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## Bounded cohomology with coefficients in uniformly convex Banach spaces

Mladen Bestvina\*, Ken Bromberg\* and Koji Fujiwara\*\*

**Abstract.** We show that for acylindrically hyperbolic groups  $\Gamma$  (with no nontrivial finite normal subgroups) and arbitrary unitary representation  $\rho$  of  $\Gamma$  in a (nonzero) uniformly convex Banach space the vector space  $H_b^2(\Gamma; \rho)$  is infinite dimensional. The result was known for the regular representations on  $\ell^p(\Gamma)$  with  $1 < p < \infty$  by a different argument. But our result is new even for a non-abelian free group in this great generality for representations, and also the case for acylindrically hyperbolic groups follows as an application.

**Mathematics Subject Classification (2010).** 20F65; 46B99.

**Keywords.** Uniformly convex Banach space, second bounded cohomology, acylindrically hyperbolic groups.

### 1. Introduction

**1.1. Quasi-cocycle and quasi-action.** Let  $G$  be a group and  $E$  a normed vector space (usually complete, either over  $\mathbb{R}$  or over  $\mathbb{C}$ ). The linear or rotational part of an isometric  $G$ -action on  $E$  determines a representation  $\rho : G \rightarrow O(E)$  where  $O(E)$  is the group of norm-preserving linear isomorphisms  $E \rightarrow E$ . We will refer to  $\rho$  as a *unitary representation*. We will usually write  $\rho(g)x$  as  $g(x)$  or  $gx$ .

The translational part of the  $G$ -action is a *cocycle* (with respect to  $\rho$ ). Namely the translational part is a function  $F : G \rightarrow E$  that satisfies

$$F(gg') = F(g) + gF(g') \tag{1.1}$$

for all  $g, g' \in G$ . Going in the other direction, if  $\rho$  is a unitary representation and  $F$  a cocycle then the map  $g \mapsto (x \mapsto \rho(g)x + F(g))$  determines an (affine) isometric  $G$ -action on  $E$ . Note that  $F(g^{-1}) = -g^{-1}F(g)$ .  $\rho(g)$  is sometimes called the linear part of the action.

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For an isometric quasi-action of  $G$  on  $E$  the linear part will still be a unitary representation. However, the translational part  $F$  will become a *quasi-cocycle* and will only satisfy (1.1) up to a uniformly bounded error so that

$$\Delta(F) := \sup_{g, g' \in G} |F(gg') - F(g) - gF(g')| < \infty. \quad (1.2)$$

The quantity  $\Delta(F)$  is the *defect* of the quasi-cocycle.

A basic question is if there are quasi-actions that are not boundedly close to an actual action. Such a quasi-action is *essential*. Since quasi-actions determine unitary representations a more refined question is if there are essential quasi-actions for a given unitary representation.

The above discussion is perhaps more familiar in its algebraic form where it can be rephrased in terms of bounded cohomology. A quasi-cocycle  $F$  can be viewed as 1-cochain in the group cohomology twisted by the representation  $\rho$ . Condition (1.2), is equivalent to the coboundary  $\delta F$  being a bounded 2-cocycle and will therefore determine a cohomology class in  $H_b^2(G; \rho)$ , the second bounded cohomology group. Now this cocycle will clearly be trivial in the regular second cohomology group  $H^2(G; \rho)$  as it is the coboundary of a 1-cochain. If the cochain  $F$  is a bounded distance from a cocycle then  $\delta F$  will also be trivial in  $H_b^2(G; \rho)$  so we are interested in the kernel of the map

$$H_b^2(G; \rho) \rightarrow H^2(G; \rho)$$

from bounded cohomology to regular cohomology. In particular this kernel is the vector space  $QC(G; \rho)$  of all quasi-cocycles modulo the subspace generated by bounded functions and cocycles. We denote this quotient space  $\widetilde{QC}(G; \rho)$ . This is the vector space of *essential* quasi-cocycles and it is the main object of study of this paper.

For the trivial representation on  $\mathbb{R}$  a cocycle is just a homomorphism to  $\mathbb{R}$  and a quasi-cocycle is usually called a quasi-morphism. When  $G = F_2$ , the free group on two generators, Brooks [7] gave a combinatorial construction of an infinite dimensional family of essential quasi-morphisms.

**1.2. Uniformly convex Banach space and main result.** Following the work of Brooks, there is a long history of generalizations of this construction to other groups. Initially, the work focused on the trivial representation. See [4, 5, 12]. This was followed by generalizations to the same groups  $G$  but with coefficients in the regular representation  $\ell^p(G)$ ,  $1 \leq p < \infty$ . See [14, 16].

In this paper we will extend this work to unitary representations in *uniformly convex* Banach spaces. Note that this essentially includes the previous cases since  $\ell^p(G)$  is uniformly convex when  $1 < p < \infty$ .

If one is a bit more careful about how the counting is done then Brooks construction of quasi-morphisms can also be used to produce quasi-cocycles. In

Brooks' original work (i.e., for trivial representations) it is easy to see that the quasi-morphisms are essential. Here we will have to work harder to get the following result.

**Theorem 1.1** (Theorem 3.9). *Let  $\rho$  be a unitary representation of  $F_2$  on a uniformly convex Banach space  $E \neq 0$ . Then  $\dim \widetilde{QC}(F_2; \rho) = \infty$ .*

To show  $\widetilde{QC}(F_2; \rho)$  is non-trivial is already hard. We will argue that for a certain Brooks' quasi-cocycle  $H$  into a Banach space  $E$ , there exists a sequence of elements in  $F_2$  on which  $H$  is unbounded. For that we use that  $E$  is uniformly convex in an essential way (Lemma 3.4). We also show those quasi-cocycles are not at bounded distance from any cocycle using that  $E$  is reflexive (using Lemma 3.6). Those two steps are the novel part of the paper. It seems that the uniform convexity is nearly a necessary assumption for the conclusion. See the examples at the end of this section.

