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Moments on Riemann surfaces and hyperelliptic Abelian integrals

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Abstract. In the present paper we solve the following different but interrelated problems: (a) the moment problem on Riemann surfaces, (b) the vanishing problem for polynomial Abelian integrals of dimension zero on the projective plane, (c) the vanishing problem for polynomial hyperelliptic Abelian integrals.

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1. Introduction

Let f be a non-constant meromorphic function on a compact Riemann surface R, ω be a meromorphic one-form on R, and $\gamma \subset R$ be a curve. In the present paper we solve the following different but interrelated problems:

(a) In Section 2 we give necessary and sufficient conditions for the "moments"

$$m_s = \int_\gamma f^s \omega, \quad s \geq 0, \tag{1}$$

to vanish for all s. These conditions are expressed in terms of the identical vanishing of a finite collection of algebraic functions, which can be interpreted as Abelian integrals of dimension zero on R.

(b) In Section 3, motivated by problem (a), we describe necessary and sufficient conditions for the identical vanishing of polynomial Abelian integrals of dimension zero on the projective plane.

(c) Finally, in Section 4 we apply the results obtained to the problem of identical vanishing of complete hyperelliptic Abelian integrals of the form

$$I(t) = \int_{\gamma(t)} P(x,y)dx + Q(x,y)dy, \quad P, Q \in \mathbb{C}[x,y], \tag{2}$$

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where $\gamma(t) \in H_1(\Gamma_t, \mathbb{Z})$ is a continuous family of 1-cycles and

$$\Gamma_t = \{(x, y) \in \mathbb{C}^2 : y^2 - f(x) = t\}, \quad f \in \mathbb{C}[x] \tag{3}$$

is a family of hyperelliptic curves.

In the particular case where f is a polynomial, $\omega = g dz$ is polynomial one form, and $\gamma \subset \mathbb{CP}^1$ is a non-closed curve, the moment vanishing problem (a), called in this case the polynomial moment problem, has been studied by several authors in a series of papers initiated by [2]. The main motivation for a study of the polynomial moment problem is its relation with the center problem for the Abel differential equation

$$\frac{dy}{dz} = p(z)y^2 + q(z)y^3, \quad p, q \in \mathbb{C}[z],$$

which in its turn is closely related to the classical center-focus problem of Poincaré (see [3] and an extensive list of references therein). A solution of problem (a) in the general case presented here is given in terms of zero-dimensional Abelian integrals and is inspired by the approach of [12], [13]. Notice that the initial polynomial moment problem admits also a more explicit solution involving compositional properties of f, g in the composition algebra of polynomials (see [15], [14]).

Polynomial zero-dimensional Abelian integrals on $\mathbb{CP}^1$, that is, algebraic functions of the form

$$I(z) = n_1 g(f_1^{-1}(z)) + n_2 g(f_1^{-1}(z)) + \cdots + n_d g(f_d^{-1}(z)), \quad n_i \in \mathbb{Z},$$

where f and g are polynomials and $f_i^{-1}(z)$ are branches of the algebraic function inverse to f, were introduced recently in the paper [7] in an attempt to verify certain conjectures concerning the 16th Hilbert problem in dimension zero. In particular, the problem of identical vanishing of such zero-dimensional integrals for simple cycles has been studied and solved in [7], [4]. Notice however that in this case the problem reduces to the finding of conditions implying that for a pair of polynomials f, g the equality $g(f_i^{-1}(z)) \equiv g(f_j^{-1}(z))$, $i \neq j$, holds, and in such a form the problem was studied and solved earlier (see e.g. [17], [10]). In the general case a solution of the problem (b) in an implicit form essentially was already done in [15] as an ingredient of the solution of the polynomial moment problem. However, having in mind possible applications, we present here a detailed and full exposition which is self-contained up to a single purely algebraic result of [15]. Notice that the problem (b) also was studied in the recent paper [1] where however only a partial solution has been achieved.

The last problem (c) solved in this paper concerns the identical vanishing of complete hyperelliptic Abelian integrals of the form (2). Although this problem is of independent interest, we are once again motivated by applications to the 16th Hilbert problem. Namely, it is well known that if a limit cycle of the perturbed plane foliation

$$d(y^2 - f(x)) + \varepsilon(P(x, y)dx + Q(x, y)dy) = 0, \quad \varepsilon \sim 0, \tag{4}$$

bifurcates from the periodic orbit $\gamma(t_0) \subset \Gamma_{t_0}$ of the non-perturbed foliation, then the Abelian integral $I(t)$ defined by (2) vanishes at t_0. This is a corollary of the representation

$$P_\varepsilon(t) = t + \varepsilon I(t) + o(\varepsilon) \tag{5}$$

of the first return map P_ε associated to the family of periodic orbits $\gamma(t)$. The situation in which $I(t) \equiv 0$ is exceptional, and this phenomenon is related to the singularities of the algebraic set of plane integrable foliation. On the other hand, the identical vanishing of $I(t)$ only shows that the foliation (4) is integrable "at a first order", and the study of the higher order terms in the expansion (5) is needed in order to solve the associated center problem on the plane [5], [8].

The key idea to solve problem (c) is to interpret the derivatives of $I(t)$ as moments (1) for a certain choice of R, f and ω. Then the identical vanishing of $I(t)$ turns out to be equivalent, according to (a), to the identical vanishing of a collection of Abelian integrals of dimension zero. Furthermore, these Abelian integrals essentially reduce to the ones studied in (b).

2. Moments on Riemann surfaces and zero-dimensional Abelian integrals

2.1. Moment problem and zero-dimensional Abelian integrals. Let f be a non-constant meromorphic function on a compact Riemann surface R, ω be a meromorphic one-form on R, and $\gamma \subset R$ be a rectifiable curve which avoids the poles of f and ω. Then the moments (1) are well defined. In this subsection we will give necessary and sufficient conditions for the generating function

$$J(t) = J(\omega, f, \gamma, t) = -\sum_{s=0}^{\infty} \frac{m_s}{t^{s+1}} = \int_\gamma \frac{\omega}{f - t}, \quad t \sim \infty, \tag{6}$$

of the moments m_s to vanish identically or more generally to be rational. Our approach to this problem is inspired by [13], where the genus zero case, $R = \mathbb{CP}^1$, was studied in details. We will suppose for simplicity that the set of poles of ω is contained in the set of poles of f and that γ is closed (for the general case see the remarks given in the end of this subsection).

Consider the induced holomorphic map $f : R \to \mathbb{CP}^1$ and let $\{c_1, c_2, \dots c_k\}$ be the set of all *finite* critical values of f. For a regular generic value $c_0 \in \mathbb{C}$, consider the "star" $S \subset \mathbb{C}$ consisting of the segments $[c_0, c_i]$, $i = 1, 2 \dots k$. Using the assumption that S contains all finite critical values of f, one can show that the path γ can be continuously deformed, without changing the corresponding function $J(t)$, in such a way that the image $f(\gamma)$ will be contained in S (the explicit construction is given below). Therefore, moments (1) may be written in the form

$$m_s = \int_\gamma f^s \omega = \int_\gamma f^s \frac{\omega}{df} df = \sum_{i=1}^{k} \int_{c_0}^{c_i} \varphi_i(z) z^s dz, \tag{7}$$

where each φ_i is an appropriate sum of branches of the algebraic function

$$\frac{\omega}{df} \circ f^{-1}$$

in some simply-connected domain U containing $S \setminus \{c_1, c_2, \dots c_k\}$.

Clearly,

$$J(t) = \sum_{i=1}^{k} J_i(t), \quad \text{where } J_i(t) = \int_{c_0}^{c_i} \frac{\varphi_i(z)}{z-t} dz. \tag{8}$$

Further, the functions $J_i(t)$ and therefore $J(t)$ allow for an analytic continuation on $\mathbb{CP}^1 \setminus \{c_1, c_2, \dots c_k\}$. On the other hand, by a well-known property of Cauchy type integrals, the limits of the function $J(t)$ when t approaches to a point $t \in [c_0, c_i]$ from the "left" and "right" sides of $[c_0, c_i]$ are related by the equality

$$J^+(t) - J^-(t) = 2\pi\sqrt{-1}\,\varphi_i(t).$$

Therefore, if the generating function $J(t)$ vanishes identically (or just allows for a single-valued analytical continuation), then the algebraic functions φ_i, $1 \le i \le k$, defined by (7) vanish identically. Of coarse, the equalities $\varphi_i \equiv 0$, $1 \le i \le k$, in their turn imply that $J \equiv 0$.

The study of conditions implying the vanishing of the algebraic functions φ_i is *a priori* a simpler problem than the initial one. Furthermore, the functions φ_i allow for the following remarkable interpretation as zero-dimensional Abelian integrals.

Consider the singular fibration $f : R \to \mathbb{CP}^1$ with fibers

$$f^{-1}(z) = \{f_1^{-1}(z), f_2^{-1}(z), \dots, f_d^{-1}(z)\} \tag{9}$$

where d is the degree of f. For $z \neq c_i, \infty$ define the (reduced) zero-homology group

$$\tilde{H}_0(f^{-1}(z), \mathbb{Z}) = \{n_1 f_1^{-1}(z) + n_2 f_1^{-1}(z) + \dots + n_d f_d^{-1}(z) : \textstyle\sum n_i = 0,\ n_i \in \mathbb{Z}\}.$$

It is a free $\mathbb{Z}$-module generated by

$$f_1^{-1}(z) - f_d^{-1}(z), f_2^{-1}(z) - f_d^{-1}(z), \dots, f_{d-1}^{-1}(z) - f_d^{-1}(z)$$

and its dual space is denoted by $\tilde{H}^0(f^{-1}(z), \mathbb{C})$. The map $f : R \to \mathbb{CP}^1$ induces homology and co-homology bundles with the base $\mathbb{C} \setminus \{c_1, \dots, c_k\}$ and fibers $\tilde{H}_0(f^{-1}(z), \mathbb{Z})$ and $\tilde{H}^0(f^{-1}(z), \mathbb{C})$. The continuous families of cycles

$$f_i^{-1}(z) - f_j^{-1}(z) \in \tilde{H}_0(f^{-1}(z), \mathbb{Z})$$

generate a basis of locally constant sections of a canonical connection on the homology bundle (the Gauss–Manin connection). Clearly, a meromorphic function g on R defines a meromorphic section of the co-homology bundle, and we may define a zero-dimensional Abelian integral as follows (see [7]).

Definition 2.1. A zero-dimensional Abelian integral is an algebraic function

$$\int_{\delta(z)} g = n_1 g(f_1^{-1}(z)) + n_2 g(f_1^{-1}(z)) + \cdots + n_d g(f_d^{-1}(z)), \tag{10}$$

where g is a meromorphic function on R and

$$\delta(z) = n_1 f_1^{-1}(z) + n_2 f_2^{-1}(z) + \cdots + n_d f_d^{-1}(z) \in \tilde{H}_0(f^{-1}(z), \mathbb{Z}) \tag{11}$$

is a continuous family of 0-cycles.

Clearly, the functions φ_i in (8) may be interpreted as zero-dimensional Abelian integrals

$$\varphi_i(z) = \int_{\delta_i(z)} \frac{\omega}{df}, \tag{12}$$

where

$$\delta_i(z) = \sum_{j=1}^{d} n_{ij} f_j^{-1}(z) \tag{13}$$

and n_{ij} are suitable integers (computed below).

Thus, we proved that the following statement is true.

