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On Serre's injectivity question and norm principle

Nivedita Bhaskhar*

Abstract. Let k be a field of characteristic not 2. We give a positive answer to Serre's injectivity question for any smooth connected reductive k-group whose Dynkin diagram contains connected components only of type A_n , B_n or C_n . We do this by relating Serre's question to the norm principles proved by Barquero and Merkurjev. We give a scalar obstruction defined up to spinor norms whose vanishing will imply the norm principle for the non-trialitarian D_n case and yield a positive answer to Serre's question for connected reductive k-groups whose Dynkin diagrams contain components of (non-trialitarian) type D_n too. We also investigate Serre's question for quasi-split reductive k-groups.

Mathematics Subject Classification (2010). 14L35, 20G10, 20G15.

Keywords. Principal homogeneous spaces, Serre's question, zero cycles, norm principles, spinor norm, Galois cohomology.

1. Introduction

Let k be a field. Then the following question of Serre, which is open in general, asks

Question 1.1 (Serre, [13, p. 233]). Let G be any connected linear algebraic group over a field k. Let L_1, L_2, \ldots, L_r be finite field extensions of k of degrees d_1, d_2, \ldots, d_r respectively such that $\gcd_i(d_i) = 1$. Then is the following sequence exact?

$$1 \to \mathrm{H}^1(k,G) \to \prod_{i=1}^r \mathrm{H}^1(L_i,G).$$

The classical result that the index of a central simple algebra divides the degrees of its splitting fields answers Serre's question affirmatively for the group PGL_n . Springer's theorem for quadratic forms answers it affirmatively for the (albeit sometimes disconnected) group O(q) and Bayer–Lenstra's theorem [2], for the groups of isometries of algebras with involutions. Jodi Black [3] answers Serre's question positively for absolutely simple simply connected and adjoint k-groups of classical type. In this paper, we use and extend Jodi's result to connected reductive k-groups whose Dynkin diagram contains connected components only of type A_n , B_n or C_n .

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Theorem 1.2. Let k be a field of characteristic not 2. Let G be a connected reductive k-group whose Dynkin diagram contains connected components only of type A_n , B_n or C_n . Then Serre's question has a positive answer for G.

We also investigate Serre's question for reductive k-groups whose derived subgroups admit quasi-split simply connected covers. More precisely, we give a uniform proof for the following:

Theorem 1.3. Let k be a field of characteristic not 2. Let G be a connected quasi-split reductive k-group whose Dynkin diagram does not contain connected components of type E_8 . Then Serre's question has a positive answer for G.

We relate Serre's question for G with the norm principles of other closely related groups following a series of reductions previously used by Barquero and Merkurjev to prove the norm principles for reductive groups whose Dynkin diagrams do not contain connected components of type D_n , E_6 or E_7 [1]. We also give a scalar obstruction defined up to spinor norms whose vanishing will imply the norm principle for the (non-trialitarian) D_n case and yield a positive answer to Serre's question for connected reductive k-groups whose Dynkin diagrams contain components of this type also.

In the next section, we begin with some lemmata and preliminary reductions. In Section 3, we introduce intermediate groups \hat{G} and \tilde{G} and relate Serre's question for G to Serre's question for \hat{G} and \tilde{G} via the norm principle. In Section 4, we investigate the norm principle for (non-trialitarian) type D_n groups and find the scalar obstruction whose vanishing will imply the norm principle for the (non-trialitarian) D_n case. In the final section, we use the reduction techniques used in Sections 2 and 3 to discuss Serre's question for connected reductive k-groups whose derived subgroups admit quasi-split simply connected covers.

2. Preliminaries

We work over the base field k of characteristic not 2. By a k-group, we mean a smooth connected linear algebraic group defined over k. And mostly, we will restrict ourselves to reductive groups. We say that a k-group G satisfies SQ if Serre's question has a positive answer for G.

2.1. Reduction to characteristic 0. Let G be a connected reductive k-group whose Dynkin diagram contains connected components only of type A_n , B_n , C_n or (non-trialitarian) D_n . Without loss of generality we may assume that k is of characteristic 0 [7, p. 47]. We give a sketch of the reduction argument for the sake of completeness.

Suppose that the characteristic of k is p > 0. Let L_1, L_2, \ldots, L_r be finite field extensions of k of degrees d_1, d_2, \ldots, d_r respectively such that $gcd_i(d_i) = 1$ and

let ξ be an element in the kernel of

$$\mathrm{H}^1(k,G) \to \prod_{i=1}^r \mathrm{H}^1(L_i,G).$$

By a theorem of Gabber, Liu and Lorenzini [5, Thm. 9.2] which was pointed out to us by O. Wittenberg, we note that any torsor under a smooth group scheme G/k which admits a zero-cycle of degree 1 also admits a zero-cycle of degree 1 whose support is étale over k. Thus without loss of generality we can assume that the given coprime extensions L_i/k are in fact separable.

By [10, Thms. 1 & 2], there exists a complete discrete valuation ring R with residue field k and fraction field K of characteristic zero. Let S_i denote corresponding étale extensions of R with residue fields L_i and fraction fields K_i .