Recently Osin [20] (see also [11]) has identified the class of *acylindrically hyperbolic groups* and this seems to be the most general context where the Brooks' construction can be applied. Osin has shown that acylindrically hyperbolic groups contain *hyperbolically embedded* copies of  $F_2$  and then applying work of Hull–Osin [17] we have the the following corollary to Theorem 3.9. See Section 4 for the proof.

**Corollary 1.2.** *Let  $\rho$  be a unitary representation of an acylindrically hyperbolic group  $G$  on a uniformly convex Banach space  $E \neq 0$  and assume that the maximal finite normal subgroup has a non-zero fixed vector. Then  $\dim \widetilde{QC}(G; \rho) = \infty$ .*

A wide variety of groups are acylindrically hyperbolic. In particular our results apply to the following examples. To apply our result, in all examples assume  $G$  has no nontrivial finite normal subgroups, or more generally that for the maximal finite normal subgroup  $N$  (see [11]) we have that  $\rho(N)$  fixes a nonzero vector in  $E$ .

**Examples 1.3** (Acylindrically hyperbolic groups).

- $G$  is non-elementary word hyperbolic,
- $G$  admits a non-elementary isometric action on a connected  $\delta$ -hyperbolic space such that at least one element is hyperbolic and WPD,
- $G = \text{Mod}(S)$ , the mapping class group of a compact surface which is not virtually abelian,
- $G = \text{Out}(F_n)$  for  $n \geq 2$ ,
- $G$  admits a non-elementary isometric action on a  $CAT(0)$  space and at least one element is WPD and acts as a rank 1 isometry.

**Remark 1.4.** Recall that a Banach space is *superreflexive* if it admits an equivalent uniformly convex norm. It is observed in [1, Proposition 2.3] that if  $\rho : G \rightarrow E$  is a unitary representation with  $E$  superreflexive, then there is an equivalent uniformly convex norm with respect to which  $\rho$  is still unitary. Thus in Corollary 1.2 we may replace “uniformly convex” with “superreflexive”.

**Remark 1.5.** There is also a more direct approach to going from Theorem 3.9 to our main theorem. The key point is that any group  $G$  covered in the main theorem acts on a quasi-tree such that there is a free group  $F \subset G$  that acts properly and co-compactly on a tree isometrically embedded in the quasi-tree. This is done using the *projection complex* of [2]. Using this one can apply the Brooks' construction to produce quasi-cocycles that when restricted to the free group are exactly the quasi-cocycles of Theorem 3.9. We carry this out in a separate paper [3].

**1.3. Known examples with certain Banach spaces.** Here are some known vanishing/non-vanishing examples in the literature.

- $E = \mathbb{R}$  and  $\rho$  is trivial. In this case  $H_b^2(G; \rho)$  is the usual bounded cohomology and quasi-cocycles are quasi-morphisms. As we said this case was known for various kinds of groups.
- $E = \ell^p(G)$  and  $\rho$  is the regular representation, see [13, 15]. When  $1 < p < \infty$ ,  $\ell^p(G)$  is uniformly convex and our theorem applies. When  $p = 1$  or  $p = \infty$  then  $\ell^p(G)$  is not uniformly, or even strictly, convex. However, for  $p = 1$  summation determines a  $\rho$ -invariant functional and one can produce a family of quasi-cocycles that when composed with the invariant functional are an infinite dimensional family of non-trivial quasi-morphisms in  $\widetilde{QH}(G)$  implying that  $\dim \widetilde{QC}(G; \ell^1(G)) = \infty$ .

On the other hand,

- When  $p = \infty$  given any quasi-cocycle one can explicitly find a cocycle a bounded distance away so  $\widetilde{QC}(G; \ell^\infty(G)) = 0$  for any group  $G$ .
- If  $G$  is countable and exact (e.g.,  $F_2$ ), then  $H_b^2(G; \ell_0^\infty(G)) = 0$ . In particular,  $\widetilde{QC}(G; \ell_0^\infty(G)) = 0$  (Example 3.10). Here  $\ell_0^\infty(G)$  is the subspace of  $\ell^\infty(G)$  consisting of sequences which are asymptotically 0.

There are also examples where  $G$  is not acylindrically hyperbolic but where  $\widetilde{QC}(G; \rho)$  is known to be non-zero for certain actions of  $G$  on  $\ell^p$  spaces.

- If  $G$  has a non-elementary action on a  $CAT(0)$  cube complex then  $\widetilde{QC}(G; \rho) \neq 0$  where  $\rho$  is the representation of  $G$  on the space of  $\ell^p$ -functions ( $1 \leq p < \infty$ ) on a certain space where  $G$  naturally acts [8]. Note that this class of groups is closed under products so it contains groups that aren't acylindrically hyperbolic.

There are other examples where essentially nothing is known.

- $E = \ell_0^1(G) \subset \ell^1(G)$  is the space of  $\ell^1$ -functions on  $G$  that sum to zero and  $\rho$  is the regular representation. Unlike with  $\ell^1(G)$ ,  $\ell_0^1(G)$  has no  $\rho$ -invariant functionals.
- $E = \mathcal{B}(\ell^2(G))$  the space of bounded linear maps of  $\ell^2(G)$  to itself. This example was suggested to us by N. Monod as the non-commutative analogue to  $\ell^\infty(G)$ .

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## 2. Quasi-cocycles from trees

Fix  $F_2 = \langle a, b \rangle$  and choose a word  $w \in F_2$ . For simplicity we will assume that  $w$  is cyclically reduced. Let  $E$  be a normed vector space and  $\rho : G \rightarrow O(E)$  a linear representation. Also choose a nonzero  $e \in E$ . We now set up some notation that will be convenient for what we will do later.