Theorem 2.1. *The moments*

$$m_s = \int_\gamma f^s \omega, \quad s \geq 0, \tag{14}$$

vanish if and only if the zero-dimensional Abelian integrals

$$\varphi_i(z) = \int_{\delta_i(z)} \frac{\omega}{df}, \quad i = 1, 2, \ldots, k,$$

vanish identically. □

Of course, in order to apply Theorem 2.1 we must define values of the integer numbers n_{ij} in (13). For this purpose, following [13], consider the preimage of the star S under f

$$\lambda_f = f^{-1}(S) \subset R$$

as a graph embedded in the Riemann surface R. This graph, called a constellation, in a sense is a "combinatorial portrait" of the corresponding covering (see [9] for details and different versions of this construction). By construction, the restriction of $f(z)$ on $R \setminus \lambda_f$ is a covering of the topological punctured disk $\mathbb{CP}^1 \setminus \{S \cup \infty\}$ and therefore $R \setminus \lambda_f$ is a disjoint union of disks. This implies that the graph λ_f is connected and the faces of λ_f are in a one-to-one correspondence with poles of $f(z)$. For each i, $1 \le i \le k$, we will mark vertices of λ_f which are preimages of the point c_i by the number i (see Figure 1). Further, define a *star* of λ_f as a subset of edges of λ_f consisting of edges adjacent to some non-marked vertex. If U is a simply-connected domain such that $S \setminus \{c_1, c_2, \dots, c_k\} \subset U$, then the set of stars of λ_P may be naturally identified with the set of single-valued branches of $f^{-1}(z)$ in U as follows: to the branch $f_j^{-1}(z)$, $1 \le j \le d$, corresponds the star S_j such that $f_j^{-1}(z)$ maps bijectively the interior of S to the interior of S_j.

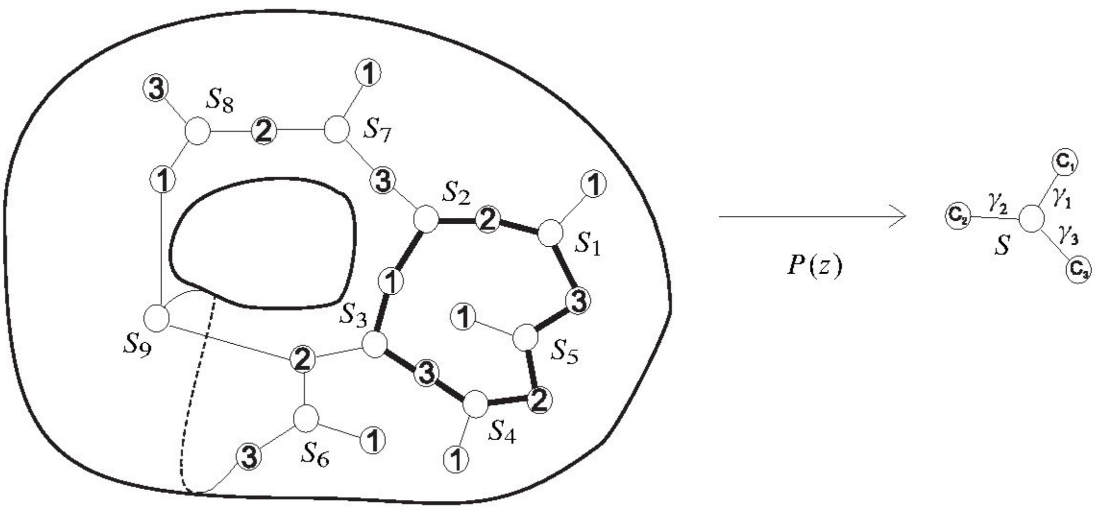

Figure 1

Since $R \setminus \lambda_f$ is a disjoint union of disks each of which contains a single pole of f, we may deform γ continuously from the interior of each of these disks to its boundary avoiding poles of f (see Figure 2). Since by assumption the set of poles of ω is contained in the set of poles of f, this deformation does not change the function $J(z)$. Keeping the same notation γ for this deformation we see that $f(\gamma) \subset S$. Furthermore, denoting by $c_{i,j}$ a unique vertex of the star S_j marked by the number i, it is easy to see that the number n_{ij} in formula (13) is equal to a sum of "signed" appearances of the vertex c_{ij} on γ. By definition, this means that an appearance is taken with the sign plus if the center of S_j is followed by c_{ij}, and minus if c_{ij} is followed by the center of S_j. For example, for the graph λ_f shown in Figure 1 and

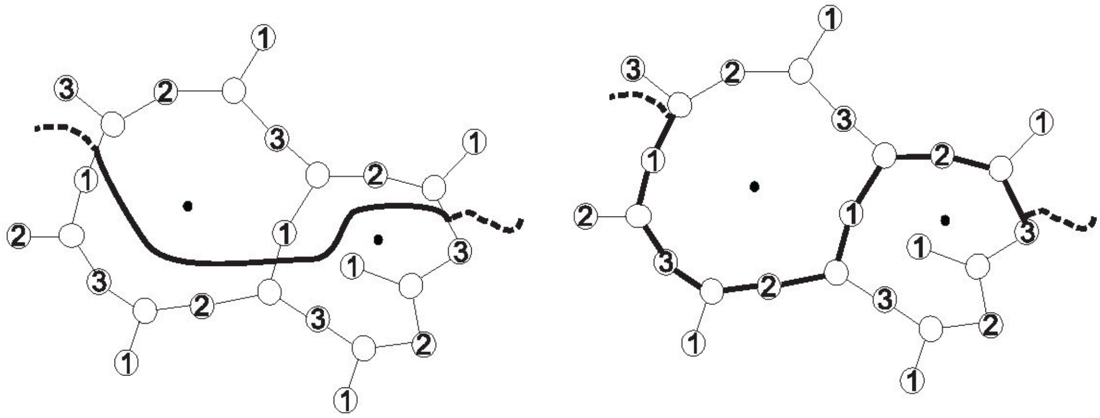

Figure 2

the path $\gamma \subset \lambda_f$ pictured by the fat line we have

$$\begin{aligned}
\delta_1(z) &= f_3^{-1}(z) - f_2^{-1}(z),\\
\delta_2(z) &= f_2^{-1}(z) - f_1^{-1}(z) + f_5^{-1}(z) - f_4^{-1}(z),\\
\delta_3(z) &= f_1^{-1}(z) - f_5^{-1}(z) + f_4^{-1}(z) - f_3^{-1}(z).
\end{aligned}$$

Since γ is a closed loop, it follows from the above construction that $\sum_j n_{ij} = 0$ implying that δ_i in (13) are contained in $\widetilde{H}_0(f^{-1}(z), \mathbb{Z})$. Furthermore, the following statement is true.

Corollary 2.1. *If the curve γ is not homological to zero in R with poles of f removed, then the vanishing of moments (14) implies that there exists a non-zero cycle $\delta \in \widetilde{H}_0(f^{-1}(z), \mathbb{Z})$ such that $\int_\delta \frac{\omega}{df} = 0$.* □

Theorem 2.1 and Corollary 2.1 remain true without the restriction that the set of poles of ω is contained in the set of poles of f if to change the condition $J(t) \equiv 0$ to the condition that $J(t)$ is rational. Indeed, we always may find a polynomial R such that the set of poles of the form $\widetilde{\omega} = R(f)\,\omega$ is contained in the set of poles of f. On the other hand, it follows from the definition of the function $J(t)$ that $J(\omega, f, \gamma, t)$ and $J(f\omega, f, \gamma, t)$ are related by the equality

$$J(f\omega, f, \gamma, t) = J(\omega, f, \gamma, t)t + \int_\gamma \omega$$

which implies inductively that the function $J(\omega, f, \gamma, t)$ is rational if and only if the function $J(\widetilde{\omega}, f, \gamma, t)$ does.

Further, observe that the above method may be applied also in the situation where the curve γ is not closed and/or is not connected (see [13], Section 3, for the rational case which extends to the general case in the same way as above). Of coarse, if γ is non-closed, then the condition $\sum_j n_{ij} = 0$ for $\delta_i(z)$ in (13) is not necessary true.

2.2. Case of generic position. Let $f: R \to \mathbb{CP}^1$ be a holomorphic function on a compact Riemann surface R, z_0 be a fixed regular value of f, and Δ be the set of critical values of f. Recall that the monodromy group G_f of the function f is defined as the image of the homomorphism

$$\pi_1(\mathbb{C} \setminus \Delta, z_0) \to \mathrm{Aut}(f^{-1}(z_0)), \tag{15}$$

where $\mathrm{Aut}(f^{-1}(z_0))$ is the full permutation group. Further, a holomorphic function $f: R \to \mathbb{CP}^1$ can be decomposed into a composition $f = p \circ q$ of holomorphic functions $q: R \to C$ and $p: C \to \mathbb{CP}^1$, where C is another compact Riemann surface, if and only if the group G_f has an imprimitivity system which consists of $l = \deg p$ blocks. Notice that the set of blocks of the imprimitivity system corresponding to the decomposition $f = p \circ q$ has the form $\mathcal{B}_i = q^{-1}\{t_i\}$, $1 \leq i \leq l$, where $\{t_1, t_2, \dots, t_l\} = p^{-1}\{z_0\}$. Finally, notice that if $f = \tilde{p} \circ \tilde{q}$, where $\tilde{q}: R \to \widetilde{C}$, $\tilde{p}: \widetilde{C} \to \mathbb{CP}^1$, is an other decomposition of f, then the corresponding imprimitivity systems coincide if and only if there exists an isomorphism $\mu: \widetilde{C} \to C$ such that

$$p = \tilde{p} \circ \mu^{-1}, \quad q = \mu \circ \tilde{q}.$$

In this case the decompositions $p \circ q$ and $\tilde{p} \circ \tilde{q}$ are called equivalent.

We say that two holomorphic functions $f, g: R \to \mathbb{CP}^1$ on a compact Riemann surface R have a non-trivial common compositional right factor if there exists a Riemann surface $\widetilde{R}$, a holomorphic function $h: R \to \widetilde{R}$ of degree greater than one, and holomorphic functions $\tilde{f}, \tilde{g}: \widetilde{R} \to \mathbb{CP}^1$ such that $f = \tilde{f} \circ h$, $g = \tilde{g} \circ h$. The property of two functions f, g to have a common compositional right factor may be expressed via the vanishing of some zero-dimensional Abelian integrals.

Proposition 2.1. *Two holomorphic functions $f, g: R \to \mathbb{CP}^1$ on a compact Riemann surface R have a common compositional right factor if and only if there exists a cycle $\delta(z) \in \widetilde{H}_0(f^{-1}(z), \mathbb{Z})$ of the form $f_i^{-1}(z) - f_j^{-1}(z)$, $i \neq j$, such that*

$$\int_{\delta(z)} g \equiv 0. \tag{16}$$

In particular, equality (16) holds for all $\delta(z) \in \widetilde{H}_0(f^{-1}(z), \mathbb{Z})$ if and only if there exists a rational function $\tilde{g}$ such that $g = \tilde{g} \circ f$.

