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Immersions associated with holomorphic germs

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Abstract. A holomorphic germ $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, singular only at the origin, induces at the links level an immersion of S^3 into S^5. The regular homotopy type of immersions $S^3 \looparrowright S^5$ are determined by their Smale invariant, defined up to a sign ambiguity. In this paper we fix a sign of the Smale invariant and we show that for immersions induced by holomorphic gems the sign-refined Smale invariant Ω is the negative of the number of cross caps appearing in a generic perturbation of Φ. Using the algebraic method we calculate Ω for some families of singularities, among others the A-D-E quotient singularities. As a corollary, we obtain that the regular homotopy classes which admit holomorphic representatives are exactly those, which have non-positive sign-refined Smale invariant. This answers a question of Mumford regarding exactly this correspondence. We also determine the sign ambiguity in the topological formulae of Hughes–Melvin and Ekholm–Szűcs connecting the Smale invariant with (singular) Seifert surfaces. In the case of holomorphic realizations of Seifert surfaces, we also determine their involved invariants in terms of holomorphic geometry.

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1. Introduction

1.1. Let S^n denote the n–sphere, the boundary of the unit ball in $\mathbb{R}^{n+1}$. The regular homotopy classes of immersions $f : S^3 \looparrowright S^5$, denoted by $\mathrm{Imm}(S^3, S^5)$, are identified with the elements of $\pi_3(V_3(\mathbb{R}_5)) \cong \pi_3(SO(5)) \cong \mathbb{Z}$ by the Hirsch-Smale theory [9, 22]. The correspondence is given by the *Smale invariant* $\Omega(f)$ of an immersion f. Besides the original definition of Smale [22], there are several equivalent definitions of $\Omega(f)$ (see [9, 10, 25]). Usually, in all these constructions there is no identification of a distinguished generator of $\pi_3(SO(5))$, hence the Smale invariant is well–defined only up to a sign.

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The subgroup $\mathrm{Emb}(S^3, S^5)$ of $\mathrm{Imm}(S^3, S^5)$ consists of the regular homotopy classes which admit embedding representatives. By [10] this is the subgroup $24\cdot\mathbb{Z} \subset \mathbb{Z} = \mathrm{Imm}(S^3, S^5)$. For embeddings the Smale invariant has the following alternative definition too, given by Hughes and Melvin. Let M^4 be a 'Seifert surface' in S^5 of $f(S^3)$, then $2\Omega(f)/3$ is the signature of F (up to a sign), cf. [10]. (This and similar identities will be reviewed in Section 8.)

Our goal is to analyse the complex analytic realizations of the elements of the above two groups. Let $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a holomorphic germ. We assume that Φ is singular only at the origin, that is $\{z : \mathrm{rank}(d\Phi_z) < 2\} \subset \{0\}$ in a small representative of $(\mathbb{C}^2, 0)$. Such a germ, at the level of links of the spacegerms $(\mathbb{C}^2, 0)$ and $(\mathbb{C}^3, 0)$, provides an immersion $f : S^3 \looparrowright S^5$ (see 2.1). If an element of $\mathrm{Imm}(S^3, S^5)$, or $\mathrm{Emb}(S^3, S^5)$ respectively, can be realized (up to regular homotopy) by such an immersion, we call it *holomorphic*. The corresponding subsets will be denoted by $\mathrm{Imm}_{hol}(S^3, S^5)$ and $\mathrm{Emb}_{hol}(S^3, S^5)$ respectively.

As we will see, $\mathrm{Imm}_{hol}(S^3, S^5)$ is not symmetric with respect to a sign change of $\mathbb{Z}$, hence, in order to identify the subset $\mathrm{Imm}_{hol}(S^3, S^5)$ without any sign-ambiguity, we will fix a 'canonical' generator of $\pi_3(SO(5))$. This will be done via the ismorphisms $\pi_3(U(3)) \to \pi_3(SO(6)) \to \pi_3(SO(5))$ and by fixing a canonical generator in $\pi_3(U(3))$ (see 4.2). Sometimes, to emphasize that we work with the Smale invariant with this fixed sign convention, we refer to it as the *sign–refined Smale invariant*. Our second goal is to determine the correct signs (compatibly with the above choice of generators) in the existing topological formulas, which were stated only up to a sign–ambiguity.

1.2. The set $\mathrm{Imm}_{hol}(S^3, S^5)$. One expects that the analytic geometry of holomorphic realization imposes some rigidity restrictions, and also provides some further connections with the properties of complex analytic spaces. Mumford already in 1961 in his seminal article [20] asked for the characterization of the Smale invariant of a holomorphic (algebraic) immersion in terms of the analytic/algebraic geometry. This article provides a complete answer to his question. A more precise formulation of our guiding questions are:

Question 1.2.1.

(a) *Which are the regular homotopy classes* $\mathrm{Imm}_{hol}(S^3, S^5)$ *and* $\mathrm{Emb}_{hol}(S^3, S^5)$ *represented by holomorphic germs?*

(b) *How can a certain regular homotopy class be identified via complex singularity theory, that is, via algebraic or analytic invariants of the involved analytic spaces? Furthermore, if some* Φ *realizes some Smale invariant (e.g., if its Smale invariant is zero), then what kind of specific analytic properties* Φ *must have?*

The main results of this paper provide the following answer in the case of immersions.

Theorem 1.2.2.

(a) $\mathrm{Imm}_{hol}(S^3, S^5)$ *is identified via the sign–refined Smale invariant* $\Omega(f)$ *by the set of non–positive integers.*

(b) *If the immersion* f *is induced by the holomorphic germ* Φ*, then* $\Omega(f) = -C(\Phi)$*, where* $C(\Phi)$ *is the number of cross cap points (complex Whitney umbrellas, or pinch points) of a generic perturbation of* Φ*.* $C(\Phi)$ *can be calculated in an algebraic way, as the codimension of the ideal generated by the determinants of the* 2×2*-minors of the Jacobian matrix of* Φ*.*

The main tool of the proof of Theorem 1.2.2 is the concept of *complex Smale invariant* of the germ Φ. We introduce it in Section 3 and then we prove that it agrees with $C(\Phi)$. Next, in Section 6 we identify the complex Smale invariant of a germ Φ with the (classical) Smale invariant of the link of Φ. The proof of the part (b) of Theorem 1.2.2 is then ready up to sign. In 4.2 we fix explicit generators of the groups $\pi_3(U)$ and $\pi_3(SO)$ and calculate the homomorphism between them. With this convention the complex Smale invariant of Φ is equal to $C(\Phi)$ and is opposite to the sign–refined Smale invariant.

Part (b) of Theorem 1.2.2 implies that the sign–refined Smale invariant of a complex analytic realization is always non–positive. The proof of part (a) is then completed by Example 7.1.1, which provides analytic representatives for all non–positive $\Omega(f)$.

Note that in the present literature the known (C^∞) realizations of certain Smale invariants $\Omega(f)$ are rather involved (similarly, as the computation of $\Omega(f)$ for any concrete f), see e.g. [11, 2]. Here we provide very simple polynomial maps realizing all non–positive Smale invariants. Furthermore, the computation of $C(\Phi)$ for any Φ is extremely simple.

Moreover, precomposing the above complex realizations with the C^∞ reflection $(s, t) \mapsto (s, \bar{t})$, we get explicit representatives for all positive Smale numbers as well, compare [2, Lemma 3.4.2.].

1.3. The set $\mathrm{Emb}_{hol}(S^3, S^5)$**.** Recall that $\mathrm{Emb}_{hol}(S^3, S^5)$ consists of regular homotopy classes (that is, sign–refined Smale invariants in $\mathbb{Z}$) represented by holomorphic germs Φ whose induced immersions $S^3 \looparrowright S^5$ might not be embeddings, but are regular homotopic with embeddings.

A more restrictive subset consists of those regular homotopy classes (Smale invariants), which can be represented by holomorphic gems, whose restrictions off origin are embeddings.

Theorem 1.3.1.

(a) $\mathrm{Emb}_{hol}(S^3, S^5) = (24 \cdot \mathbb{Z}) \cap \mathbb{Z}_{\leq 0}$.

(b) *Assume that the immersion* f *is the restriction at links level of a holomorphic germ* Φ *as above,* $f = \Phi|_{S^3}$*. Then the following facts are equivalent:*

(1) $\operatorname{rank} d\Phi_0 = 2$ *(hence* Φ *is not singular),*

(2) $\Omega(f) = 0$,

(3) $f : S^3 \hookrightarrow S^5$ *is an embedding,*

(4) $f : S^3 \hookrightarrow S^5$ *is the trivial embedding.*

Again, we wish to emphasize that the previous construction of the generator of $24 \cdot \mathbb{Z} = \operatorname{Emb}(S^3, S^5)$ (that is, of a smooth embedding with $\Omega(f) = \pm 24$) is complicated, it is more existential than constructive [10]. On the other hand, by our complex realizations, for any given $\Omega(f) \in 24 \cdot \mathbb{Z}$ we provide several easily defined germs, which are immersions, and are regular homotopic with embeddings. Moreover, part (b) says that it is impossible to find holomorphic representatives Φ such that $\Phi|_{S^3}$ is already embedding (except for $\Omega(f) = 0$).

The essential parts of Theorem 1.3.1(b) are the implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1), which conclude an analytic statement from topological ones. The proof (2) $\Rightarrow$ (1) is based on Theorem 1.2.2, which recovers the vanishing of the analytic invariant $C(\Phi)$ from the 'topological vanishing' $\Omega(f) = 0$.

A possible proof of ($\Phi|_{S^3}$ embedding) $\Rightarrow$ ($\operatorname{rank} d\Phi_0 = 2$) is based on a deep theorem of Mumford, which says that if the link of a complex normal surface singularity is S^3 then the germ should be non–singular [20]. We will provide two other possible proofs too: one of them is based on Mond's Theorem 2.2.1, the other on a theorem of Ekholm–Szűcs [3] (a generalization of the already mentioned Hughes–Melvin resuls [10]). These theorems will be discussed in connection with properties of Seifert surfaces in Section 8 as well.

1.4. The literature of singular analytic germs $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is huge with several deep and interesting results and invariants, see e.g. the articles of D. Mond and V. Goryunov [6, 7, 8, 16, 17, 18] and the references therein. In singularity classifications finitely determined or finite codimensional germs are central (with respect to some equivalence relation). For germs $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ Mond proved that the finiteness of the (holomorphic) right-left codimension is equivalent with the finiteness of three invariants: the number of (virtual) cross caps $C(\Phi)$, the number of (virtual) ordinary triple points $T(\Phi)$, and an other invariant $N(\Phi)$ measuring the non–transverse selfintersections [17]. This is more restrictive than our assumption $\{z : \operatorname{rank}(d\Phi_z) < 2\} = \{0\}$, which requires the finiteness of $C(\Phi)$ only.

However, it is advantageous to consider this larger class, since there are many key families of germs with infinite right-left codimension, but with finite $C(\Phi)$, and they produce interesting connections with other areas as well (see e.g. the next example).

Example 1.4.1. Consider a simple hypersurface singularity $(X, 0) \subset (\mathbb{C}^3, 0)$ (that is, of type A–D–E). They are quotient singularities, that is $(X, 0) \simeq (\mathbb{C}^2, 0)/G$ for certain finite subgroup $G \subset GL(2, \mathbb{C})$. Let K be the link of $(X, 0)$ (e.g.,

it is a lens space for A-type), and consider the regular G–covering $S^3 \to K$. This composed with the inclusion $K \hookrightarrow S^5$ provides an *immersion* $S^3 \looparrowright S^5$. Hence, the universal cover of each A–D–E singularity automatically provides an element of $\mathrm{Imm}_{hol}(S^3, S^5)$, which usually have infinite right–left codimension. The corresponding Smale invariants are given in Section 7. E.g., $-\Omega(A_{n-1}) = n^2 - 1$, hence A_4 represents (up to regular homotopy) a generator of $24 \cdot \mathbb{Z} = \mathrm{Emb}(S^3, S^5)$.

