Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 89 (2014)

Artikel: On minimal spheres of area 4 and rigidity
Autor: Mazet, Laurent / Rosenberg, Harold
DOI: https://doi.org/10.5169/seals-515690

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 07.02.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Comment. Math. Helv. 89 (2014), 921–928
DOI 10.4171/CMH/338

Commentarii Mathematici Helvetici
© Swiss Mathematical Society

On minimal spheres of area 4π and rigidity

Laurent Mazet and Harold Rosenberg*

Abstract. Let M be a complete Riemannian 3-manifold with sectional curvatures between 0 and 1. A minimal 2-sphere immersed in M has area at least 4π. If an embedded minimal sphere has area 4π, then M is isometric to the unit 3-sphere or to a quotient of the product of the unit 2-sphere with $\mathbb{R}$, with the product metric. We also obtain a rigidity theorem for the existence of hyperbolic cusps. Let M be a complete Riemannian 3-manifold with sectional curvatures bounded above by -1. Suppose there is a 2-torus T embedded in M with mean curvature one. Then the mean convex component of M bounded by T is a hyperbolic cusp, *i.e.*, it is isometric to $T \times \mathbb{R}$ with the constant curvature -1 metric: $e^{-2t} d\sigma_0^2 + dt^2$ with $d\sigma_0^2$ a flat metric on T.

Mathematics Subject Classification (2010). 53C24, 53C42; 35J15, 35J20.

Keywords. Area of minimal sphere, rigidity of 3-manifolds, hyperbolic cusp.

1. Introduction

Consider a smooth (C^∞) complete metric on the 2-sphere S whose curvature is between 0 and 1. It is well known that a simple closed geodesic in S has length at least 2π (see [4] or Klingenberg's theorem in higher dimension [3], [2]). It is less well known that when such an S has a simple closed geodesic of length exactly 2π, then S is isometric to the unit 2-sphere $\mathbb{S}^2_1$. This result is proved in [1], and the authors attribute the theorem to E. Calabi.

With this in mind, we consider what happens in a complete 3-manifold M with sectional curvatures between 0 and 1 (henceforth we suppose this curvature condition on M, unless stated otherwise).

Let Σ be an embedded minimal 2-sphere in M. Then the Gauss–Bonnet theorem and the Gauss equation tells us that the area of S is at least 4π: indeed we have

$$4\pi = \int_\Sigma \bar{K}_\Sigma = \int \det(A) + K_{T\Sigma} \leq \int_\Sigma 1 = A(\Sigma) \tag{1}$$

with $\det(A)$ the determinant of the shape operator which is non-positive. We prove in Theorem 1, that when the area of Σ equals 4π, then M is isometric to the unit

*The authors were partially supported by the ANR-11-IS01-0002 grant.

3-sphere $\mathbb{S}^3_1$ or to a quotient of the product of the unit 2-sphere with $\mathbb{R}$, $\mathbb{S}^2_1 \times \mathbb{R}$, with the product metric.

We remark that Theorem 1 does not hold for embedded minimal tori. Given ε greater than zero, there are Berger spheres with curvatures between 0 and 1, which contain embedded minimal tori of area less than ε. But a minimal sphere always has area at least 4π.

It would be interesting to know what happens in higher dimensions. In the unit n-sphere $\mathbb{S}^n_1$, a compact minimal hyper-surface Σ always has volume at least the volume of the equatorial $n-1$ sphere $\mathbb{S}^{n-1}_1$. Is there a rigidity theorem when one allows metrics on $\mathbb{S}^n$ $(= M)$ of sectional curvatures between 0 and 1? Two questions arise. First, does an embedded minimal hyper-sphere Σ in M have volume at least the volume of $\mathbb{S}^{n-1}_1$. If this is so, and if Σ is an embedded minimal hyper-sphere with volume exactly the volume of $\mathbb{S}^{n-1}_1$, is M isometric to $\mathbb{S}^n_1$ or to $\mathbb{S}^{n-1}_1 \times \mathbb{R}$?

