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A Sato–Tate law for GL(3)

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Abstract. We consider statistical properties of Hecke eigenvalues $A_j(p,1)$ for fixed p as ϕ_j runs through a basis of Hecke–Maaß cusp forms for the group $\mathrm{SL}_3(\mathbb{Z})$. We show that almost all of them satisfy the Ramanujan conjecture at p and that their distribution is governed by the Sato–Tate law.

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1. Introduction

Given an elliptic curve E over $\mathbb{Q}$ and a prime p of good reduction, one can write $\cos\theta_p := (p+1-\#E(\mathbb{F}_p))/2\sqrt{p}$ which defines (by Hasse's bound) an angle $\theta_p \in [0,\pi]$. It is an interesting problem to study the statistical behaviour of θ_p as p varies (or as E varies in some natural family and p is kept fixed). The Sato–Tate conjecture states that for E without complex multiplication one has the "semicircle distribution"

$$\frac{\log P}{P}\sum_{\substack{p\leq P\\ p \text{ prime}}} f(2\cos\theta_p) \to \frac{1}{2\pi}\int_{-2}^{2} f(x)\sqrt{4-x^2}dx, \quad P\to\infty,$$

for any continuous function f on $[-2,2]$. More generally, given a (non-dihedral) holomorphic Hecke cusp form $F\in S_k(N)$, its normalized Hecke eigenvalues $\lambda(p)$ are bounded by 2 in absolute value, and one expects the same distribution as p varies, that is,

$$\frac{\log P}{P}\sum_{\substack{p\leq P\\ p \text{ prime}}} f(\lambda(p)) \to \frac{1}{2\pi}\int_{-2}^{2} f(x)\sqrt{4-x^2}\,dx, \quad P\to\infty. \tag{1}$$

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This is now a theorem of Barnet-Lamb, Geraghty, Harris and Taylor [BGHT]. The analogous question for Maaß forms, however, is still wide open.

In this article we are interested in the statistical properties of Hecke eigenvalues $A_j(p,1)$ for fixed p as ϕ_j runs through a basis of Hecke–Maaß cusp forms for the group $\mathrm{SL}_3(\mathbb{Z})$. In particular, we show below in Theorems 1–3 that almost all of the $A_j(p,1)$ satisfy the Ramanujan conjecture at p and that their distribution is governed by the appropriate Sato–Tate law on $\mathrm{GL}(3)$. Before we give a more precise description of these results, we review a bit more closely the "classical" case of Hecke–Maaß cusp forms for $\mathrm{SL}_2(\mathbb{Z})$.

1.1. The rank one case. Let $\{u_j\}$ run through an orthonormal basis of Hecke–Maaß cusp forms for the modular group $\mathrm{SL}_2(\mathbb{Z})$. We denote their n-th Hecke eigenvalue by $\lambda_j(n)$ and their Laplace eigenvalue by $\lambda_j = 1/4 + t_j^2 > 1/4$. By Weyl's law there are

$$\sim \frac{1}{12}T^2 \tag{2}$$

linearly independent such eigenforms with eigenvalue $\lambda_j \leqslant T^2$. One may investigate the statistical properties of $\lambda_j(p)$ (for p prime) either for fixed j as p varies, or for fixed p as j varies. Here we take the latter point of view. The Ramanujan–Petersson conjecture predicts $|\lambda_j(p)| \leqslant 2$. In contrast to the holomorphic case, this is not known for Maaß forms, the best approximation being the Kim–Sarnak bound

$$|\lambda_j(p)| \leqslant p^{7/64} + p^{-7/64}, \tag{3}$$

but one can hope that the expected bound $|\lambda_j(p)| \leqslant 2$ cannot be violated too often. Using the Selberg trace formula, Sarnak ([Sar], Theorem 1), proved[1]

$$\frac{1}{T^2}\#\{\lambda_j \leqslant T^2 : |\lambda_j(p)| \geqslant \alpha\} \ll T^{-\frac{2\log \alpha/2}{\log p}} \tag{4}$$

for any prime p and any constant $\alpha > 2$, with an absolute implicit constant. This gives a power saving for any fixed $\alpha > 2$ and any fixed p, but also if p is tending to infinity and α is at least a small power of p. It should be viewed as a density theorem (analogous to bounding the density of zeros of the Riemann zeta-function off the critical line): the more the Ramanujan conjecture is violated, the fewer such Maaß forms exist. Often one can obtain stronger density theorems if one uses the Kuznetsov formula instead of the Selberg trace formula, see e.g. Chapter 11.4 of [Iw2]. In particular, one can improve (4) essentially by a factor 4:

Proposition 1. *For a prime p, $\alpha > 2$, $T \geqslant p$ and $\varepsilon > 0$ one has*

$$\frac{1}{T^2}\#\{\lambda_j \leqslant T^2 : |\lambda_j(p)| \geqslant \alpha\} \ll_\varepsilon T^{-\frac{8\log \alpha/2}{\log p}+\varepsilon}$$

where the implied constant depends on ε at most.

[1] His original exponent is $\frac{\log \alpha/2}{\log p}$ instead of $\frac{2\log \alpha/2}{\log p}$, but in his bound (3.6) the factors 2^k should be 2^{2k} which produces the stronger result (4).

In particular, this recovers the Selberg bound $\lambda_j(p) \ll p^{1/4+\varepsilon}$ (whereas (4) gives only $\lambda_j(p) \ll p$), since for sufficiently large p and T the proposition implies

$$\#\{\lambda_j \leqslant T^2 : |\lambda_j(p)| \geqslant p^{1/4+\varepsilon}\} < 1.$$

The Sato–Tate conjecture in the version for Hecke–Maaß cusp forms predicts (1), but a proof seems currently out of reach. However, Sarnak proved the following complementary version for a fixed prime p ([Sar], Theorem 1.2):

$$\frac{12}{T^2} \sum_{\lambda_j \leqslant T^2} f(\lambda_j(p)) \to \frac{1}{2\pi} \int_{-2}^{2} f(x)\sqrt{4-x^2} \frac{p+1}{p+2+\frac{1}{p}-x^2}\, dx, \quad T \to \infty. \tag{5}$$

Note that if p tends to infinity, this approaches the semicircle distribution.

This type of question has received much attention. In the context of holomorphic cusp forms for large weight k, this type of equidistribution result (along with applications) has been discussed in [Se], [CDF], [Ro], [Golu], [MS]. The analogue of (5) for the upper half space modulo the group $\mathrm{SL}_2(\mathcal{O})$ (where $\mathcal{O}$ is the ring of integers of an imaginary quadratic field of class number one) was established in [IR]. The case of Siegel modular forms of degree 2 and large weight k is treated in detail in the paper [KST], see in particular their Theorem 1.6. A far-reaching generalization to automorphic forms of cohomological type has recently been obtained in the monumental work [ST], partly based on [Sh] and [Sau].

The asymptotic (5) is an application of the Selberg trace formula. It is interesting to see what the Kuznetsov formula gives in this situation. The difference here is that the Kuznetsov formula naturally considers a *harmonic* average, i.e. an average over Hecke eigenvalues, weighted by the L^2-norm of the underlying cusp form which is proportional to $L(1, \mathrm{sym}^2 u_j)$. Interestingly, this slightly different counting procedure produces the semicircle distribution "on the nose" [2]:

Proposition 2. *Let f be a compactly supported continuous function and let p be a prime. Then*

$$\frac{12}{T^2} \sum_{\lambda_j \leqslant T^2} f(\lambda_j(p)) \frac{\zeta(2)}{L(1, \mathrm{sym}^2 u_j)} \to \frac{1}{2\pi} \int_{-2}^{2} f(x)\sqrt{4-x^2}\, dx$$

as $T \to \infty$.

A weighted version of this result for general congruence subgroups was proved in [KL]. As a preparation for the GL(3) case, we include a short independent proof of Proposition 2.

[2] see the discussion after Theorem 1.2 in [KST] for interesting remarks about the difference of the trace formula and the relative trace formula in the case of the group Sp_4.

1.2. The rank two case. We now turn to the main topic of this paper, namely the statistical distribution of Hecke eigenvalues on $\mathrm{GL}(3)$. This seems to be completely new and has not been investigated. The central tool here is a usable version of the Kuznetsov formula on $\mathrm{GL}(3)$ as developed in [Bl] and [Bu1], [Bu2]. Combining both works, we will present other useful versions of independent interest, and we refer in particular to the nicely packaged Theorem 5 below.

Let $\{\phi_j\}$ run through an orthonormal basis of Hecke–Maaß cusp forms for the group $\mathrm{SL}_3(\mathbb{Z})$ with Hecke eigenvalues $A_j(n,1)$ and general Fourier coefficients $A_j(n,m)$. In other words, ϕ_j lives on the quotient $\mathrm{SL}_3(\mathbb{Z})\backslash\mathcal{H}_3$ where $\mathcal{H}_3$ is the "generalized upper half plane" consisting of upper triangular matrices with right lower entry 1, a 5-dimensional space. We refer to [Gold] for an introduction to the relevant notation and theory. Each ϕ_j is an eigenfunction of two differential operators, and it comes with two spectral parameters $\nu_1^{(j)}$, $\nu_2^{(j)}$ (sometimes we drop the superscript if it is clear from the context) that we normalize to have real part 0 if ϕ_j is tempered. Then the Laplacian eigenvalue is

$$\lambda_j = 1 - 3\nu_1^2 - 3\nu_1\nu_2 - 3\nu_2^2.$$

The Weyl law for $\mathrm{SL}_3(\mathbb{Z})\backslash\mathcal{H}_3$ (see [Mi2]) tells us that there are

$$\sim \frac{18\,\mathrm{vol}(\mathrm{SL}_3(\mathbb{Z})\backslash\mathcal{H}_3)}{\Gamma(7/2)(4\pi)^{5/2}}\Big(\frac{T}{\sqrt{3}}\Big)^5 = \frac{\zeta(3)}{120\pi^3\sqrt{3}}T^5$$

Hecke–Maaß eigenfunctions ϕ_j with $\lambda_j \leqslant T^2$. (We use the Haar measure and the $\mathrm{GL}(3)$ Laplacian given in [Gold]. Note that these differ from the normalizations producing the standard Weyl law; see [SW].) Note, however, that even though the Selberg eigenvalue conjecture for $\mathrm{SL}_3(\mathbb{Z})$ is known [Mi1] (that is, the Laplacian eigenvalue of each ϕ_j is $\geqslant 1$), this does not imply the Ramanujan conjecture at ∞, i.e. that the two spectral parameters are purely imaginary.

