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Spectral theory for the Weil–Petersson Laplacian on the Riemann moduli space

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Abstract. We study the spectral geometric properties of the scalar Laplace–Beltrami operator associated to the Weil–Petersson metric g_{WP} on $\mathcal{M}_\gamma$, the Riemann moduli space of surfaces of genus $\gamma > 1$. This space has a singular compactification with respect to g_{WP}, and this metric has crossing cusp-edge singularities along a finite collection of simple normal crossing divisors. We prove first that the scalar Laplacian is essentially self-adjoint, which then implies that its spectrum is discrete. The second theorem is a Weyl asymptotic formula for the counting function for this spectrum.

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1. Introduction

This paper initiates the analytic study of the natural geometric operators associated to the Weil–Petersson metric g_{WP} on the Riemann moduli space of surfaces of genus $\gamma > 1$, denoted below by $\mathcal{M}_\gamma$. We consider the Deligne–Mumford compactification of this space, $\bar{\mathcal{M}}_\gamma$, which is a stratified space with many special features. The geometry and topology of $(\bar{\mathcal{M}}_\gamma, g_{\mathrm{WP}})$ has been extensively studied, starting from the early works of Ahlfors, and including the systematic study by Wolpert [16], [18]. More recent results include the asymptotic behaviour of the Weil–Petersson volume by Mirzakhani [15], the ergodicity of the Weil–Petersson geodesic flow by Burns–Masur–Wilkinson [5], and asymptotic bounds on the Weil–Petersson diameter of $\bar{\mathcal{M}}_\gamma$ by Cavendish–Parlier [6].

Given all these results and the many well-known close relations between various aspects of the geometry and analysis of Riemannian manifolds, it is natural to study the spectral theory of the moduli space with respect to the Weil–Petersson metric. In general, the spectral geometry of geodesically incomplete Riemannian manifolds is poorly understood. One source of examples of such spaces consists of singular algebraic varieties with certain natural Kähler metrics, and $(\bar{\mathcal{M}}_\gamma, g_{\mathrm{WP}})$ is arguably one of the most interesting of these. We hope that the results in this paper will draw

people's attention to further problems concerning analysis on moduli space and the relationship of analytic and geometric properties on this space.

This paper stands as the first step in a development of analytic techniques to study the natural elliptic differential operators associated to the Weil–Petersson metric. More generally, our results here apply to the wider class of metrics with crossing cusp-edge singularities on certain stratified Riemannian pseudomanifolds. This fits into a much broader study of geometric analysis on stratified spaces using the techniques of geometric microlocal analysis. Some of this work is directed toward studying general classes of such spaces, while other parts are focused on specific problems arising on particular singular spaces which arise 'in nature', such as compactifications of geometric moduli spaces, etc. These approaches are, of course, closely intertwined. The perspective of this paper is that $(\mathcal{M}_\gamma, g_{\mathrm{WP}})$ is inherently interesting, and that the spectral theory of its Laplacian will most likely find interesting applications; at the same time, it is an interesting challenge to develop analytic techniques which can be used to study other singular spaces with related metric structures.

Our goals here are relatively modest. As stated above, we focus on the scalar Laplacian Δ, rather than any more complicated operator, associated to the Weil–Petersson metric on $\mathcal{M}_\gamma$, and provide answers to the most basic analytic questions about this operator.

Theorem 1. *The scalar Laplace operator Δ on $(\mathcal{M}_\gamma, g_{\mathrm{WP}})$ is essentially self-adjoint, i.e. there is a unique self-adjoint extension from the core domain $\mathcal{C}^\infty_{0,\mathrm{orb}}(\mathring{\mathcal{M}}_\gamma)$. The spectrum of this operator is discrete, and if $N(\lambda)$ denotes the number of eigenvalues of Δ which are less than λ, then*

$$N(\lambda) = \frac{\omega_n}{(2\pi)^n} \mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}) \lambda^{n/2} + o(\lambda^{n/2})$$

as $\lambda \to \infty$. Here, as usual, ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Remark 1. There is a subtlety in the statement of this theorem which we point out immediately. The *interior* of the space $\mathcal{M}_\gamma$ already has singularities, but these are caused not by any properties of the Weil–Petersson metric, but rather are orbifold points which arise because the mapping class group does not act freely on Teichmüller space. While an analysis of the self-adjoint extensions near such points can be carried out, we instead restrict to an easily defined core domain which fixes the nature of the self-adjoint extension near these points. This is defined as follows. If p is a singular point in the interior of $\mathcal{M}_\gamma$, then there is a neighbourhood $\mathcal{U}_p$ around p, an open set $\widetilde{\mathcal{U}}_p$ in $\mathbb{R}^N$ ($N = 6\gamma - 6$), and a finite group Γ_p which acts on $\widetilde{\mathcal{U}}_p$ such that $\widetilde{\mathcal{U}}_p/\Gamma_p = \mathcal{U}_p$. We then define $\mathcal{C}^\infty_0(\mathcal{M}_\gamma)$ to consist of all functions f such that the restriction of f to $\mathcal{U}_p$ lifts to a $\mathcal{C}^\infty$ function $\tilde{f}$ on $\widetilde{\mathcal{U}}_p$. We refer to [11] for more on this and related other analytic constructions on orbifolds. Our main result then is

that Δ is essentially self-adjoint on *this* core domain. The arguments in this paper are essentially all local (or at least localizable), which means that for all analytic purposes, it suffices to assume that the interior is smooth and that the singular set of the compactification is of crossing cusp-edge type, as described below, even though the actual space is locally a finite quotient of this.

The emphasis on showing that a given operator is essentially self-adjoint is not so standard in geometric analysis for the simple reason that this property is automatic for all 'reasonable' elliptic operators on any complete manifold, and when the issue actually arises, e.g. on manifolds with boundary, it is so customary to impose boundary conditions that one rarely thinks of this as choosing a self-adjoint extension. On singular spaces, by contrast, the issue becomes a very real one, and a key preliminary part of the analysis on any such space is to determine whether the singular set is large enough, in some sense, to create the need for the imposition of boundary values. For other classes of singular spaces, e.g. those with cones, edges, etc., this issue is well understood. It is known that if the singular set has codimension at least 4, then there is no need to impose boundary conditions for the Laplacian on functions. For the Laplacian on differential forms, however, and for these same types of 'iterated edge metrics', the situation is more complicated, and was first considered carefully by Cheeger [8], [9], see also [2], [3] for some recent work on this. However, the Weil–Petersson metric is more singular than these spaces, which leads to the goals of this paper.

The proof of essential self-adjointness for any operator translates to a technical problem of showing that any element in the maximal domain $\mathcal{D}_{\max}$ of this operator is necessarily in the minimal domain $\mathcal{D}_{\min}$. We review the definitions of these domains in the beginning of §3. This is simply a regularity statement: we wish to show that any $u \in \mathcal{D}_{\max}$ enjoys enough regularity and decay near the singular set to allow us to prove that it can be approximated in graph norm by elements of $\mathcal{C}_0^\infty(\mathcal{M}_\gamma)$. The techniques used to prove this regularity here are somewhat ad hoc, and in particular do not use any of the heavy microlocal machinery which has proved to be very helpful for the study of more detailed analytic questions on stratified spaces. The advantage, however, is that this approach is much more self-contained.

It is worth recalling the well-known fact that the Laplacian on $\mathbb{R}^n$ is essentially self-adjoint on the core-domain $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$ if and only if $n \geq 4$, see [10], and that the 4-dimensional case has a borderline nature. The 'radial part' of the Weil–Petersson Laplacian near a divisor is essentially the same as the radial part of the Laplacian on $\mathbb{R}^4$, so in our setting too there are some borderline effects in the analysis. This motivates our introduction of a slightly broader class of crossing cusp-edge metrics of any order $k \geq 3$, for any of which we carry out this analysis. This is intended to clarify the slightly more delicate argument needed when $k = 3$.

An immediate consequence of the equivalence $\mathcal{D}_{\max} = \mathcal{D}_{\min}$ for the scalar Laplacian is the fact that this domain is compactly contained in L^2, which proves immedi-

ately that the spectrum is discrete. Our final result, concerning the Weyl asymptotics of the counting function for this spectrum, employs the classical Dirichlet–Neumann bracketing method, hence does not provide much information about the error term.

In §2 we provide a brief (and sketchy) review of the key properties of the geometric structure of the Weil–Petersson metric and of the singular set of $\bar{\mathcal{M}}_\gamma$. The key fact, that the local lifts of g_{WP} have 'crossing cusp-edge' singularities of order 3, leads to the introduction of the analogous class of metrics of any order $k \geq 3$. The rest of the paper then considers the scalar Laplacian for any metric of this more general type. Essential self-adjointness is studied in §3, and the Weyl estimate is obtained in §4.

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2. The geometry of the Weil–Petersson metric

We begin with a description of the singular structure of $\bar{\mathcal{M}}_\gamma$ and the structure of g_{WP} near the singular strata. The results about the structure of the Deligne–Mumford compactification itself are classical at this point, and we refer to [12], [1] for more detailed descriptions of all of this and references. The form of the Weil–Petersson metric traces back to a paper of Masur [14] in the early 1970s, but a far more detailed picture has emerged through the work of Yamada [20] and Wolpert [16], [17]. We point to two important recent survey articles [21] and [18] and the references therein.

The compact space $\bar{\mathcal{M}}_\gamma$ is a complex space which is singular along the union of a collection $D_0, \ldots, D_{[\gamma/2]}$ of immersed divisors with simple normal crossing. Elements of $\mathcal{M}_\gamma$ correspond to conformal structures on the underlying compact surface Σ of genus γ, where conformal structures are identified if they differ by an arbitrary diffeomorphism of Σ. Another realization of this space is as the space of hyperbolic metrics on Σ identified by the same space of diffeomorphisms. By contrast, Teichmüller space $\mathcal{T}_\gamma$ consists of the space of all conformal structures or hyperbolic metrics identified only by the smaller group of diffeomorphisms of Σ isotopic to the identity. Thus

$$\mathcal{M}_\gamma = \mathcal{T}_\gamma / \mathrm{Map}(\Sigma),$$

where the so-called mapping class group $\mathrm{Map}(\Sigma)$ is a discrete group of automorphisms of $\mathcal{T}_\gamma$ defined as the quotient of the group of all diffeomorphisms by the subgroup of those isotopic to the identity.

Let $c_1, \ldots, c_N$ be a maximal collection of homotopically nontrivial disjoint simple closed curves on Σ. It is well known that $N = 3\gamma - 3$, and that $\Sigma \setminus \{c_1, \ldots, c_N\}$ is a union of $2\gamma - 2$ pairs of pants, and moreover, exactly $[\gamma/2] + 1$ of the curves are distinct after identification by $\mathrm{Map}(\Sigma)$.

There is a simple geometric meaning to each of the divisors. Let D_j be the divisor associated to an equivalence class of curves $[c]$ (i.e. curves in this equivalence class

are identified, up to homotopy, by elements of Map (Σ)). A sequence of points $p_i \in \mathcal{M}_\gamma$ converges to D_j if the geodesics freely homotopic to c_j for the corresponding sequence of hyperbolic metrics have lengths $\ell(c_j) \to 0$. A crossing $D_{j_1} \cap \cdots \cap D_{j_\ell}$ corresponds to the independent length degeneration of some collection of equivalence classes of curves $c_{j_1}, \ldots, c_{j_\ell}$. In the following, we shall often denote such an ℓ-fold intersection by D_J where $J = \{j_1, \ldots, j_\ell\}$. Each divisor D_j can be identified with the Riemann moduli space for the (possibly disconnected) noded surface Σ' obtained by pinching the curve c_j, or equivalently, by cutting Σ along c_j and identifying each of the boundaries, which are copies of c_j, to points.

