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Long time behaviour of Ricci flow on open 3-manifolds

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Abstract. We study the long time behaviour of the Ricci flow with bubbling-off on a possibly noncompact 3-manifold of finite volume whose universal cover has bounded geometry. As an application, we give a Ricci flow proof of Thurston's hyperbolisation theorem for 3-manifolds with toral boundary that generalises Perelman's proof of the hyperbolisation conjecture in the closed case.

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1. Introduction

A Riemannian metric is *hyperbolic* if it is complete and has constant sectional curvature equal to -1. If N is a 3-manifold-with-boundary, then we say it is *hyperbolic* if its interior admits a hyperbolic metric. In the mid-1970s, W. Thurston stated his *Hyperbolisation Conjecture,* which gives a natural sufficient condition on the topology of a 3-manifold-with-boundary N which implies that it is hyperbolic. Recall that N is *irreducible* if every embedded 2-sphere in N bounds a 3-ball. It is *atoroidal* if every incompressible embedded 2-torus in N is parallel to a component of ∂N or bounds a product neighbourhood $T^2 \times [0, 1)$ of an end of N. A version of Thurston's conjecture states that if N is compact, connected, orientable, irreducible, and $\pi_1 N$ is infinite and does not have any subgroup isomorphic to $\mathbf{Z}^2$, then N is hyperbolic. If one replaces the hypotheses on the fundamental group by the assumption that N is atoroidal then one gets the conclusion that N is hyperbolic or Seifert fibred.

Thurston proved his conjecture for the case of so-called *Haken manifolds*, which includes the case where ∂N is nonempty. The case where N is closed was solved by G. Perelman [17, 18] using Ricci flow with surgery, based on ideas of R. Hamilton.

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It is natural to ask whether the Hamilton-Perelman approach works when $\partial N \neq \emptyset$. The interior M of N, on which one wishes to construct a hyperbolic metric, is then noncompact. This question can be divided into two parts: first, is it possible to construct some version of Ricci flow with surgery on such an open manifold M, under reasonable assumptions on the initial metric? Second, does it converge (modulo scaling) to a hyperbolic metric? A positive answer to both questions would give a Ricci flow proof of the full Hyperbolisation Conjecture logically independent of Thurston's results.

A positive answer to the first question was given in [5], for initial metrics of bounded geometry, i.e. of bounded curvature and positive injectivity radius. If one considers irreducible manifolds, surgeries are topologically trivial: each surgery sphere bounds a 3-ball. Hence a surgery splits off a 3-sphere. In this situation we can refine the construction of the Ricci flow with surgery so that it is not necessary to perform the surgery topologically. We obtain a solution which is a piecewise smooth Ricci flow on a fixed manifold; at singular times, one performs only a metric surgery, changing the metric on some 3-balls. This construction was defined in [4] in the case of closed irreducible nonspherical 3-manifolds, and called Ricci flow with bubbling-off. One can extend it to the setting of bounded geometry. The purpose of this paper is to answer the second question, in the situation where the initial metric has a *cusp-like structure*.

Definition 1.1. We say that a metric g on M has a *cusp-like structure*, or is a *cusp-like metric*, if M has finitely many ends (possibly zero), and each end has a neighbourhood which admits a metric g_{cusp} homothetic to a rank two cusp neighbourhood of a hyperbolic manifold such that $g - g_{\text{cusp}}$ goes to zero at infinity in C^k-norm for all positive integers k. (Thus if M is closed, any metric is cusp-like.)

Note that such a metric is automatically complete with bounded curvature and of finite volume, but its injectivity radius equals zero hence it does not have bounded geometry. However, except in the case where M is homeomorphic to a solid torus, its universal covering does have bounded geometry (see Lemma 2.21). Since solid tori are Seifert fibred, we will assume that M is not homeomorphic to a solid torus when necessary. Also note that if M admits a cusp-like metric, then M admits a manifold compactification whose boundary is empty or a union of 2-tori. This compactification is irreducible (resp. atoroidal, resp. Seifert-fibred) if and only if M is irreducible (resp. atoroidal, resp. Seifert-fibred).

In section 2 we construct a Ricci flow with bubbling-off on M, for any cusp-like initial metric, by passing to the universal cover and working equivariantly. For simplicity we restrict ourselves to the case where M is nonspherical. This is not a problem since spherical manifolds are Seifert fibred. We also prove that the cusp-like structure is preserved by this flow (cf. Theorem 2.22).

Using this tool, we can adapt Perelman's proof of geometrisation to obtain the following result:

Theorem 1.2. *Let M be a connected, orientable, irreducible, atoroidal 3-manifold and g_0 be a metric on M which is cusp-like at infinity. Then M is Seifert-fibred, or there exists a Ricci flow with bubbling-off $g(\cdot)$ on M defined on $[0, \infty)$, such that $g(0) = g_0$, and as t goes to infinity, $(4t)^{-1}g(t)$ converges smoothly in the pointed topology for appropriate base points to some finite volume hyperbolic metric on M. Moreover, $g(\cdot)$ has finitely many singular times, and there are positive constants T, C such that $|\operatorname{Rm}| < Ct^{-1}$ for all $t \geq T$.*

If N is a compact, connected, orientable 3-manifold such that ∂N is empty or a union of 2-tori, then $M = \operatorname{int} N$ always carries a cusp-like at infinity metric. Thus we obtain:

Corollary 1.3 (Thurston, Perelman). *Let N be a compact, connected, orientable 3-manifold-with-boundary such that ∂N is empty or a union of 2-tori. If N is irreducible and atoroidal, then N is Seifert-fibred or hyperbolic.*

Note that it should be possible to obtain this corollary directly from the closed case by a doubling trick. The point of this paper is to study the behaviour of Ricci flow in the noncompact case.

Let us review some results concerning global stability or convergence to finite volume hyperbolic metrics. In the case of surfaces, R. Ji, L. Mazzeo and N. Sesum [11] show that if (M, g_0) is complete, asymptotically hyperbolic of finite area with $\chi(M) < 0$, then the normalised Ricci flow with initial condition g_0 converges exponentially to the unique complete hyperbolic metric in its conformal class. G. Giesen and P. Topping [7, Theorem 1.3] show that if g_0, possibly incomplete and with unbounded curvature, is in the conformal class of a complete finite area hyperbolic metric g_{hyp}, then there exists a unique Ricci flow with initial condition g_0 which is instantaneously complete and maximaly stretched (see the precise definition in [7]), defined on $[0, +\infty)$ and such that the rescaled solution $(2t)^{-1}g(t)$ converges smoothly locally to g_0 as $t \to \infty$. Moreover, if $g_0 \leq C g_{\text{hyp}}$ for some constant $C > 0$ then the converence is global: for any $k \in \mathbf{N}$ and $\mu \in (0, 1)$ there exists a constant $C > 0$ such that for all $t \geq 1$, $|(2t)^{-1}g(t) - g_{\text{hyp}}|_{C^k(M, g_{\text{hyp}})} < \frac{C}{t^{1-\mu}}$. In dimensions greater than or equal to 3, R. Bamler [3] shows that if g_0 is a small C^0-perturbation of a complete finite volume hyperbolic metric g_{hyp}, that is if $|g_0 - g_{\text{hyp}}|_{C^0(M, g_{\text{hyp}})} < \varepsilon$ where $\varepsilon = \varepsilon(M, g_{hyp}) > 0$, then the normalised Ricci flow with initial condition g_0 is defined for all time and converges in the pointed Gromov-Hausdorff topology to g_{hyp}. In dimension 3 at least, there cannot be any global convergence result. Indeed, consider a complete finite volume hyperbolic manifold (M^3, g_{hyp}) with at least one cusp. Let g_0 be a small C^0 pertubation of g_{hyp} such that g_0 remains cusp-like at infinity but with a different hyperbolic structure in the given cusp (change the cross-sectional flat structure on the cusp). By Bamler [3] a rescaling of $g(t)$ converges in the pointed topology to g_{hyp}. The pointed convergence takes place on balls of radius R for all R; however, our stability theorem 2.22 implies that, out of these balls, the

cusp-like structure of g_0 is preserved for all time, hence is different from the one of the pointed limit. Thus the convergence cannot be global.

The paper is organised as follows. In Section 2 we introduce the necessary definitions and we prove the existence of a Ricci flow with bubbling-off which preserves cusp-like structures. Section 3 is devoted to a thick-thin decomposition theorem which shows that the thick part of $(M, (4t)^{-1}g(t))$ (sub)-converges to a complete finite volume hyperbolic manifold. We give also some estimates on the long time behaviour of our solutions. In Section 4 we prove the incompressibility of the tori bounding the thick part. Section 5 is devoted to a collapsing theorem, which is used to show that the thin part is a graph manifold. Finally the main theorem 1.2 is proved in Section 6. To obtain the curvature estimates on the thin part, we follow [1]. An overview of the proof is given at the beginning of that section.

Throughout this paper, we will use the following convention: *all 3-manifolds are connected and orientable.*

2. Ricci flow with bubbling-off on open manifolds

2.1. Definition and existence. In this section we define Ricci flow with bubbling-off and state the main existence theorem.

For convenience of the reader we recall here the most important definitions involved, and refer to Chapters 2, 4, and 5 of the monograph [4] for completeness.

Definition 2.1 (Evolving metric). Let M be an n-manifold and $I \subset \mathbf{R}$ be an interval. An *evolving metric* on M defined on I is a map $t \mapsto g(t)$ from I to the space of smooth Riemannian metrics on M. A *regular* time is a value of t such that this map is C^1-smooth in a neighbourhood of t. If t is not regular, then it is *singular*. We denote by $g_+(t)$ the right limit of g at t, when it exists. An evolving metric is *piecewise* C^1 if singular times form a discrete subset of $\mathbf{R}$ and if $t \mapsto g(t)$ is left continuous and has a right limit at each point. A subset $N \times J \subset M \times I$ is *unscathed* if $t \to g(t)$ is smooth there. Otherwise it is *scathed*.

If g is a Riemannian metric, we denote by $R_{\min}(g)$ (resp. $R_{\max}(g)$) the infimum (resp. the supremum) of the scalar curvature of g. For any $x \in M$, we denote by $\mathrm{Rm}(x) : \Lambda^2 T_x M \to \Lambda^2 T_x M$ the *curvature operator* defined by

$$\langle \mathrm{Rm}(X \wedge Y), Z \wedge T \rangle = \mathrm{Riem}(X, Y, Z, T),$$

where Riem is the Riemann curvature tensor and $\wedge$ and $\langle \cdot, \cdot \rangle$ are normalised so that $\{e_i \wedge e_j \mid i < j\}$ is an orthonormal basis if $\{e_i\}$ is. In particular, if $\lambda \geq \mu \geq \nu$ are the eigenvalues of Rm, then λ (resp. ν) is the maximal (resp. minimal) sectional curvature and $R = 2(\lambda + \mu + \nu)$. [1]

[1] This convention is different from that used by Hamilton and other authors.

Definition 2.2 (Ricci flow with bubbling-off). A piecewise C^1 evolving metric $t \mapsto g(t)$ on M defined on I is a *Ricci flow with bubbling-off* if

(i) The Ricci flow equation $\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}$ is satisfied at all regular times;

(ii) for every singular time $t \in I$ we have

(a) $R_{\min}(g_+(t)) \geqslant R_{\min}(g(t))$, and

(b) $g_+(t) \leqslant g(t)$.

Remark 2.3. If $g(\cdot)$ is a complete Ricci flow with bubbling-off of bounded sectional curvature defined on an interval of type $[0, T]$ or $[0, \infty)$, and if $g(0)$ has finite volume, then $g(t)$ has finite volume for every t.

A *parabolic neighbourhood* of a point $(x, t) \in M \times I$ is a set of the form

$$P(x, t, r, -\Delta t) = \{(x', t') \in M \times I \mid x' \in B(x, t, r), t' \in [t - \Delta t, t]\}.$$

Definition 2.4 (κ-noncollapsing). For $\kappa, r > 0$ we say that $g(\cdot)$ is *κ-collapsed* at (x, t) on the scale r if for all (x', t') in the parabolic neighbourhood $P(x, t, r, -r^2)$ we have $|\mathrm{Rm}(x', t')| \leq r^{-2}$ and $\mathrm{vol}(B(x, t, r)) < \kappa r^n$. Otherwise, $g(\cdot)$ is *κ-noncollapsed* at (x, t) on the scale r. If this is true for all $(x, t) \in M \times I$, then we say that $g(\cdot)$ is *κ-noncollapsed on the scale r*.

