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Teichmüller discs with completely degenerate Kontsevich–Zorich spectrum

David Aulicino*

Abstract. We reduce a question of Eskin–Kontsevich–Zorich and Forni–Matheus–Zorich, which asks for a classification of all $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic probability measures with completely degenerate Kontsevich–Zorich spectrum, to a conjecture of Möller's. Let $\mathcal{D}_g(1)$ be the subset of the moduli space of Abelian differentials $\mathcal{M}_g$ whose elements have period matrix derivative of rank one. There is an $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic probability measure ν with completely degenerate Kontsevich–Zorich spectrum, i.e. $\lambda_1 = 1 > \lambda_2 = \cdots = \lambda_g = 0$, if and only if ν has support contained in $\mathcal{D}_g(1)$. We approach this problem by studying Teichmüller discs contained in $\mathcal{D}_g(1)$. We show that if (X,ω) generates a Teichmüller disc in $\mathcal{D}_g(1)$, then (X,ω) is completely periodic. Furthermore, we show that there are no Teichmüller discs in $\mathcal{D}_g(1)$, for $g = 2$, and the two known examples of Teichmüller discs in $\mathcal{D}_g(1)$, for $g = 3, 4$, are the only two such discs in those genera. Finally, we prove that if there are no genus five Veech surfaces generating Teichmüller discs in $\mathcal{D}_5(1)$, then there are no Teichmüller discs in $\mathcal{D}_g(1)$, for $g = 5, 6$.

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1. Introduction

In [21], Kontsevich and Zorich introduced the Kontsevich–Zorich cocycle as a cocycle on the Hodge bundle over the moduli space of Riemann surfaces, denoted G_t^{KZ}, which is a continuous time version of the Rauzy–Veech–Zorich cocycle. They showed that this cocycle has a spectrum of $2g$ Lyapunov exponents with the property

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq -\lambda_g \geq \cdots \geq -\lambda_2 \geq -\lambda_1 = -1.$$

These exponents have strong implications about the dynamics of flows on Riemann surfaces, interval exchange transformations, rational billiards, and related systems.

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They also describe how generic trajectories of an Abelian differential distribute over a surface [39]. Furthermore, Zorich [39] proved that they fully describe the non-trivial exponents of the Teichmüller geodesic flow, denoted G_t. Veech [34] proved $\lambda_2 < 1$, which implies that G_t is non-uniformly hyperbolic. Since then, the study of the Lyapunov spectrum of the Kontsevich–Zorich cocycle has become of widespread interest. Forni [11] proved the first part of the Kontsevich–Zorich conjecture [21]: $\lambda_g > 0$ for the canonical $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic measure in the moduli space of holomorphic quadratic differentials. His result implies G_t^{KZ} is also non-uniformly hyperbolic. Avila and Viana [2] then used independent techniques to show that the spectrum is simple for the canonical measures on the strata of Abelian differentials, i.e. $\lambda_k > \lambda_{k+1}$, for all k.

Throughout this paper, the spectrum of Lyapunov exponents of the Kontsevich–Zorich cocycle will be referred to as the Kontsevich–Zorich spectrum (KZ-spectrum). Veech asked to what extent the KZ-spectrum could be degenerate. Forni [12] found an example of an $\mathrm{SL}_2(\mathbb{R})$-invariant measure supported on the Teichmüller disc of a genus three surface with completely degenerate KZ-spectrum, i.e. $\lambda_1 = 1 > \lambda_2 = \lambda_3 = 0$. In the literature, the genus three surface generating Forni's example, denoted here by (M_3, ω_{M_3}), is known as the Eierlegende Wollmilchsau for its numerous remarkable properties [18]. Forni and Matheus [13] then found an example generated by a genus four surface, denoted here by (M_4, ω_{M_4}), with $\lambda_1 = 1 > \lambda_2 = \lambda_3 = \lambda_4 = 0$. Both surfaces are Veech surfaces and in particular, square tiled cyclic covers. They will be defined and depicted in Section 8. By relating Teichmüller and Shimura curves, Möller [30] proved that these two examples are the only examples of Veech surfaces generating Teichmüller discs supporting a measure with completely degenerate KZ-spectrum except for possible examples in certain strata of Abelian differentials in genus five. In a paper of Forni, Matheus and Zorich [14], they proved that the two examples are the only square-tiled cyclic cover surfaces generating Teichmüller discs supporting a measure with completely degenerate KZ-spectrum. In the recent work of [8], it was shown that there are no regular $\mathrm{SL}_2(\mathbb{R})$-invariant suborbifolds with completely degenerate Kontsevich–Zorich spectrum for $g \geq 7$. It was recently announced by Eskin and Mirzakhani [9], that the closure of every Teichmüller disc is an $\mathrm{SL}_2(\mathbb{R})$-invariant suborbifold. The technical condition of regularity for an $\mathrm{SL}_2(\mathbb{R})$-invariant suborbifold is defined in [8, Section 1.5]. It was recently announced by Avila, Matheus, and Yoccoz that every $\mathrm{SL}_2(\mathbb{R})$-invariant suborbifold is regular [1]. Hence, the result of [8, Corollary 5] perfectly complements the results of this paper.

Both [8] and [14] asked if the two known examples generate the only Teichmüller discs whose closures support an $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic probability measure with completely degenerate Kontsevich–Zorich spectrum. In this paper we give a nearly complete answer to this question by reducing the entire problem to a conjecture of Möller that claims there are no Veech surfaces in genus five that generate a Teichmüller disc with this property. Let $\mathcal{D}_g(1)$ denote the subset of the moduli space of Abelian differentials, where the derivative of the period matrix has rank one. We address a potentially stronger problem and ask for a classification of all Teichmüller discs in $\mathcal{D}_g(1)$.

Theorem 1.1. *There are no Teichmüller discs in $\mathcal{D}_g(1)$, for $g = 2$. The surface (M_3, ω_{M_3}) generates the only Teichmüller disc in $\mathcal{D}_3(1)$ and (M_4, ω_{M_4}) generates the only Teichmüller disc in $\mathcal{D}_4(1)$. Furthermore, if there are no Teichmüller curves in $\mathcal{D}_5(1)$, then there are no Teichmüller discs in $\mathcal{D}_g(1)$, for $g = 5, 6$.*

The main techniques used in this paper include degenerating surfaces under the Deligne–Mumford compactification of the moduli space of Riemann surfaces, and an analysis of the derivative of the period matrix under such deformations. This concept has already been used successfully in [11]. Several other authors have also used this concept in other guises such as the second fundamental form of the Hodge bundle [15] and the Kodaira–Spencer map in the work of Möller and his coauthors [4, 5, 30].

To prove this theorem we show first that any surface generating a Teichmüller disc in $\mathcal{D}_g(1)$ is completely periodic, cf. Theorem 5.5. Then we show that degenerating surfaces in the closure of a Teichmüller disc in $\mathcal{D}_g(1)$ must have a very specific configuration, cf. Lemma 5.9. Proving the results requires some technical lemmas demonstrating convergence of the derivative of the period matrix, cf. Section 3.2, and a technical lemma concerning the limit of a surface with cylinders that do not fill the surface under the Teichmüller geodesic flow, cf. Lemma 4.3. These results quickly yield some applications, cf. Proposition 6.4.

Next we show that the closure of every Teichmüller disc in $\mathcal{D}_g(1)$ must contain a (possibly degenerate) surface that is a Veech surface, cf. Theorem 7.4. This leads to an analysis of punctures on a Veech surface with the goal of excluding more and more configurations of the punctures until the remainder of the results follow. Theorem 1.1 summarizes Proposition 6.4, Theorem 8.10, Theorem 9.10, and Proposition 8.16.

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2. Preliminaries

2.1. The moduli space of Riemann surfaces. Let X be a Riemann surface of genus g with n punctures (i.e. marked points). Let $R(X)$ denote the *Teichmüller space of* X or simply $R_{g,n}$ when X is understood. The surface X admits a *pants decomposition*, $X = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{3g-3+n}$, into $3g - 3 + n$ pairs of pants, where each pair of pants is homeomorphic to the sphere with a total of three punctures and disjoint boundary curves. The Fenchel–Nielsen coordinates for Teichmüller space describe surfaces in terms of the lengths and twists of curves in a pants decomposition of X. A point in Teichmüller space is given by $(\ell_1, \ldots, \ell_{3g-3+n}, \theta_1, \ldots, \theta_{3g-3+n}) \in \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$.

Let $\mathrm{Diff}^+(X)$ be the group of orientation preserving diffeomorphisms on X. Let $\mathrm{Diff}_0^+(X)$ denote the normal subgroup of $\mathrm{Diff}^+(X)$ whose elements are isotopic to the identity. Then the *mapping class group* is the quotient

$$\Gamma(X) = \mathrm{Diff}^+(X)/\mathrm{Diff}_0^+(X).$$

The *moduli space of genus g surfaces with n punctures* is defined to be

$$\mathcal{R}_{g,n} = R(X)/\Gamma(X).$$

Deligne and Mumford [7] introduced a compactification of the moduli space denoted $\overline{\mathcal{R}_{g,n}}$ of Riemann surfaces within the more general setting of compactifying the space of stable curves. Every neighborhood of a point on a *Riemann surface with nodes* is either conformally equivalent to the unit complex disc, or to the set $\{(x, y) \in \mathbb{C}^2 | xy = 0\}$. The point mapped to $(0, 0)$ with the latter property is called a *node*. We regard this as the contraction or pinching of a simple closed curve on a surface to a point. Removing a node results in two punctures on either side of the node. This may or may not disconnect the surface. After removing all nodes, each of the connected components of the punctured degenerate surface is called a *part*. A *pair of punctures*, denoted (p, p'), will specifically refer to the punctures created by removing a node. We will assume this deconstruction throughout and say that pinching a curve results in a pair of punctures unless we say otherwise. Theorem B.1 in Appendix B of [20] describes the compactification of the moduli space in terms of the Fenchel–Nielsen coordinates (or equivalently, a choice of pants decomposition) for Teichmüller space. By [20, Theorem B.1], the boundary of the moduli space $\overline{\mathcal{R}_{g,n}}$ under the Deligne–Mumford compactification is given by letting one or more of the lengths ℓ_i in the Fenchel–Nielsen coordinates be zero.

2.2. Abelian and quadratic differentials.

2.2.1. Abelian differentials. Let K be the cotangent bundle over X. A section ω of K is a complex 1-form called an *Abelian differential*. An Abelian differential ω on X

is given in local coordinates by $\omega = \phi(z)\,dz$, where $\phi(z)$ is a holomorphic function on the punctured surface possibly having poles of finite order at the punctures. Furthermore, ω obeys the change of coordinates formula

$$\phi(\sigma(z))\,d\sigma(z) = \phi(\sigma(z))\sigma'(z)\,dz.$$

The zeros and poles of ω are called *singularities* and all other points are called *regular*. The Chern formula relates the total number of zeros and poles counting multiplicity, by

$$\sharp(\text{zeros}) - \sharp(\text{poles}) = 2g - 2.$$

An Abelian differential ω determines an orientable horizontal and vertical foliation of a surface given by $\{\Im(\omega) = 0\}$ and $\{\Re(\omega) = 0\}$, respectively. Equivalently, the foliations can be defined by a pullback of the horizontal and vertical lines in the complex plane under the local coordinate chart on the surface. The Abelian differential ω determines a flat structure on the surface away from the singularities. A maximal connected subset of a foliation is called a *leaf*. If a leaf is compact and it does not pass through a singularity of ω, then it is called a *closed regular trajectory*. A closed connected subset σ of a leaf with endpoints at zeros of ω whose interior consists entirely of regular points of ω is called a *saddle connection*. Given a closed regular trajectory γ, the closure of the maximal set of parallel closed regular trajectories homotopic to γ form a *cylinder*. By definition, the boundaries of a cylinder consist of a union of saddle connections. We say that two cylinders are *homologous* (resp. *parallel*) if their core curves are homologous (resp. parallel). If every leaf of a foliation is compact, the foliation is *periodic*.

Lemma 2.1. *If C_1 and C_2 are homologous cylinders on a surface (X, ω), then C_1 and C_2 are parallel.*

Proof. Let γ_1 and γ_2 be the core curves of C_1 and C_2, respectively. Without loss of generality, assume that γ_1 is a closed curve of the vertical foliation on X by ω. Then by the definition of homologous

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

Thus

$$\int_{\gamma_1} \omega = \int_{\gamma_1} \Re(\omega) + i\Im(\omega) = i\int_{\gamma_1} \Im(\omega).$$

The last equality follows because γ_1 lies exactly in the vertical foliation so it has no horizontal holonomy. However, this implies

$$\int_{\gamma_2} \omega = i\int_{\gamma_1} \Im(\omega),$$

which implies

$$\int_{\gamma_2} \Re(\omega) = 0.$$

Therefore, γ_2 has no horizontal holonomy either, so it must be parallel to γ_1. $\square$

Call $\phi(z)$ or ω *holomorphic* if it can be continued holomorphically across all punctures of X. When $\phi(z)$ is holomorphic it naturally determines a flat metric on the surface. The length of a curve γ in this metric is given by

$$\int_{\gamma} |\phi(z)dz|.$$

Furthermore, there is an area form given by

$$A(\omega) = \frac{i}{2}\int_X \omega \wedge \bar{\omega}.$$

In the case of meromorphic differentials, the metric is still defined on compact subsets away from the punctures at which the differential has a pole though the area form is infinite.

Let $T_{g,n}$ be the Teichmüller space of Riemann surfaces carrying Abelian differentials. Define *the moduli space of Abelian differentials on Riemann surfaces of genus g with n punctures* by $\mathcal{M}_{g,n} = T_{g,n}/\Gamma(X)$. Define $\mathcal{M}_g := \mathcal{M}_{g,0}$ and $\mathcal{M}_{g,n}^{(1)} := \{(X,\omega) \in \mathcal{M}_g | A(\omega) = 1\}$.

Given a holomorphic differential ω on X, the sum of the orders of the zeros of ω is $2g-2$. This determines a stratification of the moduli space of holomorphic differentials by the multiplicities of the zeros of the Abelian differential. Denote the strata by $\mathcal{H}(\kappa)$, where κ is a vector corresponding to a partition of $2g-2$. In the case of meromorphic differentials, we list the orders of the poles in the vector κ so that the sum of the components of the vector remains $2g-2$.

The moduli space of Abelian differentials can be expanded so that limits of convergent sequences of Abelian differentials lying on degenerating surfaces exist on nodal surfaces [17]. An Abelian differential ω on a nodal Riemann surface is holomorphic everywhere except possibly at the punctures arising from removing the nodes, where ω is meromorphic with at most simple poles. At each pair of punctures (p,p'), ω satisfies

$$\mathrm{Res}_p(\omega) = -\mathrm{Res}_{p'}(\omega).$$

Let $\overline{\mathcal{M}_g}$ denote the moduli space of meromorphic Abelian differentials over the compactified base space $\overline{\mathcal{R}_g}$.

There is a natural action by $\mathbb{R}^*$ on the bundle of Abelian differentials. Let $r \in \mathbb{R}^*$ and $(X,\omega) \in \overline{\mathcal{M}_g}$, then

$$r \cdot (X,\omega) := (X, r\omega).$$

For the remainder of the paper, we abuse notation and assume that the moduli space $\mathcal{M}_g$ is always quotiented by $\mathbb{R}^*$ unless we say otherwise. Furthermore, it will often be useful to choose a representative differential of the coset $(X, \omega)[\mathbb{R}^*]$. For instance, if ω is holomorphic and nonzero, we may choose the representative so that its area form is one and if ω is not holomorphic, we may choose a representative such that the modulus of the largest residue is one. This will be called *area normalization* or *residue normalization*, respectively.

The advantage of this projectivized moduli space of Abelian differentials is that it guarantees that for every sequence of Abelian differentials converging to an Abelian differential on a degenerate surface that there is at least one part of the degenerate surface on which the limiting Abelian differential is not identically zero. Without the projectivization, no such guarantee can be made. Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ be a sequence of surfaces carrying holomorphic Abelian differentials converging to a degenerate surface (X', ω') in $\overline{\mathcal{M}_g}$. Since X_n has finite genus, there are finitely many pinching curves.

Definition. *For positive constants M, $\rho_1 < \rho_2 < 1$, an Abelian differential ω on the annulus $A_{t,c,c'} := \{\zeta \in \mathbb{C} \,\|\, |t|/c' < |\zeta| < c\}$ is* band bounded, *provided that $|\omega(d\zeta/\zeta)^{-1}| \leq M$ for ζ satisfying $|t|/(c'\rho_2) \leq |\zeta| \leq |t|/(c'\rho_1)$ and satisfying $\rho_1 c \leq |\zeta| \leq \rho_2 c$.*

A sequence of Abelian differentials ω_t on annuli $A_{t,c,c'}$ with t tending to zero, is band bounded provided the differentials ω_t on $A_{t,c,c'}$ are band bounded for positive constants M, ρ_1, ρ_2, and all small t.

We can assume that ω_n is band bounded [37, Definition 1] on the annulus around each pinching curve. If we multiply ω_n by r_n so that the constant M in the definition of band bounded is uniformly bounded away from zero and infinity for all n, then Lemma 2.2 follows from [37, Lemma 2].

Lemma 2.2. *Given a sequence $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ such that the sequence $\{X_n\}_{n=0}^{\infty}$ converges to a degenerate surface X', there exists an Abelian differential ω' on X' such that ω' is the limit of the sequence $\{\omega_n\}_{n=0}^{\infty}$ in $\overline{\mathcal{M}_g}/\mathbb{R}^*$ and ω' is not identically zero on every part of X'.*

2.2.2. Quadratic differentials. Let K be the cotangent bundle over X. The sections of the bundle $K \otimes_{\mathbb{C}} K$ are called *quadratic differentials.* A quadratic differential is given in local coordinates by $q = \phi(z)\, dz^2$ and obeys the change of coordinates formula

$$\phi(\sigma(z))\, d\sigma(z)^2 = \phi(\sigma(z))(\sigma'(z))^2\, dz^2.$$

Singularities and regular points are defined as before and in this case the Chern formula reads

$$\sharp(\text{zeros}) - \sharp(\text{poles}) = 4g - 4.$$

A quadratic differential determines a horizontal and vertical foliation of a surface given by $\{\Im(\sqrt{\phi(z)}) = 0\}$ and $\{\Re(\sqrt{\phi(z)}) = 0\}$, respectively. These foliations are not necessarily orientable. If they are, q is called an *orientable quadratic differential*. If a quadratic differential is holomorphic everywhere except for at most a finite set of simple poles, then it is called an *integrable quadratic differential*. Denote the *Teichmüller space of integrable quadratic differentials* by $Q_{g,n}$ and the corresponding *moduli space of integrable quadratic differentials* by $\mathcal{Q}_{g,n} := Q_{g,n}/\Gamma_{g,n}$.

There is a natural way of associating an Abelian differential to a given quadratic differential. If q is non-orientable, then there is a connected double covering $\pi : \hat{X} \to X$ defined as follows. For each chart U of X, let $q = \phi_U(z)\,dz^2$ and define two charts $V^{\pm}$ of $\hat{X}$ each of which maps homeomorphically to U under π and $V^{\pm}$ carry the local differentials $\pm\sqrt{\phi_U(z)}\,dz$. This lift is compatible across charts and defines a quadratic differential ω^* with the property $\hat{q} = h^2$, where h is an Abelian differential. This lifting procedure is called the *orientating double cover construction*, and it can be used to translate the terms defined for Abelian differentials above (metrics, etc.) to non-orientable quadratic differentials.

As above, the bundle of quadratic differentials can be extended to the boundary of the moduli space of Riemann surfaces as defined by the Deligne–Mumford compactification. By admitting quadratic differentials with at most double poles, limits of sequences of integrable quadratic differentials on non-degenerate surfaces exist on degenerate surfaces. Define the residue of a quadratic differential q at a point p to be the coefficient of the term $1/z^2$ in its Taylor expansion at p. Given a quadratic differential q on a degenerate surface X with a pair of punctures (p, p'), the residues of q obey the relation

$$\mathrm{Res}_p(q) = \mathrm{Res}_{p'}(q).$$

Let $\overline{\mathcal{Q}_{g,n}}$ denote the moduli space of regular quadratic differentials on the compactified base space of Riemann surfaces $\overline{\mathcal{R}_{g,n}}$.

2.3. The $\mathrm{SL}_2(\mathbb{R})$ action. We define the $\mathrm{SL}_2(\mathbb{R})$ action on quadratic differentials. It is clear that this definition applies to Abelian differentials as well. Let q be an integrable quadratic differential. Let h (resp. v) denote the horizontal (resp. vertical) foliation of q. The action by $A \in \mathrm{SL}_2(\mathbb{R})$ on an integrable quadratic differential q is defined by

$$\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h \\ v \end{bmatrix}$$

and denoted by $A \cdot (X, q)$. The action is well-defined on and between charts of X. Thus it defines an action by A globally on (X, q). It was stated in [4, Section 11] that the action is also well-defined on meromorphic Abelian differentials with at most simple poles. Furthermore, [4, Proposition 11.1] says that the action of $\mathrm{GL}_2^+(\mathbb{R})$

extends continuously to the boundary of $\overline{\mathcal{M}_g}$. We point out to the reader that the action by $\mathrm{GL}_2^+(\mathbb{R})$ on $\overline{\mathcal{M}_g}$ without the action by $\mathbb{R}^*$ is the same as considering the action of $\mathrm{SL}_2(\mathbb{R})$ on $\overline{\mathcal{M}_g}/\mathbb{R}^*$ because the action by $\mathbb{R}$ commutes with everything.

Definition. *Given a surface* $(X,q) \in \mathcal{Q}_{g,n}$, *the* Teichmüller disc *of* (X,q) *is the orbit of* (X,q) *in* $\mathcal{Q}_{g,n}$ *under the action by* $SL_2(\mathbb{R})$.

The *Teichmüller geodesic flow*, denoted G_t, on the bundle of quadratic differentials is the action by diagonal matrices:

$$G_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

We note for the convenience of the reader that the residue of the simple pole of an Abelian differential differs from the holonomy vector by a factor of $2\pi i$.

Lemma 2.3. *Let ω be an Abelian differential on a surface X with residue $c = a+ib$ at $p \in X$. Let c_{G_t} denote the residue at p after acting by G_t on (X,ω). Then*

$$c_{G_t} = ae^{-t} + ibe^t.$$

Proof. Without loss of generality, let $p = 0$ in local coordinates about p. By [32, Theorem 6.3], it suffices to look at how the differential $c\,dz/z$ changes under the action by G_t. To do this, convert to polar coordinates and integrate the differential around the curve γ defined by $r = 1$. Let $c = a + ib$. Then

$$\frac{c\,dz}{z} = (a+ib)\left(\frac{dr}{r} + i\,d\theta\right) = \frac{a\,dr}{r} - b\,d\theta + i\left(b\frac{dr}{r} + a\,d\theta\right).$$

Furthermore, $dr = 0$ because $r = 1$. So this simplifies to $(-b+ia)\,d\theta$ and acting by G_t we get $(-be^t + iae^{-t})\,d\theta$. Therefore,

$$c_{G_t} = \frac{1}{2\pi i}\int_0^{2\pi} (-be^t + iae^{-t})\,d\theta = ae^{-t} + ibe^t. \qquad \square$$

Definition. *A number $c \in \mathbb{C}$ is* ε-nearly imaginary *if* $|\arg(c) \pm \pi/2| < \varepsilon$.

Lemma 2.4. *Let (X',ω') be a degenerate surface carrying an Abelian differential with simple poles and residues $\{c_1,\dots,c_m\}$. Given $\varepsilon > 0$, there exists $A \in SL_2(\mathbb{R})$ such that if c'_j is a residue of $A\cdot(X',\omega')$, for $1 \le j \le m$, then c'_j is ε-nearly imaginary.*

Proof. It is possible that ω' has some real residues. If so, multiply ω' by a complex unit ζ so that $\zeta\omega'$ has no real residues. Given a residue ζc_j of $\zeta\omega'$, after acting on ζc_j by G_t, the real part of the resulting residue is $e^{-t}\Re(\zeta c_j)$ by Lemma 2.3. Hence, there exists T such that

$$|e^{-T}\Re(\zeta c_j)| < \varepsilon|e^{-T}\Re(\zeta c_j) + ie^T\Im(\zeta c_j)|. \qquad \square$$

Lemma 2.5. *Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ be a sequence of surfaces containing cylinders $C_n \subset X_n$ with core curves γ_n. Let w_n and h_n denote the flat length with respect to ω_n of the circumference and height of C_n, respectively. If the ratio h_n/w_n tends to infinity with n, then the hyperbolic length of γ_n converges to zero.*

Proof. The modulus of the cylinder C_n is exactly the quotient h_n/w_n. By [23, Lemma 3],

$$\mathrm{Ext}_x(\gamma_n) \le \frac{1}{\mathrm{Mod}_x(\gamma_n)},$$

where $\mathrm{Ext}_x(\gamma_n)$ is the extremal length of γ_n with respect to a Riemann surface X. By [24, Corollary 2], $\mathrm{Ext}_x(\gamma_n)$ goes to zero with the hyperbolic length of γ_n. □

Corollary 2.6. *Let (X, ω) admit a cylinder with core curve γ such that γ lies in the vertical foliation of X by ω. Then for all divergent sequences of positive times $\{t_n\}_{n=1}^{\infty}$ for which the limit*

$$\lim_{n\to\infty} G_{t_n} \cdot (X, \omega) = (X', \omega'),$$

exists, γ degenerates to a node of X'.

Proof. Let $C \subset X$ denote the cylinder with core curve γ and let w and h denote the circumference and height of C, respectively. After time t_n, the circumference and height are given by $e^{-t_n} w$ and $e^{t_n} h$. Since

$$\lim_{n\to\infty} \frac{e^{t_n} h}{e^{-t_n} w} = \infty,$$

γ pinches as n tends to infinity, by Lemma 2.5. □

Lemma 2.7. *Let D be a Teichmüller disc in $\overline{\mathcal{M}_g}/\mathbb{R}^*$. Given a sequence $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ in D converging to a degenerate surface (X', ω'), there exists a degenerate surface (X'', ω'') in the closure of D such that ω'' is not holomorphic on every part of X''. Furthermore, X'' is reached from X' by pinching additional curves of X'.*

Proof. By Lemma 2.2, we assume that there is a part $S \subset X'$ such that ω' is not identically zero on S. If ω' has simple poles on X', then we are done, so assume otherwise. By [27, Theorem 2], there is a cylinder C_1 on S. Degenerate S under the Teichmüller geodesic flow by pinching the core curve of C_1. All punctures of X' are obviously preserved under the $\mathrm{SL}_2(\mathbb{R})$ action. The new limit ω_1' carries an Abelian differential which is not identically zero everywhere by Lemma 2.2. If ω_1' is holomorphic on every part we can repeat the argument. Since the genus is finite, the repetition of this argument will terminate when we reach a differential that is not holomorphic or when the surface degenerates to a sphere, which does not carry holomorphic differentials. Since the punctures of X' are preserved under the

$\mathrm{SL}_2(\mathbb{R})$ action, X'' is reached from X' by pinching additional curves. Furthermore, it follows from the continuity of the $\mathrm{SL}_2(\mathbb{R})$ action [4, Proposition 11.1] that X'' is in the closure of D. $\square$

3. Lyapunov exponents and the rank one locus

In the first subsection, we give the precise formulation of the problem answered in this paper. In the second subsection we present all of the technical lemmas related to the derivative of the period matrix that will be used throughout the remainder of this paper.