Let  $[g, h]$  be an oriented segment in the Cayley graph for  $F_2$  with generators  $a$  and  $b$ . Then we write  $[g, h] \overset{\circ}{\subset} [g', h']$  if  $[g, h]$  is a subsegment of  $[g', h']$  and the orientations of the two segments agree. We then define

$$w_+(g) = \{h \in G \mid [h, hw] \overset{\circ}{\subset} [1, g]\}$$

and 
$$w_-(g) = \{h \in H \mid [h, hw] \overset{\circ}{\subset} [g, 1]\}.$$

Now define a function  $H = H_{w,e} : F_2 \rightarrow E$  by

$$H(g) = \sum_{h \in w_+(g)} h(e) - \sum_{h \in w_-(g)} h(e)$$

In other words, to a translate  $h \cdot w$  we assign  $h(e)$  when traversed in the positive direction, and  $-h(e)$  when traversed in negative direction. Note that it follows that  $H(g^{-1}) = -g^{-1}H(g)$ .

**Proposition 2.1.** *The function  $H$  constructed above is a quasi-cocycle.*

*Proof.* This is the standard Brooks argument. Consider the tripod spanned by  $1, g, gf$ . Call the central point  $p$ . We will see that contributions of copies of  $w$  in the tripod that do not cross  $p$  cancel out leaving only a bounded number of terms.

If  $h \cdot w \overset{\circ}{\subset} [1, p]$  then  $h(e)$  enters with positive sign in  $H(g)$  and in  $H(gf)$ , so it cancels in the expression  $H(gf) - H(g)$ . Likewise, if  $h \cdot w \overset{\circ}{\subset} [p, 1]$  then  $-h(e)$  enters both  $H(g)$  and  $H(gf)$ , so it again cancels.

If  $h \cdot w \overset{\circ}{\subset} [p, g]$  then  $h(e)$  is a summand in  $H(g)$ . Since  $h \cdot w \overset{\circ}{\subset} [gf, g]$  we also have  $g^{-1}h \cdot w \overset{\circ}{\subset} [f, 1]$ , so  $-g^{-1}h(e)$  is a summand in  $H(f)$ , and thus we have cancellation in  $-H(g) - gH(f)$ . There is similar cancellation if  $h \cdot w \overset{\circ}{\subset} [g, p]$ .

If  $h \cdot w \overset{\circ}{\subset} [p, gf]$  or  $[gf, p]$  then similarly to the previous paragraph there is cancellation in  $H(gf) - gH(f)$ .

After the above cancellations in the expression  $H(gf) - H(g) - gH(f)$  the only terms left are of the form  $\pm h(e)$  where  $h(w)$  is contained in the tripod and contains  $p$  in its interior. The number of such terms is clearly (generously) bounded by  $6|w|$  so we deduce that  $\Delta(H) \leq 6|w|||e||$ . □

**Remark 2.2.** Note that if  $h \cdot w$  does not overlap  $w$  for any  $1 \neq h \in F_2$ , then  $\Delta(H) \leq 6\|e\|$ . More generally, for a given  $w$ , write  $w = u^n v$  as a word such that  $|v| < |u|$  and  $n > 0$  is maximal. Then,  $\Delta(H) \leq 6(n + 1)\|e\|$ .

**Example 2.3.** Suppose  $w = ab$ . Then  $H(a^n) = H(b^n) = 0$ , while  $H((ab)^n) = (1 + ab + (ab)^2 + \dots + (ab)^{n-1})e \in E$ . If the operator  $1 - ab : E \rightarrow E$  has a continuous inverse (i.e. if  $1 \in \mathbb{C}$  is not in the spectrum of  $ab$ ) then  $H$  is uniformly bounded on the powers of  $ab$  since  $(1 - ab)H((ab)^n) = e - (ab)^n(e)$  has bounded norm. For example, this happens even for  $E = \mathbb{R}^2$  when  $\rho(ab)$  is a (proper) rotation.

On the other hand, for the representation  $\ell^p(F_2)$  with  $1 \leq p < \infty$  and with  $e \in \ell^p(F_2)$  defined by  $e(1) = 1$ ,  $e(g) = 0$  for  $g \neq 1$ , the quasi-cocycle  $H$  is unbounded on the powers of  $ab$ .

### 3. Nontriviality of quasi-cocycles

In Brooks' original construction of quasi-morphisms  $F_2 = \langle a, b \rangle \rightarrow \mathbb{R}$  it is easy to see that the quasi-morphisms are nontrivial. Choosing  $w$  to be a reduced word not of the form  $a^m$  or  $b^m$  it is clear that  $H(w^n)$  will be unbounded while  $H(a^n)$  and  $H(b^n)$  will be zero. By this last fact if  $G$  is a homomorphism that is boundedly close to  $H$  then  $G$  must be bounded on powers of  $a$  and  $b$  and therefore  $G(a) = G(b) = 0$ . Since any homomorphism is determined by its behavior on the generators we have  $G \equiv 0$  and the nontriviality of  $H$  follows.

When the Brooks construction is extended to quasi-cocycles it is no longer clear that the quasi-cocycle is nontrivial. In particular if  $H = H_{w,e}$  it may be that  $H(w^n)$  is bounded. See Examples 2.3 and 3.5. In fact if  $1$  is not in the spectrum of  $\rho(w)$  then  $H(w^n)$  will be bounded for all choices of vectors  $e$ . Even if  $1$  is in the spectrum, when  $e$  is chosen arbitrarily  $H(w^n)$  may be bounded. To show that the Brooks quasi-cocycles are unbounded we will need to restrict to the class of *uniformly convex* Banach spaces and to look at a wider class of words than powers of  $w$ .