There are a number of natural and interesting metrics on Teichmüller space which are invariant with respect to Map(Σ) and which thus descend to metrics on $\mathcal{M}_\gamma$. One of the most fundamental is the Weil–Petersson metric, g_{WP}, which is the one studied here. It is incomplete on $\mathcal{M}_\gamma$ and induces the corresponding Weil–Petersson metric on each of the divisors. It is simply the canonical L^2 inner product on tangent vectors: if h is a hyperbolic metric on Σ representing a point of $\mathcal{M}_\gamma$, then the tangent space $T_h\mathcal{M}_\gamma$ is identified with the space of transverse-traceless symmetric two-tensors κ on Σ, i.e. $\mathrm{tr}^h\kappa = 0$ and $\delta^h\kappa = 0$. If κ_1 and κ_2 are two such tangent vectors, then

$$\langle \kappa_1, \kappa_2 \rangle_{g_{\mathrm{WP}}} = \int_\Sigma \langle \kappa_1, \kappa_2 \rangle_h \, dA_h = \int_\Sigma (\kappa_1)_{ij} (\kappa_2)_{k\ell} h^{ik} h^{j\ell} \, dA_h.$$

It is known that $\mathcal{M}_\gamma$ is a complex orbifold and that g_{WP} is a Kähler metric with many interesting properties. Our main concern is its fine asymptotic structure near the singular divisors, which are due to Yamada [20] and Wolpert [19], with closely related results by Liu, Sun and Yau [13]. Let p be a point in some D_J and choose a local holomorphic coordinate chart $(z_1, \ldots, z_{3g-3})$ with $D_J = \{z_1 = \cdots = z_\ell = 0\}$. Setting $z_j = \rho_j e^{i\theta_j}$, $j \le \ell$, then

$$g_{\mathrm{WP}} = \pi^3 \sum_{j=1}^{\ell} (4 d\rho_j^2 + \rho_j^6 d\theta_j^2)(1 + |\rho|^3) + g_{D_J} + \mathcal{O}(|\rho|^3) \tag{1}$$

where g_{D_J} is the Weil–Petersson metric on D_J. The expression $\mathcal{O}(|\rho|^3)$ indicates that all other terms are combinations of $d\rho_j$, $\rho_j^3 d\theta_j$ and dy (where y is a local coordinate along D_J) with coefficients vanishing at this rate. This (and in fact a slightly sharper version) is proved in [17], and the same result with some further information on the first derivatives of the metric components appears in [13].

We do not belabor the precise form of the remainders in these asymptotics for the following two reasons. First, g_{WP} is Kähler, and we can invoke the standard fact in Kähler geometry, see [4], p. 252, that if g is any Kähler metric, then its Laplace operator has the particularly simple form

$$\Delta_g = \sum g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}, \tag{2}$$

with no first order terms. In particular, the coefficients of this operator do not depend on derivatives of the metric. The same is true for the Dirichlet form for this metric, which also involves only the components of the (co)metric, but not their derivatives. The proof in §3 involves various integrations by parts, but a close examination of the details shows that one needs in any case very little about the derivatives of the metric, and for Kähler metrics one needs no information about these derivatives at all.

The other reason is that current work by the second author and J. Swoboda aims at deriving a complete asymptotic expansion for g_{WP}, and this implies all the results needed here about the remainder terms and their derivatives. However, since that work has not yet appeared, we emphasize that enough is known about the asymptotics of the metric in the existing literature to justify all the calculations below.

In any case, using (1), disregarding the constants π^3 and 4 for simplicity, and using the product polar coordinates, $(\rho_1, \dots, \rho_\ell, \theta_1, \dots, \theta_\ell, y_1, \dots, y_s) \in \mathcal{U} = (0, \rho_0)^\ell \times (S^1)^\ell \times \mathcal{V}$, where $\mathcal{V}$ is an open neighbourhood in D_J, we note that

$$\|u\|_{L^2}^2 = \int |u|^2 \mathcal{J} \cdot (\rho_1 \dots \rho_\ell)^3 \, d\rho \, d\theta \, dy,$$

where the Jacobian factor $\mathcal{J}$ is uniformly bounded and uniformly positive. Similarly, the Dirichlet form for Δ_{WP} is given by

$$\int \Big(\sum_{j=1}^{\ell} |\partial_{\rho_j} u|^2 + \sum_{j=1}^{k} \rho_j^{-6} |\partial_{\theta_j} u|^2 + |\nabla_y u|^2 \Big) \mathcal{J} \cdot (\rho_1 \dots \rho_\ell)^3 \, d\rho \, d\theta \, dy$$

modulo terms vanishing like $|\rho|^3$.

In order to focus on the key analytic points of the argument, we shall work with a slightly more general class of (not necessarily Kähler) Riemannian metrics, the asymptotic structure of which models that of g_{WP}, with singularities of similar crossing cusp-edge type. We thus let $\mathcal{M}$ be any manifold which has a compactification $\bar{\mathcal{M}}$ with the same structural features as the Deligne–Mumford compactification. Specifically, $\bar{\mathcal{M}}$ is a stratified pseudomanifold, with $\mathcal{M}$ its dense top-dimensional stratum. All other strata are of even codimension, and can be locally described as finite intersections $D_{j_1} \cap \dots \cap D_{j_\ell}$, where $\dim D_j = \dim \mathcal{M} - 2$ for all j. The main point is that we can use the same sort of product polar coordinate systems as above, and we shall do so henceforth without comment. We now consider metrics which in any such local coordinate system are modelled by the product metric

$$g_{\ell,k} := \sum_{i=1}^{\ell} (d\rho_i^2 + \rho_i^{2k} d\theta_i^2) + \sum_{\mu=1}^{n-2\ell} dy_\mu^2, \tag{3}$$

for any $k \geq 3$. The corresponding Laplacian is given by

$$\Delta_{\ell,k} = -\sum_{i=1}^{\ell} \Big(\partial_{\rho_i}^2 + \frac{k}{\rho_i} \partial_{\rho_i} + \frac{1}{\rho_i^{2k}} \partial_{\theta_i}^2 \Big) + \Delta_y = \Delta_{\ell,k}^{\perp} + \Delta_y. \tag{4}$$

The first term on the right is the normal component of this model Laplacian. Note that we are restricting to $k \geq 3$. (We could even choose different orders k_j on the different D_j, but for the sake of simplicity do not do so.) As we shall see below, the case $k = 3$ is in some sense a critical value, and the analysis is slightly easier for larger values of k. One motivation for discussing this more general setting is to clarify the borderline nature of the Weil–Petersson metric.

The main point is to clarify exactly what sorts of perturbations are allowed. We phrase this by focusing on the end result, i.e. by delineating the properties of the operators for which our arguments work, and then 'backfilling' by defining the corresponding class of metrics appropriately. Thus we first assume that

$$\Delta_g = -\sum_{i=1}^{\ell} (\partial_{\rho_i}^2 + k\rho_i^{-1}\partial_{\rho_i} + \rho_i^{-2k}\partial_{\theta_i}^2) + \Delta_{D_J} + E, \tag{5}$$

where Δ_{D_J} is the Laplacian for the induced metric on the codimension 2ℓ stratum, and where E is an error term. The key structural assumptions are now as follows. First, if f is supported (or indeed just defined since differential operators are local) in one of these local coordinate systems and depends only on the ρ_i, then we shall assume that

$$\Delta_g f = -\sum_{i=1}^{\ell} (\partial_{\rho_i}^2 + (k + a_i)\rho_i^{-1}\partial_{\rho_i}) f + \sum a_{ij}\partial_{\rho_i\rho_j}^2 f, \tag{6}$$

where

$$|a_{ij}|,\ |a_i| \leq C|\rho|^{\eta}$$

for some $\eta > 0$ with $|\rho| = (\rho_1^2 + \cdots + \rho_\ell^2)^{1/2}$ the Euclidean length; moreover, if f depends only on ρ and y, but not θ, then

$$\begin{aligned} \Delta_g f = &-\sum_{i=1}^{\ell} (\partial_{\rho_i}^2 + (k + a_i)\rho_i^{-1}\partial_{\rho_i}) f + \Big(\Delta_{D_J} + \sum c_{ij}\partial_{y_i y_j}^2\Big) f \\ &+ \sum a_{ij}\partial_{\rho_i\rho_j}^2 f + \sum b_{ij}\partial_{\rho_i y_j}^2 + \sum \tilde{b}_{ij}\rho_i^{-1}\partial_{y_j}, \end{aligned} \tag{7}$$

where

$$|a_{ij}|,\ |a_i|,\ |b_{ij}|,\ |\tilde{b}_{ij}|,\ |c_{ij}| \leq C|\rho|^{\eta},$$

again for some $\eta > 0$.

If g is a Kähler metric and the coordinates (ρ, θ, y) are adapted to the complex structure, then using (2) it is easy to guarantee that (6) and (7) hold simply by requiring that

$$|g_{i\bar{j}} - \delta_{ij}|, |g_{i\bar{\mu}}|, |g_{\mu\bar{\nu}} - (g_{D_J})_{\mu\bar{\nu}}| \leq |\rho|^{\eta}, \tag{8}$$

for $i, j = 1, \ldots, \ell$, $\mu, \nu = \ell+1, \ldots, n$. More generally, if g is a real (non-Hermitian) metric, assume the convention that $i, j, \ldots$ are indices for the ρ and θ variables, and $\mu, \nu, \ldots$ are indices for the y variables. Now write

$$g = \sum g_{ij}^{\rho\rho} d\rho_i d\rho_j + \sum g_{ij}^{\rho\theta} d\rho_i \rho_j^k d\theta_j + \sum g_{ij}^{\theta\theta} \rho_i^k d\theta_i \rho_j^k d\theta_j + \sum g_{\mu\nu}^{yy} dy_\mu dy_\nu + \sum g_{i\mu}^{\rho y} d\rho_i dy_\mu + \sum g_{i\mu}^{\theta y} \rho_i^k d\theta_i dy_\mu.$$

(The superscripts ρ, θ and y have been affixed to the metric components because of obvious ambiguities in the indices.) We require first that

$$|g_{ij}^{\rho\rho} - \delta_{ij}|,\ |g_{ij}^{\rho\theta}|,\ |g_{ij}^{\theta\theta} - \delta_{ij}|,\ |g_{\mu\nu}^{yy} - (g_{D_J})_{\mu\nu}|,\ |g_{i\mu}^{\rho y}|,\ |g_{j\mu}^{\theta y}| \leq C|\rho|^\eta \tag{9}$$

for all choices of indices and for some $\eta > 0$. These conditions are sufficient to guarantee that all the coefficients of the second order terms in (6) and (7) have the right form. To control the coefficients of the first order terms, we must impose some conditions on the first derivatives of certain components of the metric. To specify these, recall the standard formula for the Laplacian in an arbitrary coordinate system $(w_1, \ldots, w_{2n})$,

$$-\Delta_g = \sum g^{\alpha\beta} \partial^2_{w_\alpha w_\beta} + \sum \left(\tfrac{1}{2} g^{\alpha\beta} \partial_{w_\alpha} \log\det(g_{\alpha\beta}) + \partial_{w_\alpha} g^{\alpha\beta}\right) \partial_{w_\beta}.$$

Write

$$\log\det g = 2k \sum \log \rho_i + A.$$

Then the coefficient of ∂_{ρ_i} is

$$\sum_j \left((g^{\rho\rho})^{ij} (k\rho_j^{-1} + \partial_{\rho_j} A + \partial_{\rho_j} (g^{\rho\rho})^{ij}\right) + \sum_j \left(\rho_j^{-2k} (g^{\rho\theta})^{ji} \partial_{\theta_j} A + \rho_j^{-2k} \partial_{\theta_j} (g^{\rho\theta})^{ji}\right) + \sum_\mu \left((g^{\rho y})^{\mu i} \partial_{y_\mu} A + \partial_{y_\mu} (g^{\rho y})^{\mu i}\right),$$

and the coefficient of ∂_{y_μ} is

$$\sum_j \left((g^{\rho y})^{j\mu} (k\rho_j^{-1} + \partial_{\rho_j} A + \partial_{\rho_j} (g^{\rho y})^{j\mu}\right) + \sum_j \left(\rho_j^{-2k} (g^{\theta y})^{j\mu} \partial_{\theta_j} A + \rho_j^{-2k} \partial_{\theta_j} (g^{\theta y})^{j\mu}\right) + \sum_\nu \left((g^{yy})^{\nu\mu} \partial_{y_\nu} A + \partial_{y_\nu} (g^{yy})^{\nu\mu}\right).$$

Comparing with (9), we see that the new conditions we must impose are that

$$\begin{aligned} &|\partial_{\rho_j} A|,\ |\partial_{\theta_j} A|,\ |\partial_{y_\mu} A|,\ |\partial_{\rho_j} (g^{\rho\rho})^{ij}|,\ |\rho_j^{-2k} \partial_{\theta_j} (g^{\rho\theta})^{ji}|, \\ &|\partial_{y_\mu} (g^{\rho y})^{\mu i}|,\ |\partial_{\rho_j} (g^{\rho y})^{j\mu}|,\ |\rho_j^{-2k} \partial_{\theta_j} (g^{\theta y})^{j\mu}|,\ |\partial_{y_\nu} (g^{yy})^{\nu\mu}| \leq |\rho|^\eta. \end{aligned} \tag{10}$$

We have written this out in some detail to indicate that it is possible to phrase the necessary conditions in terms of metric components. However, it is clearly far simpler to think of these conditions using (6) and (7).