In the same spirit as Theorem 1, we prove a rigidity theorem for hyperbolic cusps. We recall that a 3-dimensional hyperbolic cusp is a manifold of the form $T \times \mathbb{R}$ with T a 2-torus and the hyperbolic metric $e^{-2t} d\sigma_0^2 + dt^2$ with $d\sigma_0^2$ a flat metric on T. In Theorem 2, we prove that if M is a complete Riemannian manifold with sectional curvatures bounded above by -1 and T is a constant mean curvature-1 torus embedded in M then the mean convex side of T in M is isometric to a hyperbolic cusp.

2. Minimal spheres of area 4π and rigidity of 3-manifolds

In this section, we prove a rigidity result for a Riemannian 3-manifold M whose sectional curvatures are between 0 and 1. As explained in the introduction, any minimal sphere in such a manifold has area at least 4π.

We denote by $\mathbb{S}^n_1$ the sphere of dimension n with constant sectional curvature 1. We then have the following result.

Theorem 1. *Let M be a complete Riemannian 3-manifold whose sectional curvatures satisfy $0 \le K \le 1$. Assume that there exists an embedded minimal sphere Σ in M with area 4π. Then the manifold M is isometric either to the sphere $\mathbb{S}^3_1$ or to a quotient of $\mathbb{S}^2_1 \times \mathbb{R}$.*

Proof. Let Φ be the map $\Sigma \times \mathbb{R} \to M$, $(p,t) \mapsto \exp_p(tN(q))$ where N is a unit normal vector field along Σ. In the following we focus on $\Sigma \times \mathbb{R}_+$; by symmetry of the configuration, the study is similar for $\Sigma \times \mathbb{R}_-$.

Σ is compact, so there is an ε such that Φ is an immersion and even an embedding on $\Sigma \times [0, \varepsilon)$. Let us define

$$\varepsilon_0 = \sup\{\varepsilon > 0 \mid \Phi \text{ is an immersion on } \Sigma \times [0, \varepsilon)\};$$

ε_0 can be equal to $+\infty$. Using Φ, we pull back the Riemannian metric of M to $\Sigma \times [0, \varepsilon_0)$. This metric can be written $ds^2 = d\sigma_t^2 + dt^2$ where $d\sigma_t^2$ is a smooth family of metrics on Σ. With this metric, Φ becomes a local isometry from $\Sigma \times [0, \varepsilon_0)$ to M and $(\Sigma \times [0, \varepsilon_0), ds^2)$ has sectional curvatures between 0 and 1. Moreover, Σ_0 is minimal and has area 4π. Actually, we will prove the following facts.

Claim. *The metric $d\sigma_0^2$ has constant sectional curvature 1 so $(\Sigma, d\sigma_0^2)$ is isometric to $\mathbb{S}_1^2$. Moreover, we have two cases:*

(1) $\varepsilon_0 = \pi/2$ *and* $d\sigma_t^2 = \sin^2 t\, d\sigma_0^2$, *or*

(2) $\varepsilon_0 = +\infty$ *and* $d\sigma_t^2 = d\sigma_0^2$.

Let us denote by $\Sigma_t = \Sigma \times \{t\}$ the equidistant surfaces. We denote by $H(p,t)$ the mean curvature of Σ_t at the point (p,t) with respect to the unit normal vector ∂_t. We also define $\lambda(p,t) \geq 0$ such that $H + \lambda$ and $H - \lambda$ are the principal curvature of Σ_t at (p,t). We notice that $\lambda = 0$ if Σ_t is umbilical at (p,t).

The surfaces Σ_t are spheres, so, using the Gauss equation, the Gauss–Bonnet formula implies that

$$4\pi = \int_{\Sigma_t} \bar{K}_{\Sigma_t} = \int_{\Sigma_t} (H+\lambda)(H-\lambda) + K_t = \int_{\Sigma_t} H^2 - \lambda^2 + K_t$$

where $\bar{K}_{\Sigma_t}$ is the intrinsic curvature of Σ_t and K_t is the sectional curvature of the ambient manifold of the tangent space to Σ_t. Since $K_t \leq 1$, we obtain the following inequality:

$$\int_{\Sigma_t} \lambda^2 = \int_{\Sigma_t} H^2 + K_t - 4\pi \leq \int_{\Sigma_t} H^2 + A(\Sigma_t) - 4\pi \tag{2}$$

where $A(\Sigma_t)$ is the area of Σ_t. In the following, we denote by $F(t)$ the right-hand side of this inequality.