Next is the definition of *canonical neighbourhoods*. From now on and until the end of this section, M is a 3-manifold and ε, C are positive numbers.

Definition 2.5 (ε-closeness, ε-homothety). If $U \subset M$ is an open subset and g, g_0 are two Riemannian metrics on U we say that g is ε-close to g_0 on U if

$$||g - g_0||_{[\varepsilon^{-1}, U, g_0]} < \varepsilon,$$

where the norm is defined on page 26 of [4]. We say that g is ε-homothetic to g_0 on U if there exists $\lambda > 0$ such that λg is ε-close to g_0 on U. A pointed Riemannian manifold (U, g, x) is ε-close to another Riemannian manifold (U_0, g_0, x_0) if there exists a $C^{[\varepsilon^{-1}]+1}$-diffeomorphism ψ from U_0 to U sending x_0 to x and such that the pullback metric $\psi^*(g)$ is ε-close to g_0 on U. We say that (U, g, x) is ε-homothetic to (U_0, g_0, x_0) if there exists $\lambda > 0$ such that $(U, \lambda g, x)$ is ε-close to (U_0, g_0, x_0).

Definition 2.6 (ε-necks, ε-caps). Let g be a Riemannian metric on M. If x is a point of M, then an open subset $U \subset M$ is an *ε-neck centred at x* if (U, g, x) is ε-homothetic to $(S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}), g_{\mathrm{cyl}}, (*, 0))$, where g_{cyl} is the standard metric with unit scalar curvature. An open set U is an *ε-cap centred at x* if U is the union of two sets V, W such that $x \in \mathrm{int}\, V$, V is a closed 3-ball, $\bar{W} \cap V = \partial V$, and W is an ε-neck.

Definition 2.7 ((ε, C)-cap). An open subset $U \subset M$ is an (ε, C)-*cap centred at* x if U is an ε-cap centred at x and satisfies the following estimates: $R(x) > 0$ and there exists $r \in (C^{-1}R(x)^{-1/2}, CR(x)^{-1/2})$ such that

(i) $\overline{B(x,r)} \subset U \subset B(x,2r)$;

(ii) The scalar curvature function restricted to U has values in a compact subinterval of $(C^{-1}R(x), CR(x))$;

(iii) $\mathrm{vol}(U) > C^{-1}R(x)^{-3/2}$ and if $B(y,s) \subset U$ satisfies $|\mathrm{Rm}| \leqslant s^{-2}$ on $B(y,s)$ then

$$C^{-1} < \frac{\mathrm{vol}B(y,s)}{s^3};$$

(iv) On U,

$$|\nabla R| < CR^{\frac{3}{2}},$$

(v) On U,

$$|\Delta R + 2|\mathrm{Ric}|^2| < CR^2, \tag{2.1}$$

(vi) On U,

$$|\nabla \mathrm{Rm}| < C|\mathrm{Rm}|^{\frac{3}{2}},$$

Remark 2.8. If $t \mapsto g(t)$ is a Ricci flow, then $\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Ric}|^2$ hence equation (2.1) implies that $|\frac{\partial R}{\partial t}| \leq CR^2$.

Definition 2.9 (Strong ε-neck). We call *cylindrical flow* the pointed evolving manifold $(S^2 \times \mathbf{R}, \{g_{\mathrm{cyl}}(t)\}_{t\in(-\infty,0]})$, where $g_{\mathrm{cyl}}(\cdot)$ is the product Ricci flow with round first factor, normalised so that the scalar curvature at time 0 is 1. If $g(\cdot)$ is an evolving metric on M, and (x_0, t_0) is a point in spacetime, then an open subset $N \subset M$ is a *strong* ε*-neck centred at* (x_0, t_0) if there exists $Q > 0$ such that $(N, \{g(t)\}_{t\in[t_0-Q^{-1},t_0]}, x_0)$ is unscathed, and, denoting $\bar{g}(t) = Qg(t_0 + tQ^{-1})$ the parabolic rescaling with factor $Q > 0$ at time t_0, $(N, \{\bar{g}(t)\}_{t\in[-1,0]}, x_0)$ is ε-close to $(S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}), \{g_{\mathrm{cyl}}(t)\}_{t\in[-1,0]}, *)$.

Remark 2.10. A strong ε-neck satisfies the estimates (i)–(vi) of Definition 2.7 for an appropriate constant $C = C(\varepsilon)$, at all times, that is on all $N \times [t_0 - Q^{-1}, t_0]$ for any $Q > 0$ as above.

Definition 2.11 ((ε, C)-canonical neighbourhood). Let $\{g(t)\}_{t\in I}$) be an evolving metric on M. We say that a point (x,t) admits (or is centre of) an (ε, C)-*canonical neighbourhood* if x is centre of an (ε, C)-cap in $(M, g(t))$ or if (x,t) is centre of a strong ε-neck N which satisfies (i)–(vi) at all times.

In [4, Section 5.1] we fix constants ε_0, C_0. For technical reasons, we need to take them slightly different here; this will be explained in the proof of Theorem 2.17.

Definition 2.12 (Canonical Neighbourhood Property $(CN)_r$). Let $r > 0$. An evolving metric satisfies the property $(CN)_r$ if, for any (x,t), if $R(x,t) \geq r^{-2}$ then (x,t) is centre of an (ε_0, C_0)-canonical neighbourhood.

Next we define a pinching property for the curvature tensor coming from work of Hamilton and Ivey [8, 10]. We consider a familly of positive functions $(\phi_t)_{t\geqslant 0}$ defined as follows. Set $s_t := \frac{e^2}{1+t}$ and define

$$\phi_t : [-2s_t, +\infty) \longrightarrow [s_t, +\infty)$$

as the reciprocal of the increasing function

$$s \mapsto 2s(\ln(s) + \ln(1+t) - 3).$$

Compared with the expression used in [8, 10], there is an extra factor 2 here. This comes from our curvature conventions. A key property of this function is that $\frac{\phi_t(s)}{s} \to 0$ as $s \to +\infty$.

Definition 2.13 (Curvature pinched toward positive). Let $I \subset [0,\infty)$ be an interval and $\{g(t)\}_{t\in I}$ be an evolving metric on M. We say that $g(\cdot)$ has *curvature pinched toward positive at time t* if for all $x \in M$ we have

$$R(x,t) \geqslant -\frac{6}{4t+1}, \tag{2.2}$$

$$\mathrm{Rm}(x,t) \geqslant -\phi_t(R(x,t)). \tag{2.3}$$

We say that $g(\cdot)$ has *curvature pinched toward positive* if it has curvature pinched toward positive at each $t \in I$.

This allows in particular to define the notion of *surgery parameters* r, δ (cf. [4, Definition 5.2.5]). Using [4, Theorem 5.2.4] we also define their *associated cutoff parameters* h, Θ. Using the metric surgery theorem, we define the concept of a metric g_+ being *obtained from* $g(\cdot)$ *by* (r,δ)*-surgery at time* t_0 (cf. [4, Definition 5.2.7]). This permits to define the following central notion:

Definition 2.14 (Ricci flow with (r,δ)-bubbling-off). Fix surgery parameters r, δ and let h, Θ be the associated cutoff parameters. Let $I \subset [0,\infty)$ be an interval and $\{g(t)\}_{t\in I}$ be a Ricci flow with bubbling-off on M. We say that $\{g(t)\}_{t\in I}$ is a *Ricci flow with* (r,δ)*-bubbling-off* if it has the following properties:

(i) $g(\cdot)$ has curvature pinched toward positive and satisfies $R(x,t) \leqslant \Theta$ for all $(x,t) \in M \times I$;

(ii) For every singular time $t_0 \in I$, the metric $g_+(t_0)$ is obtained from $g(\cdot)$ by (r,δ)-surgery at time t_0;

(iii) $g(\cdot)$ satisfies property $(CN)_r$.

Definition 2.15 (Ricci flow with (r, δ, κ)-bubbling-off). Let $\kappa > 0$. A Ricci flow with (r, δ)-bubbling-off $g(\cdot)$ is called a *Ricci flow with (r, δ, κ)-bubbling-off* if it is κ-noncollapsed on all scales less than or equal to 1.

Definition 2.16. A metric g on a 3-manifold M is *normalised* if it satisfies $\operatorname{tr} \mathrm{Rm}^2 \leq 1$ and each ball of radius 1 has volume at least half of the volume of the unit ball in Euclidean 3-space.

Note that a normalised metric always has bounded geometry. At last we can state our existence theorem:

Theorem 2.17. *There exist decreasing sequences of positive numbers $r_k, \kappa_k > 0$ and, for every continuous positive function $t \mapsto \bar{\delta}(t)$, a decreasing sequence of positive numbers δ_k with $\delta_k \leqslant \bar{\delta}(\cdot)$ on $]k, k+1]$ with the following property. For any complete, normalised, nonspherical, irreducible Riemannian 3-manifold (M, g_0), one of the following conclusions holds:*

(i) *There exists $T > 0$ and a complete Ricci flow with bubbling-off $g(\cdot)$ of bounded geometry on M, defined on $[0, T]$, with $g(0) = g_0$, and such that every point of $(M, g(T))$ is centre of an ε_0-neck or an ε_0-cap, or*

(ii) *There exists a complete Ricci flow with bubbling-off $g(\cdot)$ of bounded geometry on M, defined on $[0, +\infty)$, with $g(0) = g_0$, and such that for every nonnegative integer k, the restriction of $g(\cdot)$ to $]k, k+1]$ is a Ricci flow with $(r_k, \delta_k, \kappa_k)$-bubbling-off.*

Definition 2.18 (Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off). We fix forever a function $r(\cdot)$ such that $r(t) = r_k$ on each interval $]k, k+1]$. Given $\delta(\cdot)$ satisfying $\delta(t) = \delta_k$ on all $]k, k+1]$, we call a solution as above a *Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off*. We define similarly $h(\cdot)$ and $\Theta(\cdot)$ their associated cutoff parameters.

Addendum 2.19 (Ricci flow with bubbling-off on the quotient). *With the same notation as in Theorem 2.17 and under the same hypotheses, if in addition (M, g_0) is a Riemannian cover of some Riemannian manifold $(X, \bar{g}_0)$, then in either case there exists a Ricci flow with bubbling-off $\bar{g}(\cdot)$ on X such that for each t, $(M, g(t))$ is a Riemannian cover of $(X, \bar{g}(t))$, and in Case (ii), the restriction of $\bar{g}(\cdot)$ to $]k, k+1]$ is a Ricci flow with (r_k, δ_k)-bubbling-off for every k.*

The only differences between Theorem 2.17 and Theorem 11.5 of [5] is that M is assumed to be irreducible, that 'surgical solution' is replaced with 'Ricci flow with bubbling-off', and that there is the alternative conclusion (i).

Theorem 2.17 follows from iteration of the following result, which is analogous to [5, Theorem 5.6]:

Theorem 2.20. *For every Q_0, ρ_0 and all $0 \leq T_A < T_\Omega < +\infty$, there exist $r, \kappa > 0$ and for all $\bar{\delta} > 0$ there exists $\delta \in (0, \bar{\delta})$ with the following property. For any complete, nonspherical, irreducible Riemannian 3-manifold (M, g_0) which satisfies*

$|\mathrm{Rm}| \leq Q_0$, *has injectivity radius at least* ρ_0, *has curvature pinched toward positive at time* T_A, *one of the following conclusions holds:*

(i) *There exists* $T \in (T_A, T_\Omega)$ *and a Ricci flow with bubbling-off* $g(\cdot)$ *on* M, *defined on* $[T_A, T]$, *with* $g(T_A) = g_0$, *and such that every point of* $(M, g(T))$ *is centre of an* ε_0*-neck or an* ε_0*-cap, or*

(ii) *There exists a Ricci flow with* (r, δ, κ)*-bubbling-off* $g(\cdot)$ *on* M, *defined on* $[T_A, T_\Omega]$, *satisfying* $g(T_A) = g_0$.