We conclude this section with the following observation. Since most of the basic arguments in the remainder of this paper are local, we take this opportunity to note the existence of a partition of unity $\{\psi_\alpha\}_{\alpha \in A}$ on $\mathcal{M}_\gamma$ with the property that each ψ_γ is supported either away from all of the divisors or else on one of the product polar coordinate charts above, and which satisfy $|\nabla \psi_\gamma|, |\Delta \psi_\gamma| \leq C$. Indeed, we need only choose these functions so that $\partial_{\theta_j} \psi_\alpha$, ∂_{ρ_i}, $\rho_i^{-k} \partial_{\theta_i}$ and ∂_{y_j}, for $y \in D_J$, applied to this function are $\mathcal{O}(\rho_j^{2k})$; one can even arrange that ψ_α is independent of θ_j when ρ_j is sufficiently small.

3. Essential self-adjointness of the Weil–Petersson Laplacian

The operator Δ_g is symmetric on $\mathcal{C}_0^\infty(\mathcal{M}_\gamma)$, but since this space is incomplete, we must consider the possibility that there is not a unique self-adjoint extension. Because Δ_g is semibounded, there is always at least one, namely the Friedrichs extension. Whether there are others besides this depends on the following considerations.

Recall the general definition of the minimal and maximal domains of the Laplacian:

$$\mathcal{D}_{\max}(\Delta) = \{u \in L^2(\mathcal{M}_\gamma; dV_{\mathrm{WP}}) : \Delta u \in L^2(\mathcal{M}_\gamma; dV_{\mathrm{WP}})\}$$

and

$$\begin{aligned} \mathcal{D}_{\min}(\Delta) = \{u \in L^2(\mathcal{M}_\gamma; dV_{\mathrm{WP}}) :\ &\text{there exists } u_j \in \mathcal{C}_0^\infty(\mathcal{M}_\gamma) \text{ such that} \\ &u_j \to u \text{ and } \Delta u_j \to f \text{ in } L^2(\mathcal{M}_\gamma; dV_{\mathrm{WP}})\} \end{aligned}$$

The operator Δ_{WP} is called essentially self-adjoint provided $\mathcal{D}_{\min}(\Delta) = \mathcal{D}_{\max}(\Delta)$, and in this case this is the unique self-adjoint extension of Δ from the core domain $\mathcal{C}_0^\infty$. If these subspaces are not equal, then the self-adjoint extensions are in bijective correspondence with the Lagrangian subspaces of $\mathcal{D}_{\max}/\mathcal{D}_{\min}$ with respect to a natural symplectic structure (coming from the classical Green identities). Choosing such an extension is tantamount to specifying a boundary condition.

We prove here the following result.

Theorem 2. *The scalar Laplace operator Δ_{WP} is essentially self-adjoint on $L^2(\mathcal{M}_\gamma, dV_{\mathrm{WP}})$.*

The rest of this section is devoted to the proof. We also obtain the

Corollary 1. *This unique self-adjoint extension has discrete spectrum.*

The key to proving these statements is to show that if $u \in \mathcal{D}_{\max}(\Delta)$, then u must decay sufficiently and have enough regularity to lie in $\mathcal{D}_{\min}(\Delta)$. We accomplish this here through a sequence of one- and multi-dimensional Hardy and interpolation estimates.

3.1. Hardy inequalities

Lemma 1. *Fix any measure space (Y, dv) and consider the measure space $(X, d\mu)$ where $X = \mathbb{R}^+ \times Y$ and $d\mu = \rho^\alpha d\rho\, dv$. Suppose that $u \in L^2_{\mathrm{loc}}(X)$ has support in $\{\rho < \rho_0\}$ and satisfies $\rho^\beta \partial_\rho u \in L^2(X; d\mu)$. If $2\beta + \alpha > 1$, then $\rho^{\beta-1} u \in L^2(X, d\mu)$ and*

$$\|\rho^{\beta-1} u\|_{L^2(X;d\mu)} \leq \frac{2}{2\beta + \alpha - 1} \|\rho^\beta \partial_\rho u\|_{L^2(X,d\mu)}. \tag{11}$$

If we drop the condition that u is supported in a finite strip in $\mathbb{R}^+$, then for any $\rho_0, \epsilon > 0$, there exists some $C > 0$ such that

$$\begin{aligned}\|\rho^{\beta-1} u\|_{L^2((0,\rho_0)\times Y;d\mu)} \\ \leq \frac{2}{2\beta + \alpha - 1} \|\rho^\beta \partial_\rho u\|_{L^2((0,\rho_0+\epsilon)\times Y;d\mu)} + C\|u\|_{L^2((\rho_0,\rho_0+\epsilon)\times Y;d\mu)}.\end{aligned}$$

Proof. Assume first that u is supported in $\rho < \rho_0$. Choose a function $\phi \in \mathcal{C}^\infty(\mathbb{R})$ which is nonnegative and monotone nondecreasing, vanishes for $\rho \leq 1/2$ and with $\phi(\rho) = 1$ for $\rho \geq 3/4$. We use $\phi(\rho/\delta)$ as a cutoff, with $\delta \searrow 0$.

By hypothesis, $u \in H^1_{\mathrm{loc}}$, and $\phi(\rho/\delta)u = 0$ near $\rho = 0$, we calculate

$$\begin{aligned}&(2\beta + \alpha - 1)\|\phi(\rho/\delta)\rho^{\beta-1} u\|^2_{L^2(X;d\mu)} \\ &= (2\beta + \alpha - 1) \int \phi(\rho/\delta)^2 \rho^{2\beta+\alpha-2} u\bar{u}\, d\rho dv \\ &\leq \int \left((2\beta + \alpha - 1)\phi(\rho/\delta)^2 \rho^{2\beta+\alpha-2} + 2\delta^{-1}\phi(\rho/\delta)\phi'(\rho/\delta)\rho^{2\beta+\alpha-1}\right) u\,\bar{u}\, d\rho dz \\ &= \int \partial_\rho(\phi(\rho/\delta)^2 \rho^{2\beta+\alpha-1}) u\,\bar{u}\, d\rho dz \\ &= \int (\partial_\rho(\phi(\rho/\delta)^2 \rho^{2\beta+\alpha-1} u) - \phi(\rho/\delta)^2 \rho^{2\beta+\alpha-1}\partial_\rho u)\,\bar{u}\, d\rho dz \\ &= -\int \left(\phi(\rho/\delta)^2 \rho^{2\beta+\alpha-1} u\,\overline{\partial_\rho u} + \partial_\rho u\,\phi(\rho/\delta)^2 \rho^{2\beta+\alpha-1}\bar{u}\right) d\rho dz \\ &= -\left\langle \phi(\rho/\delta)\rho^{\beta-1} u, \phi(\rho/\delta)\rho^\beta \partial_\rho u\right\rangle_{L^2} - \left\langle \phi(\rho/\delta)\rho^\beta \partial_\rho u, \phi(\rho/\delta)\rho^{\beta-1} u\right\rangle_{L^2}.\end{aligned}$$

Using the Cauchy–Schwartz inequality, this yields

$$(2\beta+\alpha-1)\|\phi(\rho/\delta)\rho^{\beta-1}u\|^2 \le 2\|\phi(\rho/\delta)\rho^{\beta-1}u\|\|\phi(\rho/\delta)\rho^{\beta}\partial_\rho u\|, \tag{12}$$

so dividing through by $\|\phi(\rho/\delta)\rho^{\beta-1}u\|$ gives

$$(2\beta+\alpha-1)\|\phi(\rho/\delta)\rho^{\beta-1}u\| \le 2\|\phi(\rho/\delta)\rho^{\beta}\partial_\rho u\|_{L^2} \le 2\|\rho^{\beta}\partial_\rho u\|_{L^2}. \tag{13}$$

This inequality is obvious if $\|\phi(\rho/\delta)\rho^{\beta-1}u\| = 0$. Finally, let $\delta \to 0$ to get that $\rho^{\beta-1}u \in L^2$ and that the estimate of the lemma holds.

If u is not compactly supported then replace $\phi(\rho/\delta)$ by $\phi(\rho/\delta)\psi(\rho)$ where $\psi \in \mathcal{C}^\infty(\mathbb{R})$ is nonnegative, equals 1 for $\rho \le \rho_0$ and is supported in $\rho \le \rho_0 + \epsilon$. Apply the Hardy inequality above to ψu to get

$$\frac{2\beta+\alpha-1}{2}\|\rho^{\beta-1}\psi(\rho)u\|_{L^2} \le \|\rho^{\beta}\psi\partial_\rho u\|_{L^2} + \|\rho^{\beta}\psi'(\rho)u\|_{L^2}; \tag{14}$$

in view of the support properties of ψ and ψ', this proves the lemma. □

Now suppose we are near an intersection of divisors D_J. Choose coordinates as before and, for simplicity, for any multi-indices $\sigma \in \mathbb{R}^\ell$ and $\gamma \in \mathbb{N}^\ell$, write

$$\rho^\sigma = \rho_1^{\sigma_1}\dots\rho_\ell^{\sigma_\ell}, \quad \text{and} \quad (\rho\partial_\rho)^\gamma = (\rho_1\partial_{\rho_1})^{\gamma_1}\dots(\rho_\ell\partial_{\rho_\ell})^{\gamma_\ell}.$$

If $s \in \mathbb{R}$, then we also write $\rho^s = (\rho_1\dots\rho_\ell)^s$. We also define

$$\langle\rho\rangle = \Big(\sum \rho_i^{-1}\Big)^{-1}$$

so $\langle\rho\rangle = 0$ when any of the ρ_i vanishes. Note that $\langle\rho\rangle \le \rho_j$ for any j. Then the one-dimensional Hardy inequality above immediately gives

Lemma 2. *Let $X = (\mathbb{R}^+)^\ell \times Y$ where (Y, dv) is a measure space and set $d\mu = \rho^\alpha d\rho\, dv$. Fix $\alpha, \beta \in \mathbb{R}^\ell$ such that $2\beta_i + \alpha_i > 1$ for each i. If $u \in L^2_{\mathrm{loc}}(X, d\mu)$ is supported in $\{\rho_i < \rho_0 \text{ for all } i\}$ and $\rho_i^\beta\partial_{\rho_i}u \in L^2(X; d\mu)$ for each i, then*

$$\|\rho^\beta\rho_i^{-1}u\|_{L^2(X;d\mu)} \le \frac{2}{2\beta_i+\alpha_i-1}\|\rho^\beta\partial_{\rho_i}u\|_{L^2(X;d\mu)}.$$

If we do not assume that u has compact support, then for any $\rho_0, \epsilon > 0$, there exists $C > 0$ such that

$$\|\rho^\beta\rho_i^{-1}u\|_{L^2(X\cap\{\rho_i<\rho_0\};d\mu)} \le \frac{2}{2\beta_i+\alpha_i-1}\|\rho^\beta\partial_{\rho_i}u\|_{L^2(X\cap\{\rho_i\rho_0+\epsilon\};d\mu)} + C\|\rho^\beta u\|_{L^2(X\cap\{\rho_0\le\rho_i\le\rho_0+\epsilon\};d\mu)}.$$

3.2. Interpolation inequalities. For any $\sigma \in \mathbb{R}^\ell$, define the space

$$X^{-\sigma} = \{u \in L^2_{\mathrm{loc}} : \|u\|^2_{X^{-\sigma}} := \|\rho^\sigma \langle\rho\rangle^{-1} u\|^2 + \|\rho^\sigma \langle\rho\rangle \Delta u\|^2 < \infty\}. \tag{15}$$

We write $X^{-\sigma}(\mathcal{U})$ for the space of functions with finite $X^{-\sigma}$ norm in $\mathcal{U}$.