We will also have to work harder to show that a cocycle  $G$  that is bounded on powers of the generators is bounded everywhere. In fact we cannot do this in general but instead will show that in a reflexive Banach space (which includes uniformly convex Banach spaces) either the cocycle is bounded or the original representation, when restricted to a non-abelian subgroup, has an eigenvector. In this latter case it is easy to construct many nontrivial quasi-cocycles.

**3.1. Uniformly convex and reflexive Banach spaces.** We will use basic facts about Banach spaces. General references are [6, 18]. The following concept was introduced by Clarkson [10].

**Definition 3.1.** A Banach space  $E$  is *uniformly convex* if for every  $\epsilon > 0$  there is  $\delta > 0$  such that  $x, y \in E$ ,  $|x| \leq 1$ ,  $|y| \leq 1$ ,  $|x - y| \geq \epsilon$  implies  $|\frac{x+y}{2}| \leq 1 - \delta$ .

The original definition in [10] replaces  $|x|, |y| \leq 1$  above with equalities, but it is not hard to see that the two are equivalent.

**Proposition 3.2.** (i)  $\ell^p$  spaces are uniformly convex for  $1 < p < \infty$  [10].  $\ell^1$  and  $\ell^\infty$  spaces are not uniformly convex and not reflexive.

(ii) A uniformly convex Banach space is reflexive (the Milman–Pettis theorem).

(iii) If  $E$  is uniformly convex, then for any  $R > 0$  there are  $\epsilon > 0$  and  $\mu > 0$  so that the following holds. If  $|v| \leq R$  and  $f : E \rightarrow \mathbb{R}$  is a functional of norm 1 with  $f(v) = |v|$  and if  $e$  is a vector of norm  $\geq 1/2$  with  $f(e) \geq -\mu$  then  $|v + e| \geq |v| + \epsilon$ .

*Proof.* We only prove (iii). Choose  $\delta \in (0, 1)$  so that  $|x|, |y| \leq 1, |x - y| \geq \frac{1}{2(R+1)}$  implies  $|\frac{x+y}{2}| \leq 1 - \delta$ . Then choose  $\epsilon, \mu > 0$  so that  $\epsilon < \frac{1}{8}$  and  $\frac{\frac{1}{8} - \frac{\mu}{2}}{\frac{1}{8} + \epsilon} > 1 - \delta$ . Suppose  $f, v, e$  satisfy the assumptions but  $|v + e| < R + \epsilon$ . If  $|v| \leq 1/8$  then  $|v + e| \geq |e| - |v| \geq 1/4 \geq |v| + 1/8$  and we are done. So assume that  $|v| > 1/8$ . Then for  $x = \frac{v}{|v| + \epsilon}, y = \frac{v+e}{|v| + \epsilon}$  we have  $|x|, |y| \leq 1$  and  $|x - y| \geq \frac{1}{2(|v|+1)} \geq \frac{1}{2(R+1)}$ , so we must have  $|\frac{x+y}{2}| \leq 1 - \delta$ . Thus

$$1 - \delta \geq \left| \frac{x + y}{2} \right| = \left| \frac{v + e/2}{|v| + \epsilon} \right| \geq \frac{|v| - \frac{\mu}{2}}{|v| + \epsilon} \geq \frac{\frac{1}{8} - \frac{\mu}{2}}{\frac{1}{8} + \epsilon}$$

since  $f(v + e/2) = |v| + f(e)/2 \geq |v| - \frac{\mu}{2}$  and  $|f| = 1$ . This contradicts the choice of  $\mu, \epsilon$ .  $\square$

**Lemma 3.3.** Let  $\rho$  be a unitary representation of a group  $F$  on a reflexive Banach space  $E$ . If there is a linear functional  $f$  and a vector  $e \in E$  such that the  $F$ -orbit of  $e$  lies in the half space  $\{f \geq \mu\}$  with  $\mu > 0$  then there is an  $F$ -invariant vector  $e' \neq 0 \in E$  and an  $F$ -invariant functional  $\phi$  with  $\phi(e') \geq \mu$ . If  $e$  is  $F$ -invariant, then we can take  $e' = e$ .

*Proof.* Let  $\Lambda$  be the convex hull of the  $F$ -orbit of  $e$  in the weak topology on  $E$ . Since  $E$  is reflexive,  $\Lambda$  is weakly compact. The convex hull  $\Lambda$  is also  $F$ -invariant so by the Ryll-Nardzewski fixed point theorem it will contain an  $F$ -invariant vector  $e'$ . Since  $e' \in \Lambda$ ,  $f(e') \geq \mu$  and therefore  $e' \neq 0$ .

Since  $e'$  is a functional on the reflexive Banach space  $E^*$  and the  $F$ -orbit of  $f$  will be contained in the half space  $\{e' \geq \mu\}$  we similarly get a  $F$ -invariant vector  $\phi \in E^*$  with  $e'(\phi) = \phi(e') \geq \mu$ .  $\square$

Note that if  $E$  contains a nonzero vector that is  $F$ -invariant, then the Hahn–Banach theorem supplies a functional that satisfies the conditions of the lemma and so there is also a nonzero  $F$ -invariant functional.