The proof of Theorem 2.20 is the same as [5, Theorem 5.6]. It follows from three propositions, which we do not write here, analogous to Propositions A, B, C of [5] (see the propositions page 949). The only notable difference is that we have to modify Proposition A to add the alternative conclusion that in $(M, g(b))$, every point is centre of an ε_0-cap or an ε_0-neck. Let us explain the proof of this adapted proposition A (see [5] pages 959–961). It uses the surgical procedure of the monograph [4] rather than that of [5]. If the curvature is large everywhere, that is if $R \geq 2r^{-2}$ on $(M, g(b))$ where r is the surgery parameter, then by property $(CN)_r$ (Definitions 2.10 and 2.12 (iii)) every point has a canonical neighbourhood, so the alternative conclusion holds. Otherwise, we partition M in three sets of small, large or very large curvature. Precisely, as in [4, page 89], we define $\mathcal{G}$ (resp. $\mathcal{O}$, resp. $\mathcal{R}$) as the set of points of M of scalar curvature less than $2r^{-2}$, (resp. $\in [2r^{-2}, \Theta/2)$, resp. $\geqslant \Theta/2$). By the assumption that $R_{\min}(b) < 2r^{-2}$ and $R_{\max}(b) = \Theta$, these sets are nonempty. One can find a locally finite collection of cutoff δ-necks $\{N_i\}$ in $\mathcal{O}$ which separates $\mathcal{G}$ from $\mathcal{R}$, in the sense that any connected component of $M \setminus \{N_i\}$ is contained in $\mathcal{G} \cup \mathcal{O}$ or in $\mathcal{O} \cup \mathcal{R}$. Since M is irreducible and not homeomorphic to S^3, the middle sphere of each N_i bounds a unique topological 3-ball B_i. Then one of the following cases occurs:

Case 1 Each B_i is contained in a unique maximal 3-ball B_j.

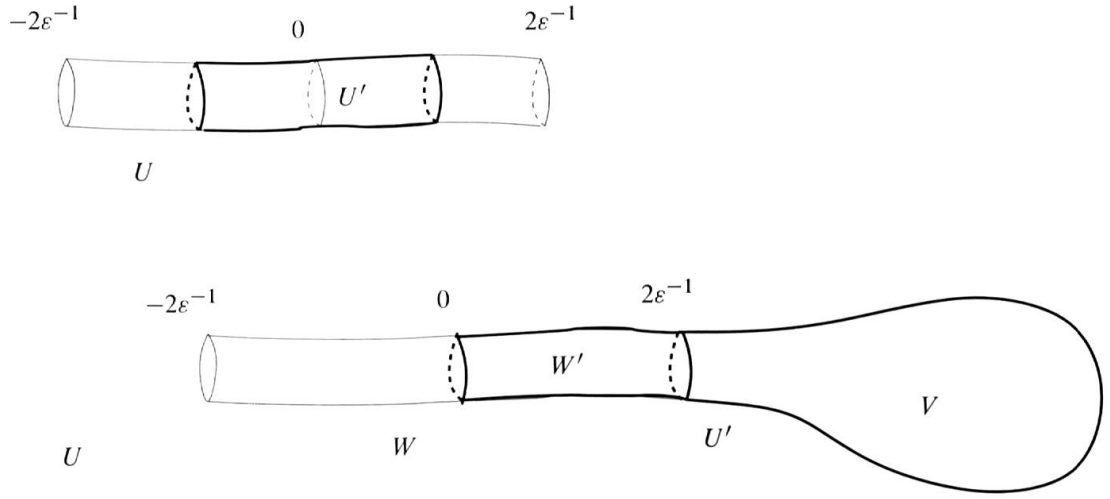
If $\mathcal{O}$ is contained in the union of maximal B_j's, we can perform the surgical procedure using the Metric surgery theorem 5.2.2 of [4] on each maximal cap B_j, yielding a metric which has the desired properties. Otherwise one can see that each point of M is centre of ε-cap. Hence the alternative conclusion holds.

Case 2 M is the union of the B_i's.

Then each point is separated from infinity by a cutoff neck, so each point is centre of a cap. Hence the alternative conclusion holds.

Finally, we need to explain how the addendum is proved. We already remarked in [5] Section 11 that the construction can be made equivariant with respect to a properly discontinuous group action, by work of Dinkelbach and Leeb [6]. The only thing to check is that we still have the Canonical Neighbourhood Property for the quotient evolving metric $\bar{g}(\cdot)$. This is not obvious, since the projection map $p : M \to X$ might not be injective when restricted to a canonical neighbourhood.

We use a classical trick: by adjusting the constants, we may assume that $g(\cdot)$ has the stronger property that each point (x,t) such that $R(x,t) \geq r^{-2}$ has an $(\varepsilon_0/2, C_0)$-canonical neighbourhood. Take now $(x,t) \in X \times I$ such that $R(x,t) \geq r^{-2}$. Choose $\bar{x} \in M$ such that $p(\bar{x}) = x$. Then $R(\bar{x},t) = R(x,t) \geq r^{-2}$, so $(\bar{x},t)$ has an $(\varepsilon_0/2, C_0)$-canonical neighbourhood U. By truncation, it also has an (ε_0, C_0)-canonical neighbourhood U' contained in U (see figure below) :



Precisely, if U is an $\varepsilon_0/2$-neck with parametrisation $\phi : S^2 \times (-2\varepsilon_0^{-1}, 2\varepsilon_0^{-1}) \to U$, we set $U' := \phi(S^2 \times (-\varepsilon_0^{-1}, \varepsilon_0^{-1}))$. If U is a cap, then U is the union of two sets V, W, where $\overline{W} \cap V = \partial V$ and W is an $\varepsilon_0/2$-neck with parametrisation ϕ. Then we set $W' := \phi(S^2 \times (0, 2\varepsilon_0^{-1}))$ and $U' := V \cup W'$.

Claim 1. *The restriction of the projection map p to U' is injective.*

Once the claim is proved, we can just project U' to X and obtain an (ε_0, C_0)-canonical neighbourhood for (x,t), so we are done.

To prove the claim we consider two cases:

Case 1. U and U' are caps.

Assume by contradiction that there is an element γ in the deck transformation group, different from the identity, and a point $y \in U'$ such that $\gamma y \in U'$. Following [6], we consider the subset N_{ε_0} of M consisting of points which are centres of ε_0-necks. According to [6, Lemmas 3.6, 3.7] there is an open supset $F \supset \overline{N_{\varepsilon_0}}$ which has an equivariant foliation $\mathcal{F}$ by almost round 2-spheres. All points sufficiently close to the centre of W are centres of ε_0-necks.

Pick a point z in $N_{\varepsilon_0} \cap W \setminus W'$ sufficiently far from W' so that the leaf S of $\mathcal{F}$ through z is disjoint from U'. By Alexander's theorem, S bounds a 3-ball $B \subset U$. Note that B contains U'. If $S = \gamma S$, then $B = \gamma B$ or $M = B \cup \gamma B$. The former possibility is ruled out by the fact that the action is free, while any self-homeomorphism of the 3-ball has a fixed point. The latter is ruled out by the assumption that M is not diffeomorphic to S^3.

Hence $S \neq \gamma S$. Since S and γS are leafs of a foliation, they are disjoint. Then we have the following three possibilities:

Subcase a. γS is contained in B.

Then we claim that $\gamma B \subset B$. Indeed, otherwise we would have $M = B \cup \gamma B$, and M would be diffeomorphic to S^3. Now γ acts by isometry, so $\mathrm{vol}B = \mathrm{vol}\gamma B$. This is impossible since the annular region between S and γS has nonzero volume.

Subcase b. S is contained in γB. This case is ruled out by a similar argument exchanging the roles of S and γS (resp. of B and γB.)

Subcase c. B and γB are disjoint.

Then since $U' \subset B$, the sets U' and $\gamma U'$ are also disjoint, contradicting the existence of y.

Case 2. U and U' are necks. Seeking a contradiction, let γ be an element of the deck transformation group, different from the identity, and y be a point of U' such that $\gamma y \in U'$. Consider again the set N_{ε_0} defined above and the equivariant foliation $\mathcal{F}$. Since U' is contained in the bigger set U, each point of U' is centre of an ε_0-neck. Let S (resp. γS) be the leaf of $\mathcal{F}$ passing through y (resp. γy.) Since M is irreducible, S (resp. γS) bounds a 3-ball B (resp. B_γ). As in the previous case, we argue that one of these balls is contained into the other, otherwise we could cover M by B, B_γ and possibly an annular region between them, and get that M is diffeomorphic to S^3. Since γ acts by an isometry, we must in fact have $B = B_\gamma$, and γ has a fixed point, contradicting our hypotheses. This finishes the proof of the claim, hence that of Addendum 2.19.

2.2. Stability of cusp-like structures. In this section, we prove the stability of cusp-like structures under Ricci flow with bubbling-off. We consider a (nonspherical, irreducible) 3-manifold M, endowed with a cusp-like metric g_0. To begin we remark that the universal cover of M has bounded geometry, except in the case of solid tori:

Lemma 2.21. *Assume that M is not homeomorphic to a solid torus. Let $(\tilde{M}, \tilde{g}_0)$ denote the universal cover of (M, g_0). Then $(\tilde{M}, \tilde{g}_0)$ has bounded geometry.*

Proof. Sectional curvature is bounded on (M, g_0), hence on the universal cover $(\tilde{M}, \tilde{g}_0)$ by the same constant. Observe that for any lift $\tilde{x} \in \tilde{M}$ of some $x \in M$, the injectivity radius at $\tilde{x}$ is not less than the injectivity radius at x. Fix a compact subset $K \subset M$ such that each connected component C of $M \setminus K$ is ε-homothetic to a hyperbolic cusp neighbourhood, for some small $\varepsilon > 0$. Let $\tilde{K}$ denote any lift of K to $\tilde{M}$. Then the 5-neighbourhood of $\tilde{K}$ has injectivity radius bounded below by $i_0 > 0$, the injectivity radius of the (compact) 5-neighbourhood of K. Now consider a lift $\tilde{C}$ of a cuspidal component C. The boundary ∂C is incompressible in M, otherwise M would be homeomorphic to a solid torus (see Theorem A.3.1 in [4]). It follows that $\tilde{C}$ is simply connected with an incomplete metric of negative sectional curvature. Arguing as in the proof of the Hadamard theorem, it follows that the

injectivity radius at a given point $p \in \tilde{C}$ is not less than $d(p, \partial\tilde{C})$. Together with the previous estimate, this implies that $\mathrm{inj}(\tilde{M}, \tilde{g}_0) \geq \min\{i_0, 5\} > 0$. □

Let us denote by g_c a metric on M which is hyperbolic on the complement of some compact subset of M, and such that, for each end E of M there is a factor $\lambda_E > 0$ such that $\lambda_E g_0 - g_c$ goes to zero at infinity in the end, in C^k-norm for each integer k. Let $g(\cdot)$ be a Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off on M such that $g(0) = g_0$, defined on $[0, T]$ for some $T > 0$. Set $\lambda_E(t) = \frac{\lambda_E}{1+4\lambda_E t}$. We then have:

Theorem 2.22 (Stability of cusp-like structures). *For each end E of M, $\lambda_E(t)g(t) - g_c$ goes to zero at infinity in this end, in C^k-norm for each integer k, uniformly for $t \in [0, T]$.*

Proof. Let us first explain the idea. It is enough to work on each cusp. The main tool is the Persistence Theorem 8.1.3 from [4], which proves that a Ricci flow remains close, on a parabolic neighbourhood where it has a priori curvature bounds, to a given Ricci flow model, if the initial data are sufficiently close on some larger balls. The model we use now is a hyperbolic Ricci flow on $T^2 \times \mathbf{R}$. To obtain the required curvature bounds, we shall consider an interval $[0, t]$ where the closeness to the hyperbolic flow holds, and $\sigma > 0$ fixed small enough so that Property $(CN)_r$, which prevents scalar curvature to explode too fast, gives curvature bounds on $[0, t + \sigma]$. The Persistence Theorem then gives closeness to the hyperbolic flow until time $t + \sigma$ on a smaller neighbourhood of the cusp. One can iterate this procedure, shrinking the neighbourhood of the cusp by a definite amount at each step, until time T.

