**Zeitschrift:** Commentarii Mathematici Helvetici
**Herausgeber:** Schweizerische Mathematische Gesellschaft
**Band:** 91 (2016)
**Heft:** 1

**Artikel:** On Serre's injectivity question and norm principle
**Autor:** Bhaskhar, Nivedita
**DOI:** https://doi.org/10.5169/seals-630555

Comment. Math. Helv. 91 (2016), 145–161
DOI 10.4171/CMH/381

**Commentarii Mathematici Helvetici**


# On Serre's injectivity question and norm principle

Nivedita Bhaskhar*

**Abstract.** Let $k$ be a field of characteristic not 2. We give a positive answer to Serre's injectivity question for any smooth connected reductive $k$-group whose Dynkin diagram contains connected components only of type $A_n$, $B_n$ or $C_n$. We do this by relating Serre's question to the norm principles proved by Barquero and Merkurjev. We give a scalar obstruction defined up to spinor norms whose vanishing will imply the norm principle for the non-trialitarian $D_n$ case and yield a positive answer to Serre's question for connected reductive $k$-groups whose Dynkin diagrams contain components of (non-trialitarian) type $D_n$ too. We also investigate Serre's question for quasi-split reductive $k$-groups.

**Mathematics Subject Classification (2010).** 14L35, 20G10, 20G15.

**Keywords.** Principal homogeneous spaces, Serre's question, zero cycles, norm principles, spinor norm, Galois cohomology.

## 1. Introduction

Let $k$ be a field. Then the following question of Serre, which is open in general, asks

**Question 1.1** (Serre, [13, p. 233]). *Let $G$ be any connected linear algebraic group over a field $k$. Let $L_1, L_2, \dots, L_r$ be finite field extensions of $k$ of degrees $d_1, d_2, \dots, d_r$ respectively such that $\gcd_i(d_i) = 1$. Then is the following sequence exact ?*

$$1 \to \mathrm{H}^1(k, G) \to \prod_{i=1}^{r} \mathrm{H}^1(L_i, G).$$

The classical result that the index of a central simple algebra divides the degrees of its splitting fields answers Serre's question affirmatively for the group $\mathrm{PGL}_n$. Springer's theorem for quadratic forms answers it affirmatively for the (albeit sometimes disconnected) group $\mathrm{O}(q)$ and Bayer–Lenstra's theorem [2], for the groups of isometries of algebras with involutions. Jodi Black [3] answers Serre's question positively for absolutely simple simply connected and adjoint $k$-groups of classical type. In this paper, we use and extend Jodi's result to connected reductive $k$-groups whose Dynkin diagram contains connected components only of type $A_n$, $B_n$ or $C_n$.

*The author acknowledges support from the NSF-FRG grant 1463882.

**Theorem 1.2.** *Let $k$ be a field of characteristic not 2. Let $G$ be a connected reductive $k$-group whose Dynkin diagram contains connected components only of type $A_n$, $B_n$ or $C_n$. Then Serre's question has a positive answer for $G$.*

We also investigate Serre's question for reductive $k$-groups whose derived subgroups admit quasi-split simply connected covers. More precisely, we give a uniform proof for the following :

**Theorem 1.3.** *Let $k$ be a field of characteristic not 2. Let $G$ be a connected quasi-split reductive $k$-group whose Dynkin diagram does not contain connected components of type $E_8$. Then Serre's question has a positive answer for $G$.*

We relate Serre's question for $G$ with the norm principles of other closely related groups following a series of reductions previously used by Barquero and Merkurjev to prove the norm principles for reductive groups whose Dynkin diagrams do not contain connected components of type $D_n, E_6$ or $E_7$ [1]. We also give a scalar obstruction defined up to spinor norms whose vanishing will imply the norm principle for the (non-trialitarian) $D_n$ case and yield a positive answer to Serre's question for connected reductive $k$-groups whose Dynkin diagrams contain components of this type also.

In the next section, we begin with some lemmata and preliminary reductions. In Section 3, we introduce intermediate groups $\hat{G}$ and $\tilde{G}$ and relate Serre's question for $G$ to Serre's question for $\hat{G}$ and $\tilde{G}$ via the norm principle. In Section 4, we investigate the norm principle for (non-trialitarian) type $D_n$ groups and find the scalar obstruction whose vanishing will imply the norm principle for the (non-trialitarian) $D_n$ case. In the final section, we use the reduction techniques used in Sections 2 and 3 to discuss Serre's question for connected reductive $k$-groups whose derived subgroups admit quasi-split simply connected covers.

## 2. Preliminaries

We work over the base field $k$ of characteristic not 2. By a $k$-group, we mean a smooth connected linear algebraic group defined over $k$. And mostly, we will restrict ourselves to reductive groups. We say that a $k$-group $G$ satisfies $SQ$ if Serre's question has a positive answer for $G$.

**2.1. Reduction to characteristic 0.** Let $G$ be a connected reductive $k$-group whose Dynkin diagram contains connected components only of type $A_n$, $B_n$, $C_n$ or (non-trialitarian) $D_n$. Without loss of generality we may assume that $k$ is of characteristic 0 [7, p. 47]. We give a sketch of the reduction argument for the sake of completeness.

Suppose that the characteristic of $k$ is $p > 0$. Let $L_1, L_2, \dots, L_r$ be finite field extensions of $k$ of degrees $d_1, d_2, \dots, d_r$ respectively such that $\gcd_i(d_i) = 1$ and let $\xi$ be an element in the kernel of

$$\mathrm{H}^1(k,G) \to \prod_{i=1}^{r} \mathrm{H}^1(L_i,G).$$

By a theorem of Gabber, Liu and Lorenzini [5, Thm. 9.2] which was pointed out to us by O. Wittenberg, we note that any torsor under a smooth group scheme $G/k$ which admits a zero-cycle of degree 1 also admits a zero-cycle of degree 1 whose support is étale over $k$. Thus without loss of generality we can assume that the given coprime extensions $L_i/k$ are in fact separable.

By [10, Thms. 1 & 2], there exists a complete discrete valuation ring $R$ with residue field $k$ and fraction field $K$ of characteristic zero. Let $S_i$ denote corresponding étale extensions of $R$ with residue fields $L_i$ and fraction fields $K_i$.