Proof. It is easy to see by the analytical continuation that, for a fixed index i, the set of indices $j \neq i$ satisfying the equality

$$g(f_i^{-1}(z)) = g(f_j^{-1}(z)) \tag{17}$$

form a block of an imprimitivity system I with respect to the action of G_f on fibers of f. Therefore, if (16) holds, then there exists a Riemann surface $\widetilde{R}$ and a meromorphic function $h\colon R \to \tilde{R}$ such that $f = \tilde{f} \circ h$ and fibers of h coincide with blocks of I. Furthermore, since any branch of f^{-1} may be written as

$$f_i^{-1} = h_j^{-1} \circ \tilde{f}_k^{-1}$$

for some branches of h^{-1} and $\tilde{f}^{-1}$, equalities (17) imply that the function g is constant on fibers of h. Therefore, the function $\tilde{g} = g \circ h^{-1}$ is well defined and satisfies the equality $g = \tilde{g} \circ h$. □

Notice that in the case where $R = \mathbb{CP}^1$ Proposition 2.1 is well known and follows easily from the Lüroth theorem (see e.g. [17], [10]).

Proposition 2.1 permits to obtain the following very implicit solution of the moment problem in the case where f is in a generic position.

Theorem 2.2. *If the monodromy group G_f of f is the full symmetric group of $d = \deg f$ elements, then the vanishing of moments (14) implies that either γ is homological to zero in R with poles of f removed, or there exists a rational function Q such that $\omega = Q(f)\,df$ and $f(\gamma)$ is homological to zero in $\mathbb{CP}^1$ with poles of Q removed.*

Proof. If γ is not homological to zero in R with poles of f removed, then by Corollary 2.1 there exist integer numbers $\alpha_1, \alpha_2, \dots, \alpha_d$ not all equal to zero such that

$$\sum_{j=1}^{d} \alpha_j \left(\frac{\omega}{df}\right)(f_j^{-1}(z)) = 0 \tag{18}$$

and $\sum_{i=1}^{n} \alpha_i = 0$. The last equality implies that the numbers $\alpha_1, \alpha_2, \dots, \alpha_d$ are not all equal between themselves. Let us assume that $\alpha_1 \neq \alpha_2$.

Since G_f is a full symmetric group, it contains the transposition $\sigma = (12)$. Acting by σ on equality (18) and subtracting we obtain the equality

$$(\alpha_1 - \alpha_2)\left(\left(\frac{\omega}{df}\right)(f_1^{-1}(z)) - \left(\frac{\omega}{df}\right)(f_2^{-1}(z))\right) = 0$$

implying the equality

$$\left(\frac{\omega}{df}\right)(f_1^{-1}(z)) = \left(\frac{\omega}{df}\right)(f_2^{-1}(z)). \tag{19}$$

Since the full symmetric group of f is primitive, the function f is indecomposable. Therefore, Proposition 2.1 applied to equality (19) implies that there exists a rational function Q such that $\frac{\omega}{df} = Q(f)$. Hence, moments (14) equal to the moments

$$\int_{f(\gamma)} Q(z) z^s dz, \quad s \geq 0,$$

and the statement follows from the classical result of the complex analysis. □

Notice that Theorem 2.2 remains true if G_f is only doubly transitive. This is a corollary of the characterization of doubly transitive groups via the structure of their irreducible subspaces over $\mathbb{C}$, see [10], [13] where this approach is used for rational f and g.

3. Vanishing of zero-dimensional polynomial Abelian integrals

In this section we give necessary and sufficient conditions for zero-dimensional Abelian integral (10) to vanish identically in the case where R is the Riemann sphere and the functions involved are polynomials. More precisely, we solve the following problem:

for a given polynomial P of degree n and a cycle $\delta(z) \in \widetilde{H}_0(P^{-1}(z), \mathbb{Z})$ describe polynomials Q such that the associated Abelian integral

$$I(z) = \int_{\delta(z)} Q = \sum_{i=1}^{n} v_i Q(P_i^{-1}(z)) \tag{20}$$

vanishes identically.

In distinction with the previous section we will not assume that $\widetilde{H}_0(P^{-1}(z), \mathbb{Z})$ is reduced. Thus $\delta(z)$ may be any expression of the form

$$\delta(z) = v_1 P_1^{-1}(z) + v_2 P_2^{-1}(z) + \cdots + v_n P_n^{-1}(z),$$

where $v_i \in \mathbb{Q}$. It is convenient to identify the cycle $\delta(z)$ with the vector

$$\vec{\delta} = (v_1, v_2, \ldots, v_n)$$

of $\mathbb{Q}^n$. Under such an identification the natural action of the monodromy group G_P of P on $\widetilde{H}_0(P^{-1}(z), \mathbb{Z})$ descends to an action on $\mathbb{Q}^n$ defining a *permutation representation* of the group G_P

$$\rho \colon G_P \to \mathrm{GL}(\mathbb{Q}^n). \tag{21}$$

The understanding of irreducible components of ρ plays a crucial role in the solution of the problem above. Indeed, let Z_δ be the vector space consisting of polynomials Q such that Abelian integral (20) vanishes identically, and V_δ be the minimal ρ-invariant vector subspace of $\mathbb{Q}^n$ containing the vector $\vec{\delta}$. Then it is easy to see by the analytical continuation that $\int_{\delta(z)} Q \equiv 0$ if and only if $\int_{\gamma(z)} Q \equiv 0$ for any $\gamma(z) \in \widetilde{H}_0(P^{-1}(z), \mathbb{Z})$ such that $\vec{\gamma} \in V_\delta$. This implies that in order to describe Z_δ it is enough to solve the following three problems:

(1) First, describe all possible irreducible G_P-invariant subspaces of $\mathbb{Q}^n$.

(2) Second, provide a method which allows for any given $\vec{\delta} \in \mathbb{Q}^n$ to decompose the invariant subspace V_δ into a direct sum of irreducible G_P-invariant subspaces.

(3) Third, for each irreducible G_P-invariant subspace U, describe the vector space Z_U consisting of polynomials Q such that $\int_{\delta(z)} Q \equiv 0$ for all $\delta \in U$.

A solution of problem (1) is given in [15] in a closed form (see Theorem 3.1 of [15]), while solutions of problems (2) and (3) can be obtained by appropriate modifications of proofs of Proposition 4.1 and Theorem 1.1 of [15] correspondingly. Below we recall the classification of irreducible G_P-invariant subspaces of $\mathbb{Q}^n$ obtained in [15] and provide self-contained solutions of problems (2) and (3) using the approach of [15].

3.1. Description of irreducible G_P-invariant subspaces of $\mathbb{Q}^n$. In this subsection we recall the description of G_P-invariant subspaces of $\mathbb{Q}^n$ obtained in [15]. More generally, we will describe G-invariant subspaces of $\mathbb{Q}^n$ for a permutation representation of an arbitrary permutation group $G \subset S_n$ containing a cycle of length n (the monodromy group of a polynomial of degree n always contains such a cycle which corresponds to a loop around infinity). For more details we refer the reader to Section 3 of [15].

Without loss of generality we may assume that the cycle of length n contained in G coincides with the cycle $(1 \dots n)$. This implies in particular that any imprimitivity system for G must coincide with residue classes modulo d for some $d \mid n$. For each $d \mid n$ we denote by V_d the subspace of $\mathbb{Q}^n$ consisting of "d-periodic" vectors that is of vectors of the form

$$(v_1, \dots, v_d, v_1, \dots, v_d, \dots, v_1, \dots, v_d).$$

It is easy to see that for given d residue classes modulo d form an imprimitivity system for G if and only if the subspace V_d is G-invariant.

Denote by $D(G)$ the set of all divisors of n for which V_d is G-invariant. Notice that $D(G)$ is a lattice with respect to the operations $\wedge, \vee$, where $d \wedge \tilde{d} := \gcd(d, \tilde{d})$ and $d \vee \tilde{d} := \operatorname{lcm}(d, \tilde{d})$. Indeed, for an element $x \in X$ the intersection of two blocks containing x and corresponding to $d, \tilde{d} \in D(G)$ is a block which corresponds to $d \vee \tilde{d}$. On the other hand, the intersection of two invariant subspaces $V_d, V_{\tilde{d}}$ is an invariant subspace which is equal to $V_{d \wedge \tilde{d}}$. We say that $d \in D(G)$ *covers* $\tilde{d} \in D(G)$ if $\tilde{d} \mid d$, $\tilde{d} < d$, and there exists no $l \in D(G)$ such that $\tilde{d} < l < d$ and $\tilde{d} \mid l$, $l \mid d$.

Theorem 3.1. ([15]) *Each irreducible G-invariant subspace of $\mathbb{Q}^n$ has the form*

$$U_d := V_d \cap \left(V_{d_1}^{\perp} \cap \dots \cap V_{d_\ell}^{\perp}\right) \tag{22}$$

where $d \in D(G)$ and $d_1, \ldots, d_\ell$ is a complete set of elements of $D(G)$ covered by d. The subspaces U_d are mutually orthogonal and every G-invariant subspace of $\mathbb{Q}^n$ is a direct sum of some U_d as above.

3.2. Decomposition of V_δ into a direct sum of irreducible subspaces. Set

$$\vec{w}_k = (1, \varepsilon_n^k, \varepsilon_n^{2k}, \ldots, \varepsilon_n^{(n-1)k}), \quad \varepsilon_n = e^{2\pi i/n},\ 1 \leq k \leq n. \tag{23}$$

Clearly, the vectors $\vec{w}_k$, $1 \leq k \leq n$, form an orthonormal basis of $\mathbb{C}^n$ with respect to the standard Hermitian inner product in $\mathbb{C}^n$, and for any divisor d of n the vectors $\vec{w}_k$ for which $(n/d)\,|\,k$ form a basis of the complexification $V_d^{\mathbb{C}}$ of the subspace V_d. Furthermore, defining Ψ_d, $d \in D(G)$, as a subset of $\{1, 2, \ldots, n\}$ consisting of numbers r such that n/d is a divisor of r but for any element $\tilde{d} \in D(G)$ covered by d the number $n/\tilde{d}$ is not a divisor of r, we see that the vectors $\vec{w}_r$, $r \in \Psi_d$, form a basis of $U_d^{\mathbb{C}}$.

Theorem 3.2. *The subspace U_d, $d \in D(G)$, is a component in the decomposition of the subspace V_δ into a sum of irreducible G-invariant subspaces of $\mathbb{Q}^n$, if and only if there exists a number $r \in \Psi_d$ such that $(\vec{\delta}, \vec{w}_r) \neq 0$.*

Proof (cf. Proposition 4.1 in [15]). Let $V_\delta = \oplus\, U_{d_j}$ be a decomposition of V_δ into a sum of irreducible G-invariant subspaces. If V_δ is orthogonal to U_d, then V_δ is orthogonal also to $U_d^{\mathbb{C}}$ implying that $(\vec{\delta}, \vec{w}_r) = 0$ for all $r \in \Psi_d$, since the vectors $\vec{w}_r$, $r \in \Psi_d$, form a basis of $U_d^{\mathbb{C}}$. Therefore, if $(\vec{\delta}, \vec{w}_r) \neq 0$ for some $r \in \Psi_d$, then U_d coincides with some U_{d_j}.

In other direction, if U_d coincides with some U_{d_j}, then in view of the minimality of V_δ the projection of δ onto U_d is distinct from zero implying that there exists a number $r \in \Psi_d$ such that $(\vec{\delta}, \vec{w}_r) \neq 0$, since $\vec{w}_r$, $r \in \Psi_d$, form a basis of $U_d^{\mathbb{C}}$. □

3.3. Description of spaces Z_{U_d}. First of all observe that if $P = A \circ B$ is a decomposition of a polynomial P into a composition of rational functions, then the corresponding equivalence class of decompositions contains a decomposition where both functions involved are *polynomials*, and below we always will consider only such decompositions. In order to keep the correspondence between imprimitivity systems of G_P and equivalence classes of decompositions of P we modify the definition of equivalence correspondingly. Namely, we will call decompositions $P = A_1 \circ W_1$ and $P = A_2 \circ W_2$ equivalent if there exists a *polynomial* ν of degree one such that $A_2 = A_1 \circ \nu$, $W_2 = \nu^{-1} \circ W_1$. Abusing of notation, usually we will mean by a decomposition a corresponding equivalence class of decompositions.