Claim 1. *F is vanishing on $[0, \varepsilon_0)$.*

Since Σ_0 is minimal and has area 4π, we have $F(0) = 0$. We notice that this implies that $\lambda(p,0) = 0$, so Σ_0 is umbilical and $K_{T\Sigma_0} = 1$. Thus $(\Sigma_0, d\sigma_0)$ is isometric to $\mathbb{S}_1^2$.

We have the usual formulae:

$$\frac{\partial}{\partial t} A(\Sigma_t) = -\int_{\Sigma_t} 2H \quad \text{and} \quad \frac{\partial H}{\partial t} = \frac{1}{2}(\mathrm{Ric}(\partial_t) + |A_t|^2) \tag{3}$$

where A_t is the shape operator of Σ_t and Ric is the Ricci tensor of $\Sigma \times [0, \varepsilon_0)$. Since the sectional curvatures of $M \times [0, \varepsilon_0)$ are non-negative, Ric is non-negative. So the

second formula above implies that H is non-decreasing and thus $H \geq 0$ everywhere. Let us now compute and estimate the derivative of F:

$$\begin{aligned}
F'(t) &= \int_{\Sigma_t} (2H\frac{\partial H}{\partial t} - 2H^3) - \int_{\Sigma_t} 2H \\
&= \int_{\Sigma_t} H(\mathrm{Ric}(\partial_t) + |A_t|^2 - 2H^2 - 2) \\
&= \int_{\Sigma_t} H\big((\mathrm{Ric}(\partial_t) - 2) + ((H+\lambda)^2 + (H-\lambda)^2 - 2H^2)\big) \\
&= \int_{\Sigma_t} H((\mathrm{Ric}(\partial_t) - 2) + 2\lambda^2) \\
&\leq 2\int_{\Sigma_t} H\lambda^2
\end{aligned}$$

where the last inequality comes from $\mathrm{Ric}(\partial_t) - 2 \leq 0$ because of the hypothesis on the sectional curvatures. If we choose $\varepsilon < \varepsilon_0$, there is a constant $C \geq 0$ such that $H \leq C$ on $\Sigma \times [0, \varepsilon]$. So for $t \in [0, \varepsilon]$, using the inequality (2), we get $F'(t) \leq 2CF(t)$. Then $F(t) \leq F(0)e^{2Ct} = 0$ on $[0, \varepsilon]$. So $F \leq 0$ on $[0, \varepsilon_0)$ and, because of (2), $F = 0$ on $[0, \varepsilon_0)$; this finishes the proof of Claim 1.

The first consequence of Claim 1 is that all the equidistant surfaces Σ_t are umbilical (see inequality (2)); so $\lambda \equiv 0$. In the computation of the derivative of F, this implies that

$$\int_{\Sigma_t} H(\mathrm{Ric}(\partial_t) - 2) = 0.$$

Since $H(\mathrm{Ric}(\partial_t) - 2) \leq 0$ everywhere, we obtain

$$H(\mathrm{Ric}(\partial_t) - 2) = 0 \quad \text{everywhere.} \tag{4}$$

Moreover the umbilicity and (3) imply that $\dfrac{\partial H}{\partial t} = \frac{1}{2}\mathrm{Ric}(\partial_t) + H^2$. We now prove the following claim.

Claim 2. *Let $(p,t) \in \Sigma \times [0, \varepsilon_0)$ $(t > 0)$ be such that $H(p,t) > 0$ then $H(q,t) > 0$ for any $q \in \Sigma$*

In other words, when the mean curvature is positive at a point of an equidistant, it is positive at any point of this equidistant. We recall that H is increasing in the t variable, so when it becomes positive it stays positive.

So assume that $H(p,t) > 0$ and consider $\Omega = \{q \in \Sigma \mid H(q,t) > 0\}$ which is a nonempty open subset of Σ. Let $q \in \Omega$. Since $H(q,t) > 0$, $\mathrm{Ric}(\partial_t)(q,t) = 2$ by (4). Thus $\mathrm{Ric}(\partial_t)(r,t) = 2$ for any $r \in \overline{\Omega}$. So if $r \in \overline{\Omega}$, then, for $s < t$, $\mathrm{Ric}(\partial_t)(r,s) > 0$ for s close to t and, by (3), this implies that $H(r,t) > 0$ and $r \in \Omega$. So Ω is closed and $\Omega = \Sigma$. This finishes the proof of Claim 2.