There exists a smooth R-group scheme \tilde{G} with special fiber G and connected reductive generic fiber \tilde{G}_K . Now given any torsor $t \in H^1(k,G)$, there exists a torsor $\tilde{t} \in H^1_{\text{\'et}}(R,\tilde{G})$ specializing to t which is unique upto isomorphism. This in turn gives a torsor \tilde{t}_K in $H^1(K,\tilde{G}_K)$ by base change, thus defining a map $i_k: H^1(k,G) \to H^1(K,\tilde{G}_K)$ [6, p. 29]. It clearly sends the trivial element to the trivial element. The map i also behaves well with the natural restriction maps, i.e., it fits into the following commutative diagram:

$$H^{1}(k,G) \xrightarrow{i_{k}} H^{1}(K,\tilde{G}_{K})$$

$$\downarrow \qquad \qquad \downarrow$$

$$\prod H^{1}(L_{i},G) \xrightarrow{\prod i_{L_{i}}} \prod H^{1}(K_{i},\tilde{G}_{K}).$$

Let $\tilde{\xi}$ denote the torsor in $H^1_{\text{\'et}}(R, \tilde{G})$ corresponding to ξ as above. Therefore $\tilde{\xi}_K := i_k(\xi)$ is in the kernel of

$$\mathrm{H}^1(K, \tilde{G}_K) \to \prod_{i=1}^r \mathrm{H}^1(K_i, \tilde{G}_K).$$

Suppose that \tilde{G}_K satisfies SQ. Then $\tilde{\xi}_K$ is trivial. However by [12], the natural map $H^1_{\text{\'et}}(R,\tilde{G}) \to H^1(K,\tilde{G}_K)$ is injective and hence $\tilde{\xi}$ is trivial in $H^1_{\text{\'et}}(R,\tilde{G})$. This implies that its specialization, ξ , is trivial in $H^1(k,G)$.

Thus from here on, we assume that the base field k has characteristic 0.

2.2. Lemmata.

Lemma 2.1. Let k-groups G and H satisfy SQ. Then $G \times_k H$ also satisfies SQ.

Proof. Let L/k be a field extension. Then the map

$$\mathrm{H}^1(k, G \times_k H) \to \mathrm{H}^1(L, G \times_k H)$$

is precisely the product of the maps

$$\mathrm{H}^1(k,G) \to \mathrm{H}^1(L,G)$$
 and $\mathrm{H}^1(k,H) \to \mathrm{H}^1(L,H)$.

This immediately shows that if G and H satisfy SQ, so does $G \times_k H$.

Lemma 2.2. Let $1 \to Q \to H \to G \to 1$ be a central extension of a k-group G by a quasi-trivial torus Q. Then H satisfies SQ if and only if G satisfies SQ.

Proof. Let L_i be field extensions of k such that $gcd[L_i : k] = 1$. Since Q is quasitrivial, $H^1(L, Q) = \{1\} \ \forall \ L/k$. From the long exact sequence in cohomology, we have the following commutative diagram.

$$1 \longrightarrow H^{1}(k, H) \longrightarrow H^{1}(k, G) \xrightarrow{\delta_{k}} H^{2}(k, Q)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$1 \longrightarrow \prod H^{1}(L_{i}, H) \longrightarrow \prod H^{1}(L_{i}, G) \xrightarrow{\prod \delta_{L_{i}}} \prod H^{2}(L_{i}, Q)$$

From the above diagram, it is clear that if G satisfies SQ, so does H.

Conversely, assume that H satisfies SQ. Let $a \in H^1(k,G)$ become trivial in $\prod H^1(L_i,G)$. Then $\delta_k(a)$ becomes trivial in each $H^2(L_i,Q)$. Hence the corestriction $\operatorname{Cor}_{L_i/k}(\delta_k(a)) = \delta_k(a)^{d_i}$ becomes trivial in $H^2(k,Q)$ where $d_i = [L_i : k]$. Since $\gcd_i(d_i) = 1$, this implies that $\delta_k(a)$ is itself trivial in $H^2(k,Q)$. Therefore a comes from an element $b \in H^1(k,H)$ which is trivial in $\prod H^1(L_i,H)$. (The fact that $H^1(L_i,Q) = \{1\}$ guarantees that b is trivial in $H^1(L_i,H)$.) Since H satisfies SQ by assumption, b is trivial in $H^1(k,H)$ which implies the triviality of a in $H^1(k,G)$.

Lemma 2.3. Let E be a finite separable field extension of k and let H be an E-group satisfying SQ. Then the k-group $R_{E/k}(H)$ also satisfies SQ.

Proof. Set $G = R_{E/k}(H)$ and let ξ be an element in the kernel of $H^1(k, G) \to \prod_{i=1}^r H^1(L_i, G)$ where $gcd_i[L_i:k] = 1$.

Since $\operatorname{char}(k) = 0$, $L_i \otimes_k E$ is an étale E-algebra and hence isomorphic to $E_{1,i} \times E_{2,i} \times \cdots \times E_{n_i,i}$ where each $E_{j,i}$ is a separable field extension of E. Thus $\sum_{j=1}^{n_i} \left[E_{j,i} : E \right] = \left[L_i : k \right]$ and therefore $\operatorname{gcd} \left[E_{j,i} : E \right] = 1$ where $1 \leq i \leq r$ and $1 \leq j \leq n_i$.

By Eckmann-Faddeev-Shapiro, we have a natural bijection of pointed sets

$$\mathrm{H}^{1}\left(k,G\right)\simeq\mathrm{H}^{1}\left(E,H\right),$$
 $\mathrm{H}^{1}\left(L_{i},G\right)\simeq\prod_{j=1}^{n_{i}}\mathrm{H}^{1}\left(E_{j,i},H\right).$

Thus we have that ξ is in the kernel of $H^1(E, H) \to \prod_{i \le r, j \le n_i} H^1(E_{j,i}, H)$. Since H satisfies SQ, we see that ξ is trivial.

