

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 88 (2013)

Artikel: Injective modules and amenable groups
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DOI: <https://doi.org/10.5169/seals-515663>

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Injective modules and amenable groups

Gerhard Racher

Abstract. We show that a locally compact group is amenable if and only if it admits a (non-zero) injective Banach module that is reflexive as a Banach space.

Mathematics Subject Classification (2010). 43A07, 46H25, 18G05.

Keywords. Amenable groups, injective Banach modules, weak compactness.

1. Introduction

Let A be a Banach algebra. By a left A -module we shall always mean a Banach left A -module satisfying $\|ax\| \leq \|a\| \|x\|$ whenever $a \in A$ and $x \in X$, and a morphism of left A -modules will be a bounded linear map commuting with the respective actions. X is called injective, cf. [H], III.1.14, p. 136, if for any morphism ι of left A -modules admitting a bounded linear left inverse, ℓ , and any morphism λ_0 from Y_0 into X , there is a morphism λ from Y into X satisfying $\lambda_0 = \lambda \circ \iota$,

$$\begin{array}{ccc} Y_0 & \xrightarrow{\iota} & Y \xrightarrow{\ell} Y_0, \\ \lambda_0 \downarrow & \swarrow \lambda & \\ X & & \end{array} \quad \ell \circ \iota = \text{id}_{Y_0}.$$

Let the essential part, X_e , of a left A -module X be defined as the closed linear hull of the set of products ax , $a \in A$, $x \in X$. X is called non-zero if $X_e \neq 0$, essential if $X_e = X$, and reflexive if X is reflexive as a Banach space. In case X is reflexive and A has a bounded two-sided approximate unit (of norm $\leq c$), there is an A -module morphism (of norm $\leq c$) projecting X onto X_e . The Banach space dual, X^* , of X becomes a right A -module under the action defined by $\langle x, x^*a \rangle = \langle ax, x^* \rangle$, for $x^* \in X^*$, $a \in A$, and $x \in X$.

Choosing a left invariant Haar measure on the locally compact group G we obtain the Banach algebra $L^1(G)$. It is well known that every essential left $L^1(G)$ -module is a left G -module such that, for any $x \in X$, the mapping $s \mapsto sx$ is continuous from G into X and $\|sx\| = \|x\|$, $s \in G$, the respective actions being related by the

formula $ax = \int a(s)sx ds$, for $a \in L^1(G)$ and $x \in X$. This same formula defines on any such left G -module an essential left $L^1(G)$ -action.

Letting G act by left translation on $L^p(G)$, $1 < p < \infty$, $L^p(G)$ becomes an essential reflexive left $L^1(G)$ -module. H. G. Dales, M. Daws, H. L. Pham and P. Ramsden recently showed the following theorem, [DDPR], Theorem 9.6.

Theorem ([DDPR]). *Let G be a locally compact group, and $1 < p < \infty$. If the left $L^1(G)$ -module $L^p(G)$ is injective, then G is amenable.* \square

Employing F. J. Yeadon's method, [Y], for establishing the existence of a trace in a finite von Neumann algebra, we show

Proposition. *Let G be a locally compact group. If G admits a non-zero injective Banach left $L^1(G)$ -module that is reflexive as a Banach space, then G is amenable.*

Combining this with known results we obtain the following characterization of compact and amenable groups, in good correspondence with Helemskii's philosophy, cf. e.g. [H], p. 262.

Corollary. *Let G be a locally compact group.*

- a) *If G admits a non-zero projective left $L^1(G)$ -module that is reflexive as a Banach space, then G is compact; if, conversely, G is compact then every essential left $L^1(G)$ -module is projective.*
- b) *If G admits a non-zero flat left $L^1(G)$ -module that is reflexive as a Banach space, then G is amenable; if, conversely, G is amenable then every left $L^1(G)$ -module is flat.*

These results are equally valid for uniformly bounded, left or right Banach $L^1(G)$ -modules. For the notion of the injective tensor product, $\check{\otimes}$, of Banach spaces we refer to the monograph of J. Cigler, V. Losert and P. Michor, [CLM]. The proof of the Proposition starts immediately after this introduction.

2. The auxiliary module $C^{bu}(G) \check{\otimes} X$

The G -action on $C^{bu}(G) \check{\otimes} X$ and the morphism ι below were already considered by P. Ramsden, [Ra], Chapter 5, p. 21; cf. also Chapter 9 of [DDPR].