Recently Kinjo, using the plumbing graphs of the links of A–D singularities and C^∞–techniques, constructed immersions with the same Smale invariants as our $-C(\Phi)$ [13]. Hence, the natural complex analytic maps $(\mathbb{C}^2, 0) \to (X, 0) \subset (\mathbb{C}^3, 0)$ provide analytic realizations of the C^∞ constructions of [13], and emphasize their distinguished nature.

1.5. Smale invariants and the geometry of Seifert surfaces. In Section 8 we review three major topological theorems, which recover the classical Smale invariant in terms of the geometry of their Seifert surfaces (namely the Hughes–Malvin theorem [10], and two theorems of Ekholm–Szűcs [3]). All of them carry the sign ambiguity of the Smale invariant (which sometimes is also caused by the nature of their proofs).

Section 9 has two goals. First, we will indicate the correct sign in all these formulae, whenever the Smale invariant is replaced by the sign–refined Smale invariant. Moreover, we also determine the Seifert type invariants in terms of $C(\Phi)$ and $T(\Phi)$, whenever the immersion is induced by a holomorphic germ Φ.

When f is a generic immersion, the invariant $L(f)$ of generic immersions introduced by Ekholm [2] is also expressed in terms of $C(\Phi)$ and $T(\Phi)$, namely $L(f) = C(\Phi) - 3T(\Phi)$.

1.6. C^∞-characterisation of $C(\Phi)$ and $T(\Phi)$. The formulae connecting the holomorphic invariants $C(\Phi)$ and $T(\Phi)$ with C^∞-invariants $\Omega(f)$ and $L(f)$ have the following consequence.

Theorem 1.6.1. *Assume that the analytic germs Φ and $\Phi' : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ are C^∞ left-right equivalent (that is, $\Phi' = \Lambda \circ \Phi \circ \psi$ holds for some germs of orientation preserving diffeomorphisms $\psi : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ and $\Lambda : (\mathbb{R}^6, 0) \to (\mathbb{R}^6, 0)$). Then*

(a) $C(\Phi) = C(\Phi')$.

(b) *If additionally the immersion f associated with Φ is a generic immersion, then $T(\Phi) < \infty$ and $T(\Phi) = T(\Phi')$.*

Part (a) of Theorem 1.6.1 is proved as Corollary 6.1.2, while part (b) in Remark 9.1.5.

We thank the anonymous referee for suggesting to us that such a consequence might follow from our characterisations of C and T in terms of $\Omega(f)$ and $L(f)$.

In fact, one might even ask for the topological analogue of Theorem 1.6.1: is it true that if $\Phi' = \Lambda \circ \Phi \circ \psi$ holds for some germs of orientation preserving *homeomorphisms* $\psi : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ and $\Lambda : (\mathbb{R}^6, 0) \to (\mathbb{R}^6, 0)$, then $C(\Phi) = C(\Phi')$ and $T(\Phi) = T(\Phi')$? In Remark 9.1.5 we show the following.

Corollary 1.6.2. *In the presence of a topological left–right equivalence as above, if f and f' are generic immersions then $L(f) = L(f')$, hence $C(\Phi) - 3T(\Phi) = C(\Phi') - 3T(\Phi')$.*

The full extension of Theorem 1.6.1 from C^∞ to topological category is obstructed by the following facts. Though we identify the analytic invariant $C(\Phi)$ with the smooth Smale invariant $\Omega(f)$, it is not known if $\Omega(f)$ is stable with respect to topological left–right equivalence. A possible way to prove this requires the extension of the formulae from Section 8 from smooth to more general Seifert surfaces, which exceeds the aims of the present note. We plan to return back to this point later in a different article. Also, we do not know how the topological left–right equivalence behaves with respect to analytic deformations used in Sections 8 and 9.

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2. Basic definitions and preliminary properties

2.1. The immersion associated with Φ. If $(X, 0)$ is a complex analytic germ with an isolated singularity $0 \in X$ then its link K can be defined as follows. Set a real analytic map $\rho : X \to [0, \infty)$ such that $\rho^{-1}(0) = \{0\}$. Then, for $\epsilon > 0$ sufficiently small, $K := \rho^{-1}(\epsilon)$ is an oriented manifold, whose isotopy class (in $X \setminus \{0\}$) is independent of the choices, cf. Lemma (2.2) and Proposition (2.5) of [15]. e.g., if $(X, 0)$ is a subset of $(\mathbb{C}^N, 0)$, then one can take the restriction of $\rho(z) = |z|$ (the norm of z). In this way, the link of $(\mathbb{C}^N, 0)$ is the sphere S^{2N-1}_ϵ. Nevertheless, the general definition is very convenient even if $(X, 0) = (\mathbb{C}^N, 0)$.

Let $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a holomorphic germ singular only at 0 (as in the introduction). Define $\rho : (\mathbb{C}^2, 0) \to [0, \infty)$ by $\rho(z) = |\Phi(z)|$. Since $\Phi^{-1}(0) = \{0\}$ (if $\Phi^{-1}(0)$ would be a positive dimensional germ, then along it the rank of $d\Phi_z$ would be < 2). Hence $\rho^{-1}(0) = \{0\}$ (in a small representative).

Lemma 2.1.1. *There exists an $\epsilon_0 > 0$ sufficiently small such that $\mathfrak{B}_\epsilon := \Phi^{-1}(\{z : |z| \leq \epsilon\})$ is a non-metric C^∞ closed ball around the origin of $\mathbb{C}^2$. Its boundary, $\Phi^{-1}(S^5_\epsilon)$ is canonically diffeomorphic to S^3 for any $\epsilon < \epsilon_0$. In fact, for $\tilde{\epsilon}$ with*

$0 < \tilde{\epsilon} \ll \epsilon$, *any standard metric sphere* $S^3_{\tilde{\epsilon}}$ *sits in* $\mathfrak{B}_\epsilon$, *and it is isotopic with* $\Phi^{-1}(S^5_\epsilon)$ *in* $\mathfrak{B}_\epsilon \setminus 0$.

In the sequel $\Phi^{-1}(S^5_\epsilon)$ and $S^3_{\tilde{\epsilon}} \subset \mathbb{C}^2$ will be identified. When it is important to differentiate them we will use the notation $\mathfrak{S}^3 := \Phi^{-1}(S^5_\epsilon)$. We write also $S^3 = S^3_{\tilde{\epsilon}}$ and $S^5 = S^5_\epsilon$.

Definition 2.1.2. We call the restriction $f = \Phi|_{\mathfrak{S}^3} : \mathfrak{S}^3 \looparrowright S^5$ the immersion associated with Φ. (It's regular homotopy class is independent of all the choices.)

2.2. The number of cross caps. Let $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a holomorphic germ singular only at 0. Consider a generic holomorphic deformation Φ_λ of $\Phi = \Phi_0$. The singular points of $\Phi_{\lambda \neq 0}$ are cross caps (or complex Whitney umbrellas), i.e. they have the local form $(s, t) \mapsto (s^2, st, t)$ in some local holomorphic coordinates. Their number does not depend on the deformation Φ_λ, it is an invariant of Φ, cf. [16, 17]. We denote it by $C(\Phi)$.

$C(\Phi)$ can be computed in an algebraic way as well. Let $M_j : Hom(\mathbb{C}^2, \mathbb{C}^3) \to \mathbb{C}$ denote the determinants of the three 2×2 minors ($j = 1, 2, 3$). Let J be the ideal of the local ring $\mathcal{O}_{\mathbb{C}^2,0}$ generated by the elements $M_j \circ d\Phi$, where $d\Phi$ is the complex Jacobian matrix. J has finite codimension exactly when Φ is immersion off the origin.

Theorem 2.2.1. [16, Proposition 1] $C(\Phi) = \dim_{\mathbb{C}} (\mathcal{O}_{\mathbb{C}^2,0}/J)$.

2.3. The number of triple points. If Φ_λ is a generic deformation as above, then the singular points of the *image* of $\Phi_{\lambda \neq 0}$ might have the following types: self-transversal double points, cross caps and triple points, cf. [16, 17]. The double point set has complex dimension 1, while triple points are isolated. If the codimension of the second fitting ideal of $\Phi_*(\mathcal{O}_{\mathbb{C}^2})$ is finite, say $T(\Phi)$, then the number of triple points $T(\Phi_{\lambda \neq 0})$ of $\Phi_{\lambda \neq 0}$ is independent of the deformation and λ, it is exactly $T(\Phi)$ (cf. [19, 16]).

2.4. The Smale-invariant. Let $f : S^3 \looparrowright \mathbb{R}^5$ be an immersion. Instead of the original definition of Smale [22] we adopt the construction of Hughes and Melvin for the *Smale invariant* of f, see [10], compare also with [25]. Let U be a tubular neighborhood of the standard $S^3 \subset \mathbb{R}^5$, and let $F : U \looparrowright \mathbb{R}^5$ be an orientation preserving immersion extending f, i.e. $F|_{S^3} = f$. Let TU be the tangent bundle of U. It inherits a global trivialization from the natural trivialization of $T\mathbb{R}^5$. In particular, there is a map (the Jacobian matrix)

$$dF|_U : U \to GL^+(5, \mathbb{R}).$$

Its homotopy class is the Smale invariant of f:

$$\Omega(f) = [dF|_{S^3}] \in \pi_3(SO(5)) \tag{2.4.1}$$

(via the homotopy equivalence induced by the inclusion $SO(5) \subset GL^+(5, \mathbb{R})$).

Remark 2.4.2. If G is a connected Lie group, or a factor of it by a closed connected subgroup, then $\pi_n(G)$ can be identified with the homotopy classes of the continuous maps $f : S^n \to G$ without any base point. Furthermore, for Lie groups, the group operation of $\pi_n(G)$ agrees with that induced by the pointwise multiplication in G; cf. [24, p. 88 and 89].

Proposition 2.4.3. *$\Omega(f)$ does not depend on the choice of U and F, it depends only on the regular homotopy class of f, and $\Omega : \mathrm{Imm}(S^3, \mathbb{R}^5) \to \pi_3(SO(5))$ is a bijection.*

Indeed, Smale proved that his original invariant gives a bijection between $\mathrm{Imm}(S^3, \mathbb{R}^5)$ and $\pi_3(V_3(\mathbb{R}^5))$, cf. [22], where $V_3(\mathbb{R}^5)$ denotes the real Stiefel manifold (the space of linear independent 3-frames of $\mathbb{R}^5$). Hughes and Melvin proved that their alternative definition (2.4.1) of the Smale invariant does not depend on the choice of F and agrees with the original Smale invariant through the natural group isomorphism $\pi_3(SO(5)) \to \pi_3(V_3(\mathbb{R}^5))$.

Note that the standard embedding $SO(5) \hookrightarrow SO$ induces a group isomorphism between $\pi_3(SO(5))$ and $\pi_3(SO)$. These groups are (a priori non–canonically) isomorphic to $\mathbb{Z}$.

We wish to emphasize the following facts regarding orientations of S^3 and $\mathbb{R}^5$ and their effects on the above definition. (This might serve also as a small guide for the next sections.)

Let us think about S^3 as the subset of $\mathbb{R}^4$, the boundary of the 4–ball B^4 in $\mathbb{R}^4$, or via embedding $\mathbb{R}^4 \subset \mathbb{R}^5$, as a subset of $\mathbb{R}^5$. We do not wish to fix any orientation on it as the orientation of ∂B^4 (that would depend on the convention how one defines the orientation of the boundary of an oriented manifold — called, say, 'boundary convention').

Note that in the above definition of the Smale invariant, not the orientation of S^3 is used, but the orientation of the tubular neighborhood $U \subset \mathbb{R}^5$ and the orientation of the target $\mathbb{R}^5$. Moreover, $\Omega(f)$ is unsensitive to the orientation change simultaneously in both $\mathbb{R}^5$. In this way we get an element $\Omega(f) \in [S^3, SO(5)]$, which is independent of the orientation of $\mathbb{R}^5$ and does not use any orientation of S^3. Furthermore, if we define (this will done in 4.2) a generator $[L]$ in $[S^3, SO(5)]$, using again only the embedding $S^3 \subset \mathbb{R}^5$ (and no other orientation data), then $\Omega(f)$ identifies with an element of $\mathbb{Z}$, such that its definition is independent of any orientations of S^3 and $\mathbb{R}^5$, hence also of the 'boundary convention'.