3. Serre's question and norm principles

3.1. Intermediate groups \hat{G} and \tilde{G} . Notations are as in Section 5 of [1].

Let G be our given connected reductive k-group whose Dynkin diagram contains connected components only of type A_n , B_n , C_n or (non-trialitarian) D_n and let G' denote its derived subgroup. Let Z(G) = T and $Z(G') = \mu$.

Let $\rho: \mu \hookrightarrow S$ be an embedding of μ into a quasi-trivial torus S. We denote the cofibre product $e(G', \rho) = \frac{G' \times S}{\mu}$ by \hat{G} . This k-group is called an *envelope* of G'.

$$\mu \xrightarrow{\delta} G'$$

$$\downarrow^{\rho} \qquad \downarrow$$

$$S \xrightarrow{\gamma} \hat{G}$$

Now the quasi-trivial torus $S = Z(\hat{G})$ and \hat{G} fit into an exact sequence as follows:

$$1 \to S \to \hat{G} \to G'^{ad} \to 1 \tag{*}$$

where G'^{ad} corresponds to the adjoint group of G'. We now recall the following result of Jodi Black which addresses Serre's question for adjoint groups of classical type.

Theorem 3.1 (Jodi Black, [3, Thm. 0.2]). Let k be a field of characteristic different from 2 and let J be an absolutely simple algebraic k-group which is not of type E_8 and which is either a simply connected or adjoint classical group or a quasi-split exceptional group. Then Serre's question has a positive answer for J.

Since every adjoint group of classical type is a product of Weil restrictions of absolutely simple adjoint groups, the above theorem, along with Lemmata 2.1 and 2.3, implies that G'^{ad} satisfies SQ. Applying Lemma 2.2 to the exact sequence (*) above, we see that \hat{G} satisfies SQ. Let us chose such an envelope \hat{G} of G' which satisfies SQ.

Define an intermediate abelian group \tilde{T} to be the cofibre product $\frac{T \times S}{\mu}$.

$$\mu \longrightarrow T$$

$$\downarrow^{\rho} \qquad \downarrow^{\alpha}$$

$$S \stackrel{\nu}{\longrightarrow} \tilde{T}$$

Let the algebraic group \tilde{G} be the cofibre product defined by the following diagram:

$$G' \times T \xrightarrow{m} G$$

$$\downarrow_{id} \times \alpha \qquad \qquad \downarrow_{\beta}$$

$$G' \times \tilde{T} \xrightarrow{\epsilon} \tilde{G}.$$

Then we have the following commutative diagram with exact rows [1, Prop. 5.1]. Note that each row is a central extension of \tilde{G} .

$$1 \longrightarrow \mu \xrightarrow{\delta,\nu\rho} G' \times \tilde{T} \xrightarrow{\epsilon} \tilde{G} \longrightarrow 1$$

$$\downarrow^{\rho} \qquad \downarrow \qquad \downarrow^{id}$$

$$1 \longrightarrow S \xrightarrow{\gamma,\nu} \hat{G} \times \tilde{T} \longrightarrow \tilde{G} \longrightarrow 1$$

$$(***)$$

Since \tilde{T} is abelian, the existence of the co-restriction map shows that \tilde{T} satisfies SQ. Since \hat{G} satisfies SQ, we can apply Lemmata 2.1 and 2.2 to (***) to see that \tilde{G} satisfies SQ.

3.2. Norm principle and weak norm principle. Let $f: G \to T$ be a map of k-groups where T is an abelian k-group. Then we have norm maps $N_{L/k}: T(L) \to T(k)$ for any separable field extension L/k.

$$G(L) \xrightarrow{f(L)} T(L)$$

$$\downarrow^{N_{L/k}}$$

$$G(k) \xrightarrow{f(k)} T(k)$$

We say that the *norm principle* holds for $f: G \to T$ if for all separable field extensions L/k,

$$N_{L/k}(\text{Image } f(L)) \subseteq \text{Image } f(k).$$

That is, we say that the *norm principle* holds for $f: G \to T$ if given any separable field extension L/k and any $t \in T(L)$ such that

$$t \in (\text{Image } f(L) : G(L) \to T(L)),$$

 $N_{L/k}(t) \in (\text{Image } f(k) : G(k) \to T(k)).$

then

then

Note that the norm principle holds for any algebraic group homomorphism between abelian groups.

We say that the *weak norm principle* holds for $f: G \to T$ if given any $t \in T(k)$ such that

$$t \in (\text{Image } f(L) : G(L) \to T(L)),$$

$$t^{[L:k]} = N_{L/k}(t) \in (\text{Image } f(k) : G(k) \to T(k)).$$

It is clear that if the norm principle holds for f, then so does the weak norm principle.

3.3. Relating Serre's question and norm principle. The deduction of SQ for G from \hat{G} and \tilde{G} follows via the (weak) norm principles.

Let $\beta: G \to \tilde{G}$ be the embedding of k-groups with the cokernel P isomorphic to the torus $\frac{S}{\mu}$ where \tilde{G} and G are as in Section 3.1. Thus we have the following exact sequence:

$$1 \to G \xrightarrow{\beta} \tilde{G} \xrightarrow{\pi} P \to 1.$$

Lemma 3.2. If the weak norm principle holds for $\pi: \tilde{G} \to P$, then G satisfies SQ.