2.1. Let G be a locally compact group, and X be an essential Banach left $L^1(G)$ -module, with $sx, s \in G, x \in X$, denoting the corresponding G -action. We let G act on the Banach space, $C^{bu}(G)$, of uniformly continuous bounded functions on G by left translation $(L_s\varphi)(t) = \varphi(s^{-1}t)$, $s \in G, \varphi \in C^{bu}(G)$, so that the injective tensor

product $C^{bu}(G) \check{\otimes} X$ becomes a continuous isometric Banach left G -module under the action $s(\varphi \otimes x) = L_s \varphi \otimes sx$, $s \in G$, $\varphi \otimes x \in C^{bu}(G) \check{\otimes} X$.

The morphism $\iota: X \rightarrow C^{bu}(G) \check{\otimes} X$ is defined by $\iota x = 1_G \otimes x$, $x \in X$, 1_G the function constant one on G , and for any $s \in G$ the bounded linear map $\ell: C^{bu}(G) \check{\otimes} X \rightarrow X$, $\ell(\varphi \otimes x) = \varphi(s)x$, $\varphi \in C^{bu}(G)$, $x \in X$, is left inverse to ι .

In case the essential left $L^1(G)$ -module X is injective, setting $Y_0 = X$, $Y = C^{bu}(G) \check{\otimes} X$, and $\lambda_0 = \text{id}_X$ in the diagram on p. 1023 yields a morphism λ of $L^1(G)$ -modules left inverse to ι ,

$$X \xrightarrow{\iota} C^{bu}(G) \check{\otimes} X \xrightarrow{\lambda} X.$$

Since λ commutes also with the G -actions, the map λ enjoys the following properties:

- (i) λ is linear and bounded;
- (ii) $\lambda(L_s \varphi \otimes sx) = s\lambda(\varphi \otimes x)$;
- (iii) $\lambda(1_G \otimes x) = 1$,

whenever $s \in G$, $\varphi \in C^{bu}(G)$, and $x \in X$.

2.2. Remark Instead of $C^{bu}(G)$ we could also take $L^\infty(G)$, Corollary 3.7 below equally applying to it. By using the module $C^{bu}(G) \check{\otimes} X$, suggested by the referee, however, we shall obtain: *If an arbitrary topological group G admits a non-zero relatively injective Banach left G -module X that is reflexive as a Banach space, then G is amenable.* For the relevant notions we refer to N. Monod's Lecture Notes, [M], Definition 4.1.2, p. 32, and the definition preceding 5.1.4, p. 46.

3. Weakly compact operators on $C(K) \check{\otimes} X$

The formulation of the main lemma, (3.5) below, is due to the referee.

3.1. Let K be a compact Hausdorff space, and X be a Banach space. It is well known that the dual space of the injective tensor product $C(K) \check{\otimes} X = C(K, X)$ is isometrically isomorphic to the Banach space, $I(C(K), X^*)$, of integral operators v from $C(K)$ into X^* , and that this again is isometrically isomorphic to the Banach space, $bvaca(B(K), X^*)$, of regular countably additive vector measures m of bounded variation on the Borel σ -algebra, $B(K)$, of K with values in X^* ,

$$(C(K) \check{\otimes} X)^* = I(C(K), X^*) = bvaca(B(K), X^*),$$

the correspondence between v and m being given by $m(A) = \tilde{v}(c_A)$, $A \in B(K)$, where $\tilde{v}: C(K)^{**} \rightarrow X^*$ denotes the unique weak*-weak* continuous extension of v

and c_A the characteristic function of A . The variation, $|m|$, of $m \in bvrca(B(K), X^*)$, defined as

$$|m|(A) = \sup \sum \|m(A_i)\| \quad (A \in B(K)),$$

the supremum being taken over all finite Borel partitions (A_i) of A , is a regular finite positive Borel measure on K . Defining the norm of $m \in bvrca(B(K), X^*)$ by $\|m\| = |m|(K)$, we have $\|m\| = I(v)$, the integral norm of $v \in I(C(K), X^*)$ corresponding to m . – The theorems involved in this discussion are due to I. Singer, [S]; cf. also VI.3.Theorem 3, p. 162, and VI.3.Theorem 12, p. 169, in [DU], and, in particular, Satz 1 in Losert's Thesis, [L], p. 7.

We shall need the following two lemmas.