Our first task is to show that the $X^{-\sigma}$ norm controls the L^2 norm of $\rho^\sigma \nabla u$. Note that $\rho^\sigma \langle\rho\rangle^{-1} u \in L^2$ is equivalent to $\rho^\sigma \rho_j^{-1} u \in L^2$ for all j. This result is local, so we fix $\rho_0 > 0$ and work in a neighbourhood $\mathcal{U}_{\rho_0} = \{\rho_j < \rho_0,\ j = 1, \ldots, k\}$.

Lemma 3. *Let $\sigma \in \mathbb{R}^\ell$, $\epsilon > 0$, and suppose that $u \in X^{-\sigma}(\mathcal{U}_{\rho_0+\epsilon})$. Then $\rho^\sigma \nabla u$ is in $L^2(\mathcal{U}_{\rho_0})$ and*

$$\|\rho^\sigma \nabla u\|^2_{L^2(\mathcal{U}_{\rho_0})} \leq C \|\rho^\sigma \langle\rho\rangle^{-1} u\|_{L^2(\mathcal{U}_{\rho_0+\epsilon})} \, ||u||_{X^{-\sigma}(\mathcal{U}_{\rho_0+\epsilon})}.$$

In particular

$$\rho\langle\rho\rangle^{-1} u,\ \rho\langle\rho\rangle \Delta u \in L^2 \Longrightarrow \rho \nabla u \in L^2.$$

Remark 2. As a consequence, if $u \in X^{-\sigma}$, then $\rho^\sigma \nabla u \in L^2$. Furthermore, if $\sigma_i \geq 1$ for all i then $u \in L^2$ implies $\rho^\sigma \langle\rho\rangle^{-1} u \in L^2$. In particular, if $u, \Delta u \in L^2$, then $u \in X^{-\sigma}$ for all σ with $\sigma_i \geq 1$ for all i.

Proof. Let f be real-valued and $\mathcal{C}^\infty$ and denote by m_f the operator of multiplication by f. We claim that there is an equality of differential operators

$$\Delta \circ m_f + m_f \circ \Delta = 2\nabla^*(m_f \circ \nabla) + m_{\Delta f}. \tag{16}$$

Indeed, both sides are symmetric and have the same principal symbol, so their difference is a symmetric, real, first order scalar operator, hence must actually have order zero. Applying both sides to the constant function 1 gives the claim.

Suppose that $f \geq 0$ is real valued and has compact support in $\mathcal{M}_\gamma$. Applying the operator in (16) to u and taking the inner product with u gives

$$2\mathrm{Re}\,\langle \Delta u, fu\rangle = 2\|f^{1/2} \nabla u\|^2 + \langle (\Delta f) u, u\rangle. \tag{17}$$

This equality extends to all $u \in H^2_{\mathrm{loc}}(\mathcal{M}_\gamma)$.

Choose $\phi, \psi \in \mathcal{C}^\infty(\mathbb{R})$ with $0 \leq \phi, \psi \leq 1$ and such that

$$\begin{aligned} \phi(t) &= 0 \quad \text{for } t \leq 1/2, \quad \phi(t) = 1 \quad \text{for } t \geq 1, \\ \psi &= 1 \quad \text{for } t \leq \rho_0, \quad \psi(t) = 0 \quad \text{for } t \geq \rho_0 + \epsilon/2. \end{aligned}$$

Furthermore, let $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^s)$ be supported in the y-coordinate chart $\mathcal{V}$. Then define

$$f_{i,\delta} = \phi(\rho_i/\delta)^2 \psi(\rho_i)^2 \rho_i^{2\sigma_i}, \quad f_\delta = f_{1,\delta} \ldots f_{\ell,\delta} \chi(y)^2.$$

In order to use this in the identity above, we must compute

$$\begin{aligned}\Delta f_\delta &= \Big(\sum_{i=1}^{\ell}(-\partial_{\rho_i}^2 - k\rho_i^{-1}\partial_{\rho_i}) - E + \Delta_{D_J}\Big) f_\delta \\ &= \sum_{i=1}^{\ell}\rho_i^{-2}\Big(-(\rho_i\partial_{\rho_i})^2 - \Big(k-1+a_i - \Big(\sum_j a_{ij}\Big)\Big)(\rho_i\partial_{\rho_i})\Big) f_\delta \\ &\quad - \sum_{i,j=1}^{\ell}\rho_i^{-1}\rho_j^{-1}a_{ij}(\rho_i\partial_{\rho_i})(\rho_j\partial_{\rho_j}) f_\delta + \Delta_{D_J} f_\delta \\ &\quad - \sum \rho_i^{-1}b_{ij}(\rho_i\partial_{\rho_i})\partial_{y_j} f_\delta - \sum c_{ij}\partial_{y_i}\partial_{y_j} f_\delta - \sum \tilde{b}_{ij}\rho_i^{-1}\partial_{y_j} f_\delta.\end{aligned}$$

Then, since

$$(\rho_i\partial_{\rho_i})^j(\phi(\rho_i/\delta)) = ((t\partial_t)^j\phi)(\rho_i/\delta),\ j=1,2,$$

we see that $\langle\rho\rangle^2\rho^{-2\sigma}\Delta f_\delta$ is uniformly bounded as $\delta \searrow 0$ (recall that $\rho_i \geq \langle\rho\rangle$ for all i). Hence, setting $f = f_\delta$ in (17), then

$$|\langle(\Delta f)u, u\rangle| \leq C\|\rho^\sigma\langle\rho\rangle^{-1}u\|^2$$

uniformly in δ. Applying Cauchy–Schwartz to (17) thus shows that

$$\begin{aligned}&2\|\prod_{i=1}^{\ell}\phi(\rho_i/\delta)\psi(\rho_i))\chi(y)\rho^\sigma\nabla u\|^2 \\ &\qquad \leq C\left(\|\rho^\sigma\langle\rho\rangle^{-1}u\|^2 + \|\rho^\sigma\langle\rho\rangle^{-1}u\|\|\rho^\sigma\langle\rho\rangle\Delta u\|\right),\end{aligned}$$

and since $f_\delta \to f_0 = \psi(\rho_1)^2\ldots\psi(\rho_\ell)^2\chi(y)^2\rho^{2\sigma}$ as $\delta \to 0$ we conclude that $\psi(\rho_1)\ldots\psi(\rho_\ell)\rho^\sigma\nabla u \in L^2$ by letting $\delta \to 0$. □

Another useful property of these spaces is that they localize.

Lemma 4. *Let $\{\psi_\alpha\}$ be the partition of unity described at the end of §2. Then $u \in X^{-\sigma}$ implies $\psi_\alpha u \in X^{-\sigma}$.*

Proof. First note that

$$\Delta(\psi_\alpha u) = \psi_\alpha\Delta u - 2(\nabla\psi_\alpha, \nabla u)_g + (\Delta\psi_\alpha)u.$$

The expression in the middle is the pointwise inner product with respect to the metric g. Since ψ_α, $\langle\rho\rangle\nabla\psi_\alpha$ and $\langle\rho\rangle^2\Delta\psi_\alpha$ are all bounded (even without the factor of $\langle\rho\rangle$) we obtain $\rho^\sigma\langle\rho\rangle\Delta(\psi_\alpha u) \in L^2$. However, $\rho^\sigma\langle\rho\rangle^{-1}\psi_\alpha u \in L^2$ as well, so $\psi_\alpha u \in X^{-\sigma}$. □

Corollary 2. *For any $\sigma \in \mathbb{R}^\ell$, $\mathcal{C}_0^\infty(\mathcal{M}_\gamma)$ is dense in $X^{-\sigma}$.*

Proof. Fix any $u \in X^{-\sigma}$. By the previous lemma, we may as well assume that u is supported in $\mathcal{U}_{\rho_0}$. Lemma 3 implies that $\rho^\sigma \nabla u \in L^2$.

Choose the cutoff function $\phi(t)$ as in the proof of Lemma 3 and set $\Phi_\delta(\rho) = \phi(\rho_1/\delta)\dots\phi(\rho_\ell/\delta)$. The function $\Phi_\delta u$ is compactly supported in $\mathcal{M}_\gamma$, and $\Delta u \in L^2_{\mathrm{loc}}$, so $\Phi_\delta u \in H_0^2(\mathcal{M}_\gamma)$, and hence clearly lies in $X^{-\sigma}$.

We claim that $\Phi_\delta u \to u$ in $X^{-\sigma}$, i.e.

$$\rho^\sigma \langle\rho\rangle^{-1}\Phi_\delta u \to \rho^\sigma \langle\rho\rangle^{-1} u, \quad \text{and} \quad \rho^\sigma \langle\rho\rangle \Delta(\Phi_\delta u) \to \rho^\sigma \langle\rho\rangle \Delta u$$

in L^2. The former follows from the dominated convergence theorem. For the latter, we use that

$$\Delta(\Phi_\delta u) = \Phi_\delta \Delta u - 2(\nabla\Phi_\delta, \nabla u)_g + (\Delta\Phi_\delta)u.$$

Since $\rho^\sigma \langle\rho\rangle \Delta u \in L^2$, dominated convergence gives that $\rho^\sigma \langle\rho\rangle \Phi_\delta \Delta u \to \rho^\sigma \langle\rho\rangle \Delta u$. In addition $\rho^\sigma \nabla u \in L^2$ and $\rho^\sigma \langle\rho\rangle^{-1} u \in L^2$, so the estimate for the remaining term follows from the fact that $\langle\rho\rangle \nabla\Phi_\delta$ and $\langle\rho\rangle^2 \Delta\Phi_\delta$ are uniformly bounded and converge to 0 pointwise. These bounds on Φ_δ follow from $(\rho_i\partial_{\rho_i})^k \phi(./\delta) = ((t\partial_t)^k\phi)(./\delta)$ (cf. the proof of Lemma 3), and the pointwise convergence is clear. This completes the proof of the convergence claim, and the corollary follows from the density of $\mathcal{C}_0^\infty(\mathcal{M}_\gamma)$ in $H_0^2(\mathcal{M}_\gamma)$. □

3.3. Improving the decay rate. The only reasonable general estimate for the weighted norm of ∇u involves the weighted norms of Δu and on u, with closely related powers of the weight function. However, there is a critical range of weights in which one can estimate $\|\rho^r \nabla u\|_{L^2}$ using $\|\rho^r \langle\rho\rangle \Delta u\|_{L^2}$ but a much weaker norm of u. We explain this now, and in particular develop localized versions of these estimates.

Lemma 5. *Let ψ_α be the partition of unity defined at the end of §2. Fix $\sigma, \sigma_0 \in \mathbb{R}^\ell$ such that $\sigma_0 - \ell^{-1} \leq \sigma < \sigma_0$. Let $u \in X^{-\sigma_0}$ satisfy $\rho^\sigma \langle\rho\rangle \Delta u \in L^2$. Then $\rho^\sigma \langle\rho\rangle \Delta(\psi_\alpha u) \in L^2$.*

Note that the conclusion is, up to one order of decay, better than the assumption $u \in X^{-\sigma_0}$ which implies that $\rho^{\sigma_0} \langle\rho\rangle \Delta(\psi_\alpha u) \in L^2$.