Let us assume that there is an $\varepsilon_1 > 0$ such that $H(p,t) = 0$ for $(p,t) \in \Sigma \times [0, \varepsilon_1]$ and $H(p,t) > 0$ for any $(p,t) \in \Sigma \times (\varepsilon_1, \varepsilon_0)$. Because of the evolution equation of H, this implies that $\mathrm{Ric}(\partial_t) = 0$ on $\Sigma \times [0, \varepsilon_1]$. On $\Sigma \times (\varepsilon_1, \varepsilon_0)$, we have $\mathrm{Ric}(\partial_t) = 2$ because of (4). So by continuity of $\mathrm{Ric}(\partial_t)$, we get a contradiction and then we have two possibilities:

(1) $H = 0$ on $\Sigma \times [0, \varepsilon_0)$ and $\mathrm{Ric}(\partial_t) = 0$ on $\Sigma \times [0, \varepsilon_0)$;

(2) $H > 0$ on $\Sigma \times (0, \varepsilon_0)$ and $\mathrm{Ric}(\partial_t) = 2$ on $\Sigma \times [0, \varepsilon_0)$.

In the first case, this implies that the sectional curvature of any 2-plane orthogonal to Σ_t is zero. Thus $d\sigma_t^2 = d\sigma_0^2$. Since the map Φ ceases to be an immersion only if $d\sigma_t^2$ becomes singular this implies that $\varepsilon_0 = +\infty$. Thus $\Sigma \times \mathbb{R}_+$ with the induced metric is isometric to $\mathbb{S}_1^2 \times \mathbb{R}_+$ and Φ is a local isometry from $\mathbb{S}_1^2 \times \mathbb{R}_+$ to M.

In the second case, the sectional curvature of any 2-plane orthogonal to Σ_t is equal to 1. The sectional curvature of Σ_t is also 1, since the inequality in (2) is an equality by Claim 1. Thus $d\sigma_t^2 = \sin^2 t\, d\sigma_0$ and $\varepsilon_0 = \pi/2$. This also implies that $\Phi(p, \pi/2)$ is a point. So $\Sigma \times [0, \pi/2]$ with the metric ds^2 is isometric to a hemisphere of $\mathbb{S}_1^3$ and the map Φ is a local isometry from that hemisphere to M.

Doing the same study for $\Sigma \times \mathbb{R}_-$, we get in the first case a local isometry $\Phi\colon \mathbb{S}_1^2 \times \mathbb{R} \to M$ and in the second case a local isometry $\Phi\colon \mathbb{S}_1^3 \to M$. Since $\mathbb{S}_1^2 \times \mathbb{R}$ and $\mathbb{S}_1^3$ are simply connected, Φ is then the universal cover of M and M is then isometric to a quotient of $\mathbb{S}_1^2 \times \mathbb{R}$ or $\mathbb{S}_1^3$. Since Φ is injective on Σ this implies that in the second case, Φ is actually injective and then a global isometry. □

Remark 1. In the proof, since Φ is injective on Σ, the possible quotients of $\mathbb{S}_1^2 \times \mathbb{R}$ are either $\mathbb{S}_1^2 \times \mathbb{R}$ or its quotient by the subgroup generated by an isometry of the form $\mathbb{S}_1^2 \times \mathbb{R} \to \mathbb{S}_1^2 \times \mathbb{R}$, $(p,t) \mapsto (\alpha(p), t + t_0)$ with α an isometry of $\mathbb{S}_1^2$ and $t_0 \neq 0$.

Remark 2. Something can be said about constant mean curvature H_0 spheres in a Riemannian 3-manifold with sectional curvatures between 0 and 1. Indeed, the computation (1) implies that the area of Σ is larger than $\frac{4\pi}{1+H_0^2}$, which is the area of a geodesic sphere in $\mathbb{S}_1^3$ of mean curvature H_0. Moreover, if Σ has area $\frac{4\pi}{1+H^2}$, the above proof can be adapted to prove that the mean convex side of Σ is isometric to a spherical cap of $\mathbb{S}_1^3$ with constant mean curvature H_0 (see Theorem 2 below, for a similar result in the hyperbolic case).