3.1. Lyapunov exponents of the KZ-cocycle. Let X be a Riemann surface of genus g. Consider the cocycle defined by the Teichmüller geodesic flow as follows

$$G_t \times \mathrm{Id} : T_g \times H^1(X, \mathbb{C}) \to T_g \times H^1(X, \mathbb{C}).$$

The mapping class group preserves the real and imaginary parts of $T_g \times H^1(X, \mathbb{C})$. The *Kontsevich–Zorich cocycle* is the quotient cocycle

$$G_t^{KZ} : \Re\left((T_g \times H^1(X, \mathbb{C}))/\Gamma_g\right) \to \Re\left((T_g \times H^1(X, \mathbb{C}))/\Gamma_g\right)$$

restricted to the real part.

Let ν denote a finite $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic measure on $\mathcal{M}_g$. The cocycle G_t^{KZ} admits a spectrum of $2g$ Lyapunov exponents with respect to ν. The natural symplectic structure on $H^1(X, \mathbb{C})$ induces a symplectic structure on the entire bundle $\Re\left((T_g \times H^1(X, \mathbb{C}))/\Gamma_g\right)$, which forces a symmetry of the $2g$ Lyapunov exponents.

$$1 = \lambda_1^\nu \geq \lambda_2^\nu \geq \cdots \geq \lambda_g^\nu \geq -\lambda_g^\nu \geq \cdots \geq -\lambda_2^\nu \geq -\lambda_1^\nu = -1.$$

We refer to these $2g$ numbers as the *spectrum of Lyapunov exponents of the Kontsevich–Zorich cocycle* or the *KZ-spectrum* for short. If $\lambda_k^\nu = 0$, for some k, then the spectrum is called *degenerate*. If $\lambda_k^\nu = 0$ for all $k > 1$, then the KZ-spectrum is *completely degenerate*.

Kontsevich and Zorich [21] as well as Forni [11] gave a formula for the sum of these exponents in terms of the eigenvalues of a Hermitian form. These eigenvalues were reinterpreted through the second fundamental form of the Hodge bundle [15]. Let $(X, \omega) \in \mathcal{M}_g$. Let $L^2_\omega(X)$ be the Hilbert space of complex-valued functions on X that are L^2 with respect to ω. Let $\langle \cdot, \cdot \rangle_\omega$ be the inner product on $L^2_\omega(X)$. Let $M^\pm_\omega \subset L^2_\omega(X)$ be the subspaces of meromorphic and anti-meromorphic functions, respectively. Define the orthogonal projections

$$\pi^\pm_\omega : L^2_\omega(X) \to M^\pm_\omega.$$

For two meromorphic functions $m_1^+, m_2^+ \in M_\omega^+$,

$$H_\omega(m_1^+, m_2^+) = \langle \pi_\omega^-(m_1^+), \pi_\omega^-(m_2^+)\rangle_\omega.$$

The eigenvalues of $H_\omega(\cdot,\cdot)$ are given by the functionals $\Lambda_k(\omega) : \mathcal{M}_g^{(1)} \to \mathbb{R}$, which are continuous for all k and ω, and obey the inequalities

$$1 \equiv \Lambda_1(\omega) \geq \Lambda_2(\omega) \geq \cdots \geq \Lambda_g(\omega) \geq 0.$$

In [12], Forni introduced a filtration of sets

$$\mathcal{D}_g(1) \subset \mathcal{D}_g(2) \subset \cdots \subset \mathcal{D}_g(g-1),$$

where

$$\mathcal{D}_g(k) = \{(X,\omega) \in \mathcal{M}_g \mid \Lambda_{k+1}(\omega) = \cdots = \Lambda_g(\omega) = 0\},$$

and $\mathcal{D}_g(k)$ is called the *rank k locus*. The set $\mathcal{D}_g(g-1) = \mathcal{D}_g$ is the *determinant locus* introduced in [11].

Let ν be a canonical $SL_2(\mathbb{R})$-invariant measure on a connected component $\mathcal{C}_\kappa$ of the stratum $\mathcal{H}(\kappa) \subset \mathcal{M}_g$ of Abelian differentials. Corollary 5.3 of [11] gives the following identity:

$$\lambda_2^\nu + \cdots + \lambda_g^\nu = \frac{1}{\nu(\mathcal{C}_\kappa)} \int_{\mathcal{C}_\kappa} \Lambda_2(\omega) + \cdots + \Lambda_g(\omega)\, d\nu.$$

In [12], Forni notes that this formula can be extended to any $SL_2(\mathbb{R})$-invariant ergodic probability measure, from which the lemma below follows.

Lemma 3.1 (Forni [12, Cor. 7.1]). *Let ν be a finite $SL_2(\mathbb{R})$-invariant ergodic measure on the moduli space $\mathcal{M}_g$. The KZ-spectrum with respect to ν is completely degenerate if and only if for almost every $(X,\omega) \in supp(\nu)$, H_ω has rank one, i.e. $supp(\nu) \subset \mathcal{D}_g(1)$.*

We introduce the derivative of the period matrix, which will be the focus of this paper. Let $\{a_1,\dots,a_g,b_1,\dots,b_g\}$ be a basis for the first homology group $H_1(X,\mathbb{C})$. Let $\{\theta_j\}_{j=1}^g$ be a basis of the complex vector space of holomorphic Abelian differentials on X normalized so that

$$\int_{a_i} \theta_j = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Under this choice of basis of Abelian differentials, the *period matrix* $\Pi(X)$ is the symmetric matrix with positive definite imaginary part whose components are given by

$$b_{ij} = \int_{b_i} \theta_j.$$

The space of Beltrami differentials, $B(X)$ is dual to the cotangent space of quadratic differentials. Every Abelian differential ω uniquely determines a Beltrami differential

$$\mu_\omega = \frac{\bar{\omega}}{\omega},$$

which is defined everywhere except at the zeros and poles of ω of which there are only finitely many. In the Teichmüller space $R(X)$ the space $B(X)$ represents the tangent space and $\mu \in B(X)$ a tangent vector at X. In $R(X)$, μ determines a direction in which we can take a derivative of $\Pi(X)$. The derivative of the period matrix at X in direction μ is denoted by $d\Pi(X)/d\mu$. Let $\omega = h(z)\,dz$ and $\theta_k = f_k(z)\,dz$, for all k. Rauch's formula, [20, Proposition A.3], gives a concise formula for the components of the derivative of the period matrix.

$$\frac{d\Pi_{ij}(X)}{d\mu_\omega} = \int_X \theta_i\theta_j\, d\mu_\omega = \int_X f_i f_j \frac{\bar{h}}{h} dz \wedge d\bar{z}$$

In the proof of Lemma 4.1 of [11], Forni defines a complex bilinear form on holomorphic Abelian differentials ω_1, ω_2 by

$$B_\omega(\omega_1, \omega_2) = \left\langle \frac{\omega_1}{\omega}, \frac{\bar{\omega}_2}{\omega} \right\rangle_\omega .$$

It was proven in [11] that $H_\omega = B_\omega B_\omega^*$ (and a typo in the equation in [11] was corrected in [15]). It is possible to choose a basis of Abelian differentials $\{\phi_1, \dots, \phi_g\}$ on X such that

$$\frac{d\Pi_{ij}(X)}{d\mu_\omega} = B_\omega(\phi_i, \phi_j).$$

Hence, H_ω has rank one if and only if $d\Pi(X)/d\mu_\omega$ has rank one. For this reason it suffices to regard $\mathcal{D}_g(1)$ as the set where $d\Pi(X)/d\mu_\omega$ has rank one for the remainder of this paper.

Since ν is an $SL_2(\mathbb{R})$-invariant measure, $\operatorname{supp}(\nu)$ must be an $SL_2(\mathbb{R})$-invariant set. Consider $(X, \omega) \in \operatorname{supp}(\nu)$. Let D be the Teichmüller disc generated by (X, ω). Then $D \subset \operatorname{supp}(\nu)$, and if the KZ-spectrum with respect to ν is completely degenerate, then $D \subset \mathcal{D}_g(1)$. This is precisely the problem that we address in this paper.

Problem. *Classify all Teichmüller discs D such that $D \subset \mathcal{D}_g(1)$.*

3.2. The derivative of the period matrix. One of the most important techniques in this paper is the use of estimates for the derivative of the period matrix near the boundary of the moduli space $\mathcal{M}_g$. In this section we introduce plumbing coordinates for a Riemann surface and express Abelian differentials in terms of those plumbing coordinates using the exposition of [38]. Unfortunately, it will not be possible to guarantee convergence of the derivative of the period matrix in every possible scenario, but it will be possible for all cases relevant to this paper. Lemma 3.2 below is a stronger statement than that of [11, Lemma 4.2] because it applies to any sequence satisfying a relatively lax set of assumptions. These convergence lemmas motivate and justify defining the rank of the derivative of the period matrix for surfaces in the boundary of $\overline{\mathcal{M}_g}$.

Plumbing coordinates have been used extensively from [26] to [11], among others. They have been used to write explicit formulas for differentials near the boundary of the moduli space. Wolpert [37] reworked the foundations of differentials on families of degenerating surfaces using the language of sheaves, and expressed the differentials on degenerating surfaces in terms of plumbing coordinates. We copy the language and notation of [11, Section 4] and [37, 38], as appropriate. Let X' be a degenerate Riemann surface in the boundary of $\overline{\mathcal{R}_g}$. Let X' have $1 \leq m \leq 3g-3$ pairs of punctures $\{(p_i, p_i')\}$, for $1 \leq i \leq m$. Let $\tau \in \mathbb{C}^{3g-3-m}$ denote the local coordinates for a neighborhood of X' in the Teichmüller space of X'. We denote surfaces in a neighborhood of $X' \in \partial\mathcal{R}_g$ by $X(0,\tau)$. We refer the reader to [38, Section 3], where the coordinates are specifically chosen to correspond to small deformations of the complex structure on X'. For our purposes, it suffices to know that such a coordinate τ exists. Let $(U_i(0,\tau), z_i)$ and $(V_i(0,\tau), w_i)$ be coordinate charts around p_i and p_i', respectively, such that $z_i(p_i) = w_i(p_i') = 0$. Following [37, 38], let c', c'' be positive constants, $V = \{|z| < c', |w| < c''\}$, $D = \{|t| < c'c''\}$, and $\pi : V \to D$ be the singular fibration with projection $\pi(z,w) = zw = t$, where $z, w, t \in \mathbb{C}$. Let $t_m = (t^{(1)}, \ldots, t^{(m)}) \in D^m$. Let $c < 1$ be a small positive constant. For $|t^{(i)}| < c^4$ and $1 \leq i \leq m$, remove the discs $\{|z_i| \leq c^2\}$ and $\{|w_i| \leq c^2\}$ from $X'(0,\tau)$ to get an open surface X_τ^*. For each i, identify a point $u_0 \in \{u | c^2 < |z_i(u)| < c\} \subset X_\tau^*$ to the point $(z_i(u_0), t^{(i)}/z_i(u_0))$ in the fiber of a i^{th} factor of $\pi_m : V^m \to D^m$ (induced by $\pi : V \to D$), and identify a point $v_0 \in \{v | c^2 < |w_i(v)| < c\} \subset X_\tau^*$ to the point $(t^{(i)}/w_i(v_0), w_i(v_0))$ in the fiber of a k^{th} factor of $\pi : V \to D$. This implies that we can write $X(t,\tau)$ to fully coordinatize a neighborhood of the degenerate surface $X' := X(0,\tau_\infty) \in \overline{\mathcal{R}_g}$.

In [26] and [11], the identification of the annuli is made directly so that if we translate their language to Wolpert's, we get

$$(z_i(u_0), t^{(i)}/z_i(u_0)) = (t^{(i)}/w_i(v_0), w_i(v_0))$$

and identify along the curve $|w_i(v_0)| = |z_i(u_0)| = \sqrt{|t^{(i)}|}$. It suffices to follow this convention throughout this paper. Following the notation of [37], we define annuli with respect to this identification. Let

$$R_z(t^{(i)}) := \left\{ \sqrt{|t^{(i)}|}/c'' < |\zeta_i| < c' \right\} \subset \left\{ |t^{(i)}|/c'' < |\zeta_i| < c' \right\}$$

and

$$R_w(t^{(i)}) := \left\{\sqrt{|t^{(i)}|/c'} < |\zeta_i| < c''\} \subset \{|t^{(i)}|/c' < |\zeta_i| < c''\right\}.$$

Let $c = c' = c''$ and define

$$X^*(t,\tau) =: X^*_\tau \cup \bigcup_{i=1}^{m} R_z(t^{(i)}) \cup R_w(t^{(i)}).$$

Next we consider Abelian differentials on Riemann surfaces. Let $D_1 \times \cdots \times D_m = D^m$ denote the m copies of D above. Following [37], every Abelian differential can be expressed in terms of local coordinates on D_j. This is done by considering the coordinate ζ_j on an annulus and the map $\zeta_j \mapsto (\zeta_j, t^{(j)}/\zeta_j)$ (resp. $\zeta_j \mapsto (t^{(j)}/\zeta_j, \zeta_j)$). As $t^{(j)}$ tends to zero this yields the convergence of the differential in local coordinates about the degenerating annuli resulting in the map $\zeta_j \mapsto (\zeta_j, 0)$ (resp. $\zeta_j \mapsto (0, \zeta_j)$).

It follows from a version of the Cartan–Serre theorem with parameters or [26, Proposition 4.1] , that there is a basis of Abelian differentials $\{\theta_1(t,\tau), \ldots, \theta_g(t,\tau)\}$ on $X(t,\tau)$, for all small t, such that $\{\theta_1(0,\tau_\infty), \ldots, \theta_g(0,\tau_\infty)\}$ spans the space of Abelian differentials on X'. We assume such a fixed basis in a neighborhood of a degenerate surface throughout this paper. Let

$$t' = (t^{(1)}, \ldots, t^{(j-1)}, t^{(j+1)}, \ldots, t^{(m)}).$$

In local coordinates on D_j, let $\theta_i(t',\tau,\zeta_j,t^{(j)}/\zeta_j) = 2 f_i(t',\tau,\zeta_j,t^{(j)}/\zeta_j)\, d\zeta_j/\zeta_j$, where

$$f_i(t',\tau,\zeta_j,t^{(j)}/\zeta_j) = \sum_{k,\ell \geq 0} a_{k\ell}(t',\tau)\zeta_j^k (t^{(j)}/\zeta_j)^\ell,$$

by [37].

Let $\{(X_n,\omega_n)\}_{n=0}^\infty$ be a sequence of surfaces carrying Abelian differentials converging to a degenerate surface (X',ω'). Without loss of generality, we can ignore the beginning of the sequence so that every element of the sequence can be expressed in terms of the local coordinates established above. Thus, let $X_n = X(t_n,\tau_n)$ and $X' = X(0,\tau_\infty)$. Let

$$\omega_n = 2A_n(t',\tau,\zeta_j,t^{(j)}/\zeta_j)\,\frac{d\zeta_j}{\zeta_j},$$

in local coordinates on D_j. Contrary to the coefficients f_i in the basis of Abelian differentials, note the dependence of the function A_n on n.

Lemma 3.2. *We follow the notation established above. Let $\{(X(t_n,\tau_n),\omega_n)\}_{n=0}^\infty$ be a sequence of surfaces converging to a degenerate surface $(X(0,\tau_\infty),\omega')$. For each n, let $\{\theta_1(t_n,\tau_n),\ldots,\theta_g(t_n,\tau_n)\}$ be a basis for the space of Abelian differentials on $X(t_n,\tau_n)$. Given i, j, for all k, if one of the following is true:*

(1) Either $f_i(0, \tau_\infty, 0, 0) = 0$ *on* D_k *or* $f_j(0, \tau_\infty, 0, 0) = 0$ *on* D_k*, or*

(2) $A_\infty(0, \tau_\infty, 0, 0) \neq 0$ *on* D_k*,*

then

$$\lim_{n\to\infty}\left(\frac{d\Pi_{ij}(X(t_n,\tau_n))}{d\mu_{\omega_n}} - \int_{X^*(t_n,\tau_\infty)} \theta_i(0,\tau_\infty)\theta_j(0,\tau_\infty)\, d\mu_{\omega'}\right) = 0.$$

Proof. On compact subsets away from the punctures, the integrand converges to an integrable real analytic function, so the dominated convergence theorem gives us the desired convergence on these compact sets. Hence, it suffices to prove convergence on each annulus $R_z(t_n^{(k)})$ and $R_w(t_n^{(k)})$. To get convergence on $R_w(t_n^{(k)})$, it suffices to show convergence on $R_z(t_n^{(k)})$ because they are symmetric up to multiplication by a constant. Using Rauch's formula, we explicitly write the expression to be estimated as t_n tends to zero in $\mathbb{C}^n$. That the following integral makes sense and proves the desired convergence follows from [37, Lemma 2].

$$4\int_{R_z(t_n^{(k)})}\left(\frac{f_i(t_n', \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)}{\zeta_k}\,\frac{f_j(t_n', \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)}{\zeta_k}\,\frac{\overline{A_n(t_n', \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t_n', \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)/\zeta_k}\right.$$
$$\left. - \frac{f_i(0, \tau_\infty, \zeta_k, 0)}{\zeta_k}\,\frac{f_j(0, \tau_\infty, \zeta_k, 0)}{\zeta_k}\,\frac{\overline{A_\infty(0, \tau_\infty, \zeta_k, 0)/\zeta_k}}{A_\infty(0, \tau_\infty, \zeta_k, 0)/\zeta_k}\right) d\zeta_k \wedge d\overline{\zeta_k}.$$

Following the proof of [11, Lemma 4.2], we split the difference in the integrand into the following three terms:

$$\text{(I)}\quad 4\int_{R_z(t_n^{(k)})}\left(\frac{f_i(t_n', \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)}{\zeta_k} - \frac{f_i(0, \tau_\infty, \zeta_k, 0)}{\zeta_k}\right)$$
$$\frac{f_j(t_n', \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)}{\zeta_k}\,\frac{\overline{A_n(t_n', \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t_n', \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)/\zeta_k}\, d\zeta_k \wedge d\overline{\zeta_k}$$

$$\text{(II)}\quad 4\int_{R_z(t_n^{(k)})}\left(\frac{f_j(t_n', \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)}{\zeta_k} - \frac{f_j(0, \tau_\infty, \zeta_k, 0)}{\zeta_k}\right)$$
$$\frac{f_i(0, \tau_\infty, \zeta_k, 0)}{\zeta_k}\,\frac{\overline{A_n(t_n', \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t_n', \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)/\zeta_k}\, d\zeta_k \wedge d\overline{\zeta_k}$$

$$\text{(III)}\quad 4\int_{R_z(t_n^{(k)})}\left(\frac{f_i(0,\tau_\infty,\zeta_k,0)}{\zeta_k}\frac{f_j(0,\tau_\infty,\zeta_k,0)}{\zeta_k}\right)$$
$$\left(\frac{\overline{A_n(t_n',\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t_n',\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k}-\frac{\overline{A_\infty(0,\tau_\infty,\zeta_k,0)/\zeta_k}}{A_\infty(0,\tau_\infty,\zeta_k,0)/\zeta_k}\right)d\zeta_k\wedge d\overline{\zeta_k}.$$

Regardless of whether Case 1) or 2) holds, convergence of the expressions (I) and (II) is guaranteed. Consider the difference

$$f_*(t_n',\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)-f_*(0,\tau_\infty,\zeta_k,0),$$

where $*$ indicates that the choice of subscript i or j does not matter here as long as the subscript is the same on both functions. By [37], f_* is holomorphic in all variables, hence, there is a constant $C_0>0$ such that

$$\frac{2}{|\zeta_k|}\left|f_*(t_n',\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)-f_*(0,\tau_\infty,\zeta_k,0)\right|\le C_0\frac{|t_n^{(k)}|}{|\zeta_k|^2}$$

and

$$2\left|\frac{f_*(t_n',\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)}{\zeta_k}\right|\le C_0\frac{1}{|\zeta_k|}.$$

Using Hölder's inequality, there is a constant $C_1>0$ such that the following inequalities hold

$$\begin{aligned}|(\mathrm{I})| &\le 4\|(f_i(t_n',\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)-f_i(0,\tau_\infty,\zeta_k,0))/\zeta_k\|_{L^2(R_z(t_n^{(k)}))}\\ &\qquad\qquad \|f_j(t_n',\tau_n,\zeta_k,t_n^{(k)}/\zeta_k)/\zeta_k\|_{L^2(R_z(t_n^{(k)}))}\\ &\le \left\|C_0\frac{|t_n^{(k)}|}{|\zeta_k|^2}\right\|_{L^2(R_z(t_n^{(k)}))}\left\|C_0\frac{1}{|\zeta_k|}\right\|_{L^2(R_z(t_n^{(k)}))}\\ &\le C_1\frac{|t_n^{(k)}|}{\sqrt{|t_n^{(k)}|}}(\log|t_n^{(k)}|)^{1/2}=C_1\sqrt{|t_n^{(k)}|}(\log|t_n^{(k)}|)^{1/2}\end{aligned}$$

and

$$\begin{aligned}|(\mathrm{II})| &\leq 4\|(f_j(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k) - f_j(0, \tau_\infty, \zeta_k, 0))/\zeta_k\|_{L^2(R_z(t_n^{(k)}))} \\ &\qquad\qquad\qquad\qquad \|f_i(0, \tau_\infty, \zeta_k, 0)/\zeta_k\|_{L^2(R_z(t_n^{(k)}))} \\ &\leq \|C_0 \frac{|t_n^{(k)}|}{|\zeta_k|^2}\|_{L^2(R_z(t_n^{(k)}))} \|C_0 \frac{1}{|\zeta_k|}\|_{L^2(R_z(t_n^{(k)}))} \\ &\leq C_1 \sqrt{|t_n^{(k)}|}(\log|t_n^{(k)}|)^{1/2}.\end{aligned}$$

The convergence for (III) remains to be shown. We split this into two cases that are resolved by Lemmas 3.3 and 3.4. Note that in Case 2), it suffices to assume that $f_i(0, \tau_\infty, 0, 0) \neq 0$ and $f_j(0, \tau_\infty, 0, 0) \neq 0$. Otherwise, Case 2) is subsumed by Case 1). □

Lemma 3.3. *Given k, if $f_i(0, \tau_\infty, 0, 0) = 0$ on D_k or $f_j(0, \tau_\infty, 0, 0) = 0$ on D_k, then (III) converges to zero as n tends to infinity.*

Proof. By the assumption that at most one of f_i and f_j has a simple pole, we have

$$\begin{aligned}|(\mathrm{III})| \leq 4 \Bigg| \int_{R_z(t_n^{(k)})} &\frac{f_i(0, \tau_\infty, \zeta_k, 0) f_j(0, \tau_\infty, \zeta_k, 0)}{\zeta_k} \\ &\left(\frac{\overline{A_n(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)/\zeta_k} - \frac{\overline{A_\infty(0, \tau_\infty, \zeta_k, 0)/\zeta_k}}{A_\infty(0, \tau_\infty, \zeta_k, 0)/\zeta_k} \right) \frac{d\zeta_k \wedge d\overline{\zeta_k}}{\zeta_k} \Bigg|.\end{aligned}$$

The quantity $f_i(0, \tau_\infty, \zeta_k, 0) f_j(0, \tau_\infty, \zeta_k, 0)/\zeta_k$ is bounded on $R_z(t_n^{(k)})$ because f_i and f_j are holomorphic (hence bounded) and $f_i(0, \tau_\infty, 0, 0) \cdot f_j(0, \tau_\infty, 0, 0) = 0$. This implies that there exists a constant $C > 0$ such that

$$|(\mathrm{III})| \leq C \int_{R_z(t_n^{(k)})} \left| \frac{\overline{A_n(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)}}{A_n(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)} - \frac{\overline{A_\infty(0, \tau_\infty, \zeta_k, 0)}}{A_\infty(0, \tau_\infty, \zeta_k, 0)} \right| \frac{d\zeta_k \wedge d\overline{\zeta_k}}{|\zeta_k|}.$$

The integrand is clearly bounded by the integrable function $2/|\zeta_k|$ for all n, and thus, the dominated convergence theorem yields the desired convergence. □

Lemma 3.4. *Given k, if $f_i(0, \tau_\infty, 0, 0) \neq 0$, $f_j(0, \tau_\infty, 0, 0) \neq 0$, and $A_\infty(0, \tau_\infty, 0, 0) \neq 0$ on D_k, then (III) converges to zero as n tends to infinity.*

Proof. By assumption, there exists N such that $A_n(0, \tau_\infty, 0, 0) \neq 0$ for all $n \geq N$. Since $A_\infty(0, \tau_\infty, 0, 0) \neq 0$, there exists $r > 0$ such that

$A_n(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k) \neq 0$ and $\overline{A_n(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)}/A_n(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)$ is a real analytic function in the polydisc $\{|t_n^{(k)}| < r, |\zeta_k| < r\} \subset \mathbb{C}^2$. Therefore, there exists a constant $C_2 > 0$ such that in the annulus $\{\sqrt{|t_n^{(k)}|} < |\zeta_k| < r/2\}$, we have

$$\begin{aligned}
&\left|\frac{\overline{A_n(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)/\zeta_k} - \frac{\overline{A_\infty(0, \tau_\infty, \zeta_k, 0)/\zeta_k}}{A_\infty(0, \tau_\infty, \zeta_k, 0)/\zeta_k}\right| \\
&\qquad = \left|\frac{\overline{A_n(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)}}{A_n(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)} - \frac{\overline{A_\infty(0, \tau_\infty, \zeta_k, 0)}}{A_\infty(0, \tau_\infty, \zeta_k, 0)}\right| \\
&\qquad \leq C_2 \frac{|t_n^{(k)}|}{|\zeta_k|} \\
&\qquad \leq C_2 \sqrt{|t_n^{(k)}|}.
\end{aligned}$$

Exactly as in the proof of [11, Lemma 4.2], there exists a constant $C_3 > 0$ such that

$$\begin{aligned}
|(\mathrm{III})| \quad &\leq -C_3 \sqrt{|t_n^{(k)}|} \log |t_n^{(k)}| + 4 \left| \int_{|\zeta_k| \geq r/2} \left(\frac{f_i(0, \tau_\infty, \zeta_k, 0)}{\zeta_k} \frac{\overline{f_j(0, \tau_\infty, \zeta_k, 0)}}{\zeta_k} \right) \right. \\
&\left. \left(\frac{\overline{A_n(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)/\zeta_k}}{A_n(t'_n, \tau_n, \zeta_k, t_n^{(k)}/\zeta_k)/\zeta_k} - \frac{\overline{A_\infty(0, \tau_\infty, \zeta_k, 0)/\zeta_k}}{A_\infty(0, \tau_\infty, \zeta_k, 0)/\zeta_k} \right) d\zeta_k \wedge d\overline{\zeta_k} \right|.
\end{aligned}$$

Since the domain of integration in the right-hand integral does not depend on t, the domain of integration is compact and the integrand is bounded by an integrable function for all n. This proof is completed by applying the dominated convergence theorem to the sequence as n tends to infinity. □

Definition. *Define the extension of the rank k locus to the boundary of $\mathcal{M}_g$ to be the closure of $\mathcal{D}_g(k)$ in $\overline{\mathcal{M}_g}$ and denote it by $\overline{\mathcal{D}_g(k)}$.*

Remark. *Since $\mathcal{D}_g(k)$ is already a closed set in $\mathcal{M}_g$, we would never need to write $\overline{\mathcal{D}_g(k)}$ to mean the closure of $\mathcal{D}_g(k)$ in $\mathcal{M}_g$.*