There exists a smooth $R$-group scheme $\tilde{G}$ with special fiber $G$ and connected reductive generic fiber $\tilde{G}_K$. Now given any torsor $t \in \mathrm{H}^1(k,G)$, there exists a torsor $\tilde{t} \in \mathrm{H}^1_{\text{ét}}(R,\tilde{G})$ specializing to $t$ which is unique upto isomorphism. This in turn gives a torsor $\tilde{t}_K$ in $\mathrm{H}^1(K,\tilde{G}_K)$ by base change, thus defining a map $i_k : \mathrm{H}^1(k,G) \to \mathrm{H}^1(K,\tilde{G}_K)$ [6, p. 29]. It clearly sends the trivial element to the trivial element. The map $i$ also behaves well with the natural restriction maps, i.e., it fits into the following commutative diagram :

$$\begin{array}{ccc}
\mathrm{H}^1(k,G) & \xrightarrow{\;i_k\;} & \mathrm{H}^1(K,\tilde{G}_K) \\
\downarrow & & \downarrow \\
\prod \mathrm{H}^1(L_i,G) & \xrightarrow{\prod i_{L_i}} & \prod \mathrm{H}^1(K_i,\tilde{G}_K).
\end{array}$$

Let $\tilde{\xi}$ denote the torsor in $\mathrm{H}^1_{\text{et}}(R,\tilde{G})$ corresponding to $\xi$ as above. Therefore $\tilde{\xi}_K := i_k(\xi)$ is in the kernel of

$$\mathrm{H}^1(K,\tilde{G}_K) \to \prod_{i=1}^{r} \mathrm{H}^1(K_i,\tilde{G}_K).$$

Suppose that $\tilde{G}_K$ satisfies $SQ$. Then $\tilde{\xi}_K$ is trivial. However by [12], the natural map $\mathrm{H}^1_{\text{ét}}(R,\tilde{G}) \to \mathrm{H}^1(K,\tilde{G}_K)$ is injective and hence $\tilde{\xi}$ is trivial in $\mathrm{H}^1_{\text{ét}}(R,\tilde{G})$. This implies that its specialization, $\xi$, is trivial in $\mathrm{H}^1(k,G)$.

Thus from here on, we assume that the base field $k$ has characteristic 0.

### 2.2. Lemmata.

**Lemma 2.1.** *Let $k$-groups $G$ and $H$ satisfy $SQ$. Then $G \times_k H$ also satisfies $SQ$.*

*Proof.* Let $L/k$ be a field extension. Then the map

$$\mathrm{H}^1(k, G\times_k H) \to \mathrm{H}^1(L, G\times_k H)$$

is precisely the product of the maps

$$\mathrm{H}^1(k,G) \to \mathrm{H}^1(L,G) \text{ and } \mathrm{H}^1(k,H) \to \mathrm{H}^1(L,H).$$

This immediately shows that if $G$ and $H$ satisfy $SQ$, so does $G \times_k H$. □

**Lemma 2.2.** *Let $1 \to Q \to H \to G \to 1$ be a central extension of a $k$-group $G$ by a quasi-trivial torus $Q$. Then $H$ satisfies $SQ$ if and only if $G$ satisfies $SQ$.*

*Proof.* Let $L_i$ be field extensions of $k$ such that $\gcd[L_i : k] = 1$. Since $Q$ is quasi-trivial, $\mathrm{H}^1(L,Q) = \{1\}\ \forall\ L/k$. From the long exact sequence in cohomology, we have the following commutative diagram.

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{H}^1(k,H) & \longrightarrow & \mathrm{H}^1(k,G) & \xrightarrow{\delta_k} & \mathrm{H}^2(k,Q) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \prod \mathrm{H}^1(L_i,H) & \longrightarrow & \prod \mathrm{H}^1(L_i,G) & \xrightarrow{\prod \delta_{L_i}} & \prod \mathrm{H}^2(L_i,Q) \end{array}$$

From the above diagram, it is clear that if $G$ satisfies $SQ$, so does $H$.

Conversely, assume that $H$ satisfies $SQ$. Let $a \in \mathrm{H}^1(k,G)$ become trivial in $\prod \mathrm{H}^1(L_i,G)$. Then $\delta_k(a)$ becomes trivial in each $\mathrm{H}^2(L_i,Q)$. Hence the corestriction $\mathrm{Cor}_{L_i/k}(\delta_k(a)) = \delta_k(a)^{d_i}$ becomes trivial in $\mathrm{H}^2(k,Q)$ where $d_i = [L_i : k]$. Since $\gcd_i(d_i) = 1$, this implies that $\delta_k(a)$ is itself trivial in $\mathrm{H}^2(k,Q)$. Therefore $a$ comes from an element $b \in \mathrm{H}^1(k,H)$ which is trivial in $\prod \mathrm{H}^1(L_i,H)$. (The fact that $\mathrm{H}^1(L_i,Q) = \{1\}$ guarantees that $b$ is trivial in $\mathrm{H}^1(L_i,H)$.) Since $H$ satisfies $SQ$ by assumption, $b$ is trivial in $\mathrm{H}^1(k,H)$ which implies the triviality of $a$ in $\mathrm{H}^1(k,G)$. □

**Lemma 2.3.** *Let $E$ be a finite separable field extension of $k$ and let $H$ be an $E$-group satisfying $SQ$. Then the $k$-group $R_{E/k}(H)$ also satisfies $SQ$.*

*Proof.* Set $G = R_{E/k}(H)$ and let $\xi$ be an element in the kernel of $\mathrm{H}^1(k,G) \to \prod_{i=1}^r \mathrm{H}^1(L_i,G)$ where $\gcd_i [L_i : k] = 1$.

Since $\mathrm{char}(k) = 0$, $L_i \otimes_k E$ is an étale $E$-algebra and hence isomorphic to $E_{1,i} \times E_{2,i} \times \cdots \times E_{n_i,i}$ where each $E_{j,i}$ is a separable field extension of $E$. Thus $\sum_{j=1}^{n_i} \left[E_{j,i} : E\right] = [L_i : k]$ and therefore $\gcd\left[E_{j,i} : E\right] = 1$ where $1 \le i \le r$ and $1 \le j \le n_i$.

By Eckmann–Faddeev–Shapiro, we have a natural bijection of pointed sets

$$\mathrm{H}^1(k,G) \simeq \mathrm{H}^1(E,H),$$
$$\mathrm{H}^1(L_i,G) \simeq \prod_{j=1}^{n_i} \mathrm{H}^1\left(E_{j,i},H\right).$$

Thus we have that $\xi$ is in the kernel of $\mathrm{H}^1(E,H) \to \prod_{i \le r,\ j \le n_i} \mathrm{H}^1(E_{j,i},H)$. Since $H$ satisfies $SQ$, we see that $\xi$ is trivial. □

## 3. Serre's question and norm principles

**3.1. Intermediate groups $\hat{G}$ and $\tilde{G}$.** *Notations are as in Section 5 of [1].*

Let $G$ be our given connected reductive $k$-group whose Dynkin diagram contains connected components only of type $A_n$, $B_n$, $C_n$ or (non-trialitarian) $D_n$ and let $G'$ denote its derived subgroup. Let $Z(G) = T$ and $Z(G') = \mu$.