Proof. From the long exact sequence of cohomology, we have the following commutative diagram:

Let $a \in H^1(k, G)$ become trivial in $\prod H^1(L_i, G)$. As \tilde{G} satisfies SQ, $\beta_k(a)$ becomes trivial in $H^1(k, \tilde{G})$. Hence $a = \delta_k(b)$ for some $b \in P(k)$ and $\delta_{L_i}(b)$ is trivial in $H^1(L_i, G)$. Therefore, there exist $c_i \in \tilde{G}(L_i)$ such that $\pi_{L_i}(c_i) = b$.

Showing that G satisfies SQ, i.e. that a is trivial, is equivalent to showing

$$b \in (\operatorname{Image} \pi_k : \tilde{G}(k) \to P(k)).$$

However $b \in (\operatorname{Image} \pi_{L_i} : \tilde{G}(L_i) \to P(L_i))$. Since the weak norm principle holds for $\pi : \tilde{G} \to P$, $b^{d_i} \in \operatorname{Image} (\pi_k : \tilde{G}(k) \to P(k))$ where $[L_i : k] = d_i$ for each i. As $\gcd_i(d_i) = 1$, this means $b \in \operatorname{Image} (\pi_k : \tilde{G}(k) \to P(k))$.

We recall now the norm principle of Merkurjev and Barquero for reductive groups of classical type.

Theorem 3.3 (Barquero–Merkurjev, [1]). Let G be a reductive group over a field k. Assume that the Dynkin diagram of G does not contain connected components D_n , $n \ge 4$, E_6 or E_7 . Let T be any commutative k-group. Then the norm principle holds for any group homomorphism $G \to T$.

This shows that the norm principle and hence the weak norm principle holds for the map $\pi: \tilde{G} \to P$ for reductive k-groups G as in the main theorem (Theorem 1.2). Thus we have concluded the proof for the following:

Theorem 1.2. Let k be a field of characteristic not 2. Let G be a connected reductive k-group whose Dynkin diagram contains connected components only of type A_n , B_n or C_n . Then Serre's question has a positive answer for G.

4. Obstruction to norm principle for (non-trialitarian) D_n

4.1. Preliminaries. Let (A, σ) be a central simple algebra of degree 2n over k and let σ be an orthogonal involution. Let $C(A, \sigma)$ denote its Clifford algebra which is a central simple algebra over its center, Z/k, the discriminant extension. Let i denote the non-trivial automorphism of Z/k and let $\underline{\sigma}$ denote the canonical involution of $C(A, \sigma)$.

Recall that, depending on the parity of n, $\underline{\sigma}$ is either an involution of the second kind (when n is odd) or of the first kind (when n is even). Let $\underline{\mu}$: Sim (C (A, σ), $\underline{\sigma}$) $\rightarrow R_{Z/k}\mathbb{G}_m$ denote the multiplier map sending similitude c to $\underline{\sigma}(c)c$.

Let $\Omega(A, \sigma)$ be the *extended Clifford group*. Note that this has center $R_{Z/k}\mathbb{G}_m$ and is an *envelope* of Spin (A, σ) [1, Ex. 4.4]. We recall below the map $\kappa: \Omega(A, \sigma)(k) \to Z^*/k^*$ as defined in [9, p. 182].

Given $\omega \in \Omega(A, \sigma)(k)$, let $g \in GO^+(A, \sigma)(k)$ be some similitude such that $\omega \to gk^*$ under the natural surjection $\Omega(A, \sigma)(k) \to PGO^+(A, \sigma)(k)$.

Let $h = \mu(g)^{-1}g^2 \in O^+(A, \sigma)(k)$ and let $\gamma \in \Gamma(A, \sigma)(k)$ be some element in the *special Clifford group* which maps to h under the vector representation $\chi': \Gamma(A, \sigma)(k) \to O^+(A, \sigma)(k)$. Then $\omega^2 = \gamma z$ for some $z \in Z^*$ and $\varkappa(\omega) = zk^*$.

Note that the map \varkappa has $\Gamma(A, \sigma)(k)$ as kernel. Also if $z \in \mathbb{Z}^*$, then $\varkappa(z) = z^2 k^*$.

By following the reductions in [1], it is easy to see that one needs to investigate whether the norm principle holds for the canonical map

$$\Omega\left(\mathbf{A},\sigma\right) \to \frac{\Omega\left(\mathbf{A},\sigma\right)}{\left[\Omega\left(\mathbf{A},\sigma\right),\Omega\left(\mathbf{A},\sigma\right)\right]}.$$

We will need to investigate the norm principle for two different maps depending on the parity of n.

The map μ_* for n odd. Let $U \subset \mathbb{G}_m \times R_{Z/k}\mathbb{G}_m$ be the algebraic subgroup defined by

$$U(k) = \{ (f, z) \in k^* \times Z^* | f^4 = N_{Z/k}(z) \}.$$

Recall the map $\mu_*: \Omega(A, \sigma) \to U$ defined in [9, p. 188] which sends

$$\omega \to \left(\underline{\mu}(\omega), ai(a)^{-1}\underline{\mu}(\omega)^2\right),$$

where $\omega \in \Omega(A, \sigma)(k)$ and $\varkappa(\omega) = a k^*$. This induces the following exact sequence [9, p. 190]

$$1 \to \mathrm{Spin}\,(\mathbf{A},\sigma) \to \Omega\ (\mathbf{A},\sigma) \xrightarrow{\mu_*} U \to 1.$$

Since the semisimple part of $\Omega(A, \sigma)$ is Spin (A, σ) , the above exact sequence shows that it suffices to check the norm principle for the map μ_* .