Lemma 3.5. *If $(X', \omega') \in \overline{\mathcal{D}_g(k)}$, ω' is holomorphic on X', and $\omega' \not\equiv 0$ on any part of X', then*

$$\operatorname{Rank}\left(\frac{d\Pi(X')}{d\mu_{\omega'}}\right) \leq k.$$

Proof. This is clear for $(X', \omega') \in \mathcal{D}_g(k)$, so we assume $(X', \omega') \in \overline{\mathcal{D}_g(k)} \cap \partial\overline{\mathcal{M}_g}$. By definition, $\overline{\mathcal{D}_g(k)}$ is the closure of $\mathcal{D}_g(k)$ in $\overline{\mathcal{M}_g}$, so there exists a sequence

$\{(X_n, \omega_n)\}_{n=1}^{\infty}$ in $\mathcal{D}_g(k)$ converging to (X', ω'). Let X' be a surface of genus $g' < g$. Let $\{\theta_1^{(n)}, \dots, \theta_{g'}^{(n)}, \dots \theta_g^{(n)}\}$ be a basis of Abelian differentials on X_n ordered so that

$$\lim_{n\to\infty} \theta_m^{(n)} = \theta_m,$$

for $1 \leq m \leq g'$, and the set $\{\theta_1, \dots, \theta_{g'}\}$ is a basis for the space of holomorphic Abelian differentials on X'. Note that for each m, $1 \leq m \leq g'$, $\{\theta_m^{(n)}\}_{n=1}^{\infty}$ is a sequence of holomorphic differentials converging to a holomorphic differential. Let $A_n = (A_{ij}^{(n)})$ denote the minor of $d\Pi(X_n)/d\mu_{\omega_n}$ defined by

$$A_{ij}^{(n)} = \int_{X_n} \theta_i^{(n)} \theta_j^{(n)} d\mu_{\omega_n},$$

for $1 \leq i, j \leq g'$, and let A denote the derivative of the period matrix of (X', ω'). Since we restricted our attention to the basis of differentials that are holomorphic on X' and $\omega' \not\equiv 0$ is holomorphic, A_n converges to A component-wise by Lemma 3.2. For any sequence of matrices $\{A_n\}_{n=1}^{\infty}$ converging to a matrix A component-wise, there exists an $\varepsilon > 0$ such that if $\|A_n - A\| < \varepsilon$, where $\|A\|$ denotes the sum of the absolute values of the components of A, then $\operatorname{Rank}(A_n) \geq \operatorname{Rank}(A)$. Also, given a matrix M with minor B, $\operatorname{Rank}(M) \geq \operatorname{Rank}(B)$. The lemma follows by letting $M = d\Pi(X_n)/d\mu_{\omega_n}$ and $B = A_n$, so that

$$k \geq \operatorname{Rank}\left(\frac{d\Pi(X_n)}{d\mu_{\omega_n}}\right) \geq \operatorname{Rank}(A_n) \geq \operatorname{Rank}(A) = \operatorname{Rank}\left(\frac{d\Pi(X')}{d\mu_{\omega'}}\right). \qquad \square$$

Lemma 3.6. *Let $\{(X(t_n, \tau_n), \omega_n)\}_{n=0}^{\infty}$ be a sequence of surfaces converging to a surface $(X', \omega') \in \overline{\mathcal{M}_g}$. For all i, j and $n \geq 0$, there exists a constant $C > 0$, such that*

$$\left| \int_{X^*_{\tau_\infty}} \theta_i(0, \tau_\infty)\theta_j(0, \tau_\infty) \frac{\bar{\omega}'}{\omega'} \right| < C.$$

Proof. The differentials $\theta_i(0, \tau_\infty)$ are holomorphic on the compact set $\overline{X^*_{\tau_\infty}}$, for all i, by the definition of $X^*_{\tau_\infty}$. Hence, $|\theta_i(0, \tau_\infty)| < C'$ for some constant C' and all i. This implies

$$\left| \int_{X^*_{\tau_\infty}} \theta_i(0, \tau_\infty)\theta_j(0, \tau_\infty) \frac{\bar{\omega}'}{\omega'} \right| \leq \int_{X^*_{\tau_\infty}} \left|\theta_i(0, \tau_\infty)\theta_j(0, \tau_\infty)\right| \leq C'^2 = C. \qquad \square$$

Lemma 3.7. *Let $D_\varepsilon = \{z \,\big|\, |\varepsilon| \leq |z| \leq 1\} \subset \mathbb{C}$. For all $N \geq 0$ and $\varepsilon > 0$,*

$$\int_{D_\varepsilon} \frac{z^N}{\bar{z}} dz \wedge d\bar{z} = 0.$$

Proof. Convert to polar coordinates by letting $z = re^{i\theta}$. For all $\varepsilon > 0$

$$\int_{D_\varepsilon} z^N/\bar{z}\, dz \wedge d\bar{z} = -2i \int_0^{2\pi} \int_\varepsilon^1 \frac{r^N e^{iN\theta}}{re^{-i\theta}} r\, dr d\theta$$
$$= -2i \int_0^{2\pi} \int_\varepsilon^1 r^N e^{i(N+1)\theta}\, dr d\theta.$$

This expression integrates to zero, for all $N \geq 0$. ☐

Lemma 3.8. *Let $D_\varepsilon = \{z \mid |\varepsilon| \leq |z| \leq 1\} \subset \mathbb{C}$. For all $N \in \mathbb{Z}$, $K \geq 0$ and $\varepsilon > 0$, there exists $C > 0$ such that*

$$\left| \int_{D_\varepsilon} z^N \bar{z}^K\, dz \wedge d\bar{z} \right| < C.$$

Proof. Convert to polar coordinates by letting $z = re^{i\theta}$. Then for all $\varepsilon > 0$

$$\int_{D_\varepsilon} z^N \bar{z}^K\, dz \wedge d\bar{z} = -2i \int_0^{2\pi} \int_\varepsilon^1 r^N e^{iN\theta} r^K e^{-iK\theta} r\, dr d\theta$$
$$= -2i \int_0^{2\pi} \int_\varepsilon^1 r^{N+1+K} e^{i(N-K)\theta}\, dr d\theta\,.$$

If $N - K \neq 0$, this expression integrates to zero. Otherwise, this expression is bounded by

$$2 \left| \int_0^{2\pi} \int_\varepsilon^1 r^{2K+1}\, dr d\theta \right| < \frac{2\pi}{K+1} + O(\varepsilon) < C,$$

for some $C > 0$. ☐

We state the following two results for the annulus $R_z(t_n^{(k)})$ and remark that the same results hold for $R_w(t_n^{(k)})$.

Lemma 3.9. *We follow the notation established above. Let $\{(X(t_n, \tau_n), \omega_n)\}_{n=0}^\infty$ be a sequence of surfaces converging to a degenerate surface $(X(0, \tau_\infty), \omega')$. For each n, let $\{\theta_1(t_n, \tau_n), \ldots, \theta_g(t_n, \tau_n)\}$ be a basis for the space of Abelian differentials on $X(t_n, \tau_n)$. Given i, j, k, if either $f_i(0, \tau_\infty, 0, 0) = 0$ on D_k or $f_j(0, \tau_\infty, 0, 0) = 0$ on D_k, then there exists $C > 0$ such that for all $n \geq 0$*

$$\left| \int_{R_z(t_n^{(k)})} \theta_i(0, \tau_\infty) \theta_j(0, \tau_\infty) d\mu_{\omega'} \right| < C.$$

In particular, if $f_i(0, \tau_\infty, 0, 0) = 0$ or $f_j(0, \tau_\infty, 0, 0) = 0$ on D_k, and $A_\infty(0, \tau_\infty, 0, 0) \neq 0$ on D_k, then

$$\lim_{n \to \infty} \int_{R_z(t_n^{(k)})} \theta_i(0, \tau_\infty) \theta_j(0, \tau_\infty)\, d\mu_{\omega'} = 0.$$

Proof. There are three cases to consider in the first claim of the lemma. It suffices to consider the case where exactly one of the differentials $\theta_i(0,\tau_\infty)$ or $\theta_j(0,\tau_\infty)$ has a simple pole. Without loss of generality, assume that $\theta_j(0,\tau_\infty)$ is holomorphic. Fix a choice of coordinates ζ_k in $R_z(t_n^{(k)})$ so that by [32, Theorem 6.3], there exists $K \geq -1$ and $c \in \mathbb{C}$ such that $\omega' = c\zeta_k^K\, d\zeta_k$. Let $\theta_i(0,\tau_\infty) = (c_i/\zeta_k + h_i(\zeta_k))\, d\zeta_k$ and $\theta_j(0,\tau_\infty) = h_j(\zeta_k)\, d\zeta_k$, where h_i and h_j are holomorphic in ζ_k. This yields

$$\begin{aligned}
&\left|\int_{R_z(t_n^{(k)})} \theta_i(0,\tau_\infty)\theta_j(0,\tau_\infty)\, d\mu_{\omega'}\right| \\
&= \left|\int_{R_z(t_n^{(k)})} (c_i/\zeta_k + h_i(\zeta_k))\, h_j(\zeta_k)\frac{\overline{c\zeta_k^K}}{c\zeta_k^K}\, d\zeta_k \wedge d\overline{\zeta_k}\right| \\
&\leq \left|\int_{R_z(t_n^{(k)})} h_j(\zeta_k)\frac{c_i\overline{c\zeta_k^K}}{c\zeta_k^{K+1}}\, d\zeta_k \wedge d\overline{\zeta_k}\right| + \left|\int_{R_z(t_n^{(k)})} h_i(\zeta_k)h_j(\zeta_k)\frac{\overline{c\zeta_k^K}}{c\zeta_k^K}\, d\zeta_k \wedge d\overline{\zeta_k}\right| \\
&\leq \left|\int_{R_z(t_n^{(k)})} h_j(\zeta_k)\frac{c_i\overline{\zeta_k^K}}{\zeta_k^{K+1}}\, d\zeta_k \wedge d\overline{\zeta_k}\right| + \left|\int_{R_z(t_n^{(k)})} \left|h_i(\zeta_k)h_j(\zeta_k)\right|\, d\zeta_k \wedge d\overline{\zeta_k}\right|.
\end{aligned}$$

By Lemma 3.7 or 3.8, the right-hand side of the inequality is bounded independently of K.

In the particular case when $K = -1$, we have

$$\begin{aligned}
&\left|\int_{R_z(t_n^{(k)})} \theta_i(0,\tau_\infty)\theta_j(0,\tau_\infty)\, d\mu_{\omega'}\right| \\
&\quad= \left|\int_{R_z(t_n^{(k)})} (c_i/\zeta_k + h_i(\zeta_k))\, h_j(\zeta_k)\frac{\overline{c}\zeta_k}{\overline{c\zeta_k}}\, d\zeta_k \wedge d\overline{\zeta_k}\right| \\
&\quad\leq \left|\int_{R_z(t_n^{(k)})} h_j(\zeta_k)\frac{c_i}{\overline{\zeta_k}}\, d\zeta_k \wedge d\overline{\zeta_k}\right| + \left|\int_{R_z(t_n^{(k)})} h_i(\zeta_k)h_j(\zeta_k)\frac{\zeta_k}{\overline{\zeta_k}}\, d\zeta_k \wedge d\overline{\zeta_k}\right|.
\end{aligned}$$

By Lemma 3.7, both terms on the right-hand side of the inequality are zero. □

Lemma 3.10. *We follow the notation established above. Let* $\{(X(t_n,\tau_n),\omega_n)\}_{n=0}^\infty$ *be a sequence of surfaces converging to a degenerate surface* $(X(0,\tau_\infty),\omega')$. *For each* n, *let* $\{\theta_1(t_n,\tau_n),\dots,\theta_g(t_n,\tau_n)\}$ *be a basis for the space of Abelian differentials on* $X(t_n,\tau_n)$. *Given* i,j,k, *if* $f_i(0,\tau_\infty,0,0) = c_i \neq 0$, $f_j(0,\tau_\infty,0,0) = c_j \neq 0$, *and* $A_\infty(0,\tau_\infty,0,0) = c \neq 0$ *on* D_k, *then for sufficiently large* n,

$$\int_{R_z(t_n^{(k)})} \theta_i(0,\tau_\infty)\theta_j(0,\tau_\infty)\, d\mu_{\omega'} = c_i c_j \frac{\overline{c}}{c} 2\pi\sqrt{-1}\log|t_n^{(k)}| + O(1).$$

Proof. We have

$$\begin{aligned}
&\int_{R_z(t_n^{(k)})} \theta_i(0,\tau_\infty)\theta_j(0,\tau_\infty)\, d\mu_{\omega'} \\
&= \int_{R_z(t_n^{(k)})} (c_i/\zeta_k + h_i(\zeta_k))\,(c_j/\zeta_k + h_j(\zeta_k))\,\frac{\overline{\zeta_k}}{\zeta_k}\left(\overline{c}/c + H(\zeta_k,\overline{\zeta_k})\right) d\zeta_k \wedge d\overline{\zeta_k},
\end{aligned}$$

where h_i and h_j are holomorphic, H is analytic in both variables, and $H(0,0) = 0$. It follows from Lemmas 3.7 and 3.8 that every term is bounded uniformly for all n with the exception of

$$\begin{aligned}
c_i c_j \frac{\overline{c}}{c}\int_{R_z(t_n^{(k)})} \frac{1}{|\zeta_k|^2}\, d\zeta_k \wedge d\overline{\zeta_k} &= -2c_i c_j \frac{\overline{c}}{c}\sqrt{-1}\int_0^{2\pi}\int_{\sqrt{|t_n^{(k)}|/c''}}^{c'} \frac{1}{r^2} r\, dr\, d\theta \\
&= -4\pi c_i c_j \frac{\overline{c}}{c}\sqrt{-1}\left(\log(c') - \log(\sqrt{|t_n^{(k)}|/c''})\right) \\
&= c_i c_j \frac{\overline{c}}{c} 2\pi\sqrt{-1}\log|t_n^{(k)}| + O(1). \qquad \square
\end{aligned}$$

4. Surgery on Abelian differentials

The goal of this section is to prove Lemma 4.3, which is a technical result essential to the proof of Theorem 5.5, which in turn forms the foundation of the remainder of this paper. We start with a general lemma concerning quadratic differentials with simple poles. Then we introduce a surgery on Riemann surfaces with Abelian differentials. This surgery allows us to use Lemma 4.1 to prove Lemma 4.3. Finally, we include Corollary 4.4, which will not be used in this paper, but the author feels it is inherently interesting.

Recall that a *saddle connection* is a trajectory between two not necessarily distinct singularities such as a zero or a simple pole of a quadratic differential. It is implicit in the definition that all saddle connections have finite length.

Lemma 4.1. *Let (X,q) be a surface carrying an integrable quadratic differential with at most simple poles. If the vertical foliation of (X,q) has no regular closed trajectories and every trajectory emanating from a simple pole is a saddle connection, then there exists a sequence of times $\{t_n\}$ such that*

$$\lim_{n\to\infty} G_{t_n}\cdot(X,q) = (X',q'),$$

and the lengths of the saddle connections in the vertical foliation converge to zero as n tends to infinity.

In particular, every saddle connection to a simple pole has length converging to zero. Furthermore, q' may have double poles and X' may be a degenerate Riemann surface.

Proof. We will refer to saddle connections in the vertical foliation as saddle connections for short because no other foliation will be considered in this proof. Let the saddle connections have length bounded above by w_1. After time t, the saddle connections will have length at most $e^{-t}w_1$ by assumption. Since they contract at the maximal rate, their length after passing to a limit is always finite because the largest possible normalization term is e^t.

We first claim that if a saddle connection has nonzero length, then q' must have double poles. If not, then q' would be integrable and the area normalization would have been used at every step of the limit causing the lengths of the saddle connections of q to be contracted to zero by e^{-t}, as t tended to infinity. Hence, q' has double poles.

By contradiction, assume that for all sequences of divergent times $\{t_n\}$, the lengths of some saddle connections on q' have nonzero length. Let $G_{t_n} \cdot (X, q) = (X_n, q_n)$. Let $w_1^{(n)}$ be the length of the longest saddle connection on (X_n, q_n), and by the contradiction assumption, let $w_1' > 0$ be the limit of the lengths. Since q' has double poles, the double poles are realized geometrically by one or more infinite cylinders. Let C_2' be one such infinite cylinder on (X', q'). For all sufficiently large n, this cylinder persists on (X_n, q_n). Denote the cylinder by $C_2^{(n)}$, and let $w_2^{(n)}$ be its circumference. Let w_2' be the circumference of C_2'. Consider the ratio $w_1^{(n)}/w_2^{(n)}$. By assumption,

$$\lim_{n\to\infty} w_1^{(n)}/w_2^{(n)} > C > 0.$$

Pass to a subsequence of times $\{t_n\}_{n=0}^{\infty}$ such that there is a constant C^L satisfying $0 < C^L \leq w_1^{(n)}/w_2^{(n)}$, for all n. Recall that under the area normalization, the lengths of the saddle connections contract by e^{-t_n} for each n. Therefore,

$$\lim_{n\to\infty} e^{t_n} w_1^{(n)} = w_1' < \infty,$$

and $w_1' > 0$ by assumption. This implies $w_2' < \infty$ because e^{t_n} is the maximal rate of expansion and

$$w_2' \leq \lim_{n\to\infty} e^{t_n} w_2^{(n)} < \infty.$$

Hence, for all sequences $\{t_n\}$ the saddle connections and the core curve of the cylinder $C_2^{(n)}$ contract for all n at the maximal rate under the area normalization. This is only possible if the saddle connections and the core curve of $C_2^{(n)}$ are parallel for all n. Otherwise, there would be an $N > 0$ sufficiently large, such that $e^{t_n} w_2^{(n)}$ increases exponentially for all $n \geq N$. However, it was assumed above that the saddle connections and $C_2^{(n)}$ are not parallel because there were no closed regular trajectories parallel to the saddle connections of q. This contradiction implies that there must exist a sequence of divergent times along the Teichmüller trajectory such that the length of every saddle connection converges to zero. □

We proceed by introducing a surgery on Riemann surfaces with Abelian differentials. For convenience, if S is a subsurface of X and carries a differential ω, then (S, ω) will mean the subsurface S with the differential that is given by the restriction of ω to S.

Definition. *Let (X, ω) be a Riemann surface carrying an Abelian differential. Let $\mathcal{C}$ be a collection of cylinders in the vertical foliation of (X, ω) such that $\cup\mathcal{C} \neq (X, \omega)$. Consider the open set $X \setminus \mathcal{C}$, and its closure $\overline{X \setminus \mathcal{C}}$, which is a (possibly disconnected) Riemann surface with boundary. Then $\partial\overline{X \setminus \mathcal{C}}$ is a union of circles (as is true for any Riemann surface with boundary), and $(\partial\overline{X \setminus \mathcal{C}}, \omega)$ is a union of saddle connections which are exactly a subset*[1] *of the saddle connections lying in the boundary of the cylinders in $\mathcal{C}$. For each of the boundaries (circles) of $\overline{X \setminus \mathcal{C}}$, choose a pair of antipodes $(\partial\overline{X \setminus \mathcal{C}}, \omega)$ with distance measured as usual with respect to the flat metric. Next, identify opposite sides so that if the antipode chosen is at a regular point p of ω, then the identification yields a simple pole of a quadratic differential at p. This procedure, which results in a (possibly disconnected) surface $(\tilde{X}, \tilde{q})$ carrying an integrable meromorphic quadratic differential, is called the* cylinder surgery of (X, ω) relative to $\mathcal{C}$.

Lemma 4.2. *Let (X, ω) be a Riemann surface carrying an Abelian differential, and $\mathcal{C}$ be a collection of cylinders in the vertical foliation of (X, ω) such that $\cup\mathcal{C} \neq (X, \omega)$. If the cylinder surgery of (X, ω) relative to $\mathcal{C}$ yields $(\tilde{X}, \tilde{q})$, then $(\tilde{X}, \tilde{q})$ is a possibly disconnected Riemann surface carrying an integrable meromorphic quadratic differential.*

Proof. It is clear that if we identify opposite sides of borders on a bordered Riemann surface, we get a Riemann surface. The fact that we get a unique quadratic differential depending only on our choice of antipodes follows from [19, Main Theorem]. □

Remark. *Note that the definition does not require the collection to be maximal. However, we do not claim that taking a non-maximal collection, performing the surgery, and then performing the surgery on the remaining cylinders in the foliation will yield the same $(\tilde{X}, \tilde{q})$ as if we performed the surgery on the maximal set of cylinders in the beginning. We will only use a maximal collection of cylinders in all of the results below. Thus, this issue will not arise.*

Furthermore, in all of the proofs below the choice of antipodes will not matter because the foliation will be fixed throughout the argument.

Lemma 4.3. *Let (X, ω) be a Riemann surface carrying an Abelian differential. Let $\mathcal{C}$ be a maximal collection of cylinders in the vertical foliation of (X, ω). If*

[1] We say subset because there could be a saddle connection on the top and bottom of a cylinder in $\mathcal{C}$ or lying between two cylinders in $\mathcal{C}$ that is permanently deleted by the excision and cannot be recovered by taking a closure.

the cylinders in $\mathcal{C}$ do not fill the surface, then there exists a sequence of times $\{t_n\}$ such that

$$\lim_{n\to\infty} G_{t_n} \cdot (X, \omega) = (X', \omega'),$$

where the circumferences of the cylinders in $\mathcal{C}$ converge to zero on (X', ω'). The surface (X', ω') will necessarily be a degenerate surface and ω' will have at most simple poles.

Proof. Consider the action of G_t on (X, ω) by decomposing (X, ω) into $(X \setminus \mathcal{C}, \omega) \sqcup (\mathcal{C}, \omega)$. This is a natural partition of the surface because the boundaries of the cylinders naturally divide the surface into these two regions, and the boundaries are preserved by G_t for all t. Let $(\tilde{X}, \tilde{q})$ denote the surface resulting from performing the cylinder surgery on (X, ω) relative to $\mathcal{C}$. The choice of antipodes in the cylinder surgery will not matter in this proof.

Note that outside of the measure zero set corresponding to the boundaries of $(\overline{X \setminus \mathcal{C}}, \omega)$ that were identified, $\tilde{q}$ and ω coincide exactly on $\tilde{X}$. Along the identification, it is possible that some zeros on opposite sides were identified, but generically, the identification at the antipodes will result in at most two simple poles of a quadratic differential. By definition of a cylinder, the boundary of a cylinder contains zeros of ω. Hence, after the identification, all of the trajectories from the simple poles on $\tilde{q}$ terminate at a zero of $\tilde{q}$ in time bounded by at most (half) of the largest circumference of a cylinder in $\mathcal{C}$. Moreover, the boundary of every cylinder in $\mathcal{C}$ is a union of saddle connections in the vertical foliation, which implies that the boundaries of the cylinders correspond to a union of saddle connections in the vertical foliation of $\tilde{q}$. Finally, since $\mathcal{C}$ was taken to be a maximal set, $(\tilde{X}, \tilde{q})$ has no closed regular trajectories. This demonstrates that the surface $(\tilde{X}, \tilde{q})$ satisfies all of the assumptions of Lemma 4.1. Hence, there exists a sequence of times $\{t_n\}$ such that the limit of $G_{t_n} \cdot (\tilde{X}, \tilde{q})$ converges to a quadratic differential $(\tilde{X}', \tilde{q}')$ where all of the lengths of the saddle connections in the vertical foliations converge to zero.

On the other hand, consider the action of G_{t_n} on $(\overline{X \setminus \mathcal{C}}, \omega)$ along the subsequence of times $\{t_n\}$. Fixing the antipodes chosen for the cylinder surgery, and for each n, identifying opposite sides of the borders of $(\overline{X \setminus \mathcal{C}}, \omega)$ relative to the choice of antipodes yields $G_{t_n} \cdot (\tilde{X}, \tilde{q})$. Since all of the saddle connections in the vertical foliation of $G_{t_n} \cdot (\tilde{X}, \tilde{q})$ converge to a point as n tends to infinity, all of the borders of $(\overline{X \setminus \mathcal{C}}, \omega)$ also converge to a point. This implies that

$$\lim_{n\to\infty} G_{t_n} \cdot (\tilde{X}, \tilde{q}) = \lim_{n\to\infty} G_{t_n} \cdot (\overline{X \setminus \mathcal{C}}, \omega).$$

To complete the proof, it suffices to prove

$$\lim_{n\to\infty} G_{t_n} \cdot (\overline{X \setminus \mathcal{C}}, \omega) = \lim_{n\to\infty} G_{t_n} \cdot (X, \omega).$$

However, this follows from the proof of Lemma 4.1. The core curves of the cylinders contract by e^{-t_n} under the Teichmüller geodesic flow. In Lemma 4.1, the sequence

$\{t_n\}$ was constructed exactly so that the normalization constant $r_n \in \mathbb{R}^+$ satisfied $\lim_{n\to\infty} r_n e^{-t_n} = 0$. Hence, the circumferences of the cylinders converge to zero. This proves the claim and produces a limit with the desired properties. □

Though the following corollary will not be used in this paper, the result elucidates the cylinder surgery via an application of it, and its proof is sufficiently short that the author feels its inclusion is merited.

Corollary 4.4. *Let (X, ω) be a Riemann surface carrying a meromorphic Abelian differential with at most simple poles. Let $\mathcal{C}$ be a maximal collection of (not necessarily finite) cylinders in the vertical foliation of (X, ω). If all of the infinite cylinders corresponding to the simple poles of ω are contained in $\mathcal{C}$*[2] *and $\cup\mathcal{C} \neq X$, then there exists $\theta \in (0, \pi)$ such that the vertical foliation of $(X, e^{i\theta}\omega)$ admits a cylinder C' such that $\mathcal{C} \cap C'$ has measure zero.*

Proof. Recall that a cylinder was defined to be a closed set. Note that the proof of Lemma 4.3, applies just as well if some or all of the cylinders in $\mathcal{C}$ had infinite height because the essential ingredient in the proof of Lemma 4.3 is the use of Lemma 4.1, where the collection of cylinders $\mathcal{C}$ is non-existent.

By Lemma 4.3, there exists a sequence of times $\{t_n\}$ such that $\lim_{n\to\infty} G_{t_n} \cdot (X, \omega) = (X', \omega')$, where all of the cylinders in $\mathcal{C}$ converged to punctures of (X', ω'). Either ω' is holomorphic or it is not, in which case it has a simple pole.

If ω' has a simple pole, then (X', ω') has an infinite cylinder, and for large n, there is a cylinder C_n on $G_{t_n} \cdot (X, \omega)$, which is not in $\mathcal{C}$. Since $\mathcal{C}$ was assumed to be maximal, C_n is a cylinder that does not lie in the vertical foliation of (X, ω).

If ω' is holomorphic, then [27, Theorem 2] implies that there is a dense set of directions containing a cylinder C'. The cylinder C' persists on $G_{t_n} \cdot (X, \omega)$ and due to our freedom to choose the foliation in which it lies, it can be chosen so that it is not parallel to the cylinders in $\mathcal{C}$. Hence, in either case, we can produce a cylinder that is not parallel to the cylinders of $\mathcal{C}$. Thus, C' can only intersect $\mathcal{C}$ at at most a finite set of points. □

5. Complete periodicity and the connectivity graph in $\mathcal{D}_g(1)$

The key results of this section are Theorem 5.5 and Lemma 5.9. They form the foundation on which the remainder of this paper rests. The former result proves that every surface generating a Teichmüller disc in the rank one locus must be completely periodic, while the latter result describes the configuration of the parts of a degenerate surface in the closure of a Teichmüller disc contained in the rank one locus. We begin by recalling some basic definitions from graph theory.