Let $\rho : \mu \hookrightarrow S$ be an embedding of $\mu$ into a quasi-trivial torus $S$. We denote the cofibre product $e(G', \rho) = \frac{G' \times S}{\mu}$ by $\hat{G}$. This $k$-group is called an *envelope* of $G'$.

$$\begin{array}{ccc} \mu & \xrightarrow{\delta} & G' \\ \downarrow{\scriptstyle \rho} & & \downarrow \\ S & \xrightarrow{\gamma} & \hat{G} \end{array}$$

Now the quasi-trivial torus $S = Z(\hat{G})$ and $\hat{G}$ fit into an exact sequence as follows:

$$1 \to S \to \hat{G} \to G'^{\,ad} \to 1 \qquad (*)$$

where $G'^{\,ad}$ corresponds to the adjoint group of $G'$. We now recall the following result of Jodi Black which addresses Serre's question for adjoint groups of classical type.

**Theorem 3.1** (Jodi Black, [3, Thm. 0.2]). *Let $k$ be a field of characteristic different from 2 and let $J$ be an absolutely simple algebraic $k$-group which is not of type $E_8$ and which is either a simply connected or adjoint classical group or a quasi-split exceptional group. Then Serre's question has a positive answer for $J$.*

Since every adjoint group of classical type is a product of Weil restrictions of absolutely simple adjoint groups, the above theorem, along with Lemmata 2.1 and 2.3, implies that $G'^{\,ad}$ satisfies $SQ$. Applying Lemma 2.2 to the exact sequence $(*)$ above, we see that $\hat{G}$ satisfies $SQ$. *Let us chose such an envelope $\hat{G}$ of $G'$ which satisfies $SQ$.*

Define an intermediate abelian group $\tilde{T}$ to be the cofibre product $\frac{T \times S}{\mu}$.

$$\begin{array}{ccc} \mu & \longrightarrow & T \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \alpha} \\ S & \xrightarrow{\nu} & \tilde{T} \end{array}$$

Let the algebraic group $\tilde{G}$ be the cofibre product defined by the following diagram:

$$\begin{array}{ccc} G' \times T & \xrightarrow{m} & G \\ \downarrow{\scriptstyle id \times \alpha} & & \downarrow{\scriptstyle \beta} \\ G' \times \tilde{T} & \xrightarrow{\epsilon} & \tilde{G}. \end{array}$$

Then we have the following commutative diagram with exact rows [1, Prop. 5.1]. Note that each row is a central extension of $\tilde{G}$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu & \xrightarrow{\delta,\nu\rho} & G' \times \tilde{T} & \xrightarrow{\epsilon} & \tilde{G} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho} & & \downarrow & & \downarrow{\scriptstyle id} & & \\ 1 & \longrightarrow & S & \xrightarrow{\gamma,\nu} & \hat{G} \times \tilde{T} & \longrightarrow & \tilde{G} & \longrightarrow & 1 \end{array} \qquad \begin{array}{c} (**) \\ \\ (***) \end{array}$$

Since $\tilde{T}$ is abelian, the existence of the co-restriction map shows that $\tilde{T}$ satisfies $SQ$. Since $\hat{G}$ satisfies $SQ$, we can apply Lemmata 2.1 and 2.2 to $(***)$ to see that $\tilde{G}$ *satisfies* $SQ$.

**3.2. Norm principle and weak norm principle.** Let $f : G \to T$ be a map of $k$-groups where $T$ is an abelian $k$-group. Then we have norm maps $N_{L/k} : T(L) \to T(k)$ for any separable field extension $L/k$.

$$\begin{array}{ccc} G(L) & \xrightarrow{f(L)} & T(L) \\ & & \downarrow{\scriptstyle N_{L/k}} \\ G(k) & \xrightarrow{f(k)} & T(k) \end{array}$$

We say that the *norm principle* holds for $f : G \to T$ if for all separable field extensions $L/k$,

$$N_{L/k}(\text{Image } f(L)) \subseteq \text{Image } f(k).$$

That is, we say that the *norm principle* holds for $f : G \to T$ if given any separable field extension $L/k$ and any $t \in T(L)$ such that

$$t \in (\text{Image } f(L) : G(L) \to T(L)),$$

then

$$N_{L/k}(t) \in (\text{Image } f(k) : G(k) \to T(k)).$$

Note that the norm principle holds for any algebraic group homomorphism between abelian groups.

We say that the *weak norm principle* holds for $f : G \to T$ if given any $t \in T(k)$ such that

$$t \in (\text{Image } f(L) : G(L) \to T(L)),$$

then

$$t^{[L:k]} = N_{L/k}(t) \in (\text{Image } f(k) : G(k) \to T(k)).$$

It is clear that if the norm principle holds for $f$, then so does the weak norm principle.

**3.3. Relating Serre's question and norm principle.** The deduction of SQ for $G$ from $\hat{G}$ and $\tilde{G}$ follows via the (weak) norm principles.

Let $\beta : G \to \tilde{G}$ be the embedding of $k$-groups with the cokernel $P$ isomorphic to the torus $\frac{S}{\mu}$ where $\tilde{G}$ and $G$ are as in Section 3.1. Thus we have the following exact sequence:

$$1 \to G \xrightarrow{\beta} \tilde{G} \xrightarrow{\pi} P \to 1.$$

**Lemma 3.2.** *If the weak norm principle holds for $\pi : \tilde{G} \to P$, then $G$ satisfies $SQ$.*

*Proof.* From the long exact sequence of cohomology, we have the following commutative diagram:

$$\begin{array}{ccccccccccc} 1 & \to & G(k) & \to & \tilde{G}(k) & \xrightarrow{\pi_k} & P(k) & \xrightarrow{\delta_k} & \mathrm{H}^1(k,G) & \xrightarrow{\beta_k} & \mathrm{H}^1(k,\tilde{G}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \to & \prod G(L_i) & \to & \prod \tilde{G}(L_i) & \xrightarrow{\prod \pi_{L_i}} & \prod P(L_i) & \xrightarrow{\prod \delta_{L_i}} & \prod \mathrm{H}^1(L_i,G) & \to & \prod \mathrm{H}^1(L_i,\tilde{G}). \end{array}$$

Let $a \in \mathrm{H}^1(k,G)$ become trivial in $\prod \mathrm{H}^1(L_i,G)$. As $\tilde{G}$ satisfies $SQ$, $\beta_k(a)$ becomes trivial in $\mathrm{H}^1(k,\tilde{G})$. Hence $a = \delta_k(b)$ for some $b \in P(k)$ and $\delta_{L_i}(b)$ is trivial in $\mathrm{H}^1(L_i,G)$. Therefore, there exist $c_i \in \tilde{G}(L_i)$ such that $\pi_{L_i}(c_i) = b$.