The Hecke eigenvalues $A_j(p,1)$ are the sum of the three Satake parameters $\alpha_1^{(j)}(p), \alpha_2^{(j)}(p), \alpha_3^{(j)}(p)$. The Ramanujan conjecture predicts that they are of absolute value 1, in particular $|A_j(p,1)| \leqslant 3$ for a prime p, but this is unknown. Again one may ask how often this is violated.

Theorem 1. *For a prime p, $\alpha > 3$, $T > p$ and $\varepsilon > 0$ one has*

$$\frac{1}{T^5}\#\{\lambda_j \leqslant T^2 : |A_j(p,1)| \geqslant \alpha\} \ll_\varepsilon T^{-\frac{3\log\alpha/3}{\log p}+\varepsilon}$$

where the implied constant depends on ε at most.

At the archimedean place, a corresponding density result was proved in Theorem 2 of [Bl]. Unlike in the GL(2) situation, Theorem 1 does not immediately tell us something about the individual Satake parameters, and in particular we cannot immediately conclude that the Ramanujan conjecture is violated "not too often". However, a modification of the argument gives the following:

Theorem 2. *Fix a prime p, let $\delta > 0$ and let T be sufficiently large (in terms of p and δ). Then there is $\eta > 0$ (depending on δ and p) such that*

$$\frac{1}{T^5}\#\Big\{\lambda_j \leqslant T^2 : \max\{|\alpha_1^{(j)}(p)|, |\alpha_2^{(j)}(p)|, |\alpha_3^{(j)}(p)|\} \geqslant 1+\delta\Big\} \ll T^{-\eta}.$$

Informally speaking, this shows that the Ramanujan conjecture at p is satisfied for almost all Hecke–Maaß cusp forms.

Next we turn to an analogue of Proposition 2, the Sato–Tate distribution. It is easiest to describe the Sato–Tate distribution in terms of the Satake parameters. We parametrize the circle as e^{it}, $0 \leqslant t < 2\pi$, and write $\alpha_1(p) = e^{it_1}$, $\alpha_2(p) = e^{it_2}$, $\alpha_3(p) = e^{-i(t_1+t_2)}$. The Sato–Tate measure is then given by [Sar]

$$d\alpha(t_1,t_2) := \frac{1}{24\pi^2}|e^{it_1} - e^{it_2}|^2|e^{it_1} - e^{-i(t_1+t_2)}|^2|e^{it_2} - e^{-i(t_1+t_2)}|^2 dt_1\, dt_2. \quad (6)$$

Let W be the group of 6 maps $S^1 \times S^1 \to S^1 \times S^1$ generated by $(e^{it_1}, e^{it_2}) \mapsto (e^{it_2}, e^{it_1})$ and $(e^{it_1}, e^{it_2}) \mapsto (e^{it_1}, e^{-i(t_1+t_2)})$. Then the map

$$\Phi\colon (S^1 \times S^1)/W \to \mathbb{C}, \quad (e^{it_1}, e^{it_2}) \mapsto e^{it_1} + e^{it_2} + e^{-i(t_1+t_2)},$$

is injective and hence bijective onto its image $\mathscr{R}$ that is the region inside the disc of radius 3 that is surrounded by the curve $2e^{it} + e^{-2it}$, $t \in [0, 2\pi]$.

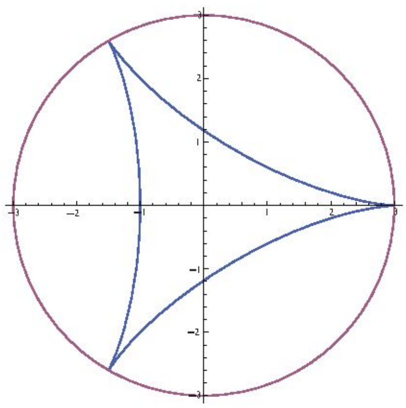

Figure 1. The region $\mathscr{R} \subseteq \mathbb{C}$ where the Sato–Tate measure is supported.

For functions on $S^1 \times S^1$ that are symmetric under W we will not distinguish between the function and its projection onto $(S^1 \times S^1)/W$. We now define a measure

$d\mu(z)$ on $\mathcal{R}$ as the push-forward of (6), i.e.

$$\int_{\mathcal{R}} f(z)\,d\mu(z) := \int_{S^1\times S^1} (f\circ\Phi)(t_1,t_2)\,d\alpha(t_1,t_2). \tag{7}$$

With this notation we have

Theorem 3. *Let f be a compactly supported continuous function, and let p be a prime. Then*

$$\frac{1}{c_1T^5}\sum_{\lambda_j\leqslant T^2}\frac{f(A_j(p,1))}{\operatorname*{res}_{s=1} L(s,\phi_j\times\bar{\phi}_j)} \to \int_{\mathcal{R}} f(z)\,d\mu(z)$$

as $T\to\infty$, where $c_1 = \sqrt{3}/(240\pi^5)$.

We remark that a lot of technical work in this paper is devoted to the treatment of the exact shape of the sum over all eigenforms with eigenvalue $\lambda_j \leqslant T^2$, as opposed to a weighted and re-normalized count of the shape

$$\Bigg(\sum_j \frac{h_T(\nu_1^{(j)},\nu_2^{(j)})}{\operatorname*{res}_{s=1} L(s,\phi_j\times\bar{\phi}_j)}\Bigg)^{-1}\sum_j \frac{f(A_j(p,1))h_T(\nu_1^{(j)},\nu_2^{(j)})}{\operatorname*{res}_{s=1} L(s,\phi_j\times\bar{\phi}_j)}$$

for some sufficiently nice test function h_T with support roughly on $\lambda_j \leqslant T^2$. We believe that the corresponding results and techniques are of independent interest.

2. Proofs in the rank 1 case

Let $l,k\in\mathbb{N}$. We use the Hecke recurrence relation $\lambda_j(p)\lambda_j(p^l) = \lambda_j(p^{l+1}) + \lambda_j(p^{l-1})$ to write

$$\lambda_j(p)^k = \sum_{l=0}^{k}\alpha_{l,k}\lambda_j(p^l)$$

for certain integers $\alpha_{l,k}$. It follows trivially by induction that

$$\sum_{l=0}^{k}|\alpha_{l,k}| \leqslant 2^k. \tag{8}$$

The Hecke relation can be expressed in terms of Chebychev polynomials which readily leads to the following integral representation ([CDF], Lemma 3):

$$\alpha_{l,k} = \frac{1}{2\pi}\int_{-2}^{2} x^k U_l(x/2)\sqrt{4-x^2}dx$$

where $U_l(x)$ is the usual Chebyshev polynomial of the second kind. In particular

$$\alpha_{0,k} = \frac{1}{2\pi}\int_{-2}^{2} x^k\sqrt{4-x^2}dx. \tag{9}$$

The numbers $\alpha_{0,2k}$ are known to be the Catalan numbers, but we do not need this information.

Next we prepare for an application of the Kuznetsov formula which is usually given in terms of Fourier coefficients $\rho_j(n)$ of L^2-normalized eigenforms. The coefficients $\rho_j(n)$ are proportional to the Hecke eigenvalues $\lambda_j(n)$, and we need to compute the proportionality constant. Let u_j be a Hecke–Maaß cusp form with Fourier expansion

$$u_j(x+iy) = y^{1/2}\sum_{n\neq 0}\lambda_j(n)e(nx)K_{it_j}(2\pi|n|y) \tag{10}$$

as in [Ku], (2.10). By a standard Rankin–Selberg unfolding argument we can compute its norm: for $\Re s > 1$ we have

$$\begin{aligned}\langle |u_j|^2, E(.,s)\rangle &= \int_0^\infty \sum_{n\neq 0}|\lambda_j(n)|^2 K_{it_j}(2\pi|n|y)^2 y^s\frac{dy}{y}\\ &= \frac{2}{(2\pi)^s}\sum_{n>0}\frac{|\lambda_j(n)|^2}{n^s}\int_0^\infty K_{it_j}(y)^2y^s\frac{dy}{y}\\ &= \frac{2}{(2\pi)^s}\sum_{n>0}\frac{|\lambda_j(n)|^2}{n^s}\frac{\sqrt{\pi}\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}-it_j)\Gamma(\frac{s}{2}+it_j)}{4\Gamma(\frac{1+s}{2})}\end{aligned}$$

by [GR], 6.576.4. Comparing residues at $s=1$ on both sides, we find

$$\|u_j\|^2 = \frac{\pi}{3}\cdot\frac{2}{2\pi}\cdot\frac{L(1,\mathrm{sym}^2u_j)}{\zeta(2)}\cdot\frac{\pi\Gamma(\frac{1}{2}-it_j)\Gamma(\frac{1}{2}+it_j)}{4} = \frac{L(1,\mathrm{sym}^2u_j)}{2\cosh(\pi t_j)}.$$