All our discussions are in this spirit (except Sections 8 and 9, where oriented Seifert surfaces are treated): we run orientation and 'boundary convention' free definitions and statements (associated with S^3, regarded as a subset of $\mathbb{R}^5$, and immersions $S^3 \looparrowright \mathbb{R}^5$).

However, if we fix a 'boundary convention', then S^3 (in $\mathbb{R}^5$) will get an orientation (as ∂B^4). Then, for any *oriented abstract* S^3, let us denote it by $\mathbf{S}^3$, and immersion $\mathbf{S}^3 \looparrowright \mathbb{R}^5$, we can define the Smale invariant $\Omega^a(f) \in \mathbb{Z}$ (here 'a'

refers to the 'abstract' $\mathbf{S}^3$) by identifying $\mathbf{S}^3$ with the embedded $S^3 \subset \mathbb{R}^5$ by an orientation preserving diffeomorphism and taking $\Omega(S^3 \to \mathbf{S}^3 \looparrowright \mathbb{R}^5)$. This $\Omega^a(f)$ depends on the 'boundary convention', since the identification $S^3 \to \mathbf{S}^3$ depends on it: changing the convention we change $\Omega^a(f)$ by a sign.

This point of view should be adapted when $\mathbf{S}^3$ will be the (oriented) boundary of an oriented Seifert surface. But till Section 8 we will focus on the first version, $\Omega(f)$.

Next, in the definition of $\Omega(f)$, one can replace $\mathbb{R}^5$ by S^5, where S^5 is the boundary of the ball in $\mathbb{R}^6$, and S^3 is embedded naturally in S^5. By taking a generic point $P \in S^5$ we identify $S^5 \setminus \{P\}$ with $\mathbb{R}^5$, and U will be replaced by a tubular neighborhood of S^3 in S^5. Then the previous definition of $\Omega(f)$ can be repeated for any immersion $S^3 \looparrowright S^5$ (where $S^3 \subset S^5$) providing an element $[S^3, SO(5)]$, which becomes an integer once a generator $[L]$ is constructed from the embedding $S^3 \subset S^5$. Again, this Smale invariant $\Omega(f)$ will be independent of the orientations of S^3 and S^5, hence of the 'boundary convention' as well.

For immersions defined in Subsection 2.1, $\mathfrak{S}^3$ evidently sits naturally in $\mathbb{C}^2 = \mathbb{R}^4$ (hence also in a certain $\mathfrak{S}^5 = S^5 \subset \mathbb{C}^3 = \mathbb{R}^6$, cf. 6.1). This together with definition 2.4.1 provide $\Omega(f)$ (which becomes an integer once $[L]$ will be constructed in 4.2).

3. The complex Smale invariant

3.1. In this section we define the *complex Smale invariant* $\Omega_{\mathbb{C}}(\Phi)$ for a holomorphic germ $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, singular only at 0. It will be the bridge between $C(\Phi)$ and $\Omega(f)$.

Definition 3.1.1. Consider the map (with target the complex Stiefel variety $V_2(\mathbb{C}^3)$):

$$d\Phi|_{S^3} : S^3 \to V_2(\mathbb{C}^3)$$

defined via the natural trivialization of the complex tangent bundles $T\mathbb{C}^2$ and $T\mathbb{C}^3$. By definition, the complex Smale invariant of Φ is the homotopy class:

$$\Omega_{\mathbb{C}}(\Phi) = [d\Phi|_{S^3}] \in \pi_3(V_2(\mathbb{C}^3)).$$

By the connectivity of the group of local coordinate transformations, $\Omega_{\mathbb{C}}(\Phi)$ is independent of the choice of local coordinates in $(\mathbb{C}^2, 0)$ and $(\mathbb{C}^3, 0)$.

Remark 3.1.2. The projection $U(3) \to V_2(\mathbb{C}^3)$ induces an isomorphism between $\pi_3(V_2(\mathbb{C}^3))$ and $\pi_3(U(3)) = \pi_3(U)$ (see e.g. [12]). Hence, if we choose a complex normal vector field N_Φ of Φ, then the map

$$(d\Phi, N_\Phi)|_{S^3} : S^3 \to GL(3, \mathbb{C})$$

represents $\Omega_{\mathbb{C}}(\Phi)$ in $\pi_3(GL(3, \mathbb{C})) = \pi_3(U(3)) = \pi_3(U)$.

A canonical choice of N_Φ could be the complex conjugate of the cross product of the partial derivatives of Φ:

$$N_\Phi(s,t) = \overline{\partial_s\Phi(s,t) \times \partial_t\Phi(s,t)}\,.$$

Remark 3.1.3. $\pi_3(U) \cong \mathbb{Z}$ and in 4.2 we identify them through a fixed isomorphism. In this way $\Omega_{\mathbb{C}}(\Phi)$ becomes a well-defined integer without any sign–ambiguity.

4. Distinguished generators and sign conventions

4.1. There is a natural map $\tau : U(3) \to SO(6)$, which replaces any entry $M_{ij} = a + bi$ of a matrix $M \in U(3)$ by the real 2×2–matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. A map $F : \mathbb{C}^3 \to \mathbb{C}^3$ can be regarded as a map $\tilde{F} : \mathbb{R}^6 \to \mathbb{R}^6$: if we denote by $z_j = x_j + iy_j$ $(j = 1,2,3)$ the coordinates of $\mathbb{C}^3$, then for the components of F and $\tilde{F}$ one has

$$F_j(z_1,z_2,z_3) = \tilde{F}_{2j-1}(x_1,y_1,x_2,y_2,x_3,y_3) + i\,\tilde{F}_{2j}(x_1,y_1,x_2,y_2,x_3,y_3).$$

Then $\tau(d_{\mathbb{C}}F) = d_{\mathbb{R}}\tilde{F}$ holds for the complex Jacobian of F and the real Jacobian of $\tilde{F}$.

Let $j : SO(5) \hookrightarrow SO(6)$ denote the inclusion. It is well known (see e.g. [12]) that

$$\pi_3(j) : \pi_3(SO(5)) \to \pi_3(SO(6)) \tag{4.1.1}$$

is an isomorphism.

Lemma 4.1.2. *The homomorphism $\pi_3(\tau) : \pi_3(U(3)) \to \pi_3(SO(6))$ is an isomorphism too.*

Proof. First, we provide a more conceptual proof, which does not identify distinguished generators. Both sides are in the stable range (see [12]), hence we can switch to the homomorphism $\pi_3(U) \to \pi_3(O)$ induced by the embedding $\tau : U \hookrightarrow O$. By (a proof of) Bott periodicity, the factor O/U is homotopically equivalent to the loopspace ΩO of O, cf. [1]. Hence $\pi_i(O/U) = \pi_i(\Omega O) = \pi_{i+1}(O) = 0$ for $i = 3$ and 4. Then the isomorphism follows from the homotopy exact sequence of the fibration $O \to O/U$ with fibre U. □

In 4.2 we will give another, more computational proof, where we will be able to fix distinguished generators for $\pi_3(U(3))$ and $\pi_3(SO(6))$, and via these generators we identify $\pi_3(\tau)$ with multiplication by -1.

4.2. Conventions and identifications. First, we identify $\mathbb{H}$ and $\mathbb{R}^4$ and $\mathbb{C}^2$ in the obvious way: we identify the quaternion $q = a + bi + cj + dk = z + wj \in \mathbb{H}$ with $(a, b, c, d) \in \mathbb{R}^4$ and with the complex pair $(z, w) \in \mathbb{C}^2$, where $z = a + bi$ and $w = c + di$. Also, we identify S^3 with the quaternions of unit length: $S^3 = \{q = a + bi + cj + dk \in \mathbb{H} \mid a^2 + b^2 + c^2 + d^2 = 1\}$.

Notation 4.2.1. We define the following maps. Set

$$u : S^3 \to U(2)\,,\, u_q = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix},$$

where $q = z + wj$. u_q is the (complex) matrix of the *right* (quaternionic) multiplication with q, that is, of the map $\mathbb{H} \to \mathbb{H}$, $p \mapsto pq$. Note that the left multiplication by q is not a complex unitary transformation, in general. Next, set

$$L : S^3 \to SO(4), \quad L_q = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix},$$

where $q = a + bi + cj + dk$. L_q is the (real) matrix of the *left* multiplication with q (i.e., of the map $\mathbb{H} \to \mathbb{H}$, $p \mapsto qp$).

Let $R : S^3 \to SO(4)$ be the map which assigns for a $q \in S^3$ the (real) matrix R_q of the righ multiplication with q (i.e., of the map $\mathbb{H} \to \mathbb{H}$, $p \mapsto pq$).

Let $\rho : S^3 \to SO(4)$ be the map which assigns for a $q \in S^3$ the (real) matrix ρ_q of the conjugation with q (i.e., of the map $\mathbb{H} \to \mathbb{H}$, $p \mapsto qpq^{-1}$).

We use the same notation for the compositions of these maps with the inclusions $SO(4) \hookrightarrow SO$ and $U(2) \hookrightarrow U$. Note that these inclusions commute with τ.

Proposition 4.2.2. [12, Section 7, Subsection 12]

(a) $\pi_3(U(2)) = \pi_3(U) = \mathbb{Z}\langle[u]\rangle$.

(b) $\pi_3(SO(4)) = \mathbb{Z}\langle[L]\rangle \oplus \mathbb{Z}\langle[\rho]\rangle$.

(c) $\pi_3(SO) = \mathbb{Z}\langle[L]\rangle$ *and* $[\rho] = 2[L]$ *in* $\pi_3(SO)$.

In the sequel, *using these base choices* $[u]$ *and* $[L]$ *we identify the groups* $\pi_3(U)$ *and* $\pi_3(SO)$ *with* $\mathbb{Z}$. Now we can state the explicit version of Proposition 4.1.2.

Proposition 4.2.3. $\pi_3(\tau)([u]) = -[L] \in \pi_3(SO)$ *holds for* $[u] \in \pi_3(U)$.

Proof. From definitions $\tau \circ u = R$ and $\rho R = L$, thus $\pi_3(\tau)([u]) = [R] = [L] - [\rho] = -[L]$ by part (c) of Proposition 4.2.2. □

Remark 4.2.4. Let $p : U(2) \to S^3$ be the projection (choosing the first or the second column of the matrix). Then $[u] \in \pi_3(U(2))$ is the unique generator for which $\deg(u \circ p) = 1$.