Notice that since any imprimitivity system for a group $G \subset S_n$ containing the cycle $(12\ldots n)$ coincides with residue classes modulo d for some $d\,|\,n$, any two decompositions $P = P_1 \circ W_1$ and $P = P_2 \circ W_2$ of P such that $\deg P_1 = \deg P_2$ are equivalent. Notice also that in the above notation the set $D(G_P)$ consists of numbers d for which there exists a decomposition $P = A \circ W$ with $\deg A = d$.

The structure of Z_{U_d} or, more generally, of any G_P-invariant subspace of $\mathbb{Q}^n$ is closely related to the compositional properties of polynomials. For example, for any $\delta(z) \in \widetilde{H}_0(P^{-1}(z), \mathbb{Q})$ and polynomial Q of the form $Q = R \circ P$ we have

$$\int_{\delta(z)} Q = R(z) \sum_{i=1}^{n} v_i, \tag{24}$$

implying that $\int_{\delta(z)} Q$ vanishes identically whenever $\delta(z)$ is contained in the reduced homology group, or equivalently the vector $\vec{\delta}$ is contained in $V_1^{\perp}$.

Further, if $P = A \circ W$, $\deg A = d$, is a decomposition of P corresponding to $d \in D(G_P)$, then for any branch $P_i^{-1}(z)$ of $P^{-1}(z)$ there exist a branch $W_j^{-1}(z)$ of $W^{-1}(z)$ and a branch $A_k^{-1}(z)$ of $A^{-1}(z)$ such that

$$P_i^{-1} = W_j^{-1} \circ A_k^{-1}. \tag{25}$$

Therefore, for any cycle $\delta(z) \in \widetilde{H}_0(P^{-1}(z), \mathbb{Q})$ and polynomial Q we have:

$$\int_{\delta(z)} Q = \sum_{k=1}^{d} \left(\int_{\delta_{k,W}(z)} Q \right) \circ A_k^{-1}, \tag{26}$$

where $\delta_{k,W}(z) \in \widetilde{H}_0(W^{-1}(z), \mathbb{Q})$, implying that the integral $\int_{\delta(z)} Q$ vanishes identically whenever all the integrals $\int_{\delta_{k,W}(z)} Q$, $1 \leq k \leq d$, do. In particular, if $\delta(z) \in \widetilde{H}_0(P^{-1}(z), \mathbb{Q})$ is a cycle such that all cycles $\delta_{k,W}(z)$, $1 \leq k \leq d$, are in the reduced homology group $\widetilde{H}_0(W^{-1}(z), \mathbb{Q})$, then $\int_{\delta(z)} Q$ vanishes identically for any polynomial Q of the form $Q = B \circ W$.

In the following we always will assume that the numeration of roots $P_i^{-1}(z)$ of $P^{-1}(z)$ satisfies the requirement that the cycle in G_P corresponding to a loop around infinity coincides with the cycle $(1\,2\,\ldots\,n)$. In particular, such a choice of the numeration yields that without loss of generality we may assume that when k in formula (25) remains fixed, the corresponding i runs the set of numbers equal to k by modulo d, implying that a cycle $\delta_{k,W}(z) \in \widetilde{H}_0(W^{-1}(z), \mathbb{Q})$ in (26) is reduced if and only if

$$(\vec{\delta}, \vec{e}_{k,d}) = 0, \tag{27}$$

where $\vec{e}_{k,d}$, $1 \leq k \leq d$, denotes a vector of $\mathbb{Q}^n$ with coordinates $v_1, v_2, \ldots, v_n$ such that $v_i = 1$ if $i = k \bmod d$, and $v_i = 0$ otherwise. Since vectors $\vec{e}_{k,d}$, $1 \leq k \leq d$, obviously form a basis of V_d, this implies that all cycles $\delta_{k,W}(z)$, $1 \leq k \leq d$, are reduced if and only if $\vec{\delta}$ is orthogonal to V_d.

Returning to the description of the space Z_{U_d}, $d \in D(G_P)$, observe that it always contains the space Z_{V_d} in view of the inclusion $U_d \subset V_d$. Furthermore, if

$\tilde{d} \in D(G_P)$ is covered by d and $P = \widetilde{A} \circ \widetilde{W}$ is a decomposition corresponding to $\tilde{d}$, then, since U_d is orthogonal to $V_{\tilde{d}}$, the cycles $\delta_{k,\widetilde{W}}(z)$, $1 \le k \le \tilde{d}$, are reduced for any $\delta \in U_d$. Therefore, for any such $\tilde{d}$, the ring $\mathbb{C}[\widetilde{W}]$ of polynomials in $\widetilde{W}$ is contained in the space Z_{U_d}.

Theorem 3.3. *Let d be an element of $D(G_P)$. Furthermore, let $d_1, \dots, d_\ell$ be a complete set of elements of $D(G_P)$ covered by d and $P = A_i \circ W_i$, $1 \le i \le \ell$, be the corresponding decompositions. Then*

$$Z_{U_d} = Z_{V_d} + \mathbb{C}[W_1] + \mathbb{C}[W_2] + \cdots + \mathbb{C}[W_l]. \tag{28}$$

Proof (cf. Theorem 1.1 in [15]). In view of the above remarks, the right part of (28) is contained in Z_{U_d}. So, we only must establish the inverse inclusion.

First, observe that the numeration of branches of $P^{-1}(z)$ implies that at points close enough to infinity the functions $Q(P_i^{-1}(z))$, $1 \le i \le n$, may be represented by converging Puiseux series

$$Q(P_i^{-1}(z)) = \sum_{k=-q}^{\infty} s_k \varepsilon_n^{(i-1)k} z^{-\frac{k}{n}}, \tag{29}$$

where $q = \deg Q(z)$ and $\varepsilon_n = \exp(2\pi i/n)$. Furthermore, substituting (29) to (20) we see that the integral $\int_{\delta(z)} Q$ vanishes identically if and only if for any $k \ge -q$ the equality

$$\sum_{i=1}^{n} v_i s_k \varepsilon_n^{(i-1)k} = (\vec{\delta}, \overline{\vec{w}_k}) s_k = 0 \tag{30}$$

holds. In particular, if $Q(z) \in Z_{U_d}$, $d \in D(G_P)$, then the equalities

$$(\vec{v}, \overline{\vec{w}_k}) s_k = 0, \quad k \ge -q, \tag{31}$$

hold for any $\vec{v} \in U_d$ and therefore they hold also for any $\vec{v} \in U_d^{\mathbb{C}}$. Since $U_d^{\mathbb{C}}$ is generated by the set of vectors $\vec{w}_r$, $r \in \Psi_d$, and this set transforms to itself under the complex conjugation, this implies that if $Q(z) \in Z_{U_d}$, then for any $r \in \Psi_d$ the equality $s_k = 0$ holds for any k such that $k \equiv r \bmod n$. Furthermore, clearly the inverse is also true. Similarly, it is easy to see that $Q(z) \in Z_{V_d}$ if and only if $s_k = 0$ for any k such that $(n/d)|k$.

Assume now that $Q(z) \in Z_{U_d}$ and consider series (29). If $s_k = 0$ for any k such that $(n/d)|k$, then $Q(z) \in Z_{V_d}$ and we are done. Thus, suppose that there exists k such that $(n/d)|k$ but $s_k \neq 0$. It follows from the definition of U_d and the above characterization of coefficients of (29) for $Q(z) \in Z_{U_d}$ that in this case necessarily $(n/\tilde{d})|k$ for some $\tilde{d} \in D(G_P)$ covered by d, and without loss of generality we may assume that $\tilde{d} = d_1$. Set

$$\psi(z) = \sum_{\substack{k \geq -q \\ k \equiv 0 \bmod n/d_1}} s_k z^{-\frac{k}{n}}, \tag{32}$$

where s_k, $k \geq -q$, are coefficients of series (29). Clearly, we have:

$$\left(\frac{n}{d_1}\right)\psi(z) = Q(P_1^{-1}(z)) + Q(P_{d_1+1}^{-1}(z)) + Q(P_{2d_1+1}^{-1}(z)) + \cdots + Q(P_{n-d_1+1}^{-1}(z)). \tag{33}$$

Since indices appearing in the right part of (33) form a block, the function $\psi(z)$ is invariant with respect to the subgroup of G_P which stabilizes $P_1^{-1}(z)$. Therefore, by the main theorem of Galois theory, $\psi(z)$ is contained in the field $\mathbb{C}(z)(P_1^{-1}(z))$. Further, since $z = P((P_1^{-1}(z))$ the equality $\mathbb{C}(z)(P_1^{-1}(z)) = \mathbb{C}(P_1^{-1}(z))$ holds and hence $\psi(z) = R_1(P_1^{-1}(z))$ for some rational function R_1. Moreover, R_1 is actually a polynomial since the right part of (33) may have a pole only at infinity. Finally, since (33) implies by analytical continuation that

$$R_1(P_1^{-1}(z)) = R_1(P_{d_1+1}^{-1}(z)) = R_1(P_{2d_1+1}^{-1}(z)) = \cdots = R(P_{n-d_1+1}^{-1}(z)),$$

reasoning now as in Proposition 2.1 we conclude that R_1 is constant on fibers of W_1 and $R_1 = S_1 \circ W_1$ for some polynomial S_1 (cf. Lemma 4.3 in [15]).

Define now a polynomial $T_1(z)$ by the equality

$$T_1(z) = Q(z) - R_1(z).$$

Then by construction the Puiseux series of $T_1(P_1^{-1}(z))$ contains no non-zero coefficients with indices which are multiple of n/d_1. If $T_1(z)$ is contained in Z_{V_d}, then

$$Q(z) = T_1(z) + S_1(W_1(z))$$

and we are done. Otherwise arguing as above we may find polynomials R_2, S_2 such that $R_2 = S_2 \circ W_2$ and the Puiseux expansion of $T_2(P_1^{-1}(z))$, where

$$T_2(z) = T_1(z) - R_2(z),$$

contains no non-zero coefficients whose indices are multiple of n/d_1 or n/d_2. It is clear that continuing this process we eventually will arrive to some $T_s(z)$ which is contained in Z_{V_d} and therefore to a representation

$$Q(z) = T_s(z) + S_1(W_1(z)) + S_2(W_2(z)) + \cdots + S_l(W_l(z)). \qquad \square$$

In view of Theorem 3.3, in order to complete the description of the space Z_{U_d} we only must describe the space Z_{V_d}. Observe first that the vectors $\vec{e}_{j,d}$, $1 \leq j \leq d$, defined above satisfy the equality

$$(\vec{e}_{j,d}, \vec{w}_k) = \varepsilon_n^{k(j-1)}(\vec{e}_{1,d}, \vec{w}_k).$$

Therefore, in order to check that equality (31) holds for any $\vec{v} \in Z_{V_d}$ it is enough to check that it holds for one single vector $\vec{e}_{1,d}$. In other words, the space Z_{V_d} consists of polynomials $Q(z)$ satisfying the equality

$$Q(P_1^{-1}(z)) + Q(P_{d+1}^{-1}(z)) + Q(P_{2d+1}^{-1}(z)) + \cdots + Q(P_{n-d+1}^{-1}(z)) \equiv 0. \quad (34)$$

Furthermore, if $P = A \circ W$ is a decomposition corresponding to $d \in D(G_P)$, then in view of (25) equality (34) reduces to the equality

$$Q(W_1^{-1}(z)) + Q(W_2^{-1}(z)) + Q(W_3^{-1}(z)) + \cdots + Q(W_{n/d}^{-1}(z)) \equiv 0. \quad (35)$$

The Newton formulae imply that whenever $\deg Q < \deg W$ the sum in the left hand side of (35) is a constant. Therefore, setting $\mu_i = W_i^{-1}(c)$, $i = 1, 2, \ldots, n/d$, for some generic $c \in \mathbb{C}$, we see that the intersection $Z_{V_d} \cap T_W$, where T_W is a vector space of polynomials of degree less than $\deg W$, has codimension one in T_W and is described by the relation

$$Q(\mu_1) + Q(\mu_2) + \cdots + Q(\mu_{n/d}) = 0.$$

On the other hand, using W-adic decomposition, it is easy to see that that for $Q(z)$ of arbitrary degree the sum in (35) is a polynomial, and that a polynomial $Q(z)$ satisfies (35) if and only if all coefficients in its W-adic decomposition satisfy it.