**3.2. Detecting unboundedness.**

**Lemma 3.4.** *Let  $\rho$  be any unitary representation of  $F_2 = \langle a, b \rangle$  into a uniformly convex Banach space  $E$ . Then one of the following holds:*

- (i) *for every  $e \neq 0 \in E$  and any  $1 \neq w \in F_2$  not of the form  $a^m b^n$  nor  $b^m a^n$  the quasi-cocycle  $H = H_{w,e}$  is unbounded on  $F_2$ , or*
- (ii) *there is a free subgroup  $F \subset F_2$  with  $F \cong F_2$ , a linear functional  $g$ , a vector  $e$  and a  $\mu > 0$  such that the  $F$ -orbit of  $e$  is contained in the half-space  $\{g \leq -\mu\}$ . In particular, there is an  $F$ -invariant vector  $e' \neq 0$  in the half space.*

*Proof.* We first make some observations about words in  $F_2$ . Given a word  $w$  as in (i) we can find buffer words  $B$  and  $B'$  of the form  $a^\ell b^\ell$  or  $b^\ell a^\ell$  and a subgroup  $F = \langle a^m, b^m \rangle$  with  $m \gg \ell, |w|$  such that if  $w' = BwB'$  and  $y_1, y_2, \dots, y_n \in F$  then in the reduced word for the element  $x = y_1 w' y_2 w' \dots y_n w'$  there is exactly one copy of  $w$  for each  $w'$  and no other copies of either  $w$  or  $w^{-1}$ . Note that the word  $y_1 w' y_2 w' \dots y_n w'$  may not be reduced and in its reduced version there may be cancellations in the  $w'$ . However, the buffer words will prevent these cancellations from reaching  $w$ . The restrictions on  $w$  ensure that  $w$  does not appear as a subword of some  $y_i$ . In particular,  $|H(w')| = |e|$  and  $H(xyw') = H(x) + xH(yw') = H(x) + xyH(w')$  for any  $y \in F$ .

For simplicity, normalize so that  $|e| = 1$ , so  $|H(w')| = 1$ . Assume that (ii) doesn't hold, and that  $H$  is bounded on  $F_2$ . Let  $F_w$  be the set of words of the form

$$y_1 w' y_2 w' \dots y_n w', (y_i \in F)$$

and let  $R = \sup_{x \in F_w} |H(x)| < \infty$ . Let  $\epsilon, \mu > 0$  be as in Proposition 3.2(iii). Choose an  $x \in F_w$  such that  $|H(x)| > R - \epsilon$ . We will find a  $y \in F$  such that  $|H(xyw')| > R$  to obtain a contradiction since  $xyw' \in F_w$ .

Let  $\phi$  be a linear functional of norm 1 such that  $\phi(H(x)) = |H(x)|$ . Let  $\psi = \phi \circ x$ . Since (ii) doesn't hold, there exists a  $y \in F$  with  $\psi(yH(w')) > -\mu$ . (We are applying the negation of (ii) not to  $e$  but to  $H(w')$ , which is in the  $F_2$ -orbit of  $e$ , but it is easy to see that this follows from the corresponding fact for  $e$  by replacing  $F$  with a conjugate.) So,  $\phi(xyH(w')) > -\mu$ . Then by Proposition 3.2(iii),  $|H(xyw')| = |H(x) + xyH(w')| \geq |H(x)| + \epsilon > R$ , contradiction.

For an  $F$ -invariant vector in (ii), see the proof of Lemma 3.3. □

We give an application of Lemma 3.4.

**Example 3.5.** Choose an embedding  $\rho : F_2 \subset U(2)$  so that every nontrivial element is conjugate to a matrix of the form

$$\begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix}$$

with  $t, s, \frac{t}{s}$  all irrational.

(Such representations can be constructed by noting that they form the complement of countably many proper subvarieties in  $\text{Hom}(F_2, U(2))$ .) Put  $E = \mathbb{C}^2$ .

Then any  $H = H_{w,e}$  with  $0 \neq e \in E, 1 \neq w \in F_2$  is bounded on any cyclic subgroup, but many are globally unbounded. The second statement follows by noting that the orbit of any unit vector under a nontrivial cyclic subgroup is dense in a torus  $S^1 \times S^1 \subset \mathbb{C}^2$ , so (ii) of Lemma 3.4 fails, and (i) must hold. For the first statement, observe that for a fixed  $g \in F$  the values  $H(g^n)$  can be computed, up to a bounded error, by adding sums of the form

$$U_n = u(e) + gu(e) + \cdots + g^{n-1}u(e)$$

one for every  $g$ -orbit of occurrences of  $w$  or  $w^{-1}$  along the axis of  $g$ . Applying  $g$  we have

$$g(U_n) = gu(e) + \cdots + g^n u(e)$$

and so  $|g(U_n) - U_n| \leq 2|e|$ , which implies that  $|U_n|$  is bounded, since  $g : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  moves every unit vector a definite amount. It follows  $H(g^n)$  is bounded on  $n$ . This gives an isometric quasi-action of  $F_2$  on  $\mathbb{C}^2$  or  $\mathbb{R}^4$  with unbounded orbits, but with every cyclic subgroup having bounded orbits.

In fact, since  $H^1(F_2; \rho) \neq 0$ , it follows that there are *isometric* actions of  $F_2$  on  $\mathbb{R}^4$  with unbounded orbits and with every element fixing a point.

The following is our basic method of detecting bounded cocycles. In the presence of reflexivity of the Banach space, bounded isometric actions have fixed points. Thus a cocycle  $G : F_2 \rightarrow E$  is bounded if and only if for some  $v \in E$  (a fixed point of the action) we have  $G(g) = v - \rho(g)v$  for every  $g \in F_2$ .

**Lemma 3.6.** *Let  $\rho$  be a unitary representation of  $F_2$  on a reflexive Banach space  $E$  and  $G$  a cocycle that is bounded on  $\langle a^2, b \rangle$  and  $\langle a^3, b \rangle$ . Then one of the following holds.*

- (i)  $G$  is bounded on  $F_2$ , or
- (ii) There is a free subgroup  $F \subset F_2$  with  $F \cong F_2$  such that  $\rho|_F$  fixes a nonzero vector in  $E$ .