[2] This is equivalent to the condition that there exists $\alpha \in \mathbb{R}$ such that for each simple pole of ω with residue c, $e^{i\alpha}c \in \mathbb{R}$.

Let G be a graph consisting of a vertex set $V(G)$ and an edge set $E(G)$. A *path* is a graph with vertex set $\{v_1, \ldots, v_n\}$ such that there is an edge from v_i to v_{i+1}, for all $1 \leq i \leq n-1$. A *cycle* is a path with an additional edge connecting v_1 to v_n. Consider the set of all cycles contained in G. This set forms a finite dimensional vector space over the field $\mathbb{F}_2$ called the *cycle space of* G. Denote the dimension of the cycle space by $\dim^C(G)$. All the graphs in the discussion below may be multigraphs, i.e. we permit multiple edges between the same pair of vertices and there may be edges from a vertex to itself.

Definition. *Let $G(X')$ be the following multigraph associated to the degenerate surface X', or simply G when the surface is understood. There is a bijection sending $V(G)$ to the parts of X' by $v_i \mapsto S_i$. For all i, j and all pairs of punctures (p, p') from parts S_i to S_j of X', with i not necessarily distinct from j, there is a unique edge of G from v_i to v_j representing (p, p'). The graph G is called the* connectivity graph. *Let $G^P(X', \omega')$ be the subgraph of $G(X')$ such that $V(G^P) = V(G)$ and the edges of G^P correspond to the pairs of punctures at which ω' has simple poles.*

Remark. *We will be using Lemma 3.2 implicitly throughout this section. It is extremely important to note that nowhere in these results do we require that every component of the derivative of the period matrix has a limit as we take sequences in $\mathcal{M}_g$ converging to a degenerate surface. We are very careful to choose minors of the derivative of the period matrix such that the limit exists. This will suffice to provide the requisite lower bounds on the rank of the derivative of the period matrix near the boundary of the moduli space.*

Throughout this section, it will be advantageous to choose a basis of Abelian differentials with very specific properties depending on the surface to which a sequence of Abelian differentials is converging. Most importantly, the choice of basis we make in the following lemma will facilitate the application of the convergence lemmas from Section 3.2.

Lemma 5.1. *Given a degenerate surface $X(0, \tau_\infty)$ in the boundary of $\overline{\mathcal{R}_g}$, there exists a set of Abelian differentials $\{\theta_1(0, \tau_\infty), \ldots, \theta_g(0, \tau_\infty)\}$ on $X(0, \tau_\infty)$ such that for all $t = (t_1, \ldots, t_m)$, with $t_j \neq 0$ for all j, $\{\theta_1(t, \tau), \ldots, \theta_g(t, \tau)\}$ is a basis for the space of holomorphic Abelian differentials on $X(t, \tau)$. Moreover, this set can be constructed so that $\{\theta_1(0, \tau_\infty), \ldots, \theta_g(0, \tau_\infty)\}$ has the following properties:*

(1) *For some $1 \leq g_1 \leq g$, $\theta_i(0, \tau_\infty)$ is holomorphic if and only if $1 \leq i \leq g_1$.*

(2) *For all (p_i, p_i') such that $(p_i, p_i') \in S$ for some part $S \subset X'$, $\theta_i(0, \tau_\infty)$ has simple poles at (p_i, p_i'), $\theta_i(0, \tau_\infty)$ is holomorphic across all other punctures of S, and $\theta_i(0, \tau_\infty) \equiv 0$ on $X' \setminus S$.*

(3) *For each cycle $C_i \in G(X')$ consisting of more than one edge, $\theta_i(0, \tau_\infty)$ has poles at the pairs of punctures corresponding to the edges of C_i and $\theta_i \equiv 0$ for all $S \subset X'$ such that S does not correspond to a vertex of C_i.*

(4) *For any puncture $p \in X'$ and for all i, j, if $Res_p(\theta_i) \neq 0$ and $Res_p(\theta_j) \neq 0$, then $Res_p(\theta_i) = Res_p(\theta_j) = \pm 1$.*

Proof. The first claim follows from the Cartan–Serre theorem or [26, Proposition 4.1]. We proceed by explicitly constructing a basis of Abelian differentials on X' with the desired properties. The first g_1 differentials can be taken as a union of the bases of holomorphic differentials on each part such that if θ_i is an element of the basis of Abelian differentials on a part $S \subset X'$, then define $\theta_i \equiv 0$ on $X' \setminus S$.

Let the parts of X' be given by $S_1 \sqcup \cdots \sqcup S_n$. By [10, Theorem II.5.1 b.], given two punctures (p, p') on a connected Riemann surface S, there exists a meromorphic Abelian differential on S which is holomorphic everywhere on S and across all punctures of S except p and p', where it can be expressed as dz/z and $-dw/w$, in terms of local coordinates z and w, respectively. Hence, for each part S_j carrying a pair of punctures (p, p') we can take a basis element to be a differential which has simple poles only at those two punctures and is zero on every other part. Let the basis of Abelian differentials on X' consist of g_2 such differentials with exactly two simple poles, where $0 \leq g_2 \leq g$.

Finally, let G_1 be the subgraph of $G(X', \omega')$ such that G_1 has no edges from a vertex to itself. We claim $\dim^C(G_1) = g - g_1 - g_2$. This follows because each basis differential on X' corresponds to a closed horizontal homology curve on a surface near X' in the interior of the moduli space $\mathcal{R}_g$. The only horizontal homology curves that have not been accounted for in the description above are those that split over several parts. Define the remaining basis differentials as follows. For each j, with $0 \leq j \leq g - g_1 - g_2$, let C_j be an element of the cycle basis of G. Define θ_j to be zero on every part which does not correspond to a vertex of C_j. Each vertex v of C_j corresponds to a part S of X' such that S has two punctures p_1 and p_2 corresponding to edges of C_j incident to v. The punctures p_1 and p_2 are not paired. By [10, Theorem II.5.1 b.], there is a meromorphic differential holomorphic everywhere on S and across all punctures of S except for p_1 and p_2 at which it has simple poles with residues 1 and -1, respectively. Define the differential θ_j to have two poles on each part corresponding to a vertex in the cycle C_j. The only restriction is given by the rule that if the residue of the simple pole at p_1 is ± 1, then the residue of the simple pole at p_1' is ∓ 1. This construction completes the proof that such a basis exists.

By construction, the residues of each differential at every pole are ± 1. In order to satisfy the final property, it may be necessary to multiply some of the differentials by -1 so that the residues at each puncture are equal. □

Lemma 5.2. *Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ be a sequence of surfaces in a Teichmüller disc D converging to a degenerate surface (X', ω'). Let $S \subset X'$ be a part of X'. If ω' has k_1 pairs of poles on S, then*

$$\sup_n Rank\left(\frac{d\Pi(X_n)}{d\mu_{\omega_n}}\right) \geq k_1.$$

Proof. We show that a single pair of poles on X' corresponds to a divergent diagonal term of $d\Pi(X_n)/d\mu_{\omega_n}$ as n tends to infinity, while the off-diagonal terms in the row and column of that unbounded diagonal term are bounded for all n. Let $b_{ij}^{(n)}$ be the ij component of $d\Pi(X_n)/d\mu_{\omega_n}$. Let (p_i, p_i') be a pair of punctures on S such that ω' has a pair of poles at (p_i, p_i'), for $1 \le i \le k_1$. As in Lemma 5.1, let θ_i have a pair of poles with residue ± 1 at (p_i, p_i') and let θ_i be holomorphic everywhere else on X', for $1 \le i \le k_1$. We consider the $k_1 \times k_1$ minor of $d\Pi(X_n)/d\mu_{\omega_n}$ given by $(b_{ij}^{(n)})$, for $1 \le i, j \le k_1$, and show that it has full rank for sufficiently large n. By Lemmas 3.6 and 3.9, all of the off-diagonal terms $b_{ji}^{(n)} = b_{ij}^{(n)}$ are bounded, for all n, because θ_i and θ_j do not have any poles at the same pair of punctures for $i \neq j$. Furthermore, for each i, the contribution of the integral in Rauch's formula to the diagonal term $b_{ii}^{(n)}$ is bounded everywhere outside of the discs around p_i and p_i' by Lemmas 3.6 and 3.9. By Lemma 3.10, the contribution to the integral in Rauch's formula on $R_z(t_n^{(k)})$ diverges with n. Recall that if ω' has residue c at p_i, then it has residue $-c$ at p_i'. Since the quotient $\bar{c}/c = -\bar{c}/-c$, the sum of the two divergent terms coming from Lemma 3.10 do not cancel and $b_{ii}^{(n)}$ diverges to infinity with n. $\square$

Lemma 5.3. *Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ be a sequence of surfaces in a Teichmüller disc D converging to a degenerate surface (X', ω'). Let G'^P be the subgraph of G^P formed by removing all edges from each vertex to itself. Let $k_2 = \min(\dim^C(G'^P), 2)$. Then*

$$\sup_n \mathit{Rank}\left(\frac{d\Pi(X_n)}{d\mu_{\omega_n}}\right) \ge k_2.$$

Proof. If $\dim^C(G'^P) = 0$, we are done. If $\dim^C(G'^P) = 1$, then we claim that $d\Pi(X_n)/d\mu_{\omega_n}$ is not the zero matrix, for some choice of n. Let θ_1 be the differential with poles along the cycle of G'^P. Let (p_1, p_1') be a pair of poles of ω' in the cycle. The claim follows from Lemma 3.10 by letting $c_1 = \pm 1$, $\lim_{n\to\infty} c^{(n)} = c_1 = \pm 1$, where $c^{(n)}$ is the residue of ω_n in local coordinates about p_1, and considering the $1,1$ component of $d\Pi(X_n)/d\mu_{\omega_n}$.

Assume $\dim^C(G'^P) \ge 2$. Let $C \subset G'^P$ be a cycle. Using Lemma 2.4 assume that the residues of ω' are δ-nearly imaginary. It can be shown that given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $c \in \mathbb{C}$ that are δ-nearly imaginary

$$\left|\frac{\bar{c}}{c} + 1\right| < \varepsilon.$$

Hence, the coefficients of the unbounded $\log|t_n^{(k)}|$ terms in Lemma 3.10, for all k, differ from each other by at most 2ε.

By Lemma 5.1, there is a basis $\{\theta_1, \dots, \theta_g\}$ such that for all $1 \le i \le g$, θ_i has residue ± 1 at all of its simple poles. Without loss of generality, let θ_1 be an element of the basis of Abelian differentials that has pairs of simple poles corresponding to all of the edges of C. Again, let $b_{ij}^{(n)}$ denote the ij component of the derivative of the period matrix on X_n with respect to ω_n. By Lemma 3.6, the integral in Rauch's formula for the derivative of the period matrix is bounded outside of all discs around the punctures of X'. However, it is possible that two different elements in the basis of differentials have simple poles at the same pairs of punctures at which ω' has a simple pole.

Let $C' \subset G'^P$ be a cycle distinct from C (though it may have non-trivial intersection with C). Let θ_2 be the differential with poles at the pairs of punctures corresponding to edges of C'. Every edge of both C and C' corresponds to a pair of poles of ω'. (Note that Lemma 3.2 guarantees that we can apply all of the lemmas of Section 3.2 to the 2×2 minor $(b_{ij}^{(n)})$, for $1 \leq i, j \leq 2$, because ω' has poles at every puncture where θ_1 or θ_2 have poles.) We claim that for all n sufficiently large, $|b_{11}^{(n)}| > |b_{12}^{(n)}| = |b_{21}^{(n)}|$. Lemma 3.10 implies that each of these three terms is a sum of divergent terms. However, $\sharp(E(C \cap C')) < \sharp(E(C))$ implies that $b_{12}^{(n)}$ is a sum of fewer divergent terms than $b_{11}^{(n)}$, and there is no cancellation between the divergent terms by the δ-nearly imaginary assumption. For the exact same reason, $|b_{22}^{(n)}| > |b_{12}^{(n)}| = |b_{21}^{(n)}|$. Thus the diagonal term of each row and column is strictly larger than the off-diagonal terms in its row and column, for n sufficiently large. This implies that the derivative of the period matrix has a 2×2 minor of full rank. □

Lemma 5.4. *Let D be a Teichmüller disc contained in $\mathcal{D}_g(1)$. If (X', ω') is a degenerate surface in the closure of D and ω' is not holomorphic, then $G^P(X', \omega')$ is the union of a cycle (possibly on just one vertex) and a finite (possibly empty) set of isolated vertices.*

Proof. Since every Abelian differential with a simple pole on a Riemann surface S has at least two simple poles on S, no vertex in G'^P has degree one. Using the notation of Lemmas 5.2 and 5.3, we must have $k_1 + k_2 \leq 1$. The case where $k_1 + k_2 = 0$ is excluded by the assumption that ω' is not holomorphic, so we assume $k_1 + k_2 = 1$. If $k_1 = 1$, then G^P has a vertex with an edge forming a loop and Lemmas 5.2 and 5.3 imply that there are no other edges. If $k_2 = 1$, then G^P contains a cycle C. However, we claim G^P cannot contain any other edges. There are no additional paths in G^P between any two vertices in C because $k_2 = 1$. Since $k_1 = 0$ implies there are no edges from a vertex to itself, there are no additional paths emanating from a vertex in C because any such path would have to end in a vertex of degree one in G'^P. Hence, $k_2 = 1$ implies $E(G^P) = E(C)$. □

Definition. *Given (X, ω), let $\mathcal{F}_\theta$ denote the vertical foliation of $(X, e^{i\theta}\omega)$. For all $\theta \in \mathbb{R}$, if the existence of a closed regular trajectory of $\mathcal{F}_\theta$ implies that every trajectory of $\mathcal{F}_\theta$ is closed, then (X, ω) is* completely periodic.

Theorem 5.5. *If the Teichmüller disc D generated by (X,ω) is contained in $\mathcal{D}_g(1)$, then (X,ω) is completely periodic.*

Proof. By [27, Theorem 2], there exists a real number θ such that $(X, e^{i\theta}\omega)$ admits a cylinder in the vertical foliation. Without loss of generality, let (X,ω) admit a cylinder C_1 in its vertical foliation. By contradiction, suppose that (X,ω) is not completely periodic. Acting on (X,ω) by the Teichmüller geodesic flow, there exists a sequence of times $\{t_n\}$ by Lemma 4.3 such that every cylinder parallel to C_1 has circumference converging to zero and the limit surface (X',ω') has punctures in place of the cylinders parallel to C_1. Define $(X_n,\omega_n) = G_{t_n}\cdot(X,\omega)$, and let $C_1^{(n)} \subset (X_n,\omega_n)$ be the cylinder on (X_n,ω_n) corresponding to C_1.

Either ω' is holomorphic, or it is not. If ω' is not holomorphic, then there is a simple pole which corresponds to an infinite cylinder C_2'. Let $C_2^{(n)} \subset (X_n,\omega_n)$ be the sequence of cylinders converging to C_2'. By Lemma 4.3, $C_1^{(n)}$ is not parallel to $C_2^{(n)}$. On the other hand, if ω' is holomorphic, then there is a part of X' with positive genus and there exists a choice of direction on (X',ω') that admits a finite cylinder C_2' by [27, Theorem 2]. As before, let $C_2^{(n)} \subset (X_n,\omega_n)$ be the sequence of cylinders converging to C_2'. In this case pinch the core curve of the cylinder C_2' under the Teichmüller geodesic flow while normalizing the largest residue. The new degenerate surface, denoted (X',ω') by abuse of notation, either has (Case A:) poles resulting from an infinite cylinder C_2', or (Case B:) neither C_1' nor C_2' (the limits of $C_1^{(n)}$ and $C_2^{(n)}$) exist. By the continuity of the $\mathrm{SL}_2(\mathbb{R})$ action to the boundary of the moduli space [4, Proposition 11.1], there is a sequence $\{(X_n,\omega_n)\}_{n=0}^{\infty}$ in D converging to (X',ω'). We address Cases A and B in the course of the remainder of the proof.

By Lemma 2.1, $C_1^{(0)}$ is not homologous to $C_2^{(0)}$ because $C_1^{(0)}$ is not parallel to $C_2^{(0)}$. Since the $\mathrm{SL}_2(\mathbb{R})$ action preserves homology, $C_1^{(n)}$ is not homologous to $C_2^{(n)}$ for all $n \geq 0$. The remainder of this proof is dedicated to finding a degenerate surface (X',ω') in the closure of D such that $G^P(X',\omega')$ contradicts the conclusion of Lemma 5.4.

Consider the case when ω' has one or more pairs of simple poles arising from pinching a set of cylinders that are pairwise homologous. In this case, let C_2' be an infinite cylinder, while C_1' does not exist because the circumferences of the cylinders in the sequence $\{C_1^{(n)}\}_{n=0}^{\infty}$ converge to zero. Given $\varepsilon' > 0$, we can find a surface $(X_n,\omega_n) \in D$, where n depends on ε', such that (X_n,ω_n) has two non-homologous cylinders of equal circumference at most $\sqrt{\varepsilon'}$ and the moduli of the cylinders tend to infinity as ε' tends to zero. Choose $\varepsilon < \varepsilon'$ such that the circumference of $C_1^{(N)}$ is equal to ε for a sufficiently large value of N. Since the sequence $\{C_2^{(n)}\}_{n=0}^{\infty}$ converges to a cylinder of finite nonzero circumference, the circumferences of the cylinders $C_2^{(n)}$, denoted $w_2^{(n)}$ satisfy $0 < w_2^L \leq w_2^{(n)} \leq w_2^U < \infty$, for all n. The core curves of $C_1^{(n)}$ and $C_2^{(n)}$ are not parallel for all n, so for each n there exists a matrix $B_n \in \mathrm{SL}_2(\mathbb{R})$ that transforms the core curve of $C_1^{(n)}$ into a leaf of the vertical foliation and transforms the core curve of $C_2^{(n)}$ into a leaf of the horizontal foliation. For each N, consider the one parameter family of matrices, $G_t B_N \in \mathrm{SL}_2(\mathbb{R})$. Action by $G_t B_N$ on (X_N, ω_N) results in the core curve of $C_1^{(N)}$ expanding at the maximal rate e^t, while the core curve of $C_2^{(N)}$ contracts at the maximal rate e^{-t}. At time t, the circumference of $C_1^{(N)}$ is given by $e^t \varepsilon$, and the circumference of $C_2^{(N)}$ is given by $e^{-t} w_2^{(N)}$. Let T_N be the time satisfying the equation $e^{T_N}\varepsilon = e^{-T_N} w_2^{(N)}$. At time T_N, the circumference of each cylinder is given by $\sqrt{w_2^{(N)}\varepsilon}$. Define a sequence by

$$(X^{(N)}, \omega^{(N)}) := G_{T_N} B_N \cdot (X_N, \omega_N)$$

and consider $C_1^{(N)}, C_2^{(N)}$ to be cylinders in $X^{(N)}$. We claim the moduli of $C_1^{(N)}$ and $C_2^{(N)}$ diverge to infinity with N. Let h denote the height of a cylinder C, w its circumference, $A(C)$ its area, and $\mathrm{Mod}(C)$ its modulus. By the definition of the modulus,

$$\mathrm{Mod}(C) = \frac{h}{w} = \frac{A(C)}{w^2}.$$

In the case at hand, the areas of the cylinders $C_1^{(N)}$ and $C_2^{(N)}$ are bounded below for all N because $\mathrm{SL}_2(\mathbb{R})$ preserves area. Both cylinders have circumference $\sqrt{w_2^{(N)}\varepsilon}$, so their core curves pinch because

$$\lim_{\varepsilon' \to 0} \sqrt{w_2^{(N)}\varepsilon} \le \lim_{\varepsilon' \to 0} \sqrt{w_2^U \varepsilon'} = 0.$$

Note that this argument can be applied to Case A above. Let $(X'^{(2)}, \omega'^{(2)})$ be the limit of the sequence $\{(X^{(N)}, \omega^{(N)})\}_{N=0}^{\infty}$. As N tends to infinity, the cylinders $C_1^{(N)}$ and $C_2^{(N)}$ degenerate to cylinders of equal circumference. If that circumference is non-zero, then $\omega'^{(2)}$ has two pairs of simple poles coming from non-homologous cylinders. By Lemma 5.4, $G^P(X', \omega')$ has a cycle with the pair of punctures represented by C_2' corresponding to an edge of G^P. Since cylinders with pinched core curves remain pinched under this procedure, $G^P(X'^{(2)}, \omega'^{(2)})$ must contain an edge e corresponding to C_1' in addition to the cycle of $G^P(X', \omega')$. It is impossible for e and the edges of $G^P(X', \omega')$ to be part of a larger cycle in $G^P(X'^{(2)}, \omega'^{(2)})$ because that would imply that e represents a cylinder whose core curve, a posteriori, must be parallel to the core curves of the cylinders represented by the edges of $G^P(X', \omega')$. This contradicts Lemma 5.4. However, it is still possible that the circumferences of both cylinders converge to zero in which case neither C_1' nor C_2' exist and $\omega'^{(2)}$ is holomorphic at both pairs of punctures. We address this possibility.

By Lemma 2.7, we can assume without loss of generality, that $\omega'^{(2)}$ has a pair of simple poles. We proceed by induction, where each step of the induction is to perform the argument of the preceding paragraph until we reach a contradiction. The first step is already done. We present the j^{th} step of the procedure. Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ denote the sequence of surfaces converging to a degenerate surface $(X'^{(j)}, \omega'^{(j)})$ such that (X_n, ω_n) has j pairwise non-homologous cylinders all of whose circumferences converge to zero while another sequence of cylinders $\{C_{j+1}^{(n)}\}_{n=0}^{\infty}$ converges to a pair of poles of $\omega'^{(j)}$. Let $\{C_k^{(n)}\}_{n=0}^{\infty}$, for $1 \leq k \leq j$, denote the j distinct sequences of cylinders whose circumferences converge to zero as n tends to infinity. Without loss of generality, let $\{C_1^{(n)}\}_{n=0}^{\infty}$ be a sequence of cylinders such that for infinitely many values of n and all $k \neq 1$, the circumference of $C_k^{(n)}$ is less than or equal to the circumference of $C_1^{(n)}$. This may require the sequences to be renamed. We pass to a subsequence such that this holds for all n. Recall that $\varepsilon' > 0$ was fixed in the preceding paragraph and an appropriate $\varepsilon > 0$ was chosen. Furthermore, the circumference of the cylinder $C_1^{(n)}$ is $w_1^{(n)} \varepsilon^{1/(2^j)}$, where $w_1^{(n)}$ is a constant satisfying $0 < w_1^L \leq w_1^{(n)} \leq w_1^U < \infty$ for all n. Let $w_{j+1}^{(n)}$ denote the circumference of $C_{j+1}^{(n)}$, which also satisfies $0 < w_{j+1}^L \leq w_{j+1}^n \leq w_{j+1}^U < \infty$ for all n. We highlight the differences that arise in the course of repeating the argument of the preceding paragraph. Solving the equation $e^{T_N} w_1^{(N)} \varepsilon^{1/(2^j)} = e^{-T_N} w_{j+1}^{(N)}$ shows that at time T_N the lengths of the circumferences are $\sqrt{w_{j+1}^{(N)} w_1^{(N)}} \varepsilon^{1/(2^{j+1})}$. To see that the core curves of all $j+1$ cylinders still pinch as ε' tends to zero, note that, as before, the areas of all of the cylinders are fixed under the $\mathrm{SL}_2(\mathbb{R})$ action and thus their areas are bounded from below. Finally,

$$\lim_{\varepsilon' \to 0} \sqrt{w_{j+1}^{(N)} w_1^{(N)}} \varepsilon^{1/(2^{j+1})} \leq \lim_{\varepsilon' \to 0} \sqrt{w_{j+1}^U w_1^U} \varepsilon'^{1/(2^{j+1})} = 0.$$

Note that this induction procedure includes Case B that was left unaddressed above. Let $(X'^{(j+1)}, \omega'^{(j+1)})$ denote the degenerate surface formed by letting N tend to infinity in the sequence $\{G_{T_N} B_N \cdot (X_N, \omega_N)\}_{N=0}^{\infty}$. As above, the cylinders $C_1^{(N)}$ and $C_{j+1}^{(N)}$ degenerate to cylinders of equal circumference. If that circumference is non-zero, then $\omega'^{(j+1)}$ has at least two pairs of simple poles coming from non-homologous cylinders, namely C_1' and C_{j+1}'. By Lemma 5.4, $G^P(X'^{(j)}, \omega'^{(j)})$ has a cycle with the pair of punctures represented by C_{j+1}' corresponding to an edge of G^P. Since cylinders with pinched core curves remain pinched under this procedure, $G^P(X'^{(j+1)}, \omega'^{(j+1)})$ must contain an edge e corresponding to C_1' in addition to the cycle from $G^P(X'^{(j)}, \omega'^{(j)})$. As before, e and the edges of $G^P(X'^{(j)}, \omega'^{(j)})$ cannot be edges of a larger cycle. This contradicts Lemma 5.4. However, it is still possible that the circumferences of all $j+1$ cylinders converge to zero in which case $\omega'^{(j+1)}$ is holomorphic at $j+1$ pairs of punctures. In that case, repeat this argument.

This procedure must terminate at worst when $j = g$ because the core curves of the cylinders chosen at each step are pairwise non-homologous, and one can pinch at most g such curves. Hence, performing this procedure at the $g-1$ iteration guarantees at least two poles from sequences of non-homologous cylinders and results in a contradiction. This contradiction demonstrates that $\tilde{X}$ must in fact be the empty set. In other words, the surface is filled by cylinders, and the vertical foliation of X by ω is periodic. Since this argument holds for all $\theta \in \mathbb{R}$ such that $(X, e^{i\theta}\omega)$ admits a cylinder in the vertical foliation, (X, ω) is completely periodic. □

Theorem 5.5 is used implicitly in the following corollary to guarantee that it is not a vacuous statement. Compare this statement with [30, Lemma 5.3].