Showing that $G$ satisfies $SQ$, i.e. that $a$ is trivial, is equivalent to showing

$$b \in \left(\mathrm{Image}\, \pi_k : \tilde{G}(k) \to P(k)\right).$$

However $b \in \left(\mathrm{Image}\, \pi_{L_i} : \tilde{G}(L_i) \to P(L_i)\right)$. Since the weak norm principle holds for $\pi : \tilde{G} \to P$, $b^{d_i} \in \mathrm{Image}\left(\pi_k : \tilde{G}(k) \to P(k)\right)$ where $[L_i : k] = d_i$ for each $i$. As $\gcd_i(d_i) = 1$, this means $b \in \mathrm{Image}\left(\pi_k : \tilde{G}(k) \to P(k)\right)$. □

We recall now the norm principle of Merkurjev and Barquero for reductive groups of classical type.

**Theorem 3.3** (Barquero–Merkurjev, [1]). *Let $G$ be a reductive group over a field $k$. Assume that the Dynkin diagram of $G$ does not contain connected components $D_n$, $n \geq 4$, $E_6$ or $E_7$. Let $T$ be any commutative $k$-group. Then the norm principle holds for any group homomorphism $G \to T$.*

This shows that the norm principle and hence the weak norm principle holds for the map $\pi : \tilde{G} \to P$ for reductive $k$-groups $G$ as in the main theorem (Theorem 1.2). Thus we have concluded the proof for the following:

**Theorem 1.2.** *Let $k$ be a field of characteristic not 2. Let $G$ be a connected reductive $k$-group whose Dynkin diagram contains connected components only of type $A_n$, $B_n$ or $C_n$. Then Serre's question has a positive answer for $G$.*

## 4. Obstruction to norm principle for (non-trialitarian) $D_n$

**4.1. Preliminaries.** Let $(\mathrm{A}, \sigma)$ be a central simple algebra of degree $2n$ over $k$ and let $\sigma$ be an orthogonal involution. Let $\mathrm{C}(\mathrm{A}, \sigma)$ denote its Clifford algebra which is a central simple algebra over its center, $Z/k$, the discriminant extension. Let $i$ denote the non-trivial automorphism of $Z/k$ and let $\underline{\sigma}$ denote the canonical involution of $\mathrm{C}(\mathrm{A}, \sigma)$.

Recall that, depending on the parity of $n$, $\underline{\sigma}$ is either an involution of the second kind (when $n$ is odd) or of the first kind (when $n$ is even). Let $\underline{\mu} : \mathrm{Sim}(\mathrm{C}(\mathrm{A}, \sigma), \underline{\sigma}) \to R_{Z/k}\mathbb{G}_m$ denote the multiplier map sending similitude $c$ to $\underline{\sigma}(c)c$.

Let $\Omega(\mathrm{A}, \sigma)$ be the *extended Clifford group.* Note that this has center $R_{Z/k}\mathbb{G}_m$ and is an *envelope* of $\mathrm{Spin}(\mathrm{A}, \sigma)$ [1, Ex. 4.4]. We recall below the map $\varkappa : \Omega(\mathrm{A}, \sigma)(k) \to Z^*/k^*$ as defined in [9, p. 182].

Given $\omega \in \Omega(\mathrm{A}, \sigma)(k)$, let $g \in \mathrm{GO}^+(\mathrm{A}, \sigma)(k)$ be some similitude such that $\omega \rightsquigarrow gk^*$ under the natural surjection $\Omega(\mathrm{A}, \sigma)(k) \to \mathrm{PGO}^+(\mathrm{A}, \sigma)(k)$.

Let $h = \mu(g)^{-1}g^2 \in \mathrm{O}^+(\mathrm{A}, \sigma)(k)$ and let $\gamma \in \Gamma(\mathrm{A}, \sigma)(k)$ be some element in the *special Clifford group* which maps to $h$ under the vector representation $\chi' : \Gamma(\mathrm{A}, \sigma)(k) \to \mathrm{O}^+(\mathrm{A}, \sigma)(k)$. Then $\omega^2 = \gamma z$ for some $z \in Z^*$ and $\varkappa(\omega) = zk^*$.

Note that the map $\varkappa$ has $\Gamma(\mathrm{A}, \sigma)(k)$ as kernel. Also if $z \in Z^*$, then $\varkappa(z) = z^2k^*$.

By following the reductions in [1], it is easy to see that one needs to investigate whether the norm principle holds for the canonical map

$$\Omega(\mathrm{A}, \sigma) \to \frac{\Omega(\mathrm{A}, \sigma)}{[\Omega(\mathrm{A}, \sigma), \Omega(\mathrm{A}, \sigma)]}.$$

We will need to investigate the norm principle for two different maps depending on the parity of $n$.

**The map $\mu_*$ for $n$ odd.** Let $U \subset \mathbb{G}_m \times R_{Z/k}\mathbb{G}_m$ be the algebraic subgroup defined by

$$U(k) = \{(f, z) \in k^* \times Z^* | f^4 = \mathrm{N}_{Z/k}(z)\}.$$

Recall the map $\mu_* : \Omega(\mathrm{A}, \sigma) \to U$ defined in [9, p. 188] which sends

$$\omega \rightsquigarrow \left(\underline{\mu}(\omega), ai(a)^{-1}\underline{\mu}(\omega)^2\right),$$

where $\omega \in \Omega(\mathrm{A}, \sigma)(k)$ and $\varkappa(\omega) = a\ \ k^*$. This induces the following exact sequence [9, p. 190]

$$1 \to \mathrm{Spin}(\mathrm{A}, \sigma) \to \Omega(\mathrm{A}, \sigma) \xrightarrow{\mu_*} U \to 1.$$

Since the semisimple part of $\Omega(\mathrm{A}, \sigma)$ is $\mathrm{Spin}(\mathrm{A}, \sigma)$, the above exact sequence shows that it suffices to check the norm principle for the map $\mu_*$.

**The map $\underline{\mu}$ for $n$ even.** Recall the following exact sequence induced by restricting $\underline{\mu}$ to $\Omega\,(\mathrm{A},\sigma)$ [9, p. 187]

$$1 \to \mathrm{Spin}\,(\mathrm{A},\sigma) \to \Omega\,(\mathrm{A},\sigma) \xrightarrow{\mu} R_{Z/k}\mathbb{G}_m \to 1.$$

Since the semisimple part of $\Omega\,(\mathrm{A},\sigma)$ is $\mathrm{Spin}\,(\mathrm{A},\sigma)$, the above exact sequence shows that it suffices to check the norm principle for the map $\underline{\mu}$.

**4.2. An obstruction to being in the image of $\mu_*$ for $n$ odd.** Given $(f,z) \in U(k)$, we would like to formulate an obstruction which prevents $(f,z)$ from being in the image $\mu_*\,(\Omega\,(\mathrm{A},\sigma)\,(k))$. Note that for $z \in Z^*$, $\mu_*(z) = (\mathrm{N}_{Z/k}(z), z^4)$ and hence the algebraic subgroup $U_0 \subseteq U$ defined by

$$U_0(k) = \{(N_{Z/k}(z), z^4) | z \in Z^*\}$$

has its $k$-points in the image $\mu_*\,(\Omega\,(\mathrm{A},\sigma)\,(k))$.