Finally, notice that Theorem 3.3 provides a description of the space Z_V for any G_P-invariant subspace V of $\mathbb{Q}^n$ since by Theorem 3.1 any such a subspace has the form $V = \oplus U_{d_j}$ implying that $Z_V = \cap Z_{U_{d_j}}$.

3.4. Corollaries. In this subsection we discuss some particular cases of the above results which may be useful for applications. Below, we always will assume that Q is a non-zero polynomial and δ is a non-zero element of $\widetilde{H}_0(P^{-1}(z), \mathbb{Q})$.

Proposition 3.1. *Let P be an indecomposable polynomial. If an Abelian integral $\int_{\delta(z)} Q$ vanishes identically, then either Q is a polynomial in P and the cycle $\delta(z)$ is reduced, or $Q \in Z_{V_1}$ and there exists a rational number a such that*

$$\delta(z) = a(P_1^{-1}(z) + P_2^{-1}(z) + \cdots + P_n^{-1}(z)). \quad (36)$$

Proof. First, observe that V_δ does not coincide with whole $\mathbb{Q}^n$ since otherwise $\vec{e}_{1,n} \in V_\delta$ would imply that $Q(z) \equiv 0$. Therefore, by Theorem 3.1 either $V_\delta = U_1$ or $V_\delta = U_n = U_1^\perp$. Obviously, in the first case $Q \in Z_{V_1}$ and (36) holds, while in the second case the cycle $\delta(z)$ is reduced. Furthermore, by Theorem 3.3 in the second case Q is contained in $Z_{U_n} = Z_{V_n} + \mathbb{C}[P]$, implying that $Q \in \mathbb{C}[P]$, since $Z_{V_n} = \{0\}$ by (35). Alternatively, one can observe that in the second case V_δ contains vectors $e_i - e_j$, $1 \le i, j \le n$. Therefore,

$$Q(P_1^{-1}(z)) = Q(P_2^{-1}(z)) = \cdots = Q(P_n^{-1}(z)) \quad (37)$$

and hence $Q \in \mathbb{C}[P]$ by Proposition 2.1. $\square$

Notice that the conclusion of Proposition 3.1 holds for any polynomial $P(z)$ in generic position since decomposable polynomials obviously form a proper algebraic subset in the set of all polynomials of degree n. Notice also that in order to prove Proposition 3.1 one can use instead of Theorem 3.1 the classical result, relating the doubly transitivity of a group with the structure of its permutation representation over $\mathbb{C}$, combined with the Schur theorem, relating the doubly transitivity and the primitivity for a group containing a transitive cyclic subgroup (see [10] for such an approach).

The conclusion similar to the one in Proposition 3.1 is true for arbitrary P if to impose some limitations on $\delta(z)$.

Proposition 3.2. *If an Abelian integral $\int_{\delta(z)} Q$ vanishes identically and for any $d \in D(G_P)$, $d \neq 1$, there exists $r \in \Psi_d$ such that $(\vec{\delta}, \vec{w}_r) \neq 0$, then Q is a polynomial in P and the cycle $\delta(z)$ is reduced.*

Proof. Indeed, it follows from Theorem 3.1 and Theorem 3.2 that $V_\delta = U_1^{\perp}$. □

A finer version of Proposition 3.2 is the following statement.

Proposition 3.3. *If an Abelian integral $\int_{\delta(z)} Q$ vanishes identically and there exists $r \in \Psi_n$ such that $(\vec{\delta}, w_r) \neq 0$, then $Q(z)$ may be represented in the form*

$$Q(z) = S_1(W_1(z)) + S_2(W_2(z)) + \cdots + S_l(W_l(z)), \tag{38}$$

where $S_1, S_2, \ldots, S_l$ are polynomials and $W_1, W_2, \ldots W_l$ are compositional right factors of $P(z)$ corresponding to elements $d_1, \ldots, d_\ell$ of $D(G_P)$ covered by n.

Proof. It follows from Theorem 3.2 that V_δ contains U_n. Therefore, $Z_\delta \subseteq Z_{U_n}$ and the statement follows from Theorem 3.3. □

Notice that it follows from (32) and the characterization of polynomials Q satisfying $\int_{\delta(z)} Q = 0$ via their Puiseux expansions (30) that for any polynomial $Q_j = S_j(W(z))$, $1 \leq j \leq l$, appearing in representation (38) the integral $\int_{\delta(z)} Q_j$ vanishes. However, unless $V_\delta = U_n$, it is not true that for any polynomial W_j, $1 \leq j \leq l$, the corresponding cycles

$$\delta_{k,W_j}(z) \in \widetilde{H}_0(W_j^{-1}(z), \mathbb{Q}), \quad 1 \leq k \leq d_j, \tag{39}$$

are reduced. Still, the following statement is true.

Proposition 3.4. *If an Abelian integral $\int_{\delta(z)} Q$ vanishes identically and there exist $d_1, d_2, \ldots d_l \in D(G_P)$ such that for any $d \in D(G_P)$ the inequality $(\vec{\delta}, w_r) \neq 0$*

holds for some $r \in \Psi_d$ *if and only if* d *is not a divisor of one of the numbers* $d_1, d_2, \dots, d_l$, *then* $Q(z)$ *may be represented in the form* (38), *where* $W_1, W_2, \dots W_l$ *are compositional right factors of* $P(z)$ *corresponding to* $d_1, \dots, d_\ell$ *and all cycles* (39) *are reduced.*

Proof. Since the condition of the theorem implies by Theorem 3.2 that V_δ coincides with the orthogonal complement to the sum of $V_{d_1}, V_{d_2}, \dots, V_{d_l}$ in $\mathbb{Q}^n$, the proof is obtained by an obvious modification of the proof of Theorem 3.3. □

Remark. The results similar to Propositions 3.3, 3.4 (without a solution of the general problem) were obtained in the recent paper [1] (Theorem 2.2) where they also were deduced from Theorem 3.1 by the method of [15]. Notice however that the corresponding statements in [1] are weaker. For example, the second part of Theorem 2.2 in [1] which is an analog of our Proposition 3.3 contains an additional assumption which in our settings means that $r \in \Psi_n$ for which $(\vec{\delta}, \vec{w}_r) \neq 0$ is coprime with n.

3.5. Polynomial moment problem on a system of intervals with weights. Recall that the polynomial moment problem, recently solved in [15], [14], asks to describe, for a given polynomial P and $a, b \in \mathbb{C}$, all polynomials Q satisfying the system of equations

$$\int_a^b P^s dQ = 0, \quad s \geq 0. \tag{40}$$

It is easy to see using a change of variable that if W_j is a right compositional factor of P such that $W_j(a) = W_j(b)$, then for any polynomial $S_j(z)$ the polynomial $Q_j = S_j(W_j(z))$ is a solution of (40), and it is shown in [15] that any solution Q may be represented in the form (38), where W_j are compositional right factors of $P(z)$ satisfying the condition $W_j(a) = W_j(b)$. In the above notation the proof given in [15] may be sketched as follows.

First, by the method of Section 2 it is shown that there exists a collection of cycles $\delta_i(z)$, $1 \leq i \leq k$, in $\widetilde{H}_0(P^{-1}(z), \mathbb{Q})$ such that equalities (40) hold if and only if the equalities

$$\int_{\delta_i(z)} Q = 0, \quad 1 \leq i \leq k, \tag{41}$$

hold. Then, it is shown that the minimal G_P-invariant subspace containing the cycles $\delta_i(z)$, $1 \leq i \leq k$, contains a vector $\vec{v}$ such that $(\vec{v}, w_r) \neq 0$ for some $r \in \Psi_n$ (this is done in [12] by means the so-called "monodromy lemma" which uses, in contrast to Theorem 3.1, topological properties of polynomials). Further, by the method of Section 3 it is proved that $Q(z)$ may be represented in the form (38), where for any polynomial $Q_j = S_j(W_j(z))$ integrals (41) vanish implying that moments (40) also vanish (although it is not necessary true that $W_j(a) = W_j(b)$). Finally, since

$$\int_a^b P^s dQ_j = \int_{W_j(a)}^{W_j(b)} R_j^s dS_j, \quad s \geq 0,$$

where R_j is a polynomial such that $\deg R_j < \deg P$, representation (38) with $W_j(a) = W_j(b)$ is obtained by the recursive use of the above construction for indices j with $W_j(a) \neq W_j(b)$.

It is not hard to see that the results of the current section may be interpreted as a solution of the polynomial moment problem "on a system of intervals with weights". More precisely, for any collection consisting of a polynomial P, complex numbers a_i, b_i, $1 \leq i \leq l$, and rational numbers c_i, $1 \leq i \leq l$, using approach of Section 1 (see also [15], [13] where more attention to non-closed curves is given) one can construct a finite collection of cycles $\delta_i(z)$, $1 \leq i \leq k$, in $\widetilde{H}_0(P_i^{-1}(z), \mathbb{Q})$ such that the equalities

$$c_1 \int_{a_1}^{b_1} P^s dQ + c_2 \int_{a_2}^{b_2} P^s dQ + \cdots + c_l \int_{a_l}^{b_l} P^s dQ = 0, \quad s \geq 0, \tag{42}$$

hold if and only if equalities (41) hold. Since the results of this section provide a description of Q satisfying (41), they provide also a description of solutions of (42).

As a simple illustration take P equal to T_6, where T_n denotes nth Chebyshev polynomial, $T_n(\cos\varphi) = \cos(n\varphi)$. Notice that it follows from the definition that for ant $d|n$ the equality $T_n = T_d \circ T_{n/d}$ holds. In particular, $D(G_{T_n})$ consists of all divisors of n. Furthermore, it is easy to see that T_n has only two finite critical values and that the corresponding constellation is a "chain" (see e.g. [11]). For $n = 6$ the corresponding constellation is shown in Figure 3, where the "middle" vertices of stars are omitted and the numeration of stars is chosen in such a way that a permutation at

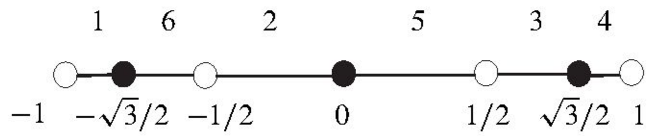

Figure 3

infinity coincides with the cycle (123456).