Proposition 4.2.5. *Let $\Phi : \mathbb{C}^2 \to \mathbb{C}^3$ be the cross cap, i.e. $\Phi(s,t) = (s^2, st, t)$. Then $\Omega_{\mathbb{C}}(\Phi) = [u]$.*

Proof. The map $d\Phi|_{S^3} : S^3 \to V_2(\mathbb{C}^3)$ represents $\Omega_{\mathbb{C}}(\Phi)$. We should compose this with $V_2(\mathbb{C}^3) \to U(3)$, then with the inverse of the inclusion $U(2) \to U(3)$, and finally with the projection $U(2) \to S^3$; and then calculate the degree of the resulting map $S^3 \to S^3$. In fact, along these compositions we will use (the homotopically equivalent groups) $GL(2,\mathbb{C})$ and $GL(3,\mathbb{C})$ instead of $U(2)$ and $U(3)$. Therefore, we will arrive in $\mathbb{C}^2 \setminus \{0\}$ instead of S^3.

$$d\Phi|_{S^3} : S^3 \to V_2(\mathbb{C}^3), \quad (s,t) \mapsto \begin{pmatrix} 2s & 0 \\ t & s \\ 0 & 1 \end{pmatrix}.$$

The first composition gives the map

$$S^3 \to GL(3,\mathbb{C}), \quad (s,t) \mapsto \begin{pmatrix} 2s & 0 & N_1 \\ t & s & N_2 \\ 0 & 1 & N_3 \end{pmatrix},$$

where $N_1 = \bar{t}$, $N_2 = -2\bar{s}$ and $N_3 = 2\bar{s}^2$ are the coordinates of the normal vector N_Φ (see Remark 3.1.2). This is modified by the homotopy

$$S^3 \times [0,1] \to GL(3,\mathbb{C}), \quad (s,t,h) \mapsto \begin{pmatrix} 2s & 0 & N_1 \\ t & hs & N_2 \\ 0 & 1 & hN_3 \end{pmatrix},$$

which maps $(s,t,0)$ into $GL(2,\mathbb{C}) \subset GL(3,\mathbb{C})$. [Note that the determinant is $|t|^2 + 4|s|^2 + 4h^2|s|^4 \neq 0$, thus the image is indeed in $GL(3,\mathbb{C})$.] Hence, we obtain the map

$$S^3 \to GL(2,\mathbb{C}), \quad (s,t) \mapsto \begin{pmatrix} 2s & N_1 \\ t & N_2 \end{pmatrix},$$

which composes with the projection (first column) provides $S^3 \to \mathbb{C}^2 \setminus \{0\}$, $(s,t) \mapsto (2s,t)$. After a normalisation, the degree of the resulting map is 1. □

Remark 4.2.6. The proof of Proposition 4.2.5 works for all germs of the form

$$\Phi(s,t) = (g_1(s,t), g_2(s,t), t) \tag{4.2.7}$$

and implies that

$$\Omega_{\mathbb{C}}(\Phi) = \deg\left(S^3 \to S^3\,,\, (s,t) \mapsto \frac{(\partial_s g_1(s,t), \partial_s g_2(s,t))}{|(\partial_s g_1(s,t), \partial_s g_2(s,t))|}\right).$$

This degree agrees with the intersection multiplicity in $(\mathbb{C}^2, 0)$ of $\partial_s g_1$ and $\partial_s g_2$, i.e.

$$\Omega_{\mathbb{C}}(\Phi) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2,0}}{(\partial_s g_1, \partial_s g_2)}. \tag{4.2.8}$$

This also equals $C(\Phi)$ by Theorem 2.2.1. This proves $\Omega_{\mathbb{C}}(\Phi) = C(\Phi)$ for maps of corank 1. (All germs which satisfy $\operatorname{rank}(d\Phi_0) = 1$ are right–left equivalent with germs of type (4.2.7)). This identity will be proved in the general case in Section 5.

Conversely, the identity $\Omega_{\mathbb{C}}(\Phi) = C(\Phi)$ is proved in Section 5 independently of Theorem 2.2.1, therefore (4.2.8) together with Theorem 5.1.1 give a new proof for Mond's Theorem 2.2.1 in the case of germs which satisfy $\operatorname{rank}(d\Phi_0) = 1$.

Remark 4.2.9. The conventions we use are not universal. For example, Kirby and Melvin in [14] have chosen the same generators of $\pi_3(U)$ and $\pi_3(SO)$ (these are $[u]$ and $[L]$ with our notations), but they identified $\mathbb{R}^4$ and $\mathbb{C}^2$ differently than us. Namely, they identified the quaternion $q = a + bi + cj + dk = z + jw \in \mathbb{H}$ with $(a, b, c, d) \in \mathbb{R}^4$ and the complex pair $(z, w) \in \mathbb{C}^2$, where $z = a + bi$ and $w = c - di$. With that identification u_q becomes the (complex) matrix of the quaternionic *left* multiplication with q. In that identification $\pi_3(\tau)[u]$ would be equal to $-[R]$, since that is the homotopy class of the map $S^3 \to SO(4)$ given by the composition $\tau \circ u \circ \kappa$, where κ is the reflection $\kappa(z, w) = (z, \bar{w})$.

5. The identity $\Omega_{\mathbb{C}}(\Phi) = C(\Phi)$.

5.1. Next we identify the complex Smale invariant with the number of cross caps.

Theorem 5.1.1. $\Omega_{\mathbb{C}}(\Phi) = C(\Phi)$.

Proof. Consider the following diagram:

$$\begin{array}{rccc} d\Phi: & \mathbb{C}^2 & \longrightarrow & Hom(\mathbb{C}^2, \mathbb{C}^3) \\ & \cup & & \cup \\ d\Phi|_{\mathbb{C}^2\setminus\{0\}}: & \mathbb{C}^2 \setminus \{0\} & \longrightarrow & Hom(\mathbb{C}^2, \mathbb{C}^3) \setminus \mathcal{D} = V_2(\mathbb{C}^3) \end{array},$$

where $\mathcal{D} = \{M \in Hom(\mathbb{C}^2, \mathbb{C}^3) \mid \operatorname{rank}(M) < 2\}$. $\mathcal{D}$ is an irreducible algebraic variety of complex codimension 2, its Zariski open set $\mathcal{D}^1 = \{M \in \mathcal{D} \mid \operatorname{rank}(M) = 1\}$ is smooth.

First we prove that $\Omega_{\mathbb{C}}(\Phi)$ is equal to the linking number of $d\Phi|_{S^3}$ and $\mathcal{D}$ in $Hom(\mathbb{C}^2, \mathbb{C}^3)$. This is defined as follows. If $g : S^3 \to Hom(\mathbb{C}^2, \mathbb{C}^3) \setminus \mathcal{D}$ is a smooth map, and $\tilde{g}$ is a smooth extension defined on the ball such that $\tilde{g}|_{S^3} = g$ and $\tilde{g}$ intersects $\mathcal{D}$ transversally along $\mathcal{D}^1$, then the linking number of g and $\mathcal{D}$ is the algebraic number of the intersection points of $\tilde{g}$ and $\mathcal{D}$. By standard argument it is a homotopy invariant of maps $S^3 \to Hom(\mathbb{C}^2, \mathbb{C}^3) \setminus \mathcal{D}$.

The linking number gives a group homomorphism $\operatorname{lk} : \pi_3(V_2(\mathbb{C}^3)) \to \mathbb{Z}$. Next lemma shows that this homomorphism is surjective.

Lemma 5.1.2. *Let $\Phi(s,t) = (s^2, st, t)$ be the cross cap. If $g = d\Phi|_{S^3}$ and $\tilde{g} = d\Phi|_{B^4}$, then $\tilde{g}(0) \in \mathcal{D}$ is the only intersection point and the intersection is transversal at that point.*

Proof. This is a straightforward local computation left to the reader. The transversality follows also from the conceptual fact that the cross cap is a stable map. □

The sign of the intersection multiplicity at the intersection point of two complex submanifolds is always positive. For g described in 5.1.2 the linking number of $g(S^3)$ and $\mathcal{D}$ is 1. This shows not only that the homomorphism given by the linking number is surjective (hence an isomorphism too), but also that this isomorphism agrees with the chosen one in 4.2. This follows from the fact that the complex Smale invariant of the cross cap is exactly the chosen generator, see Proposition 4.2.5. Hence, the homomorphisms $\Omega_{\mathbb{C}}$ and lk coincide.

Next, we show that $\mathrm{lk}_{Hom(\mathbb{C}^2,\mathbb{C}^3)}(d\Phi|_{S^3}(S^3), \mathcal{D}) = C(\Phi)$. Take a generic perturbation Φ_ϵ of Φ. $d\Phi_\epsilon|_{S^3}$ is homotopic to $d\Phi|_{S^3}$, hence their linking numbers are the same. Φ_ϵ has only cross cap singularities, their number is $C(\Phi)$. This means that $d\Phi_\epsilon|_{B^4}$ intersects transversally $\mathcal{D}$ in $C(\Phi)$ points. Intersection of complex manifolds provides positive signs. □

Corollary 5.1.3. $\Omega_{\mathbb{C}}(\Phi) \geq 0$.

6. The proof of Theorems 1.2.2 and 1.3.1

6.1. Theorem 1.2.2 follows from Theorem 5.1.1, Proposition 4.2.3 and the next identity.

Proposition 6.1.1. $\pi_3(\tau)(\Omega_{\mathbb{C}}(\Phi)) = \pi_3(j)(\Omega(f))$.

Proof. By the definition of the Smale invariant, one has to extend f to a neigbourhood of the standard embedding of $\mathfrak{S}^3$ in an $\mathbb{R}^5$ (cf. 2.4). On the other hand Φ extends f in the $\mathbb{C}^2$ direction. We will compare these two extensions using a common extension $F : W \to \mathbb{C}^3$, where W is a suitable neighborhood of $\mathfrak{S}^3$ in $\mathbb{C}^3$.

Let us consider a fixed ϵ which satisfies the properties of Corollary 2.1.1. We also write $B^6_\epsilon = \{z : |z| \leq \epsilon\} \subset \mathbb{C}^3$, $S^5_\epsilon = \partial B^6_\epsilon$, $\mathfrak{B}^4_\epsilon := \Phi^{-1}(B^6_\epsilon)$ for the C^∞ ball in $\mathbb{C}^2$, and $\mathfrak{S}^3_\epsilon := \Phi^{-1}(S^5_\epsilon)$ for its boundary. (Late we will drop some of the ϵ's.) For positive ϵ_1, ϵ_2 sufficiently closed to ϵ, $\epsilon_1 < \epsilon < \epsilon_2$, and for $0 < \rho \ll \epsilon$ one defines $F(s,t,r) = \Phi(s,t) + r \cdot N_\Phi(s,t)$, where $(s,t,r) \in W := \Phi^{-1}(z : \epsilon_1 < |z| < \epsilon_2) \times D^2_\rho$, D^2_ρ is the ρ-disc in $\mathbb{C}$, and N_Φ is the complex normal vector of Φ, see Remark 3.1.2 . Since the normal bundle of f in S^5_ϵ is trivial (and since the transversality is an open property), we get that $F^{-1}(S^5_\epsilon)$ is diffeomorphic to $\mathfrak{S}^3_\epsilon \times D^2_\rho$. In fact, if $p : \mathbb{C}^2 \times D^2_\rho \to D^2_\rho$ is the natural projection, then for any $r \in D^2_\rho$ we can define $\mathfrak{S}^3_{\epsilon,r} := F^{-1}(S^5_\epsilon) \cap p^{-1}(r)$. Then each $\mathfrak{S}^3_{\epsilon,r}$ is a C^∞ 3-sphere, being

the boundary of the C^∞ 4-ball $\mathfrak{B}^4_{\epsilon,r} \subset p^{-1}(r)$. Then $F^{-1}(S^5_\epsilon) = \cup_{r\in D^2_\rho}\mathfrak{S}^3_{\epsilon,r}$. Moreover, $\mathfrak{B}^6 := \cup_{r\in D^2_\rho}\mathfrak{B}^4_{\epsilon,r} \subset \mathbb{C}^2\times\mathbb{C}$ is a thickened tubular neighbourhood of $\mathfrak{B}^4_\epsilon \subset \mathbb{C}^2\times 0$, homeomorphic to the real 6-ball. Its corners can be smoothed, hence we think about it as a C^∞ ball. Its boundary $\mathfrak{S}^5 := \partial\mathfrak{B}^6$ (diffeomorphic to the 5-sphere) is the union of $F^{-1}(S^5_\epsilon)$ (diffeomorphic to $S^3\times D^2$) and $\cup_{r\in\partial D^2_\rho}\mathfrak{B}^4_{\epsilon,r}$ (diffeomorphic to $B^4\times S^1$).

In a point $(s,t,0)\in\mathfrak{S}^3_\epsilon\times\{0\}$ the differential of F is

$$\begin{aligned} dF(s,t,0) &= (\partial_s F(s,t,0), \partial_t F(s,t,0), \partial_r F(s,t,0)) \\ &= (\partial_s\Phi(s,t), \partial_t\Phi(s,t), N_\Phi(s,t))\,. \end{aligned}$$

Thus, the homotopy class of $dF|_{\mathfrak{S}^3_\epsilon}$ equals $\Omega_{\mathbb{C}}(\Phi)$ (cf. 3.1.2). Therefore, taking the *real* function $\tilde{F} : W \to \mathbb{R}^6$ (cf. 4.1), its real Jacobian satisfies $[d\tilde{F}|_{\mathfrak{S}^3_\epsilon}] = \pi_3(\tau)(\Omega_{\mathbb{C}}(\Phi))$.