Corollary 5.6. *Let (X, ω) generate a Teichmüller disc $D \subset \mathcal{D}_g(1)$. For each $\theta \in \mathbb{R}$ such that the vertical foliation of $(X, e^{i\theta}\omega)$ is periodic, $(X, e^{i\theta}\omega)$ decomposes into a union of cylinders $C_1, \dots, C_k$ such that all of the saddle connections on the top of C_i are identified to the saddle connections on the bottom of C_{i+1} and vice versa, for all $i \leq k-1$, and all of the saddle connections on the top of C_k are identified to the saddle connections on the bottom of C_1 and vice versa. Furthermore, the circumference of C_i equals the circumference of C_j, for all i, j.*

Proof. Without loss of generality, assume that the vertical foliation of (X, ω) is periodic. Consider a divergent sequence of times $\{t_n\}$ such that the sequence $G_{t_n} \cdot (X, \omega)$ converges to a degenerate surface (X', ω'). By [25, Theorem 3], the limit of this sequence is given by pinching the core curves of every cylinder in the cylinder decomposition of (X, ω). Furthermore, ω' has a pair of simple poles at all of the pairs of punctures of X'. Hence, $G(X', \omega') = G^P(X', \omega')$. Since $G(X', \omega')$ is a connected graph, $G(X', \omega')$ must be a cycle by Lemma 5.4. This implies that the cylinders must be arranged in exactly the configuration described in the statement of the corollary. Clearly this argument does not depend on θ, so the result follows. □

Lemma 5.7. *Let (X, ω) generate a Teichmüller disc $D \subset \mathcal{D}_g(1)$. If (X', ω') is a degenerate surface in the closure of D and ω' is not holomorphic, then on every part of X', either ω' has simple poles, or $\omega' \equiv 0$.*

Proof. Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ be a sequence in D converging to the degenerate surface (X', ω') as n tends to infinity. Since ω' is not holomorphic, there is a sequence of cylinders $\{C_1^{(n)}\}_{n=0}^{\infty}$, such that $C_1^{(n)} \subset X_n$ and the core curve of $C_1^{(n)}$ pinches to form a pair of simple poles of ω'. By Theorem 5.5, the foliation in which (X_n, ω_n) admits the cylinder $C_1^{(n)}$ is periodic. Therefore, there is a collection of cylinders $\{C_1^{(n)}, \dots, C_k^{(n)}\}$ that fill X_n. Let $w_i^{(n)}$ denote the circumference of $C_i^{(n)}$. By Corollary 5.6, the ratios $w_i^{(n)}/w_1^{(n)} = 1$ for all $i \leq k$ and $n \geq 0$. Hence, if the core curve of $C_1^{(n)}$ pinches, then the core curve of every cylinder in that foliation with height $h_i^{(n)}$ pinches if it satisfies the condition that $h_i^{(n)}/w_1^{(n)}$ diverges to infinity. Since the ratios between the circumferences are constant, every sequence of cylinders contains one or more cylinders converging to an infinite cylinder on X', and ω' must have simple poles on every part with the exception of parts corresponding to the collapsing of saddle connections in the boundary of the cylinders. However, since the saddle connections have zero area, any part of X' corresponding to their collapse must also have zero area, i.e. $\omega' \equiv 0$. □

Definition. *An edge e of a connectivity graph $G(X')$ is called a* holomorphic edge *with respect to ω' if ω' is holomorphic at the pair of punctures corresponding to e.*

Lemma 5.8. *Let (X, ω) generate a Teichmüller disc $D \in \mathcal{D}_g(1)$ and let (X', ω') be a degenerate surface in the closure of D. If e is an edge in the connectivity graph $G(X')$ between two distinct vertices corresponding to parts carrying a nonzero differential, then e is not a holomorphic edge with respect to ω'.*

Proof. By contradiction, assume there is a holomorphic edge e between two distinct vertices corresponding to parts on which ω' is not the zero differential. First, we claim that ω' cannot be holomorphic on a surface with two or more parts. By Lemma 2.7, we can act by the $\mathrm{SL}_2(\mathbb{R})$ action on (X', ω') to reach a surface (X'', ω'') such that ω'' has a pair of simple poles. By Lemma 5.7, on every part of X'', ω'' must have simple poles or be identically zero. However, for every pair of punctures (p, p') on X' where ω' is holomorphic, ω'' must also be holomorphic at the corresponding pair of punctures on X''. This forces $G^P(X'', \omega'')$ to be a disconnected graph with at least two connected components such that each of the two components contains a vertex of degree at least two. This contradicts Lemma 5.4, hence ω' is not holomorphic on every part of X'.

If ω' is not holomorphic, then by assumption and Lemmas 5.4 and 5.7 imply that e is an edge between two vertices of the cycle $G^P(X', \omega')$. Let C_1 be a cylinder corresponding to an edge of $G^P(X', \omega')$. Let (X_1, ω_1) be a surface whose vertical foliation contains the core curve of C_1. The vertical foliation of (X_1, ω_1) is periodic by Theorem 5.5, and [25, Theorem 3] implies that the core curves of all of the cylinders parallel to C_1 pinch under G_t. Let (X'', ω'') be the resulting degenerate surface. Note that ω'' has simple poles at every pair of punctures on X''. Moreover, since we pinched the core curve of every cylinder parallel to C_1, ω'' must have poles at all of the same punctures at which ω' has poles on X'. However, the edge e is no longer in the graph $G(X'', \omega'')$, which implies that the two vertices it joined are a single vertex in $G(X'', \omega'')$. This is impossible because it would imply that $\dim_C(G^P) \geq 2$. Therefore, $G(X')$ has no holomorphic edges with respect to ω'. □

Lemma 5.9. *If (X', ω') is a degenerate surface in the closure of a Teichmüller disc $D \subset \mathcal{D}_g(1)$, then (X', ω') has one of the following three configurations:*

(1) (X', ω') has exactly one part on which $\omega' \not\equiv 0$ with at most *two simple poles.*

(2) (X', ω') has exactly two parts on which $\omega' \not\equiv 0$ that are joined by exactly two pairs of poles.

(3) $X' = S_1 \sqcup \cdots \sqcup S_n$ has $n \geq 3$ parts on which $\omega' \not\equiv 0$ such that ω' has exactly one pair of poles joining S_j to S_{j+1}, for $1 \leq j \leq n-1$, and exactly one pair of poles joining S_n to S_1.

Furthermore, there are no pairs of punctures joining two distinct parts in the second and third configuration above such that ω' is holomorphic at those pairs of punctures.

Proof. By Lemma 5.2, if X' has one part, then ω' has at most one pair of poles. If X' has more than one part, then this lemma follows from Lemmas 5.4, 5.7, and Lemma 5.8. □

Remark. *Case (2) describes a cycle on two vertices that is simply a degenerate version of Case (3). We distinguished it from Case (3) for clarity.*

Convention. *For the remainder of this paper, we will ignore parts of a degenerate surface (X', ω') carrying the zero differential. For example, we may say that a degenerate surface has two parts, when we mean that it has two parts on which $\omega' \not\equiv 0$, but it may have many more parts on which $\omega' \equiv 0$.*

6. Applications of complete periodicity in $\mathcal{D}_g(1)$

The property of complete periodicity imposes very strong restrictions on a surface. With little effort we prove that there are no Teichmüller discs in $\mathcal{D}_g(1)$ in certain strata of Abelian differentials and apply this to genus two.

Lemma 6.1. *Given a completely periodic surface $(X, \omega) \in \mathcal{M}_g$, $g \geq 2$, there exists $\theta \in \mathbb{R}$ such that the cylinder decomposition of $(X, e^{i\theta}\omega)$ has at least two cylinders.*

Proof. Assume that (X, ω) is filled by a single cylinder C. We show that there exists a direction such that (X, ω) is not filled by a single cylinder. The top and bottom of C consist of a union of saddle connections. Choose one such saddle connection σ on the bottom of C joining zeros z_1 to z_2, which are not necessarily distinct. Let σ' be the saddle connection on the top of C to which σ is identified. Let σ' have endpoints z_1' and z_2' such that z_i is identified to z_i', for $i = 1, 2$. Consider the family of trajectories in C parallel to a trajectory from z_1 to z_1'. This determines a cylinder $C' \subset X$ with z_1 on its top and z_2 on its bottom formed by identifying σ to σ'. Since σ is a proper subset of the top of cylinder C, the cylinder C' does not fill (X, ω). Furthermore, (X, ω) is completely periodic, so the complement of C' must contain at least one cylinder. □

Proposition 6.2. *There are no Teichmüller discs contained in $\mathcal{D}_g(1) \cap \mathcal{H}(2g-2)$.*

Proof. By contradiction, assume that there is a surface (X,ω) generating a Teichmüller disc in $\mathcal{D}_g(1) \cap \mathcal{H}(2g-2)$. By Lemma 6.1, choose a direction θ such that $(X, e^{i\theta}\omega)$ decomposes into two or more cylinders. Under the Teichmüller geodesic flow, $(X, e^{i\theta}\omega)$ degenerates to a surface (X', ω') with two or more parts by Lemma 5.9 and [25, Theorem 3]. Moreover, the zero of order $2g-2$ must lie on exactly one of the parts because [25, Theorem 3] implies that only the core curves of cylinders are pinched. This implies that there is a part of X' with two simple poles and no zeros, i.e. a twice punctured sphere. This is not admissible under the Deligne–Mumford compactification, thus we get a contradiction. □

Proposition 6.3. *Let n and m be odd numbers such that $n+m = 2g-2$. There are no Teichmüller discs contained in $\mathcal{D}_g(1) \cap \mathcal{H}(n,m)$.*

Proof. By contradiction, assume that there is a surface (X,ω) generating a Teichmüller disc in $\mathcal{D}_g(1) \cap \mathcal{H}(n,m)$. By Lemma 6.1, choose a direction θ such that $(X, e^{i\theta}\omega)$ decomposes into two or more cylinders. Under the Teichmüller geodesic flow, $(X, e^{i\theta}\omega)$ degenerates to a surface (X', ω') with two or more parts by Lemma 5.9 and [25, Theorem 3]. Moreover, the zeros must lie on one or two of the parts of X' because [25, Theorem 3] implies that only the core curves of cylinders were pinched. If they lie on the same part, then as before, every other part must be a twice punctured sphere, which is impossible. However, if they lie on different parts, then there is a part with two simple poles and a zero of order n. Since there does not exist an integer $g' \geq 0$ such that $n-2 = 2g'-2$, the Chern formula cannot be satisfied and we have a contradiction. □

Though Proposition 6.4 is well-known, we provide an original proof that there are no Teichmüller discs contained in $\mathcal{D}_2(1)$. The best possible result for the Lyapunov exponents of genus two surfaces was proven by Bainbridge [3], who used McMullen's [29] classification of $SL_2(\mathbb{R})$-invariant ergodic measures in genus two to calculate the Lyapunov exponents of the Kontsevich–Zorich cocycle explicitly. Bainbridge found $\lambda_2 = 1/2$, for all $SL_2(\mathbb{R})$-invariant ergodic measures with support in $\mathcal{H}(1,1)$, and $\lambda_2 = 1/3$, for all $SL_2(\mathbb{R})$-invariant ergodic measures with support in $\mathcal{H}(2)$.

Proposition 6.4. *There are no Teichmüller discs contained in $\mathcal{D}_2(1)$.*

Proof. This follows from Propositions 6.2 and 6.3 because $\mathcal{M}_2 = \mathcal{H}(2) \cup \mathcal{H}(1,1)$. □

Note that $\mathcal{D}_2(1)$ is the determinant locus in genus two. We remark that the author has another proof of Proposition 6.4 using more direct methods than those in this paper and more elementary than those of [3].

7. Convergence to Veech surfaces

The goal of this section is to prove Theorem 7.4, which will serve as the first step toward bridging the gap between the problem of classifying all Teichmüller discs in $\mathcal{D}_g(1)$ and Möller's [30] nearly complete classification of Teichmüller *curves* in $\mathcal{D}_g(1)$.

Lemma 7.1. *Given a surface (X,ω) generating a Teichmüller disc $D_1 \subset \mathcal{D}_g(1)$, let $\{(X_n,\omega_n)\}_{n=1}^{\infty}$ be a sequence of surfaces in D_1 converging to $(X',\omega') \in \overline{\mathcal{M}_g}$, where $(X',\omega') \notin D_1$ and ω' is holomorphic. If D_2 is the Teichmüller disc generated by (X',ω'), then $D_2 \subset \overline{\mathcal{D}_g(1)}$. Furthermore, $D_2 \subset \overline{D_1}$.*

Proof. We recall that the $\mathrm{SL}_2(\mathbb{R})$ action on $\overline{\mathcal{M}_g}$ is continuous by [4, Proposition 11.1]. Since $\overline{\mathcal{D}_g(1)}$ is closed, the closure of D_1 in $\overline{\mathcal{M}_g}$ is also contained in $\overline{\mathcal{D}_g(1)}$. Furthermore, every point in D_2 is the limit of a sequence of points in D_1. This can be seen by taking a sufficiently small neighborhood of (X',ω'), which contains points in D_1 by assumption. By the continuity of the $\mathrm{SL}_2(\mathbb{R})$ action on $\overline{\mathcal{M}_g}$, there is an arbitrarily small neighborhood of any point in D_2 that also contains points in D_1. Hence, $D_2 \subset \overline{D_1} \subset \overline{\mathcal{D}_g(1)}$. □

Definition. *A surface (X,ω) is called a* Veech surface *if its group $SL(X,\omega)$ of affine diffeomorphisms is a lattice in $SL_2(\mathbb{R})$. The Teichmüller disc generated by a Veech surface in the moduli space $\mathcal{M}_g$ is called a* Teichmüller curve.

The reason for the term Teichmüller curve follows from a result of Smillie, which states that the $\mathrm{SL}_2(\mathbb{R})$ orbit of a Veech surface projected into $\mathcal{R}_g$ is closed. This result was never published by John Smillie. However, it was communicated to William Veech, who outlined a proof of it in [35] (see also [31]). Moreover, when projected into $\mathcal{R}_g$, Teichmüller curves are algebraic curves. One striking property of Veech surfaces is the *Veech dichotomy*. The Veech dichotomy completely describes the dynamics of the trajectory of any point on the surface X [33]. It says that the geodesic flow on X with respect to the flat structure induced by ω is either periodic or uniquely ergodic. The following definition was introduced in [6].

Definition. *A completely periodic surface satisfies* topological dichotomy *if any direction that admits a saddle connection is periodic.*

Lemma 7.2. *Given a Teichmüller disc $D \subset \mathcal{D}_g(1)$ of a completely periodic surface $(X_0,\omega_0) \in \mathcal{M}_g$, which does not satisfy topological dichotomy, there exists a sequence of surfaces $\{(X_n,\omega_n)\}_{n=0}^{\infty}$ in D converging to a surface $(X',\omega') \in \overline{\mathcal{D}_g(1)}$ such that X' has one part, ω' is holomorphic, and a saddle connection of (X_0,ω_0) contracts to a point on (X',ω').*

Proof. By assumption, there exists a saddle connection σ_0 lying in a nonperiodic foliation of the surface (X_0,ω_0). Without loss of generality, let σ_0 lie in the vertical foliation of (X_0,ω_0). Act by the Teichmüller geodesic flow G_t on (X_0,ω_0) so that σ_0 contracts by e^{-t} as t tends to infinity. We prove that we can choose a divergent sequences of times $\{t_n\}_{n=0}^{\infty}$ such that the corresponding sequence of surfaces $\{(X_n, \omega_n)\}_{n=0}^{\infty}$, defined by

$$(X_n, \omega_n) = G_{t_n} \cdot (X_0, \omega_0),$$

converges to a degenerate surface (X', ω'), where ω' is holomorphic. Let $t_0 = 0$.

Let σ_t be the saddle connection on $G_t \cdot (X_0, \omega_0)$ defined by contracting the saddle connection σ by e^{-t}. If ω' is not holomorphic, then, by Corollary 5.6, for all $\varepsilon > 0$, there exists an N and θ_N, such that the vertical foliation of $(X_N, e^{i\theta_N}\omega_N)$ determines a decomposition of $(X_N, e^{i\theta_N}\omega_N)$ into a union of cylinders $C_1, \dots, C_p$, with waist lengths ε and heights $h_1, \dots, h_p$, respectively, such that $\sum_k h_k = 1/\varepsilon$. This follows from the assumption that the area of every surface in the sequence is one. This sequence of surfaces defines a sequence of closed curves $\{\gamma_{n,t_n}\}_{n=0}^{\infty}$ whose lengths tend to zero as n tends to infinity, where γ_{n,t_n} is the waist curve of a cylinder on $(X_n, e^{i\theta_n}\omega_n)$. Furthermore, for each n, the curve γ_{n,t_n} corresponds to a closed curve γ_{n,t_0} on (X_0, ω_0) with the property that the image of γ_{n,t_0} under G_{t_n} is γ_{n,t_n}. Note that for all n and t_n, no curve γ_{n,t_n} is parallel to σ_{t_n} because σ_{t_n} does not lie in a periodic foliation while γ_{n,t_n} always lies in a periodic foliation.

We claim that we can pass to a subsequence such that γ_{n,t_n} is transverse to γ_{n+1,t_n}. Let $0 \leq \alpha_{n,t} < \pi$ denote the angle between $\gamma_{n,t}$ and $\sigma_{n,t}$. For all n and t_n, $\alpha_{n,t_n} \neq 0$ because γ_{n,t_n} is not parallel to σ_{t_n}. Fixing n and letting t tend to infinity, $|\alpha_{n,t}|$ tends to $\pi/2$ because γ_{n,t_n} has nontrivial length in the maximally expanding direction of G_t, so for sufficiently large t, $\gamma_{n,t}$ converges to the direction of maximum expansion, which is orthogonal to the direction of minimal expansion in which σ_0 lies. We prove that the set $\Gamma = \{\gamma_{n,0} | n \geq 0\}$ is infinite. If not, the previous comment would imply that given $\delta > 0$, there exists a time $T > 0$, such that for all n and $t > T$,

$$\sup_n ||\alpha_{n,t}| - \pi/2| < \delta.$$

This would contradict the fact that the lengths of the curves $\{\gamma_{n,t}\}_{n=0}^{\infty}$ tend to zero. Hence, the set Γ is infinite and we can pass to a subsequence such that γ_{n,t_n} is transverse to γ_{n+1,t_n}. Equivalently, $\gamma_{n,t_{n+1}}$ is transverse to $\gamma_{n+1,t_{n+1}}$.

Now we can construct a sequence of surfaces corresponding to a divergent sequence of times $\{t'_n\}_{n=0}^{\infty}$ such that the limit is holomorphic and the saddle connection σ_n degenerates to a point. Let $\varepsilon_N > 0$ be the infimum, taken over all cylinder decompositions of (X_N, ω_N), of the length of the waist curves of the cylinders at time t_N. By passing to a subsequence of times, we can assume $\gamma_{N+1,t_{N+1}}$ has length $\varepsilon_{N+1} < \varepsilon_N$. However, γ_{N,t_N} has length ε_N and γ_{N+1,t_N} is transverse to γ_{N,t_N}. For any surface (X, ω), whose Teichmüller disc is contained in $\mathcal{D}_g(1)$, let γ be the waist curve of a cylinder C_j which is an element of a cylinder decomposition $\mathcal{C}$ of (X, ω). It follows from Corollary 5.6 that every closed regular trajectory transverse to γ must pass through every cylinder in $\mathcal{C}$ at least once.

Thus, in this case, γ_{N+1,t_N} has length at least $1/\varepsilon_N$. Since $\gamma_{N+1,t_{N+1}}$ has length $\varepsilon_{N+1} < \varepsilon_N$ and γ_{N+1,t_N} can be pinched under the Teichmüller geodesic flow so that the direction of σ_N contracts, then there is a time t'_{N+1} such that $t_N < t'_{N+1} < t_{N+1}$ and $\gamma_{N+1,t'_{N+1}}$ has length one. Furthermore, if $\gamma_{N+1,t'_{N+1}}$ has length one, then by the assumption that the area of (X_0, ω_0) is one, the fact that G_t preserves area, and the Teichmüller disc of (X_0, ω_0) is contained in $\mathcal{D}_g(1)$, we have that the minimum length of any curve transverse to $\gamma_{N+1,t'_{N+1}}$ is also one. This implies that there are no short closed curves which are not unions of saddle connections. This defines a divergent sequence of times $\{t'_n\}_{n=0}^{\infty}$ such that the corresponding sequence of surfaces $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ converges to a degenerate surface (X', ω'), where ω' is holomorphic and σ_0 contracts to a point on X'. Finally, by Lemma 5.9, the only admissible boundary points of a Teichmüller disc contained in $\mathcal{D}_g(1)$, which carry holomorphic Abelian differentials, must have exactly one part. □

The following definition was introduced by Vorobets [36]. In [31, Theorem 1.3, Parts (i) and (ii)], Smillie and Weiss prove that a surface is uniformly completely periodic if and only if it is a Veech surface.

Definition. *Let $\mathcal{S}_\theta$ denote the set of saddle connections of the vertical foliation of $(X, e^{i\theta}\omega)$. A surface is called* uniformly completely periodic *if it satisfies topological dichotomy and there exists a real number $s > 0$ such that for all θ, where $\mathcal{S}_\theta \neq \emptyset$, the ratio of the length of the longest saddle connection in $\mathcal{S}_\theta$ to the shortest saddle connection in $\mathcal{S}_\theta$ is bounded by s.*

Lemma 7.3. *Given a Teichmüller disc $D \subset \mathcal{D}_g(1)$ of a surface satisfying topological dichotomy $(X_0, \omega_0) \in \mathcal{M}_g$ that is not uniformly completely periodic, there exists a sequence of surfaces $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ in D converging to a surface $(X', \omega') \in \overline{\mathcal{D}_g(1)}$ such that X' has one part, ω' is holomorphic, and a saddle connection of (X_0, ω_0) contracts to a point on (X', ω').*

Proof. Since the surface (X_0, ω_0) is not uniformly completely periodic, given a divergent sequence of positive real numbers $\{s_j\}_{j=0}^{\infty}$, there exists a corresponding sequence of angles $\{\theta_j\}_{j=0}^{\infty}$ such that the ratio of the longest saddle connection to the shortest saddle connection on $(X_0, e^{i\theta_j}\omega_0)$ is greater than s_j, for all j. We show that there exists a sequence of times $\{t_n\}_{n=0}^{\infty}$ such that the sequence of surfaces $\{G_{t_n} \cdot (X_0, e^{i\theta_n}\omega_0)\}_{n=0}^{\infty}$ converges to a surface (X', ω'), where ω' is holomorphic. Moreover, there is a sequence of saddle connections on $G_{t_n} \cdot (X_0, e^{i\theta_n}\omega_0)$ converging to a point as n tends to infinity.

Pass to a subsequence of $\{\theta_j\}_{j=0}^{\infty}$ defined as follows. Since there is a finite number of zeros, there is a finite number of pairs of zeros. Choose a pair of zeros z_1 and z_2 that occur infinitely often in the sequence $\{(X_0, e^{i\theta_n}\omega_0)\}_{n=0}^{\infty}$ as the pairs of zeros which are joined by the shortest saddle connection. By Corollary 5.6, all of the cylinders in the cylinder decomposition of a surface in $\mathcal{D}_g(1)$ have equal circumference and we can assume that (X_0, ω_0) has unit area and cylinders of unit circumference. For each angle θ_j, denote by $w_j \geq 1$ the length of the circumference of the cylinders in that direction. Then define the times t_j by $e^{-t_j} w_j = 1$, for all j. Then $G_{t_n} \cdot (X_0, e^{i\theta_n}\omega_0) = (X_n, \omega_n)$ is the action on the surface such that the waist curves of the cylinders of circumference w_j contract at the maximal rate. Furthermore, since the length of each saddle connection is bounded above by the circumference of the cylinders, the length of the shortest saddle connection on (X_n, ω_n) is bounded above by $1/s_n$. Note that $\lim_{n\to\infty} 1/s_n = 0$. The Teichmüller geodesic flow preserves area, so the surface (X_n, ω_n) also has unit area for all n. This implies that the sum of the heights of the cylinders is equal to one, as well. It follows from Corollary 5.6 that any closed curve transverse to the horizontal direction has length at least one because any such curve must travel the heights of every cylinder in the cylinder decomposition. Since in this situation the minimum length of a closed curve transverse to the vertical direction is the waist curve of a cylinder which has length one, there are no closed curves that can pinch that are not unions of saddle connections, i.e. no core curves of cylinders can pinch.

If a closed curve, which is a union of saddle connections, degenerates as n tends to infinity, then the limit is a degenerate surface carrying a holomorphic Abelian differential. By Lemma 5.9, the only such degenerate surfaces in the boundary of $\mathcal{D}_g(1)$ have one part. □

Theorem 7.4. *If the Teichmüller disc D of (X, ω) is contained in $\overline{\mathcal{D}_g(1)}$, then either there is a Veech surface $(X', \omega') \in \overline{\mathcal{M}_g}$, or a punctured torus $(S, dz) \in \overline{\mathcal{M}_g}$ such that the Teichmüller disc D' generated by it is contained in $\overline{\mathcal{D}_g(1)}$. Furthermore, every surface in D' is the limit of a sequence of surfaces in D.*

Proof. If (X, ω) is a Veech surface, let $(X, \omega) = (X', \omega')$. Otherwise, assume that $(X, \omega) = (X_{0,1}, \omega_{0,1})$ is not a Veech surface and let D_1 be its Teichmüller disc. Since (X, ω) is not a Veech surface, but its Teichmüller disc is contained in $\overline{\mathcal{D}_g(1)}$, (X, ω) is completely periodic by Theorem 5.5. Furthermore, (X, ω) is not uniformly completely periodic by [31, Theorem 1.3, Parts (i) and (ii)]. By Lemmas 7.2 and 7.3, there exists a sequence $\{(X_{n,1}, \omega_{n,1})\}_{n=1}^{\infty}$ converging to a surface $(X_{0,2}, \omega_{0,2}) \in \overline{\mathcal{M}_g}$ with one part carrying a holomorphic Abelian differential with $(X_{0,2}, \omega_{0,2}) \in \mathcal{D}_g(1)$ and a saddle connection on $\omega_{0,1}$ degenerates to a point on $X_{0,2}$. A degenerate saddle connection implies either two or more zeros of $\omega_{0,1}$ converge to a single zero of $\omega_{0,2}$ or a closed curve of $X_{0,1}$ converges to a pair of punctures on $X_{0,2}$. Then $(X_{0,2}, \omega_{0,2})$ has Teichmüller disc D_2 and by Lemma 7.1, $D_2 \subset \overline{\mathcal{D}_g(1)}$. By Theorem 5.5, $(X_{0,2}, \omega_{0,2})$ is also completely periodic. If it is a Veech surface, then we are done. Otherwise, we proceed by induction using Lemmas 7.2 and 7.3 to create a sequence of surfaces $\{(X_{0,j}, \omega_{0,j})\}_{j=1}^{N}$ in $\overline{\mathcal{M}_g}$ such that each surface in the sequence carries a differential either with fewer distinct zeros or lower genus than the previous surface in the sequence. Since both the number of zeros as well as the genus are finite, this process will terminate at some step N resulting in a surface $(X_{0,N}, \omega_{0,N}) \in \overline{\mathcal{D}_g(1)}$ with Teichmüller disc D_N. By Lemma 7.1, $D_N \subset \overline{\mathcal{D}_g(1)}$. The surface $X_{0,N}$ cannot be a sphere because $\omega_{0,N}$ is holomorphic and $\omega_{0,N}$ is nonzero by Lemma 2.2. Hence, there are three possibilities. Either $\omega_{0,N}$ has a single zero, $X_{0,N}$ is a punctured torus, or $(X_{0,N}, \omega_{0,N})$ is a Veech surface. By Lemma 6.2, $\omega_{0,N}$ cannot have a single zero. Thus, the only remaining possibility is that $(X_{0,N}, \omega_{0,N})$ is a Veech surface or a punctured torus carrying a holomorphic Abelian differential.

Let D' be the Teichmüller disc generated by $(X_{0,N}, \omega_{0,N})$. Lemma 7.1 implies that every surface in D' is the limit of a sequence of surfaces in D_1. □

8. Punctured Veech surfaces

There are several key results that give a nearly complete picture of Teichmüller curves in $\mathcal{D}_g(1)$. We recall all of the results here for the sake of completeness and convenience of the reader. There are two similarly named, related concepts: a square-tiled covering and a square-tiled cyclic cover. A *square-tiled covering* is a specific type of Veech surface introduced by Thurston formed by gluing unit squares together to form a genus g surface. Naturally, such a surface comes with a covering of the unit square, i.e. the torus. A surface is a square-tiled covering if and only if it has affine group commensurable to $\mathrm{SL}_2(\mathbb{Z})$, by [16, Theorem 5.9].