Proof. Expanding

$$\Delta(\psi_\alpha u) = \psi_\alpha \Delta u - 2(\nabla\psi_\alpha, \nabla u)_g + (\Delta\psi_\alpha)u,$$

and using that $\rho^\sigma \langle\rho\rangle \Delta u \in L^2$ and $\nabla\psi_\alpha$ and $\Delta\psi_\alpha$ are bounded, we see that it suffices to check that

$$\rho^{\sigma-\sigma_0} \langle\rho\rangle \leq C.$$

However, this follows since $\sigma_j - (\sigma_0)_j \geq -1/\ell$ for every j and

$$\langle \rho \rangle = (\rho_1 \dots \rho_\ell)^{1/\ell} R,$$

where $R \in L^\infty$. □

We now prove a stronger version of this, that a relatively weak weighted bound on u and a stronger weighted bound on Δu imply We can now prove that just information on the weighted space Δu sits in suffices to obtain information on u and ∇u if at least weak a priori weighted information on u is available.

Lemma 6. *Let* $-\frac{k-2}{4} < \sigma < \sigma_0 < \frac{k-1}{2}$, *and suppose that* $\rho^{\sigma_0}\langle \rho \rangle^{-1} u \in L^2$ *and* $\rho^\sigma \langle \rho \rangle \Delta u \in L^2$; *then* $u \in X^{-\sigma}$.

Proof. It suffices to prove the lemma assuming only that $\sigma \geq \sigma_0 - 1/\ell$, for once we have established this, then we may iterate a finite number of times to obtain the result as stated. Furthermore, we can also replace u by $\psi_\alpha u$, so as to work in a single coordinate chart.

Recall from Lemma 3 that if $u \in X^{-\sigma_0'}$ for any σ_0', then $\rho^{\sigma_0'} \nabla u \in L^2$. Now, consider the first and third terms in (17), and rewrite these as follows:

$$2\mathrm{Re}\,\langle \Delta u, fu \rangle = 2\mathrm{Re}\,\left\langle \rho^{\sigma_0'}\langle \rho \rangle \Delta u, (\rho^{-2\sigma_0'} f)\rho^{\sigma_0'}\langle \rho \rangle^{-1} u \right\rangle$$

and

$$\langle (\Delta f)u, u \rangle = \left\langle (\rho^{-2\sigma_0'}\langle \rho \rangle^2 \Delta f)\rho^{\sigma_0'}\langle \rho \rangle^{-1} u, \rho^{\sigma_0'}\langle \rho \rangle^{-1} u \right\rangle.$$

From this it follows that if f is smooth but not necessarily compactly supported, and

$$\rho^{-2\sigma_0'} f,\ \rho^{-2\sigma_0'}\langle \rho \rangle^2 \Delta f \in L^\infty, \tag{18}$$

then these two terms yield continuous bilinear forms on $X^{-\sigma_0'}$. Hence, since (17) holds for all u lying in the dense subspace $\mathcal{C}_0^\infty(\mathcal{M}_\gamma)$, it extends by continuity to all of $X^{-\sigma_0'}$; in other words, (17) holds for all elements of this space.

The next step is a judicious choice of the function f, or rather, a family of such functions, satisfying these properties. We define

$$f = f_\delta = \Phi_\delta(\rho)^2 \rho^{2\sigma}, \quad \text{where } \phi_j(t) = \left(\frac{t}{1+t}\right)^{\beta_j},\ \beta_j > 0,$$

where, as before, Φ_δ is the product of the functions $\phi_j(\rho_j/\delta)$ over $j = 1, \dots, \ell$. It is clear that both conditions in (18) hold provided $\sigma + \beta \geq \sigma_0'$.

We claim now that

$$\Delta f_\delta + \sum_{i=1}^{\ell} C_i \rho_i^{-2} f_\delta \geq 0 \quad \text{for some } C_i < \frac{(2\sigma_i + k - 1)^2}{2}. \tag{19}$$

This will be established below. For later use, fix $b_1 > 0$ and $b_2 \in (0,2)$ such that

$$C_i + b_1 < b_2 \frac{(2\sigma_i + k - 1)^2}{4} \iff -b_2 \frac{(2\sigma_i + k - 1)^2}{4} + b_1 < -C_i \tag{20}$$

for all $i = 1, \dots, \ell$.

Since the functions ϕ_j defined here satisfy $\phi_j' \geq 0$, the proof of the *regularized* Hardy inequality (13) carries through exactly as before, giving

$$\sum_{i=1}^{\ell} \frac{(2\sigma_i + k - 1)^2}{4} \|\Phi_\delta(\rho)\rho^\sigma \rho_i^{-1} u\|^2 \leq \sum_{i=1}^{\ell} \|\Phi_\delta(\rho)\rho^\sigma \partial_{\rho_i} u\|^2 \leq \|\Phi_\delta(\rho)\rho^\sigma \nabla u\|^2.$$

Rearranging this and using (20), we deduce

$$\begin{aligned}(2 - b_2)\|\Phi_\delta(\rho)\rho^\sigma \nabla u\|^2 + b_1 \sum_{i=1}^{\ell} \|\Phi_\delta(\rho)\rho^\sigma \rho_i^{-1} u\|^2 \\ \leq 2\|\Phi_\delta(\rho)\rho^\sigma \nabla u\|^2 + \langle (\Delta f_\delta) u, u\rangle,\end{aligned}$$

and hence, recognizing this as the right side of (17) with our particular choice of f and using the Cauchy–Schwartz inequality, we bound this expression by

$$\begin{aligned}2\|\Phi_\delta(\rho)\rho^\sigma \langle\rho\rangle^{-1} u\| \|\Phi_\delta(\rho)\rho^\sigma \langle\rho\rangle \Delta u\| \\ \leq \frac{b_1}{2\ell}\|\Phi_\delta(\rho)\rho^\sigma \langle\rho\rangle^{-1} u\|^2 + \frac{2\ell}{b_1}\|\Phi_\delta(\rho)\rho^\sigma \langle\rho\rangle \Delta u\|^2.\end{aligned}$$

Now, $\langle\rho\rangle^{-2} \leq \ell \sum_{i=1}^{\ell} \rho_i^{-2}$, so the first term on the right hand side can be absorbed in the second term on the left hand side. Bounding this new term from below by exactly the same inequality, we obtain finally that

$$(2 - b_2)\|\Phi_\delta(\rho)\rho^\sigma \nabla u\|^2 + \frac{b_1}{2\ell}\|\Phi_\delta(\rho)\rho^\sigma \langle\rho\rangle^{-1} u\|^2 \leq \frac{2\ell}{b_1}\|\Phi_\delta(\rho)\rho^\sigma \langle\rho\rangle \Delta u\|^2.$$

To conclude the argument, let $\delta \searrow 0$; this shows that $\rho^\sigma \nabla u$ and $\rho^\sigma \langle\rho\rangle^{-1} u$ lie in L^2, as desired.

It remains to show that (19) holds for some $C_i < \frac{(2\sigma_i + k - 1)^2}{2}$. First,

$$\begin{aligned}\Delta f_\delta = \sum_{i=1}^{\ell} \rho_i^{-2}\Big(-(\rho_i \partial_{\rho_i})^2 - \Big(k - 1 + a_i - \Big(\sum_j a_{ij}\Big)\Big)(\rho_i \partial_{\rho_i})\Big) f_\delta \\ - \sum_{i,j=1}^{\ell} \rho_i^{-1}\rho_j^{-1} a_{ij} (\rho_i \partial_{\rho_i})(\rho_j \partial_{\rho_j}) f_\delta\end{aligned}$$

with $|a_i|, |a_{ij}| \leq C|\rho|^\eta$. A straightforward but somewhat lengthy computation gives that

$$\Delta f_\delta + \sum_{i=1}^{l} C_i \rho_i^{-2} f_\delta$$
$$= -\Phi_\delta(\rho)^2 \rho^{2\sigma} \Big(\sum_{j=1}^{\ell} \rho_j^{-2} g_j(\rho_j/\delta) + \sum_{i,j=1}^{\ell} \rho_i^{-1}\rho_j^{-1} \tilde{h}_{ij}(\rho_i/\delta, \rho_j/\delta) \Big).$$

Here

$$g_j(t) = \phi_j(t)^{-2} \left((t\partial_t)^2 + (4\sigma_j + k - 1) t\partial_t + 2\sigma_j(2\sigma_j + k - 1) \right) \phi_j(t)^2$$
$$= \frac{2\beta_j(2\beta_j + 1)}{(1+t)^2} + \frac{(4\sigma_j + k - 2)2\beta_j}{1+t} + 2\sigma_j(2\sigma_j + k - 1) - C_j$$

results from the model part of the Laplacian, while $\tilde{h}_{ij}$ comes from applying the remainder terms a_i and a_{ij}, so that

$$|\rho|^{-\eta} |\tilde{h}_{ij}(s,t)| \leq \widetilde{C}, \quad s, t \geq 0,$$

for some $\widetilde{C} > 0$.

To do this computation efficiently, say for the g_j term, note first that commuting the factor $\rho^{2\sigma}$ in f through each differential expression $\rho_i \partial_{\rho_i}$ replaces this expression by $\rho_i \partial_{\rho_i} + 2\sigma_i$. Next,

$$(\rho_i \partial_{\rho_i} + 2\sigma_i)(\phi_i(\rho_i/\delta)^2) = (t\partial_t + 2\sigma_i)\phi_i^2|_{t=\rho_i/\delta},$$

so we have reduced to computing the action of a second order regular singular ordinary differential operator

$$-(t\partial_t + 2\sigma_i)^2 - (k-1)(t\partial_t + 2\sigma_i).$$

on the function $(t/(1+t))^{\beta_i}$, which is straightforward.

Now, if $-g_j(t) \geq \delta > 0$ for all j and for some $C_j < \frac{(2\sigma_j + k - 1)^2}{2}$, then using $\rho_i^{-1}\rho_j^{-1} \leq \frac{1}{2}(\rho_i^{-2} + \rho_j^{-2})$, one can estimate

$$\Big| \sum_{i,j=1}^{\ell} \rho_i^{-1}\rho_j^{-1} \tilde{h}_{ij}(\rho_i/\delta, \rho_j/\delta) \Big| \leq |\rho|^\eta \widetilde{C} \ell \sum_{i=1}^{\ell} \rho_i^{-2}.$$

Consequently, assuming that $|\rho|$ is sufficiently small, which is possible here by adjusting the partition of unity, we deduce that (19) holds.

We now wish to show that $g_j(t) \leq 0$ for $t \geq 0$. To this end, note that

$$g_j(0) = 2(\beta_j + \sigma_j)(2\beta_j + 2\sigma_j + k - 1) - C_j,$$

so this must certainly be nonpositive. Furthermore,

$$\begin{aligned} g_j'(t) &= -\frac{4\beta_j(2\beta_j+1)}{(1+t)^3} - \frac{2\beta_j(4\sigma_j+k-2)}{(1+t)^2} \\ &= -\frac{2\beta_j}{(1+t)^3}\big(2(2\beta_j+1)+(4\sigma_j+k-2)(1+t)\big), \end{aligned}$$

so we wish that

$$4\beta_j + 2 + (4\sigma_j + k - 2)(1+t) \geq 0.$$

Since $4\sigma_j + k - 2 > 0$, this is bounded below by $4(\beta_j+\sigma_j)+k$.

Setting $\gamma_j = \beta_j + \sigma_j$, we have now shown that $g_j(t) \leq 0$ provided

$$4\gamma_j + k \geq 0 \quad \text{and} \quad 2\gamma_j(2\gamma_j + k - 1) - C_j \leq 0.$$

Calculating the roots of this quadratic equation, we see that

$$-\frac{k}{4} \leq \gamma_j \leq -\frac{k-1}{4} + \frac{1}{2}\sqrt{\frac{(k-1)^2}{4} + C_j}$$

implies $g_j(t) \leq 0$ for $t \geq 0$. The leftmost term here is always less than the rightmost provided $C_j > 0$, so there is always a permissible range for γ_j. Indeed, combining the various restrictions above, we see that we must choose

$$2(\sigma_j+\beta_j)(2(\sigma_j+\beta_j)+k-1) < C_j < \frac{(2\sigma_j+k-1)^2}{2}. \tag{21}$$

Once again, setting $\beta_j = 0$, the leftmost term is less than the rightmost term precisely when

$$-\frac{k-1}{2} < \sigma_j < \frac{k-1}{2},$$

so there is certainly some allowable interval for C_j provided β_j is sufficiently small.