On the other hand we show that $[d\tilde{F}|_{\mathfrak{S}^3}] = \pi_3(j)(\Omega(f))$. In order to recover the Smale invariant $\Omega(f)$ of $f = \Phi|_{\mathfrak{S}^3} : \mathfrak{S}^3 \looparrowright S^5$, first we need to fix a global coordinate system in a contractible neighbourhood of the source $\mathfrak{S}^3$ in $\mathfrak{S}^5$ and also in $\mathbb{R}^5 \approx S^5_\epsilon \setminus \{\text{a point}\}$ containing $\mathrm{im}(f)$. Let us introduce the 'outward normal at the end' convention to orient compatibly a manifold and its boundary. In this way we fix an orientation of $\mathfrak{S}^5 = \partial\mathfrak{B}^6$ and $S^5 = \partial B^6$. (According to 2.4, the output of the proof is independent of the convention choice.)

In the first case we introduce a coordinate system in $\mathfrak{S}^5\setminus\{Q\}\simeq\mathbb{R}^5$ compatibly with the orientation, where $Q\in\mathfrak{S}^5\setminus\mathfrak{S}^3$ is an arbitrary point (e.g. $(0,0,\rho)$). Let ν' denote the framing of $T(\mathfrak{S}^5\setminus\{Q\})\simeq T\mathbb{R}^5$ induced by this coordinate system. We can extend the outward normal frame ν_6 of $\mathfrak{S}^3$ in $\mathbb{C}^2$ to the rest of $\mathfrak{S}^5\setminus\{Q\}$ (as the outward normal vector of $\mathfrak{S}^5$). This framing can be extended to a neighbourhood V of $\mathfrak{S}^5\setminus\{Q\}$ in $\mathbb{C}^3$. Let $\nu : V\to GL^+(6,\mathbb{R})$ denote this framing (or more precisely, ν is the transition function from the standard framing inherited from $\mathbb{R}^6$ to the one just constructed).

The target is the standard $S^5\subset\mathbb{R}^6$. We can choose a point $P\in S^5\setminus f(\mathfrak{S}^3)$ and a coordinate system on $S^5\setminus\{P\}$ compatibly with the orientation. The coordinate system induces a framing η' of the tangent bundle $T(S^5\setminus\{P\})$ of $S^5\setminus\{P\}$. In the points of the target of $\tilde{F}$ the vectors of η' and $d\tilde{F}(\nu_6)$ are linearly independent, that is, $d\tilde{F}(\nu_6)$ behaves like a normal framing (this follows from the transversality property of 2.1). We can extend it to a normal framing η_6 of $S^5\setminus\{P\}$ in $\mathbb{R}^6$. In this way we get a framing of the tangent bundle of a neighbourhood V' of $S^5\setminus\{P\}$ in $\mathbb{R}^6$. Let $\eta : V'\to GL^+(6,\mathbb{R})$ denote the transition from the framing on V' inherited from $\mathbb{R}^6$ to the framing just defined.

The Smale invariant $\Omega(f)$ is constructed in the following way, cf. 2.4. Take

$$\mathcal{J}_{(\nu',\eta')}(\tilde{F}|_{\mathfrak{T}}),$$

the Jacobian of $\tilde{F}$ restricted to $\mathfrak{T} = F^{-1}(S^5_\epsilon)$ prescribed in the framings ν' and η'.

The homotopy class of this matrix restricted to $\mathfrak{S}^3$ (as a map $\mathfrak{S}^3 \to GL^+(6,\mathbb{R})$) equals to $\Omega(f)$. (Since $\tilde{F}$ preserves the orientation, $\tilde{F}|_{\mathfrak{T}}$ does as well.)

$$\mathcal{J}_{(\nu,\eta)}(\tilde{F})|_{\mathfrak{T}} = j(\mathcal{J}_{(\nu',\eta')}(\tilde{F}|_{\mathfrak{T}}))$$

because $d\tilde{F}(\nu_6) = \eta_6$, thus the homotopy class of $\mathcal{J}_{(\nu,\eta)}(\tilde{F})|_{\mathfrak{S}^3}$ equals $\pi_3(j)(\Omega(f))$.

On the other hand $\mathcal{J}_{(\nu,\eta)}(\tilde{F}) = (\eta^{-1} \circ \tilde{F}) \cdot d\tilde{F} \cdot \nu$. As maps $\mathfrak{S}^3 \to GL^+(6,\mathbb{R})$, $(\eta^{-1} \circ \tilde{F})|_{\mathfrak{S}^3}$ and $\nu|_{\mathfrak{S}^3}$ are nullhomotopic because the vector fields are defined on the contractible spaces $\mathfrak{S}^5 \setminus \{Q\}$ and $S^5 \setminus \{P\}$. Therefore (cf. Remark 2.4.2)

$$[\mathcal{J}_{(\nu,\eta)}(\tilde{F})|_{\mathfrak{S}^3}] = [(\eta^{-1} \circ \tilde{F})|_{\mathfrak{S}^3}] + [d\tilde{F}|_{\mathfrak{S}^3}] + [\nu|_{\mathfrak{S}^3}] = [d\tilde{F}|_{\mathfrak{S}^3}]\,.$$

The left hand side of this identity is $\pi_3(j)(\Omega(f))$, while the right hand side $\pi_3(\tau)(\Omega_{\mathbb{C}}(\Phi))$. □

Corollary 6.1.2. *Assume that the analytic germs Φ and $\Phi' : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ are $\mathcal{C}^\infty$ left-right equivalent, that is, $\Phi' = \Lambda \circ \Phi \circ \psi$ holds for some germs of orientation preserving diffeomorphisms $\psi : (\mathbb{R}^4,0) \to (\mathbb{R}^4,0)$ and $\Lambda : (\mathbb{R}^6,0) \to (\mathbb{R}^6,0)$. Then $C(\Phi) = C(\Phi')$.*

Proof. For a sufficiently small ϵ take $\mathfrak{S}'^3 = (\Phi')^{-1}(S^5_\epsilon) \cong S^3$ and let $f' : \mathfrak{S}'^3 \hookrightarrow S^5$ be the immersion associated with the germ Φ' (cf. 2.1).

As in the proof of Proposition 6.1.1, let $\tilde{F} : W \to \mathbb{R}^6$ be the extension of Φ viewed as a real function. Then, by that proof, $[d\tilde{F}|_{\mathfrak{S}^3}] = \pi_3(j)(\Omega(f)) \in \pi_3(GL^+(6,\mathbb{R}))$, which is $-C(\Phi)$ under the above identification.

Let us define $\Psi : (\mathbb{R}^6,0) \to (\mathbb{R}^6,0)$ by

$$\Psi(x_1,x_2,x_3,x_4,x_5,x_6) = (\psi(x_1,x_2,x_3,x_4),x_5,x_6).$$

Ψ is a germ of orientation preserving diffeomorphism extending ψ.

Then $\Lambda \circ \Phi \circ \Psi$ restricted on some neighborhood W' of $\mathfrak{S}'$plays the role of a real extension of $\Phi'|_{\mathfrak{S}'^3} = f'$. One can verify that the proof of Proposition 6.1.1 works for this extension as well, since it sends the normal vector of $\mathfrak{S}'^5$ into a non-tangent vector of S^5. Therefore,

$$[d(\Lambda \circ \Phi \circ \Psi)|_{\mathfrak{S}'^3}] = \pi_3(j)(\Omega(f')).$$

Finally, note that $[d(\Lambda \circ \Phi \circ \Psi)|_{\mathfrak{S}'^3}] = [d\tilde{F}|_{\mathfrak{S}^3}]$. This follows from the fact that the functions $d\Lambda \circ \Phi \circ \Psi$ and $d\Psi$ (with images in $(GL^+(6,\mathbb{R}))$ extend to the ball $\mathfrak{B}'^4_\epsilon$. □

6.2. Proof of Theorem 1.3.1. Part (a) follows from Theorem 1.2.2 and [10]. In part (b), the implications (1) ⇒ (2,3,4), and (4) ⇒ (3) are clear.

Proof of (2) $\Rightarrow$ (1): (2) implies $C(\Phi) = 0$ by Theorem 1.2.2, while this vanishing implies (1) via Mond's Theorem 2.2.1. For (3) $\Rightarrow$ (1) we provide three proofs, each of them emphasize a different geometrical/topological aspect.

(A) (*Based on Mumford's Theorem.*) If f is an embedding then the image $(X, 0)$ of Φ is an isolated hypersurface singularity in $(\mathbb{C}^3, 0)$. Moreover, its link is S^3, hence by Mumford's theorem [20] $(X, 0)$ is smooth. Hence its normalization Φ is an isomorphism.

(B) (*Based on Mond's Theorem.*) Let us take the generic deformation Φ_λ, and consider the closure D of the preimage of the the set of double values. It is a 1–dimensional closed complex analytic subspace of the disc in $\mathbb{C}^2$. The preimages of cross cap and triple points are interior points of the closure of D, while its boundary is $D \cap S^3$ is the preimage of the double points of the immersion of $f : S^3 \looparrowright S^5$. If f is an embedding then $\partial D = \emptyset$, hence D is a compact analytic curve in (the disc of) $\mathbb{C}^2$, hence it should be empty. This shows that Φ_λ has no cross cap and triple points either. Hence $C(\Phi) = 0$, which implies (1) by 2.2.1 as before.

(C) (*Based on Ekholm–Szűcs Theorem.*) As above, we get that Φ_λ is an embedding. Since $\Phi|S^3$ is an embedding, this embedding is regular homotopic to $\Phi_\lambda|S^3$, hence they have the same Smale invariant. In the second case it can be determined by an Ekholm–Szűcs formula [3] (recalled as Theorem 8.1.9 here): since $\mathrm{im}(\Phi_\lambda)$ is an embedded Seifert surface with signature zero we get $\Omega(f) = 0$. This basically proves (3) $\Rightarrow$ (2). Then we continue with the already shown (2) $\Rightarrow$ (1).

In fact, the main point of this last proof is already coded in Hughes–Melvin Theorem [10] (8.1.1 here), but in that statement the Seifert surface is in $\mathbb{R}^5$ and not in $\mathbb{R}^6_+$ (or in the 6–ball). But 8.1.9 shows that that Hughes–Melvin Theorem is true even if the 4–manifold M^4 with boundary in $\mathbb{R}^5$ is embedded in $\mathbb{R}^6_+$ (instead of $\mathbb{R}^5$).

7. Examples

7.1. This section contains the first list of the promised examples.

Example 7.1.1. Fix $k \in \mathbb{Z}_{\geq 0}$. $\Phi_{-k}(s, t) = (s, t^2, t^3 + s^k t)$. The ideal J (cf. 2.2) is generated by $(2t, 3t^2 + s^k, -2kt^2 s^{k-1}) = (t, s^k)$. Hence $\Omega(f) = -C(\Phi) = -k$.

This family gives representatives for every regular homotopy class with non-positive sign–refined Smale invariant. Furthermore, we can represent any regular homotopy class with Smale invariant k in the form $\Phi_{-k} \circ \kappa$, where κ is the reflection $\kappa(z, w) = (z, \bar{w})$ (c.f. [2, Lemma 3.4.2.]).

Example 7.1.2 (Singularities of type A). These are quotient singularities of the form $(X, 0) = (\mathbb{C}^2, 0)/\mathbb{Z}_k$, where $\mathbb{Z}_k = \{\xi \in \mathbb{C} \mid \xi^k = 1\}$ denotes the cyclic group of order k, and the action is $\xi * (s, t) = (\xi s, \xi^{-1} t)$ for $\xi \in \mathbb{Z}_k$. $(X, 0)$ is the image of a map Φ, whose components are the generators of the invariant algebra $\mathbb{C}\{s, t\}^{\mathbb{Z}_k}$, see [23, page 95], namely $\Phi(s, t) = (s^k, t^k, st)$. One can easily compute

that $J = (s^k, t^k, s^{k-1}t^{k-1})$ and $\Omega(f) = -C(\Phi) = -(k^2-1)$. ($(X,0)$ is the A_{k-1}–singularity.)