Let us now give the details. Let E be an end of M and U be a neighbourhood of E such that (U, g_c) is isometric to $(\mathbf{T}^2 \times [0, +\infty), g_{\mathrm{hyp}} = e^{-2r} g_{\mathbf{T}^2} + dr^2)$, where $g_{\mathbf{T}^2}$ is flat. Let $\phi : \mathbf{T}^2 \times [0, +\infty) \to U$ be an isometric parametrisation (between g_c and g_{hyp}.) Then $\lambda_E \phi^* g_0 - g_{\mathrm{hyp}}$ and its derivatives go to zero at infinity. We may assume for simplicity that $\lambda_E = 1$, and we define $\bar{g}(t) := \phi^* g(t)$ to be the pullback Ricci flow with bubbling-off on $\mathbf{T}^2 \times [0, +\infty)$. Let $g_{\mathrm{hyp}}(\cdot)$ denote the Ricci flow on $\mathbf{T}^2 \times \mathbf{R}$ such that $g_{\mathrm{hyp}}(0) = e^{-2r} g_{\mathbf{T}^2} + dr^2$, i.e. $g_{\mathrm{hyp}}(t) = (1 + 4t) g_{\mathrm{hyp}}$. We use it as the Ricci flow model, in the sense of [4, Theorem 8.1.3.]. Our goal is to compare $g_{\mathrm{hyp}}(t)$ to $\bar{g}(t)$.

By definition of our Ricci flow with bubbling-off, $r(\cdot)$ and $\Theta(\cdot)$ are piecewise constant. More precisely, there exist $0 = t_0 < t_1 < \cdots < t_N = T$ such that $r(t) = r_i$ and $\Theta(t) = \Theta_i$ on $(t_i, t_{i+1}]$. In fact, we can choose $t_i = i$ for $i < N$ (cf. Definition 2.18). In particular, $g(t)$ satisfies the canonical neighbourhood property at scale r_i on this interval (every point at which the scalar curvature is greater than r_i^{-2} is centre of an (ε_0, C_0) canonical neighbourhood) and the scalar curvature is bounded above by Θ_i. The pinching assumption (cf. Definition 2.13) then implies that the full curvature tensor is bounded by some K_i on the same interval.

Set $K := \sup_{i=1,\dots,N-1}\{K_i\}$. Define a small number $\sigma > 0$ by setting

$$\sigma := \frac{r_{N-1}^2}{2C_0} \leq \frac{r_i^2}{2C_0} \quad \forall i = 0, \dots, N-1 .$$

This number is small enough so that $g(\cdot)$ cannot develop a singularity on a cusp on $[t, t+\sigma]$ if $R \leq 0$ at time t. Precisely, let us put $\mathcal{C}_s := \mathbf{T}^2 \times [s, +\infty)$, for $s \geq 0$. Then we have:

Lemma 2.23. *If $\bar{g}(\cdot)$ is unscathed on $\mathcal{C}_s \times [0, \Delta]$ and has scalar curvature $R \leqslant 0$ there, then it is also unscathed on $\mathcal{C}_s \times [0, \Delta + \sigma]$ and has curvature tensor bounded by K.*

Proof. We know that singular times are discrete. Let $t \in [0, \sigma]$ be maximal such that $\mathcal{C}_s \times [0, \Delta + t]$ is unscathed for $\bar{g}(\cdot)$ (possibly $t = 0$).

We prove first that for $x \in \mathcal{C}_s$ and $t' \in [\Delta, \Delta + t]$ we have

$$R(x, t') \leq 2r(t')^{-2} << h(t')^{-2}.$$

Indeed, since $r(\cdot)$ is nonincreasing, $g(\cdot)$ satisfies $(CN)_{r(\Delta+t)}$ on $[\Delta, t']$. If $R(x, \Delta) \leq 0$ and $R(x, t') > 2r(t')^{-2}$, then we can find a subinterval $[t_1, t_2] \subset [\Delta, t']$ such that for $u \in [t_1, t_2]$, $R(x, u) \geq r(t')^{-2}$, $r(x, t_1) = r(t')^{-2}$, and $r(x, t_2) = 2r(t')^{-2}$.

Then the inequality $|\frac{\partial R}{\partial t}| < C_0 R^2$ holds on $\{x\} \times [t_1, t_2]$, thanks to Property (2.1) of canonical neighbourhoods (cf Remark 2.8). The contradiction follows by integrating this inequality and using the fact that $t_2 - t_1 < \sigma$.

Assume now that $t < \sigma$. Then there is a surgery at time $\Delta + t$ and, by definition of the maximal time, $\phi(\mathcal{C}_s)$ is scathed at time $\Delta + t$. The surgery spheres are disjoint from $\phi(\mathcal{C}_s)$, as they have curvature $\approx (h(\Delta + t))^{-2}$, where $h(\Delta + t)$ is the cutoff parameter, and curvature on $\phi(\mathcal{C}_s)$ is less than $2r(t')^{-2} << (h(\Delta + t))^{-2}$. By definition of our surgery, this means that $\phi(\mathcal{C}_s) \subset M$ is contained in a 3-ball where the metric surgery is performed. But a cusp of M cannot be contained in a 3-ball of M, hence we get a contradiction. We conclude that $t = \sigma$ and $R(x, t') \leq 2r(t')^{-2}$, $\forall t' \in [\Delta, \Delta + \sigma]$. The pinching assumption then implies $|\operatorname{Rm}| < K$ there. □

For every $A > 0$, let $\rho_A = \rho(A, T, K)$ be given by the Persistence Theorem 8.1.3 of [4]. The proof of Theorem 2.22 is obtained by iteration of Lemma 2.23 and the Persistence Theorem as follows.

Fix $A > 0$. Let $s_0 > 0$ be large enough so that $\bar{g}(0)$ is ρ_A^{-1}-close to $g_{\text{hyp}}(0)$ on $\mathcal{C}_{s_0}$. In particular $R \leq 0$ there, so by Lemma 2.23, $\bar{g}(\cdot)$ is unscathed on $\mathcal{C}_{s_0} \times [0, \sigma]$, with curvature tensor bounded by K. The above-mentioned Persistence Theorem applied to $P(q, 0, A, \sigma)$, for all $q \in \mathcal{C}_{s_0+\rho_A}$, shows that $\bar{g}(t)$ is A^{-1}-close to $g_{\text{hyp}}(t)$ there. Hence on $\mathcal{C}_{s_0+\rho_A-A} \times [0, \sigma]$, $\bar{g}(\cdot)$ is A^{-1}-close to $g_{\text{hyp}}(\cdot)$, and in particular $R \leqslant 0$ there. We then iterate this argument, applying Lemma 2.23 and the

Persistence Theorem, $n = [T/\sigma]$ times and get that $\bar{g}(\cdot)$ is A^{-1}-close to $g_{\mathrm{hyp}}(\cdot)$ on $\mathcal{C}_{s_0+n(\rho_A-A)} \times [0, T]$.

By letting A go to infinity and rescaling appropriately, this finishes the proof of Theorem 2.22. □

3. Thick-thin decomposition theorem

Let (X, g) be a complete Riemannian 3-manifold and ε be a positive number. The *ε-thin part* of (X, g) is the subset $X^-(\varepsilon)$ of points $x \in X$ for which there exists $\rho \in (0, 1]$ such that on the ball $B(x, \rho)$ all sectional curvatures are at least $-\rho^{-2}$ and the volume of this ball is less than $\varepsilon\rho^3$. Its complement is called the *ε-thick part* of (X, g) and denoted by $X^+(\varepsilon)$. The aim of this section is to gather curvature and convergence estimates on the ε-thick part of $(M, (4t)^{-1}g(t))$ as $t \to \infty$, when $g(\cdot)$ is a Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off for suitably chosen surgery parameters $r(\cdot)$ and $\delta(\cdot)$. In fact for simplicity we will consider the rescaling $t^{-1}g(t)$. *Here, we assume M irreducible, nonspherical and not Seifert fibred. We assume also that M is not homeomorphic to* $\mathbf{R}^3$, which does not have cusp-like metrics. As a consequence, *M does not have a complete metric with* $\mathrm{Rm} \geq 0$. In the compact case, this follows from Hamilton's classification theorem (Theorem B.2.5 in Appendix B of [4]). In the noncompact case, this follows from the Cheeger-Gromoll theorem and the Soul theorem (cf. B.2.3 in [4]).

Recall that $r(\cdot)$ has been fixed in Definition 2.18. In [4, Definition 11.1.4], we define a positive nonincreasing function $\bar{\delta}(\cdot)$ such that any Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off satisfies some technical theorems — Theorems 11.1.3 and 11.1.6, analoguous to [18, Propositions 6.3 and 6.8] — if $\delta \leq \bar{\delta}$ and the initial metric is normalised.

Both Theorems 11.1.3 and 11.1.6 remain true for a Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off on a noncompact nonspherical irreducible manifold, with the weaker assumption that the metric has normalised curvature at time 0, i.e. $\mathrm{tr}\,\mathrm{Rm}^2 \leqslant 1$ for the initial metric, instead of being normalised in the sense of Definition 2.16. In particular it applies to metrics which are cusp-like at infinity. Indeed, the proofs of theorems 11.1.3 and 11.1.6 do not use the assumption on the volume of unit balls for the initial metric; it only uses the assumption on the curvature, mainly through the estimates (2.2)–(2.3). It uses neither the compactness of the manifold, the finiteness of the volume nor the particular manifold. We recall that the core of Theorem 11.1.3 is to obtain κ-noncollapsing property, canonical neighbourhoods and curvature controls relatively to a distant ball satisfying a lower volume bound assumption. The parameters then depend on the distance to the ball and on its volume, not on time or initial data. These estimates are then used to control the thick part (Theorem 11.1.6).

We gather below results following mainly from Perelman [18, 6.3, 6.8, 7.1–3]. We need some definitions.

Given a Ricci flow with bubbling-off on M, we define

$$\rho(x,t) := \max\{\rho > 0 \,:\, \mathrm{Rm} \geqslant -\rho^{-2} \text{ on } B(x,t,\rho)\}$$

and $\rho_{\sqrt{t}} := \min\{\rho(x,t), \sqrt{t}\}$. We denote by $\tilde{M}$ the universal cover of M and $\tilde{g}(t)$ the lifted evolving metric, which is by Addendum 2.19 a Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off if $g(t)$ is. If $x \in M$, we denote by $\tilde{x} \in \tilde{M}$ a lift of x and by $\tilde{B}(\tilde{x}, t, r)$ the r-ball in $(\tilde{M}, \tilde{g}(t))$ centered at $\tilde{x}$. An evolving metric $\{g(t)\}_{t\in I}$ on M is said to have *finite volume* if $g(t)$ has finite volume for every $t \in I$. We denote this volume by $V(t)$. We then have:

Proposition 3.1. *For every $w > 0$ there exists $0 < \bar{\rho}(w) < \bar{r}(w) < 1$, $\bar{T} = \bar{T}(w)$, $\bar{K} = \bar{K}(w) > 0$ such that for any Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off $g(\cdot)$ on M such that $\delta(\cdot) \leq \bar{\delta}(\cdot)$ and with normalised curvature at time 0, the following holds:*

(i) *For all $x \in M$, $t \geqslant \bar{T}$ and $0 < r \leq \min\{\rho(x,t), \bar{r}\sqrt{t}\}$, if $\mathrm{vol}\tilde{B}(\tilde{x}, t, r) \geqslant wr^3$ for some lift $\tilde{x}$ of x then $|\mathrm{Rm}| \leqslant Kr^{-2}$, $|\nabla \mathrm{Rm}| \leqslant Kr^{-3}$ and $|\nabla^2 \mathrm{Rm}| \leqslant Kr^{-4}$ on $B(x,t,r)$.*

(ii) *For all $x \in M$ and $t \geqslant \bar{T}$, if $\mathrm{vol}\tilde{B}(\tilde{x}, t, r) \geq wr^3$ for some lift $\tilde{x}$ of x where $r = \rho(x,t)$, then $\rho(x,t) \geqslant \bar{\rho}\sqrt{t}$.*

(iii) *If $g(\cdot)$ has finite volume, then:*

   (a) *There exists $C > 0$ such that $V(t) \leq Ct^{3/2}$.*

   (b) *Let $w > 0$, $x_n \in M$ and $t_n \to +\infty$. If x_n is in the w-thick part of $(M, t_n^{-1}g(t_n))$ for every n, then the sequence of pointed manifolds $(M, t_n^{-1}g(t_n), x_n)$ subconverges smoothly to a complete finite volume pointed 'hyperbolic' 3-manifold of sectional curvature $-1/4$.*

Proof. Note that $\mathrm{vol}\tilde{B}(\tilde{x}, t, r) \geq \mathrm{vol}B(x,t,r)$. Properties (i), (ii) with the stronger assumption $\mathrm{vol}B(x,t,r) \geq wr^3$ correspond to Perelman [18, 6.8, 7.3]). For the extension to the universal cover see [1, propositions 4.1, 4.2]. We remark that we extend the curvature controls to the full ball, as in [4, Sec. 11.2.3] (cf. [4, Remark 11.2.12]). Property (iii) follows from Perelman [18, 7.1,7.2]. For more details one can see Section 11.2 in [4], using technical theorems 11.1.3 and 11.1.6. The assumption on the volume is used to prove that limits of rescaled parabolic neighbourhoods are hyperbolic (cf Proposition 11.2.3). □

Remark 3.2. The hypothesis that M is irreducible is not essential here, but since our Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off is defined for this situation, it makes sense to keep this assumption throughout.