3.2 Lemma ([Gro], Théorème 2). *Let K be a compact Hausdorff space. A bounded subset C of $C(K)^*$ is relatively weakly compact if and only if for every sequence (A_n) of pairwise disjoint open subsets of K we have*

$$\lim_n \mu(A_n) = 0$$

uniformly for μ in C . □

3.3 Lemma. *Let K be a compact Hausdorff space, and X be a Banach space. If D is a relatively weakly compact subset of $(C(K) \otimes X)^*$, then the set, $|D|$, of variations of its corresponding vector measures is relatively weakly compact in $C(K)^*$.*

Proof. Let D be a relatively weakly compact subset of $(C(K) \otimes X)^*$. Using the identification in (3.1), we may assume D to be relatively weakly compact in $bvrca(B(K), X^*)$; being a closed subspace of the Banach space $bvca(B(K), X^*)$ of all countably additive measures of bounded variation, it is relatively weakly compact also there. Theorem 1.ii) in [B], p. 288, yields a finite positive measure v on $B(K)$ such that the set $|D| = \{|m| : m \in D\}$ is v -equicontinuous. For any sequence (A_n) of disjoint Borel subsets of K , $\lim v(A_n) = 0$ therefore implies $\lim |m|(A_n) = 0$ uniformly for m in D . The elements of the set $|D|$ being all regular, its relative weak compactness in $C(K)^*$ results now, for instance, from (3.2). □

3.4. Let X and Y be Banach spaces, and $u: C(K) \otimes X \rightarrow Y$ a bounded linear map with adjoint $u^*: Y^* \rightarrow (C(K) \otimes X)^* = I(C(K), X^*)$. Any pair of elements (x, y^*) in $X \times Y^*$ defines an element u_{x,y^*} of $C(K)^*$ by

$$u_{x,y^*}(\varphi) = \langle u(\varphi \otimes x), y^* \rangle, \quad \varphi \in C(K), x \in X, y^* \in Y^*.$$

Denoting by $(u^*y^*)^{\sim}: B(K) \rightarrow X^*$ the vector measure corresponding to $u^*y^*: C(K) \rightarrow X^*$, we deduce from

$$u_{x,y^*}(\varphi) = \langle \varphi \otimes x, u^*y^* \rangle = \langle x, u^*y^*(\varphi) \rangle, \quad \varphi \in C(K),$$

that

$$u_{x,y^*}(A) = \langle x, (u^* y^*)^\sim(A) \rangle, \quad A \in B(K),$$

for all $x \in X$, $y^* \in Y^*$.

3.5 Lemma. *Let K be a compact Hausdorff space, X and Y be Banach spaces, and u be a weakly compact linear map from $C(K) \otimes X$ into Y . Then the set*

$$\{u_{x,y^*} : \|x\| \leq 1, \|y^*\| \leq 1\}$$

is relatively weakly compact in $C(K)^$.*

Proof. Let (A_n) be a sequence of pairwise disjoint open subsets of K , and $\varepsilon > 0$. As $u^* : Y^* \rightarrow (C(K) \otimes X)^*$ is equally weakly compact, the image, $u^*(OY^*)$, of the unit ball of Y^* is relatively weakly compact in $(C(K) \otimes X)^*$, and so is the set, $|u^*(OY^*)|$, of variations of its corresponding vector measures in $C(K)^*$, by (3.3). Lemma (3.2) furnishes an index n_0 such that

$$|(u^* y^*)^\sim|(A_n) \leq \varepsilon \quad (\|y^*\| \leq 1, n \geq n_0),$$

implying, for all $x \in X$ and $y^* \in Y^*$ of norm ≤ 1 ,

$$\begin{aligned} |u_{x,y^*}(A_n)| &= |\langle x, (u^* y^*)^\sim(A_n) \rangle| \\ &\leq \|x\| \|(u^* y^*)^\sim(A_n)\| \\ &\leq |(u^* y^*)^\sim|(A_n) \\ &\leq \varepsilon \quad (n \geq n_0), \end{aligned}$$

thus proving the assertion, again by (3.2). \square

3.6. Each of the following conditions on X and Y assures the weak compactness of any bounded linear map from $C(K) \otimes X$ into Y :

- (a) X is arbitrary and Y reflexive;
- (b) X^* has the Radon–Nikodym property and Y is weakly sequentially complete, cf. [G];
- (c) X is a C^* -algebra and Y is weakly sequentially complete, cf. [ADG], Theorem 4.2, p. 449.