*Proof.* The cocycle  $G$  induces an action of  $F_2$  on  $E$  by affine isometries and the image of  $G$  is the orbit of 0 under this action. If the restriction of this action to  $\langle a^2, b \rangle$  is bounded (with respect to the norm topology) then the convex hull of the orbit (in the weak topology) will be  $\langle a^2, b \rangle$ -invariant and compact since  $E$  is reflexive so by the Ryll-Nardzewski fixed point theorem  $\langle a^2, b \rangle$  will have a fixed point. Thus  $\text{Fix}(a^2) \cap \text{Fix}(b) \neq \emptyset$ . If this intersection is not a single point then (ii) holds since the representation  $\rho$  restricted to  $F = \langle a^2, b \rangle$  fixes the difference of any two vectors in the intersection. ( $\rho$  is the derivative!) Similarly, (ii) holds if  $\text{Fix}(a^3) \cap \text{Fix}(b) \neq \emptyset$  is not a single point. Now suppose each intersection is a single point. If the two intersections coincide then the intersection point is fixed by both  $a = a^3(a^2)^{-1}$  and  $b$ ,

thus by the whole group  $F_2$ , which implies that  $G$  is bounded. If the intersections are distinct then  $F = \langle a^6, b \rangle$  fixes two distinct points, so (ii) holds as before.  $\square$

**3.3. Detecting essentiality and proof of Theorem 1.1.** We now show that under suitable conditions our quasi-cocycles are essential. We consider two cases. If there is a free subgroup that fixes a nonzero vector  $e \in E$ , the argument essentially goes back to Brooks, since in this case we restrict to the trivial representation. This case is presented first.

**Proposition 3.7.** *Let  $\rho$  be a unitary representation of  $F_2$  in a reflexive Banach space  $E$  and let  $F$  be a rank two free subgroup such that  $\rho|_F$  has an invariant vector  $e \neq 0$ . Then quasi-cocycles of the form  $H_{w,e}$  where  $w$  is a reduced word span an infinite dimensional subspace of  $\widetilde{QC}(F_2; \rho)$ .*

*Proof.* After possibly conjugating  $F$  we can assume that the minimal  $F$ -tree contains the identity in the Cayley graph for  $F_2$  and allows us to find cyclically reduced words  $\alpha$  and  $\beta$  in  $F$  such that the concatenation

$$w_k = \alpha^k \beta^k \alpha^k \beta^k$$

is cyclically reduced. Furthermore we can assume that  $\alpha$  and  $\beta$  generate  $F$ . Let  $H_k = H_{w_k,e}$ . By Lemma 3.3 there exists an  $F$ -invariant (continuous) linear functional  $\phi$  with  $\phi(e) \geq \mu > 0$ .

Then the restriction to  $F$  of the composition  $\phi \circ G$  with any co-cycle  $G$  is a homomorphism, and similarly the restriction of the composition  $\phi \circ H$  to  $F$  with any quasi-co-cycle  $H$  is a quasi-morphism.

We will show that the sequence  $H_1, H_2, \dots$  represents linearly independent elements in  $\widetilde{QC}(F_2; \rho)$ . Indeed, if  $H = H_k - c_1 H_1 - \dots - c_{k-1} H_{k-1}$ , with  $1 < k$ , for any constants  $c_i$  then the quasi-morphism  $\phi \circ H$  on  $F$  is 0 on the powers of  $\alpha$  and  $\beta$ , so if a co-cycle  $G$  is boundedly close  $H$ , then the homomorphism  $\phi \circ G$  on  $F$  must be bounded, and therefore zero, on powers of  $\alpha$  and  $\beta$ . Therefore  $\phi \circ G$  is trivial when restricted to  $F$ . On the other hand a straightforward calculation shows that  $\phi \circ H(w_k^n) \geq n\mu$  so  $\phi \circ H$  is unbounded on  $F$  and  $H$  and  $G$  cannot be boundedly close. We showed that  $H$  is non-trivial in  $\widetilde{QC}(F_2; \rho)$ , so  $H_1, H_2, \dots, H_k$  are linearly independent.  $\square$

We now consider the opposite case when no reduction to the trivial representation is possible.

**Proposition 3.8.** *Let  $\rho$  be a unitary representation of  $F_2 = \langle a, b \rangle$  on a uniformly convex Banach space and assume that no nonabelian subgroup of  $F_2$  fixes a nonzero vector. Then for any fixed  $e \neq 0$  the quasi-cocycles of the form  $H_{w,e}$  span an infinite dimensional subspace of  $\widetilde{QC}(F_2; \rho)$ , where  $w$  ranges over cyclically reduced words.*

*Proof.* Let  $w_m = a^{5m}b^{5m}a^{7m}b^{7m}$ ,  $m \geq 1$ , and  $\gcd(m, 6) = 1$ . By Lemma 3.4,  $H_m = H_{w_m, e}$  is unbounded. Furthermore  $H_m$  is 0 on the subgroups  $\langle a^2, b \rangle$  and  $\langle a^3, b \rangle$  listed in Lemma 3.6.

We claim that those  $H_m$ 's are linearly independent in  $\widetilde{QC}(F_2; \rho)$ . Fix  $m$  and let  $H = H_m - \sum_{i < m} c_i H_i$  for constants  $c_i$ . Then  $H$  is also unbounded, since the  $H_i$  for  $i < m$  are visibly 0 on all words in  $F_{w_m}$ , the set given in the proof of Lemma 3.4, but  $H_m$  is unbounded on  $F_{w_m}$ .  $H$  is bounded on  $\langle a^2, b \rangle$  and  $\langle a^3, b \rangle$ .

Suppose  $H$  differs from a cocycle  $G$  by a bounded function. Then  $G$  is also bounded on the subgroups  $\langle a^2, b \rangle$  and  $\langle a^3, b \rangle$ , therefore  $G$  is bounded on  $F_2$  since (i) must hold in Lemma 3.6. So,  $H$  is bounded on  $F_2$ , contradiction. We showed that  $H_i, i \leq m$  are linearly independent in  $\widetilde{QC}(F_2; \rho)$ . □

Theorem 1.1 now follows immediately.