Since we wish to iterate the argument above, it is useful to estimate how large we can choose β_j so that there still exists an admissible C_j. One half the difference between the left and right hand side of (21) equals

$$\begin{aligned} &\frac{1}{2}\left(2(\sigma_j+\beta_j)(2(\sigma_j+\beta_j)+k-1) - \frac{(2\sigma_j+k-1)^2}{2}\right) \\ &\quad = \left(\sigma_j - \frac{k-1}{2}\right)\left(\sigma_j + \frac{k-1}{2}\right) + 4\beta_j\left(\sigma_j + \frac{k-1}{4}\right) + 2\beta_j^2. \end{aligned}$$

Taking $\beta_j = \gamma(\frac{k-1}{2} - \sigma_j)$, which is positive provided $\gamma > 0$, gives

$$\left(\sigma_j - \frac{k-1}{2}\right)\left\{\left(\sigma_j + \frac{k-1}{2}\right) - 4\gamma\left(\sigma_j + \frac{k-1}{4}\right) + 2\gamma^2\left(\sigma_j - \frac{k-1}{2}\right)\right\}.$$

Since $\sigma_j - \frac{1}{2}(k-1) < 0$, we need the other factor to be positive. Let us write this factor as $2A\gamma^2 - 4B\gamma + C$. Thus

$$A = \sigma_j - \frac{k-1}{2} < 0, \quad B = \sigma_j + \frac{k-1}{4} > \frac{1}{4} \quad \text{and} \quad C = \sigma_j + \frac{k-1}{2} > \frac{k}{4}.$$

These sign conditions and the quadratic formula show that there is one positive and one negative root of this quadratic equation, and since the leading coefficient $2A$ is negative, we see that there is a $\gamma_0 > 0$ depending only on k such that if $\gamma \in (0, \gamma_0)$, and with β_j chosen as above, then there is indeed a gap between the left and right sides of (21). In other words, we have now proved that if $-\frac{k-2}{4} < \sigma_j < \frac{k-1}{2}$ and if $\rho^{\sigma_0'}\langle\rho\rangle^{-1}u, \rho^{\sigma_0'}\langle\rho\rangle\Delta u \in L^2$, with $\sigma_0' = \sigma + \beta = \sigma + \gamma\big(\frac{k-1}{2} - \sigma\big)$, for any fixed $\gamma \in (0, \gamma_0)$, then $\rho^\sigma \nabla u$ and $\rho^\sigma \langle\rho\rangle^{-1}u$ both lie in L^2. This shows that starting with $\sigma_0 < \frac{k-1}{2}$, then we can iterate the entire argument a finite number of times to conclude that $\rho^\sigma \nabla u, \rho^\sigma \langle\rho\rangle^{-1}u \in L^2$ for any $\sigma > -\frac{k-2}{4}$. □

There is a variant of this result which holds even in the borderline case $\sigma_0 = \frac{k-1}{2}$. We lose a bit, however, in that we can only estimate some combination of ∇u and $\langle\rho\rangle^{-1}u$, but not these two terms separately.

Lemma 7. *Set $\sigma_0 = \frac{k-1}{2}$ and suppose that $\max(0, \sigma_0 - 1/\ell) \le \sigma < \sigma_0$, and $\sigma \neq 0$. Suppose furthermore that the rate of decay η for the error terms in the metric satisfies $\eta \ge 1$ (note, this is certainly true for g_{WP}). If $\rho^{\sigma_0}\langle\rho\rangle^{-1}u \in L^2$ and $\rho^{2\sigma-\sigma_0}\langle\rho\rangle\Delta u \in L^2$ then $\rho^{\sigma-2\sigma_0}\nabla\rho^{2\sigma_0}u \in L^2$.*

Proof. By assumption, $u \in X^{-\sigma_0}$, and in view of Lemma 5, we may replace u by some $\psi_\alpha u$ so as to assume that u has compact support in some chart.

We now claim that acting on $\mathcal{C}_c^\infty(\mathcal{M}_\gamma)$, one has

$$\begin{aligned} f\Delta + \Delta f = {} & 2(\rho^{-(k-1)}\nabla\rho^{k-1})^* f(\rho^{-(k-1)}\nabla\rho^{k-1}) \\ & + \Delta f - 2(\rho^{-(k-1)}\nabla\rho^{k-1})^*(f\rho^{-(k-1)}(\nabla\rho^{k-1})), \end{aligned} \tag{22}$$

where the two occurrences of f on the left as well as the last two terms on the right are multiplication operators. Indeed, both sides are formally self-adjoint, real, and have the same principal symbol, so the difference is an operator of order 0; evaluation on the constant function 1 then gives the result. The last term on the right in (22) can be rewritten in a more useful way as follows. First write this term as $-2\rho^{k-1}\nabla^*\rho^{-2(k-1)}f\nabla\rho^{k-1}$, then commute to the front the middle factor $\rho^{-2(k-1)}f$; omitting the initial minus sign, this yields

$$\begin{aligned} & 2\rho^{-(k-1)}f\Delta\rho^{k-1} - 2\rho^{k-1}[\nabla, \rho^{-2(k-1)}f]^*\nabla\rho^{k-1} \\ & = 2\rho^{-(k-1)}f\Delta\rho^{k-1} - 2\rho^{-(k-1)}\langle\nabla f, \nabla\rho^{k-1}\rangle_g - 2f\rho^{k-1}\langle\nabla\rho^{-2(k-1)}, \nabla\rho^{k-1}\rangle_g \\ & = -2(k-1)\rho^{-1}\langle\nabla\rho, \nabla f\rangle_g + 2\rho^{-(k-1)}f\Delta\rho^{k-1} + 4(k-1)f\rho^{-k}\langle\nabla\rho, \nabla\rho^{k-1}\rangle_g. \end{aligned}$$

Using (6), we expand and then combine the last two terms on the right; this produces a cancellation, from which all that remains is $f\sum_{i,j} a'_{ij}\rho_i^{-2}$, for some coefficients a'_{ij} which satisfy $|a'_{ij}| \leq C|\rho|^\eta$.

On the other hand, assuming that f is independent of y and θ, we also combine the first term of this last equation (remember to reinsert the minus sign!) with the penultimate term Δf in (22). These together yield

$$\sum_{i=1}^{\ell} \big(-\partial_{\rho_i}^2 + (k-2+a_i'')\rho_i^{-1}\partial_{\rho_i}\big) f + \sum_{i,j=1}^{\ell} a_{ij}''\partial_{\rho_i\rho_j}^2 f,$$

where $|a_i''|, |a_{ij}''| \leq C|\rho|^\eta$.

We shall use, as before, the specific function $f = f_\delta = (\prod_{i=1}^{\ell}\phi_i(\rho_i/\delta)^2)\rho^{2\sigma}$, with $\phi_i(t) = (1+t^{-1})^{-\beta_i}$. Thus

$$-\partial_{\rho_i}^2 f_i + \frac{k-2}{\rho_i}\partial_{\rho_i} f_i = f_i\rho_i^{-2}\big(-2\beta_i(2\beta_i+1)(1+t)^{-2} + 2\beta_i(k-4\sigma_i)(1+t)^{-1} + 2\sigma_i(k-1-2\sigma_i)\big)\big|_{t=\rho_i/\delta};$$

the remainder terms $a_i''\rho_i^{-1}\partial_{\rho_i} f$, $a_{ij}''\partial_{\rho_i\rho_j}^2 f$ are all bounded by $|\rho|^\eta$ times a linear combination of the three terms on the right. Taking $\beta_i = \frac{k-1}{2} - \sigma_i$, then for $\delta > 0$,

$$\rho^{-2\sigma_0} f, \quad \rho^{-2\sigma_0}\langle\rho\rangle\partial_{\rho_i} f, \quad \text{and} \quad \rho^{-2\sigma_0}\langle\rho\rangle^2\Delta f$$

are all bounded, though not uniformly in δ. Furthermore,

$$\begin{aligned}\sum_{i=1}^{\ell}\Big(-\partial_{\rho_i}^2 f + \frac{k-2}{\rho_i}\partial_{\rho_i} f\Big) &= \sum_{i=1}^{\ell} 2\beta_i f\rho_i^{-2}\big(-(2\beta_i+1)(1+t)^{-2} \\ &\qquad + (2\beta_i+1-2\sigma_i)(1+t)^{-1} + 2\sigma_i\big)|_{t=\rho_i/\delta} \\ &= \sum_{i=1}^{\ell} 2\beta_i f\rho_i^{-2}\frac{\rho_i/\delta}{1+\rho_i/\delta}\Big(\frac{2\beta_i+1}{1+\rho_i/\delta} + 2\sigma_i\Big).\end{aligned}$$

The right side of this equation is nonnegative if $\beta_i > 0$ and $\sigma_i \geq 0$ for all i, and bounded from below by

$$\sum_i 4\beta_i\sigma_i f\rho_i^{-2}\frac{\rho_i/\delta}{1+\rho_i/\delta}.$$

Finally, using Cauchy–Schwarz,

$$\begin{aligned}&\sum_{i,j}|\partial_{\rho_i\rho_j}^2 f| + \sum_i|\rho_i^{-1}\partial_{\rho_i} f| \\ &\qquad \leq Cf\sum_{i,j}(\rho_i\rho_j)^{-1}\big((1+t)^{-2} + (1+t)^{-1} + 1\big)|_{t=\rho_i/\delta} \leq Cf\sum_i\rho_i^{-2}.\end{aligned}$$

We have now proved that the last two terms of (22) are bounded from below by $-C|\rho|^\eta \sum(\rho_i\rho_j)^{-1}$. Since we are assuming that $\sigma_i > 0$ for every i and $\eta \geq 1$, we can refine this since then $|\rho|^\eta \leq |\rho| \leq \sum_j \rho_j$. Indeed, using these two conditions, then for any specified $\epsilon_0 > 0$,Q

$$\sum_j \rho_j \left((1+t)^{-2} + (1+t)^{-1} + 1\right)|_{t=\rho_i/\delta} \leq 3\sum_j \rho_j \leq \sum_j \epsilon_0 4\beta_j\sigma_j \frac{\rho_j/\delta}{1+\rho_j/\delta}$$

when both δ and $|\rho|$ are sufficiently small. Taking $\operatorname{supp} u$ sufficiently small so that both of these last conditions hold, this shows that the last two terms of (22) are actually non-negative.

Now, when $\delta > 0$, (22) gives

$$\begin{aligned}\langle f_\delta \Delta u, u\rangle + \langle \Delta(f_\delta u), u\rangle \\ = 2\| f_\delta^{1/2}(\rho^{-(k-1)}\nabla\rho^{k-1})u\|^2 \\ + \left\langle \left(\Delta f_\delta - 2(\rho^{-(k-1)}\nabla\rho^{k-1})^*(f_\delta\rho^{-(k-1)}(\nabla\rho^{k-1}))\right)u, u\right\rangle\end{aligned} \tag{23}$$

if $u \in \mathcal{C}_c^\infty(\mathring{\mathcal{M}})$. Both sides are continuous bilinear forms on $X^{-\sigma_0}$, so since $\mathcal{C}_c^\infty(\mathring{\mathcal{M}})$ is dense in $X^{-\sigma_0}$, the identity holds in this larger space. Since $\eta \geq 1$, the last term is nonnegative, hence

$$\| f_\delta^{1/2}(\rho^{-(k-1)}\nabla\rho^{k-1})u\|^2 \leq \|\rho^{\sigma_0}\langle\rho\rangle^{-1}u\|\,\|\rho^{2\sigma-\sigma_0}\langle\rho\rangle\Delta u\|.$$

Letting $\delta \to 0$ shows $\rho^\sigma(\rho^{-(k-1)}\nabla\rho^{k-1})u \in L^2$ and

$$\|\rho^\sigma(\rho^{-(k-1)}\nabla\rho^{k-1})u\|^2 \leq \|\rho^{2\sigma-\sigma_0}\langle\rho\rangle\Delta u\|\|\rho^{\sigma_0}\langle\rho\rangle^{-1}u\|.$$

This completes the proof. □

Corollary 3. *If $\sigma_0 = \frac{k-1}{2} > 0$ and $0 \leq \sigma < \sigma_0$, $\eta \geq 1$, and if $\rho^{\sigma_0}\langle\rho\rangle^{-1}u \in L^2$ and $\rho^\sigma\langle\rho\rangle\Delta u \in L^2$ then $u \in X^\sigma$, so $\rho^\sigma\nabla u \in L^2$ and $\rho^\sigma\langle\rho\rangle^{-1}u \in L^2$.*

In particular, if $\rho^{\sigma_0}\langle\rho\rangle^{-1}u \in L^2$ and $\langle\rho\rangle\Delta u \in L^2$ then $\nabla u \in L^2$.