Applying the above results, it is easy to see that if we are searching for solutions of the moment problem

$$\int_{-\sqrt{3}/2}^{\sqrt{3}/2} T_6^s dQ = 0, \quad s \geq 0, \tag{43}$$

on a single segment $[-\sqrt{3}/2, \sqrt{3}/2]$, then we arrive to the vanishing problem for the Abelian integral

$$I(z) = Q(T_{6,6}^{-1}(z)) - Q(T_{6,2}^{-1}(z)) + Q(T_{6,5}^{-1}(z)) - Q(T_{6,3}^{-1}(z)),$$

where by $T_{6,i}^{-1}(z)$, $1 \leq i \leq 6$, are denoted the branches of $T_6^{-1}(z)$ (formally, we should obtain a cycle for each critical value, however, since we have only two critical values, the corresponding cycles are proportional). Clearly, the corresponding vector $(0,-1,-1,0,1,1) \in \mathbb{Q}^6$ is orthogonal to both V_2 and V_3 implying that $V_\delta = U_6$. Therefore, by Theorem 3.3 any solution of (43) has the form

$$Q(z) = A(T_3(z)) + B(T_2(z)),$$

where $A, B \in \mathbb{C}[z]$.

On the other hand, the "generalized" moment problem

$$\int_{-1}^{-1/2} T_6^s dQ - \int_{-1/2}^{1/2} T_6^s dQ + \int_{1/2}^{1} T_6^s dQ = 0, \quad s \geq 0, \tag{44}$$

leads to the vanishing problem for the Abelian integral

$$\begin{aligned} I(z) = {} & Q(T_{6,1}^{-1}(z)) - Q(T_{6,2}^{-1}(z)) + Q(T_{6,3}^{-1}(z)) \\ & - Q(T_{6,4}^{-1}(z)) + Q(T_{6,5}^{-1}(z)) - Q(T_{6,6}^{-1}(z)). \end{aligned}$$

Since the corresponding vector $\vec{\delta} = (1,-1,1,-1,1,-1)$ is contained in U_2, the subspace V_δ coincides with U_2, and Theorem 3.3 implies that any solution of (44) has the form

$$Q(z) = A(T_6(z)) + B(z),$$

where A is an arbitrary polynomial and B is a polynomial such that

$$B(T_{3,1}^{-1}(z)) + B(T_{3,2}^{-1}(z)) + B(T_{3,3}^{-1}(z)) \equiv 0.$$

4. Vanishing of hyperelliptic Abelian integrals

Let $f \in \mathbb{C}[x]$ be a polynomial and $\Gamma_t = \{(x,y) \in \mathbb{C}^2 : y^2 - f(x) = t\}$ a family of hyperelliptic curves. Consider the Abelian integral

$$I(t) = \int_{\gamma(t)} \omega \tag{45}$$

where $\omega = P(x,y)dx + Q(x,y)dy$ is a polynomial one form, and $\gamma(t) \in H_1(\Gamma_t, \mathbb{Z})$ is a continuous family of 1-cycles.

The purpose of this section is to determine necessary and sufficient conditions for the Abelian integral I to be single valued, polynomial, or rational function. These three conditions are in fact equivalent. Indeed, $I(t)$ is a function of moderate growth, with a bounded modulus in any sector, centered at a singularity. Thus $I(t)$ is single-valued if and only if it is a rational, in fact polynomial function.

4.1. Reduction to the moment problem. The derivatives of I can be seen as moments on a Riemann surface and this permits to apply the results of the preceding section. Indeed, every polynomial one-form ω can be written as

$$\omega = k(x)ydx + dA + Bd(y^2 - f(x)), \quad A, B \in \mathbb{C}[x, y],\ k \in \mathbb{C}[x].$$

Therefore,

$$I(t) = \int_{\gamma(t)} k(x)ydx, \quad I'(t) = \frac{1}{2}\int_{\gamma(t)} \frac{k(x)dx}{y} \tag{46}$$

and more generally

$$I^{(k+1)}(t) = (1/2)(-1/2)(-3/2)\dots(-k+1/2)\int_{\gamma(t)} \frac{k(x)}{y^{2k+1}}dx, \quad k \geq 0. \tag{47}$$

Thus,

$$I^{(k+1)}(0) = m_k = (1/2)(-1/2)(-3/2)\dots(-k+1/2)\int_{\gamma(0)} g^{k+1}\omega$$

where

$$\omega = k(x)ydx, \quad g = \frac{1}{f},$$

implying that the Abelian integral $I(t)$ vanishes identically if and only if the moments $\int_{\gamma(0)} g^k\omega$, $k \geq 0$, vanish. Furthermore, if we replace ω by $g^k\omega$, then, for k sufficiently big, the set of poles of $g^k\omega$ will be a subset of the set of poles of g and the results of Section 1 apply.

The zero-dimensional integrals described in Theorem 2.1 take the form

$$\varphi_i(z) = \int_{\delta_i} \frac{\omega}{f^k df} = \int_{\delta_i(z)} \frac{k(x)}{f^{(2k-1)/2}f'(x)} = z^{-(2k-1)/2}\frac{d}{dz}\int_{\delta_i(z)} K(x)$$

where $K(x) = \int k(x)dx$ is a primitive of k, and the zero-cycles δ_i are constructed from the constellation $\lambda_f = f^{-1}(S)$ as explained in Section 2. The above gives necessary and sufficient conditions for the moments m_i, $i \geq k$ to vanish or, equivalently, for $I(t)$ to be a polynomial. Thus, we have proved

Theorem 4.1. *The Abelian integral (45) is a rational function if and only if the zero-dimensional integrals*

$$\int_{\delta_i(z)} K(x), \quad i = 1, 2, \ldots, k,$$

are identically constant.

Remark. Consider the polynomial $f(x) = (x^2/2 - 1)^2$ as in Figure 8 bellow and the family of 1-cycles $\gamma(t)$, represented on the x-plane by a big loop surrounding the four roots of $f(x) + t$, on the family of elliptic curves Γ_t. Further, consider the complete elliptic integral $I(t) = \int_{\gamma(t)} xy\,dx$. A simple computation shows that the associated zero-dimensional Abelian integral is $\int_{\delta(z)} x^2$, where $\delta(z) = x_1(z) + x_2(z) + x_3(z) + x_4(z)$, $f(x_i(z)) \equiv z$. On the other hand

$$I'(t) = \int_{\gamma(t)} \frac{x\,dx}{2y}$$

is a complete elliptic integral of third kind, and $\gamma(t)$ is homologous to a small loop around one of the two "infinite" point of the affine curve Γ_t. The conclusion is that $I'(t)$ is a residue, in fact a non zero constant. The Abelian integral $I(t)$ is therefore linear in t. This example shows that the claim of Theorem 4.1 can not be improved.

Note that the zero-cycles $\delta_i(z)$ are by no means unique, they depend on the mutual position of the segments $[c_0, c_i]$. If all the zero-cycles $\delta_i(z)$ are in the orbit of a given cycle δ_{i_0}, obtained after a continuation with respect to z, then the vanishing of φ_{i_0} implies the vanishing of all the φ_i, and hence of all the moments. Finally, the orbit of a given δ_{i_0} may contain other cycles, more suitable for our purposes. In the next subsection we propose an alternative construction of such a cycle, by using a residue calculus. As we shall see, this will be more natural for the applications.

4.2. The Cauchy integral related to I. In this section we give an alternative computation of a convenient necessary condition for the identical vanishing of the Abelian integral $I(t)$, defined in (45), (46). Our result will hold under the additional assumption that there is a path along which the cycle $\gamma(t)$ vanishes. More precisely, let $\gamma(t) \subset \Gamma_t$ be a continuous family of closed continuous curves defined in a neighborhood of some regular value t_0 of f. Consider a path

$$[0, 1] \to \mathbb{C} : s \mapsto t(s) \tag{48}$$

such that $s(0) = t_0$, $s(1) = t_1$, $t(s)$ is a regular value of f for $0 \le s < 1$, and t_1 is a singular value of f. We shall say that the continuous family of closed loops $\gamma(t)$ vanishes along the path (48) if it can be extended to a continuous family of loops along this path such that $\gamma(t_1)$ is homologous to zero on the singular affine curve Γ_{t_1}. This implies in particular that $I(t_1) = 0$ as well as that the corresponding zero-dimensional Abelian integral vanishes at t_1.

Without loss of generality we suppose that y restricted to $\gamma(t_0)$ does not vanish. Then, for all (t, z) such that $|z|$ and $|t - t_0|$ are sufficiently small, the Cauchy type integral

$$J_t(z) = \int_{\gamma(t)} \frac{k(x)y}{y^2 - z} dx, \quad z \sim 0 \tag{49}$$

is well defined and analytic in t, z. The definition of $J_t(z)$ is illustrated in Figure 4, where a closed loop $\gamma(t)$ projected on the x-plane is shown, which makes one turn around two roots of the polynomial $f(x) + t$. The roots of $f(x) + t$ are represented

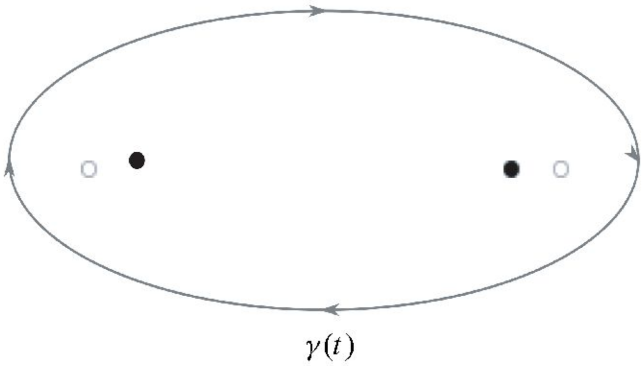

Figure 4. The definition of the Cauchy type integral $J_t(z)$.

by small black bullet circles, while the roots of $f(x) + t - z$ by small empty circles. Note that in Section 2 we supposed that $z \sim \infty$, while in this section that $z \sim 0$, and this is essential for what follows.

Since $I'(t) = \frac{1}{2} J_t(0)$, $J_t(z)$ is a deformation of the Abelian integral $I'(t)$. At the same time, for a fixed t, $J_t(z)$ is a generating function of the moments $I^k(t)$, $k \geq 1$, in the sense of Section 2 and

$$J_t(z) = J(z),$$

where the Riemann surface $R = \Gamma_t$ depends on the parameter t. For $(x, y) \in \gamma(t)$ and $|z|$ sufficiently small the series

$$\sum_{k=0}^{\infty} \left(\frac{z}{y^2}\right)^k$$

converges uniformly and hence

$$\begin{aligned} J_t(z) &= \int_{\gamma(t)} \frac{k(x)y}{y^2(1 - \frac{z}{y^2})} dx \\ &= \int_{\gamma(t)} \frac{k(x)}{y} dx + z \int_{\gamma_0} \frac{k(x)}{y^3} dx + z^2 \int_{\gamma_0} \frac{k(x)}{y^5} dx + \cdots . \end{aligned}$$

Taking into consideration (47) we conclude

Proposition 4.1. *For every regular value t of f the equalities*

$$\binom{-1/2}{k}\frac{d^k}{dz^k}J_t(0) = 2I^{(k+1)}(t), \quad k = 0, 1, 2\dots$$

hold.

The above proposition implies the following corollary.

