is bounded on any cyclic subgroup, but many are globally unbounded. The second statement follows by noting that the orbit of any unit vector under a nontrivial cyclic subgroup is dense in a torus $S^1 \times S^1 \subset \mathbb{C}^2$, so (ii) of Lemma 3.4 fails, and (i) must hold. For the first statement, observe that for a fixed $g \in F$ the values $H(g^n)$ can be computed, up to a bounded error, by adding sums of the form

$$U_n = u(e) + gu(e) + \cdots + g^{n-1}u(e)$$

one for every g -orbit of occurrences of w or w^{-1} along the axis of g . Applying g we have

$$g(U_n) = gu(e) + \cdots + g^n u(e)$$

and so $|g(U_n) - U_n| \leq 2|e|$, which implies that $|U_n|$ is bounded, since $g : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ moves every unit vector a definite amount. It follows $H(g^n)$ is bounded on n . This gives an isometric quasi-action of F_2 on \mathbb{C}^2 or \mathbb{R}^4 with unbounded orbits, but with every cyclic subgroup having bounded orbits.

In fact, since $H^1(F_2; \rho) \neq 0$, it follows that there are *isometric* actions of F_2 on \mathbb{R}^4 with unbounded orbits and with every element fixing a point.

The following is our basic method of detecting bounded cocycles. In the presence of reflexivity of the Banach space, bounded isometric actions have fixed points. Thus a cocycle $G : F_2 \rightarrow E$ is bounded if and only if for some $v \in E$ (a fixed point of the action) we have $G(g) = v - \rho(g)v$ for every $g \in F_2$.

Lemma 3.6. *Let ρ be a unitary representation of F_2 on a reflexive Banach space E and G a cocycle that is bounded on $\langle a^2, b \rangle$ and $\langle a^3, b \rangle$. Then one of the following holds.*

- (i) G is bounded on F_2 , or
- (ii) There is a free subgroup $F \subset F_2$ with $F \cong F_2$ such that $\rho|_F$ fixes a nonzero vector in E .

Proof. The cocycle G induces an action of F_2 on E by affine isometries and the image of G is the orbit of 0 under this action. If the restriction of this action to $\langle a^2, b \rangle$ is bounded (with respect to the norm topology) then the convex hull of the orbit (in the weak topology) will be $\langle a^2, b \rangle$ -invariant and compact since E is reflexive so by the Ryll-Nardzewski fixed point theorem $\langle a^2, b \rangle$ will have a fixed point. Thus $\text{Fix}(a^2) \cap \text{Fix}(b) \neq \emptyset$. If this intersection is not a single point then (ii) holds since the representation ρ restricted to $F = \langle a^2, b \rangle$ fixes the difference of any two vectors in the intersection. (ρ is the derivative!) Similarly, (ii) holds if $\text{Fix}(a^3) \cap \text{Fix}(b) \neq \emptyset$ is not a single point. Now suppose each intersection is a single point. If the two intersections coincide then the intersection point is fixed by both $a = a^3(a^2)^{-1}$ and b ,

thus by the whole group F_2 , which implies that G is bounded. If the intersections are distinct then $F = \langle a^6, b \rangle$ fixes two distinct points, so (ii) holds as before. \square

3.3. Detecting essentiality and proof of Theorem 1.1. We now show that under suitable conditions our quasi-cocycles are essential. We consider two cases. If there is a free subgroup that fixes a nonzero vector $e \in E$, the argument essentially goes back to Brooks, since in this case we restrict to the trivial representation. This case is presented first.

Proposition 3.7. *Let ρ be a unitary representation of F_2 in a reflexive Banach space E and let F be a rank two free subgroup such that $\rho|_F$ has an invariant vector $e \neq 0$. Then quasi-cocycles of the form $H_{w,e}$ where w is a reduced word span an infinite dimensional subspace of $\widetilde{QC}(F_2; \rho)$.*

Proof. After possibly conjugating F we can assume that the minimal F -tree contains the identity in the Cayley graph for F_2 and allows us to find cyclically reduced words α and β in F such that the concatenation

$$w_k = \alpha^k \beta^k \alpha^k \beta^k$$

is cyclically reduced. Furthermore we can assume that α and β generate F . Let $H_k = H_{w_k,e}$. By Lemma 3.3 there exists an F -invariant (continuous) linear functional ϕ with $\phi(e) \geq \mu > 0$.

Then the restriction to F of the composition $\phi \circ G$ with any co-cycle G is a homomorphism, and similarly the restriction of the composition $\phi \circ H$ to F with any quasi-co-cycle H is a quasi-morphism.