Remark 3. Let M be a Riemannian n-manifold whose sectional curvatures are between 0 and 1 and let Σ be a minimal 2-sphere in M. A computation similar to (1) proves also that the area of Σ is larger than 4π. It also implies that, if Σ has area 4π, Σ is totally geodesic and isometric to $\mathbb{S}_1^2$.

3. Existence of hyperbolic cusps

Let $(\mathbb{T}^2, g)$ be a flat 2 torus, the manifold $\mathbb{T}^2 \times \mathbb{R}_+$ with the complete Riemannian metric $e^{-2t}g + dt^2$ is a hyperbolic 3-dimensional cusp. $\mathbb{T}^2 \times \mathbb{R}$ is actually isometric to the quotient of a horoball of $\mathbb{H}^3$ by a $\mathbb{Z}^2$ subgroup of isometries of $\mathbb{H}^2$ leaving the horoball invariant. Any $\mathbb{T}^2 \times \{t\}$ has constant mean curvature 1. The following theorem says that, in certain 3-manifolds, a constant mean curvature 1 torus is necessarily the boundary of a hyperbolic cusp.

Theorem 2. *Let M be a complete Riemannian 3-manifold with its sectional curvatures satisfying $K \leq -1$. Assume that there exists a constant mean curvature 1 torus T embedded in M. Then T separates M and its mean convex side is isometric to a hyperbolic cusp.*

As a consequence, the existence of this torus implies that M can not be compact. The proof uses the same ideas as in Theorem 1

Proof. Let us consider the map $\Phi\colon T \times \mathbb{R}_+ \to M$, $(p,t) \mapsto \exp_p(tN(p))$ where N is the unit normal vector field normal to T such that N is the mean curvature vector of T. Let us define

$$\varepsilon_0 = \sup\{\varepsilon > 0 \mid \Phi \text{ is an immersion on } T \times [0,\varepsilon)\}.$$

Using Φ, we pull back the Riemannian metric of M to $T \times [0,\varepsilon_0)$; it can be written $ds^2 = dt^2 + d\sigma_t^2$. We define $T_t = T \times \{t\}$ the equidistant surfaces to T_0. We also denote by $H(p,t)$ the mean curvature of the equidistant surfaces at (p,t) with respect to ∂_t. We finally define $\lambda(p,t)$ such that $H + \lambda$ and $H - \lambda$ are the principal curvatures of T_t at (p,t).

The surfaces T_t are tori so, by the Gauss equation and the Gauss–Bonnet formula, we have

$$0 = \int_{T_t} \bar{K}_{T_t} = \int_{T_t} H^2 - \lambda^2 + K_t$$

where K_t is the sectional curvature of the ambient manifold of the tangent space to T_t. Since $K_t \leq -1$, we obtain the inequality

$$\int_{T_t} \lambda^2 = \int_{T_t} H^2 + K_t \leq \int_{T_t} H^2 - A(T_t).$$

We denote by $F(t)$ the right-hand term of the above inequality. By hypothesis, $H(p,0) = 1$ so $F(0) = 0$ and $F(t) \geq 0$ for any $t \geq 0$. Let us compute the derivative of F:

$$\begin{aligned} F'(t) &= \int_{T_t} \left(2H\frac{\partial H}{\partial t} - 2H^3\right) + \int_{T_t} 2H \\ &= \int_{T_t} H\left(\mathrm{Ric}(\partial_t) + |A_t|^2 - 2H^2 + 2\right) = \int_{T_t} H\left((\mathrm{Ric}(\partial_t) + 2) + 2\lambda^2\right). \end{aligned}$$