We define a *square-tiled cyclic cover* using the exposition of [14]. A square-tiled cyclic cover is a specific type of square-tiled covering. Let $N > 1$ be an integer and $(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$ such that they satisfy

$$0 < a_i \leq N; \ \gcd(N, a_1, \ldots, a_4) = 1; \ \sum_{i=1}^{4} a_i \equiv 0(\mod N).$$

Then the algebraic equation

$$w^N = (z - z_1)^{a_1}(z - z_2)^{a_2}(z - z_3)^{a_3}(z - z_4)^{a_4}$$

defines a closed, connected and nonsingular Riemann surface denoted by $M_N(a_1, a_2, a_3, a_4)$. By construction, $M_N(a_1, a_2, a_3, a_4)$ is a ramified cover over the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ branched over the points $z_1, \ldots, z_4$. Consider the meromorphic quadratic differential

$$q_0 = \frac{dz^2}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)}$$

on $\mathbb{P}^1(\mathbb{C})$. It has simple poles at $z_1, \ldots, z_4$ and no other zeros or poles. Then the canonical projection

$$p : M_N(a_1, a_2, a_3, a_4) \to \mathbb{P}^1(\mathbb{C})$$

induces a quadratic differential $q = p^*q_0$ by pull-back. Lemma 8.1 follows from [30, Cor. 3.3, Sect. 3.6].

Remark. *The name cyclic cover comes from the fact that the group of deck transformations of a cyclic cover is the cyclic group $\mathbb{Z}/N\mathbb{Z}$.*

Lemma 8.1 (Möller). *If (X, ω) is a Veech surface whose Teichmüller disc is contained in $\mathcal{D}_g(1)$, then (X, ω) is a square-tiled covering.*

We recall the two known examples of surfaces that generate Teichmüller discs in $\mathcal{D}_g(1)$. The genus three example, denoted here by (M_3, ω_{M_3}), is commonly known as the Eierlegende Wollmilchsau for its numerous remarkable properties [18]. Forni [12] discovered that its Kontsevich–Zorich spectrum is indeed completely degenerate. The surface (M_3, ω_{M_3}) is a square-tiled surface given by the algebraic equation

$$w^4 = (z - z_1)(z - z_2)(z - z_3)(z - z_4).$$

Its differential, given in [28], can be written explicitly as

$$\omega_{M_3} = \frac{dz}{w^2}.$$

It is easy to see that this lies in the principal stratum of genus three, $\mathcal{H}(1, 1, 1, 1)$. The surface is pictured in Figure 1 and the zeros lie at the corners of the squares and are denoted by $v_1, \dots, v_4$. For completeness, note that the stratum $\mathcal{H}(1, 1, 1, 1)$ is connected by [22].

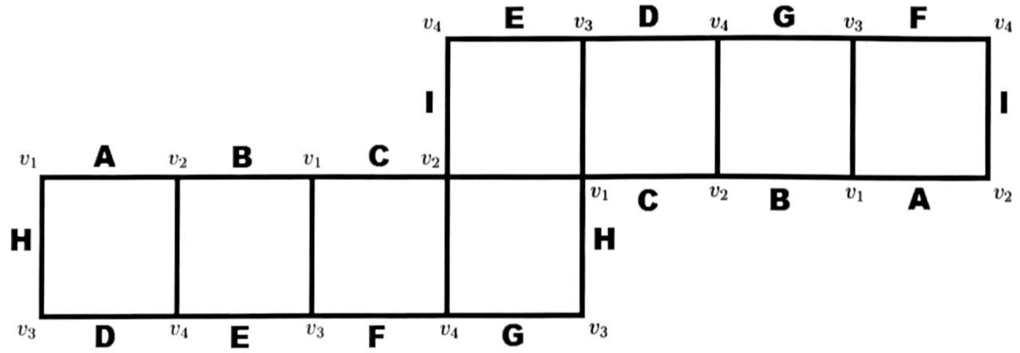

Figure 1. The Eierlegende Wollmilchsau (M_3, ω_{M_3})

Proposition 8.2 (Forni). *The square-tiled surface (M_3, ω_{M_3}) generates a Teichmüller curve in $\mathcal{D}_3(1)$.*

The genus four example was discovered by Forni and Matheus [13] and we denote it by (M_4, ω_{M_4}). Recently, Vincent Delecroix and Barak Weiss have proposed to Carlos Matheus that (M_4, ω_{M_4}) be named the Ornithorynque (Platypus in French). We adopt this terminology here. The surface (M_4, ω_{M_4}) is a square-tiled surface given by the algebraic equation

$$w^6 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)^3.$$

Its differential, see [28], can be written explicitly as

$$\omega_{M_4} = \frac{z\,dz}{w^2}.$$

It is easy to see that this lies in the stratum $\mathcal{H}(2,2,2)$. The surface is pictured in Figure 2 and the zeros, denoted by v_1, v_2, v_3, lie at the corners of the squares. For completeness, note that $\mathcal{H}(2,2,2)$ has two connected components by [22], and it was proven in [28] and again in [14] that (M_4, ω_{M_4}) lies in the connected component $\mathcal{H}^{even}(2,2,2)$ where the spin-structure has even parity.

Proposition 8.3 (Forni-Matheus). *The square-tiled surface (M_4, ω_{M_4}) generates a Teichmüller curve in $\mathcal{D}_4(1)$.*

Möller [30] showed that Teichmüller curves in $\mathcal{D}_g(1)$ must also be Shimura curves. This allowed him to give a nearly complete classification of Teichmüller curves in $\mathcal{D}_g(1)$.

Theorem 8.4 (Möller). *Other than possible examples in certain strata of $\mathcal{M}_5$, listed in the table in [30, Corollary 5.15], and the examples of Propositions 8.2 and 8.3, there are no other Teichmüller curves contained in $\mathcal{D}_g(1)$, for $g \geq 2$.*

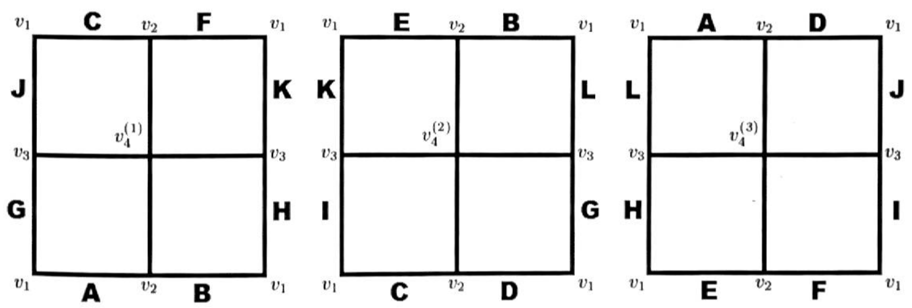

Figure 2. The Ornithorynque (M_4, ω_{M_4})

These results are key to the remainder of the paper. Theorem 7.4 implies that for any Teichmüller disc in $\mathcal{D}_g(1)$ there is a sequence of surfaces converging to a Veech surface. This Veech surface may arise from pinching curves to pairs of punctures thereby resulting in a punctured Veech surface, or possibly from degenerating unions of saddle connections that form a surface of positive genus. Moreover, Lemma 8.1 implies that this punctured Veech surface is, in fact, a punctured square-tiled surface. The strategy will be to proceed by contradiction and assume that there is such a sequence of surfaces converging to a punctured square-tiled surface. The theme of the remainder of this paper is captured in the following question.

Question. *Given a sequence of surfaces in a Teichmüller disc contained in $\mathcal{D}_g(1)$ converging to a degenerate surface X', which is square-tiled and carries a holomorphic Abelian differential ω', at which points of X' can the punctures lie?*

Definition. *Let $(X, \omega) \in \mathcal{M}_g$ and $p \in X$. Let $\Gamma_p(X)$ denote the set of all closed regular trajectories γ passing through p with respect to $e^{i\theta}$, for all $\theta \in \mathbb{R}$. Define the set*

$$C_p(X) = \bigcap_{\gamma \in \Gamma_p(X)} \gamma.$$

It should be obvious to the reader that for any compact Riemann surface X and any $p \in X$, $C_p(X)$ is a finite set. Otherwise, it would have an accumulation point on X, which is impossible.

Theorem 8.5. *Let D be a Teichmüller disc in $\mathcal{D}_g(1)$. Let (X', ω') be a degenerate surface in the closure of D such that ω' is holomorphic and X' has exactly one part. If (p, p') is a pair of punctures on (X', ω'), then $p' \in C_p(X')$.*

Proof. We proceed by contradiction and assume $p' \notin C_p(X')$. By definition of $C_p(X')$, there exists a $\theta \in \mathbb{R}$ such that $(X', e^{i\theta}\omega')$ has a closed leaf γ passing through p and not through p'. We act on $(X', e^{i\theta}\omega')$ by G_t and claim that we can find a divergent sequence of times $\{t_n\}_{n=1}^{\infty}$ such that $G_{t_n} \cdot (X', e^{i\theta}\omega')$ converges to a degenerate surface which cannot be a boundary point of a Teichmüller disc in $\overline{\mathcal{D}_g(1)}$. Since (X', ω') is completely periodic, by Theorem 5.5, all of the leaves of the vertical foliation of $(X', e^{i\theta}\omega')$ are closed. By Corollary 5.6, all of the leaves have the same length ℓ. After time t, they have length $e^{-t}\ell$. Furthermore, since p and p' do not both lie on γ, the distance between them tends to infinity exponentially with t. Let (X', ω') degenerate to (X'', ω'') under the action by G_t. This implies that (p, p') are a pair of holomorphic punctures paired between two distinct parts of (X'', ω''). However, Lemma 5.9 says that there cannot be a pair of holomorphic punctures on two distinct parts of a degenerate surface whose Teichmüller disc is contained in $\mathcal{D}_g(1)$. This contradiction implies that $p' \in C_p(X')$. □

Lemma 8.6. *Let $\mathbb{T}^2$ denote the torus. For all $p \in \mathbb{T}^2$, $C_p(\mathbb{T}^2) = \{p\}$.*

Proof. Identify $\mathbb{T}^2$ with the unit square S. Consider the horizontal and vertical lines intersecting at $p \in S$. It is obvious that these two lines have no other intersection point. Hence, $C_p(\mathbb{T}^2) = \{p\}$. □

Corollary 8.7. *Let D be a Teichmüller disc in $\mathcal{D}_g(1)$ such that (X', ω') is a degenerate surface carrying a holomorphic Abelian differential, and (X', ω') is a square-tiled surface with covering map $\pi : X' \to \mathbb{T}^2$. If (p, p') is a pair of punctures on X', then $\pi(p) = \pi(p')$.*

Proof. Closed trajectories on X' descend to closed trajectories on $\mathbb{T}^2$ under π. Hence, it follows from Theorem 8.5 and Lemma 8.6 that

$$\pi(p') \in \pi(C_p(X')) = C_{\pi(p)}(\mathbb{T}^2) = \{\pi(p)\}.$$

$\square$

Remark. *Corollary 8.7 is weaker than Theorem 8.5 because* $\pi(p) = \pi(p')$ *does* not *imply* $p' \in C_p(X)$.

In order to proceed, we need to introduce some terminology to help us work with the parts of a degenerate surface carrying zero differentials.

Definition. *Let* $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ *be a sequence of surfaces in a Teichmüller disc* D *converging to a degenerate surface* (X', ω') *in the closure of* D *with exactly one part carrying a non-zero holomorphic differential and every other part carries the zero differential. If a saddle connection* σ *between any two zeros of* ω_n *collapses to a point as* n *tends to infinity, then we call* σ *a* short saddle connection.

In the Deligne–Mumford compactification, every part of a degenerate surface carries an Abelian differential with at most simple poles. It is permitted for some parts of the degenerate surface to carry the zero differential. Since the limits we take will be of holomorphic differentials on surfaces converging to holomorphic differentials on degenerate surfaces, the zero differential represents a loss of information that we do not wish to consider. From the algebraic perspective, these surfaces carrying the zero differential are essential to preserve "stability." However, this perspective will not be relelvant to this paper. Throughout, we use the following convention. The surface (X', ω') could have several parts, where $\omega' \equiv 0$ on all but one part. We abuse notation and in the proofs below and let (X', ω') refer to the part carrying a non-zero holomorphic differential. Most of the time, (X', ω') will actually be a Veech surface with completely degenerate KZ-spectrum, or a torus.

Definition. *Let* $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ *be a sequence of surfaces converging to a degenerate surface* (X', ω') *such that* ω' *is non-zero and holomorphic on exactly one part of* X' *and identically zero on every other part of* X'*. By abuse of notation, let* X' *denote the part carrying the non-zero holomorphic differential. Let* $\{p_1, \dots, p_k\}$ *be a maximal set of punctures on* X' *with the following property. Let* γ_i *be a small curve homotopic to* p_i*, for all* i*. Then* γ_i *can also be considered as a small curve on* (X_n, ω_n) *for large* n*, and it bounds a region that degenerates to a point as* n *tends to infinity. If the set of interiors of the curves* γ_i *on* (X_n, ω_n) *is connected, then* $\{p_1, \dots, p_k\}$ *is called a* connected set of punctures.

In particular, when $k = 1$*, we say that* p_1 *is an* isolated puncture.

Lemma 8.8. *Let* (X, ω) *generate a Teichmüller disc* $D \subset \mathcal{D}_g(1)$*. If* $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ *is a sequence of surfaces in* D *converging to a degenerate surface* (X', ω') *in the closure of* D *with exactly one part carrying a non-zero holomorphic differential and* *every other part carries the zero differential, then no puncture on X' at a regular point of ω' is isolated.*

Proof. By Theorem 7.4, it suffices to assume (X', ω') is a Veech surface or a punctured torus. By contradiction, assume that p is an isolated puncture on (X', ω'). By [30], the surface (X', ω') is a branched covering of a torus (or just the torus itself if X' is a torus). On (X', ω') there is a regular trajectory η_1 that does not pass through any of the finitely many punctures of X'. Furthermore, there are at least two transverse choices for η_1. (In fact, there are infinitely many choices.) We take η_1 so that the cylinder C it represents is bounded by the isolated puncture p on at least one of its sides, and so that the intersection number of it with the transverse direction of η_2 specified below is equal to the degree of the cover, which is at most 36 by [30].

Consider (X_n, ω_n) in a small neighborhood of (X', ω'). By [27], there exists a periodic direction transverse to the foliation in which the short saddle connections lie on (X_n, ω_n). By Theorem 5.5, (X_n, ω_n) is completely periodic and therefore we get a decomposition of (X_n, ω_n) into cylinders. The trajectory η_1 persists on (X_n, ω_n), for n sufficiently large. The cylinder determined by η_1 is incident with a zero z_0 of ω_n that converges to the puncture p. The zero z_0 must be incident with a short saddle connection σ transverse to η_1. Transversality is guaranteed by the fact that there is more than one choice of direction for η_1 above.

In a sufficiently small neighborhood of z_0, there is a trajectory η_1' parallel to η_1 such that the cylinders represented by η_1 and η_1' share a common boundary as described in Corollary 5.6. Therefore, both copies of σ must be incident with the boundary of C. However, both copies of σ must also lie in the same small neighborhood by the definition of an isolated puncture. Since the puncture on X' lies at a regular point, the interior of the separating curve, from the definition of an isolated puncture, can be embedded in the plane.

By Corollary 5.6, all copies of z_0 lie on the bottom of the same cylinder. Thus there is another trajectory η_2 transverse to η_1 with the same property that every copy of z_0 lies on the boundary between two cylinders with core curves parallel to η_2. Since the intersection number of η_2 and η_1' is a finite fixed number, the two copies of σ cannot get arbitrarily close as n tends to infinity. This implies that as n tends to infinity, they converge to two distinct punctures, which contradicts the fact that they came from a single isolated puncture. □

Lemma 8.9. *Given a Teichmüller disc $D \subset \mathcal{D}_g(1)$, for $g \geq 2$, there is no degenerate surface in the closure of $D \subset \overline{\mathcal{D}_g(1)}$ of the form (S, ω), where S is a punctured torus and ω is holomorphic.*

Proof. By contradiction, assume there is a degenerate surface of the form (S, ω) in the boundary of D. We claim that every puncture on S must be isolated. By the assumption that S arises from pinching curves on a higher genus surface, S has an even, nonzero, number of punctures. Let p and p' be punctures on S in the same connected set of punctures. By Lemma 8.6, there are two parallel curves on S, η_1 and η_2 passing through p and p', respectively. There are two more curves η_1' and η_2' parallel to η_1 that do not pass through punctures of S such that η_1' is not homotopic to η_2' (because S is not a torus, but a *punctured* torus). Pinching the curves η_1' and η_2' degenerates the torus S into a union of two or more spheres S' such that p and p' do not lie on the same sphere. Any edge (possibly contained in a path in the graph theoretic sense) corresponding to the punctures between p and p' cannot be holomorphic by Lemma 5.9. Moreover, every point of S is regular and therefore, there cannot be any isolated punctures either by Lemma 8.8. Hence, no degenerate surface in the closure of D is a punctured torus. □

Theorem 8.10. *The Eierlegende Wollmilchsau (M_3, ω_{M_3}) generates the only Teichmüller disc in $\mathcal{D}_3(1)$.*

Proof. By [30] (restated in Theorem 8.4 above), the Eierlegende Wollmilchsau is the only Veech surface that generates a Teichmüller disc in $\mathcal{D}_3(1)$. By contradiction, assume that there is a genus three surface (X, ω) that generates a Teichmüller disc $D \subset \mathcal{D}_3(1)$. Then X is not a Veech surface, but it is completely periodic by Theorem 5.5. By Theorem 7.4, there is a sequence of surfaces in D converging to a Veech surface (X', ω') contained in $\overline{\mathcal{D}_3(1)}$. Since Theorem 7.4 guarantees that either the Abelian differential ω' has fewer zeros than ω, which implies (X', ω') cannot lie in the principal stratum of $\mathcal{M}_3$, or X' has lower genus than X. However, X' cannot have lower genus by Lemma 8.9, Proposition 6.4, and the fact that the sphere carries no nonzero holomorphic differentials. Moreover, Theorem 8.4 implies that (X', ω') cannot be a Veech surface because (X', ω') does not lie in the principal stratum. This contradiction implies that no other Teichmüller disc is contained in $\mathcal{D}_3(1)$. □

Before proving that the Ornithorynque is the only example of a surface in genus four with completely degenerate KZ-spectrum, we address genus five and six and then return to genus four in the following section.

If the connected set of punctures consists of exactly two punctures, then we call those punctures a *generalized pair of punctures*. Once again recall that we are abusing notation so that (X', ω') refers to both the degenerate surface and the part carrying the nonzero holomorphic differential. Note that a pair of punctures is certainly a generalized pair of punctures, but not necessarily vice versa because a generalized pair of punctures could represent the collapse of a surface with arbitrarily high genus and not just a closed curve.

Lemma 8.11. *Let (X, ω) generate a Teichmüller disc $D \subset \mathcal{D}_g(1)$. Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ be a sequence of surfaces in D converging to a degenerate surface (X', ω') in the closure of D with exactly one part carrying a non-zero holomorphic differential. If (X', ω') has a generalized pair of punctures such that both punctures lie at regular points of ω', then either the generalized pair of punctures represent a surface with* *genus strictly greater than two, or there are at least four connected sets of punctures on* (X', ω').

Figure 3. Simple Zero Saddle Connections

Proof. We consider how (X', ω') descends to the torus (T, dz), or T for short, under the branched covering $\pi : (X', \omega') \to T$. By Lemma 8.8, none of the punctures at regular points is isolated. Also, if p and q are paired punctures on (X', ω'), then they descend to the same point on the torus by Corollary 8.7.

Recall that for the torus to have the property that every trajectory passing through one marked point passes through another marked point before closing, requires the torus to have a minimum of four marked points lying at the 2-torsion points of the torus. Assume by contradiction, that T has strictly fewer than four punctures that lift to twice as many punctures on (X', ω'), and that the punctures represent the degeneration of a surface of genus at most two. We first assume that one of the generalized pairs of punctures has a degenerate genus two surface between them because the same argument will hold for genus one and a node, and in fact, be much simpler. The goal will be to produce a trajectory with length shorter than the circumferences of cylinders to which it is parallel thereby contradicting Corollary 5.6.

Recall from the definition of a connected set of punctures, that γ_i is homotopic to the puncture p_i. By the assumption that the punctures lie at regular points, the picture can be regarded as two sheets with some identifications between them that vanish to a node (or generalized pair of punctures) as n tends to infinity. This implies that γ_i traverses an angle of 2π around each puncture p_i, for $i = 1, 2$. We claim that no saddle connection in the interior of γ_i can be paired with a saddle connection in the interior of γ_i. The proof is identical to the proof of Lemma 8.8 because we can consider the same transverse curves constructed in the proof of that lemma to get the same contradiction. In other words, each saddle connection in the interior of γ_1 is paired with a saddle connection in the interior of γ_2. However, this also implies that we cannot have any zeros that are not simple in the interior of γ_i. Any zero that is not simple has angle strictly greater than 4π. Each sheet is a Euclidean plane and can contribute an angle of at most 2π, which implies that for some i, a saddle connection in the interior of γ_i is paired with a saddle connection in the same interior and we have a contradiction.

Since all of the zeros in the interior of the γ_i are simple and we have assumed that we are degenerating a surface of genus two between the two punctures, there must be at most six saddle connections in the interior of each γ_i coming from the six simple zeros. Note that zeros of total order six are needed to produce a genus two surface because the two regular points at which the genus two surface is joined each appear as double poles on the genus two surface after a blowup of the genus two surface.

By the assumption that there are fewer than four generalized pairs of punctures, we have that there are infinitely many trajectories between a generalized pair of punctures that do not meet a different generalized pair of punctures because closed trajectories on (X', ω') descend to closed trajectories on the torus and by assumption not all of the 2-torsion points of the torus have punctures over them. Each simple zero must have short saddle connections on each side of it as depicted in Figure 3. In particular, if both short saddle connections meeting a simple zero have equal length, then the result is a slit construction. However, we will not assume equality of their lengths.

We claim that not every trajectory leaving one of the short saddle connections in the interior of γ_1 meets a different short saddle connection in the interior of γ_1 before leaving the region bounded by γ_1. To see this, consider the convex hull homotopic to γ_1 with extreme points given by the zeros of ω_n in the interior of γ_1. Let η' be a closed flat trajectory passing through exactly one extreme point of this convex hull. Consider a closed regular trajectory η parallel to η' passing through the convex region such that its boundary is exactly η'. Since η' passes through the simple zero z_0, it must also pass through the other copy of z_0 by Corollary 5.6, and the trajectory η will close when it reaches the short saddle connection incident with z_0 in the interior of γ_2. This forces η to be shorter than the circumferences of cylinders parallel to it contradicting Corollary 5.6. □

Proposition 8.12. *Let (X, ω) generate a Teichmüller disc $D \subset \mathcal{D}_g(1)$. If $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ is a sequence of surfaces in D converging to a degenerate surface (X', ω') in the closure of D with exactly one part carrying a non-zero holomorphic differential and every other part carries the zero differential, and the non-zero holomorphic part of (X', ω') is (M_3, ω_{M_3}) with punctures, then $g \geq 7$.*

Proof. As above, we abuse notation and ignore any other parts of the degenerate surface carrying the zero differential and let (X', ω') denote (M_3, ω_{M_3}) with punctures. By Lemma 8.8, (X', ω') does not have isolated punctures at regular points, and by Lemma 8.11, if it has a generalized pair of punctures at regular points, then it has at least four pairs or a genus three surface between a generalized pair of punctures, which implies that (X, ω) has genus at least seven.

Therefore, to prove this proposition, it suffices to examine the possibility of punctures at the zeros of ω'. Note that by Theorem 8.5 no puncture at a zero of ω' can be connected to a puncture at any other zero. Therefore, each of the possible punctures at the zeros are isolated. If all four zeros were punctured, then there would have to be a surface of genus at least one at each puncture and we would have degenerated a surface of genus at least seven. Therefore, we assume that at most three zeros are punctured, and each puncture has at most a surface of genus three attached to it.

Note that the proof that there is no isolated puncture at a regular point fails when the isolated puncture is at a simple zero exactly because the local picture is not of a single plane but of two planes joined by a branch cut. Due to the branch cut, the curve γ homotopic to the puncture traverses an angle of 4π in this case. Therefore, this case is actually similar to the case of two punctures at regular points with the difference being the branch cut emanating from the simple zero and joining the two sheets. We claim that the proof of Lemma 8.11 that there are no generalized pairs of punctures at regular points still holds even in this case with the branch cut. The key is that the direction of the branch cut is not distinguished, so we can choose any direction in which to take it without a problem. Therefore, by choosing it parallel to the trajectory η chosen in the proof of Lemma 8.11, we get the same contradiction because η cannot pass through a branch cut parallel to itself and move to a different sheet of the cover of the torus. Therefore, we still have a contradiction that proves that no surface of genus less than seven in $\mathcal{D}_g(1)$ can degenerate to the (M_3, ω_{M_3}). □

Following the lines of terminology of an isolated puncture and a generalized pair of punctures, we define a *triple of punctures* to be a connected set of exactly three punctures.

Lemma 8.13. *Let (X, ω) generate a Teichmüller disc $D \subset \mathcal{D}_g(1)$. If $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ is a sequence of surfaces in genus five or six in D converging to (X', ω') in the closure of D, and the non-zero holomorphic part of (X', ω') is (M_4, ω_{M_4}) with a triple of punctures or an isolated puncture at a zero, then every zero of ω_n is a double zero.*

Proof. First we claim that there can be no zeros of order higher than two. Any zeros of order higher than two have a cone angle strictly greater than 6π, which implies that they must be realized by a minimum of four sheets joined by a branch cut. However, this would imply that more than three sheets have a puncture in the limit. This is impossible by Corollary 8.7 and the fact that (M_4, ω_{M_4}) is a three sheeted cover of the torus.

Next, assume that the triple of punctures on (X', ω') does not lie over a 2-torsion point on the torus covered by (M_4, ω_{M_4}). Then there can be no simple zeros on (X_n, ω_n) for sufficiently large n because every cylinder has zeros of even order on

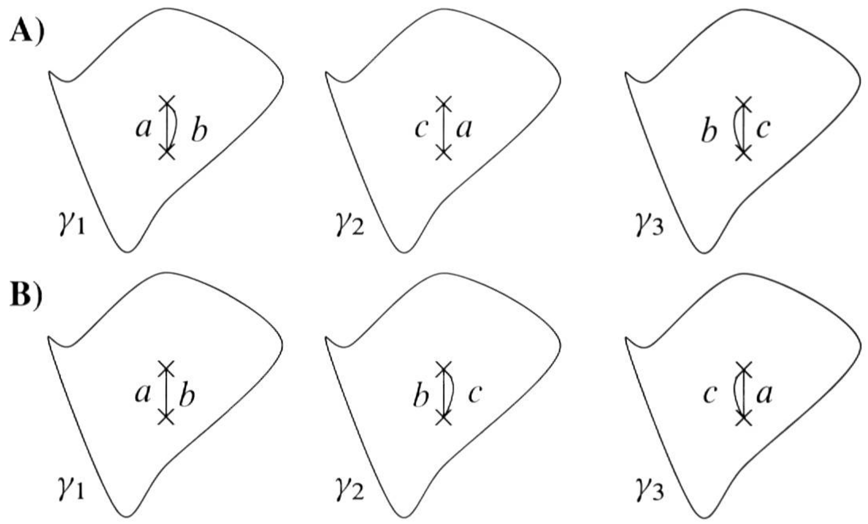

Figure 4. Double Zero Saddle Connections

its boundary. Therefore, if there is a simple zero on the boundary this implies that there must be another simple zero on the boundary. By contradiction, let z_0 be one of the simple zeros. Since z_0 converges to a puncture that does not lie over a 2-torsion point, there must be infinitely many trajectories that pass through z_0 without passing through any of the zeros that limit to the double zeros of (M_4, ω_{M_4}). However, each one of these trajectories must pass through another simple zero z_1 in a neighborhood of z_0. Since z_0 and z_1 can be taken arbitrarily close, it is impossible for an infinite set of transverse trajectories to pass through both of them. Hence, every zero converging to the puncture must be a double zero.