We will show that the sequence H_1, H_2, \dots represents linearly independent elements in $\widetilde{QC}(F_2; \rho)$. Indeed, if $H = H_k - c_1 H_1 - \dots - c_{k-1} H_{k-1}$, with $1 < k$, for any constants c_i then the quasi-morphism $\phi \circ H$ on F is 0 on the powers of α and β , so if a co-cycle G is boundedly close H , then the homomorphism $\phi \circ G$ on F must be bounded, and therefore zero, on powers of α and β . Therefore $\phi \circ G$ is trivial when restricted to F . On the other hand a straightforward calculation shows that $\phi \circ H(w_k^n) \geq n\mu$ so $\phi \circ H$ is unbounded on F and H and G cannot be boundedly close. We showed that H is non-trivial in $\widetilde{QC}(F_2; \rho)$, so H_1, H_2, \dots, H_k are linearly independent. \square

We now consider the opposite case when no reduction to the trivial representation is possible.

Proposition 3.8. *Let ρ be a unitary representation of $F_2 = \langle a, b \rangle$ on a uniformly convex Banach space and assume that no nonabelian subgroup of F_2 fixes a nonzero vector. Then for any fixed $e \neq 0$ the quasi-cocycles of the form $H_{w,e}$ span an infinite dimensional subspace of $\widetilde{QC}(F_2; \rho)$, where w ranges over cyclically reduced words.*

Proof. Let $w_m = a^{5m}b^{5m}a^{7m}b^{7m}$, $m \geq 1$, and $\gcd(m, 6) = 1$. By Lemma 3.4, $H_m = H_{w_m, e}$ is unbounded. Furthermore H_m is 0 on the subgroups $\langle a^2, b \rangle$ and $\langle a^3, b \rangle$ listed in Lemma 3.6.

We claim that those H_m 's are linearly independent in $\widetilde{QC}(F_2; \rho)$. Fix m and let $H = H_m - \sum_{i < m} c_i H_i$ for constants c_i . Then H is also unbounded, since the H_i for $i < m$ are visibly 0 on all words in F_{w_m} , the set given in the proof of Lemma 3.4, but H_m is unbounded on F_{w_m} . H is bounded on $\langle a^2, b \rangle$ and $\langle a^3, b \rangle$.

Suppose H differs from a cocycle G by a bounded function. Then G is also bounded on the subgroups $\langle a^2, b \rangle$ and $\langle a^3, b \rangle$, therefore G is bounded on F_2 since (i) must hold in Lemma 3.6. So, H is bounded on F_2 , contradiction. We showed that $H_i, i \leq m$ are linearly independent in $\widetilde{QC}(F_2; \rho)$. \square

Theorem 1.1 now follows immediately.

Theorem 3.9. *Let ρ be a unitary representation of F_2 on a uniformly convex Banach space $E \neq 0$. Then $\dim \widetilde{QC}(F_2; \rho) = \infty$.*

Proof. If there is a rank two free subgroup F in F_2 with an F -invariant vector $e \neq 0$, then use Proposition 3.7 to produce an infinite dimensional subspace. Otherwise, use Proposition 3.8. \square

We remark that Pascal Rolli has a new construction, different from the Brooks construction, that he showed in [22] produces nontrivial quasi-cocycles on F_2 (and some other groups) when the Banach space E is an ℓ^p -space (or finite dimensional).

Example 3.10. To see the importance of uniform convexity we will look more closely at the examples $\ell^\infty(F_2)$ of bounded functions and $\ell_0^\infty(F_2)$ of bounded functions that vanish at infinity.

(1) For the regular representation on $\ell^\infty(F_2)$ (or any group G) the constant functions determine a one-dimensional invariant subspace. In particular, any quasi-morphism canonically determines a quasi-cocycle with image in this invariant subspace. If the original quasi-morphism is essential one may expect that the associated quasi-cocycle is also essential. However, for any quasi-cocycle H we can define the function $H_0 : F_2 \rightarrow \ell^\infty(F_2)$ by

$$H_0(g)(f) = H(f)(f) - \rho(g)H(g^{-1}f)(f) = H(f)(f) - H(g^{-1}f)(g^{-1}f)$$

and then we can check that H_0 is a cocycle (essentially it is the coboundary of the 0-cochain defined by the function $f \mapsto H(f)(f)$) and that $\|H - H_0\|_\infty \leq \Delta(H)$. In particular, $\widetilde{QC}(F_2; \ell^\infty(F_2)) = 0$ and $H_b^2(F_2; \ell^\infty(F_2)) = 0$.