**Theorem 3.9.** *Let  $\rho$  be a unitary representation of  $F_2$  on a uniformly convex Banach space  $E \neq 0$ . Then  $\dim \widetilde{QC}(F_2; \rho) = \infty$ .*

*Proof.* If there is a rank two free subgroup  $F$  in  $F_2$  with an  $F$ -invariant vector  $e \neq 0$ , then use Proposition 3.7 to produce an infinite dimensional subspace. Otherwise, use Proposition 3.8. □

We remark that Pascal Rolli has a new construction, different from the Brooks construction, that he showed in [22] produces nontrivial quasi-cocycles on  $F_2$  (and some other groups) when the Banach space  $E$  is an  $\ell^p$ -space (or finite dimensional).

**Example 3.10.** To see the importance of uniform convexity we will look more closely at the examples  $\ell^\infty(F_2)$  of bounded functions and  $\ell_0^\infty(F_2)$  of bounded functions that vanish at infinity.

(1) For the regular representation on  $\ell^\infty(F_2)$  (or any group  $G$ ) the constant functions determine a one-dimensional invariant subspace. In particular, any quasi-morphism canonically determines a quasi-cocycle with image in this invariant subspace. If the original quasi-morphism is essential one may expect that the associated quasi-cocycle is also essential. However, for any quasi-cocycle  $H$  we can define the function  $H_0 : F_2 \rightarrow \ell^\infty(F_2)$  by

$$H_0(g)(f) = H(f)(f) - \rho(g)H(g^{-1}f)(f) = H(f)(f) - H(g^{-1}f)(g^{-1}f)$$

and then we can check that  $H_0$  is a cocycle (essentially it is the coboundary of the 0-cochain defined by the function  $f \mapsto H(f)(f)$ ) and that  $\|H - H_0\|_\infty \leq \Delta(H)$ . In particular,  $\widetilde{QC}(F_2; \ell^\infty(F_2)) = 0$  and  $H_b^2(F_2; \ell^\infty(F_2)) = 0$ .

(2) For the regular representation of  $F_2$  on  $\ell_0^\infty(F_2)$  neither  $F_2$  nor any non-trivial subgroup fixes a non-trivial subspace so we cannot, as in the  $\ell^\infty(F_2)$  case, use quasi-morphisms to construct unbounded quasi-cocycles. Furthermore for some choices

of the vector  $e$ , the quasi-cocycle  $H_{w,e}$  will be bounded. For example if  $e \in \ell_0^\infty(F_2)$  is defined by

$$e(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$$

then  $\|H_{w,e}(x)\|_\infty = 0$  or  $1$  depending on whether  $x$  does or doesn't contain a copy of  $w$ . More generally if  $e \in \ell^1(F_2) \subset \ell_0^\infty(F_2)$  we have that  $\|H_{w,e}(x)\|_\infty \leq \|e\|_1$ . On the other hand if we define  $f \in \ell_0^\infty(F_2)$  by

$$f(x) = \begin{cases} 1/n & x = w^{-n}, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

then  $|H_{w,f}(w^n)(id)| = \sum_{i=1}^n 1/i$  so  $\|H_{w,f}(w^n)\|_\infty$  is unbounded. We can still construct the cocycle  $H_0$  as in the previous paragraph where  $H = H_{w,f}$  but this cocycle will not lie in  $\ell_0^\infty(F_2)$ . This example emphasizes an inherent difficulty in extending our results to a wider class of Banach spaces.

Note that  $H_b^n(G; \ell^\infty(G)) = 0 (n \geq 1)$  for any group  $G$  [19, Proposition 7.4.1] since  $\ell^\infty(G)$  is a ‘‘relatively injective’’ Banach  $G$ -module [19, Chapter II], so some assumption on the Banach space is necessary.

(3) We also note that  $H_b^2(G; \ell_0^\infty(G)) = 0$  for any countable, *exact* group (e.g.  $G = F_2$ , see [21]). This can be seen as follows. First, since  $\ell^\infty(G)$  is a relatively injective Banach  $G$ -module,  $H_b^n(G; \ell^\infty(G)) = 0$  for all  $n > 0$ . From the long exact sequence in bounded cohomology [19, Proposition 8.2.1] induced by the short exact sequence  $0 \rightarrow \ell_0^\infty(G) \rightarrow \ell^\infty(G) \rightarrow \ell^\infty(G)/\ell_0^\infty(G) \rightarrow 0$ , it suffices to show  $H_b^1(G, \ell^\infty(G)/\ell_0^\infty(G)) = 0$ . But this holds if  $G$  is countable and exact [9, Theorem 3]. We thank Narutaka Ozawa for pointing out his work to us.

To show  $H_b^n(G; \ell_0^\infty(G)) = 0$  for all  $n > 1$  it suffices to know

$$H_b^n(G, \ell^\infty(G)/\ell_0^\infty(G)) = 0$$

for all  $n > 0$ . Ozawa informs us that this is also true.

#### 4. Hyperbolically embedded subgroups

Before proving our main theorem we need a couple of straightforward lemmas.

**Lemma 4.1.** *Let  $\rho$  be a unitary representation of a group  $G$  on  $E$  and  $K$  a finite normal subgroup. Let  $E' \subset E$  be the closed subspace of  $K$ -invariant vectors and  $\rho'$  the unitary representation of  $G$  on  $E'$  obtained by restriction. Then every (quasi)-cocycle in  $QC(G; \rho)$  is a bounded distance from a (quasi)-cocycle in  $QC(G; \rho')$*

*Proof.* We first define a linear projection  $\pi : E \rightarrow E'$  by

$$\pi(x) = \frac{1}{|K|} \sum_{k \in K} \rho(k)x.$$

If  $H$  is a (quasi)-cocycle in  $QC(G; \rho)$  then  $\tilde{H} = \pi \circ H$  is a (quasi)-cocycle in  $QC(G; \rho')$ . We need to show that  $\tilde{H}$  is at bounded distance from  $H$ .