Let $\mu_{n[Z]}$ denote the kernel of the norm map $R_{K/k}\mu_n \xrightarrow{N} \mu_n$ where $K/k$ is a quadratic extension. Note that $\mu_{4[Z]}$ is the center of $\mathrm{Spin}\,(\mathrm{A},\sigma)$ as $n$ is odd. Also recall that [9, Prop. 30.13, p. 418]

$$\mathrm{H}^1\left(k, \mu_{4[Z]}\right) \cong \frac{U(k)}{U_0(k)}.$$

Thus, we can construct the map $S : \mathrm{PGO}^+\,(\mathrm{A},\sigma)\,(k) \to \mathrm{H}^1\left(k, \mu_{4[Z]}\right)$ induced by the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z^* & \longrightarrow & \Omega\,(\mathrm{A},\sigma)\,(k) & \xrightarrow{\chi'} & \mathrm{PGO}^+\,(\mathrm{A},\sigma)\,(k) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle S} & & \\ 1 & \longrightarrow & U_0(k) & \longrightarrow & U(k) & \longrightarrow & \mathrm{H}^1\left(k, \mu_{4[Z]}\right) & \longrightarrow & 1 \end{array}$$

The map $S$ also turns out to be the connecting map from $\mathrm{PGO}^+\,(\mathrm{A},\sigma)\,(k) \to \mathrm{H}^1\left(k, \mu_{4[Z]}\right)$ [9, Prop. 13.37, p. 190] in the long exact sequence of cohomology corresponding to the exact sequence

$$1 \to \mu_{4[Z]} \to \mathrm{Spin}\,(\mathrm{A},\sigma) \to \mathrm{PGO}^+\,(\mathrm{A},\sigma) \to 1.$$

Since the maps $\mu_* : Z^* \to U_0(k)$ and $\chi' : \Omega\,(\mathrm{A},\sigma)\,(k) \to \mathrm{PGO}^+\,(\mathrm{A},\sigma)\,(k)$ are surjective, an element $(f,z) \in U(k)$ is in the image $\mu_*\,(\Omega\,(\mathrm{A},\sigma)\,(k))$ if and only if its image $[f,z] \in \mathrm{H}^1\left(k, \mu_{4[Z]}\right)$ is in the image $S\left(\mathrm{PGO}^+\,(\mathrm{A},\sigma)\,(k)\right)$.

Therefore we look for an obstruction preventing $[f, z]$ from being in the image $S(\mathrm{PGO}^+(\mathrm{A},\sigma)(k))$. Recall the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mu_2 & & \\
 & & & & & & \downarrow & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Spin}(\mathrm{A},\sigma) & \xrightarrow{\chi} & \mathrm{O}^+(\mathrm{A},\sigma) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \pi} & & \\
1 & \longrightarrow & \mu_{4[Z]} & \longrightarrow & \mathrm{Spin}(\mathrm{A},\sigma) & \xrightarrow{\chi'} & \mathrm{PGO}^+(\mathrm{A},\sigma) & \longrightarrow & 1 \\
 & & & & & & \downarrow & & \\
 & & & & & & 1 & &
\end{array}$$

The long exact sequence of cohomology induces the following commutative diagram (Figure 1) with exact columns [9, Prop. 13.36, p. 189], where

$$\begin{array}{ccc}
\mathrm{O}^+(\mathrm{A},\sigma)(k) & \xrightarrow{\quad Sn \quad} & \frac{k^*}{k^{*2}} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle i} \\
\mathrm{PGO}^+(\mathrm{A},\sigma)(k) & \xrightarrow{\quad \mathrm{S} \quad} & \mathrm{H}^1\left(k, \mu_{4[Z]}\right) \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle j} \\
\frac{k^*}{k^{*2}} & = & \frac{k^*}{k^{*2}}
\end{array}$$

Figure 1. Spinor norms and S for $n$ odd

$\mu$ : $\mathrm{PGO}^+(\mathrm{A},\sigma)(k) \to \frac{k^*}{k^{*2}}$ is induced by the multiplier map $\mu : \mathrm{GO}^+(\mathrm{A},\sigma) \to \mathbb{G}_m$

$i$ : $\frac{k^*}{k^{*2}} \to \mathrm{H}^1\left(k, \mu_{4[Z]}\right) = \frac{U(k)}{U_0(k)}$ is the map sending $f k^{*2} \rightsquigarrow [f, f^2]$

$j$ : $\frac{U(k)}{U_0(k)} = \mathrm{H}^1\left(k, \mu_{4[Z]}\right) \to \frac{k^*}{k^{*2}}$ is the map sending $[f, z] \rightsquigarrow \mathrm{N}(z_0) k^{*2}$, where $z_0 \in Z^*$ is such that $z_0 i(z_0)^{-1} = f^{-2} z$.

**Definition 4.1.** We call an element $(f, z) \in U(k)$ to be *special* if there exists a $[g] \in \mathrm{PGO}^+(\mathrm{A},\sigma)(k)$ such that $j([f, z]) = \mu([g])$.

Let $(f, z) \in U(k)$ be a special element and let $[g] \in \mathrm{PGO}^+(\mathrm{A}, \sigma)(k)$ be such that $j([f, z]) = \mu([g])$. From the discussion above, it is clear that $(f, z)$ is in the image $\mu_*(\Omega(\mathrm{A}, \sigma)(k))$ if and only if $[f, z]$ is in the image $S\left(\mathrm{PGO}^+(\mathrm{A}, \sigma)(k)\right)$.

Thus $S([g])[f, z]^{-1}$ is in kernel $j = \mathrm{Image}\, i$ and hence there exists some $\alpha \in k^*$ such that

$$[f, z] = S([g])[\alpha, \alpha^2] \in \frac{U(k)}{U_0(k)}.$$

Note that if $g$ is changed by an element in $\mathrm{O}^+(\mathrm{A}, \sigma)(k)$, then $\alpha$ changes by a spinor norm by Figure 1 above. *Thus given a special element, we have produced a scalar $\alpha \in k^*$ which is well defined upto spinor norms.*

$$\begin{aligned}[f, z] \in S\left(\mathrm{PGO}^+(\mathrm{A}, \sigma)(k)\right) &\iff [\alpha, \alpha^2] \in S\left(\mathrm{PGO}^+(\mathrm{A}, \sigma)(k)\right) \\ &\iff (\alpha, \alpha^2) \in \mu_*(\Omega(\mathrm{A}, \sigma)(k)).\end{aligned}$$

This happens if and only if there exists $w \in \Omega(\mathrm{A}, \sigma)(k)$ such that

$$\begin{aligned}\alpha &= \underline{\mu}(w) \\ \alpha^2 &= \varkappa(w) i(\varkappa(w))^{-1} \underline{\mu}(w)^2\end{aligned}$$

This implies $\varkappa(w) \in k^*$ and hence $w \in \Gamma(\mathrm{A}, \sigma)(k)$. Thus $\alpha$ is a spinor norm, being the similarity of an element in the special Clifford group. Also note if $\alpha$ is a spinor norm, then $\alpha = \underline{\mu}(\gamma)$ for some $\gamma \in \Gamma(\mathrm{A}, \sigma)(k)$ and $\mu_*(\gamma) = \left(\underline{\mu}(\gamma), \underline{\mu}(\gamma)^2\right)$.