Corollary 4.1. *The Abelian integral $I'(t)$ vanishes identically, if and only if the Cauchy type integral $J_t(z)$ vanishes identically.*

The main advantage of using $J_t(z)$ instead of $I'(t)$ is the possibility to extend it analytically with respect to z. The result is a function algebraic in z.

Proposition 4.2 ([16]). *For every fixed regular value t the Cauchy type integral $J_t(z)$ extends to an algebraic function in z with singularities at $z = 0$ and at the critical values of f.*

Indeed, for a fixed regular t, $J_t(z)$ allows for an analytic continuation along any path which does not contain critical values of $f - t$ or the value $z = 0$. In a neighborhood of a critical value of $f - t$ or at $z = 0$, the Cauchy theorem implies that, up to an addition of a holomorphic function, $J_t(z)$ is a linear combination of residues of $\frac{k(x)y}{y^2-z}dx$ at the roots $f_i^{-1}(z-t)$ of $f(x)+t-z$. Thus, $J_t(z)$ is a function of moderate growth in z with a finite number of branches, and hence is algebraic in z. □

Our next goal is to extend analytically $J_t(z)$ in a neighborhood of $(t_1, 0)$ under the condition that t_1 is a critical value of f. To simplify the notation put $t_1 = 0$, $f(0) = 0$. Consider the domain

$$D_\delta = \{(t, z) : |t| < \delta, |z| < \delta,\ t \neq z,\ t \neq 0,\ z \neq 0\}$$

and assume that $\delta > 0$ is so small that $t = 0$ is the only critical value of f in the disc $\{t : |t| < 2\delta\}$. Take some $(t, 0) \in D_\delta$ and consider the germ of the analytic function $J = J_t(z)$ in a neighborhood of this point.

Proposition 4.3. *The germ of $J = J_t(z)$ at $(t, 0) \in D_\delta$ allows for an analytic continuation along any path starting at $(t, 0)$ and contained in D_δ.*

Indeed, the affine curve Γ_t is regular, provided that $t \neq 0$, and the differential $\frac{k(x)y}{y^2-z}dx$ has simple poles if and only if $z \neq t, z \neq 0$. Therefore the closed curve $\gamma(t)$ can be deformed in a way to avoid these simple poles. □

Although the function $J_t(z)$ might be not analytic in t near the line $t = 0$, it has a finite limit there which we compute next. For this purpose, let l be a closed smooth path connecting the point $(t, 0)$ to $(0, z)$, $t, z \neq 0$, and contained in D_δ (except the ends), see Figure 5. Suppose that the homology class of the limiting loop $\gamma(0) \subset \Gamma_0$ is zero and hence is a linear combination of vanishing cycles.

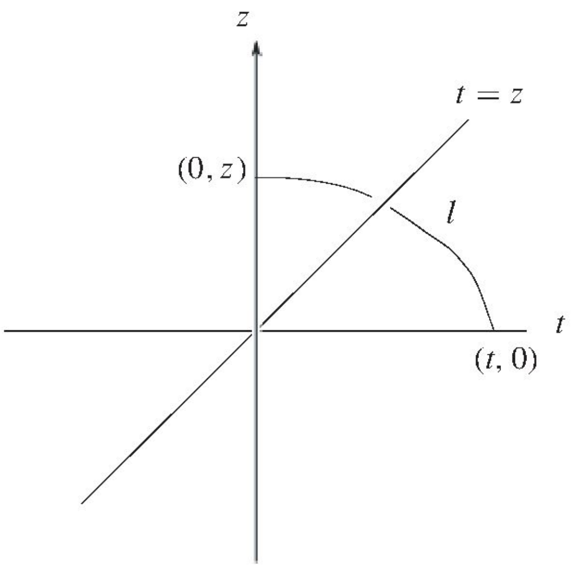

Figure 5. The domain D_δ.

Theorem 4.2. *If $\gamma(0) \subset \Gamma_0$ is homologous to zero, then the limiting value of $J_t(z)$ at $(0, z)$ along l is a zero-dimensional Abelian integral*

$$J_0(z) = 2\pi\sqrt{-z}\frac{d}{dz}\int_{\delta(z)} K(x)$$

where $K(x)$ is a primitive of $k(x)$, $\delta(z) = \sum_i n_i f_i^{-1}(z)$, $f_i^{-1}(z)$ are the roots of the polynomial $f(x) - z$, and the numbers n_i depend only on the homology class represented by the loop $\gamma(0)$ in $H_1(\check{\Gamma}_0, \mathbb{Z})$, $\check{\Gamma}_0 = \{(x, y) : y^2 = f(x), f(x) \neq z\}$.

Corollary 4.2. *If $I(t) = \int_{\gamma(t)} \omega \equiv 0$ then $\int_{\delta(z)} K \equiv 0$*

Corollary 4.3. *According to Proposition 4.1, if $I'(t) = 0$ for some regular t, then the multiplicity of this zero is the same as the multiplicity of $J_t(z)$ with respect to z at $z = 0$. In the particular case where $t = 0$ is a Morse critical point, the Abelian integral I' is analytic at $t = 0$, and the multiplicity of the zero of I' at $t = 0$ is just the multiplicity of the zero of the analytic function $J_0(z)$ at $z = 0$. Thus, the multiplicity of the one-dimensional Abelian integral at a Morse critical point equals essentially the corresponding multiplicity of the one-dimensional Abelian integral.*

Proof of Theorem 4.2. We can deform the loop $\gamma(t)$ along the interior of the path l in a way to avoid the poles of $\frac{k(x)y}{y^2-z}dx$. Taking the limit $t \to 0$ along l we obtain that $\gamma(0)$ is homologous to a sum of closed loops around the poles of $\frac{k(x)y}{y^2-z}dx$, as it is shown in Figure 6.

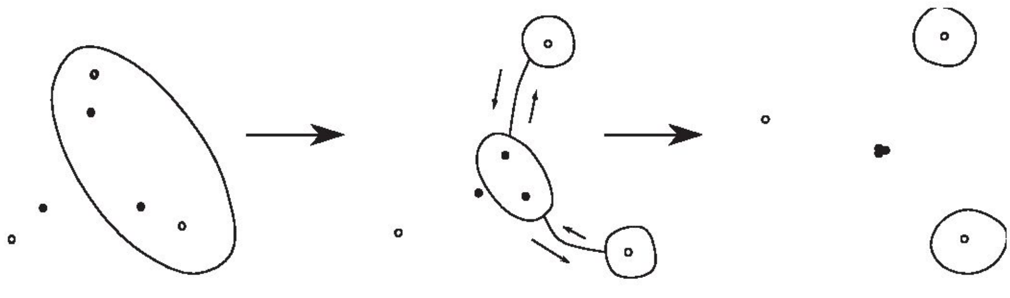

Figure 6. Computing the limit of $\gamma(t)$ at a singular value.

Therefore,

$$\begin{aligned}
J_0(z) &= 2\pi\sqrt{-1}\sum_i n_i \mathrm{Res}_{f_i^{-1}(z)} \frac{k(x)y}{f(x)-z}dx \\
&= 2\pi\sqrt{-z}\sum_i \pm n_i \frac{k(f_i^{-1}(z))}{f'(f_i^{-1}(z))} \\
&= 2\pi\sqrt{-z}\int_{\delta(z)} \frac{k(x)}{f'(x)} \\
&= 2\pi\sqrt{-z}\frac{d}{dz}\int_{\delta(z)} K(x). \qquad \square
\end{aligned}$$

Computation of the reduced 0-cycle $\delta(z)$. For simplicity, suppose that $\gamma(t)$ vanishes as t tends to 0 at the origin $(0,0)$. Thus $\gamma(t)$ is a linear combination of cycles vanishing at $(0,0)$. The standard basis of such cycles can be described as follows. Let $f_i^{-1}(t)$, $i = 1,2,\dots,n$, be the roots of the polynomial $f(x)+t$ which tend to 0 as t tends to 0, ordered cyclically with respect to the monodromy action. We denote by $\gamma_{ij}(t) \subset \Gamma_t$ a simple closed loop which is projected to the segment $[f_i^{-1}(t), f_j^{-1}(t)]$. The loops $\gamma_{i,i+1}(t)$, $i = 1,2,\dots,n-1$, form a basis of the local homology group of the Milnor fiber of $y^2 - f(x)$. We fix the orientations of these cycles by the convention

$$\gamma_{i,i+1} \cdot \gamma_{i+1,i+2}(t) = 1.$$

It is easy to check that then

$$\gamma_{i,i+1}(t) + \gamma_{i+1,i+2}(t) = \gamma_{i,i+2}(t),$$

where the orientation of $\gamma_{i,i+2}(t)$ is appropriately chosen. Therefore the orientations of the remaining cycles can be chosen to satisfy

$$\gamma_{ij} \circ \gamma_{jk} = +1, \gamma_{ij} + \gamma_{jk} = \gamma_{ik} \quad \text{for all } i < j < k. \tag{50}$$

As a by product we have also

$$\gamma_{1,2} + \gamma_{2,3} + \cdots + \gamma_{n,1} = 0.$$

Obviously this fixes the orientation of all cycles γ_{ij} up to simultaneous multiplication by -1, which have no incidence on the result claimed in Corollary 4.2. The standard basis of vanishing cycles of the singularity $y^2 + x^5$ is shown in Figure 7. We shall

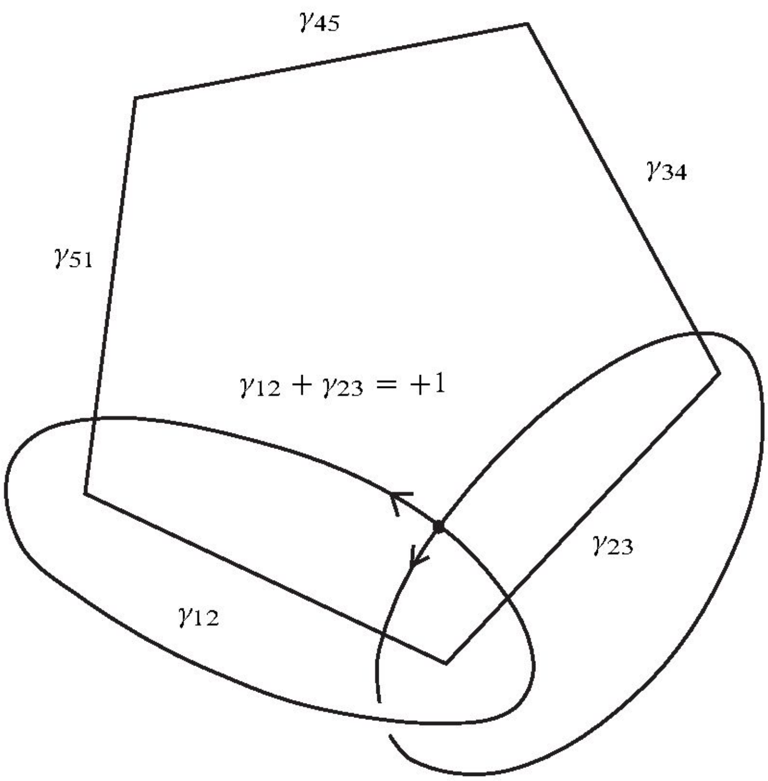

Figure 7. The standard basis of vanishing cycles of the singularity $y^2 + x^5$.

construct an isomorphism

$$\begin{aligned} H_1(\Gamma_t, \mathbb{Z}) &\to \widetilde{H}_0(f^{-1}(z), \mathbb{Z}), \\ \gamma(t) &\mapsto \delta(z), \end{aligned}$$

having the property announced in Corollary 4.2. According to the proof of Theorem 4.2 this should be a linear map which associates to the one-cycle $\gamma_{ij}(t)$ the reduced 0-cycle (see also Section 2) $\delta_{ij}(t) = \pm(f_i^{-1}(t) - f_j^{-1}(t))$ and should be therefore compatible to the relations

$$\gamma_{ij} + \gamma_{jk} = \gamma_{ik}, \quad \delta_{ij} + \delta_{jk} = \delta_{ik}.$$

It follows that the orientation of the 0-cycles $\delta_{ij}(t)$ can be fixed as

$$\delta_{ij}(t) = f_i^{-1}(t) - f_j^{-1}(t) \quad \text{for all } i < j.$$

Note that the above isomorphism is not compatible to the monodromy action.