The map $\underline{\mu}$ for n even. Recall the following exact sequence induced by restricting $\underline{\mu}$ to $\Omega(A, \sigma)$ [9, p. 187]

$$1 \to \operatorname{Spin}(A, \sigma) \to \Omega(A, \sigma) \xrightarrow{\underline{\mu}} R_{Z/k} \mathbb{G}_m \to 1.$$

Since the semisimple part of $\Omega(A, \sigma)$ is Spin (A, σ) , the above exact sequence shows that it suffices to check the norm principle for the map μ .

4.2. An obstruction to being in the image of μ_* for n odd. Given $(f, z) \in U(k)$, we would like to formulate an obstruction which prevents (f, z) from being in the image $\mu_* (\Omega(A, \sigma)(k))$. Note that for $z \in Z^*$, $\mu_*(z) = (N_{Z/k}(z), z^4)$ and hence the algebraic subgroup $U_0 \subseteq U$ defined by

$$U_0(k) = \{(N_{Z/k}(z), z^4) | z \in Z^*\}$$

has its k-points in the image $\mu_* (\Omega (A, \sigma) (k))$.

Let $\mu_{n[Z]}$ denote the kernel of the norm map $R_{K/k}\mu_n \xrightarrow{N} \mu_n$ where K/k is a quadratic extension. Note that $\mu_{4[Z]}$ is the center of Spin (A, σ) as n is odd. Also recall that [9, Prop. 30.13, p. 418]

$$\mathrm{H}^{1}\left(k,\mu_{4[Z]}\right)\cong\frac{U(k)}{U_{0}(k)}.$$

Thus, we can construct the map $S: PGO^+(A, \sigma)(k) \to H^1(k, \mu_{4[Z]})$ induced by the following commutative diagram with exact rows:

$$1 \longrightarrow Z^* \longrightarrow \Omega(A, \sigma)(k) \xrightarrow{\chi'} PGO^+(A, \sigma)(k) \longrightarrow 1$$

$$\downarrow^{\mu_*} \qquad \downarrow^{\mu_*} \qquad \downarrow^{S}$$

$$1 \longrightarrow U_0(k) \longrightarrow U(k) \longrightarrow H^1(k, \mu_{4[Z]}) \longrightarrow 1$$

The map S also turns out to be the connecting map from PGO⁺ $(A, \sigma)(k) \rightarrow H^1(k, \mu_{4[Z]})$ [9, Prop. 13.37, p. 190] in the long exact sequence of cohomology corresponding to the exact sequence

$$1 \to \mu_{4[Z]} \to \operatorname{Spin}(A, \sigma) \to \operatorname{PGO}^+(A, \sigma) \to 1.$$

Since the maps $\mu_*: Z^* \to U_0(k)$ and $\chi': \Omega(A, \sigma)(k) \to PGO^+(A, \sigma)(k)$ are surjective, an element $(f, z) \in U(k)$ is in the image $\mu_*(\Omega(A, \sigma)(k))$ if and only if its image $[f, z] \in H^1(k, \mu_{4\lceil Z \rceil})$ is in the image $S(PGO^+(A, \sigma)(k))$.

Therefore we look for an obstruction preventing [f, z] from being in the image $S(PGO^+(A, \sigma)(k))$. Recall the following commutative diagram with exact rows and columns:

$$1 \longrightarrow \mu_{2} \longrightarrow \operatorname{Spin}(A, \sigma) \xrightarrow{\chi} O^{+}(A, \sigma) \longrightarrow 1$$

$$\downarrow \qquad \qquad \downarrow^{id} \qquad \downarrow^{\pi}$$

$$1 \longrightarrow \mu_{4[Z]} \longrightarrow \operatorname{Spin}(A, \sigma) \xrightarrow{\chi'} \operatorname{PGO^{+}}(A, \sigma) \longrightarrow 1$$

$$\downarrow \qquad \qquad \downarrow$$

The long exact sequence of cohomology induces the following commutative diagram (Figure 1) with exact columns [9, Prop. 13.36, p. 189], where

$$O^{+}(A, \sigma)(k) \xrightarrow{Sn} \xrightarrow{k^{*}} \frac{k^{*}}{k^{*2}}$$

$$\downarrow^{\pi} \qquad \qquad \downarrow^{i}$$

$$PGO^{+}(A, \sigma)(k) \xrightarrow{S} H^{1}(k, \mu_{4[Z]})$$

$$\downarrow^{\mu} \qquad \qquad \downarrow^{j}$$

$$\frac{k^{*}}{k^{*2}} \qquad = \qquad \frac{k^{*}}{k^{*2}}$$

Figure 1. Spinor norms and S for n odd

 $\mu: \operatorname{PGO}^+(A,\sigma)(k) \to \frac{k^*}{k^{*2}} \text{ is induced by the multiplier map } \mu: \operatorname{GO}^+(A,\sigma) \to \mathbb{G}_m$ $i: \frac{k^*}{k^{*2}} \to \operatorname{H}^1\left(k,\mu_{4[Z]}\right) = \frac{U(k)}{U_0(k)} \text{ is the map sending } fk^{*2} \to [f,f^2]$ $j: \frac{U(k)}{U_0(k)} = \operatorname{H}^1\left(k,\mu_{4[Z]}\right) \to \frac{k^*}{k^{*2}} \text{ is the map sending } [f,z] \to \operatorname{N}(z_0)k^{*2},$ $\text{where } z_0 \in Z^* \text{ is such that } z_0i(z_0)^{-1} = f^{-2}z.$

Definition 4.1. We call an element $(f, z) \in U(k)$ to be *special* if there exists a $[g] \in PGO^+(A, \sigma)(k)$ such that $j([f, z]) = \mu([g])$.