Thus a special element $(f, z)$ is in the image of $\mu_*$ if and only if the produced scalar $\alpha$ is a spinor norm. We call the class of $\alpha$ in $\frac{k^*}{\mathrm{Sn}(\mathrm{A}, \sigma)}$ to be the scalar obstruction preventing the *special* element $(f, z) \in U(k)$ from being in the image $\mu_*(\Omega(\mathrm{A}, \sigma)(k))$.

**4.3. An obstruction to being in the image of $\underline{\mu}$ for $n$ even.** Given $z \in Z^*$, we would like to formulate an obstruction which prevents $z$ from being in the image $\underline{\mu}(\Omega(\mathrm{A}, \sigma)(k))$. Note that for $z \in Z^*$, $\underline{\mu}(z) = z^2$ and hence the subgroup $Z^{*2}$ is in the image $\underline{\mu}(\Omega(\mathrm{A}, \sigma)(k))$.

Like in the case of odd $n$, we can construct the map $S : \mathrm{PGO}^+(\mathrm{A}, \sigma)(k) \to \frac{Z^*}{Z^{*2}}$ induced by the following commutative diagram with exact rows [9, Def. 13.32, p. 187]:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z^* & \longrightarrow & \Omega(\mathrm{A}, \sigma)(k) & \xrightarrow{\chi'} & \mathrm{PGO}^+(\mathrm{A}, \sigma)(k) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \underline{\mu}} & & \downarrow{\scriptstyle \underline{\mu}} & & \downarrow{\scriptstyle S} & & \\ 1 & \longrightarrow & Z^{*2} & \longrightarrow & Z^* & \longrightarrow & \frac{Z^*}{Z^{*2}} & \longrightarrow & 1 \end{array}$$

Again by the surjectivity of the maps, $\underline{\mu} : Z^* \to Z^{*2}$ and $\chi' : \Omega(\mathrm{A}, \sigma)(k) \to \mathrm{PGO}^+(\mathrm{A}, \sigma)(k)$, an element $z \in Z^*$ is in the image $\underline{\mu}(\Omega(\mathrm{A}, \sigma)(k))$ if and only if its image $[z] \in \frac{Z^*}{Z^{*2}}$ is in the image $S\left(\mathrm{PGO}^+(\mathrm{A},\sigma)(k)\right)$. Therefore we look for an obstruction preventing $[z]$ from being in the image $S(\mathrm{PGO}^+(\mathrm{A},\sigma)(k))$. And as before, we arrive at the the following commutative diagram (Figure 2) with exact rows and columns [9, Prop. 13.33, p. 188], where

$$\begin{array}{ccc}
\mathrm{O}^+(\mathrm{A},\sigma)(k) & \xrightarrow{\quad Sn \quad} & \frac{k^*}{k^{*2}} \\
\downarrow \pi & & \downarrow i \\
\mathrm{PGO}^+(\mathrm{A},\sigma)(k) & \xrightarrow{\quad \mathrm{S} \quad} & \frac{Z^*}{Z^{*2}} \\
\downarrow \mu & & \downarrow j \\
\frac{k^*}{k^{*2}} & = & \frac{k^*}{k^{*2}}
\end{array}$$

Figure 2. Spinor norms and S for $n$ even

$\mu : \mathrm{PGO}^+(\mathrm{A},\sigma)(k) \to \frac{k^*}{k^{*2}}$ is induced by the multiplier map $\mu : \mathrm{GO}^+(\mathrm{A},\sigma) \to \mathbb{G}_m$

$i : \frac{k^*}{k^{*2}} \to \frac{Z^*}{Z^{*2}}$ is the inclusion map

$j : \frac{Z^*}{Z^{*2}} \to \frac{k^*}{k^{*2}}$ is induced by the norm map from $Z^* \to k^*$.

**Definition 4.2.** We call an element $z \in Z^*$ to be *special* if there exists a $[g] \in \mathrm{PGO}^+(\mathrm{A},\sigma)(k)$ such that $j([z]) = \mu([g])$.

Let $z \in Z^*$ be a special element and let $[g] \in \mathrm{PGO}^+(\mathrm{A},\sigma)(k)$ be such that $j([z]) = \mu([g])$. As before a *special* element $z \in Z^*$ is in the image $\underline{\mu}(\Omega(\mathrm{A},\sigma)(k))$ if and only if $[z]$ is in the image $S\left(\mathrm{PGO}^+(\mathrm{A},\sigma)(k)\right)$.

Thus $S([g])[z]^{-1}$ is in kernel $j$ = Image $i$ and hence there exists some $\alpha \in k^*$ such that

$$[z] = S([g])[\alpha] \in \frac{Z^*}{Z^{*2}}.$$

Note that if $g$ is changed by an element in $\mathrm{O}^+(\mathrm{A},\sigma)(k)$, then $\alpha$ changes by a spinor norm by Figure 2 above. *Thus given a special element, we have produced a scalar $\alpha \in k^*$ which is well defined up to spinor norms.*

$$\begin{aligned}
[z] \in S\left(\mathrm{PGO}^+(\mathrm{A},\sigma)(k)\right) &\iff [\alpha] \in S\left(\mathrm{PGO}^+(\mathrm{A},\sigma)(k)\right) \\
&\iff (\alpha) \in \underline{\mu}\left(\Omega(\mathrm{A},\sigma)(k)\right).
\end{aligned}$$

Since $\alpha \in k^*$ also, this is equivalent to $\alpha$ being a spinor norm [9, Prop. 13.25, p. 184].

We call the class of $\alpha$ in $\frac{k^*}{\mathrm{Sn}(\mathrm{A},\sigma)}$ to be the scalar obstruction preventing the *special* element $z \in Z^*$ from being in the image $\underline{\mu}(\Omega(\mathrm{A},\sigma)(k))$.

**4.4. Scharlau's norm principle for $\mu : \mathrm{GO}^+(\mathrm{A}, \sigma) \to \mathbb{G}_m$.** Let $\mu : \mathrm{GO}^+(\mathrm{A}, \sigma) \to \mathbb{G}_m$ denote the multiplier map and let $L/k$ be a separable field extension of finite degree. Let $g_1 \in GO^+(\mathrm{A}, \sigma)(L)$ be such that $\mu(g_1) = f_1 \in L^*$. Let $f$ denote $\mathrm{N}_{L/k}(f_1)$. We would like to show that $f$ is in the image $\mu\left(\mathrm{GO}^+(\mathrm{A}, \sigma)(k)\right)$.