Recall that  $H$  is the translational part of an isometric  $G$ (-quasi)-action on  $E$ . By the normality of  $K$  if two points in  $E$  are in the same  $G$ -orbit then their  $K$ -orbits are (quasi)-isometric. Since  $H(G)$  is the  $G$ -orbit of 0 under this (quasi)-action and  $H(Kg)$  is at bounded distance from the  $K$ -orbit of  $H(g)$  we have that the  $K$ -orbits of points in the image of  $H$  are uniformly bounded, and so  $\pi$  moves points in  $Im(H)$  a uniformly bounded amount.  $\square$

**Corollary 4.2.** *The natural map  $\widetilde{QC}(G; \rho') \rightarrow \widetilde{QC}(G; \rho)$  is an isomorphism.*

**Lemma 4.3.** *Let  $\rho$  be a unitary representation of  $G \times K$  on  $E$  such that  $K$  is finite and  $\rho$  restricted to the  $K$ -factor is trivial. Then there is a natural isomorphism from  $\widetilde{QC}(G \times K; \rho) \rightarrow \widetilde{QC}(G; \rho)$ .*

*Proof.* Given  $H \in QC(G \times K; \rho)$  define  $\tilde{H} \in QC(G; \rho)$  by  $\tilde{H}(g) = H(g, id)$ . The linear map defined by  $H \mapsto \tilde{H}$  descends to a linear map  $\widetilde{QC}(G \times K; \rho) \rightarrow \widetilde{QC}(G; \rho)$ . Any quasi-cocycle in  $QC(G; \rho)$  determines a quasi-cocycle in  $QC(G \times K; \rho)$  by extending it to be constant on the  $K$ -factor. This also descends to a map  $\widetilde{QC}(G; \rho) \rightarrow \widetilde{QC}(G \times K; \rho)$ , which is an inverse of our first map since  $\|H(g, k) - H(g, id)\| \leq \Delta(H) + C$  where  $C = \max\{\|H(id, k)\| \mid k \in K\}$ . Hence we have the desired isomorphism.  $\square$

In [11], Dahmani, Guiradel and Osin defined the notion of a *hyperbolically embedded subgroup*. For convenience we recount the definition here. Let  $G$  be a group,  $H$  a subgroup and  $X \subset G$  such that  $X \cup H$  generates  $G$ . Let  $\Gamma(G, X \sqcup H)$  be the Cayley graph with generating set  $X \sqcup H$ . Then  $H$  is hyperbolically embedded in  $G$  if

- $\Gamma(G, X \sqcup H)$  is hyperbolic;
- For all  $n > 0$  and  $h \in H$  there are at most finitely many  $h' \in H$  that can be connected to  $h$  in  $\Gamma(G, X \sqcup H)$  by a path of length  $\leq n$  with no edges in  $H$ .

A quasi-cocycle is *anti-symmetric* if

$$H(g^{-1}) = -\rho(g^{-1})H(g).$$

A cocycle automatically satisfies this condition. Furthermore every quasi-cocycle is a bounded distance from an anti-symmetric quasi-cocycle. (Simply replace  $H(g)$  with  $\frac{1}{2}(H(g) - \rho(g)H(g^{-1}))$ .) We have the following important theorem of Hull and Osin.

**Theorem 4.4** ([17]). *Let  $G$  be a group and  $F$  a hyperbolically embedded subgroup. Then there exists a linear map*

$$\iota : QC_{as}(F; \rho) \rightarrow QC_{as}(G; \rho)$$

*such that if  $H \in QC_{as}(F; \rho)$  then  $H = \iota(H)|_F$ . In particular,  $\dim \widetilde{QC}(F; \rho) \leq \dim \widetilde{QC}(G; \rho)$ .*

The action of a group  $G$  on a metric space  $X$  is *acylindrical* if for all  $B > 0$  there exist  $D, N$  such that if  $x, y \in X$  and with  $d(x, y) > D$  then there are at most  $N$  elements  $g \in G$  with  $d(x, gx) < B$  and  $d(y, gy) < B$ . A group  $G$  is *acylindrically hyperbolic* if it has an acylindrical, non-elementary, action on a  $\delta$ -hyperbolic space. To apply the previous theorem we need the following result of Dahmani–Guirardel–Osin and Osin:

**Theorem 4.5** ([11, 20]). *Let  $G$  be an acylindrically hyperbolic group and  $K$  the maximal finite normal subgroup. Then  $G$  contains a hyperbolically embedded copy of  $F_2 \times K$ .*

**Remark 4.6.** Theorem 4.5 is a combination of two theorems. In [20, Theorem 1.2], Osin proves that an acylindrically hyperbolic group contains a non-degenerate hyperbolically embedded subgroup. In [11, Theorem 2.24], Dahmani–Guirardel–Osin show that if  $G$  contains a non-degenerate hyperbolically embedded subgroup then it contains a hyperbolically embedded copy of  $F_2 \times K$ . We note that this latter theorem relies on the projection complex defined in [2].

*Proof of Corollary 1.2.* Let  $E' \subset E$  be the subspace fixed by  $K$  and  $\rho'$  the restriction of  $\rho$  to  $E'$ . By assumption  $\dim E' > 0$ . By Theorem 4.5 there is a copy of  $F_2 \times K$  hyperbolically embedded in  $G$ . By Lemma 4.3 and Theorem 3.9 we have that  $\dim \widetilde{QC}(F_2 \times K; \rho') = \dim \widetilde{QC}(F_2, \rho') = \infty$ . Corollary 4.2 implies that  $\dim \widetilde{QC}(F_2 \times K; \rho) = \infty$ . The corollary then follows from Theorem 4.4.  $\square$

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