Hence

$$\rho_j(n) = \frac{\lambda_j(n)(2\cosh(\pi t_j))^{1/2}}{\sqrt{L(1,\mathrm{sym}^2u_j)}}. \tag{11}$$

A standard application of the Kuznetsov formula gives the following:

Lemma 1. *For $m,n\in\mathbb{N}$, $T\geqslant 1$ and $\varepsilon>0$ we have*

$$\sum_j \lambda_j(n)\lambda_j(m)\frac{\zeta(2)}{L(1,\mathrm{sym}^2u_j)}e^{-t_j/T} = \delta_{m=n}\frac{T^2}{6} + O_\varepsilon\Big(T(mnT)^\varepsilon + (mn)^{1/4+\varepsilon}\Big).$$

Proof. We use a pre-Kuznetsov formula[3] [Ku], Theorem 2, with the test function

$$\phi(x) = \frac{\sinh(i/(2T))}{\pi} x \exp\big(ix\cosh(i/(2T))\big)$$

as in [DI], (7.2). As shown in Lemma 8 of [DI], the pre-Kuznetsov formula is valid for this function. Combining (7.10), (7.11), (7.14) in [DI] with (11) and trivial bounds for the Eisenstein spectrum, we obtain

$$\begin{aligned} &2\sum_j \frac{\lambda_j(n)\lambda_j(m)}{L(1,\mathrm{sym}^2 u_j)}(e^{-t_j/T} + O(e^{-t_j})) + O_\varepsilon(T(mnT)^\varepsilon) \\ &\quad = \delta_{m=n}\frac{2T^2}{\pi^2} + O\Big(1 + \sum_{c\geqslant 1}\frac{|S(m,n,c)|}{c}\min\Big(\frac{\sqrt{mn}}{c}, \Big(\frac{\sqrt{mn}}{c}\Big)^{1/2}\Big)\Big). \end{aligned}$$

The lemma follows from Weil's bound $|S(n,m,c)| \ll_\varepsilon c^{1/2+\varepsilon}(n,m,c)^{1/2}$. $\square$

We are now ready to prove Propositions 1 and 2. Recall that $L(1,\mathrm{sym}^2 u_j) \ll_\varepsilon t_j^\varepsilon$ (see e.g. [Iw1], Theorem 2, or [Iw2], Theorem 8.3). Let $k \in \mathbb{N}$. By positivity, (8) and the upper bound contained in Lemma 1 we have

$$\begin{aligned} \sum_{\lambda_j \leqslant T^2} \lambda_j(p)^{2k} &\ll_\varepsilon T^\varepsilon \sum_j \frac{\lambda_j(p)^{2k}}{L(1,\mathrm{sym}^2 u_j)} e^{-t_j/T} \\ &= T^\varepsilon \sum_{l=0}^{2k} \alpha_{l,2k} \sum_j \frac{\lambda_j(p^l)}{L(1,\mathrm{sym}^2 u_j)} e^{-t_j/T} \\ &\ll_\varepsilon 2^{2k}(T^2 + p^{k/2})^{1+\varepsilon}. \end{aligned}$$

We choose $k := \lfloor 4\log T/\log p\rfloor \geqslant 4$, and Proposition 1 follows.

For the proof of Proposition 2 we use Lemma 1 to compute

$$\begin{aligned} &\frac{12}{T^2}\sum_j \lambda_j(p)^k \frac{\zeta(2)}{L(1,\mathrm{sym}^2 u_j)} e^{-\sqrt{\lambda_j}/T} \\ &\quad = \sum_{l=0}^{k} \alpha_{l,k} \frac{12}{T^2}\sum_j \lambda_j(p^l)\frac{\zeta(2)}{L(1,\mathrm{sym}^2 u_j)} e^{-t_j/T} + O_{k,p,\varepsilon}(T^{-1+\varepsilon}) \\ &\quad = 2\alpha_{0,k} + O_{k,p,\varepsilon}(T^{-1+\varepsilon}). \end{aligned}$$

[3]The corresponding Lemma 6 in [DI] would do the same job, but it is wrongly normalized; in view of the formula $\Gamma(1/2+it)\Gamma(1/2-it) = \pi/\cosh(\pi t)$, the Whittaker function on p. 52 of [DI] should have an extra factor $\pi^{1/2}$.

Writing for the moment

$$A(t) = \sum_{\sqrt{\lambda_j} \leqslant t} \lambda_j(p)^k \frac{\zeta(2)}{L(1, \mathrm{sym}^2 u_j)},$$

we have shown

$$12\delta^2 \int_0^\infty e^{-\delta v} dA(v) \to 2\alpha_{0,k}, \quad \delta \to 0.$$

By a standard Tauberian theorem ([Te], Theorem II.7.5) and (9) we find for fixed k and p,

$$\frac{12}{T^2} \sum_{\lambda_j \leqslant T^2} \lambda_j(p)^k \frac{\zeta(2)}{L(1, \mathrm{sym}^2 u_j)} = \frac{12}{T^2} A(T) \to \frac{2\alpha_{0,k}}{\Gamma(3)} = \frac{1}{2\pi} \int_{-2}^{2} x^k \sqrt{4 - x^2}\, dx$$

as $T \to \infty$. Therefore Theorem 2 holds for power functions and hence for polynomials. Since polynomial functions are dense in the space of continuous, compactly supported functions, the proof of Theorem 2 is complete.

3. Combinatorics of Hecke eigenvalues

We compile some results on the Fourier coefficients $A(m,n)$, see e.g. Section 6 in [Gold], Chapters 4 & 9 in [Bum] or [HM]. First we recall that $A(m,n) = \overline{A(n,m)}$. A basic (but not trivial) approximation to the Ramanujan conjecture (Jacquet–Shalika bounds) is

$$A_j(n,m) \ll_\varepsilon (nm)^{1/2+\varepsilon}. \tag{12}$$

Better bounds are available (due to Luo–Rudnick–Sarnak), but we do not need them. We have the Hecke relations ([Gold], Theorem 6.4.11)

$$\begin{aligned} A(n,1)A(m_1,m_2) &= \sum_{\substack{d_0 d_1 d_2 = n \\ d_1 \mid m_1,\, d_2 \mid m_2}} A\left(\frac{m_1 d_0}{d_1}, \frac{m_2 d_1}{d_2}\right), \\ A(1,n)A(m_1,m_2) &= \sum_{\substack{d_0 d_1 d_2 = n \\ d_1 \mid m_1,\, d_2 \mid m_2}} A\left(\frac{m_1 d_2}{d_1}, \frac{m_2 d_0}{d_2}\right). \end{aligned} \tag{13}$$

Given two integers l, k, we can write

$$|A(p^l,1)|^{2k} = (A(p^l,1)A(1,p^l))^k = \sum_{r+s \leqslant 2lk} \alpha_{r,s,l,k} A(p^r, p^s)$$

for certain integers $\alpha_{r,s,l,k}$. All we need is the bound

$$\sum_{r+s\leqslant 2lk} |\alpha_{r,s,l,k}| \leqslant \tau_3(p^l)^{2k} = \left(\frac{(l+1)(l+2)}{2}\right)^{2k} \tag{14}$$

which follows by induction from (13). The Hecke eigenvalues $A(p,1)$ are the sum of the Satake parameters $\alpha_1(p), \alpha_2(p), \alpha_3(p)$, and one can express $A(p^r, p^s)$ as a symmetric function in $\alpha_1(p)$, $\alpha_2(p)$, $\alpha_3(p)$ by Schur polynomials [Gold], 7.4.14:

$$\begin{aligned} A(p^r, p^s) = {} & \frac{\alpha_1(p)^{r+s+2}(\alpha_2(p)^{r+1} - \alpha_3(p)^{r+1})}{(\alpha_2(p)-\alpha_1(p))(\alpha_3(p)-\alpha_2(p))(\alpha_1(p)-\alpha_3(p))} \\ & + \frac{\alpha_2(p)^{r+s+2}(\alpha_3(p)^{r+1} - \alpha_1(p)^{r+1})}{(\alpha_2(p)-\alpha_1(p))(\alpha_3(p)-\alpha_2(p))(\alpha_1(p)-\alpha_3(p))} \\ & + \frac{\alpha_3(p)^{r+s+2}(\alpha_1(p)^{r+1} - \alpha_2(p)^{r+1})}{(\alpha_2(p)-\alpha_1(p))(\alpha_3(p)-\alpha_2(p))(\alpha_1(p)-\alpha_3(p))} \\ =: {} & Q_{r,s}(\alpha_1(p), \alpha_2(p), \alpha_3(p)), \end{aligned} \tag{15}$$

say. By a simple brute force computation one checks that

$$\int_{S^1\times S^1} Q_{r,s}(e^{it_1}, e^{it_2}, e^{-i(t_1+t_2)})\, d\alpha(t_1,t_2) = \delta_{r=s=0}. \tag{16}$$

(Note that the denominator of (15) combines nicely with the measure (6).) The Satake parameters satisfy $\alpha_1(p)\alpha_2(p)\alpha_3(p) = 1$ as well as the unitarity condition

$$\{\alpha_1(p), \alpha_2(p), \alpha_3(p)\} = \left\{1/\overline{\alpha_1(p)}, 1/\overline{\alpha_2(p)}, 1/\overline{\alpha_3(p)}\right\}.$$