In conclusion, if

$$\gamma(t) = \sum n_{ij}\gamma_{ij}(t)$$

and $I(t) = \int_{\gamma(t)} \omega \equiv 0$ then $\int_{\delta(z)} K \equiv 0$ where

$$\delta(z) = \sum n_{ij}(f_i^{-1}(z) - f_j^{-1}(z)).$$

4.3. Hyperelliptic Abelian integrals along ovals. Let $f(x) \in \mathbb{R}[x]$ be an arbitrary non-linear real polynomial. Consider a family of ovals $\{\gamma(t)\}_t$

$$\gamma(t) \subset \{(x,y) \in \mathbb{R}^2 : y^2 - f(x) = t\}, \quad t \in \mathbb{R},$$

depending continuously on the real parameter t. Each oval $\gamma(t)$ can be parameterized as

$$y = \pm\sqrt{f(x)+t}, \quad x_1(t) \le x \le x_2(t),$$

where $x_1(t) < x_2(t)$ are two real roots of $f(x)+t$. The purpose of this last section is to solve, by making use of Theorem 4.1 and Theorem 4.2, the following problem: under what conditions the Abelian integral (46),

$$I(t) = \int_{\gamma(t)} k(x)y\,dx = 2\int_{x_1(t)}^{x_2(t)} k(x)y\,dx,$$

is identically zero?

Theorem 4.3. *The integral $I(t)$ vanishes identically if and only if there exists a polynomial $r \in \mathbb{R}[x]$, such that both f and $K = \int k$ are polynomials in r, and $r(x_1(t)) \equiv r(x_2(t))$.*

Proof. First of all, note that if K and f have a right compositional factor identifying $x_1(t)$ and $x_2(t)$, then the Abelian integral $\int_{\gamma(t)} k(x)y\,dx$ is a pull back of an integral along a cycle homologous to zero, and hence vanishes identically.

Suppose further that $I(t)$ vanishes identically. It is enough to show that this implies $K(x_1(t)) \equiv K(x_2(t))$ since in this case by Proposition 2.1 (or by the Lüroth theorem) f and K will have a right compositional factor identifying $x_1(t)$ and $x_2(t)$. If there exists a path on the complex t-plane along which the cycle $\gamma(t)$ vanishes, then Theorem 4.2 applies and we conclude that $K(x_1(t)) \equiv K(x_2(t))$.

As an example, consider a real polynomial f of degree $n \geq 2k$, $f = -x^{2k} + \cdots$. Let $x_1(t) < x_2(t)$ be the two real roots of $f(x) + t$ which tend to 0 as t tends to zero and $\{\gamma(t)\}$ be the continuous family of ovals vanishing at the origin as t tends to zero.

$$\gamma(t) \subset \{(x, y) \in \mathbb{R}^2 : y^2 + x^{2k} + \cdots = t\}$$

Then Theorem 4.2 applies and hence the result of Theorem 4.3 follows. In the Morse case ($k = 1$), this has been proved by Christopher and Mardesic [4].

The condition that $\gamma(t)$ vanishes along a suitable path is essential, and holds for arbitrary real polynomials of degree four or five, see for instance [6], Section 3.1, where the case $f(x) = (x^2-1)^2$ is studied. We do not know whether this condition is fulfilled for arbitrary polynomial f and family of ovals $\gamma(t)$. See Figure 8 (continuous families of ovals). However, using Theorem 4.1 instead of Theorem 4.2 we can prove the theorem in its full generality.

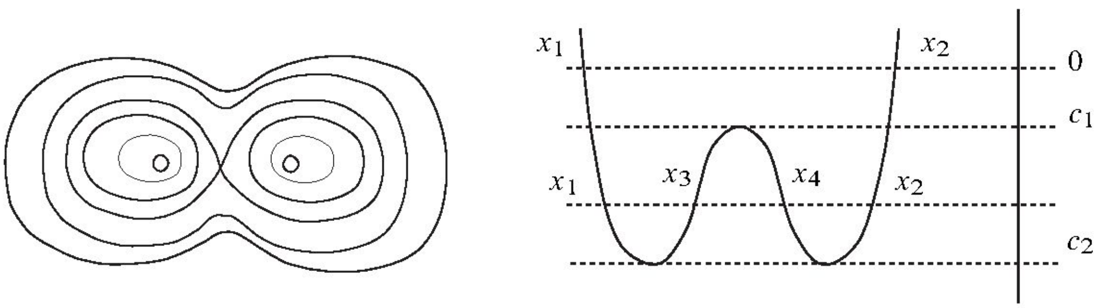

Figure 8. The continuous families of ovals of $y^2 + (x^2/2-1)^2$ and the graph of $(x^2/2-1)^2 - 1$.

Indeed, let f be an arbitrary real polynomial of degree $n > 1$ and $I(t)$ be an identically vanishing Abelian integral as before. Let us apply Theorem 4.1. For this purpose, let us fix a regular real value t of f, and consider the moment problem associated to the oval $\gamma(t)$ on the Riemann surface Γ_t. Following the method described in Section 2 we have to consider a constellation $\lambda_f \subset \mathbb{P}^1$ and to deform the image of $\gamma(t)$ under $f + t$ on λ_f. The closed loop $\gamma(t)$ being an oval, its image is just a real interval connecting 0 to a critical value c_k of $f + t$. Suppose for instance that $0 > c_1 > c_2 > \cdots > c_k$ are the remaining critical values of $f + t$ contained in $[c_k, 0]$. We have therefore

$$[c_k, 0] = [c_k, c_{k-1}] \cup \cdots \cup [c_1, 0]$$

which, without loss of generality, will be used on the place of the constellation λ_f. To each segment $[c_{i-1}, c_i]$ we associate a 0-cycle δ_i and $I(t)$ is a rational function if and only if $\int_{\delta_i(z)} K \equiv 0$. Fortunately in general we do not need to compute all of δ_i. We note that the image of $[x_1(t), x_2(t)]$ is a closed curve covering $[c_k, 0]$. The pre-image of each point $z \in (0, c_1)$ consists of two points $x_1(z)$ and $x_2(z)$ (roots of $f(x)+t-z$) and hence $\delta_1(z) = x_1(z) - x_2(z)$. We conclude that $K(x_1(t)) \equiv K(x_2(t))$ which completes the proof of Theorem 4.3. □

Example. The critical values of the polynomial $(x^2/2-1)^2-1$ are -1 and $-3/4$. The relevant constellation associated to the exterior family of ovals shown in Figure 8 is $[-1,-3/4] \cup [-3/4, 0]$. To the segment $[-3/4, 0]$ we associate the 0-cycle $\delta_1(z) = x_1(z) - x_2(z)$ and to the segment $[-1,-3/4]$ the 0-cycle $\delta_2(z) = x_1(z) - x_3(z) + x_4(z) - x_2(z)$.

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References

[1] A. Alvarez, J. Bravo, and P. Mardesic, Vanishing abelian integrals on zero-dimensional cycles. *Proc. Lond. Math. Soc.* (3) **107** (2013), no. 6, 1302–1330. Zbl 1279.30045

[2] M. Briskin, J.-P. Françoise, and Y. Yomdin, Center conditions, compositions of polynomials and moments on algebraic curve. *Ergodic Theory Dynam. Systems* **19** (1999), no. 5, 1201–1220. Zbl 0990.34017 MR 1721616

[3] M. Briskin, N. Roytvarf, and Y. Yomdin, Center conditions at infinity for Abel differential equations. *Ann. of Math.* (2) **172** (2010), no. 1, 437–483. Zbl 1216.34025 MR 2680423

[4] C. Christopher and P. Mardeshich, The monodromy problem and the tangential focus problem. *Funktsional. Anal. i Prilozhen.* **44** (2010), no. 1, 27–43. Zbl 1271.34035 MR 2656383

[5] L. Gavrilov, Higher order Poincaré–Pontryagin functions and iterated path integrals. *Ann. Fac. Sci. Toulouse Math.* (6) **14** (2005), no. 4, 663–682. Zbl 1104.34024 MR 2188587

[6] L. Gavrilov and I. D. Iliev, The displacement map associated to polynomial unfoldings of planar Hamiltonian vector fields. *Amer. J. Math.* **127** (2005), 1153–1190. Zbl 1093.34015 MR R2183522

[7] L. Gavrilov and H. Movasati, The infinitesimal 16th Hilbert problem in dimension zero. *Bull. Sci. Math.* **131** (2007), no. 3, 242–257. Zbl 1119.14040 MR 2309864

[8] L. Gavrilov, H. Movasati, and I. Nakai, On the non-persistence of Hamiltonian identity cycles. *J. Differential Equations* **246** (2009), no. 7, 2706–2723. Zbl 1175.37052 MR 2503019

[9] S. Lando and A. Zvonkin, *Graphs on surfaces and their applications.* Low-Dimensional Topology, II, Encyclopaedia Math. Sci. 141, Springer, Berlin 2004. Zbl 1040.05001 MR 2036721

[10] F. Pakovich, On the polynomial moment problem. *Math. Res. Lett.* **10** (2003), 401–410. Zbl 1043.30021 MR 1983032

[11] F. Pakovich, On polynomials orthogonal to all powers of a Chebyshev polynomial on a segment. *Israel J. Math.* **142** (2004), 273–283 Zbl 1058.30007 MR 2085720

[12] F. Pakovich, On polynomials orthogonal to all powers of a given polynomial on a segment. *Bull. Sci. Math.* **129** (2005), 749–774. Zbl 1089.30039 MR 2172140

[13] F. Pakovich, On rational functions orthogonal to all powers of a given rational function on a curve. *Moscow Math. J.* **13** (2013), no. 4, 693–731.

[14] F. Pakovich, Generalized 'second Ritt theorem' and explicit solution of the polynomial moment problem. *Compos. Math.* **149** (2013), no. 4, 705–728. Zbl 06165763 MR 3049701

[15] F. Pakovich and M. Muzychuk, Solution of the polynomial moment problem. *Proc. London Math. Soc.* **99** (2009), no. 3, 633–657. Zbl 1177.30046 MR 2551466

[16] F. Pakovich, N. Roytvarf, and Y. Yomdin, Cauchy-type integrals of algebraic functions. *Israel J. Math.* **144** (2004), 221–291. Zbl 1078.30034 MR 2121542

[17] N. Roytvarf, Generalized moments, composition of polynomials and Bernstein classes. In *Entire functions in modern analysis* (Tel-Aviv, 1997), Israel Math. Conf. Proc. 15, Bar-Ilan University, Ramat Gan 2001, 339–355. Zbl 0996.30025 MR 1890547

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