Let $(f, z) \in U(k)$ be a special element and let $[g] \in PGO^+(A, \sigma)(k)$ be such that $j([f, z]) = \mu([g])$. From the discussion above, it is clear that (f, z) is in the image $\mu_*(\Omega(A, \sigma)(k))$ if and only if [f, z] is in the image $S(PGO^+(A, \sigma)(k))$.

Thus $S([g])[f,z]^{-1}$ is in kernel $j=\operatorname{Image} i$ and hence there exists some $\alpha \in k^*$ such that

$$[f, z] = S([g])[\alpha, \alpha^2] \in \frac{U(k)}{U_0(k)}.$$

Note that if g is changed by an element in $O^+(A, \sigma)(k)$, then α changes by a spinor norm by Figure 1 above. Thus given a special element, we have produced a scalar $\alpha \in k^*$ which is well defined upto spinor norms.

$$[f, z] \in S\left(\operatorname{PGO}^+(A, \sigma)(k)\right) \iff [\alpha, \alpha^2] \in S\left(\operatorname{PGO}^+(A, \sigma)(k)\right)$$

 $\iff (\alpha, \alpha^2) \in \mu_*\left(\Omega\left(A, \sigma\right)(k)\right).$

This happens if and only if there exists $w \in \Omega(A, \sigma)(k)$ such that

$$\alpha = \underline{\mu}(w)$$

$$\alpha^2 = \varkappa(w)i(\varkappa(w))^{-1}\underline{\mu}(w)^2$$

This implies $\kappa(w) \in k^*$ and hence $w \in \Gamma(A, \sigma)(k)$. Thus α is a spinor norm, being the similarity of an element in the special Clifford group. Also note if α is a spinor norm, then $\alpha = \underline{\mu}(\gamma)$ for some $\gamma \in \Gamma(A, \sigma)(k)$ and $\mu_*(\gamma) = (\underline{\mu}(\gamma), \underline{\mu}(\gamma)^2)$.

Thus a special element (f,z) is in the image of μ_* if and only if the produced scalar α is a spinor norm. We call the class of α in $\frac{k^*}{\operatorname{Sn}(A,\sigma)}$ to be the scalar obstruction preventing the *special* element $(f,z) \in U(k)$ from being in the image $\mu_*(\Omega(A,\sigma)(k))$.

4.3. An obstruction to being in the image of $\underline{\mu}$ for n even. Given $z \in Z^*$, we would like to formulate an obstruction which prevents z from being in the image $\underline{\mu}(\Omega(A, \sigma)(k))$. Note that for $z \in Z^*$, $\underline{\mu}(z) = z^2$ and hence the subgroup Z^{*2} is in the image $\mu(\Omega(A, \sigma)(k))$.

Like in the case of odd n, we can construct the map $S: PGO^+(A, \sigma)(k) \to \frac{Z^*}{Z^{*2}}$ induced by the following commutative diagram with exact rows [9, Def. 13.32, p. 187]:

$$1 \longrightarrow Z^* \longrightarrow \Omega (A, \sigma) (k) \xrightarrow{\chi'} PGO^+ (A, \sigma) (k) \longrightarrow 1$$

$$\downarrow \underline{\mu} \qquad \qquad \downarrow \underline{\mu} \qquad \qquad \downarrow S$$

$$1 \longrightarrow Z^{*2} \longrightarrow Z^* \longrightarrow Z^* \longrightarrow 1$$

Again by the surjectivity of the maps, $\underline{\mu}: Z^* \to Z^{*2}$ and $\chi': \Omega(A, \sigma)(k) \to PGO^+(A, \sigma)(k)$, an element $z \in Z^*$ is in the image $\underline{\mu}(\Omega(A, \sigma)(k))$ if and only

if its image $[z] \in \frac{Z^*}{Z^{*2}}$ is in the image $S(PGO^+(A, \sigma)(k))$. Therefore we look for an obstruction preventing [z] from being in the image $S(PGO^+(A, \sigma)(k))$. And as before, we arrive at the following commutative diagram (Figure 2) with exact rows and columns [9, Prop. 13.33, p. 188], where

$$O^{+}(A, \sigma)(k) \xrightarrow{Sn} \frac{k^{*}}{k^{*2}}$$

$$\downarrow^{\pi} \qquad \qquad \downarrow^{i}$$

$$PGO^{+}(A, \sigma)(k) \xrightarrow{S} \frac{Z^{*}}{Z^{*2}}$$

$$\downarrow^{\mu} \qquad \qquad \downarrow^{j}$$

$$\frac{k^{*}}{k^{*2}} \qquad = \qquad \frac{k^{*}}{k^{*2}}$$

Figure 2. Spinor norms and S for *n* even

 $\mu: \operatorname{PGO}^+(A,\sigma)(k) \to \frac{k^*}{k^{*2}}$ is induced by the multiplier map $\mu: \operatorname{GO}^+(A,\sigma) \to \mathbb{G}_m$

 $i: \frac{k^*}{k^{*2}} \to \frac{Z^*}{Z^{*2}}$ is the inclusion map

 $j: \frac{Z^*}{Z^{*2}} \to \frac{k^*}{k^{*2}}$ is induced by the norm map from $Z^* \to k^*$.

Definition 4.2. We call an element $z \in Z^*$ to be *special* if there exists a $[g] \in PGO^+(A, \sigma)(k)$ such that $j([z]) = \mu([g])$.