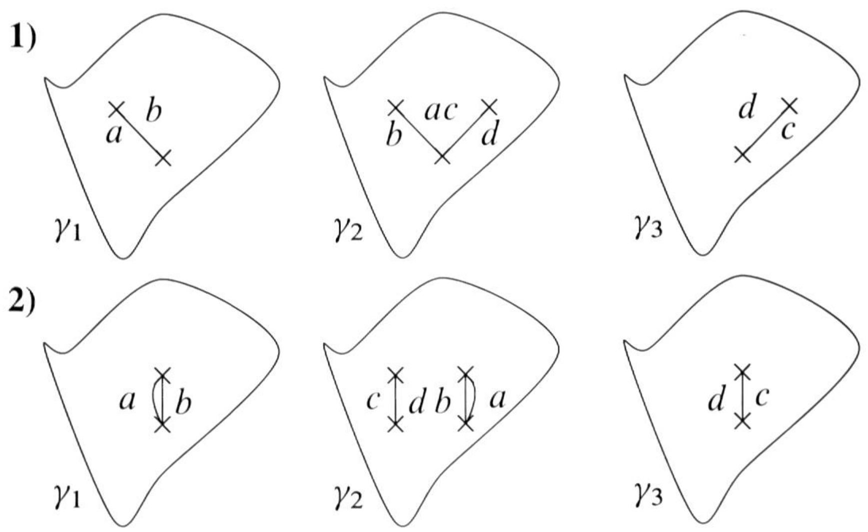

Figure 5. Cases 1) and 2) in the proof of Lemma 8.13

Next we claim that if the triple of punctures lies over the only 2-torsion point that is unramified, then there must be two double zeros converging to a triple of punctures as in Figure 4. Note that the complication here that prevents the argument above from working is that every double zero of ω' could result from two simple zeros colliding and thereby preventing the construction of a closed trajectory that avoids all simple zeros outside of the ones in question. In this case, there are either 1) two simple zeros and a double zero, or 2) four simple zeros converging to the triple of punctures. We claim that Cases 1) and 2) correspond to the configurations of saddle connections in Figure 5.

In Case 1), this is the only admissible configuration (up to permutation of the labels). Consider the three interiors of the curves γ_i, for $i = 1, 2, 3$. Each one must contain a copy of the double zero. Then, if there are two copies of each of the two simple zeros, place them in the picture as in Figure 5 without loss of generality. However, there is a unique way to connect the zeros in these pictures so that each zero has its prescribed order and the saddle connections lie entirely within the regions bounded by the γ_i.

In Case 2), note that the interior of each γ_i must contain at least two simple zeros and cannot contain more than one copy of the same zero. Furthermore, if two zeros z_1 and z_2 appear in the interior of γ_1, then they must both appear in the same copy of γ_i, for $i = 2, 3$ in order for all of the saddle connections between the zeros to lie entirely within the region bounded by the curves γ_i and for all of the zeros to be simple.

We claim that Cases 1) and 2) can be excluded because they contradict the fact that on every surface in a periodic direction, the core curves of every cylinder in that direction are homologous. In each case, pick the longest saddle connection σ incident with a simple zero, or just one of them if they all have equal length. Then there is a trajectory going directly from σ to its copy forming a closed cylinder with circumference shorter than the cylinders to which it is parallel. Such a trajectory exists because in the interior of γ_2 in Figure 5, there are not enough saddle connections to block every trajectory emanating from σ. Hence, there can be no simple zero converging to the 2-torsion point that is unramified over the torus.

Finally, assume that the puncture is isolated at one, or at most two, of the double zeros of ω'. Note that on (X', ω') by considering the horizontal, vertical, and a diagonal direction, we see cylinders that isolate each of the three zeros so that each one is the unique zero lying between two cylinders. In fact, there are infinitely many such directions. Therefore, for sufficiently large n, these cylinders persist where the boundaries of the cylinders vary by an arbitrarily small quantity dependent on n. Recall that every pair of adjacent cylinders must have zeros with even total order between them. This implies that both simple zeros must lie in the boundary of the same cylinder. By considering a different periodic direction that isolates the same zero on (X', ω') and considering this cylinder on (X_n, ω_n), we see that the small saddle connection between the simple zeros must lie in both transverse directions.

This contradiction implies that it must be a double zero. Since we can repeat this argument for each of the three double zeros of (X', ω'), the stratum in genus five or six must contain zeros, which are all of even order. Hence, if (X_n, ω_n) is in genus five or six and degenerates in this way or via a triple of punctures, then all of the zeros of ω_n have order two. □

Lemma 8.14. *Let (X, ω) generate a Teichmüller disc $D \subset \mathcal{D}_g(1)$. Let $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ be a sequence of surfaces in D converging to a degenerate surface (X', ω') in the closure of D with exactly one part carrying a non-zero holomorphic differential, which is the Ornithorynque. If (X', ω') has a triple of punctures such that every puncture does not lie over a 2-torsion point of the torus, then (X_n, ω_n) has genus at least seven.*

Proof. By contradiction, assume that the (X', ω') has a triple of punctures after degenerating a surface (X_n, ω_n) of genus six. A genus five surface degenerating to a genus four surface cannot possibly have a triple of punctures. By Corollary 8.7, all three punctures must descend to the same point on the torus below. We claim the arrangement of the zeros must be as depicted in Figure 4. This follows from Lemma 8.13, which implies that there must be five double zeros on the surface. Therefore it suffices to consider the configurations of Figure 4[A and B].

Assume by contradiction that there is a triple of punctures that descend to the same puncture on the torus below, which is not a 2-torsion point. If we consider a surface (X_n, ω_n) in a neighborhood of (X', ω'), the puncture opens to a union of saddle connections with two double zeros lying along each curve as pictured in Figure 4. Let z_0 be one of the zeros. The saddle connections are identified in one of two possible ways pictured in Figure 4 as either Configurations A or B. Without loss of generality, we assume Configuration A and note that the argument below is identical for Configuration B. In fact, the only difference between A and B is the order in which a trajectory visits the punctures.

The first important fact to note is that we can often get the same contradiction we achieved above in the case of a generalized pair of punctures. Specifically, if the geometry of (X_n, ω_n) forces a trajectory to travel from a saddle connection, say b, to its copy in Configuration A, without passing through a and c first, we see a cylinder with circumference that is too short, i.e., different from the circumference of a parallel cylinder, which is a contradiction. Furthermore, after an appropriate twisting and action by the Teichmüller geodesic flow, we can assume that the saddle connections a, b, c of Figure 4 lie in a direction which converges to the vertical direction on (X', ω') as n tends to infinity. By inspection of the identifications in (M_4, ω_{M_4}) in Figure 2, we see two cylinders in the horizontal direction and the triple of punctures are visited by a horizontal trajectory γ in a different order in each cylinder. This eliminates the possibility of having punctures in the interior of the lower horizontal cylinder in Figure 2. Since (M_4, ω_{M_4}) decomposes into two

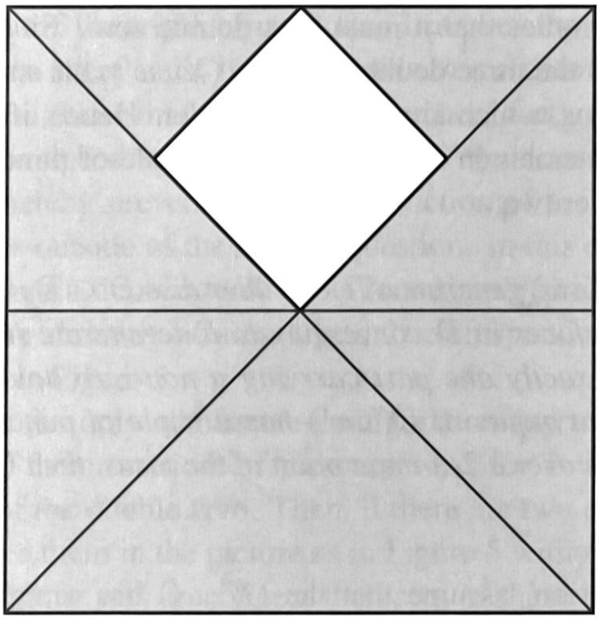

Figure 6. An Open Set of Points (the Shaded Region) on the Torus to which the Triple of Punctures Cannot Descend

cylinders in every direction, this argument eliminates the possibility of the punctures lying in the interior of one of the two cylinders in each direction except for the vertical one.

As we said above, assume without loss of generality that we have the identification of Figure 4[A]. Then if we consider the horizontal direction and the slope ± 1 directions, relative to Figure 2, and map (X', ω') to the torus, we see that the open shaded region in Figure 6 can be excluded as an admissible location for the punctures. Acting by an element of $SL_2(\mathbb{R})$ is equivalent to considering different directions on (X', ω'). Let each 2×2 square in Figure 2 have unit side length. Then (M_4, ω_{M_4}) is fixed under action by the matrix

$$h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

because (M_4, ω_{M_4}) has Veech group $SL_2(\mathbb{Z})$. However, the non-shaded region is obviously not fixed by the action, but an admissible point to which a triple of punctures on (X', ω') must descend, must lie in the intersection of the non-shaded regions we see after acting by h^n for all $n \in \mathbb{Z}$. This infinite intersection can only contain the 2-torsion points of the torus. This concludes the proof that a triple of punctures must lie over the 2-torsion points. $\square$

Proposition 8.15. *Let (X, ω) generate a Teichmüller disc $D \subset \mathcal{D}_g(1)$, $g > 4$. If $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ is a sequence of surfaces in D converging to (X', ω') in the closure of D, and the non-zero holomorphic part of (X', ω') is (M_4, ω_{M_4}) with punctures, then $g \geq 7$.*

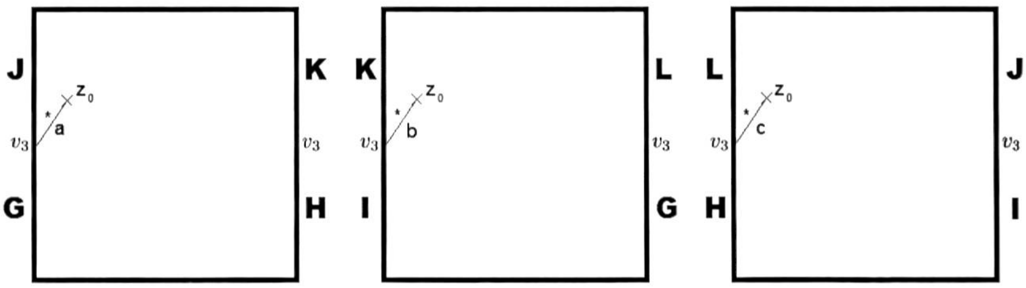

Figure 7. A genus 5 or 6 surface in a small neighborhood of the Ornithorynque with two double zeros collapsing to a double zero

Proof. We consider all possible degenerations of a surface of genus five and six to the punctured (M_4, ω_{M_4}) and exclude each one. By Lemma 8.8, we cannot have isolated punctures at regular points, and by Lemma 8.11, generalized pairs of punctures at regular points must come in quadruples or from degenerating a surface of genus higher than two. In either case, the surface (X, ω) from which it would have to degenerate would have genus eight and the claim would be complete. Furthermore, Lemma 8.14 says there cannot be a triple of punctures over a point that is not a 2-torsion point of the torus covered by (M_4, ω_{M_4}). The remainder of this proof is devoted to excluding the only remaining possibilities using proof by contradiction: one or two punctures at a zero of (X', ω'), or a triple of punctures over the 2-torsion point with regular points above it. By Lemma 8.13, every zero of (X_n, ω_n) is a double zero and if any two double zeros collide to form a higher order zero, then Lemma 8.13 also implies that we have reached a contradiction.

Assume that there are five double zeros on a genus six surface because the genus five case will be a simplification of the following proof. We enumerate the possible degenerations in terms of zeros colliding on the genus six surface. I) First, it is possible for three double zeros to collide and degenerate to a single puncture at a double zero. II) Secondly, it is possible for two pairs of double zeros to collide to two punctured double zeros of (M_4, ω_{M_4}). III) Finally, it is possible to have a triple of punctures over the only 2-torsion point at which (M_4, ω_{M_4}) is unramified.

I) Assume by contradiction that three double zeros collide to form a puncture. We claim that it is possible to construct a sequence of surfaces such that only two of the three double zeros collapse. Assume that the three double zeros denoted z_0, z_1, and z_2 collapsing to a zero of (X', ω') do not all lie on the same side of a cylinder by choosing a transverse direction if they do and declaring this direction to be horizontal. Let

$$h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

For each n, we claim there exists an element h_{t_n} of the horocycle flow such that exactly two of z_0, z_1, z_2 of (X_n, ω_n) converge for $h_{t_n} \cdot (X_n, \omega_n)$. There are two cases to consider: 1) there exists $h_{t'_n}$ such that no pair of z_0, z_1, z_2 converges, or 2) there exists $h_{t'_n}$ such that a pair of zeros from $\{z_0, z_1, z_2\}$ converge while the third zero is isolated.

In Case 1), this implies that the three zeros separate for some value $h_{t'_n}$. We claim there must be a choice of h_{t_n} so that one zero, say z_0, converges to another zero, say z_2, while z_1 is bounded away from z_0, z_2. This will happen by taking t_n large enough so that z_2 wraps back around and becomes close to z_0 again. There are two possibilities within this collapse of two double zeros. Either two double zeros collapse to form a punctured double zero, or several saddle connections simultaneously collapse and yield a triple of saddle connections collapsing as in Figure 4[A or B]. The latter case is subsumed by II) below. We claim that either scenario yields a contradiction.

Consider (X_n, ω_n) in place of $h_{t_n} \cdot (X_n, \omega_n)$, and without loss of generality let z_0 and z_2 be the two zeros that collide in the sequence $h_{t_n} \cdot (X_n, \omega_n)$ as n tends to infinity. Then there are saddle connections in the direction from z_0 to z_2 on (X_n, ω_n) of length approximately $1/3$, where the side length of each square in Figure 2 is $1/3$, and approximately in the horizontal direction on (X_n, ω_n). In particular, this implies that the triple of saddle connections must be arranged as in Figure 4[A or B] because they are the only two arrangements that can exist. If we consider the periodic direction on (X_n, ω_n) that converges to the vertical direction of (X', ω') in the limit, we see two cylinders crossing the triple of saddle connections. However, this implies that a vertical trajectory in one of the two vertical cylinders must close and have circumference less than the other vertical direction contradicting Corollary 5.6.

In Case 2) a pair of zeros from $\{z_0, z_1, z_2\}$ converges while the third zero is isolated in the sequence $h_{t_n} \cdot (X_n, \omega_n)$. If two double zeros collapse to a triple of punctures, then we can choose h_{t_n} so that the triple of punctures does not lie over the 2-torsion points of (X', ω'). This contradicts Lemma 8.14 and implies that two of the double zeros collapse to a punctured double zero. If two double zeros collapse to a double zero, then the surface degenerates from genus six to genus five. The possibility of such a degeneration will be excluded by the argument below.

II) It is a priori possible for two double zeros to collapse to a punctured double zero. In fact, on a genus six surface this degeneration can occur among at most two different pairs of double zeros, while in genus five, at most one such pair can exist. Assume the labels as in Figure 2. Since the resulting genus four surface will have one or two punctures depending on the genus we degenerate from, we make a careful analysis of the picture in a neighborhood of one of the double zeros, say v_3 to which a double zero is collapsing. We claim that up to three possible choices of labels for the saddle connections labeled with an asterisk in Figure 7, the picture locally around v_3 must appear as in Figure 7. Without loss of generality, we assume that the horizontal direction on (X_n, ω_n) is transverse to the short saddle connections connecting the double zero z_0 and v_3. Then after acting by either a horocycle flow that fixes either the vertical or horizontal direction (upper or lower triangular matrices) we can assume that the saddle connections are in the direction pictured in Figure 7. Label the bottom saddle connections a, b, c without loss of generality. Then there are three choices for the asterisked saddle connections listed in the order they would appear in Figure 7: (b, a, c), (b, c, a), (c, a, b). It is easily checked that (b, c, a) is not permitted because with this identification, v_3 is no longer a double zero. However, the other two identifications, which are both of the form $(*, a, *)$ are not possible either because a horizontal trajectory from a to itself closes too quickly and contradicts Corollary 5.6.

III) Using the argument from I) Case 1) above where the horocycle flow is used to stretch a saddle connection, we can consider the case of two double zeros collapsing to a triple of punctures at regular points that descend to a 2-torsion point on the torus. By stretching the short saddle connections so that each of the three pairs of saddle connections wraps around a third of the total surface, exactly as in I) Case 1), we get the same contradiction as above. This completes the claim that a (X_n, ω_n) must have at least genus seven to degenerate to (X', ω'). □

Define the natural poset on partitions of an integer where if κ and κ' are partitions of n, then $\kappa \prec \kappa'$ if κ' is strictly a refinement of κ. Note that $\kappa \neq \kappa'$.

Theorem 8.16. *If Möller's conjecture is true, i.e. there are no Teichmüller curves in $\mathcal{D}_5(1)$, then there are no Teichmüller discs in $\mathcal{D}_g(1)$, for $g = 5, 6$. Moreover, the stratum $\mathcal{H}(\kappa') \subset \mathcal{M}_5$ does not contain a Teichmüller disc in $\mathcal{D}_5(1)$ if there does not exist κ in [30, Corollary 5.15: Table] such that $\kappa \prec \kappa'$.*

Proof. First we prove the claim for genus six. By [30], any such Teichmüller disc is not generated by a Veech surface. Therefore, by Theorem 7.4, we can collapse zeros and converge to a Veech surface with completely degenerate KZ-spectrum. By Lemma 8.9, Proposition 6.2, and Lemmas 8.12 and 8.15, a genus six surface cannot degenerate to a surface of genus one, two, three, or four, respectively. Since there are no Teichmüller discs in $\mathcal{D}_6(1)$ in $\mathcal{H}(10)$ by Proposition 7.1, the closure of a Teichmüller disc $D \subset \mathcal{D}_6(1)$ must contain a Veech surface in genus five with completely degenerate KZ-spectrum. Thus, the claim about genus six follows.

Next, we address genus five. By contradiction, any such Teichmüller disc in $\mathcal{D}_5(1)$ could not be generated by a Veech surface. Therefore we can collapse zeros by Theorem 7.4. By Lemma 8.9, Proposition 6.2, and Lemmas 8.12 and 8.15, the surface cannot degenerate to genus one, two, three, or four, respectively. Therefore, we can continue to collapse zeros until we reach the stratum $\mathcal{H}(8)$. However, this stratum does not contain any Teichmüller discs in $\mathcal{D}_5(1)$ by Proposition 7.1 and we have the desired contradiction.

In the absence of Möller's conjecture, the only obstruction to the argument above is the existence of a Teichmüller curve in $\mathcal{D}_5(1)$. Hence, the second claim of the lemma follows. □

Remark. *In fact, much more can be said about the degeneration of a genus six surface to a genus five Veech surface with completely degenerate KZ-spectrum. Topologically speaking, the only way for a genus six surface to degenerate to a genus five surface, is for a curve to pinch, or a torus with zero area to separate resulting in a genus five surface with exactly one or two punctures. By Lemma 8.8, there are no isolated punctures at regular points, and so such a genus six surface would have to degenerate to a genus five surface by pinching a closed curve of saddle connections, or having an isolated puncture at a zero.*

We claim it is also impossible to have a pair of punctures both lying at regular points. Let (X', ω') be the degenerate surface of genus five. If (X', ω') has a pair of punctures (p, q), then there is a closed trajectory γ' joining them. If $\{(X_n, \omega_n)\}_{n=0}^{\infty}$ is a sequence of surfaces in D converging to (X', ω') in the closure of D, then there is a closed regular trajectory γ on (X_n, ω_n) parallel to γ' defining a cylinder C with boundary containing a zero z_0, for sufficiently large n, that converges to p or q as n tends to infinity. However, the saddle connections incident with z_0 must be as depicted in Figure 3 because z_0 is a simple zero. Otherwise, X_n would have genus strictly greater than six. Hence, the trajectory through saddle connection a in Figure 3 closes too quickly, and we get the same contradiction as in the proof of Lemma 8.12. Hence, all punctures on (X', ω') must lie over the branch points on the torus it covers.

It is not clear how to rule out this last possibility due to the combinatorics involved in possibly having multiple higher order zeros collapsing to one or two punctures with zeros, and the lack of precise information about the identifications on a theoretical genus five example.

9. The Teichmüller Disc in $\mathcal{D}_4(1)$

The goal of this section is to prove Theorem 9.10, which says that the Ornithorynque (M_4, ω_{M_4}), discovered by [13], and depicted in Figure 2, generates the only Teichmüller disc in $\mathcal{D}_4(1)$. Throughout this section we adopt the standard shorthand for strata, e.g. $\mathcal{H}(1^4, 2) := \mathcal{H}(1, 1, 1, 1, 2)$.

Lemma 9.1. *If (X, ω) is not a Veech surface, (X, ω) generates a Teichmüller disc D in $\mathcal{D}_4(1)$, and a sequence of surfaces in D converges to a Veech surface (X', ω'), then X' has genus four.*

Proof. The surface X' cannot have positive genus less than four by Lemma 8.9, Proposition 8.12, and Proposition 6.4. Recall that X' cannot be a sphere either because ω' is holomorphic by Theorem 7.4 and ω' is nonzero by Lemma 2.2. □

Lemma 9.2. *If (X, ω) generates a Teichmüller disc in $\mathcal{D}_4(1)$, then (X, ω) decomposes into at most three cylinders.*

Proof. The top of every cylinder must have a positive, even number of zeros counted with multiplicity, and the total order of the zeros of ω is six. □

Ideally, we would like to use the same exact proof as Theorem 8.10 to show that the genus four surface (M_4, ω_{M_4}) generates the only Teichmüller disc in $\mathcal{D}_4(1)$. However, this is not possible because (M_4, ω_{M_4}) does not lie in the principal stratum as the genus three example does. A priori, it is possible for zeros to converge under the conditions of Theorem 7.4 without reaching a contradiction. On the other hand, this technique can prove the result in most of the strata of $\mathcal{M}_4$.

Lemma 9.3. *There are no Teichmüller discs in $\mathcal{D}_4(1)$ except possibly in the strata $\mathcal{H}(2^3)$, $\mathcal{H}(1^2, 2^2)$, $\mathcal{H}(1^4, 2)$, and $\mathcal{H}(1^6)$. Furthermore, (M_4, ω_{M_4}) generates the only Teichmüller disc in $\mathcal{H}(2^3) \cap \mathcal{D}_4(1)$.*

Proof. By [30] (see Theorem 8.4), (M_4, ω_{M_4}) generates the only Teichmüller curve in $\mathcal{D}_4(1)$. Hence, any other Teichmüller disc D must be generated by a surface (X, ω), which is completely periodic by Theorem 5.5, but not Veech. By Theorem 7.4, there exists a sequence of surfaces in D converging to a Veech surface (X', ω') in $\overline{\mathcal{D}_4(1)}$. The surface X' cannot have genus less than four by Lemma 9.1. Moreover, it is impossible to collapse zeros in any strata other than $\mathcal{H}(1^2, 2^2)$, $\mathcal{H}(1^4, 2)$, and $\mathcal{H}(1^6)$, which are excluded in the statement of the lemma, and converge to the Veech surface in $\mathcal{H}(2^3)$.

Since any other Teichmüller disc in $\mathcal{H}(2^3) \cap \mathcal{D}_4(1)$ must be generated by a non-Veech surface (X, ω), the zeros of (X, ω) can be collapsed to reach a contradiction. □

Lemma 9.3 says that the classification problem is complete in genus four except for three strata. The remainder of this section is dedicated to addressing those strata. The strategy is similar to the one used to prove Theorem 7.4.

Lemma 9.4. *If (X, ω) generates a Teichmüller disc D in $\mathcal{H}(1^2, 2^2) \cap \mathcal{D}_4(1)$, then (X, ω) satisfies topological dichotomy.*

Proof. By Theorem 5.5, (X, ω) is completely periodic. We show that every saddle connection between two zeros must lie in a periodic foliation. First, consider any saddle connection σ from a double zero, denoted by z, to any other zero, denoted z'. If σ does not lie in a periodic foliation, then we can act on it by the Teichmüller geodesic flow so that it contracts at the maximal rate and choose a subsequence of times $\{t_n\}_n$ as in the proof of Lemma 7.2 such that $G_{t_n} \cdot (X, \omega)$ converges to a surface (X', ω'), where ω' is holomorphic. The surface X' cannot degenerate to a lower genus surface by Lemma 9.1, so σ must degenerate to a point resulting in a zero of order strictly greater than two. However, there are no such Teichmüller discs in a stratum with a zero of order strictly greater than two by Lemma 9.3. This contradiction implies the saddle connection σ, which does *not* lie in a periodic foliation, can only lie between the two simple zeros denoted by z_1 and z_2.

Without loss of generality, assume σ has length $\varepsilon > 0$ and (X,ω) has a cylinder decomposition consisting of cylinders with unit circumference. By Lemma 9.2, (X,ω) is a union of one to three cylinders. Degenerating the cylinders under the Teichmüller geodesic flow results in a surface as described in Lemma 5.9. This implies that the total order of the zeros on the top (and bottom) of every cylinder in the cylinder decomposition must be even because every part of the degenerate surface has two poles. In the stratum $\mathcal{H}(1^2,2^2)$, this forces the two simple zeros to lie on the top of the same cylinder in every cylinder decomposition of (X,ω). As usual, assume the area of the surface is one and the lengths of the waists of the cylinders are also one, so that the total heights of the cylinders is one. Since σ does not lie in a periodic foliation, it must leave z_1 and travel up the entire height of all the cylinders before reaching z_2. However, this implies that σ has length at least $1 > \varepsilon$ and this contradiction implies that all saddle connections of (X,ω) must lie in a periodic foliation. □

Lemma 9.5. *There are no Teichmüller discs contained in $\mathcal{H}(1^2,2^2) \cap \mathcal{D}_4(1)$.*

Proof. We prove this lemma by showing that if $(X,\omega) \in \mathcal{H}(1^2,2^2)$ generates a Teichmüller disc in $\mathcal{D}_4(1)$, then (X,ω) is uniformly completely periodic. By [31], (X,ω) is a Veech surface and by [30], there are no Veech surfaces in $\mathcal{H}(1^2,2^2)$ that generate a Teichmüller disc in $\mathcal{D}_4(1)$. This contradiction will imply the lemma.