(2) For the regular representation of F_2 on $\ell_0^\infty(F_2)$ neither F_2 nor any non-trivial subgroup fixes a non-trivial subspace so we cannot, as in the $\ell^\infty(F_2)$ case, use quasi-morphisms to construct unbounded quasi-cocycles. Furthermore for some choices

of the vector e , the quasi-cocycle $H_{w,e}$ will be bounded. For example if $e \in \ell_0^\infty(F_2)$ is defined by

$$e(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$$

then $\|H_{w,e}(x)\|_\infty = 0$ or 1 depending on whether x does or doesn't contain a copy of w . More generally if $e \in \ell^1(F_2) \subset \ell_0^\infty(F_2)$ we have that $\|H_{w,e}(x)\|_\infty \leq \|e\|_1$. On the other hand if we define $f \in \ell_0^\infty(F_2)$ by

$$f(x) = \begin{cases} 1/n & x = w^{-n}, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

then $|H_{w,f}(w^n)(id)| = \sum_{i=1}^n 1/i$ so $\|H_{w,f}(w^n)\|_\infty$ is unbounded. We can still construct the cocycle H_0 as in the previous paragraph where $H = H_{w,f}$ but this cocycle will not lie in $\ell_0^\infty(F_2)$. This example emphasizes an inherent difficulty in extending our results to a wider class of Banach spaces.

Note that $H_b^n(G; \ell^\infty(G)) = 0$ ($n \geq 1$) for any group G [19, Proposition 7.4.1] since $\ell^\infty(G)$ is a “relatively injective” Banach G -module [19, Chapter II], so some assumption on the Banach space is necessary.

(3) We also note that $H_b^2(G; \ell_0^\infty(G)) = 0$ for any countable, *exact* group (e.g. $G = F_2$, see [21]). This can be seen as follows. First, since $\ell^\infty(G)$ is a relatively injective Banach G -module, $H_b^n(G; \ell^\infty(G)) = 0$ for all $n > 0$. From the long exact sequence in bounded cohomology [19, Proposition 8.2.1] induced by the short exact sequence $0 \rightarrow \ell_0^\infty(G) \rightarrow \ell^\infty(G) \rightarrow \ell^\infty(G)/\ell_0^\infty(G) \rightarrow 0$, it suffices to show $H_b^1(G, \ell^\infty(G)/\ell_0^\infty(G)) = 0$. But this holds if G is countable and exact [9, Theorem 3]. We thank Narutaka Ozawa for pointing out his work to us.

To show $H_b^n(G; \ell_0^\infty(G)) = 0$ for all $n > 1$ it suffices to know

$$H_b^n(G, \ell^\infty(G)/\ell_0^\infty(G)) = 0$$

for all $n > 0$. Ozawa informs us that this is also true.

4. Hyperbolically embedded subgroups

Before proving our main theorem we need a couple of straightforward lemmas.

Lemma 4.1. *Let ρ be a unitary representation of a group G on E and K a finite normal subgroup. Let $E' \subset E$ be the closed subspace of K -invariant vectors and ρ' the unitary representation of G on E' obtained by restriction. Then every (quasi)-cocycle in $QC(G; \rho)$ is a bounded distance from a (quasi)-cocycle in $QC(G; \rho')$*

Proof. We first define a linear projection $\pi : E \rightarrow E'$ by

$$\pi(x) = \frac{1}{|K|} \sum_{k \in K} \rho(k)x.$$

If H is a (quasi)-cocycle in $QC(G; \rho)$ then $\tilde{H} = \pi \circ H$ is a (quasi)-cocycle in $QC(G; \rho')$. We need to show that \tilde{H} is at bounded distance from H .

Recall that H is the translational part of an isometric G (-quasi)-action on E . By the normality of K if two points in E are in the same G -orbit then their K -orbits are (quasi)-isometric. Since $H(G)$ is the G -orbit of 0 under this (quasi)-action and $H(Kg)$ is at bounded distance from the K -orbit of $H(g)$ we have that the K -orbits of points in the image of H are uniformly bounded, and so π moves points in $Im(H)$ a uniformly bounded amount. \square

Corollary 4.2. *The natural map $\widetilde{QC}(G; \rho') \rightarrow \widetilde{QC}(G; \rho)$ is an isomorphism.*

Lemma 4.3. *Let ρ be a unitary representation of $G \times K$ on E such that K is finite and ρ restricted to the K -factor is trivial. Then there is a natural isomorphism from $\widetilde{QC}(G \times K; \rho) \rightarrow \widetilde{QC}(G; \rho)$.*