Since $H(p,0) = 1$, we can consider $\varepsilon \in (0,\varepsilon_0)$ such that $0 < H \leq C$ on $T \times [0,\varepsilon]$. Since $\mathrm{Ric}(\partial_t) + 2 \leq 0$ we get

$$F'(t) \leq \int_{T_t} 2H\lambda^2 \leq 2CF(t).$$

Thus $F(t) \leq F(0)e^{2Ct}$ for $t \in [0,\varepsilon]$; this implies $F(t) = 0$ on that segment. We then obtain $\lambda = 0$ on $T \times [0,\varepsilon]$ (the equidistant surfaces are umbilical) and $\mathrm{Ric}(\partial_t) = -2$ since $H > 0$. Thus H satisfies the differential equation $\dfrac{\partial H}{\partial t} = -2 + 2H^2$. This gives that $H = 1$ on $T \times [0,\varepsilon]$ since $H = 1$ on T_0. Thus we can let ε tend to ε_0 to obtain that $F(t) = 0$ on $[0,\varepsilon_0)$ and $\mathrm{Ric}(\partial_t) = -2$ and $H = 1$ on $T \times [0,\varepsilon_0)$. Since $0 = \int_{T_t} H^2 + K_t$ and $K_t \leq -1$, it follows that $K_t = -1$ for all t in the interval. We then have proved that the sectional curvature of $T \times [0,\varepsilon_0)$ with the metric ds^2 is equal to -1 for any 2-plane. Moreover, we get that $d\sigma_0^2$ is flat and that $d\sigma_t^2 = e^{-2t}d\sigma_0^2$. This implies that Φ is actually an immersion on $T \times \mathbb{R}_+$ ($\varepsilon_0 = +\infty$) and $T \times \mathbb{R}_+$ is isometric to a hyperbolic cusp. Φ is then a local isometry from this hyperbolic cusp to M.

To finish the proof, let us prove that Φ is in fact injective. If this is not the case, let $\varepsilon_1 > 0$ be the smallest ε such that Φ is not injective on $T \times [0,\varepsilon]$. This implies that there exist p and q in T such that either

- $\Phi(p,0) = \Phi(q,\varepsilon_1)$, or
- $\Phi(p,\varepsilon_1) = \Phi(q,\varepsilon_1)$ (with $p \neq q$ in this case).

Let U and V be respective neighborhoods of $(p,0)$ (or (p,ε_1)) in T_0 (or T_{ε_1}) and (q,ε_1) in T_{ε_1} such that Φ is injective on them. Since ε_1 is the smallest one, $\Phi(U)$ and $\Phi(V)$ are two constant mean curvature 1 surfaces in M that are tangent at $\Phi(q,\varepsilon_1)$. Moreover, in the first case, $\Phi(U)$ is included in the mean convex side of $\Phi(V)$ so by the maximum principle $\Phi(U) = \Phi(V)$. Thus $\Phi(T_0)$ would be equal to $\Phi(T_{\varepsilon_1})$ which is impossible since these two surfaces do not have the same area. In the second case, $\Phi(U)$ is included in the mean convex side of $\Phi(V)$ and then Φ is not injective on T_s for s near t, $s < t$, which is a contradiction. □

References

[1] Lars Andersson and Ralph Howard, Comparison and rigidity theorems in semi-Riemannian geometry. *Comm. Anal. Geom.* **6** (1998), 819–877. Zbl 0963.53038 MR 1664893

[2] Jeff Cheeger and David G. Ebin, *Comparison theorems in Riemannian geometry.* Revised reprint of the 1975 original, AMS Chelsea Publishing, Providence, RI, 2008. Zbl 1142.53003 MR 2394158

[3] W. Klingenberg, Contributions to Riemannian geometry in the large. *Ann. of Math.* (2) **69** (1959), 654–666. Zbl 0133.15003 MR 0105709

[4] A. Pogorelov, A theorem regarding geodesics on closed convex surfaces. *Rec. Math.* [*Mat. Sbornik*] *N.S.* **18** (60) (1946), 181–183 (in Russian). Zbl 0061.37612 MR 0017557

Received December 12, 2012

Laurent Mazet, Université Paris-Est, Laboratoire d'Analyse et Mathématiques Appliquées, CNRS UMR8050, UFR des Sciences et Technologie, 61 avenue du Général de Gaulle, 94010 Créteil cedex, France

E-mail: laurent.mazet@math.cnrs.fr

Harold Rosenberg, Instituto de Matematica Pura y Aplicada, 110 Estrada Dona Castorina, Rio de Janeiro 22460-320, Brazil

E-mail: hrosen@free.fr