This equality of sets implies that if the Ramanujan conjecture at p is violated, that is, if not all three parameters have absolute value 1, then we must have

$$\{\alpha_1(p), \alpha_2(p), \alpha_3(p)\} = \{\rho e^{-it}, \rho^{-1}e^{-it}, e^{2it}\} \tag{17}$$

for some $\rho > 1$ and some $t \in \mathbb{R}$. Combining this with (15) with $r = l$, $s = 0$, we see that in this case

$$|A(p^l,1)| \geqslant \frac{\rho^{2+l}(1-1/\rho) - 4\rho}{2(\rho+1)^2}. \tag{18}$$

Remark. Equation (16) is a special case of the orthogonality relation (also verified by direct computation)

$$\begin{aligned} &\langle Q_{r,s}, Q_{\tilde r,\tilde s}\rangle \\ &:= \int_{S^1\times S^1} Q_{r,s}(e^{it_1}, e^{it_2}, e^{-i(t_1+t_2)})\overline{Q_{\tilde r,\tilde s}(e^{it_1}, e^{it_2}, e^{-i(t_1+t_2)})}d\alpha(t_1,t_2) = \delta_{\substack{r=\tilde r \\ s=\tilde s}}. \end{aligned}$$

In particular, the Schur polynomials $Q_{r,s}$ for $r, s \in \mathbb{N}_0$ form an orthonormal basis of the space $(S^1 \times S^1)/W$, and every smooth function f on $(S^1 \times S^1)/W$ has a Fourier expansion

$$f(t_1, t_2) = \sum_{r,s \geqslant 0} \langle f, Q_{r,s} \rangle Q_{r,s}(e^{it_1}, e^{it_2}, e^{-i(t_1+t_2)}).$$

4. The Kuznetsov formula for GL(3)

We write the Fourier expansion of a Hecke–Maaß cusp form for $\mathrm{SL}_3(\mathbb{Z})$ as

$$\phi(z) = \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A(m_1, m_2)}{|m_1 m_2|} \sum_{\gamma \in U_2 \backslash \mathrm{SL}_2(\mathbb{Z})} \mathcal{W}^{\mathrm{sgn}(m_2)}_{\nu_1,\nu_2} \left(\begin{pmatrix} |m_1 m_2| & & \\ & m_1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z \right)$$

with $U_2 = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mid x \in \mathbb{Z} \right\}$ and $\mathcal{W}^{\pm}_{\nu_1,\nu_2}(z) = e(x_1 \pm x_2) W_{\nu_1,\nu_2}(y_1, y_2)$ for $z = \begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}$ where

$$\begin{aligned} &W_{\nu_1,\nu_2}(y_1, y_2) \\ &\quad = 8 y_1 y_2 \left(\frac{y_1}{y_2} \right)^{\frac{\nu_1 - \nu_2}{2}} \\ &\qquad \cdot \int_0^\infty K_{\frac{3}{2}\nu_0}(2\pi y_2 \sqrt{1 + 1/u^2}) K_{\frac{3}{2}\nu_0}(2\pi y_2 \sqrt{1 + u^2}) u^{\frac{3}{2}(\nu_1 - \nu_2)} \frac{du}{u} \end{aligned}$$

with

$$\nu_0 = \nu_1 + \nu_2$$

is the completed Whittaker function[4]. As mentioned in the introduction, we do not know if ν_1, ν_2 are purely imaginary, but if the Ramanujan conjecture is violated, then it follows by unitarity that

$$(\nu_1, \nu_2, \nu_0) = (2\rho/3, -\rho/3 + i\gamma, \rho/3 - i\gamma) \quad \text{or} \quad (\rho/3 + i\gamma, \rho/3 - i\gamma, 2\rho/3) \tag{19}$$

for $0 < |\rho| < 1/2$ and $\gamma \in \mathbb{R}$, cf. e.g. (2.8) in [Bl]. This is the archimedean analogue of (17).

Again we can compute the norm of ϕ by Rankin–Selberg theory and Stade's formula [Sta]:

$$\int_0^\infty \int_0^\infty |W_{\nu_1,\nu_2}(y_1, y_2)|^2 (y_1^2 y_2)^s \frac{dy_1\, dy_2}{(y_1 y_2)^3} = \frac{\Gamma(s/2)^3 \prod_{j=0}^{2} \Gamma(\frac{s+3\nu_j}{2}) \Gamma(\frac{s-3\nu_j}{2})}{4\pi^{3s} \Gamma(3s/2)}$$

[4] This is the standard definition of the completed Whittaker function as in [Gold], p. 154. Note that the leading constant in [Gold], (6.1.3), should be 8 instead of 4.

(as an equality of meromorphic functions in s). Note that by (19) this holds even in the non-tempered case. Let

$$E(z,s;\mathbf{1}) = \frac{1}{2}\sum_{\gamma\in P\backslash \mathrm{SL}_3(\mathbb{Z})} \det(\gamma z)^s$$

with $P = \left\{\left(\begin{smallmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1\end{smallmatrix}\right)\right\} \subseteq \mathrm{SL}_3(\mathbb{Z})$ be the maximal parabolic Eisenstein series. As in Section 3 of [Fr] or [Gold], p. 227–229, we unfold the Eisenstein series:

$$\langle \phi, \phi E(.,\bar{s},\mathbf{1})\rangle = \frac{1}{2}\int_{P\backslash \mathfrak{h}^3} |\phi(z)|^2 (y_1^2 y_2)^s \, dx_1\, dx_2\, dx_3\, \frac{dy_1\, dy_2}{(y_1y_2)^3}.$$

Let $\mathcal{F}$ denote a fundamental domain for $\left\{\left(\begin{smallmatrix} 1 & & f \\ & 1 & e \\ & & 1\end{smallmatrix}\right) \mid e, f\in\mathbb{Z}\right\}\backslash \mathfrak{h}^3$. Then $P\backslash\mathfrak{h}^3$ is in 2-to-1 correspondence with $\left\{\left(\begin{smallmatrix} \gamma & \\ & 1\end{smallmatrix}\right) \mid \gamma \in \mathrm{GL}_2(\mathbb{Z})\right\}\backslash \mathcal{F}$. Inserting the Fourier expansion of one factor and unfolding once again, we obtain

$$\begin{aligned}\langle \phi, \phi E(.,\bar{s},\mathbf{1})\rangle &= \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty} \frac{|A_\phi(m_1,m_2)|^2}{|m_1m_2|^2} \\ &\qquad \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |W_{\nu_1,\nu_2}(m_1y_1, |m_2|y_2)|^2 (y_1^2y_2)^s \frac{dy_1\, dy_2}{(y_1y_2)^3} \\ &= \frac{L(s,\phi\times\bar{\phi})}{\zeta(3s)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |W_{\nu_1,\nu_2}(y_1,y_2)|^2 (y_1^2y_2)^s \frac{dy_1\, dy_2}{(y_1y_2)^3}\end{aligned}$$

for $\Re s > 1$. Comparing residues at both sides, we find with Stade's formula that

$$\frac{1}{\zeta(3)}\operatorname*{res}_{s=1} L(s,\phi\times\bar{\phi})\, \frac{\pi}{2}\prod_{j=0}^{2}\cos\left(\frac{3}{2}\pi\nu_j\right)^{-1} = \|\phi\|^2 \operatorname*{res}_{s=1} E(.,\bar{s},\mathbf{1}) = \frac{2\pi}{3\zeta(3)}\|\phi\|^2, \tag{20}$$

see Corollary 2.5 of [Fr] and observe that his definition of the Eisenstein series differs from ours by a factor 2 (the index of $\mathrm{SL}_3(\mathbb{Z})$ in $\mathrm{GL}_3(\mathbb{Z})$). We conclude that the orthonormalized Fourier coefficients are given by

$$A(m_1,m_2)\left(\frac{4\prod_{j=0}^{2}\cos(3\pi\nu_j/2)}{3\operatorname*{res}_{s=1} L(s,\phi\times\bar{\phi})}\right)^{1/2}. \tag{21}$$

(Note that by (20) the product of the cosines is a positive real number.) It is known ([Li], Theorem 2, or [Br], Corollary 2) that

$$\operatorname*{res}_{s=1} L(s,\phi\times\bar{\phi}) \ll_\varepsilon (1+|\nu_1|+|\nu_2|)^\varepsilon. \tag{22}$$

Theorem 4. *Let n_1, n_2, m_1, m_2 be positive integers and write $P = n_1 n_2 m_1 m_2$. Let $T > 1$. Then there is a non-negative function h_T and a constant $c > 0$ satisfying $h_T \gg 1$ on the set $\{(\nu_1, \nu_2) \mid c \leqslant \Im \nu_1, \Im \nu_2 \leqslant T, |\Re \nu_1|, |\Re \nu_2| \leqslant 1/2\}$ such that*

$$\sum_j \frac{A_j(n_1, m_1)\overline{A_j(n_2, m_2)} h_T(\nu_1^{(j)}, \nu_2^{(j)})}{\operatorname{res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)} \ll_\varepsilon (T^5 + T^2 P^{1/2} + T^3 P^{7/64} + P^{5/3})^{1+\varepsilon}.$$

Proof. This is Theorem 5 of [Bl], where we invoke [Bl], Lemma 1, as well as [Bl], Theorem 1, to estimate the main term on the right-hand side of [Bl], Theorem 5. □

Theorem 4 suffices for the proof of Theorems 1 and 2. For the proof of Theorem 3 we need a more precise, but less uniform version. Its proof that we postpone to the end of the paper features a completely explicit version of the Kuznetsov formula (Theorem 6).