Example 7.1.3 (Singularities of type D). These are the quotient singularities of form $(\mathbb{C}^2,0)/D_n$ where D_n denotes the binary dihedral group, [23, page 89]. $\Phi(s,t) = (s^2t^2, s^{2n}+t^{2n}, st(s^{2n}-t^{2n}))$ [23, page 95]. By a computation $J = (st(s^{2n}-t^{2n}), s^2t^2(s^{2n}+t^{2n}), (s^{2n}-t^{2n})^2 - 4ns^{2n}t^{2n})$. In singularity theory the quotient is known as the D_{n+2}–singularity.

A possible computation of $\dim\left(\mathcal{O}_{\mathbb{C}^2,0}/J\right)$ is based on the following facts.

Lemma 7.1.4.

(a) *Take $f_1, f_2, h \in \mathcal{O}_{\mathbb{C}^2,0}$ such that f_1f_2 and h are relative primes. Then one has the following exact sequence:*

$$0 \to \mathcal{O}_{\mathbb{C}^2,0}/(f_2,h) \to \mathcal{O}_{\mathbb{C}^2,0}/(f_1f_2,h) \to \mathcal{O}_{\mathbb{C}^2,0}/(f_1,h) \to 0\,.$$

(b) *Take $f_1, f_2, g, h \in \mathcal{O}_{\mathbb{C}^2,0}$ such that the ideal (f_1f_2, g, h) has finite codimension, and $h = f_1h'$ for some $h' \in \mathcal{O}_{\mathbb{C}^2,0}$. Then one has the following exact sequence:*

$$0 \to \mathcal{O}_{\mathbb{C}^2,0}/(f_2,g,h') \to \mathcal{O}_{\mathbb{C}^2,0}/(f_1f_2,g,h) \to \mathcal{O}_{\mathbb{C}^2,0}/(f_1,g) \to 0\,.$$

Proof. Part (a) is well known as the additivity property of the local intersection number of plane curves, see e.g. [5]. The proof of part (b) is similar. □

Using these lemmas the codimension of J can be calculated, and it is $4n^2+12n-1$. Hence, the Smale invariant of the covering $S^3 \to \{\text{link of the } D_{n+2}\text{–singularity}\}$ is $-(4n^2+12n-1)$.

Example 7.1.5. Assume that the three components of Φ are weighted homogeneous of weights w_1 and w_2 and degree d_1, d_2 and d_3. Then, cf. [18],

$$C(\Phi) = \{d_1d_2+d_2d_3+d_3d_1-(w_1+w_2)(d_1+d_2+d_3-w_1-w_2)-w_1w_2\}/w_1w_2.$$

Mond proved this identity for germs with finite right–left codimension, but the same proof works for germs with finite $\mathcal{O}_{\mathbb{C}^2,0}/J$.

For example, if $\Phi : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)/G \hookrightarrow (\mathbb{C}^3,0)$ is as in Example 1.4.1, then all three components are homogeneous ($w_1 = w_2 = 1$). In the case of A_{k-1} and D_{n+2} the degrees are $(k,k,2)$ and $(4,2n,2n+2)$ respectively. Hence the values $C(\Phi)$ from Examples 7.1.2 and 7.1.3 follow in this way as well.

For E_6, E_7 and E_8 singularities the degrees are $(6,8,12)$, $(8,12,18)$ and $(12,20,30)$ respectively, see [23, 4.5.3–4.5.5], hence the corresponding values $-\Omega(f)$ are 167, 383, 1079.

8. Smale invariant via Seifert surfaces

8.1. In this section we review three important topological formulae targeting the Smale invariant in terms of the geometry of oriented Seifert surfaces. They are stated and proved only up to a sign ambiguity. In the next section we will show that the sign–refined Smale invariant appears in all these expressions with a unique well–defined sign, and we determine it simultaneously for all formulae. The discussion has an extra output as well: the topological ingredients in the formulae below will get reinterpretations in terms of complex analytic invariants, provided that the immersion is induced by a holomorphic germ Φ.

In the spirit of the discussion of Subsection 2.4, in this section we will write $\mathbf{S}^3$ for an 'oriented abstract S^3'. $\Omega(f)$ will denote the Smale invariant (given by any of its definitions, still having its sign–ambiguity). Note that in the next statements we need to fix a 'boundary convention', in order to have the notion of oriented ∂M. (Nevertheless, the sign–corrected formulae will be 'boundary convention' free, cf. Theorem 9.1.6.)

Theorem 8.1.1 (Hughes, Melvin [10]). *Let $f : \mathbf{S}^3 \hookrightarrow \mathbb{R}^5$ be an embedding and $\tilde{f} : M^4 \hookrightarrow \mathbb{R}^5$ be a Seifert surface of f, i.e. M^4 is a compact oriented 4-manifold with boundary $\partial M^4 = \mathbf{S}^3$ and $\tilde{f}$ is an embedding such that $\tilde{f}|_{\partial M^4} = f$. Let $\sigma(M^4)$ be the signature of M^4. Then*

$$\Omega(f) = \pm\frac{3}{2}\sigma(M^4)\,. \tag{8.1.2}$$

For arbitrary immersions Ekholm and Szűcs generalized the formula via generic *singular Seifert surfaces*, and in two different ways: mapped either in $\mathbb{R}^5$ or in $\mathbb{R}^6_+$ [3], see also [4, 21].

If M^4 is a compact oriented 4-manifold and $g : M^4 \to \mathbb{R}^5$ is a generic C^∞ map, then g has isolated $\Sigma^{1,1}$*–points (cusps)*, each endowed with a well–defined sign. Let $\#\Sigma^{1,1}(g)$ be their 'algebraic' number (cf. [3]).

Theorem 8.1.3 (Ekholm, Szűcs [3]). *Let $f : \mathbf{S}^3 \looparrowright \mathbb{R}^5$ be an immersion and M^4 be a compact oriented 4-manifold with boundary $\mathbf{S}^3$. Let $\tilde{f} : M^4 \to \mathbb{R}^5$ be a generic map such that $\tilde{f}|_{\partial M^4}$ is regular homotopic to f and $\tilde{f}$ has no singular points near the boundary. Then*

$$\Omega(f) = \pm\frac{1}{2}(3\sigma(M^4) + \#\Sigma^{1,1}(\tilde{f}))\,. \tag{8.1.4}$$

The last formula, the most important from the point of view of this note, uses generic C^∞ maps $g : M^4 \to \mathbb{R}^6$ defined on compact oriented 4-manifolds M^4. It involves three topological invariants associated with such a map. Next we review their definitions. They will be computed for two concrete holomorphic maps in order to identify the missing sign.

If g is as above, then it has isolated triple values (three local sheets of M^4 intersecting in general position). Such a point is endowed with a well–defined sign [3, 2.3]).

Definition 8.1.5 ([3]). $t(g)$ denotes the algebraic number of the triple values of g.

Next, assume that $\partial M^4 = \mathbf{S}^3$ and $g : (M^4, \partial M^4) \to (\mathbb{R}^6_+, \partial\mathbb{R}^6_+)$ is generic, it is nonsingular near the boundary, and $\tilde{f}^{-1}(\partial\mathbb{R}^6_+) = \partial M^4$. Here $\mathbb{R}^6_+$ is the closed half–space of $\mathbb{R}^6$. The set of double values of g is an immersed oriented 2-manifold, denoted by $D(g)$. Its oriented boundary consists of two parts, the intersection of $D(g) \cap \partial\mathbb{R}^6_+$, and the other, disjoint with $\partial\mathbb{R}^6_+$, is the set of singular values $\Sigma(g)$ of g. Let $\Sigma'(g)$ be a copy of $\Sigma(g)$ shifted slightly along the outward normal vector field of $\Sigma(g)$ in $D(g)$. Then $\Sigma'(g) \cap g(M^4) = \emptyset$.

Definition 8.1.6 ([3]). $l(g)$ denotes the linking number of $g(M^4)$ and $\Sigma'(g)$ in $(\mathbb{R}^6_+, \partial\mathbb{R}^6_+)$.

For a generic (self-transverse) immersion $f : \mathbf{S}^3 \looparrowright \mathbb{R}^5$ one defines an integer $L(f)$ as follows [3, 2.2], [21, 2.2]. f has a normal framing (v_1, v_2) which is unique up to homotopy. In any double value $y = f(x_1) = f(x_2)$ set $N(y) = v_1(x_1) + v_1(x_2)$. Let $D'(f)$ be a copy of the set of double values $D(f)$ of f shifted slightly along the vector field N. $D(f)$ (hence $D'(f)$ too) is a 1-manifold and $D'(f) \cap f(\mathbf{S}^3) = \emptyset$.

Definition 8.1.7 ([2, 3, 21]). $L(f)$ is a the linking number of $f(\mathbf{S}^3)$ and $D'(f)$ in $\mathbb{R}^5$.

Remark 8.1.8. One can define $L(f)$ without any reference to v_2 in the following way. Take an arbitrary normal vector field w of f, and in a double value $y = f(x_1) = f(x_2)$ one defines $N(y) = w(x_1) + w(x_2)$ and $D'(f)$ and $L(f)$ as above. Then $L(f)$ does not depend on the choice of w. Indeed, take two normal vector fields w_0 and w_1. In a certain trivialisation of the normal bundle of f they are represented by two maps $w_0, w_1 : \mathbf{S}^3 \to S^1$, thus there is a homotopy w_t connecting them ($w_t : \mathbf{S}^3 \to S^1$, $t \in [0, 1]$). We can define in a continuous way $N_t(y)$ and $D'_t(f)$ using the normal vector field w_t and the compactness of $[0, 1]$. Then $D'_t(f) \cap f(\mathbf{S}^3) = \emptyset$, thus the linking number of $f(\mathbf{S}^3)$ and $D'_t(f)$ in $\mathbb{R}^5$ is the same for all $t \in [0, 1]$.

Theorem 8.1.9 (Ekholm, Szűcs [3]). *Let $f : \mathbf{S}^3 \looparrowright \mathbb{R}^5$ be an immersion and M^4 be a compact oriented 4-manifold with boundary $\partial M^4 = \mathbf{S}^3$. Let $\tilde{f} : (M^4, \partial M^4) \to (\mathbb{R}^6_+, \partial\mathbb{R}^6_+)$ be a generic map nonsingular near the boundary, such that $\tilde{f}^{-1}(\partial\mathbb{R}^6_+) = \partial M^4$ and $\tilde{f}|_{\partial M^4}$ is regular homotopic to f. Then*

$$\Omega(f) = \pm\frac{1}{2}(3\sigma(M^4) + 3t(\tilde{f}) - 3l(\tilde{f}) + L(\tilde{f}|_{\partial M^4})). \tag{8.1.10}$$

9. Ekholm-Szűcs formulae for holomorphic germs Φ

In this section, from a holomorphic deformation of Φ we construct a singular Seifert surface, and we express the topological summands of (8.1.10) in terms of holomorphic invariants. As a corollary we specify the sign in the formulae (8.1.2), (8.1.4) and (8.1.10).

9.1. Singular Seifert surface associated with an analytic deformation.

Let $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a holomorphic germ singular only at the origin and let $f : S^3 \looparrowright S^5$ be the immersion associated with Φ. We take an ϵ as in Corollary 2.1.1, that is, we fix in the target a ball B^6_ϵ. We also consider a holomorphic generic deformation Φ_λ of $\Phi_0 = \Phi$, and we fix λ sufficiently small, $0 < |\lambda| \ll \epsilon$, such that the cross caps and (if $T(\Phi) < \infty$) the triple points of Φ_λ sit in B^6_ϵ. We set $\mathfrak{B}^4_{\epsilon,\lambda} := \Phi_\lambda^{-1}(B^6_\epsilon)$, it is a C^∞ non–metric ball in $\mathbb{C}^2$. Its boundary is $\mathfrak{S}^3_{\epsilon,\lambda} := \Phi_\lambda^{-1}(S^5_\epsilon)$, it is canonically diffeomorphic to S^3.