For later purposes, namely to prove that cuspidal tori in the appearing hyperbolic pieces are incompressible in M, we need the following improvement of Proposition 3.1(iii)(b), which gives convergence of flows rather than metrics. With the notations of Proposition 3.1, we define $g_n := t_n^{-1} g(t_n)$ and $g_n(t) := t_n^{-1} g(t t_n)$, the latter being a Ricci flow with bubling-off such that $g_n(1) = g_n$. If g_{hyp} denotes the 'hyperbolic' metric of sectional curvature $-1/4$, then the Ricci flow $g_{\mathrm{hyp}}(t)$ satisfying $g_{hyp}(1) = g_{\mathrm{hyp}}$ is simply $g_{\mathrm{hyp}}(t) = t g_{\mathrm{hyp}}$. Consider $w > 0$, $t_n \to \infty$ and x_n in the w-thick part of (M, g_n). By Proposition 3.1 there exists a (sub)-sequence of (M, g_n, x_n) converging smoothly to $(H, g_{\mathrm{hyp}}, x_\infty)$. By relabeling, we can assume that the sequence converges. Then we have:

Proposition 3.3. *The sequence* $(M \times [1,2], g_n(t), (x_n, 1))$ *converges smoothly to* $(H \times [1,2], g_{\mathrm{hyp}}(t), (x_\infty, 1))$.

Proof. We need to show that, for all $A > 0$, for all n large enough, the rescaled parabolic ball $B(\bar{x}_n, 1, A) \times [1,2]$ is A^{-1}-close to $B(x_\infty, 1, A) \times [1,2]$. In what follows we put a bar on x_n to indicate that the ball is w.r.t $g_n(t)$.

We use the Persistence Theorem [4, Theorem 8.1.3], the hyperbolic limit $(H \times [1,2], g_{\mathrm{hyp}}(t), (x_\infty, 1))$ being the model $\mathcal{M}_0$ in the sense of [4, page 89]. Fix $A > 1$ and let $\rho := \rho(\mathcal{M}_0, A, 1) \geq A$ be the parameter from the Persistence Theorem. By definition of $(H, g_{\mathrm{hyp}}, x_\infty)$, note that $(B(\bar{x}_n, 1, \rho), g_n)$ is ρ^{-1}-close to $(B(x_\infty, 1, \rho), g_{\mathrm{hyp}})$ for all sufficiently large n, satisfying assumption (ii) of [4, Theorem 8.1.3]. To verify the other assumptions, we adapt arguments of [12, Lemma 88.1] to our situation. In particular we have to take care of hyperbolic pieces appearing in a large 3-ball affected by a metric surgery. This is ruled out by a volume argument.

So we consider for each n, $T_n \in [t_n, 2t_n]$ maximal such that

(i) $B(x_n, t_n, \rho\sqrt{t_n}) \times [t_n, T_n]$ is unscathed,

(ii) $|2t \operatorname{Ric}(x,t) + g(x,t)|_{g(t)} \leqslant 10^{-6}$ there.

The case $T_n = t_n$, where t_n is a singular time and a surgery affects the ball just at that time, is not *a priori* excluded. Note that (ii) implies $|\operatorname{Rm}_{g_n}| \leqslant 1$ on the considered neighbourhood: one has $\operatorname{Ric}_{g(t)} \approx -\frac{1}{2t} g(t)$ for $t \in [t_n, T_n]$, or $\operatorname{Ric}_{g(tt_n)} \approx -\frac{1}{2tt_n} g(tt_n)$ for $t \in [1, T_n/t_n]$, and then $\operatorname{Ric}_{g_n(t)} = \operatorname{Ric}_{t_n^{-1} g(tt_n)} \approx -\frac{1}{2tt_n} g(tt_n) = -\frac{1}{2t} g_n(t)$. Thus the sectional curvatures of $g_n(t)$ remain in $[-\frac{1}{4} - \frac{1}{100}, -\frac{1}{8} + \frac{1}{100}]$ for A large enough.

We let $\bar{T}_n := T_n/t_n \in [1,2]$ denote the rescaled final time. The assumptions of [4, Theorem 8.1.3] being satisfied on $B(\bar{x}_n, 1, \rho) \times [1, \bar{T}_n]$, the conclusion holds on $B(\bar{x}_n, 1, A) \times [1, \bar{T}_n]$, that is $(B(\bar{x}_n, 1, A) \times [1, \bar{T}_n], g_n(t))$ is A^{-1}-close to $(B(x_\infty, 1, A) \times [1, \bar{T}_n], g_{\mathrm{hyp}}(t))$.

Claim 2. *For all n large enough,* $\bar{T}_n = 2$.

Proof of Claim 2. We first prove that there are at most finitely many integers n such that T_n is a singular time where $B(x_n, t_n, \rho\sqrt{t_n})$ is scathed, that is $g_+(x, T_n) \neq g(x, T_n)$ for some $x \in B(x_n, t_n, \rho\sqrt{t_n})$.

We first describe the idea of the proof. Assume that T_n is such a singular time. By definition of our (r, δ)-surgery, there is a surgery 3-ball $B \ni x$ whose boundary ∂B is the middle sphere of a strong δ-neck with scalar curvature $\approx h^{-2}(T_n) >> 0$, where $h(T_n)$ is the cutoff parameter at time T_n. By assumption (ii) above, $R < 0$ at time T_n on $B(x_n, t_n, \rho\sqrt{t_n})$, hence $\partial B \cap B(x_n, t_n, \rho\sqrt{t_n}) = \emptyset$. It follows that $B(x_n, t_n, \rho\sqrt{t_n}) \subset B$, which is an almost standard cap for $g_+(T_n)$. For the pre-surgery metric, the persistence theorem implies that $(B(x_n, t_n, A\sqrt{t_n}), g(T_n))$ is almost homothetic to a (large) piece of the hyperbolic manifold H. Hence the surgery shrinks this piece to a small standard cap, decreasing volume by a definite amount. As moreover $t^{-1}g(t)$ is volume decreasing along time, volume would become negative if there were too many such singular times, yielding a contradiction. We now go into the details.

Let $\mu > 0$ be the volume of the unit ball in $(H, g_{\text{hyp}}(1))$ centred at x_∞, $B_{\text{hyp}} := B(x_\infty, 1, 1)$. For any $t \geqslant 1$ we then have $\text{vol}_{g_{\text{hyp}}(t)}(B_{\text{hyp}}) = t^{3/2}\text{vol}_{g_{\text{hyp}}}(B_{\text{hyp}}) = t^{3/2}\mu$. We assume $A > 1$, so that for n large enough, by closeness at time $\bar{T}_n$ between $g_n(\cdot)$ and $g_{\text{hyp}}(\cdot)$ we have:

$$\text{vol}_{g_n(\bar{T}_n)}(B(\bar{x}_n, 1, A)) \geqslant \frac{1}{2}\text{vol}_{g_{\text{hyp}}(\bar{T}_n)}(B_{\text{hyp}}) = (\bar{T}_n)^{3/2}\frac{\mu}{2}.$$

Assume that T_n is a singular time such that $g_+(x, T_n) \neq g(x, T_n)$ for some $x \in B(x_n, t_n, \rho\sqrt{t_n})$ and let $B \ni x$ be a surgery 3-ball as discussed above. As B contains $B(\bar{x}_n, 1, \rho)$ and $\rho \geq A$, we also have

$$\text{vol}_{g_n(\bar{T}_n)}(B) > \bar{T}_n^{3/2}\frac{\mu}{2}.$$

For the unscaled metric $g(T_n) = t_n g_n(T_n/t_n) = t_n g_n(\bar{T}_n)$ we then have, before surgery, $\text{vol}_{g(T_n)}(B) = t_n^{3/2}\text{vol}_{g_n(\bar{T}_n)}(B) \geq (t_n\bar{T}_n)^{3/2}\frac{\mu}{2} = T_n^{3/2}\frac{\mu}{2}$. After surgery, $\text{vol}_{g_+(T_n)}(B)$ is comparable to $h^3(T_n)$. Computing the difference of volumes gives:

$$\text{vol}_{g_+(T_n)}(B) - \text{vol}_{g(T_n)}(B) \quad \leq \quad c.h^3(t_n') - T_n^{3/2}\frac{\mu}{2} < -T_n^{3/2}\frac{\mu}{4}.$$

for all n large enough. Since $g_+(t) \leqslant g(t)$ on the whole manifold, we have

$$\text{vol}_{g_+(T_n)}(M) - \text{vol}_{g(T_n)}(M) < -T_n^{3/2}\frac{\mu}{4}, \tag{3.1}$$

for all n large enough. Now the proof of [4, Proposition 11.2.1] shows that $(t + \frac{1}{4})^{-1}g(t)$ is volume non-increasing along a smooth Ricci flow. Since $g_+ \leqslant g$ at singular times, this monotonicity holds for a Ricci flow with bubbling-off. One easily

deduces by comparing the $(t+1/4)^{-1}$ and the t^{-1} scaling that $t^{-1}g(t)$ is also volume decreasing. Precisely, let us now set $\bar{g}(t) := t^{-1}g(t)$, then for all $t' > t$:

$$\mathrm{vol}_{\bar{g}(t')}(M) \leqslant \left(\frac{t'+1/4}{t'}\frac{t}{t+1/4}\right)^{3/2} \mathrm{vol}_{\bar{g}(t)}(M) < \mathrm{vol}_{\bar{g}(t)}(M).$$

It particular, the sequence $\mathrm{vol}_{\bar{g}(t_n)}(M)$ is decreasing. Moreover, if $[t_n, t_m]$ contains a singular time T_n as above, then using (3.1) in the second inequality, we get:

$$\mathrm{vol}_{\bar{g}(t_m)}(M) \leqslant \mathrm{vol}_{\bar{g}_+(T_n)}(M) < \mathrm{vol}_{\bar{g}(T_n)}(M) - \frac{\mu}{4} \leqslant \mathrm{vol}_{\bar{g}(t_n)}(M) - \frac{\mu}{4}.$$

On the other hand, $\mathrm{vol}_{\bar{g}(t_n)}(M) > 0$. Thus there are at most finitely many such singular times. We conclude that $B(x_n, t_n, \rho\sqrt{t_n})$ is unscathed at time T_n for all n large enough.

From now on we suppose n large enough such that $B(x_n, t_n, \rho\sqrt{t_n}) \times [t_n, T_n]$ is unscathed. Recall that singular times form a discrete subset of $\mathbf{R}$, hence there exists $\sigma_n > 0$ such that $B(x_n, t_n, \rho\sqrt{t_n})$ is unscathed on $[t_n, T_n + \sigma_n]$. By maximality of $\bar{T}_n$, when $\bar{T}_n < 2$ we must have $|2t\,\mathrm{Ric}(x,t) + g(x,t)|_{g(t)} = 10^{-6}$ at time T_n for some $x \in \overline{B(x_n, t_n, \rho\sqrt{t_n})}$. Otherwise by continuity we find σ_n small enough such that (ii) holds on $[t_n, T_n + \sigma_n] \subset [t_n, 2t_n]$, contradicting the maximality of $\bar{T}_n$.