Theorem 5. *Let $n_1, n_2, m_1, m_2 \in \mathbb{Z} \setminus \{0\}$, $P = m_1 m_2 n_1 n_2$ and $T > 1$. Then*

$$\sum_j \frac{A_j(n_1, m_1)\overline{A_j(n_2, m_2)}}{\operatorname{res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)} e^{-\lambda_j/T^2} = \delta_{\substack{|n_1|=|m_1| \\ |n_2|=|m_2|}} c_2 T^5 + O_\varepsilon\big((P T^{37/8})^{1+\varepsilon}\big)$$

where $c_2 = \sqrt{3}/(2^7 \pi^{9/2})$.

The error term is not optimized. In order to keep the argument as simple as possible we only tried to obtain explicit polynomial dependence on P and a nontrivial exponent in T. The best possible error term in this situation is $O(T^3)$ coming from the Eisenstein contribution. Theorem 5 is a direct GL(3)-analogue of [IK], (16.56).

5. Proofs in the rank 2 case

The proofs of Theorems 1 and 2 are very similar to the proof of Proposition 1. Let l, k be two integers. Combining (22), (14) and Theorem 4 (with $10T$ instead of T), we obtain

$$\begin{aligned} \sum_{\lambda_j \leqslant T^2} & |A_j(p^l, 1)|^{2k} \\ & \ll_\varepsilon T^\varepsilon \sum_j \frac{|A_j(p^l, 1)|^{2k} h_{10T}(\nu_1^{(j)}, \nu_2^{(j)})}{\operatorname{res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)} \\ & = T^\varepsilon \sum_{r+s \leqslant 2lk} \alpha_{r,s,k,l} \sum_j \frac{A_j(p^r, p^s) h_{10T}(\nu_1^{(j)}, \nu_2^{(j)})}{\operatorname{res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)} \\ & \ll_\varepsilon T^\varepsilon \Big(\frac{(l+1)(l+2)}{2}\Big)^{2k} (T^5 + T^2 p^{lk} + T^3 p^{7lk/32} + p^{10lk/3})^{1+\varepsilon} \end{aligned} \tag{23}$$

where the implied constant depends only on ε. Choosing $k = \lfloor (3/2) \log T / \log p \rfloor$ and $l = 1$ gives

$$\sum_{\lambda_j \leqslant T^2} |A_j(p,1)|^{2k} \ll_\varepsilon 3^{2k} T^{5+\varepsilon},$$

and Theorem 1 follows easily:

$$\begin{aligned} \frac{1}{T^5} \#\{\lambda_j \leqslant T^2 : |A_j(p,1)| \geqslant \alpha\} &\leqslant \frac{\alpha^{-2k}}{T^5} \sum_{\lambda_j \leqslant T^2} |A_j(p,1)|^{2k} \\ &\ll_\varepsilon \left(\frac{\alpha}{3}\right)^{-2k} T^\varepsilon \ll T^{-\frac{3 \log \alpha/3}{\log p} + \varepsilon}. \end{aligned}$$

In order to prove Theorem 2 we consider cusp forms ϕ_j with

$$\max\{|\alpha_1^{(j)}(p)|, |\alpha_2^{(j)}(p)|, |\alpha_3^{(j)}(p)|\} \geqslant 1 + \delta.$$

By (17) and (18) the Hecke eigenvalues of such a form satisfy

$$|A_j(p^l, 1)| \geqslant \frac{(1+\delta)^{1+l}\delta - 4(1+\delta)}{2\delta^2} \geqslant (l+1)(l+2) \tag{24}$$

for some sufficiently large $l = l(\delta)$. For T sufficiently large in terms of δ and p we choose $k = \lfloor \frac{3 \log T}{2l \log p} \rfloor \geqslant 1$. From (24) and (23) we conclude

$$\begin{aligned} &\frac{1}{T^5} \#\{\lambda_j \leqslant T^2 : \max\{|\alpha_1^{(j)}(p)|, |\alpha_2^{(j)}(p)|, |\alpha_3^{(j)}(p)|\} \geqslant 1 + \delta\} \\ &\quad \ll \frac{1}{T^5} \sum_{\lambda_j \leqslant T^2} \frac{|A_j(p^l,1)|^{2k}}{((l+1)(l+2))^{2k}} \ll_\varepsilon \frac{T^\varepsilon}{2^{2k}} \end{aligned}$$

and Theorem 2 follows with $\eta < \frac{3}{l \log p}$.

Finally we prove Theorem 3 analogously to Proposition 2. Let r, s be fixed integers and p a fixed prime. By Theorem 5 we have

$$\frac{1}{c_2 T^5} \sum_j \frac{A_j(p^r, p^s)}{\operatorname{res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)} e^{-\lambda_j / T^2} \to \delta_{r=s=0}$$

as $T \to \infty$, and hence by the same Tauberian argument as in the proof of Proposition 2 we conclude

$$\frac{1}{c_2 T^5} \sum_{\lambda_j \leqslant T^2} \frac{A_j(p^r, p^s)}{\operatorname{res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)} \to \frac{\delta_{r=s=0}}{\Gamma(\frac{5}{2}+1)}.$$

By the Hecke relations we can write $A_j(p^r, p^s) = q_{r,s}(A_j(p,1))$ where $q_{r,s}(z)$ is a polynomial in z and $\bar{z}$. By (15), (16) and the definition (7) of the measure $d\mu$, Theorem 3 holds for the functions $q_{r,s}$ (restricted to $\mathcal{R}$). Again by the Hecke relations we can write $A(p,1)^k A(1,p)^l$ as a linear combination of the $A(p^r, p^s)$. Hence by linearity of the integral, Theorem 3 holds for the functions $f(z) = z^k \bar{z}^l$. Since every continuous function on $\mathcal{R}$ can be approximated by polynomials in z and $\bar{z}$, Theorem 3 follows.

6. The Kuznetsov formula on GL(3) – continued

6.1. Preliminaries. We introduce some notation. We start with the definition of the relevant Kloosterman sums for the Weyl elements

$$w_4 = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \quad w_5 = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, \quad w_6 = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}.$$

For $n_1, n_2, m_1, m_2 \in \mathbb{Z} \setminus \{0\}$, $D_1, D_2 \in \mathbb{N}$ we define

$$\begin{aligned} &S_{w_6}(m_1, m_2, n_1, n_2, D_1, D_2) \\ &:= \sum_{\substack{B_1, C_1 \ (\mathrm{mod}\ D_1) \\ B_2, C_2 \ (\mathrm{mod}\ D_2) \\ (D_1, B_1, C_1) = (D_2, B_2, C_2) = 1 \\ D_1 C_2 + B_1 B_2 + C_1 D_2 \equiv 0 \ (D_1 D_2)}} \sum\sum\sum e\left(\frac{m_1 B_1 + n_1(Y_1 D_2 - Z_1 B_2)}{D_1}\right) \\ &\qquad\qquad \cdot e\left(\frac{m_2 B_2 + n_2(Y_2 D_1 - Z_2 B_1)}{D_2}\right) \end{aligned}$$

where Y_1, Y_2, Z_1, Z_2 are chosen such that

$$Y_1 B_1 + Z_1 C_1 \equiv 1 \ (\mathrm{mod}\ D_1), \quad Y_2 B_2 + Z_2 C_2 \equiv 1 \ (\mathrm{mod}\ D_2).$$

For $D_1 \mid D_2$, we put

$$\begin{aligned} &S_{w_5}(m_1, m_2, n_1, n_2, D_1, D_2) \\ &:= \sum_{\substack{C_1\ (D_1),\ C_2\ (D_2) \\ (C_1, D_1) = (C_2, D_2/D_1) = 1}} \sum e\left(\frac{n_1 C_1 + m_1 \overline{C_1} C_2}{D_1}\right) e\left(\frac{m_2 \overline{C_2}}{D_2/D_1}\right). \end{aligned}$$

The right-hand side does not depend on n_2, but it is nevertheless convenient to keep n_2 on the left-hand side. For $D_2 \mid D_1$ we put

$$S_{w_4}(m_1, m_2, n_1, n_2, D_1, D_2) := S_{w_5}(m_2, m_1, n_2, n_1, D_2, D_1).$$

Many properties of these Kloosterman sums have been derived in [BFG]. Here we only need to know the upper bounds

$$\begin{aligned} &S_{w_6}(m_1, m_2, n_1, n_2, D_1, D_2) \\ &\qquad \ll_\varepsilon (D_1 D_2)^{1/2+\varepsilon}\big((D_1, D_2)(m_1 m_2, [D_1, D_2])(n_1 n_2, [D_1, D_2])\big)^{1/2}, \\ &S_{w_5}(m_1, m_2, n_1, n_2, D_1, D_2) \ll_\varepsilon \min\left(\left(m_1, \frac{D_2}{D_1}\right) D_1^2, (m_2, n_1, D_1) D_2\right)^{1+\varepsilon}, \\ &S_{w_4}(m_1, m_2, n_1, n_2, D_1, D_2) \ll_\varepsilon \min\left(\left(m_1, \frac{D_1}{D_2}\right) D_2^2, (m_1, n_2, D_2) D_1\right)^{1+\varepsilon} \end{aligned} \tag{25}$$

where $[.,.]$ denotes the least common multiple. The last two bounds are due to Larsen (see [BFG], Appendix), the first bound is essentially due to Stevens (see [Ste], Theorem 5.1). The dependence on m_1, m_2, n_1, n_2 has been worked out in [Bu1], p. 39, by analyzing Stevens' proof.