The map Φ_λ is generic as a holomorphic map, but it is not generic as a C^∞ map. The C^∞ genericity is obstructed by its cross cap points. We will modify Φ_λ in the neighborhood of these points according to the following local model.

Let us fix local holomorphic coordinate systems in the source and the target such that Φ_λ in the neighborhood of a cross cap has local equation $\Phi^{loc}(s,t) = (s^2, st, t)$. We consider its real smooth deformation (with $0 \le \tau \ll |\lambda|$):

$$\Phi^{loc}_\tau(s,t) = (s^2 + 2\tau\bar{s}, st + \tau\bar{s}, t). \tag{9.1.1}$$

Since the restriction of Φ^{loc} near the boundary of the local 4-ball is stable, by a C^∞ bump function the local deformation can be glued to the trivial deformation of Φ_λ outside of local neighborhoods of the cross caps. This gives a C^∞ global deformation $\Phi_{\lambda,\tau}$ of Φ_λ and Φ. The map $\tilde{f} = \Phi_{\lambda,\tau} : (\mathfrak{B}^4_{\epsilon,\lambda}, \mathfrak{S}^3_{\epsilon,\lambda}) \to (B^6_\epsilon, S^5_\epsilon)$ is the singular Seifert surface we will consider. Its restriction, $f_\lambda = \Phi_{\lambda,\tau}|_{\mathfrak{S}^3_{\epsilon,\lambda}} = \Phi_\lambda|_{\mathfrak{S}^3_{\epsilon,\lambda}}$ is the immersion associated with Φ_λ.

Proposition 9.1.2.

(a) *$\tilde{f} : \mathfrak{B}^4_{\epsilon,\lambda} \to \mathbb{C}^3$ is a generic smooth map, nonsingular near the boundary.*

(b) *f_λ is a generic immersion and it is regular homotopic to f.*

(c) *If f is a generic immersion, then f_λ is regular homotopic to f through generic immersions. In this case $L(f) = L(f_\lambda)$.*

Proof. (a) First one checks that the local Φ^{loc}_τ is generic. This follows from the computation from Section 10.1. Its most complicated singularities are $\Sigma^{1,0}$ (fold) points, the singular values constitute an S^1, which — together with the double values of the image of the boundary of the local ball — bounds the 2-manifold of the double values. Cf. [3, 2.3.].

In the complement of local balls $\Phi_{\lambda,\tau}$ agrees with Φ_λ, hence it has only simple points, self-transverse double points and isolated triple points. All of them are generic. Hence $\tilde{f}$ has all the local property of a generic map (and, in fact, this is enough in the determination of all the invariants, cf. [3]).

(b) $\Phi_\lambda|_{\mathfrak{S}^3_{\epsilon,\lambda}}$ is generic in real sense too: it has only simple points and generic self-transversal double points. $\Phi_{h\lambda}|_{\mathfrak{S}^3_{\epsilon,\lambda}}$ is a regular homotopy between f and f_λ ($h \in [0,1]$).

(c) Being a generic immersion is an open condition (cf. [3, 2.1.]). Furthermore, L is constant along a regular homotopy through generic immersions, cf. [2]. □

Next, we return back to the formula (8.1.10), applied for $\tilde{f}$. Clearly, $\sigma(M^4) = 0$.

Theorem 9.1.3. *Let Φ_λ be a holomorphic generic deformation of Φ with fixed $\lambda \neq 0$ and the corresponding maps $\tilde{f}$ and f_λ as above. Then the following facts hold.*

(a) $t(\tilde{f}) = T(\Phi_\lambda)$ *(cf. 2.3).*

(b) $l(\tilde{f}) = C(\Phi)$.

(c) $L(f_\lambda) = C(\Phi) - 3T(\Phi_\lambda)$.

For the proof see 9.2.

Note that $L(f_\lambda)$ is an *analytic* invariant of Φ, since it is defined as a (topological) invariant of an *analytic* deformation. Recall from Subsection 2.3 that if $T(\Phi) < \infty$, then $T(\Phi_\lambda)$ is independent of the deformation Φ_λ and is equal to $T(\Phi)$.

Corollary 9.1.4. *If $T(\Phi) < \infty$, then $t(\tilde{f}) = T(\Phi)$ and $L(f_\lambda) = C(\Phi) - 3T(\Phi)$ is also independent of the analytic deformation Φ_λ of Φ.*

Remark 9.1.5. Assume that $\Phi|_{\mathfrak{S}^3_\epsilon} = f$ is a *generic* immersion. Then $T(\Phi)$ is finite, by [17], and for any holomorphic deformation Φ_λ one has $L(f_\lambda) = L(f)$ (cf. Proposition 9.1.2), hence $C(\Phi) - 3T(\Phi) = L(f)$.

Note that if the restriction $\Phi|_{\mathfrak{S}^3_\epsilon} = f$ of a germ Φ is a generic immersion, and the holomorphic germ Φ' is C^∞ left-right equivalent with Φ (see Corollary 6.1.2 for precise definition), then the immersion f' associated with Φ' is also generic immersion and $L(f) = L(f')$. Therefore by Corollaries 6.1.2 and 9.1.4 we have $T(\Phi) = T(\Phi')$ too.

More generally, Φ and Φ' are topological left–right equivalent, and also f and f' are generic immersions, then $L(f) = L(f')$, hence Corollary 1.6.2 follows too.

Theorem 9.1.6. *With our sign–convention, if in the left hand side of the formulae (8.1.2), (8.1.4) and (8.1.10) we put the sign–refined Smale invariant $\Omega^a(f)$, then the formulae are valid if we put the positive sign on the right hand sides.*

In particular, the validity of these sign–corrected formulae (e.g., $\Omega^a(f) = \frac{3}{2}\sigma(M^4)$) is independent of the 'boundary convention': changing the boundary convention changes the sign in both sides of the formulae simultaneously.

The proof will appear in 9.2.

Remark 9.1.7. The formula (8.1.4), involving the (algebraic) number of real cusps of maps $g : M^4 \to \mathbb{R}^5$ is the real analogue of our theorem $\Omega(f) = -C(\Phi)$, involving the number of cross caps of holomorphic deformations. This suggests that if we replace a holomorphic deformation by a smooth generic map, then we trade each cross cup by -2 real cusps.

9.2. We prove Theorems 9.1.3 and 9.1.6 simultanously (see also the discussion from Subsection 2.4).

Proof. In the definitions of the invariants t, l and L one uses very specific sign/orientation conventions, based on the orientation of the involved subspaces in their definition.

For a triple value, the sign is determined in such a way that it is $+1$ whenever the triple value is obtained from a holomorphic triple point (hence the orientations agree with the complex orientations).

Since in the local deformation Φ_τ^{loc} we do not create any new triple value, see e.g. the computation of Section 10, all the triple values of $\tilde{f}$ come from the complex triple points of the holomorphic Φ_λ, hence (a) follows.

The proof of the remaining parts is based on computations of the invariants $C(\Phi)$, $T(\Phi)$, $l(\tilde{f})$ and $L(f)$ for two concrete cases. For the integers l and L the definitions (orientation conventions) are not immediate even in simple cases. Therefore, in our computation we determine them only up to a sign. The point is that computing 'sufficiently many' examples, the formula (8.1.10), even with its sign ambiguity in front of the right hand side, and even with the (new) sign ambiguities of the integers l and L, determine uniquely all these signs. (This also shows that, in fact, there is a unique universal way to fix the orientation conventiones and signs in the definitions of l and L such that (8.1.10) works universally.)

In Section 10 we will determine the following data:

$$\begin{aligned} &\text{(i) For cross cup: } C(\Phi) = 1, \quad T(\Phi) = 0, \quad l = \pm 1, \quad L = \pm 1. \\ &\text{(ii)} \qquad \text{For } A_1\text{: } C(\Phi) = 3, \quad T(\Phi) = 1, \quad L = 0. \end{aligned} \tag{9.2.1}$$

(b) The singular values of $\tilde{f}$ are concentrated near the cross caps of Φ_λ. For Φ_τ^{loc} the value l is ± 1, see (i). Since the sign is the same for all cross caps, $l(\tilde{f}) = \pm C(\Phi)$.

We introduce the notation

$$\Omega'(f_\lambda) := \frac{1}{2}(3t(\tilde{f}) - 3l(\tilde{f}) + L(f_\lambda)). \tag{9.2.2}$$

$\Omega'(f_\lambda)$ agrees with $\Omega(f)$ up to sign, thus $\Omega'(f_\lambda) = \pm C(\Phi)$. Substituting this and the data (i) of the cross cap in (9.2.2) we conclude that $l(\tilde{f}) = -\Omega'(f_\lambda)$ and $L(f_\lambda) = \pm C(\Phi) - 3T(\Phi_\lambda)$.

Next, using the date (ii) for A_1, all the remaining sign ambuguities can be eliminated: $L(f_\lambda) = C(\Phi) - 3T(\Phi_\lambda)$, $l(\tilde{f}) = C(\Phi)$ and $\Omega'(f_\lambda) = -C(\Phi) = \Omega(f)$.

The universal signs in formulae (8.1.2), (8.1.4) and (8.1.10) are related by common examples, hence one of them determines all of them.

10. Calculations. The proof of (9.2.1).

We show the main steps of the computations, with their help the reader can fill in the details. Note that if the germ Φ is weighted homogeneous, then $\epsilon_0 = 1$ can be chosen.

10.1. The case of cross cap. For the computation of $T(\Phi)$ see e.g. [16, 17]; $C(\Phi)$ is clear. Next we compute l and L. Set $\Phi(s,t) = (s^2, st, t)$ and the smooth perturbation $\tilde{f}(s,t) = (s^2 + 2\epsilon\bar{s}, st + \epsilon\bar{s}, t)$. The singular locus is $\tilde{\Sigma} = \{(s,t) \mid s = t\ , |s| = |t| = \epsilon\} \cong S^1$.

$\tilde{f}|_{\tilde{\Sigma}}$ has no singular point, hence $\tilde{f}$ has no cusp points. The most complicated singularities of $\tilde{f}$ are $\Sigma^{1,0}$ (or fold) points. The closure of the set of the double points $\tilde{D}$ of $\tilde{f}$ is

$$\mathrm{cl}(\tilde{D}) = \{(s,t) \in \mathbb{C}^2 \mid (s-t)t + \epsilon(\bar{s} - \bar{t}) = 0\}$$

with the involution $(s,t) \mapsto (s',t) = (2t - s, t)$. The fix point set of the involution is $\{s = t\}$. Thus the set of the double points is

$$\tilde{D} = \{(s,t) \in \mathbb{C}^2 \mid (s-t)t + \epsilon(\bar{s} - \bar{t}) = 0\} \setminus \{s = t\}.$$

Each double point has exactly one pair with the same value, hence $\tilde{f}$ has no triple point.

A parametrization of $\tilde{D}$ is $(\rho, \alpha) \mapsto (-\epsilon e^{-2\alpha i} + \rho e^{i\alpha}, -\epsilon e^{-2\alpha i})$, where $\rho \in \mathbb{R}_+$, $\alpha \in [0, 2\pi)$.