Let $z \in Z^*$ be a special element and let $[g] \in PGO^+(A, \sigma)(k)$ be such that $j([z]) = \mu([g])$. As before a *special* element $z \in Z^*$ is in the image $\underline{\mu}(\Omega(A, \sigma)(k))$ if and only if [z] is in the image $S(PGO^+(A, \sigma)(k))$.

Thus $S([g])[z]^{-1}$ is in kernel j = Image i and hence there exists some $\alpha \in k^*$ such that

$$[z] = S([g])[\alpha] \in \frac{Z^*}{Z^{*2}}.$$

Note that if g is changed by an element in $O^+(A, \sigma)(k)$, then α changes by a spinor norm by Figure 2 above. Thus given a special element, we have produced a scalar $\alpha \in k^*$ which is well defined up to spinor norms.

$$[z] \in S\left(\operatorname{PGO}^{+}(A,\sigma)(k)\right) \iff [\alpha] \in S\left(\operatorname{PGO}^{+}(A,\sigma)(k)\right)$$

 $\iff (\alpha) \in \mu\left(\Omega\left(A,\sigma\right)(k)\right).$

Since $\alpha \in k^*$ also, this is equivalent to α being a spinor norm [9, Prop. 13.25, p. 184].

We call the class of α in $\frac{k^*}{\operatorname{Sn}(A,\sigma)}$ to be the scalar obstruction preventing the *special* element $z \in Z^*$ from being in the image $\underline{\mu}(\Omega(A,\sigma)(k))$.

4.4. Scharlau's norm principle for $\mu : GO^+(A, \sigma) \to \mathbb{G}_m$. Let $\mu : GO^+(A, \sigma) \to \mathbb{G}_m$ denote the multiplier map and let L/k be a separable field extension of finite degree. Let $g_1 \in GO^+(A, \sigma)$ (L) be such that $\mu(g_1) = f_1 \in L^*$. Let f denote $N_{L/k}(f_1)$. We would like to show that f is in the image $\mu(GO^+(A, \sigma)(k))$.

Note that by a generalization of Scharlau's norm principle ([9, Prop. 12.21]; [3, Lemma 4.3]) there exists a $\tilde{g} \in GO(A, \sigma)(k)$ such that $f = \mu(\tilde{g})$. However we would like to find a *proper* similitude $g \in GO^+(A, \sigma)(k)$ such that $\mu(g) = f$.

We investigate the cases when the algebra A is non-split and split separately.

Case I: A is non-split. Note that $g_1 \in GO^+(A, \sigma)(L)$. If $\tilde{g} \in GO^+(A, \sigma)(k)$, we are done. Hence assume $\tilde{g} \notin GO^+(A, \sigma)(k)$. By a generalization of Dieudonné's theorem [9, Thm. 13.38, p. 190], we see that the quaternion algebras

$$B_1 = (Z, f_1) = 0 \in Br(L),$$

 $B_2 = (Z, f) = A \in Br(k).$

Since A is non-split, $B_2 \neq 0 \in Br(k)$. However co-restriction of B_1 from L to k gives a contradiction, because

$$0 = \operatorname{Cor} B_1 = (Z, N_{L/k}(f_1)) = B_2 \in \operatorname{Br}(k).$$

Hence $\tilde{g} \in GO^+(A, \sigma)(k)$.

Case II: A is split. Since A is split, $A = \operatorname{End} V$ where (V, q) is a quadratic space and σ is the adjoint involution for the quadratic form q. Again, if $\tilde{g} \in \operatorname{GO}^+(A, \sigma)(k)$, we are done. Hence assume $\tilde{g} \notin \operatorname{GO}^+(A, \sigma)(k)$. That is

$$\det(\tilde{g}) = -f^{2n/2} = -(f^n).$$

Since A is of even degree (2n) and split, there exists an isometry h of determinant -1. Set $g = \tilde{g}h$. Then $\det(g) = f^n$ where $\mu(g) = f$. Thus we have found a suitable $g \in \mathrm{GO}^+(A,\sigma)(k)$ which concludes the proof of the following:

Theorem 4.3. The norm principle holds for the map $\mu : GO^+(A, \sigma) \to \mathbb{G}_m$.

4.5. Spinor obstruction to norm principle for non-trialitarian D_n . Let L/k be a separable field extension of finite degree. And let $w_1 \in \Omega(A, \sigma)(L)$ be such that for

$$n \text{ odd}: \mu_*(w_1) = \theta \text{ which is equal to } (f_1, z_1) \in U(L),$$

 $n \text{ even}: \underline{\mu}(w_1) = \theta \text{ which is equal to } z_1 \in (R_{Z/k}\mathbb{G}_m)(L).$

Since V is of even dimension 2n, h can be chosen to be a hyperplane reflection for instance

We would like to investigate whether $N_{L/k}(\theta)$ is in the image of $\mu_*(\Omega(A,\sigma)(k))$ (resp. $\underline{\mu}(\Omega(A,\sigma)(k))$) when n is odd (resp. even) in order to check if the norm principle holds for the map $\mu_*:\Omega(A,\sigma)\to U$ (resp. $\underline{\mu}:\Omega(A,\sigma)\to R_{Z/k}\mathbb{G}_m$).

Let $[g_1] \in PGO^+(A, \sigma)(L)$ be the image of w_1 under the canonical map $\chi': \Omega(A, \sigma)(L) \to PGO^+(A, \sigma)(L)$. Clearly θ is *special* and let $g_1 \in GO^+(A, \sigma)(L)$ be such that $\mu([g_1]) = j([\theta])$.