Proof. Given $H \in QC(G \times K; \rho)$ define $\tilde{H} \in QC(G; \rho)$ by $\tilde{H}(g) = H(g, id)$. The linear map defined by $H \mapsto \tilde{H}$ descends to a linear map $\widetilde{QC}(G \times K; \rho) \rightarrow \widetilde{QC}(G; \rho)$. Any quasi-cocycle in $QC(G; \rho)$ determines a quasi-cocycle in $QC(G \times K; \rho)$ by extending it to be constant on the K -factor. This also descends to a map $\widetilde{QC}(G; \rho) \rightarrow \widetilde{QC}(G \times K; \rho)$, which is an inverse of our first map since $\|H(g, k) - H(g, id)\| \leq \Delta(H) + C$ where $C = \max\{\|H(id, k)\| | k \in K\}$. Hence we have the desired isomorphism. \square

In [11], Dahmani, Guiradel and Osin defined the notion of a *hyperbolically embedded subgroup*. For convenience we recount the definition here. Let G be a group, H a subgroup and $X \subset G$ such that $X \cup H$ generates G . Let $\Gamma(G, X \sqcup H)$ be the Cayley graph with generating set $X \sqcup H$. Then H is hyperbolically embedded in G if

- $\Gamma(G, X \sqcup H)$ is hyperbolic;
- For all $n > 0$ and $h \in H$ there are at most finitely many $h' \in H$ that can be connected to h in $\Gamma(G, X \sqcup H)$ by a path of length $\leq n$ with no edges in H .

A quasi-cocycle is *anti-symmetric* if

$$H(g^{-1}) = -\rho(g^{-1})H(g).$$

A cocycle automatically satisfies this condition. Furthermore every quasi-cocycle is a bounded distance from an anti-symmetric quasi-cocycle. (Simply replace $H(g)$ with $\frac{1}{2}(H(g) - \rho(g)H(g^{-1}))$.) We have the following important theorem of Hull and Osin.

Theorem 4.4 ([17]). *Let G be a group and F a hyperbolically embedded subgroup. Then there exists a linear map*

$$\iota : QC_{as}(F; \rho) \rightarrow QC_{as}(G; \rho)$$

such that if $H \in QC_{as}(F; \rho)$ then $H = \iota(H)|_F$. In particular, $\dim \widetilde{QC}(F; \rho) \leq \dim \widetilde{QC}(G; \rho)$.

The action of a group G on a metric space X is *acylindrical* if for all $B > 0$ there exist D, N such that if $x, y \in X$ and with $d(x, y) > D$ then there are at most N elements $g \in G$ with $d(x, gx) < B$ and $d(y, gy) < B$. A group G is *acylindrically hyperbolic* if it has an acylindrical, non-elementary, action on a δ -hyperbolic space. To apply the previous theorem we need the following result of Dahmani–Guirardel–Osin and Osin:

Theorem 4.5 ([11, 20]). *Let G be an acylindrically hyperbolic group and K the maximal finite normal subgroup. Then G contains a hyperbolically embedded copy of $F_2 \times K$.*

Remark 4.6. Theorem 4.5 is a combination of two theorems. In [20, Theorem 1.2], Osin proves that an acylindrically hyperbolic group contains a non-degenerate hyperbolically embedded subgroup. In [11, Theorem 2.24], Dahmani–Guirardel–Osin show that if G contains a non-degenerate hyperbolically embedded subgroup then it contains a hyperbolically embedded copy of $F_2 \times K$. We note that this latter theorem relies on the projection complex defined in [2].

Proof of Corollary 1.2. Let $E' \subset E$ be the subspace fixed by K and ρ' the restriction of ρ to E' . By assumption $\dim E' > 0$. By Theorem 4.5 there is a copy of $F_2 \times K$ hyperbolically embedded in G . By Lemma 4.3 and Theorem 3.9 we have that $\dim \widetilde{QC}(F_2 \times K; \rho') = \dim \widetilde{QC}(F_2, \rho') = \infty$. Corollary 4.2 implies that $\dim \widetilde{QC}(F_2 \times K; \rho) = \infty$. The corollary then follows from Theorem 4.4. \square