Note that by a generalization of Scharlau's norm principle ([9, Prop. 12.21]; [3, Lemma 4.3]) there exists a $\tilde{g} \in \mathrm{GO}(\mathrm{A}, \sigma)(k)$ such that $f = \mu(\tilde{g})$. However we would like to find a *proper* similitude $g \in \mathrm{GO}^+(\mathrm{A}, \sigma)(k)$ such that $\mu(g) = f$.

We investigate the cases when the algebra $A$ is non-split and split separately.

**Case I: $A$ is non-split.** Note that $g_1 \in \mathrm{GO}^+(\mathrm{A}, \sigma)(L)$. If $\tilde{g} \in \mathrm{GO}^+(\mathrm{A}, \sigma)(k)$, we are done. Hence assume $\tilde{g} \notin \mathrm{GO}^+(\mathrm{A}, \sigma)(k)$. By a generalization of Dieudonné's theorem [9, Thm. 13.38, p. 190], we see that the quaternion algebras

$$B_1 = (Z, f_1) = 0 \in \mathrm{Br}(L),$$
$$B_2 = (Z, f) = A \in Br(k).$$

Since $A$ is non-split, $B_2 \neq 0 \in \mathrm{Br}(k)$. However co-restriction of $B_1$ from $L$ to $k$ gives a contradiction, because

$$0 = \mathrm{Cor}\, B_1 = \left(Z, \mathrm{N}_{L/k}(f_1)\right) = B_2 \in \mathrm{Br}(k).$$

Hence $\tilde{g} \in \mathrm{GO}^+(\mathrm{A}, \sigma)(k)$.

**Case II: $A$ is split.** Since $A$ is split, $A = \mathrm{End}\, V$ where $(V, q)$ is a quadratic space and $\sigma$ is the adjoint involution for the quadratic form $q$. Again, if $\tilde{g} \in \mathrm{GO}^+(\mathrm{A}, \sigma)(k)$, we are done. Hence assume $\tilde{g} \notin \mathrm{GO}^+(\mathrm{A}, \sigma)(k)$. That is

$$\det(\tilde{g}) = -f^{2n/2} = -(f^n).$$

Since $A$ is of even degree $(2n)$ and split, there exists an isometry[1] $h$ of determinant $-1$. Set $g = \tilde{g}h$. Then $\det(g) = f^n$ where $\mu(g) = f$. Thus we have found a suitable $g \in \mathrm{GO}^+(\mathrm{A}, \sigma)(k)$ which concludes the proof of the following:

**Theorem 4.3.** *The norm principle holds for the map* $\mu : \mathrm{GO}^+(\mathrm{A}, \sigma) \to \mathbb{G}_m$.

**4.5. Spinor obstruction to norm principle for non-trialitarian $D_n$.** Let $L/k$ be a separable field extension of finite degree. And let $w_1 \in \Omega(\mathrm{A}, \sigma)(L)$ be such that for

$$n \text{ odd} : \mu_*(w_1) = \theta \text{ which is equal to } (f_1, z_1) \in U(L),$$
$$n \text{ even} : \underline{\mu}(w_1) = \theta \text{ which is equal to } z_1 \in \left(R_{Z/k}\mathbb{G}_m\right)(L).$$

[1]Since $V$ is of even dimension $2n$, $h$ can be chosen to be a hyperplane reflection for instance

We would like to investigate whether $\mathrm{N}_{L/k}(\theta)$ is in the image of $\mu_*\ (\Omega\ (\mathrm{A},\sigma)\ (k))$ (resp. $\underline{\mu}\ (\Omega\ (\mathrm{A},\sigma)\ (k)))$ when $n$ is odd (resp. even) in order to check if the norm principle holds for the map $\mu_* : \Omega\ (\mathrm{A},\sigma) \to U$ (resp. $\underline{\mu} : \Omega\ (\mathrm{A},\sigma) \to R_{Z/k}\mathbb{G}_m$).

Let $[g_1] \in \mathrm{PGO}^+\ (\mathrm{A},\sigma)\ (L)$ be the image of $w_1$ under the canonical map $\chi' : \Omega(\mathrm{A},\sigma)(L) \to \mathrm{PGO}^+(\mathrm{A},\sigma)(L)$. Clearly $\theta$ is *special* and let $g_1 \in \mathrm{GO}^+\ (\mathrm{A},\sigma)\ (L)$ be such that $\mu([g_1]) = j([\theta])$.

By Theorem 4.3, there exists a $g \in \mathrm{GO}^+\ (\mathrm{A},\sigma)\ (k)$ such that[2]

$$\mu([g]) = \mathrm{N}_{L/k}\ (j\,[\theta]) = j\ \left([\mathrm{N}_{L/k}\,\theta]\right).$$

Hence $\mathrm{N}_{L/k}(\theta)$ is *special.*

By Subsection 4.2 (resp. 4.3) , $\mathrm{N}_{L/k}(\theta)$ is in the image of $\mu_*$ (resp $\underline{\mu}$) if and only if the scalar obstruction $\alpha \in \frac{k^*}{\mathrm{Sn}(\mathrm{A},\sigma)}$ defined for $\mathrm{N}_{L/k}(\theta)$ vanishes. Thus we have a spinor norm obstruction given below.

**Theorem 4.4** (Spinor norm obstruction). *Let $L/k$ be a finite separable extension of fields. Let $f$ denote the map $\mu_*$ (resp $\underline{\mu}$) in the case when $n$ is odd (resp. even). Given $\theta \in f\ (\Omega\ (\mathrm{A},\sigma)\ (L))$, there exists scalar obstruction $\alpha \in k^*$ such that*

$$N_{L/k}(\theta) \in f\ (\Omega\ (\mathrm{A},\sigma)\ (k)) \iff \alpha = 1 \in \frac{k*}{\mathrm{Sn}(\mathrm{A},\sigma)}.$$

Thus the norm principle for the canonical map

$$\Omega\ (\mathrm{A},\sigma) \to \frac{\Omega\ (\mathrm{A},\sigma)}{[\Omega\ (\mathrm{A},\sigma)\ ,\Omega\ (\mathrm{A},\sigma)]}$$

and hence for non-trialitarian $D_n$ holds if and only if the scalar obstructions are spinor norms.

## 5. Quasi-split groups

Let $G$ be a connected reductive $k$-group whose Dynkin diagram does not contain connected components of type $E_8$ and let $G'$ denote its derived subgroup. Let $G^{sc}$ denote the simply connected cover of $G'$. Then one has the exact sequence $1 \to C \to G^{sc} \to G' \to 1$, where $C$ is a finite $k$-group of multiplicative type, central in $G^{sc}$. Assuming that $G^{sc}$ is quasi-split, we would like to show that $G$ satisfies $SQ$ by following the reduction techniques used in Sections 2 and 3.