By Theorem 4.3, there exists a $g \in GO^+(A, \sigma)(k)$ such that²

$$\mu([g]) = \mathcal{N}_{L/k} (j[\theta]) = j \left([\mathcal{N}_{L/k} \theta] \right).$$

Hence $N_{L/k}(\theta)$ is special.

By Subsection 4.2 (resp. 4.3), $N_{L/k}(\theta)$ is in the image of μ_* (resp $\underline{\mu}$) if and only if the scalar obstruction $\alpha \in \frac{k^*}{\operatorname{Sn}(A,\sigma)}$ defined for $N_{L/k}(\theta)$ vanishes. Thus we have a spinor norm obstruction given below.

Theorem 4.4 (Spinor norm obstruction). Let L/k be a finite separable extension of fields. Let f denote the map μ_* (resp $\underline{\mu}$) in the case when n is odd (resp. even). Given $\theta \in f(\Omega(A, \sigma)(L))$, there exists scalar obstruction $\alpha \in k^*$ such that

$$N_{L/k}(\theta) \in f\left(\Omega\left(\mathbf{A},\sigma\right)(k)\right) \iff \alpha = 1 \in \frac{k*}{\mathrm{Sn}(\mathbf{A},\sigma)}.$$

Thus the norm principle for the canonical map

$$\Omega\left(\mathbf{A},\sigma\right)\rightarrow\frac{\Omega\left(\mathbf{A},\sigma\right)}{\left[\Omega\left(\mathbf{A},\sigma\right),\Omega\left(\mathbf{A},\sigma\right)\right]}$$

and hence for non-trialitarian D_n holds if and only if the scalar obstructions are spinor norms.

5. Quasi-split groups

Let G be a connected reductive k-group whose Dynkin diagram does not contain connected components of type E_8 and let G' denote its derived subgroup. Let G^{sc} denote the simply connected cover of G'. Then one has the exact sequence $1 \to C \to G^{sc} \to G' \to 1$, where C is a finite k-group of multiplicative type, central in G^{sc} . Assuming that G^{sc} is quasi-split, we would like to show that G satisfies SQ by following the reduction techniques used in Sections 2 and 3.

Lemma 5.1. Let G be a connected reductive k-group. If G^{sc} is quasi-split, then there exists an extension $1 \to Q \to H \xrightarrow{\psi} G \to 1$, where Q is a quasi-trivial k-torus, central in reductive k-group H with H' simply connected and quasi-split.

²The map j commutes with $N_{L/k}$ in both cases.

Proof. Recall that there is a central extension (called a z-extension) of G by a quasitrivial torus Q such that H' is semisimple and simply connected ([11, Prop. 3.1] and [4, Lemma 1.1.4]).

$$1 \to Q \to H \xrightarrow{\psi} G \to 1.$$

The restriction $\psi|_{H'}: H' \to G$ yields the fact that H' is the simply connected cover of G' and hence is quasi-split.

Lemmata 2.2 and 5.1 imply that we can restrict ourselves to connected reductive k-groups G such that G' is simply connected and quasi-split.

Lemma 5.2. Let H be any reductive k-group such that its derived subgroup H' is semisimple simply connected and quasi-split. Let T denote the k-torus H/H'. Then the natural exact sequence $1 \to H' \to H \to T \to 1$ induces surjective maps $\phi(L): H(L) \to T(L)$ for all field extensions L/k. In particular, the norm principle holds for $\phi: H \to T$.

Proof. There exists a quasi-trivial maximal torus Q_1 of H' defined over k [8, Lem. 6.7]. Let $Q_1 \subset Q_2$, where Q_2 is a maximal torus of H defined over k. The proof of [8, Lem. 6.6] shows that $\phi|_{Q_2}:Q_2\to T$ is surjective and that $Q_2\cap H'$ is a maximal torus of H'. Since $Q_2\cap H'\subseteq Q_1$, we get the following extension of k-tori

$$1 \rightarrow Q_1 \rightarrow Q_2 \rightarrow T \rightarrow 1$$

Since Q_1 is quasitrivial, $H^1(L,Q_1)=0$ for any field extension L/k which gives the surjectivity of $\phi(L):Q_2(L)\to T(L)$ and hence of $\phi(L):H(L)\to T(L)$. \square

Let \hat{G} be an envelope of G' defined using an embedding of $\mu = Z(G')$ into a quasi-trivial torus S. Note that G' is assumed to be simply connected and quasi-split and is also the derived subgroup of \hat{G} by construction.

$$\mu \xrightarrow{\delta} G' \\
\downarrow^{\rho} \qquad \downarrow \\
S \xrightarrow{\gamma} \hat{G}$$

Thus, we get an exact sequence $1 \to G' \to \hat{G} \to \hat{G}/G' \to 1$ to which we can apply Lemma 5.2 to conclude that the norm principle holds for the canonical map $\hat{G} \to \frac{\hat{G}}{\left[\hat{G},\hat{G}\right]}$.

Constructing the intermediate group \tilde{G} as in Section 3.1, we see that the norm principle also holds for the natural map $\tilde{G} \to \tilde{G}/G$ [1, Prop. 5.1]. Then using Theorem 3.1 [3], Lemma 3.2, and a remark from Gopal Prasad that G^{sc} is quasi-split if and only if G is quasi-split, we can conclude that Theorem 1.3 (restated below) holds.

Theorem 1.3. Let k be a field of characteristic not 2. Let G be a connected quasi-split reductive k-group whose Dynkin diagram does not contain connected components of type E_8 . Then Serre's question has a positive answer for G.

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