We now show that for all large n, $|2t\,\mathrm{Ric}(x,t) + g(x,t)|_{g(t)} < 10^{-6}$ at time T_n on $\overline{B(x_n, t_n, \rho\sqrt{t_n})}$, which will imply that $\bar{T}_n = 2$ by the discussion above. Using the A^{-1}-closeness of the rescaled parabolic ball $B(\bar{x}_n, 1, A) \times [1, \bar{T}_n]$ with $B(\bar{x}_\infty, 1, A) \times [1, \bar{T}_n]$, one can check that x_n is in the w'-thick part of $(M, {T_n}^{-1}g(T_n))$, for some fixed $w' > 0$, for all n large enough. Proposition 3.1(b) then implies that ${T_n}^{-1}g(T_n)$ becomes arbitrarily close to being hyperbolic on any fixed ball (w.r.t ${T_n}^{-1}g(T_n)$) centred at x_n, when $n \to \infty$. Controlling the distortion of distances on $B(x_n, t_n, \rho\sqrt{t_n}) \times [t_n, T_n]$ (with the estimates (ii)), one can conclude that $|2t\,\mathrm{Ric}(x,t) + g(x,t)|_{g(t)} < 10^{-6}$ on $\overline{B(x_n, t_n, \rho\sqrt{t_n})}$ at time T_n for n large enough. The details are left to the reader. Together with the first part of the proof and the maximality of $\bar{T}_n$, this implies that $\bar{T}_n = 2$ for n large enough, proving Claim 2. □

As already noted, we then have, by the Persistence Theorem, that $B(x_n, 1, A) \times [1, 2]$, with the rescaled flow $g_n(t)$, is A^{-1}-close to $B(x_\infty, 1, A) \times [1, 2]$ for all n large enough. This concludes the proof of Proposition 3.3. □

From Proposition 3.3 one easily obtains:

Corollary 3.4. *Given $w > 0$ there exist a number $T = T(w) > 0$ and a nonincreasing function $\beta = \beta_w : [T, +\infty) \to (0, +\infty)$ tending to 0 at $+\infty$ such*

that if (x,t) is in the w-thick part of $(M, t^{-1}g(t))$ with $t \geq T$, then there exists a pointed hyperbolic manifold $(H, g_{\mathrm{hyp}}, *)$ such that:

(i) *$P(x,t,\beta(t)^{-1}\sqrt{t},t)$ is $\beta(t)$-homothetic to $P(*,1,\beta(t)^{-1},1) \subset H \times [1,2]$, endowed with $g_{\mathrm{hyp}}(s) = s g_{\mathrm{hyp}}(1))$,*

(ii) *For all $y \in B(x,t,\beta(t)^{-1}\sqrt{t})$ and $s \in [t,2t]$,*

$$\|\bar{g}(y,s) - \bar{g}(y,t)\| < \beta,$$

where the norm is in the $C^{[\beta^{-1}]}$-topology w.r.t the metric $\bar{g}(t) = t^{-1}g(t)$.

4. Incompressibility of the boundary tori

We prove that under the hypotheses of the previous section the tori that separate the thick part from the thin part are incompressible.

More precisely, we consider M nonspherical, irreducible, not homeomorphic to $\mathbf{R}^3$, endowed with a complete finite volume Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off $g(\cdot)$ such that $\delta(\cdot) \leq \bar{\delta}(\cdot)$, *and whose universal cover has bounded geometry* (for each time slice). We call *hyperbolic limit* a pointed 'hyperbolic' manifold of finite volume and sectional curvature $-1/4$ that appears as a pointed limit of $(M, t_n^{-1}g(t_n), x_n)$ for some sequence $t_n \to \infty$. *In this section, we assume the existence of at least one hyperbolic limit $(H, g_{\mathrm{hyp}}, *)$, which is supposed not to be closed.*

Given a hyperbolic limit H, we call *compact core of H*, a compact submanifold $\bar{H} \subset H$ whose complement consists of finitely many product neighbourhoods of the cusps. Then for large n, we have an approximating embedding $f_n : \bar{H} \to M$ which is almost isometric with respect to the metrics g_{hyp} and $t_n^{-1}g(t_n)$. The goal of this section is to prove the following result:

Proposition 4.1. *If n is large enough, then for each component T of $\partial\bar{H}$, the image $f_n(T)$ is incompressible in M.*

We argue following Hamilton's paper [8]. A key tool is the stability of the hyperbolic limit H: it is a limit along the flow, not just along a sequence of times. We give a statement following Kleiner–Lott (cf. [12, Proposition 90.1].)

Proposition 4.2 (Stability of thick part). *There exist a number $T_0 > 0$, a nonincreasing function $\alpha : [T_0, +\infty) \to (0, +\infty)$ tending to 0 at $+\infty$, a finite collection $\{(H_1, *_1), \ldots, (H_k, *_k)\}$ of hyperbolic limits and a smooth family of smooth maps*

$$f(t) : B_t = \bigcup_{i=1}^{k} B(*_i, \alpha(t)^{-1}) \to M$$

defined for $t \in [T_0, +\infty)$, such that

(i) *The $C^{[\alpha(t)^{-1}]}$-norm of $t^{-1} f(t)^* g(t) - g_{\mathrm{hyp}}$ is less than $\alpha(t)$;*

(ii) *For every* $t_0 \geq T_0$ *and every* $x_0 \in B_{t_0}$, *the time-derivative at* t_0 *of the function* $t \mapsto f(t)(x_0)$ *is less than* $\alpha(t_0)t_0^{-1/2}$.

(iii) $f(t)$ *parametrises more and more of the thick part: the* $\alpha(t)$*-thick part of* $(M, t^{-1}g(t))$ *is contained in* $\mathrm{im}(f(t))$.

The proof of [12] transfers directly to our situation, using Corollary 3.4.

Remark 4.3. Any hyperbolic limit H is isometric to one of the H_i. Indeed, let $* \in H$ and $w > 0$ be such that $* \in H^+(w)$. Then x_n is in the $w/2$-thick part of $(M, t_n^{-1}g(t_n))$ for n large enough. Assume that $f(t_n)^{-1}(x_n) \in B(*_i, \alpha(t_n)^{-1})$ for a subsequence. Then $f(t_n)^{-1}(x_n)$ remains at bounded distance of $*_i$, otherwise it would go into a cusp contradicting the $w/2$-thickness of x_n. It follows that (M, x_n) and $(M, f(t_n)(*_i))$ will have the same limit, up to an isometry.

4.1. Proof of Proposition 4.1. The proof of Hamilton [8] is by contradiction. Assuming that some torus is compressible, one finds an embedded compressing disk for each time further. Using Meeks and Yau [15, 16], the compressing disks can be chosen of least area. By controlling the rate of change of area of these disks, Hamilton shows that the area must go to zero in finite time — a contradiction.

Due to the possible noncompactness of our manifold, the existence of the least area compressing disks is not ensured: an area minimising sequence of disks can go deeper and deeper in an almost hyperbolic cusp. We will tackle this difficulty by considering the universal cover, which has bounded geometry (cf. Lemma 2.21 and Addendum 2.19), when necessary.

Let us fix some notation. For all small $a > 0$ we denote by $\bar{H}_a$ the compact core in H whose boundary consists of horospherical tori of diameter a. By Proposition 4.2 and Remark 4.3, we can assume that the map $f(t)$ is defined on $B(*, \alpha(t)^{-1}) \supset \bar{H}_a$ for t larger than some $T_a > 0$. For all $t \geqslant T_a$ the image $f(t)(\bar{H}_a)$ is well defined and the compressibility in M of a given boundary torus $f(t)(\partial \bar{H}_a)$ does not depend on t or a. We assume that some torus $\mathbf{T}$ of $\partial \bar{H}_a$ has compressible image in M. Below we refine the choice of the torus $\mathbf{T}$.

We define, for some fixed $a > 0$,

$$Y_t := f(t)(\bar{H}_a), \quad \mathbf{T}_t := f(t)(\mathbf{T}) \quad \text{and} \quad W_t := M - \mathrm{int}(Y_t).$$

Our first task is to find a torus in ∂Y_t which is compressible in W_t. Note that $\mathbf{T}_t$ is compressible in M, incompressible in Y_t which is the core of a hyperbolic 3-manifold, but not necessarily compressible in W_t: for example Y_t could be contained in a solid torus and $\mathbf{T}_t$ compressible on this side.

Consider the surface $\partial Y_t \subset M$ (not necessarily connected). As the induced map $\pi_1(\mathbf{T}_t) \to \pi_1(M)$, with base point choosen in $\mathbf{T}_t$, is noninjective by assumption, Corollary 3.3 of Hatcher [9] tells that there is a compressing disk $D \subset M$, with $\partial D \subset \partial Y_t$ homotopically non trivial and $\mathrm{int}(D) \subset M - \partial Y_t$. As $\mathrm{int}(D)$ is not

contained in Y_t, one has $\mathrm{int}(D) \subset W_t$. Rename $\mathbf{T}_t$ the connected component of ∂Y_t which contains ∂D and $\mathbf{T} \subset \partial \bar{H}_a$ its $f(t)$-preimage. Then $\mathbf{T}_t$ is compressible in W_t.

Let X_t be the connected component of W_t which contains D. Using [4, Lemma A.3.1] we have two exclusive possibilities:

(i) X_t is a solid torus. It has convex boundary, hence Meeks-Yau [15, Theorem 3] provide a least area compressing disk $D_t^2 \subset X_t$ where $\partial D^2 \subset \mathbf{T}_t$ is in a given nontrivial free homotopy class.

(ii) $\mathbf{T}_t$ does not bound a solid torus and Y_t is contained in a 3-ball B. Then Y_t lifts isometrically to a 3-ball in the universal cover $(\tilde{M}, \tilde{g}(t))$. Let $\tilde{Y}_t$ be a copy of Y_t in $\tilde{M}$. By [9] again, there is a torus $\tilde{\mathbf{T}}_t \subset \partial \tilde{Y}_t$ compressible in $\tilde{M} - \partial \tilde{Y}_t$, hence in $\tilde{M} - \tilde{Y}_t$. We denote by $\tilde{X}_t$ the connected component of $\tilde{M} - \mathrm{int}(\tilde{Y}_t)$ in which $\tilde{\mathbf{T}}_t$ is compressible. As $(\tilde{M}, \tilde{g}(t))$ has bounded geometry, by [16, Theorem 1] there is a compressing disk $D_t^2 \subset \tilde{X}_t$ of least area with $\partial D_t^2 \subset \tilde{\mathbf{T}}_t$ in a given nontrivial free homotopy class.

We define a function $A : [T_a, +\infty) \to (0, +\infty)$ by letting $A(t)$ be the infimum of the areas of such embedded disks. Similarly to [12, Lemma 91.12] we have

Lemma 4.4. *For every $D > 0$, there is a number $a_0 > 0$ with the following property. Given $a \in (0, a_0)$ there exists $T'_a > 0$ such that for all $t_0 \geqslant T'_a$ there is a piecewise smooth function $\bar{A}$ defined in a neighbourhood of t_0 such that $\bar{A}(t_0) = A(t_0)$, $\bar{A} \geqslant A$ everywhere, and*

$$\bar{A}'(t_0) < \frac{3}{4}\left(\frac{1}{t_0 + \frac{1}{4}}\right) A(t_0) - 2\pi + D$$

if $\bar{A}$ is smooth at t_0, and $\lim_{t \to t_0, t > t_0} \bar{A}(t) \leqslant \bar{A}(t_0)$ if not.

Proof. The proof is similar to the proof of [12, Lemma 91.12], and somewhat simpler as we don't have topological surgeries. Recall that our Ricci flow with bubbling-off $g(t)$ is non increasing at singular times, hence the unscathedness of least area compressing disks ([12, Lemma 91.10]) is not needed: we have $\lim_{t \to t_0, t > t_0} A(t) \leqslant A(t_0)$ if t_0 is singular. However, something must be said about [12, Lemma 91.11]. This lemma asserts that given $D > 0$, there is $a_0 > 0$ such that for $a \in (0, a_0)$ and $\mathbf{T} \subset H$ a horospherical torus of diameter a, for all t large enough $\int_{\partial D_t^2} \kappa_{\partial D_t^2} ds \leqslant \frac{D}{2}$ and $\mathrm{length}(\partial D_t^2) \leqslant \frac{D}{2}\sqrt{t}$, where $\kappa_{\partial D_t^2}$ is the geodesic curvature of ∂D_t^2. Its proof relies on the fact that an arbitrarily large collar neighbourhood of $\mathbf{T}_t$ in W_t is close (for the rescaled metric $t^{-1}g(t)$) to a hyperbolic cusp if t is large enough. In case (1) above, this holds on $X_t \cap f(t)B(*, \alpha(t)^{-1}))$ by Proposition 4.2. In case (2) observe that $f(t)(B(*, \alpha(t)^{-1}))$ is homotopically equivalent to the compact core $\bar{H}_t$, hence lifts isometrically to $(\tilde{M}, \tilde{g}(t))$. It follows that $\tilde{X}_t$ also has an arbitrarily large collar neighbourhood of $\tilde{\mathbf{T}}_t$ close to a hyperbolic cusp.