Next we define the normalized Fourier coefficients of minimal and maximal parabolic Eisenstein series. We refer to Section 10 of [Gold] or Section 5 of [Bl] for more details. For $\nu_1, \nu_2 \in i\mathbb{R}$ and $m_1, m_2 \in \mathbb{Z} \setminus \{0\}$ we define

$$A_{\nu_1,\nu_2}(m_1, m_2) = |m_1|^{\nu_1+2\nu_2}|m_2|^{2\nu_1+\nu_2}\sigma_{-3\nu_2,-3\nu_1}(|m_1|, |m_2|)$$

where $\sigma_{\nu_1,\nu_2}(m_1, m_2)$ is the multiplicative function defined by

$$\sigma_{\nu_1,\nu_2}(p^{k_1}, p^{k_2}) = p^{-\nu_2 k_1} \frac{\left|\begin{pmatrix} 1 & p^{\nu_2(k_1+k_2+2)} & p^{(\nu_1+\nu_2)(k_1+k_2+2)} \\ 1 & p^{\nu_2(k_1+1)} & p^{(\nu_1+\nu_2)(k_1+1)} \\ 1 & 1 & 1 \end{pmatrix}\right|}{\left|\begin{pmatrix} 1 & p^{2\nu_2} & p^{2(\nu_1+\nu_2)} \\ 1 & p^{\nu_2} & p^{\nu_1+\nu_2} \\ 1 & 1 & 1 \end{pmatrix}\right|}.$$

Moreover, for $\mu \in i\mathbb{R}$ and u_j a Hecke–Maaß cusp form for $\mathrm{SL}_2(\mathbb{Z})$ with eigenvalues $\lambda_j(n)$, we define

$$B_{\mu,u_j}(1, m) = \sum_{d_1 d_2 = |m|} \lambda_j(d_1) d_1^{-\mu} d_2^{2\mu}$$

and extend this definition to all pairs of integers by the Hecke relations

$$\begin{aligned} B_{\mu,u_j}(m, 1) &= \overline{B_{(\mu,u_j)}(1, m)} = B_{\mu,u_j}(1, m), \\ B_{\mu,u_j}(m_1, m_2) &= \sum_{d \mid (m_1, m_2)} \mu(d) B_{\mu,u_j}\left(\frac{m_1}{d}, 1\right) B_{\mu,u_j}\left(1, \frac{m_2}{d}\right). \end{aligned}$$

It follows from the Kim–Sarnak bound (3) that

$$A_{\nu_1,\nu_2}(m_1, m_2),\ B_{\mu,u_j}(m_1, m_2) \ll |m_1 m_2|^{7/64+\varepsilon}. \tag{26}$$

Up to a normalizing factor proportional to

$$\frac{\prod_{j=0}^{2}\cos(3\pi\nu_j/2)}{|\zeta(1+3\nu_j)|^2} \quad \text{resp.} \quad \frac{\prod_{j=0}^{2}\cos(3\pi\nu_j/2)}{L(1,\mathrm{sym}^2 u_j)|L(1+3\mu)|^2}$$

the quantities $|A_{\nu_1,\nu_2}(m_1,m_2)|^2$ resp. $|B_{\mu,u_j}(m_1,m_2)|^2$ are the squares of the Fourier coefficients of the minimal resp. maximal Eisenstein series in the spectral decomposition, see e.g. Section 5 and Proposition 4 in [Bl].

We have the following formula for the (slightly re-normalized) double Mellin transform of the Whittaker function [Gold], (6.1.4):

$$\begin{aligned}
&\widehat{W}_{\nu_1,\nu_2}(u_1,u_2)\\
&:= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty W_{\nu_1,\nu_2}(y_1,y_2)(\pi y_1)^{u_1-1}(\pi y_2)^{u_2-1}\frac{dy_1\,dy_2}{y_1y_2}\\
&= \frac{\Gamma\left(\frac{u_1+2\nu_1+\nu_2}{2}\right)\Gamma\left(\frac{u_1-\nu_1+\nu_2}{2}\right)\Gamma\left(\frac{u_1-\nu_1-2\nu_2}{2}\right)\Gamma\left(\frac{u_2-2\nu_1-\nu_2}{2}\right)\Gamma\left(\frac{u_2+\nu_1-\nu_2}{2}\right)\Gamma\left(\frac{u_2+\nu_1+2\nu_2}{2}\right)}{\Gamma\left(\frac{u_1+u_2}{2}\right)}.
\end{aligned}$$

For $\Delta > 0$, $y_1, y_2 \in \mathbb{R}$ and $-9/8 < \Re u_j < -1$ we define the following auxiliary functions:

$$\begin{aligned}
&T_{w_4,\Delta}(u_1,u_2;y_1,y_2;\nu_1,\nu_2)\\
&= (\pi|y_1|)^{-u_1}(\pi|y_2|)^{-u_2}\int_0^\infty\int_0^\infty W_{-\nu_1,-\nu_2}(t_1,t_2)\\
&\quad t_1^{3+u_1-u_2+2\Delta}t_2^{2+u_1+\Delta}\int_{-\infty}^\infty\int_{-\infty}^\infty e\left(-\frac{y_1}{t_1t_2}\frac{x_3}{1+x_2^2+x_3^2}+t_1\frac{x_2x_3}{1+x_2^2}+t_2x_2\right)\\
&\quad (1+x_2^2)^{\frac{-1-u_1+2u_2}{2}}(1+x_2^2+x_3^2)^{\frac{-1+2u_1-u_2}{2}}dx_2dx_3\,\frac{dt_1\,dt_2}{(t_1t_2)^3},
\end{aligned}$$

$$\begin{aligned}
&T_{w_5,\Delta}(u_1,u_2;y_1,y_2;\nu_1,\nu_2)\\
&= (\pi|y_1|)^{-u_1}(\pi|y_2|)^{-u_2}\int_0^\infty\int_0^\infty W_{-\nu_1,-\nu_2}(t_1,t_2)\\
&\quad t_1^{3+u_2+2\Delta}t_2^{2+u_2-u_1+\Delta}\int_{-\infty}^\infty\int_{-\infty}^\infty e\left(-\frac{y_2}{t_1t_2}\frac{x_3}{1+x_1^2+x_3^2}+t_2\frac{x_1x_3}{1+x_1^2}+t_1x_1\right)\\
&\quad (1+x_1^2)^{\frac{-1-u_2+2u_1}{2}}(1+x_1^2+x_3^2)^{\frac{-1+2u_2-u_1}{2}}dx_1dx_3\,\frac{dt_1\,dt_2}{(t_1t_2)^3},
\end{aligned}$$

$$T_{w_6,\Delta}(u_1,u_2;y_1,y_2;\nu_1,\nu_2)$$
$$= (\pi|y_1|)^{-u_1}(\pi|y_2|)^{-u_2}\int_0^\infty\int_0^\infty W_{-\nu_1,-\nu_2}(t_1,t_2)t_1^{3+u_2+2\Delta}t_2^{2+u_1+\Delta}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e\left(\frac{y_1}{t_2}\frac{x_2+x_1x_3}{1+x_2^2+x_3^2}+\frac{y_2}{t_1}\frac{x_1+x_2(x_1x_2-x_3)}{1+x_1^2+(x_1x_2-x_3)^2}+t_1x_2+t_2x_2\right)$$
$$(1+x_2^2+x_3^2)^{\frac{-1+2u_1-u_2}{2}}(1+x_1^2+(x_1x_2-x_3)^2)^{\frac{-1-u_1+2u_2}{2}}dx_1dx_2dx_3\ \frac{dt_1\,dt_2}{(t_1t_2)^3},$$

as well as

$$M_\Delta(\nu_1,\nu_2)=\frac{\pi^{3\Delta}\Gamma\left(\frac{3+3\Delta}{2}\right)}{\pi^5\Gamma\left(\frac{1+\Delta}{2}\right)^3}\prod_{j=0}^{2}\frac{\frac{-3\pi\nu_j}{2}\tan\left(\frac{-3\pi\nu_j}{2}\right)}{\Gamma\left(\frac{1+\Delta+3\nu_j}{2}\right)\Gamma\left(\frac{1+\Delta-3\nu_j}{2}\right)}.$$

Let $-9/8 < c < -1$. For $w \in \{w_4, w_5, w_6\}$ we define the following integral kernel

$$J_{w,\Delta}(y_1,y_2;\nu_1,\nu_2)$$
$$= M_\Delta(\nu_1,\nu_2)\Bigg(\int_{(c)}\int_{(c)}\widehat{W}_{\nu_1,\nu_2}(u_1,u_2)T_{w,\Delta}(u_1,u_2;y_1,y_2;\nu_1,\nu_2)\frac{du_1\,du_2}{(2\pi i)^2}$$
$$+3\int_{(c)}\operatorname*{res}_{u_1=\nu_1+2\nu_2}\widehat{W}_{\nu_1,\nu_2}(u_1,u_2)T_{w,\Delta}(\nu_1+2\nu_2,u_2;y_1,y_2;\nu_1,\nu_2)\frac{du_2}{2\pi i}$$
$$+3\int_{(c)}\operatorname*{res}_{u_2=\nu_1-\nu_2}\widehat{W}_{\nu_1,\nu_2}(u_1,u_2)T_{w,\Delta}(u_1,\nu_1-\nu_2;y_1,y_2;\nu_1,\nu_2)\frac{du_1}{2\pi i}$$
$$+6\operatorname*{res}_{\substack{u_1=\nu_1+2\nu_2\\u_2=\nu_1-\nu_2}}\widehat{W}_{\nu_1,\nu_2}(u_1,u_2)T_{w,\Delta}(\nu_1+2\nu_2,\nu_1-\nu_2;y_1,y_2;\nu_1,\nu_2)\Bigg).$$