**Lemma 5.1.** *Let $G$ be a connected reductive $k$-group. If $G^{sc}$ is quasi-split, then there exists an extension $1 \to Q \to H \xrightarrow{\psi} G \to 1$, where $Q$ is a quasi-trivial $k$-torus, central in reductive $k$-group $H$ with $H'$ simply connected and quasi-split.*

[2]The map $j$ commutes with $\mathrm{N}_{L/k}$ in both cases.

*Proof.* Recall that there is a central extension (called a $z$-extension) of $G$ by a quasi-trivial torus $Q$ such that $H'$ is semisimple and simply connected ([11, Prop. 3.1] and [4, Lemma 1.1.4]).

$$1 \to Q \to H \xrightarrow{\psi} G \to 1.$$

The restriction $\psi|_{H'} : H' \to G$ yields the fact that $H'$ is the simply connected cover of $G'$ and hence is quasi-split. □

Lemmata 2.2 and 5.1 imply that we can restrict ourselves to connected reductive $k$-groups $G$ such that $G'$ is simply connected and quasi-split.

**Lemma 5.2.** *Let $H$ be any reductive $k$-group such that its derived subgroup $H'$ is semisimple simply connected and quasi-split. Let $T$ denote the $k$-torus $H/H'$. Then the natural exact sequence $1 \to H' \to H \xrightarrow{\phi} T \to 1$ induces surjective maps $\phi(L) : H(L) \to T(L)$ for all field extensions $L/k$. In particular, the norm principle holds for $\phi : H \to T$.*

*Proof.* There exists a quasi-trivial maximal torus $Q_1$ of $H'$ defined over $k$ [8, Lem. 6.7]. Let $Q_1 \subset Q_2$, where $Q_2$ is a maximal torus of $H$ defined over $k$. The proof of [8, Lem. 6.6] shows that $\phi|_{Q_2} : Q_2 \to T$ is surjective and that $Q_2 \cap H'$ is a maximal torus of $H'$. Since $Q_2 \cap H' \subseteq Q_1$, we get the following extension of $k$-tori

$$1 \to Q_1 \to Q_2 \to T \to 1$$

Since $Q_1$ is quasitrivial, $\mathrm{H}^1(L, Q_1) = 0$ for any field extension $L/k$ which gives the surjectivity of $\phi(L) : Q_2(L) \to T(L)$ and hence of $\phi(L) : H(L) \to T(L)$. □

Let $\hat{G}$ be an envelope of $G'$ defined using an embedding of $\mu = Z(G')$ into a quasi-trivial torus $S$. Note that $G'$ is assumed to be simply connected and quasi-split and is also the derived subgroup of $\hat{G}$ by construction.

$$\begin{array}{ccc} \mu & \xrightarrow{\delta} & G' \\ \downarrow{\scriptstyle\rho} & & \downarrow \\ S & \xrightarrow{\gamma} & \hat{G} \end{array}$$

Thus, we get an exact sequence $1 \to G' \to \hat{G} \to \hat{G}/G' \to 1$ to which we can apply Lemma 5.2 to conclude that the norm principle holds for the canonical map $\hat{G} \to \frac{\hat{G}}{[\hat{G},\hat{G}]}$.

Constructing the intermediate group $\tilde{G}$ as in Section 3.1, we see that the norm principle also holds for the natural map $\tilde{G} \to \tilde{G}/G$ [1, Prop. 5.1]. Then using Theorem 3.1 [3], Lemma 3.2, and a remark from Gopal Prasad that $G^{sc}$ is quasi-split if and only if $G$ is quasi-split, we can conclude that Theorem 1.3 (restated below) holds.

**Theorem 1.3.** *Let $k$ be a field of characteristic not 2. Let $G$ be a connected quasi-split reductive $k$-group whose Dynkin diagram does not contain connected components of type $E_8$. Then Serre's question has a positive answer for $G$.*

**Acknowledgements.** The author thanks Professors A. S. Merkurjev, R. Parimala, O. Wittenberg and the anonymous referee for their many valuable suggestions and critical comments.

## References

[1] P. Barquero and A. Merkurjev, *Norm Principle for Reductive Algebraic Groups*, J. Proceedings of the International Colloquium on Algebra, Arithmetic and Geometry TIFR, Mumbai, 2000.

[2] E. Bayer-Fluckiger and H. W. Lenstra Jr., Forms in odd degree extensions and self-dual normal bases, *Amer. J. Math.*, **112** (1990), no. 3, 359–373. Zbl 0729.12006 MR 1055648

[3] J. Black, Zero cycles of degree one on principal homogeneous spaces, *Journal of Algebra*, **334** (2011), 232–246. Zbl 1271.11046 MR 2787661

[4] M. Borovoi and B. Kunyavskiĭ, Formulas for the unramified Brauer group of a principal homogeneous space of a linear algebraic group, *Journal of Algebra*, **225** (2000), no. 2, 804–821. Zbl 1054.14510 MR 1741563

[5] O. Gabber, Q. Liu and D. Lorenzini, The index of an algebraic variety, *Inventiones mathematicae*, **192** (2013), no. 3, 567–626. Zbl 1268.13009 MR 3049930

[6] S. Garibaldi, A. Merkurjev and J.-P. Serre, *Cohomological invariants in Galois cohomology*, University Lecture Series, 28, American Mathematical Society, Providence, RI, 2003. Zbl 1159.12311 MR 1999383

[7] P. Gille, Serre's conjecture II: a survey, in *Quadratic forms, linear algebraic groups, and cohomology*, 41–56, Springer, New York, 2010. Zbl 1239.11046 MR 2648719

[8] D. Harari and T. Szamuely, Local-global questions for tori over $p$-adic function fields, *preprint*, 2013. arXiv:1307.4782

[9] M.-A. Knus, A. Merkurjev, M. Rost and J.-P. Tignol, *The book of involutions*, American Mathematical Society Colloquium Publications, 44, AMS, 1998. Zbl 0955.16001 MR 1632779

[10] S. MacLane, Subfields and automorphism groups of $p$-adic fields, *Annals of Mathematics*, **40** (1939), 423–442. Zbl 0021.00501 MR 1503470

[11] J. S. Milne and K. Y. Shih, Conjugates of Shimura varieties, in *Hodge cycles, motives, and Shimura varieties*, 280–356, Springer, Berlin Heidelberg, 1981.

[12] Y. Nisnevich, Rationally trivial principal homogeneous spaces and arithmetic of reductive group schemes over Dedekind rings, *C. R. Acad. Sci. Paris, Série I*, **299** (1984), no. 1, 5–8. Zbl 0587.14033 MR 0756297

[13] J.-P. Serre, Cohomologie galoisienne: progrès et problèmes, *Séminaire Bourbaki* (1993/94), *Astérisque*, **227** (1995), Exp. No. 783, 4, 229–257. Zbl 0837.12003 MR 1321649

Received November 1, 2015

N. Bhaskhar, Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA

E-mail: nbhaskh@emory.edu