By contradiction, assume that there exists a surface $(X,\omega) \in \mathcal{H}(1^2,2^2)$ generating a Teichmüller disc in $\mathcal{D}_4(1)$. By [30], (X,ω) is not a Veech surface and by Lemma 9.4, (X,ω) satisfies topological dichotomy. As in the proof of the previous lemma, the only two zeros that are permitted to converge in the context of Theorem 7.4 are the simple zeros z_1 and z_2. Without loss of generality, let σ be a saddle connection between z_1 and z_2 of length $\varepsilon > 0$. Consider a cylinder decomposition of (X,ω), $C_1,\dots,C_n$, with $1 \leq n \leq 3$, such that z_1 and z_2 lie on the bottom of C_1. Let z_3 be a double zero on the top of C_1. Consider the saddle connection σ_1 from z_1 to z_3. We can take σ_1 to have length less than two because the total height of all of the cylinders is one and σ_1 connects the top and bottom of a single cylinder. Then σ_1 lies on the top of a cylinder C_1' in a different cylinder decomposition $\mathcal{C}'$ of (X,ω) because (X,ω) satisfies topological dichotomy by Lemma 9.4. Since the total order of the zeros on the top of every cylinder must be even, z_2 must also lie along the top of C_1'. Furthermore, $\mathcal{C}'$ consists of at most two cylinders because the total order of the zeros along the top of one of the cylinders is four. This implies that the total height of the cylinders in the decomposition $\mathcal{C}'$ is at most ε because σ is transverse to σ_1 and σ must join the top of C_2 to the bottom of C_1. The total area of the cylinders is still one, so the waist length of the cylinders in $\mathcal{C}'$ must be at least $1/\varepsilon$. Act by the Teichmüller geodesic flow so that the waist of the cylinders in $\mathcal{C}'$ is reduced to one and the total height of the cylinders is expanded to one. In the process of the expansion and contraction, the saddle connection σ_1 of length at most two is contracted to length at most 2ε. Since this argument holds for all $\varepsilon > 0$, σ_1 can be contracted to a point resulting in a zero of order three. By Lemma 9.1, the surface will not degenerate and we get a surface generating a Teichmüller disc in a stratum that does not contain a Teichmüller disc. This contradiction completes the proof. $\square$

Lemma 9.6. *If (X, ω) generates a Teichmüller disc D in $\mathcal{H}(1^4, 2) \cap \mathcal{D}_4(1)$, then (X, ω) satisfies topological dichotomy.*

Proof. By Theorem 5.5, (X, ω) is completely periodic. We show that every saddle connection between two zeros must lie in a periodic foliation. First, consider any saddle connection σ from the double zero, denoted by z, to any other zero, denoted by z'. By contradiction, if σ does not lie in a periodic foliation, then we can act on it by the Teichmüller geodesic flow so that it contracts at the maximal rate and choose a subsequence of times $\{t_n\}_n$ as in the proof of Lemma 7.2 such that $G_{t_n} \cdot (X, \omega)$ converges to a surface (X', ω'), where ω' is holomorphic. The surface cannot degenerate to a lower genus surface by Lemma 9.1, so σ must degenerate to a point resulting in a zero of order strictly greater than two. However, there are no such Teichmüller discs in a stratum with a zero of order strictly greater than two by Lemma 9.3. This contradiction implies a saddle connection σ, which does *not* lie in a periodic foliation, can only lie between two of the simple zeros.

Let $z_1, \dots, z_4$ denote the simple zeros of ω and let z_5 denote the double zero. By contradiction, let σ denote the saddle connection that does not lie in a periodic foliation. In light of the argument above, let σ be a saddle connection from z_1 to z_3 and let it have length ε, while there is a cylinder decomposition $\mathcal{C}$ such that the cylinders have circumference one. The zeros z_1 and z_5 cannot lie on the top or bottom of the same cylinder in $\mathcal{C}$ because this would imply $\mathcal{C}$ consists of two cylinders, one of which has height less than ε. As ε tends to zero, the resulting sequence of surfaces would have to converge to (M_4, ω_{M_4}) because this generates the only Teichmüller disc in $\mathcal{D}_4(1)$ in a lower stratum. The cylinder decomposition would only consist of one cylinder, which contradicts the cylinder decomposition of (M_4, ω_{M_4}). With respect to $\mathcal{C}$, and without loss of generality, let z_1 and z_5 be on the top and bottom of a cylinder. Consider the shortest saddle connection σ_1 from z_1 to z_5, which has length less than two. Then σ_1 lies in a periodic foliation by the argument above, and in particular, it is not parallel to σ. Since the total order of the zeros on the bottom of a cylinder must be even, the leaf of the periodic foliation containing σ_1 must contain at least one other simple zero. We show that this will lead to a contradiction.

Let C_1 denote the cylinder with the saddle connection σ_1 on its bottom. Then the bottom of C_1 must also contain either z_2, z_3, or z_4. We only consider z_2 and z_3 here because the argument for z_4 will be identical to the argument for z_2. First assume that the bottom of C_1 contains the zero z_3. Then the saddle connection σ cannot be a subset of the bottom of C_1 because it does not lie in a periodic foliation, so it must traverse the heights of every cylinder in the cylinder decomposition before it reaches z_3. This implies that the total height of the cylinders is less than ε. By contracting the waist of the cylinders in this direction to unit length, σ_1 contracts to a saddle connection of length 2ε. Since this argument holds for all ε, we converge to a degenerate surface with a zero of order at least three and reach a contradiction with Lemma 9.3.

Next we assume that the bottom of C_1 contains the zeros z_1, z_5, and z_2. In this case, it is clear that the surface decomposes into at most two cylinders. Furthermore, (X, ω) cannot consist of exactly one cylinder because z_3 would lie on its top and that would imply that the height of C_1 is ε while its circumference is one, which would contradict that the area of the surface is one. Though there are two cylinders, this argument shows that one of them, say C_2 has height ε because both cylinders have z_1 and z_3 on different sides and since the distance between them is ε, the height of the cylinder must be less than ε. Lemma 9.5 implies that both z_1 and z_3 must converge to z_2 and z_4, simultaneously and respectively, (though we make no claims about the rates at which this happens) because otherwise we would have a contradiction with Lemma 9.5. However, as we consider the sequence of surfaces resulting from letting ε vary over a sequence decreasing to zero, we get that the cylinder C_2 must vanish in the limit so that (X', ω') lies in $\mathcal{H}(2^3)$ and consists of one cylinder. However, this directly contradicts the fact that (M_4, ω_{M_4}) decomposes into exactly two cylinders in every direction. This contradiction implies that any Teichmüller disc satisfying the assumptions of this lemma is generated by a surface satisfying topological dichotomy. □

Lemma 9.7. *There are no Teichmüller discs contained in $\mathcal{H}(1^4, 2) \cap \mathcal{D}_4(1)$.*

Proof. As in the proof of Lemma 9.5, we show that if $(X, \omega) \in \mathcal{H}(1^4, 2)$ generates a Teichmüller disc in $\mathcal{D}_4(1)$, then (X, ω) is uniformly completely periodic. By [31], (X, ω) is a Veech surface and by [30], there are no Veech surfaces in $\mathcal{H}(1^4, 2)$ that generate a Teichmüller disc in $\mathcal{D}_4(1)$.

By contradiction, assume that there exists a surface $(X, \omega) \in \mathcal{H}(1^4, 2)$ generating a Teichmüller disc in $\mathcal{D}_4(1)$. By [30], (X, ω) is not a Veech surface and by Lemma 9.6, (X, ω) satisfies topological dichotomy. As in the proof of the previous lemma, only simple zeros can converge. Let $z_1, \dots, z_4$ denote the simple zeros, and let z_5 denote the double zero. Without loss of generality, let σ be a saddle connection between two such zeros z_1 and z_3 of length $\varepsilon > 0$. By Lemma 9.6, consider a cylinder decomposition $\mathcal{C}$ of (X, ω) such that z_1 and z_3 lie on the bottom of $C_1 \in \mathcal{C}$. It is possible to choose C_1 so that the double zero z_5 lies on its top because z_5 must lie at the top of some cylinder, and the cylinder will have simple zeros on its bottom of distance ε. It was noted in the previous proof that there must always be two pairs of simple zeros, say z_1, z_3 and z_2, z_4, such that each zero in the pair has distance ε

from the other zero in the pair because the pairs of simple zeros must converge to double zeros simultaneously. Consider the saddle connection σ' from z_1 to z_5. We can take σ' to have length less than two because the total height of all the cylinders is one and σ' connects the top and bottom of a single cylinder. Then σ' lies on the top of a cylinder C_1' in a different cylinder decomposition $\mathcal{C}'$ of (X,ω). The top of C_1' must contain exactly one of z_2, z_3 or z_4 because the total order of the zeros on the top of C_1' is even. If it contains z_3, then the total height of the cylinders in $\mathcal{C}'$ is less than ε. We claim that if it contains either z_2 or z_4, then the total height of the cylinders in $\mathcal{C}'$ is at most 2ε. To see this, note that z_1, z_5, and say z_2, without loss of generality, lie on the top of C_1'. Then z_3 and z_4 must lie on the bottom of C_1'. Hence, C_1' has height at most ε. If the height of C_2' is bounded away from zero by a constant $C > 0$, for all $\varepsilon > 0$, then as ε tends to zero, we get a sequence converging to a surface that must be (M_4, ω_{M_4}), but with a cylinder decomposition consisting of exactly one cylinder. This contradicts the fact that every cylinder decomposition of (M_4, ω_{M_4}) has two cylinders, so C_2' must have height ε'.

We abuse notation and set $\varepsilon = \max(\varepsilon, \varepsilon')$. Furthermore, $\mathcal{C}'$ consists of at most two cylinders because the total order of the zeros along the top of one of the cylinders is four. The total area of the cylinders is one, so the waist length of the cylinders in $\mathcal{C}'$ must be at least $1/(2\varepsilon)$. Act by the Teichmüller geodesic flow so that the circumference of the cylinders in $\mathcal{C}'$ is reduced to one and the total height of the cylinders is expanded to one. In the process of the expansion and contraction, the saddle connection σ' of length at most two is contracted to length at most 4ε. Since this argument holds for all $\varepsilon > 0$, σ' can be contracted to a point resulting in a zero of order three. By Lemma 9.1, the surface will not degenerate and we get a surface generating a Teichmüller disc in a stratum that does not contain a Teichmüller disc. This shows (X,ω) must be uniformly completely periodic and yields the desired contradiction. □

Let

$$H_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

denote the horocycle flow.

Lemma 9.8. *If (X,ω) generates a Teichmüller disc D in $\mathcal{H}(1^6) \cap \mathcal{D}_4(1)$, then (X,ω) satisfies topological dichotomy.*

Proof. Let z_i, for $1 \le i \le 6$, denote the simple zeros of ω. Assume by contradiction that (X,ω) does not satisfy topological dichotomy. Let σ_1 be a saddle connection from z_1 to z_2, without loss of generality, that does not lie in a periodic foliation. If σ_1 converges to a point, then the other zeros must also converge to each other in pairs because there are no Teichmüller discs in $\mathcal{D}_4(1)$ in any lower stratum other than $\mathcal{H}(2^3)$ by Lemmas 9.5 and 9.7. Setting notation, let z_3 and z_5 converge to z_4

and z_6, respectively. Let $d_\omega(\cdot,\cdot)$ denote flat length with respect to ω. We assume that

$$\varepsilon = \max_{i\in\{1,3,5\}} \{d_\omega(z_i, z_{i+1})\},$$

and the length of σ_1 is at most ε, and (X,ω) admits a cylinder decomposition into cylinders of unit circumference. We define a sequence of surfaces (X_n,ω_n) converging to $(X',\omega') = (M_4,\omega_{M_4})$ letting $\varepsilon = 1/n$. For each (X_n,ω_n) fix a cylinder decomposition $\mathcal{C}_n$ such that the cylinders have unit circumference. Pass to a subsequence, such that $\mathcal{C}_n$ has the same number of cylinders as $\mathcal{C}_m$, for all $n,m \geq 0$.

First we claim that the cylinder decompositions $\mathcal{C}_n$ do not consist of exactly one cylinder. Assume by contradiction that it does consist of exactly one cylinder. Since σ_1 does not lie in a periodic foliation, σ_1 must traverse the height of the cylinder. However, this would imply that the height of the cylinder is at most $1/n$ while the circumference is one, which contradicts the fact that the area of each surface in the sequence is one.

Secondly, we claim that the cylinder decompositions $\mathcal{C}_n$ do not consist of exactly two cylinders. To see this, we use the same argument as above to see that if there are two cylinders, then one of them must have height at most $1/n$. As we let n tend to infinity, the surface converges to a surface (X',ω'), which must have a single cylinder because the height of one of the two cylinders in (X_n,ω_n) converged to zero. However, (M_4,ω_{M_4}) decomposes into two cylinders in every periodic direction so we have a contradiction that implies that there cannot be two cylinders.

Finally, we assume that for all n, $\mathcal{C}_n$ consists of exactly three cylinders, the maximum possible by Lemma 9.2. The saddle connection σ_1 cannot lie in the foliation of the cylinder of $\mathcal{C}_n$ because it does not lie in a periodic foliation. Therefore, z_1 and z_2 lie on the top and bottom of a cylinder, say C_3. By the assumption that there are three cylinders, there must be another pair of zeros between the top and bottom of C_3. If not, the total height of the three cylinders would be at most $3/n$, which would contradict the assumption that the surface has area one, for large n. Therefore, we have that the saddle connection σ_3 from z_5 to z_6 of length at most ε lies on the top of the cylinder C_1. This arrangement of the zeros must hold for all n in the sequence $\{(X_n,\omega_n)\}_n$ because this argument did not depend on the value of n.

Now we make an elementary observation. If we consider the action of H_s on (X_n,ω_n), then the heights and boundaries of the three cylinders in $\mathcal{C}_n$ are preserved, though the cylinders themselves are twisted (in the sense of Dehn twists). This implies that the saddle connection σ_3 is preserved under the action of H_s, while the distance between z_1 and z_2 can be increased to some constant bounded away from zero. Therefore, for each n, there exists a number s_n, where $0 < s_n < 1$ such that the sequence $\{H_{s_n}\cdot(X_n,\omega_n)\}_n$ converges to a surface which does not degenerate because the cylinders have circumference one. However, at least one pair of simple zeros remain simple zeros in the limit, while at least one pair of simple zeros converge to a double zero. This contradicts either Lemma 9.5 or 9.7, and implies that the arrangement of the cylinders and the zeros described above for the case where $\mathcal{C}_n$ consists of three cylinders cannot occur. However, since this was the only remaining potentially admissible arrangement of the zeros in such a cylinder decomposition, we have a contradiction which implies (X,ω) satisfies topological dichotomy. □

Lemma 9.9. *There are no Teichmüller discs contained in* $\mathcal{H}(1^6)\cap\mathcal{D}_4(1)$.

Proof. The idea of this proof is identical to Lemmas 9.5 and 9.7. Assume by contradiction that such a surface (X,ω) exists. By Lemma 9.8, (X,ω) satisfies topological dichotomy, but [30] implies that (X,ω) is not uniformly completely periodic. We show that there is a sequence of surfaces in the Teichmüller disc D generated by (X,ω) converging to a surface in a stratum other than $\mathcal{H}(2^3)$.

As in the proof of the previous lemma, let z_i denote the simple zeros of ω, for $1\le i\le 6$. By Theorem 7.4, there is a sequence of surfaces $\{(X_n,\omega_n)\}_n$ in D converging to (M_4,ω_{M_4}). For $i=1,3,5$, we can assume that z_i converges to z_{i+1} in this sequence because there are no Teichmüller discs in $\mathcal{D}_4(1)$ in any lower stratum other than $\mathcal{H}(2^3)$ by Lemmas 9.5 and 9.7. Let $d_\omega(\cdot,\cdot)$ denote flat length with respect to ω. As before, assume that

$$\varepsilon=\max_{i\in\{1,3,5\}}\{d_\omega(z_i,z_{i+1})\},$$

and (X,ω) admits a cylinder decomposition into cylinders of unit circumference. We define a sequence of surfaces $\{(X_n,\omega_n)\}_n$ converging to $(X',\omega')=(M_4,\omega_{M_4})$ by letting $\varepsilon=1/n$. For each (X_n,ω_n) fix a cylinder decomposition $\mathcal{C}_n$ such that the cylinders have unit circumference and pass to a subsequence, such that $\mathcal{C}_n$ has the same number of cylinders as $\mathcal{C}_m$, for all $n,m\ge 0$.

First note that $\mathcal{C}_n$ cannot contain exactly one cylinder, for all n, because the sequence converges to a surface that decomposes into two cylinders in every periodic direction. If we assume that $\mathcal{C}_n$ splits into three cylinders, then there are two possible arrangements of the zeros. Either one or more of the saddle connections of length at most ε lies between the top and bottom of a cylinder, or, after renaming the zeros, σ_i lies on the top of C_i, for $1\le i\le 3$. If one or more of the saddle connections lies across a cylinder, then we have the same arrangement as in the proof of Lemma 9.8: C_1 has the saddle connection σ_3 along its top, and C_3 has z_1 and z_3 on its bottom and z_2 and z_4 on its top. In fact, to exclude the possibility of this case from occurring, it suffices to use the "horocycle trick" from the previous lemma to get a contradiction. Therefore, we are left with the other case where σ_i lies on the top of C_i, for all i. Since we know that the limit (M_4,ω_{M_4}) decomposes into two cylinders, one of the three cylinders, say C_3 must have height h_n converging to zero. Let σ_2 lie on the bottom of C_3, and σ_3 lie on the top of C_3. Let $(X_n',\omega_n'):=H_{s_n}\cdot(X_n,\omega_n)$. For each n, there is a number s_n satisfying $0 \leq s_n \leq 1$ such that the

$$d_{\omega'_n}(z_3, z_5) = h_n.$$

As n tends to infinity, the limit must lie in the stratum $\mathcal{H}(2,4)$, which does not contain a Teichmüller disc in $\mathcal{D}_4(1)$ by Lemma 9.3. This contradiction implies that the the cylinder decomposition $\mathcal{C}_n$ must consist of exactly two cylinders for all n.

Finally, we assume that $\mathcal{C}_n$ contains exactly two cylinders, for all n. Consider sufficiently large n so that $1/n << 1$. The heights of the cylinders must be bounded away from zero so that the sequence converges to a surface with two cylinders. This implies that the three saddle connections σ_i, for $1 \leq i \leq 3$, of length less than $1/n$ lie on the boundaries of the cylinders, C_1 and C_2, i.e. the short saddle connections are parallel. Consider a straight trajectory γ of length less than two from z_1 on the bottom of C_1 to itself on the top of C_2. Such a trajectory can be found by considering a saddle connection from a double zero to itself at the limit (X', ω') and using this saddle connection to find a saddle connection between a simple zero and itself on (X_n, ω_n) for sufficiently large n. We permit γ to pass through another zero. By Lemma 9.8, this saddle connection determines a periodic foliation, thus, a cylinder decomposition $\mathcal{C}'_n$. We claim that the total height of the cylinders in $\mathcal{C}'_n$ is at most $3/n$. We consider three cases. If $\mathcal{C}'_n$ consists of exactly one cylinder, then this is clear because there is a saddle connection of length less than $1/n$ transverse to the foliation. If $\mathcal{C}'_n$ consists of exactly two cylinders, then there is at least one cylinder of height at most $1/n$. In fact, both cylinders must have height at most $1/n$ because each cylinder has four zeros on one side and two zeros on the other, which implies that one of the saddle connections of length at most $1/n$ must traverse the heights of both cylinders. Finally, if $\mathcal{C}'_n$ consists of exactly three cylinders, then each cylinder has two zeros on each side. Since σ_i does not lie in the foliation of $\mathcal{C}'_n$ for all i, every cylinder has height at most $1/n$.

If we form a new sequence of surfaces in D by acting on the foliation $\mathcal{C}'_n$ by the Teichmüller geodesic flow so that the cylinders have unit circumference, then the curve γ must have length at most $6/n$ in this new sequence. However, as n tends to infinity, this would imply that a curve from z_1 to itself contracts to a point. This forces the surface to degenerate because z_1 can no longer be a zero in the limit. This directly contradicts Lemma 9.1 and implies that there is no surface generating a Teichmüller disc in $\mathcal{D}_4(1)$ in the principal stratum. $\square$

We summarize Lemmas 9.3, 9.5, 9.7, and 9.9 in the following theorem.

Theorem 9.10. *The Ornithorynque (M_4, ω_{M_4}) generates the only Teichmüller disc in $\mathcal{D}_4(1)$.*

References

[1] A. Avila, C. Matheus, and J.-C. Yoccoz. $SL(2, \mathbb{R})$-invariant probability measures on the moduli spaces of translation surfaces are regular. *Geom. Funct. Anal.*, **23** (2013), no. 6, 1705–1729. Zbl 06259094 MR 3132901

[2] A. Avila and M. Viana. Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture. *Acta Math.*, **198** (2007), no. 1, 1–56. Zbl 1143.37001 MR 2316268

[3] M. Bainbridge. Euler characteristics of Teichmüller curves in genus two. *Geom. Topol.*, **11** (2007), 1887–2073. Zbl 1131.32007 MR 2350471

[4] M. Bainbridge and M. Möller. The Deligne-Mumford compactification of the real multiplication locus and Teichmüller curves in genus 3. *Acta Math.*, **208** (2012), no. 1, 1–92. Zbl 1250.14014 MR 2910796

[5] D. Chen and M. Möller. Nonvarying sums of Lyapunov exponents of Abelian differentials in low genus. *Geom. Topol.*, **16** (2012), no. 4, 2427–2479. Zbl 1266.14018 MR 3033521

[6] Y. Cheung, P. Hubert, and H. Masur. Topological dichotomy and strict ergodicity for translation surfaces. *Ergodic Theory Dynam. Systems*, **28** (2008), no. 6, 1729–1748. Zbl 1167.37005 MR 2465598

[7] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, **36** (1969), 75–109. Zbl 0181.48803 MR 262240

[8] A. Eskin, M. Kontsevich, and A. Zorich. Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow. *Publ. Math. Inst. Hautes Études Sci.*, **120** (2014), 207–333. Zbl 1305.32007 MR 3270590

[9] A. Eskin and M. Mirzakhani. Invariant and stationary measures for the $\mathrm{SL}(2, \mathbb{R})$ action on moduli space. *preprint*, 2013. arXiv:1302.3320

[10] H. M. Farkas and I. Kra. *Riemann surfaces*, Graduate Texts in Mathematics, 71. Springer-Verlag, New York, 1980. Zbl 0475.30001 MR 583745

[11] G. Forni. Deviation of ergodic averages for area-preserving flows on surfaces of higher genus. *Ann. of Math. (2)*, **155** (2002), no. 1, 1–103. Zbl 1034.37003 MR 1888794

[12] G. Forni. On the Lyapunov exponents of the Kontsevich-Zorich cocycle in *Handbook of dynamical systems*, 1B, 549–580. Elsevier B. V., Amsterdam, 2006. Zbl 1130.37302 MR 2186248

[13] G. Forni and C. Matheus. An example of a Teichmüller disk in genus 4 with degenerate Kontsevich-Zorich spectrum. *preprint*, 2008. arXiv:0810.0023

[14] G. Forni, C. Matheus, and A. Zorich. Square-tiled cyclic covers. *J. Mod. Dyn.*, **5** (2011), no. 2, 285–318. Zbl 1276.37021 MR 2820563

[15] G. Forni, C. Matheus, and A. Zorich. Lyapunov spectrum of invariant subbundles of the Hodge bundle. *Ergodic Theory Dynam. Systems*, **34** (2014), no. 2, 353–408. Zbl 1290.37002 MR 3233697

[16] E. Gutkin and C. Judge. Affine mappings of translation surfaces: geometry and arithmetic. *Duke Math. J.*, **103** (2000), no. 2, 191–213. Zbl 0965.30019 MR 1760625

[17] J. Harris and I. Morrison. *Moduli of curves*, Graduate Texts in Mathematics, 187. Springer-Verlag, New York, 1998. Zbl 0913.14005 MR 1631825

[18] F. Herrlich and G. Schmithüsen. An extraordinary origami curve. *Math. Nachr.*, **281** (2008), no. 2, 219–237. Zbl 1159.14012 MR 2387362

[19] J. Hubbard and H. Masur. Quadratic differentials and foliations. *Acta Math.*, **142** (1979), no. 3–4, 221–274. Zbl 0415.30038 MR 523212

[20] Y. Imayoshi and M. Taniguchi. *An introduction to Teichmüller spaces*. Translated and revised from the Japanese by the authors. Springer-Verlag, Tokyo, 1992. Zbl 0754.30001 MR 1215481

[21] M. Kontsevich. Lyapunov exponents and Hodge theory in *The mathematical beauty of physics (Saclay, 1996)*, Adv. Ser. Math. Phys., 24, 318–332. World Sci. Publ., River Edge, NJ, 1997. Zbl 1058.37508 MR 1490861

[22] M. Kontsevich and A. Zorich. Connected components of the moduli spaces of Abelian differentials with prescribed singularities. *Invent. Math.*, **153** (2003), no. 3, 631–678. Zbl 1087.32010 MR 2000471

[23] A. Lenzhen and K. Rafi. Length of a curve is quasi-convex along a Teichmüller geodesic. *J. Differential Geom.*, **88** (2011), no. 2, 267–295. Zbl 1243.30089 MR 2838267

[24] B. Maskit. Comparison of hyperbolic and extremal lengths. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **10** (1985), 381–386. Zbl 0587.30043 MR 802500

[25] H. Masur. On a class of geodesics in Teichmüller space. *Ann. of Math. (2)*, **102** (1975), no. 2, 205–221. Zbl 0322.32010 MR 385173

[26] H. Masur. Extension of the Weil-Petersson metric to the boundary of Teichmuller space. *Duke Math. J.*, **43** (1976), no. 3, 623–635. Zbl 0358.32017 MR 417456

[27] H. Masur. Closed trajectories for quadratic differentials with an application to billiards. *Duke Math. J.*, **53** (1986), no. 2, 307–314. Zbl 0616.30044 MR 850537

[28] C. Matheus and J.-C. Yoccoz. The action of the affine diffeomorphisms on the relative homology group of certain exceptionally symmetric origamis. *J. Mod. Dyn.*, **4** (2010), no. 3, 453–486. Zbl 1220.37004 MR 2729331

[29] C. T. McMullen. Dynamics of $SL_2(\mathbb{R})$ over moduli space in genus two. *Ann. of Math. (2)*, **165** (2007), no. 2, 397–456. Zbl 1131.14027 MR 2299738

[30] M. Möller. Shimura and Teichmüller curves. *J. Mod. Dyn.*, **5** (2011), no. 1, 1–32. Zbl 1221.14033 MR 2787595

[31] J. Smillie and B. Weiss. Characterizations of lattice surfaces. *Invent. Math.*, **180** (2010), no. 3, 535–557. Zbl 1195.57041 MR 2609249

[32] K. Strebel. *Quadratic differentials*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 5. Springer-Verlag, Berlin, 1984. Zbl 0547.30001 MR 743423

[33] W. A. Veech. Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Invent. Math.*, **97** (1989), no. 3, 553–583. Zbl 0676.32006 MR 1005006

[34] W. A. Veech. The Teichmüller geodesic flow. *Ann. of Math. (2)*, **124** (1986), no. 3, 441–530. Zbl 0658.32016 MR 866707

[35] W. A. Veech. Geometric realizations of hyperelliptic curves in *Algorithms, fractals, and dynamics (Okayama/Kyoto, 1992)*, 217–226. Plenum, New York, 1995. Zbl 0859.30039 MR 1402493

[36] Ya. B. Vorobets. Plane structures and billiards in rational polygons: the Veech alternative. *Uspekhi Mat. Nauk*, **51** (1996), no. 5(311), 3–42. Zbl 0897.58029 MR 1436653

[37] S. A. Wolpert. On families of holomorphic differentials on degenerating annuli in *Quasiconformal mappings, Riemann surfaces, and Teichmüller spaces*, 363–370. Contemp. Math., 575. Amer. Math. Soc., Providence, RI, 2012. Zbl 1267.32011 MR 2933910

[38] S. A. Wolpert. Infinitesimal deformations of nodal stable curves. *Adv. Math.*, **244** (2013), 413–440. Zbl 1290.14019 MR 3077878

[39] A. Zorich. How do the leaves of a closed 1-form wind around a surface? in *Pseudoperiodic topology*, 135–178. Amer. Math. Soc. Transl. Ser. 2, 197. Amer. Math. Soc., Providence, RI, 1999. Zbl 0976.37012 MR 1733872

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