The parametrization shows that the closure of $\tilde{D}$ is a Möbius band. For $\rho = 0$ we get $\tilde{\Sigma}$, which is the midline of the Möbius band. The set of double values is

$$\begin{aligned} D = \tilde{f}(\tilde{D}) &= \{(s^2 + 2\epsilon\bar{s}, st + \epsilon\bar{s}, t) \mid (s,t) \in \tilde{D}\} \\ &= \{(\rho^2 e^{2i\alpha} + \epsilon^2 e^{2i\alpha}(e^{-6i\alpha} - 2), \epsilon^2(e^{-4i\alpha} - e^{2i\alpha}), -\epsilon e^{-2\alpha i}) \\ &\qquad \mid \rho \in \mathbb{R}_+\ , \alpha \in [0, 2\pi)\}. \end{aligned}$$

Writing $\rho = 0$ we get the singular values of $\tilde{f}$,

$$\Sigma = \tilde{f}(\tilde{\Sigma}) = \{(\epsilon^2 e^{2i\alpha}(e^{-6i\alpha} - 2), \epsilon^2(e^{-4i\alpha} - e^{2i\alpha}), -\epsilon e^{-2\alpha i})\}\ .$$

The inward normal field of Σ in D is the derivative of the curve

$$\gamma(t) = (te^{2i\alpha} + \epsilon^2 e^{2i\alpha}(e^{-6i\alpha} - 2), \epsilon^2(e^{-4i\alpha} - e^{2i\alpha}), -\epsilon e^{-2\alpha i})$$

at $t = 0$, that is $\gamma'(t)|_{t=0} = (e^{2i\alpha}, 0, 0)$. The pushing out of Σ (cf. Definition 8.1.6) is

$$\Sigma' = \Sigma - \delta \cdot \gamma'(t)|_{t=0} = \{(-\delta e^{2i\alpha} + \epsilon^2 e^{2i\alpha}(e^{-6i\alpha} - 2), \epsilon^2(e^{-4i\alpha} - e^{2i\alpha}), -\epsilon e^{-2\alpha i})\},$$

where $0 < \delta \ll \epsilon$. By Definition 8.1.6 we need the linking number of $\tilde{f}(\mathbb{R}^4)$ and Σ' in $\mathbb{R}^6$. To calculate it we fill in $\Sigma' \cong S^1$ with a 'membrane', which here will be the disc

$$H = \{(-\delta w + \epsilon^2(\bar{w}^2 - 2w), \epsilon^2(\bar{w}^2 - w), -\epsilon\bar{w}) \mid w \in \mathbb{C}, |w| \leq 1\}.$$

$l(\tilde{f})$ is the algebraic number of the intersection points of H and $\tilde{f}(\mathbb{R}^4)$. The only solution is $w = 0$, $(s, t) = (0, 0)$, and the intersection at this point is transversal. Hence, for the smooth perturbation $\tilde{f}$ of the cross cap $l(\tilde{f}) = \pm 1$.

Next we compute L. The set of the double points of Φ is $\tilde{D} = \{(s, 0) \mid s \neq 0\} \subset \mathbb{C}^2$.

The set of the double values is $D = \Phi(\tilde{D}) = \{(s^2, 0, 0) \mid s \neq 0\} \subset \mathbb{C}^3$, and the set of the double values of f is $D_f = D \cap S^5 = \{(s^2, 0, 0) \mid |s| = 1\} \subset S^5$.

The sum of the normal vectors at $(s^2, 0, 0)$ is $(0, 0, \bar{s}^2)$. Hence the shifted copy of D along N is $D' = D_f + \delta N = \{(s^2, 0, \delta\bar{s}^2) \mid |s| = 1\}$.

Since D' does not intersect $\Phi(\mathbb{C}^2)$ for $\delta \in (0, 1]$, we can choose $\delta = 1$. An injective parametrization of $D_f + \delta N$ is $D' = \{(z, 0, \bar{z}) \mid |z| = 1\}$, where $z = s^2$. To calculate the linking number of $\Phi(\mathbb{C}^2)$ and D' in $\mathbb{R}^6$, we need a membrane which fills in D. We take

$$H = \{(z, \sqrt{1 - |z|^2}, \bar{z}) \mid |z| \leq 1\} \cong D^2.$$

$L(f)$ is the algebraic number of the intersection points of $\Phi(\mathbb{C}^2)$ and H. But there is only one such point, namely $P := \Phi(\sqrt{\xi}, \xi) = (\xi, \xi\sqrt{\xi}, \xi)$, where ξ is the real root of $g(z) := z^3 + z^2 - 1 = 0$. Moreover, this intersection is transversal.

10.2. The A_1 singularity. By 7.1.2 it is given by $\Phi_0(s, t) = (s^2, t^2, st)$. The immersion f_0 associated with Φ_0 is not generic, f_0 is the 2-fold covering of the projective space composed with the inclusion. Thus all points of S^3 are double points of the immersion f.

On the other hand, by 7.1.2, $C(\Phi_0) = 3$, and a similar calculation of the codimension of the second fitting ideal shows that $T(\Phi_0) = 1$. The finiteness of these invariants shows that the number of cross caps and triple points of a generic deformation of Φ_0 are independent of the chosen deformation. Below we give a concrete deformation Φ_ϵ of Φ_0 and we calculate the invariant L of the generic immersion f_ϵ associated with Φ_ϵ.

The deformation is $\Phi_\epsilon(s,t) = ((s-\epsilon)s, (t-\epsilon)t, st)$. The vector field

$$\tilde{N}(s,t) = \overline{\partial_s \Phi_\epsilon(s,t) \times \partial_t \Phi_\epsilon(s,t)} = \begin{pmatrix} \bar{t}(2\bar{t}-\epsilon) \\ -\bar{s}(2\bar{s}-\epsilon) \\ (2\bar{s}-\epsilon)(2\bar{t}-\epsilon) \end{pmatrix}.$$

is 0 at the points $(0, \epsilon/2)$, $(\epsilon/2, 0)$ and $(\epsilon/2, \epsilon/2)$. These are the cross caps.

The defining equation $\Phi_\epsilon(s,t) = \Phi_\epsilon(s',t')$ (where $(s,t) \neq (s',t')$) of the double points leads to the system of equations

$$(s-s')(s+s'-\epsilon) = 0, \quad (t-t')(y+y'-\epsilon) = 0, \quad st = s't'.$$

Thus the double locus $\tilde{D}$ has three parts and these parts correspond to the three cross caps. The first part comes from the solution $s' = s$ and $t' = \epsilon - t$, which implies $s = 0$, hence $\tilde{D}_1 = \{(0,t) \mid t \neq \epsilon/2\}$ with $\Phi_\epsilon(0,t) = \Phi_\epsilon(0, \epsilon - t)$. This provide the double value set

$$D_1 = \Phi_\epsilon(\tilde{D}_1) = \{(0, t(t-\epsilon), 0) \mid t \neq \epsilon/2\}.$$

The second part comes from the solution $s' = \epsilon - s$ and $t' = t$, which implies $t = 0$, and $\tilde{D}_2 = \{(s,0) \mid s \neq \epsilon/2\}$ with $\Phi_\epsilon(s,0) = \Phi_\epsilon(\epsilon - s, 0)$. The set of double values is

$$D_2 = \Phi_\epsilon(\tilde{D}_2) = \{(s(s-\epsilon), 0, 0) \mid s \neq \epsilon/2\}.$$

The third part comes from the solution $s' = \epsilon - s$ and $t' = \epsilon - t$, which implies $s + t = \epsilon$, and $\tilde{D}_3 = \{(s, \epsilon - s)\} \mid s \neq \epsilon/2\}$ with $\Phi_\epsilon(s, \epsilon - s) = \Phi_\epsilon(\epsilon - s, s)$. The set of double values is

$$D_3 = \Phi_\epsilon(\tilde{D}_3) = \{(s(s-\epsilon), s(s-\epsilon), -s(s-\epsilon)) \mid s \neq \epsilon/2\}.$$

D_1, D_2 and D_3 intersect each other in the unique triple value $\Phi_\epsilon(0,0) = \Phi_\epsilon(\epsilon, 0) = \Phi_\epsilon(0, \epsilon) = (0,0,0)$.

Let $D_i(f) = D_i \cap S^5$ $(i = 1,2,3)$ denote the disjoint components of the set of the double values of f. Clearly $L(f) = L_1(f) + L_2(f) + L_3(f)$, where $L_i(f)$ is the linking number corresponding to the component $D_i(f)$. But $L_1(f) = L_2(f) = L_3(f)$. Indeed, D_1 and D_2 is interchanged via the transformations $\phi(s,t) = (t,s)$ (of $\mathbb{C}^2$) and $\psi(X,Y,Z) = (Y,X,Z)$ (of $\mathbb{C}^3$), and D_3 and D_2 via $\phi(s,t) = (\epsilon - s - t, t)$ and $\psi(X,Y,Z) = (X + Y + 2Z, Y, -Y - Z)$. Thus, it is enough to calculate $L_1(f)$. The needed vector field along D_1 is

$$N(0, t(t-\epsilon), 0) = \tilde{N}(0,t) + \tilde{N}(0, \epsilon - t) = ((2\bar{t} - \epsilon)^2, 0, 0).$$

The set of the double values of f corresponding to D_1 is

$$D_1(f) = D_1 \cap S^5 = \{(0, t(t-\epsilon), 0) \mid |t(t-\epsilon)| = 1\}.$$

The shifted $D_1(f)$ along N is

$$D_1' = D_1(f) + \delta N = \{(\delta(2\bar{t} - \epsilon)^2, t(t-\epsilon), 0) \mid |t(t-\epsilon)| = 1\},$$

where δ is small enough. Nevertheless, we can choose $\delta = 1$, because $D_1' \cap \Phi(\mathbb{C}^2) = \emptyset$ for any $\delta \in (0, 1]$. With the notation $z = t(t-\epsilon)$ we give an injective parametrization $D_1' = \{(4\bar{z} + \epsilon^2, z, 0) \mid |z| = 1\}$. We fill it with the membrane

$$H = \{(4\bar{z} + \epsilon^2, z, i\sqrt{1 - |z|^2}) \mid |z| \leq 1\}.$$

Computing the intersection points of H and $\Phi(\mathbb{C}^2)$ leads to the equations

$$4\bar{z} + \epsilon^2 = a(a-\epsilon), \quad z = b(b-\epsilon), \quad i\sqrt{1-|z|^2} = ab,$$

with $|z| \leq 1$ and ϵ small. The first two equations imply that $|a| < 5$ and $|b| < 2$. Multiplying the first two equations one gets

$$z(4\bar{z} + \epsilon^2) = a^2b^2 - a^2b\epsilon - ab^2\epsilon + ab\epsilon^2.$$

>From the third equation follows $a^2b^2 = |z|^2 - 1$, hence

$$3|z|^2 = -1 - z\epsilon^2 - a^2b\epsilon - ab^2\epsilon + ab\epsilon^2,$$

and the right hand side is negative if ϵ is small enough. Hence $H \cap \Phi(\mathbb{C}^2) = \emptyset$, and $L(f) = 0$.

11. Final remark. The real version in arbitrary dimension.

11.1. There is a real version of part (b) of Theorem 1.2.2 which follows directly from the result of Whitney and Smale.

Let $\Phi : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{2n+1}, 0)$ be a real analytic germ singular only at 0. With the same method as in the complex case we can associate an immersion $f : S^n \looparrowright S^{2n}$ with Φ (see 2.1). A generalization of Whitney's double point formula valid for plane curve immersions [26] shows that the Smale invariant of f (more precisely, of a generic immersion regular homotopic to f) equals the algebraic number of self-intersection points (mod 2 if n is odd).

A generic perturbation Φ' of Φ has only cross cap type singularities, i.e. locally right–left equivalent with germs of the form $(s, t) \mapsto (s^2, st, t)$, where $s \in \mathbb{R}$ and $t \in \mathbb{R}^n$. These cross caps are isolated, and if n is even, we can associate a sign for each of them. Φ' restricted to the boundary is a generic immersion $f' : S^n \looparrowright S^{2n}$. f' and f are regular homotopic, and f' has two kinds of double values:

(a) double values related to a cross cap (that is, they are connected by a segment consisting of double values of Φ'),

(b) double values not related to a cross cap.

When n is even, the sign associated to a cross cap agrees with the sign associated with the self intersection point of f' related to the cross cap. Thus the algebraic number of such cross caps is equal to the algebraic number of double values of type (a) (mod 2 if n is odd). The double points of type (b) are pairwise joined up by segments of the double values of Φ', thus the algebraic number of them is 0. Moreover, it can happen that two cross caps are joined by a segment consisting of double values of Φ', but then they will have different algebraic sign, hence they will not contribute in the sum. Hence, we proved:

Proposition 11.1.1. *The Smale invariant of f agrees with the algebraic number of the cross cap points appearing in a generic perturbation of Φ (*mod 2 *if n is odd).*

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