The rest of the proof is identical to the proof of [12, Lemma 91.12] and hence omitted. □

In particular A is upper semi-continous from the right. Note also that as A is defined as an infimum and $g(t_k)$ and $g(t)$ are $(1+\varepsilon_k)$-bilischitz when times $t_k \nearrow t$, for some $\varepsilon_k \to 0$, A is lower semi-continuous from the left.

Fix $D < 2\pi$, $a \in (0, a_0)$ and T'_a as in Lemma 4.4. Then consider the solution $\hat{A} : [T'_a, +\infty) \to \mathbf{R}$ of the ODE

$$\hat{A}' = \frac{3}{4}\left(\frac{1}{t+\frac{1}{4}}\right)\hat{A} - 2\pi + D$$

with initial condition $\hat{A}(T'_a) = A(T'_a)$. By a continuity argument, $A(t) \leqslant \hat{A}(t)$ for all $t \geqslant T'_a$. However, from the ODE we have

$$\hat{A}(t)\left(t+\frac{1}{4}\right)^{-3/4} = 4(-2\pi + D)\left(t+\frac{1}{4}\right)^{1/4} + \text{const},$$

which implies that $\hat{A}(t) < 0$ for large t, contradicting the fact that $A(t) > 0$.

This finishes the proof of Proposition 4.2.

5. A Collapsing Theorem

In this section we state a version of the collapsing theorem [14, Theorem 0.2] in the context of manifolds with cusp-like metrics.

Let (M_n, g_n) be a sequence of Riemannian 3-manifolds.

Definition 5.1. We say that g_n has *locally controlled curvature in the sense of Perelman* if for all $w > 0$ there exist $\bar{r}(w) > 0$ and $K(w) > 0$ such that for n large enough , if $0 < r \leq \bar{r}(w)$, if $x \in (M_n, g_n)$ satisfies $\mathrm{vol}B(x,r) \geq wr^3$ and $\sec \geq -r^{-2}$ on $B(x,r)$ then $|\mathrm{Rm}(x)| \leqslant Kr^{-2}$, $|\nabla \mathrm{Rm}(x)| \leqslant Kr^{-3}$ and $|\nabla^2 \mathrm{Rm}(x)| \leqslant Kr^{-4}$ on $B(x,r)$.

Remark 5.2. Note that if $g_n = {t_n}^{-1} g(t_n)$, where $g(\cdot)$ is as in Proposition 3.1 and $t_n \to \infty$, then g_n has locally controlled curvature in the sense of Perelman.

Definition 5.3. We say that (g_n) *collapses* if there exists a sequence $w_n \to 0$ of positive numbers such that (M_n, g_n) is w_n-thin for all n.

From [14, Theorem 0.2] we obtain:

Theorem 5.4. *Assume that (M_n, g_n) is a sequence of complete Riemannian oriented 3-manifolds such that*

(i) *g_n is a cusp-like metric for each n,*

(ii) *(g_n) collapses,*

(iii) *(g_n) has locally controlled curvature in the sense of Perelman,*

then for all n large enough M_n is a graph manifold.

The manifolds in [14, Theorem 0.2] are assumed to be compact and may have convex boundary. Our cusp-like assumption (i) allows to apply their result by the following argument. First we deform each g_n so that the sectional curvature is $-\frac{1}{4}$ on some neighbourhood of the ends, assumptions (ii), (iii) remaining true. Let $w_n \to 0$ be a sequence of positive numbers such that g_n is w_n-thin. For each n, we can take a neighbourhood U_n of the ends of M_n, with horospherical boundary, small enough so that the complement $M_n' = M_n \setminus \operatorname{int} U_n$ satisfies assumptions of [14, Theorem 0.2] with collapsing numbers w_n, except for the convexity of the added boundary. Then we deform the metric on M_n' near the boundary into a reversed hyperbolic cusp so that the boundary becomes convex. It follows that M_n', hence M_n, is a graph manifold for all n large enough. In fact it should be clear from Morgan–Tian's proof that the convexity assumption is not necessary in this situation (see the more general [2, Proposition 5.1]).

6. Proof of the main theorem

Here we prove Theorem 1.2. We sketch the organisation of the proof. Let (M, g_0) be a Riemannian 3-manifold satisfying the hypotheses of this theorem. We also assume that M is not a solid torus, is nonspherical and does not have a metric with $\mathrm{Rm} \geq 0$, otherwise it would be Seifert fibred and conclusion of Theorem 1.2 holds. We first define on M a Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off $g(\cdot)$, issued from g_0 and defined on $[0, +\infty)$. As mentioned before, we may have to pass to the universal cover. By existence Theorem 2.17 $g(\cdot)$ exists on a maximal interval $[0, T_{\max})$. The case $T_{\max} < +\infty$ is ruled out using the fact that $(M, g(T_{\max}))$ is covered by canonical neighbourhoods (see claim 3 below). Proposition 3.1 then provides a sequence $t_n \nearrow +\infty$ and connected open subsets $H_n \subset M_n = (M, {t_n}^{-1} g(t_n))$, diffeomorphic to a complete, finite volume hyperbolic manifold H (possibly empty) of sectional curvature $-1/4$. We set $G_n := M_n \setminus H_n$. Proposition 4.1 proves that the tori of $\partial \overline{H}_n$ (if $H \neq \emptyset$) are incompressible in M for large n. In this case, the atoroidality assumption on M implies that H_n is diffeomorphic to M and that each component of G_n is a cuspidal end $T^2 \times [0, \infty)$ of M_n. Then $t^{-1} g(t)$ converges (in the pointed topology) to a complete, finite volume hyperbolic metric on M of sectional curvature $-1/4$. In both cases ($H = \emptyset$ or $H \neq \emptyset$), G_n collapses with curvature locally controlled in the sense of Perelman. If $H = \emptyset$, we conclude by collapsing theorem 5.4 that $M_n = G_n$ is a graph manifold (hence Seifert fibred) for all n large enough. If $H_n \neq \emptyset$, Proposition 4.2 gives a continuous decomposition $M = H_t \cup G_t$ where H_t is diffeomorphic to M, $g(t)$ is smooth and $|\mathrm{Rm}| \leq C t^{-1}$ there, and G_t is $\alpha(t)$-thin. We then use the topological/geometric description of the thin part presented in [1, 2] to obtain that $|\mathrm{Rm}| \leq C t^{-1}$ on G_t, by the same argument as in [1, Theorem 1.1].

6.1. Setting up the proof. Let $(\tilde{M}, \tilde{g}_0)$ be the Riemannian universal cover of (M, g_0). By Lemma 2.21 it has bounded geometry. Without loss of generality, we assume that it is normalised. If M is compact, we can even assume that g_0 itself is normalised.

We now define a Riemannian 3-manifold $(\bar{M}, \bar{g}_0)$ by setting $(\bar{M}, \bar{g}_0) := (M, g_0)$ if M is compact, and $(\bar{M}, \bar{g}_0) := (\tilde{M}, \tilde{g}_0)$ otherwise. In either case, $\bar{g}_0$ is complete and normalised. By [13], $\bar{M}$ is irreducible. If $\bar{M}$ is spherical, then M is spherical, contrary to the assumption. Henceforth, we assume that $\bar{M}$ is nonspherical.

Thus Theorem 2.17 applies to $(\bar{M}, \bar{g}_0)$, where $\bar{\delta}(\cdot)$ is chosen from Theorem 3.1. Let $\bar{g}(\cdot)$ be a Ricci flow with bubbling-off on $\bar{M}$ with initial condition $\bar{g}_0$. By Addendum 2.19, we also have a Ricci flow with bubbling-off $g(\cdot)$ on M with initial condition g_0 covered by $\bar{g}(\cdot)$.

Claim 3. *The evolving metrics $g(\cdot)$ and $\bar{g}(\cdot)$ are defined on $[0, +\infty)$.*

Proof. If this is not true, then they are only defined up to some finite time T, and every point of $(\bar{M}, \bar{g}(T))$ is centre of an ε_0-neck or an ε_0-cap. By Theorem 7.4 of [5], $\bar{M}$ is diffeomorphic to S^3, $S^2 \times S^1$, $S^2 \times \mathbf{R}$ or $\mathbf{R}^3$.[2] Since $\bar{M}$ is irreducible and nonspherical, $\bar{M}$ is diffeomorphic to $\mathbf{R}^3$. The complement of the neck-like part (cf. again [6]) is a 3-ball, which must be invariant by the action of the deck transformation group. Since this group acts freely, it is trivial. Thus $M = \bar{M}$.

Being covered by $\bar{g}(\cdot)$, the evolving metric $g(\cdot)$ is complete and of bounded sectional curvature. Hence by Remark 2.3, $(M, g(T))$ has finite volume. By contrast, $(\bar{M}, \bar{g}(T))$ contains an infinite collection of pairwise disjoint ε_0-necks of controlled size, hence has infinite volume. This contradiction completes the proof of Claim 3. □

It follows from Claim 3 that $\bar{M}$ carries an equivariant Ricci flow with bubbling-off $\bar{g}(\cdot)$ defined on $[0, +\infty)$ with initial condition $\bar{g}_0$. We denote by $g(\cdot)$ the quotient evolving metric on M. By Addendum 2.19, it is also a Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off. By Theorem 2.22, $g(\cdot)$ remains cusp-like at infinity for all time. Now consider the alternative that follows the conclusion of Proposition 3.1 part (iii): Either

(i) there exist $w > 0$, $t_n \to \infty$ such that the w-thick part of $(M, t_n^{-1} g(t_n))$ is nonempty for all n, or

(ii) there exist $w_n \to 0$, $t_n \to \infty$ such that the w_n-thick part of $(M, t_n^{-1} g(t_n))$ is empty for all n.

We refer to the first case as the *noncollapsing case* and to the second as the *collapsing case*.

[2] This list is shorter than the corresponding list in [5] since we do not consider caps diffeomorphic to the punctured RP^3.

We denote by g_n the metric ${t_n}^{-1} g(t_n)$. Note that g_n has curvature locally controlled in the sense of Perelman (cf. Remark 5.2). We denote by M_n the Riemannian manifold (M, g_n), $M_n^+(w)$ its w-thick part, and $M_n^-(w)$ its w-thin part. In the collapsing case, $M_n = M_n^-(w_n)$ fits the assumptions of Theorem 5.4. Hence it is a graph manifold for n large enough.

Let us consider the other case.

6.2. The noncollapsing case. By assumption, there exist $w > 0$ and a sequence $t_n \to \infty$ such that the w-thick part of M_n is nonempty for all n. Choose a sequence $x_n \in M_n^+(w)$. Up to extracting a subsequence, by part (iii) of Proposition 3.1, (M_n, x_n) converges to a complete hyperbolic manifold $(H, *)$ of finite volume whose metric g_H has sectional curvature $-1/4$. By definition of the convergence, there exist an exhaustion of H by compact cores $\bar{H}_n \subset H$ and embeddings $f_n : (\bar{H}_n, *) \to (M, x_n)$ such that $|g_H - f_n^* g_n|$ goes to zero. Proposition 4.2 (stability of the thick part) gives $T_0 > 0$ and a nonincreasing function $\alpha : [T_0, \infty) \to (0, \infty)$ tending to zero at infinity, and for $t \geq T_0$ embeddings $f(t) : B(*, \alpha(t)^{-1}) \subset H \to M$ satisfying conclusions (i)–(iii) of this proposition. If H is closed, the desired conclusion follows. From now on we assume that H is not closed. By Proposition 4.1, for each $m \in \mathbf{N}$, for all n large enough, each component of $f_n(\partial \bar{H}_m)$ is an incompressible torus in M. Relabeling the f_n we can assume that the property holds for $f_m(\partial \bar{H}_m)$ for all m. By atoroidality of M, it follows that $H_n := \operatorname{int} f_n(\bar{H}_n) \subset M$ is diffeomorphic to M for all n, and $G_n := M \setminus H_n$ is a disjoint union of neighbourhoods of cuspidal ends of M_n. For large $t \geq T_0$, choose a compact core $\bar{H}_t \subset B(*, \alpha(t)^{-1})$ such that $\partial \bar{H}_t$ consists of horospherical tori whose diameter goes to zero as $t \to \infty$. We assume moreover that $t \to \bar{H}_t$ is smooth. Set $H_t := f(t)(\bar{H}_t) \subset M$ and $G_t := M \setminus H_t$. Then H_t is diffeomorphic to M, $t \mapsto g(t)$ is smooth there and $|\operatorname{Rm}| \leq Ct^{-1}$ by closeness with H. On the other hand, G_t is $w(t)$-thin for some $w(t) \to 0$ as $t \to \infty$. There remains to prove that G_t satisfies $|\operatorname{Rm}| \leq Ct^{-1}$ also, which will imply its unscathedness.