Proof. By Lemma 7, if $\sigma' = \max(\sigma_0 - 1/\ell, \sigma_0 - \frac{1}{2}(\sigma_0 - \sigma)) < \frac{k-1}{2}$, $\sigma' > 0$, then in particular $2\sigma' - \sigma_0 \geq \sigma$, and hence $\rho^{2\sigma'-\sigma_0}\langle\rho\rangle\Delta u \in L^2$, and we also have $\rho^{\sigma'}\rho^{-(k-1)}\nabla\rho^{k-1}u \in L^2$. This implies that

$$\Big(\prod_{i\neq j}\rho_i^{\sigma'}\Big)\rho_j^{\sigma'}\rho_j^{-(k-1)}\partial_{\rho_j}\rho_j^{k-1}u \in L^2.$$

Since ρ_j^{1-k} barely fails to lie in L^2, while $u \in L^2$, we can use techniques of regular singular operator theory to get that $\rho^{\sigma'}\partial_{\rho_j}u \in L^2$ for every j, and hence using the Hardy inequality, $\rho^{\sigma'}\langle\rho\rangle^{-1}u \in L^2$. This means that $u \in X^{-\sigma'}$. Finally, by Lemma 6 with $\sigma_0 = \sigma'$, $\rho^\sigma\nabla u \in L^2$ and $\rho^\sigma\langle\rho\rangle^{-1}u \in L^2$, and this completes the proof. □

We now show how to use the results above to establish that Δ is essentially self-adjoint. It is well known that essential self-adjointness is equivalent to the density of $\mathrm{Ran}_{\mathcal{C}_c^\infty}(\Delta \pm i)$ in L^2. If either of these spaces fail to be dense, then there exists an element $\psi \in L^2$ such that $\psi \perp \mathrm{Ran}_{\mathcal{C}_c^\infty}(\Delta \pm i)$, i.e. $\langle (\Delta \pm i)\phi, \psi\rangle = 0$ for all $\phi \in \mathcal{C}_c^\infty$. This implies, in particular, that ψ is a distributional solution of $(\Delta \pm i)\psi = 0$, and hence $\psi \in \mathcal{C}^\infty$, and in addition $\Delta\psi \in L^2$.

If we can integrate by parts to justify the identity

$$0 = \langle (\Delta \pm i)\psi, \psi\rangle = \|d\psi\|^2 \pm i\|\psi\|^2,$$

we could then conclude that $\psi = 0$. Thus it remains to prove the

Lemma 8. *Suppose that $k \geq 3$, and let g be a metric which satisfies the conditions (9) and (10), with $\eta \geq 1$ if $k = 3$. If $u \in L^2$ and $\Delta u \in L^2$, then $\nabla u \in L^2$ and $\langle \Delta u, u\rangle = \langle \nabla u, \nabla u\rangle$.*

Proof. If $k > 3$ then, by Lemma 6 with $\sigma_0 = 1$ and $\sigma = 0$, we see that $\nabla u \in L^2$ and $\langle \rho\rangle^{-1} u \in L^2$. This proves the first claim.

Now suppose that $k = 3$, which is the most relevant case. Then by Lemma 3 with $\sigma_0 = 1$ and $\sigma = 0$, we see that $\nabla u \in L^2$ and $\langle \rho\rangle^{-1} u \in L^2$. This completes the proof of the first claim in all cases.

Finally, if $v \in \mathcal{C}_c^\infty(\mathring{\mathcal{M}})$, then

$$\langle \Delta v, v\rangle = \|\nabla v\|^2.$$

Both sides are continuous bilinear forms on X^0, so by the density of $\mathcal{C}_c^\infty(\mathring{\mathcal{M}}_\gamma)$, this identity remains valid in X^0. In part one we proved that the assumption $u \in L^2$ and $\Delta u \in L^2$ implies that $\nabla u \in L^2$ and $\langle \rho\rangle^{-1} u \in L^2$. Thus $u \in X^0$ and hence, the above equality holds for u. □

In summary, we proved:

Theorem 3. *Δ is essentially self-adjoint.*

To conclude this section, we note that the argument of Lemma 8 shows that the maximal domain $\mathcal{D}_{\max}(\Delta)$ is contained in $\langle \rho\rangle L^2 \cap H^1$, which is certainly compactly contained in L^2. This proves Corollary 1.

4. The Weyl estimate for the eigenvalues

In this final section we address the question of estimating the growth rate of the counting function

$$N(\lambda) = \#\{j : \lambda_j \leq \lambda\}.$$

The classical formula, valid for the Laplacian on a compact smooth manifold (M, g), states that

$$N(\lambda) = \frac{\omega_n}{(2\pi)^n}\mathrm{Vol}_g(M)\lambda^{n/2} + o(\lambda^{n/2}), \tag{24}$$

where $n = \dim M$ and ω_n is the volume of the unit ball in $\mathbb{R}^n$. It is now well understood, of course, that a good estimate of the error term, for example showing that it has the form $\mathcal{O}(\lambda^{n/2-\epsilon})$ for some $\epsilon > 0$, depends on the dynamical properties of the geodesic flow. The same is certainly true here, but we content ourselves with the most basic result in this direction, which is the extension of this Weyl counting formula to the present setting. Our main result is the

Theorem 4. *Let (M, g) be a compact stratified space with crossing cusp singularities of multi-order k, where each $k_i \geq 3$. Then the spectrum of Δ_g is discrete, and the counting function $N(\lambda)$ for this spectrum satisfies the asymptotic formula (24).*

Proof. We follow the most classical method, known as Dirichlet–Neumann bracketing. We briefly recall the idea, but refer to [7] for details. Consider a partition of $\mathcal{M}$ into the region

$$\mathcal{W}_0 = \{p : \mathrm{dist}_g(p, \cup D_j) \geq \delta\}$$

and a finite number of regions $\mathcal{W}_\alpha$, each of the form

$$\{p = (r_1, \ldots, r_\ell, \theta_1, \ldots, \theta_\ell, y) : 0 \leq r_i \leq \delta,\ i = 1, \ldots, \ell,\ \theta_i \in S^1 \text{ for all } i, \\ \text{and } y \in \mathcal{V}_\alpha \subset \mathbb{R}^{6\gamma-6-2\ell}\}.$$

Next, define the Rayleigh quotient for Δ_g by $R(u) = D(u)/L(u)$, where $L(u) = \int |u|^2\, dV_g$ and $D(u) = \int |\nabla u|^2\, dV_g$ are the L^2 norm and the Dirichlet form. We also consider the restrictions of these forms to various subdomains $\mathcal{W}$, and will denote these by $R^{\mathcal{W}}(u)$, etc.

For each α, including $\alpha = 0$, restrict R to functions u which lie in $H^1(\mathcal{W}_\alpha)$ or to $H^1_0(\mathcal{W}_\alpha)$. The critical values of this restricted functional are the Neumann and Dirichlet eigenvalues for Δ_g on $\mathcal{W}_\alpha$, which we list in order, and with multiplicity, as

$$\mu^\alpha_{1,N} \leq \mu^\alpha_{2,N} \leq \cdots \quad \text{and} \quad \mu^\alpha_{1,D} \leq \mu^\alpha_{2,D} \leq \cdots,$$

respectively. The union of these lists of eigenvalues over all α, reindexed into non-decreasing sequences, become

$$\mu_{1,N} \leq \mu_{2,N} \leq \cdots \quad \text{and} \quad \mu_{1,D} \leq \mu_{2,D} \leq \cdots.$$

Since

$$\bigoplus_\alpha H^1_0(\mathcal{W}_\alpha) \subset H^1(\mathcal{M}) \subset \bigoplus_\alpha H^1(\mathcal{W}_\alpha),$$

the minimax characterization of eigenvalues shows that the eigenvalues $\{\lambda_j\}$ of Δ_g on $\mathcal{M}$, listed with multiplicity, satisfy

$$\mu_{j,N} \leq \lambda_j \leq \mu_{j,D}$$

for all j. This implies, in turn, that if we denote the counting functions for the sequences $\{\mu_{j,N}\}$ and $\{\mu_{j,D}\}$ by $N_N(\lambda)$ and $N_D(\lambda)$, respectively, then for $\lambda \gg 0$,

$$N_D(\lambda) \leq N(\lambda) \leq N_N(\lambda).$$

Thus to prove the main assertion, it suffices to prove that both N_N and N_D satisfy this same asymptotic law.

We take as given that

$$N_{N/D}^{\mathcal{W}_0}(\lambda) = \frac{\omega_n}{(2\pi)^n}\mathrm{Vol}_g(\mathcal{W}_0)\lambda^{n/2} + o(\lambda^{n/2}).$$

This is just the standard Weyl law for the region $\mathcal{W}_0$ with either Neumann or Dirichlet eigenvalues. Since $\mathrm{Vol}_g(\mathcal{W}_0) \to \mathrm{Vol}_g(M)$ as $\delta \to 0$, this is nearly the entire leading term. On the other hand, since $N_D^{\mathcal{W}_\alpha}(\lambda) \leq N_N^{\mathcal{W}_\alpha}(\lambda)$ for $\alpha \neq 0$, it suffices to prove that for any $\epsilon > 0$ we can choose a sufficiently small δ so that for all such α,

$$N_N^{\mathcal{W}_\alpha}(\lambda) \leq \epsilon\lambda^{n/2} + \mathcal{O}(\lambda^\beta)$$

for some $\beta < n/2$. Thus we concentrate on this last estimate for any fixed α.

Since the Rayleigh quotient is changed by at most a bounded factor if we replace g by any quasi-isometric metric, we may replace $\mathcal{W}_\alpha$ by the product $(0,\delta)^\ell \times Z$, $Z = (S^1)^\ell \times B$, where B is a piecewise smooth compact domain in $\mathbb{R}^{n-2\ell}$, endowed with the warped product metric

$$g_\ell = \sum_{i=1}^{\ell}(d\rho_i^2 + \rho_i^{2k_i}\,d\theta^2) + dy^2.$$

Using radial and angular variables ρ_i and θ_i, $i = 1,\dots,\ell$, as before, as well as the multi-index notation, so that for example $\rho^k = \prod_{i=1}^{l}\rho_i^{k_i}$, then

$$L^{\mathcal{W}_\alpha}(u) = \int_0^\delta \cdots \int_0^\delta \int_Z |u|^2\rho^k\,d\rho\,d\theta\,dy,$$

and

$$D^{\mathcal{W}_\alpha}(u) = \int_0^\delta \cdots \int_0^\delta \int_Z \Big(\sum_{j=1}^{\ell}|\partial_{\rho_j}u|^2 + \sum_{j=1}^{\ell}\rho_j^{-2k_j}|\partial_{\theta_j}u|^2 + |\partial_y u|^2\Big)\rho^k\,d\rho\,d\theta\,dy.$$

Next, suppose that $\delta = 2^{-m_0}$ for some m_0 which will be fixed later, and decompose the cube $(0,\delta)^\ell$ into a finite union of subregions

$$(0, 2^{-m_0})^\ell = \bigsqcup_\mu I^\mu, \quad \text{where } I_\mu = I_{\mu_1} \times \cdots \times I_{\mu_\ell}.$$

Here μ varies over all multi-indices $(\mu_1, \ldots, \mu_\ell)$ with $\mu_i \in \{m_0, \ldots, m+1\}$ for some m to be chosen momentarily, and

$$I_j = (2^{-j-1}, 2^{-j}) \text{ for } m_0 \leq j \leq m, \quad \text{and} \quad I_{m+1} = (0, 2^{-m-1}).$$

In other words, this is just the dyadic decomposition of $(2^{-m-1}, 2^{-m_0})$ along with the 'terminal' interval $(0, 2^{-m-1})$.