References

- [1] U. Bader, A. Furman, T. Gelander and N. Monod, Property (T) and rigidity for actions on Banach spaces, *Acta Math.*, **198** (2007), no. 1, 57–105. Zbl 1162.22005 MR 2316269
- [2] M. Bestvina, K. Bromberg and K. Fujiwara, Constructing group actions on quasi-trees and applications to mapping class groups, *Publ. Math. Inst. Hautes Études Sci.*, **122** (2015), 1–64. MR 3415065
- [3] M. Bestvina, K. Bromberg and K. Fujiwara, Bounded cohomology via quasi-trees, 2015. arXiv:1306.1542
- [4] M. Bestvina and K. Fujiwara, Bounded cohomology of subgroups of mapping class groups, *Geom. Topol.*, **6** (2002), 69–89 (electronic). Zbl 1021.57001 MR 1914565
- [5] M. Bestvina and K. Fujiwara, A characterization of higher rank symmetric spaces via bounded cohomology, *Geom. Funct. Anal.*, **19** (2009), no. 1, 11–40. Zbl 1203.53041 MR 2507218

- [6] N. Bourbaki, *Topological vector spaces. Chapters 1–5*, Translated from the French by H. G. Eggleston and S. Madan, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1987. Zbl 0622.46001 MR 0910295
- [7] R. Brooks, Some remarks on bounded cohomology, in *Riemann surfaces and related topics. Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y. (1978))*, 53–63, Ann. of Math. Stud, 97, Princeton Univ. Press, Princeton, N.J., 1981. Zbl 0457.55002 MR 0624804
- [8] I. Chatterji, T. Fernos and A. Iozzi, *The median class and superrigidity of actions on $CAT(0)$ cube complexes*, 2015. arXiv:1212.1585
- [9] Y. Choi, I. Farah and N. Ozawa, A nonseparable amenable operator algebra which is not isomorphic to a C^* -algebra, *Forum Math. Sigma* 2, (2014), e2, 12pp. Zbl 1287.47057 MR 3177805
- [10] J. A. Clarkson, Uniformly convex spaces, *Trans. Amer. Math. Soc.*, **40** (1936), no. 3, 396–414. Zbl 0015.35604 MR 1501880
- [11] F. Dahmani, V. Guirardel and D. Osin, Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces, 2014. arXiv:1111.7048
- [12] D. B. A. Epstein and K. Fujiwara, The second bounded cohomology of word-hyperbolic groups, *Topology*, **36** (1997), no. 6, 1275–1289. Zbl 0884.55005 MR 1452851
- [13] U. Hamenstädt, Bounded cohomology and isometry groups of hyperbolic spaces, *J. Eur. Math. Soc. (JEMS)*, **10** (2008), no. 2, 315–349. Zbl 1139.22006 MR 2390326
- [14] U. Hamenstädt, Geometry of the mapping class groups. I. Boundary amenability, *Invent. Math.*, **175** (2009), no. 3, 545–609. Zbl 1197.57003 MR 2471596
- [15] U. Hamenstädt, Isometry groups of proper hyperbolic space, *Geom. Funct. Anal.*, **19** (2009), no. 1, 170–205. Zbl 1273.53037 MR 2507222
- [16] U. Hamenstädt, Isometry groups of proper $CAT(0)$ -spaces of rank one, *Groups Geom. Dyn.*, **6** (2012), no. 3, 579–618. Zbl 1275.20047 MR 2961285
- [17] M. Hull and D. Osin, Induced quasicocycles on groups with hyperbolically embedded subgroups, *Algebr. Geom. Topol.*, **13** (2013), no. 5, 2635–2665. Zbl 1297.20045 MR 3116299
- [18] R. E. Megginson, *An introduction to Banach space theory*, Graduate Texts in Mathematics, 183, Springer-Verlag, New York, 1998. Zbl 0910.46008 MR 1650235
- [19] N. Monod, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Mathematics, 1758, Springer-Verlag, Berlin, 2001. Zbl 0967.22006 MR 1840942

- [20] D. Osin, *Acylically hyperbolic groups*, *Trans. Amer. Math. Soc.*, **368** (2016), no. 2, 851–888. MR 3430352
- [21] N. Ozawa, Amenable actions and applications, in *International Congress of Mathematicians. Vol. II*, 1563–1580, Eur. Math. Soc., Zürich, 2006. Zbl 1104.46032 MR 2275659
- [22] P. Rolli, *Split quasicocycles*, 2013. arXiv:1305.0095

Received February 17, 2015

M. Bestvina, Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA

E-mail: bestvina@math.utah.edu

K. Bromberg, Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA

E-mail: bromberg@math.utah.edu

K. Fujiwara, Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

E-mail: kfujiwara@math.kyoto-u.ac.jp