Consider a connected component $\mathcal{C}(t)$ of G_t. For all large t, $\partial \mathcal{C}(t)$ is an incompressible torus in M with a collar neighbourhood $\alpha(t)$-close, w.r.t $t^{-1} g(t)$, to a collar neighbourhood of a horospherical torus in H. On the other hand, $\mathcal{C}(t)$ is diffeomorphic to $T^2 \times [0, \infty)$ and its end has a cusp-like structure, hence curvature also bounded by Ct^{-1}. There remains to control what happens in the middle of $\mathcal{C}(t)$.

We apply the topological/geometric description of the thin part obtained in [2, Proposition 5.1] to a compact subset $\mathcal{C}'(t) \subset \mathcal{C}(t)$ which we define as follows.

By Theorem 2.22 there is an embedding $f_{\text{cusp}} : T^2 \times [0, +\infty) \to M$ and a function $b : [0, +\infty) \to [0, \infty)$ such that

$$|(4t)^{-1} f_{\text{cusp}}^* g(t) - g_{\text{hyp}}|_{T^2 \times [b(t), +\infty)} < w(t)$$

and $f_{\text{cusp}}(T^2 \times [b(t), +\infty)) \subset \mathcal{C}(t)$ is a neighbourhood of its end. Here g_{hyp} denotes a hyperbolic metric $e^{-2s} g_{\text{eucl}} + ds^2$ (with sectional curvature -1) on $T^2 \times [0, +\infty)$.

The metric $4g_{\mathrm{hyp}}$ may differ from the one on H. We can assume $b(t) \to \infty$. We define $\mathcal{C}_{\mathrm{cusp}}(t) := f_{\mathrm{cusp}}(T^2 \times [b(t)+2, +\infty))$ and

$$\mathcal{C}'(t) := \mathcal{C}(t) \setminus \operatorname{int} \mathcal{C}_{\mathrm{cusp}}(t).$$

Now we fix functions $\bar{r}$, K given by Proposition 3.1, $\mu_1 > 0$ given by [2, Lemma 5.2], $w_1 = w_1(\mu_1, \bar{r}, K) > 0$ given by [2, Proposition 5.1].

The closed subset $\mathcal{C}'(t)$ satisfies the assumptions of the latter proposition for $t \geq T_1$ large enough such that $w(t) < w_1$. We now follow the proof of [1, Theorem 1.1 on p. 23]. Decompose $\mathcal{C}'(t)$ into closed subsets V_1, V_2, V_2' as given by the proposition. The two boundary components of $\mathcal{C}'(t)$ have to bound components of V_1. Either $\mathcal{C}'(t) = V_1$ or the boundary components of $\mathcal{C}'(t)$ bound components $\mathcal{C}_1, \mathcal{C}_2$ of V_1, which are diffeomorphic to $T^2 \times I$ and there is a component $\mathcal{C}_3$ of V_2 adjacent to $\mathcal{C}_1$. We prove that only the first case occurs, for all t large enough.

Lemma 6.1. *For all t large enough, $\mathcal{C}'(t) = V_1$.*

Before proving this lemma, we explain how to conclude the proof of the theorem. First [2, Lemma 5.2(ii)] applies to any $x \in V_1$, giving $w_1 = w_1(\mu_1, \bar{r}, K) > 0$ such that

$$\operatorname{vol} \tilde{B}(\tilde{x}, t, \rho_{\sqrt{t}}(x,t)) \geq w_1(\rho_{\sqrt{t}}(x,t))^3,$$

for any lift $\tilde{x} \in \tilde{M}$ of x. Let $\bar{\rho} = \bar{\rho}(w_1) > 0$ be given by Proposition 3.1. If $\rho_{\sqrt{t}}(x,t) < \rho(x,t)$ then $\rho(x,t) \geq \sqrt{t} > \bar{\rho}\sqrt{t}$. If not, $\rho(x,t) = \rho_{\sqrt{t}}(x,t)$ and Proposition 3.1 (ii) implies

$$\rho_{\sqrt{t}}(x,t) \geq \bar{\rho}\sqrt{t}$$

if t is large enough (larger than $\bar{T} = \bar{T}(w_1)$). In both cases, $\rho_{\sqrt{t}} \geq \bar{\rho}\sqrt{t}$. Then Proposition 3.1(i) with $r = \rho_{\sqrt{t}}(x,t)$ implies $|\operatorname{Rm}| \leq C(w_1)t^{-1}$ at (x,t) for some $C = C(w_1) > 0$. Thus the proof of the theorem is finished if $\mathcal{C}'(t) = V_1$ for all large t.

We now prove Lemma 6.1, arguing by contradiction. Set $T_2 := \max\{T_0, T_1, \bar{T}\}$. Assume that there exist arbitrarily large times $t \geq T_2$ such that $\mathcal{C}'(t) \neq V_1$. At any of these times, the S^1-fibres of $\mathcal{C}_3$ are homotopic to a fibre of $\partial\mathcal{C}_1$, by [2, Proposition 5.1(b2)]. By incompressibility of $\partial\mathcal{C}_1$ in M, this curve generates an infinite cyclic subgroup in $\pi_1(M)$. Then [2, Lemma 5.2(i)] applies to any $x \in \mathcal{C}_3 \cap V_{2,\mathrm{reg}}$ and gives

$$\operatorname{vol} \tilde{B}(\tilde{x}, t, \rho_{\sqrt{t}}(x,t)) \geq w_1(\rho_{\sqrt{t}}(x,t))^3,$$

for any lift $\tilde{x}$ of x, and hence $\rho_{\sqrt{t}}(x,t) \geq \bar{\rho}\sqrt{t}$ as above. Moreover [2, Proposition 5.1(c3)] gives $s = s_2(\mu_1, \bar{r}, K) \in (0, 1/10)$, an open set U such that

$$B(x, t, \frac{1}{2} s\rho_{\sqrt{t}}(x,t)) \subset U \subset B(x, t, s\rho_{\sqrt{t}}(x,t)), \tag{6.1}$$

and a 2-Lipschitz map $p : U \to \mathbf{R}^2$ whose image contains $B(0, \frac{1}{4}s\rho_{\sqrt{t}}(x,t)) \subset \mathbf{R}^2$ and whose fibres are homotopic to fibres of $\mathcal{C}_3$, hence noncontractible in M.

Now consider any noncontractible loop $\gamma \subset \mathcal{C}'(T_2)$. Define for all $t \geq T_2$, $\gamma_1(t) \subset \partial\mathcal{C}(t)$ freely homotopic to γ such that $f(t)^{-1} \circ \gamma_1(t)$ is geodesic in $\partial\bar{H}_t$ and evolves by parallel transport in H w.r.t. t. On the side of the cusp, define $\gamma_2(t) \subset \partial\mathcal{C}_{\text{cusp}}(t)$ freely homotopic to γ such that $f_{\text{cusp}}^{-1} \circ \gamma_2(t) \subset T^2 \times \{b(t) + 2\}$ is geodesic in (T^2, g_{eucl}) and evolves by parallel transport (at speed b').

In particular $\gamma_1(t) \subset \mathcal{C}_1$ and $\gamma_2(t) \subset \mathcal{C}_2$ at each time when these sets are defined (that is when $\mathcal{C}'(t) \neq V_1$) and these loops are freely homotopic in $\mathcal{C}'(t)$. Let $A(t)$ be the infimum of the areas of all smooth homotopies $H : S^1 \times [0,1] \to \mathcal{C}'(t)$ connecting $\gamma_1(t)$ to $\gamma_2(t)$.

Claim 4. *$t^{-1}A(t) \to 0$ as $t \to \infty$.*

Proof. It is identical to [1, Lemma 8.2], except that we have to account for the fact that $\partial_t\gamma_2(t)$ may a priori not be bounded. This estimate appears when we compute the area added to the homotopy by moving the boundary curves. The infinitesimal added area to the homotopy due to the deplacement of γ_1 is negative (we can assume $\alpha' > 0$), hence neglected. The contribution of γ_2, by closeness with the hyperbolic cusp, is bounded by $Ct.e^{-b}b'$. On the other hand, the normalised length $t^{-1/2}\ell(\gamma_i) \to 0$ and the normalised geodesic curvature $t\kappa(\gamma_i(t)) < C$, by closeness with the hyperbolic situation. Let us denote $L(t) = t^{-1/2}(\ell(\gamma_1(t)) + \ell(\gamma_2(t))$. Computations in [1, Lemma 8.2] give (compare with equation (8.1) there)

$$\frac{d}{dt^+}(t^{-1}A(t)) \leq -\frac{A(t)}{4t^2} + C\left(\frac{L(t)}{t} + e^{-b}b'\right).$$

Denoting $y(t) = t^{-1}A(t)$ this gives the differential inequality

$$\frac{d}{dt^+}y \leq -y/4t + C(t^{-1}L + e^{-b}b').$$

Using the standard method, one obtains that $y(t) = K(t)t^{-1/4}$ where

$$\frac{d}{dt^+}K \leq Ct^{1/4}\left(t^{-1}L + e^{-b}b'\right).$$

We can assume that $L(t)$ is almost nonincreasing, that is that for any $T_2 \leq a \leq t$, one has $L(t) \leq 2L(a)$. Then for $T_2 \leq a \leq t$,

$$\begin{aligned} K(t) - K(a) &\leq C\left(\int_a^t \left(u^{-3/4}L(u) + u^{1/4}e^{-b}b'\right)du\right) \\ &\leq C\left(2L(a)\int_a^t u^{-3/4}\,du + t^{1/4}\int_a^t e^{-b}b'\,du\right) \\ &\leq C\left(8L(a)t^{1/4} + t^{1/4}e^{-b(a)}\right), \end{aligned}$$

hence

$$\begin{aligned} y(t) &\leq \frac{K(a)}{t^{1/4}} + \frac{K(t)-K(a)}{t^{1/4}} \\ &\leq \frac{K(a)}{t^{1/4}} + C\left(8L(a) + e^{-b(a)}\right), \end{aligned}$$

which is arbitrary small by taking a then t large enough. □

We conclude the proof of Lemma 6.1. The argument is the same as the one given in [1]. Consider smooth loops γ, β in $\mathcal{C}'(t)$ generating $\pi_1\mathcal{C}'(t)$. Let $\gamma_i(t)$, resp. $\beta_i(t)$, $i = 1, 2$, defined as above, freely homotopic to γ, resp. β. Let $A(t)$, resp. $B(t)$, be the infimum of the areas of all smooth homotopies connecting $\gamma_1(t)$ to $\gamma_2(t)$, resp. $\beta_1(t)$ to $\beta_2(t)$. By Claim 4,

$$t^{-1}A(t) + t^{-1}B(t) \to 0 \tag{6.2}$$

as $t \to \infty$. On the other hand let H_γ, resp. H_β, be any of these homotopies. At any time t where $\mathcal{C}_3$ is defined, any fibre of the projection $p : U \to \mathbf{R}^2$ is a noncontractible loop $\subset \mathcal{C}_3$, hence it intersects at least once the homotopies H_γ,H_β. For all such times t large enough one has, using the fact that p is 2-bilipschitz and equation (6.1), that

$$\text{area}(H_\gamma) + \text{area}(H_\beta) \geq \frac{1}{4}\text{vol}(p(U)) \geq cs^2\bar{\rho}t,$$

for some constant $c = c(s, \bar{\rho}) > 0$. This contradicts (6.2).

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