For better comparison with [Bu2], we recall that the Langlands parameters μ_1, μ_2, μ_3 are related to the spectral parameters ν_0, ν_1, ν_2 by

$$\begin{aligned}\mu_1&=2\nu_1+\nu_2, &\quad \mu_2&=-\nu_1+\nu_2, &\quad \mu_3&=-\nu_1-2\nu_2,\\ \nu_1&=(\mu_1-\mu_2)/3, & \nu_2&=(\mu_2-\mu_3)/3, & \nu_0&=(\mu_1-\mu_3)/3.\end{aligned}\tag{27}$$

Then the functions $J_{w,\boldsymbol{\mu}}(\boldsymbol{y})$ in [Bu2] are related to the above defined function $J_{w,\Delta}(y_1,y_2;\nu_1,\nu_2)$ by

$$J_{w,\Delta}(y_1,y_2;\nu_1,\nu_2)$$
$$=\frac{4\pi^{3+3\Delta}\Gamma\left(\frac{3+3\Delta}{2}\right)}{k_{\mathrm{adj}}(\nu_1,\nu_2)\Gamma(\frac{1+\Delta}{2})^3\prod_{j=0}^2\cos\left(\frac{3\pi\nu_j}{2}\right)\Gamma(\frac{1+\Delta+3\nu_j}{2})\Gamma(\frac{1+\Delta-3\nu_j}{2})}J_{w,\boldsymbol{\mu}}(\boldsymbol{y})\tag{28}$$

where the function $k_{\mathrm{adj}}(\nu_1,\nu_2)$ is constructed so that the denominator is $\asymp 1$ away from poles (that is, when ν_j is an odd multiple of $1/3$). Specifically,

$$k_{\mathrm{adj}}(\nu_1, \nu_2) = \prod_{j=0}^{2} ((3 + 2\Delta)^2 - 9\nu_j^2)^{-\Delta/2}.$$

The key to estimating the Kloosterman terms in the Kuznetsov formula is the following bound for the integral kernel $J_{w,\Delta}$ which is part of [Bu2], Proposition 1.

Lemma 2. *Let* $\Delta = 5/4$.

a) *If* $\Re\nu_1 = -1/2$ *and* $\Re\nu_2 = 0$, *then*

$$J_{w_4,\Delta}(y, 1; \nu_1, \nu_2) \ll |y| \frac{\big((1 + |\nu_0|)(1 + |\nu_1|)(1 + |\nu_2|)\big)^{29/16}}{\min\big((1 + |\nu_0|), (1 + |\nu_1|), (1 + |\nu_2|)\big)^{55/16}}.$$

b) *If* $\Re\nu_1 = -1/2$ *and* $\Re\nu_2 = 1/2$, *then*

$$J_{w_5,\Delta}(1, y; \nu_1, \nu_2) \ll |y| \frac{\big((1 + |\nu_0|)(1 + |\nu_1|)(1 + |\nu_2|)\big)^{25/16}}{\min\big((1 + |\nu_0|), (1 + |\nu_1|), (1 + |\nu_2|)\big)^{43/16}}.$$

c) *If* $\Re\nu_1 = -2/3$ *and* $\Re\nu_2 = 1/3$, *then*

$$J_{w_6,\Delta}(y_1, y_2; \nu_1, \nu_2) \ll |y_1 y_2| \frac{\big((1 + |\nu_0|)(1 + |\nu_1|)(1 + |\nu_2|)\big)^{29/16}}{\min\big((1 + |\nu_0|), (1 + |\nu_1|), (1 + |\nu_2|)\big)^{55/16}}.$$

Proof. We use Proposition 1 of [Bu2] and choose

- $(\Re\mu_1, \Re\mu_2, \Re\mu_3) = (-1, 1/2, 1/2)$ for J_{w_4},
- $(\Re\mu_1, \Re\mu_2, \Re\mu_3) = (-1/2, 1, -1/2)$ for J_{w_5}, and
- $(\Re\mu_1, \Re\mu_2, \Re\mu_3) = (-1, 1, -1/2)$ for J_{w_6}. □

6.2. An explicit Kuznetsov formula. We are now ready to state the Kuznetsov formula. The following theorem is a restatement of Theorem 8 of [Bu2] which is based on Li's approach in Section 11 of [Gold] together with the spherical inversion formula for $\mathrm{SL}_3(\mathbb{R})$.

Theorem 6. *Let* $h\colon \mathbb{C}^2 \to \mathbb{C}$ *be a function that is holomorphic on* $R := \{|\Re\nu_j| < 1, 0 \leqslant j \leqslant 2\}$ *and that is symmetric under the Weyl group*

$$(\nu_1, \nu_2) \to (-\nu_1, \nu_0) \to (\nu_2, -\nu_0) \to (-\nu_2, -\nu_1) \to (-\nu_0, \nu_1) \to (\nu_0, -\nu_2).$$

Assume that $h(\nu_1, \nu_2) = 0$ *if* $(\nu_1^2 - 1/9)(\nu_2^2 - 1/9)(\nu_0^2 - 1/9) = 0$ *and that*

$$h(\nu_1, \nu_2) \ll ((1 + |\nu_1|)(1 + |\nu_2|)(1 + |\nu_1 + \nu_2|))^{-5/3-\varepsilon}$$

in R. Let $n_1, n_2, m_1, m_2 \in \mathbb{Z} \setminus \{0\}$. Then the following formula holds:

$$\frac{4}{3}\sum_j A_j(n_1,n_2)\overline{A_j(m_1,m_2)}\frac{h(\nu_1^{(j)},\nu_2^{(j)})}{\operatorname*{res}_{s=1} L(s,\phi_j\times\bar{\phi}_j)}$$
$$+ C_1\int_{(0)}\int_{(0)} A_{\nu_1,\nu_2}(n_1,n_2)\overline{A_{\nu_1,\nu_2}(m_1,m_2)}$$
$$\cdot\frac{h(\nu_1,\nu_2)}{|\zeta(1+\nu_0)\zeta(1+\nu_1)\zeta(1+\nu_2)|^2}\frac{d\nu_1\,d\nu_2}{(2\pi i)^2}$$
$$+ C_2\sum_j\int_{(0)} B_{\mu,u_j}(n_1,n_2)\overline{B_{\mu,u_j}(m_1,m_2)}\frac{h(\mu-\frac{1}{3}it_j,\frac{2}{3}it_j)}{|L(1+3\mu,u_j)|^2L(1,\mathrm{sym}^2u_j)}\frac{d\mu}{2\pi i}$$
$$= \delta_{\substack{|n_1|=|m_1|\\|n_2|=|m_2|}}\frac{1}{2\pi^6}\int_{(0)}\int_{(0)} h(\nu_1,\nu_2)\prod_{j=0}^{2}\frac{(-3\pi\nu_j)}{2}\tan\frac{3\pi\nu_j}{2}\frac{d\nu_1 d\nu_2}{(2\pi i)^2}$$
$$+ \sum_{\substack{\epsilon_1=\mathrm{sign}(m_2n_1)\\ \epsilon_2\in\{\pm1\}}}\sum_{D_1|m_2|=D_2^2|n_1|}\frac{S_{w_4}(m_1,m_2,\epsilon_1n_1,\epsilon_2n_2,D_1,D_2)}{D_1D_2}$$
$$\int_{(c_{4,2})}\int_{(c_{4,1})} h(\nu_1,\nu_2)J_{w_4,\Delta}\left(\frac{\epsilon_2m_1m_2n_2}{D_1D_2},1;\nu_1,\nu_2\right)\frac{d\nu_1\,d\nu_2}{(2\pi i)^2}$$
$$+ \sum_{\substack{\epsilon_1\in\{\pm1\}\\ \epsilon_2=\mathrm{sign}(m_1n_2)}}\sum_{D_2|m_1|=D_1^2|n_2|}\frac{S_{w_5}(m_1,m_2,\epsilon_1n_1,\epsilon_2n_2,D_1,D_2)}{D_1D_2}$$
$$\int_{(c_{5,2})}\int_{(c_{5,1})} h(\nu_1,\nu_2)J_{w_5,\Delta}\left(1,\frac{\epsilon_1m_1m_2n_1}{D_1D_2};\nu_1,\nu_2\right)\frac{d\nu_1\,d\nu_2}{(2\pi i)^2}$$
$$+ \sum_{(\epsilon_1,\epsilon_2)\in\{\pm1\}^2}\sum_{D_1,D_2}\frac{S_{w_6}(\epsilon_2n_2,\epsilon_1n_1,m_1,m_2,D_1,D_2)}{D_1D_2}$$
$$\int_{(c_{6,2})}\int_{(c_{6,1})} h(\nu_1,\nu_2)J_{w_6,\Delta}\left(\frac{D_2m_1\epsilon_2n_2}{D_1^2},\frac{D_1m_2\varepsilon_1n_1}{D_2^2};\nu_1,\nu_2\right)\frac{d\nu_1\,d\nu_2}{(2\pi i)^2}$$

where

$$(c_{4,1},c_{4,2})=(-1/2,0),\quad (c_{5,1}c_{5,2})=(-1/2,1/2),\quad (c_{6,1},c_{6,2})=(-2/3,1/3),$$

and C_1 and C_2 are absolute constants.

For the conversion from [Bu2], (8) and Theorem 8, to the present version we used (21) for the cuspidal term.