Now fix $\lambda \gg 0$ and set $m \sim \frac{1}{2}\log_2 \lambda$. We claim that if the multi-index μ has $\mu_j = m+1$ for some j, then the number of Neumann eigenvalues of Δ_{g_ℓ} on the region $I_\mu \times Z$ is bounded by the number of Neumann eigenvalues on the adjacent domain $I_{\mu'} \times Z$, where $\mu'_j = m$ and all other $\mu'_i = \mu_i$. As we show below, it is possible to directly estimate the counting functions on these non-terminal regions, and since we now show that the counting functions on the terminal regions are estimated in terms of these, we will have accounted for every block in this decomposition.

To prove this we integrate in $\rho_j \in (0, 2^{-m})$ (for the same value of j). Writing $Y = \prod_{i \neq j} I_{\mu_i} \times Z$, then the second part of Lemma 1 with $\beta = 0$, $\rho_0 = 2^{-m-1}$ and $\rho_0 + \epsilon = 2^{-m}$ gives that

$$\begin{aligned} 2^{2m+2} \int_0^{2^{-m-1}} |u|^2 \, d\rho_j &\leq \int_0^{2^{-m}} |\rho_j^{-1} u|^2 \, d\rho_j \\ &\leq C \int_0^{2^{-m}} |\partial_{\rho_j} u|^2 \, d\rho_j + 2^{2m} \int_{2^{-m-1}}^{2^{-m}} |u|^2 \, d\rho_j. \end{aligned}$$

The constant in front of the last term on the right comes from the square of the derivative of the cutoff function used to reduce back to a function which vanishes near $\rho_j = \rho_0 + \epsilon$ so as to apply the Hardy inequality for such functions. Integrating over the other factor Y, we obtain

$$2^{2m+2} \int_{I_\mu \times Y} |u|^2 \leq C \int_{(I_\mu \cup I_{\mu'}) \times Y} |\nabla u|^2 + C 2^{2m} \int_{I_{\mu'} \times Y} |u|^2.$$

Now suppose that the restriction of u to $I_{\mu'} \times Y$ is orthogonal to all the Neumann eigenfunctions with eigenvalues less than $C 2^{2m}$ on this region. We can then estimate the second term on the right by the Dirichlet integral of u on $I_{\mu'} \times Y$. So altogether, for such u,

$$2^{2m+2} \int_{(I_\mu \cup I_{\mu'}) \times Y} |u|^2 \leq C \int_{(I_\mu \cup I_{\mu'}) \times Y} |\nabla u|^2,$$

hence the Rayleigh quotient for all such u satisfies $R(u) \geq C2^{2m+2}$. This proves that

$$N_N^{(I_\mu \cup I_{\mu'})\times Y}(2^{2m+2}) \leq N_N^{I_{\mu'}\times Y}(C2^{2m+2}),$$

which proves the claim since $\lambda \sim 2^{2m}$.

We have now reduced to estimating $N_N^{I_\mu \times Y}(\lambda)$ for any multi-index μ where $\mu_j \neq m+1$ for any j. To accomplish this, we consider the Rayleigh quotient on this region. Writing $\mu \cdot k = \sum \mu_j k_j$, then

$$D(u) = \int \Big(\sum_{j=1}^{\ell} (|\partial_{\rho_j} u|^2 + 2^{2k_j\mu_j} |\partial_{\theta_j} u|^2) + |\partial_y u|^2 \Big) 2^{-\mu\cdot k}\, d\rho\, d\theta\, dy,$$

and $L(u) = \int |u|^2 2^{-\mu\cdot k}\, d\rho\, d\theta\, dy$. Up to a μ-independent constant factor, the Rayleigh quotient in this region is the same as the one for

$$\sum_{j=1}^{\ell} \partial_{\rho_j}^2 + \sum_{j=1}^{\ell} 2^{2k_j\mu_j} \partial_{\theta_j}^2 + \partial_y^2.$$

Up to another constant factor, the eigenvalues of this problem are simply

$$\sum_{j=1}^{\ell} 2^{2\mu_j} \xi_j^2 + \sum_{j=1}^{\ell} 2^{2\mu_j k_j} \zeta_j^2 + |\eta|^2, \qquad \xi, \zeta \in \mathbb{Z}^\ell,\ \eta \in \mathbb{Z}^{n-2\ell}.$$

The number of these which are no larger than λ is estimated from above by the number of multi-indices ξ, ζ, η such that the individual summands themselves are less than λ. Thus

$$\#\{\xi_j : 2^{2\mu_j}\xi_j^2 \leq \lambda\} \leq C\sqrt{\lambda} 2^{-\mu_j},$$
$$\#\{\zeta_j : 2^{2k_j\mu_j}\zeta_j^2 \leq \lambda\} \leq C\sqrt{\lambda} 2^{-\mu_j k_j},$$

and

$$\#\{\eta_j : \eta_j^2 \leq \lambda\} \leq C\sqrt{\lambda}.$$

Thus, summing over all μ (with no $\mu_j = m$), and recalling that λ is large, the number of eigenvalues of these model problems on the various regions $I_\mu \times Z$ which are less than λ is bounded by

$$C\Big(\sum_{\mu} 2^{-(1+k_j)\mu_j} \Big) \lambda^{n/2}.$$

The coefficient breaks into the product

$$\prod_{j=1}^{\ell} \Big(\sum_{\mu_j = m_0}^{m} 2^{-(1+k_j)\mu_j} \Big) \leq \prod_{j=1}^{\ell} 2^{-(1+k_j)m_0} = 2^{-(\ell+|k|)m_0} = \delta^{\ell+|k|}$$

Altogether, we have proved that the counting function on each $\mathcal{W}_\alpha$, $\alpha \neq 0$, is bounded by $C\delta^\beta \lambda^{n/2}$ where $\beta = \ell + |k|$.

We have now shown that

$$N(\lambda) = \frac{\omega_n}{(2\pi)^n}\big(\mathrm{Vol}_g(M) + \mathcal{O}(\delta^\beta)\big)\lambda^{n/2} + o(\lambda^{n/2}).$$

If we choose $\delta = (\log_2 \lambda)^{-1/\beta}$, or equivalently, $m_0 = (1/\beta)\log_2 \lambda$, then since this m_0 is still far less than $m = \frac{1}{2}\log_2 \lambda$, we conclude that

$$N(\lambda) = \frac{\omega_n}{(2\pi)^n}\mathrm{Vol}_g(M)\lambda^{n/2} + o(\lambda^{n/2}),$$

which finishes the proof. □

References

[1] E. Arbarello, M. Cornalba and P. Griffiths, *Geometry of algebraic curves*. Volume II, with a contribution by Joseph Daniel Harris, Grundlehren Math. Wiss. 268, Springer-Verlag, Berlin 2011. Zbl 1235.14002 MR 2807457

[2] P. Albin, E. Leichtnam, R. Mazzeo and P. Piazza, The signature package on Witt spaces. *Ann. Sci. Éc. Norm. Supér.* (4) **45** (2012), no. 2, 241–310. Zbl 1260.58012 MR 2977620

[3] P. Albin, E. Leichtnam, R. Mazzeo and P. Piazza, The signature operator on Cheeger spaces. In preparation.

[4] T. Aubin, *Some nonlinear problems in Riemannian geometry*. 2nd ed., Springer Monogr. Math., Springer-Verlag, Berlin 2007. Zbl 0896.53003 MR 1636569

[5] K. Burns, H. Masur and A. Wilkinson, The Weil–Petersson geodesic flow is ergodic. *Ann of Math.* (2) **175** (2012), 835–908. Zbl 1254.37005 MR 2993753

[6] W. Cavendish and H. Parlier, Growth of the Weil–Petersson diameter of moduli space. *Duke Math. J.* **161** (2012), no. 1, 139–171. Zbl 1244.32008 MR 2872556

[7] I. Chavel, *Eigenvalues in Riemannian geometry*. Including a chapter by Burton Randol, with an appendix by Jozef Dodziuk, Pure Appl. Math. 115, Academic Press, Inc., Orlando, FL, 1984. Zbl 0551.53001 MR 0768584

[8] J. Cheeger, On the Hodge theory of Riemannian pseudomanifolds. In *Geometry of the Laplace operator* (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math. XXXVI, Amer. Math. Soc., Providence, RI, 1980, 91–146. Zbl 0461.58002 MR 0573430

[9] J. Cheeger, Spectral geometry of singular Riemannian spaces. *J. Differential Geom.* **18** (1983), no. 4, 575–657. Zbl 0529.58034 MR 0730920

[10] Y. Colin de Verdiere, Pseudo-Laplaciens, I and II. *Ann. Inst. Fourier* **32** (1982), no. 3, 275–286; **33** (1983), no. 2, 87–113. Zbl 0489.58034 Zbl 0496.58016 MR 0688031 MR 0699488

[11] E. Dryden, C. Gordon, S. Greenwald and D. Webb, Asymptotic expansion for the heat kernel for orbifolds. *Michigan Math. J.* **56** (2008), no. 1, 205–238. Zbl 1175.58010 MR 2433665

[12] J. Harris and I. Morrison, *Moduli of curves*. Grad. Texts in Math. 187, Springer-Verlag, New York 1998. Zbl 0913.14005 MR 1631825

[13] K. Liu, X. Sun and S. T. Yau, Good geometry on the curve moduli. *Publ. Res. Inst. Math. Sci.* **44** (2008), no. 2, 699–724. Zbl 1219.14012 MR 2426362

[14] H. Masur, Extension of the Weil–Petersson metric to the boundary of Teichmüller space. *Duke Math. J.* **43** (1976), no. 3, 623–635. Zbl 0358.32017 MR 0417456

[15] M. Mirzakhani, On Weil–Petersson volumes and geometry of random hyperbolic surfaces. In *Proceedings of the International Congress of Mathematicians*. Vol. II, Hindustan Book Agency, New Delhi 2010, 1126–1145. Zbl 1239.32013 MR 2827834

[16] S. Wolpert, Geometry of the Weil–Petersson completion of Teichmüller space. In *Surveys in differential geometry*, Vol. VIII, (Boston, MA, 2002), Surv. Differ. Geom. VIII, International Press, Somerville, MA, 2003, 357–393. Zbl 1049.32020 MR 2039996

[17] S. Wolpert, Understanding Weil–Petersson curvature. In *Geometry and analysis*. No. 1, Adv. Lect. Math. (ALM) 17, International Press, Somerville, MA, 2011, 495–515 Zbl 1271.32017 MR 2882436

[18] S. Wolpert, *Families of Riemann surfaces and Weil–Petersson geometry*. CBMS Regional Conf. Ser. in Math. 113, Amer. Math. Soc., Providence, RI, 2010. Zbl 1198.30049 MR 2641916

[19] S. Wolpert, Behavior of geodesic length functions on Teichmüller space. *J. Differential Geom.* **79** (2008), 277–334. Zbl 1147.30032 MR 2420020

[20] S. Yamada, On the geometry of Weil–Petersson completion of Teichmüller spaces. *Math. Res. Lett.* **11** (2004), no. 2–3, 327–344. Zbl 1060.32005 MR 2067477

[21] S. Yamada, Weil–Petersson completion of Teichmüller spaces and mapping class group actions. Preprint, arXiv:math.DG/0112001.

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