Remarks. 1) The spectral side of the formula does not depend on Δ, hence the arithmetic side is independent of Δ, too. The individual terms on the arithmetic side, however, do depend on Δ. We need this somewhat artificial parameter for convergence reasons.

2) The requirement that $h(\nu_1, \nu_2) = 0$ if $(\nu_1^2 - 1/9)(\nu_2^2 - 1/9)(\nu_0^2 - 1/9) = 0$ comes from the poles of $\tan 3\pi\nu_j/2$ in the spectral measure. It allows us to choose contours with $-1 < \Re\nu_j < 1$ on the Kloosterman side. This is an analogue of the fact that the GL(2) Kuznetsov formula has better performance if the test function cancels the poles of the spectral measure $t\tanh(\pi t)dt$.

3) There is some flexibility in choosing the lines of integration, but there are also some constraints due to convergence, see Proposition 1 of [Bu2] for more details. In our situation the lines given in the theorem are, in view of Lemma 2, most useful.

The following corollary shows how to apply this rather complicated formula in practice.

Corollary 7. *Keep the notation and assumptions of Theorem 6. We write* $P = |m_1 m_2 n_1 n_2| \neq 0$ *and*

$$H_{\eta_1,\eta_2}(\alpha_1,\alpha_2) = \int_{(\eta_2)}\int_{(\eta_1)} |h(\nu_1,\nu_2)|(1+|\nu_1|)^{\alpha_1+\varepsilon}(1+|\nu_2|)^{\alpha_2+\varepsilon}|d\nu_1|\,|d\nu_2|$$

and

$$H^*_{\eta_1,\eta_2}(\alpha_1,\alpha_2) = \int_{(\eta_2)}\int_{(\eta_1)} |h(\nu_1,\nu_2)| \frac{\big((1+|\nu_0|)(1+|\nu_1|)(1+|\nu_2|)\big)^{\alpha_1+\varepsilon}}{\min\big((1+|\nu_0|),(1+|\nu_1|),(1+|\nu_2|)\big)^{\alpha_2}}|d\nu_1|\,|d\nu_2|$$

(for fixed small $\varepsilon > 0$*). Then one has*

$$\sum_j A_j(n_1,n_2)\overline{A_j(m_1,m_2)}\frac{h(\nu_1^{(j)},\nu_2^{(j)})}{\operatorname{res}_{s=1} L(s,\phi_j\times\bar{\phi}_j)}$$
$$= \delta_{\substack{|n_1|=|m_1|\\|n_2|=|m_2|}} \frac{3^2}{2^9\pi^6}\int_{(0)}\int_{(0)} h(\nu_1,\nu_2)\prod_{j=0}^{2}\frac{(-3\pi\nu_j)}{2}\tan\frac{3\pi\nu_j}{2}\frac{d\nu_1 d\nu_2}{(2\pi i)^2}$$
$$+ O_\varepsilon\Big(P^{1+\varepsilon}\Big(H^*_{-\frac{1}{2},0}\big(\tfrac{29}{16},\tfrac{55}{16}\big) + H^*_{-\frac{1}{2},\frac{1}{2}}\big(\tfrac{25}{16},\tfrac{43}{16}\big) + H^*_{-\frac{2}{3},\frac{1}{3}}\big(\tfrac{29}{16},\tfrac{55}{16}\big)\Big) + P^{7/64+\varepsilon}H_{0,0}(0,1)\Big).$$

Proof. We estimate the remaining terms in Theorem 6. For the Eisenstein series we use the bounds (26) together with the lower bounds $\zeta(1+it) \gg_\varepsilon |t|^{-\varepsilon}$ for $t \neq 0$, $L(1+it,u_j) \gg_\varepsilon (1+|t|+|t_j|)^{-\varepsilon}$ ([HR], Theorem C) and $L(1,\operatorname{sym}^2 u_j) \gg_\varepsilon (1+|t_j|)^{-\varepsilon}$ ([HL]) and Weyl's law (2). In this way we bound the maximal parabolic contribution by

$$P^{\theta+\varepsilon}\sum_j(1+|t_j|)^\varepsilon\int_{(0)}|h(\mu-\tfrac{1}{3}it_j,\tfrac{2}{3}it_j)|(1+|\mu|)^\varepsilon|d\mu| \ll_\varepsilon P^{\theta+\varepsilon}H_{0,0}(0,1),$$

and this majorizes also the minimal parabolic contribution.

For the Kloosterman terms we combine Lemma 2 with (25) and observe that the D_1-, D_2-sums on the Kloosterman side are absolutely convergent. □

6.3. Proof of Theorem 5. For the proof of Theorem 5 we choose

$$\begin{aligned} h(\nu_1,\nu_2) &= e^{-(1-3\nu_1^2-3\nu_1\nu_2-3\nu_2^2)/T^2}\prod_{j=0}^{2}\frac{(\frac{1}{9}-\nu_j^2)^2}{(1-\nu_j^2)^2} \\ &= e^{-(1-3\nu_1^2-3\nu_1\nu_2-3\nu_2^2)/T^2}\Big(1+O\Big(\sum_{j=0}^{2}\frac{1}{1+|\nu_j|^2}\Big)\Big). \end{aligned}$$

This satisfies the assumptions of Theorem 6. Note that by (19) and the truth of the Selberg eigenvalue conjecture $\lambda_j \geqslant 1$ the function h is positive on the spectrum. We compute

$$\begin{aligned} H_{0,0}(0,1) &\ll_\varepsilon T^{3+\varepsilon}, \\ H^*_{-\frac{1}{2},0}\left(\frac{29}{16},\frac{55}{16}\right)+H^*_{-\frac{2}{3},\frac{1}{3}}\left(\frac{29}{16},\frac{55}{16}\right) &\ll_\varepsilon T^{2\cdot\frac{29}{16}+1+\varepsilon}=T^{\frac{37}{8}+\varepsilon}, \\ H^*_{-\frac{1}{2},\frac{1}{2}}\left(\frac{25}{16},\frac{43}{16}\right) &\ll_\varepsilon T^{2\cdot\frac{25}{16}+1+\varepsilon}=T^{\frac{33}{8}+\varepsilon}. \end{aligned} \tag{29}$$

By a weak form of a local Weyl law for $\mathrm{SL}_3(\mathbb{Z})$ (e.g. [Bl], Theorem 1) and the bound (12) we have

$$\begin{aligned} &\sum_j A_j(n_1,n_2)\overline{A_j(m_1,m_2)}\frac{h(\nu_1^{(j)},\nu_2^{(j)})}{\operatorname{res}_{s=1}L(s,\phi\times\bar\phi)} \\ &\qquad -\sum_j A_j(n_1,n_2)\overline{A_j(m_1,m_2)}\frac{e^{-\lambda_j/T^2}}{\operatorname{res}_{s=1}L(s,\phi\times\bar\phi)} \\ &\ll |n_1n_2m_1m_2|^{1/2}\sum_j\frac{e^{-\lambda_j/T^2}\sum_{j=0}^{2}(1+|\nu_j|^2)^{-2}}{\operatorname{res}_{s=1}L(s,\phi\times\bar\phi)} \\ &\ll_\varepsilon |n_1n_2m_1m_2|^{1/2+\varepsilon}T^{3+\varepsilon}. \end{aligned} \tag{30}$$

Finally we compute

$$\begin{aligned}
&\int_{(0)}\int_{(0)} h(\nu_1,\nu_2)\prod_{j=0}^{2}\frac{3\pi\nu_j}{2}\tan\frac{(-3\pi\nu_j)}{2}\frac{d\nu_1 d\nu_2}{(2\pi i)^2}\\
&= 6\int_0^{i\infty}\int_0^{i\infty} h(\nu_1,\nu_2)\prod_{j=0}^{2}\frac{(-3\pi\nu_j)}{2}\tan\frac{3\pi\nu_j}{2}\frac{d\nu_1 d\nu_2}{(2\pi i)^2}\\
&= 6i^3\int_0^{i\infty}\int_0^{i\infty} e^{(3\nu_1^2+3\nu_1\nu_2+3\nu_2^2)/T^2}\prod_{j=0}^{2}\frac{(-3\pi\nu_j)}{2}\frac{d\nu_1 d\nu_2}{(2\pi i)^2}+O_\varepsilon(T^{3+\varepsilon})\\
&= 6\left(\frac{3\pi}{2}\right)^3\frac{1}{(2\pi)^2}\frac{\sqrt{\pi/3}}{81}T^5+O_\varepsilon(T^{3+\varepsilon})=\frac{\pi^{3/2}}{16\sqrt{3}}T^5+O(T^{3+\varepsilon}).
\end{aligned} \tag{31}$$

The double integral in the penultimate line can be computed by diagonalizing the quadratic form via $\nu_1 \mapsto \nu_1 - \nu_2/2$ (or by Mathematica). Theorem 5 follows from injecting (29)–(31) into Corollary 7.

Note added in proof. 1) After this paper was accepted for publication, similar results were obtained independently by F. Zhou ("Weighted Sato–Tate vertical distribution of the Satake parameter of Maass forms on PGL(N)", to appear in *Ramanujan Journal*, Doi 10.1007/s11139-013-9535-6).

2) The second author would like to take the opportunity to correct some misprints in [Bu2]: the leading constant in Theorem 7 should be $-1/(48\pi^4)$ instead of $-1/(64\pi^4)$, and the changes propagated. The function $C^*(\mu)$ above Theorem 9 is missing a factor $4\pi^{3/2+3\Delta}\Gamma(\frac{3+3\Delta}{2})$. The leading constant on H_I in Theorem 9 becomes $-1/(24\pi^8)$, and the leading constant in (14) should be $1/(12\pi^3)$.

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