3.7 Corollary. *Let G be a locally compact group, X a reflexive Banach space, and u a bounded linear map from $C^{bu}(G) \otimes X$ into X . Then the set*

$$\{u_{x,x^*} : \|x\| \leq 1, \|x^*\| \leq 1\}$$

is relatively weakly compact in $C^{bu}(G)^$.*

Proof. $C^{bu}(G)$ being a commutative C^* -algebra with unit, there exist a compact Hausdorff space K and an isomorphism from $C^{bu}(G)$ onto $C(K)$ so that (3.5) applies. \square

3.8 Remark (by the referee). In case X is reflexive (and therefore X and X^* enjoy the Radon–Nikodym property), one can deduce (3.5) directly from the vector-valued version of Grothendieck’s criterion (3.2), as stated in the middle of p. 117 in [DU].

4. Proof of the Proposition

Let G be a locally compact group and X a non-zero injective left $L^1(G)$ -module, reflexive as a Banach space. Since $L^1(G)$ possesses bounded approximate units, the essential part of X – being $L^1(G)$ -module complemented in X – is equally injective, and reflexive, so that we may assume X from the outset to be essential itself. Let then $\lambda: C^{bu}(G) \otimes X \rightarrow X$ be a map satisfying (2.1)(i), (ii), (iii). For any fixed pair $(x, x^*) \in X \times X^*$, $\langle x, x^* \rangle = 1$, the element λ_{x,x^*} in $C^{bu}(G)^*$, $\lambda_{x,x^*}(\varphi) = \langle \lambda(\varphi \otimes x), x^* \rangle$, $\varphi \in C^{bu}(G)$, enjoys the following two properties:

- (iv) $\lambda_{x,x^*}(1_G) = 1$;
- (v) $\{L_s^* \lambda_{x,x^*} : s \in G\}$ is relatively weakly compact in $C^{bu}(G)^*$.

(iv) follows immediately from (2.1.iii); to see (v), we use (2.1.ii) to compute, with $\varphi \in C^{bu}(G)$ and $s \in G$,

$$\begin{aligned}
 L_s^* \lambda_{x,x^*}(\varphi) &= \lambda_{x,x^*}(L_s \varphi) \\
 &= \langle \lambda(L_s \varphi \otimes x), x^* \rangle \\
 &= \langle \lambda(L_s \varphi \otimes s s^{-1} x), x^* \rangle \\
 &= \langle s \lambda(\varphi \otimes s^{-1} x), x^* \rangle \\
 &= \langle \lambda(\varphi \otimes s^{-1} x), x^* s \rangle \\
 &= \lambda_{s^{-1} x, x^* s}(\varphi) \quad (s \in G, \varphi \in C^{bu}(G)).
 \end{aligned}$$

Since $\|s^{-1} x\| = \|x\|$ and $\|x^* s\| = \|x^*\|$, $s \in G$, the assertion now follows from (3.7).

It ensues that the closed convex hull, C , of $\{L_s^* \lambda_{x,x^*} : s \in G\}$ is a weakly compact convex subset of $C^{bu}(G)^*$. Being invariant under the group of linear isometries L_s^* , $s \in G$, Ryll-Nardzewski’s fixed point theorem yields an element M of C satisfying $L_s^* M = M$, $s \in G$, and, in virtue of (iv), $M(1_G) = 1$. Decomposing M into its selfadjoint parts and these into their positive ones, we obtain, possibly after rescaling, a positive linear functional on $C^{bu}(G)$, left invariant and taking the value one at the constant function 1_G , thus establishing the amenability of G ; cf. [Gr], Theorem 2.2.1, p. 26. \square

5. Proof of the Corollary

For the definition of projective and flat Banach modules over a Banach algebra we refer to [H], III.1.14, p. 136, and [H], VII.1.2, p. 239, respectively. Rather than reproducing them here, we note only that every projective module is flat, and that a module X is flat if and only if its dual module, X^* , is injective, cf. [H], VII.1.14, p. 243.

5.1. Proof of Corollary a. Let X be a non-zero projective left $L^1(G)$ -module that is reflexive as a Banach space. Since X_e is module-complemented in X , X_e is also projective, and reflexive, so that G is compact, by [R1], 1.4, p. 316. (It is shown there that a locally compact group is already compact, if it admits a non-zero essential projective left $L^1(G)$ -module X whose dual Banach space, X^* , is weakly sequentially complete or norm separable.) The second statement is also proved there, [R1], 1.2, p. 316. \square

The second part of Corollary b is equally well known. In [H], VII.2.29, p. 257, it is deduced from the vanishing of the Tor functor over an amenable algebra, or can be seen, more directly, from B. E. Johnson's original definition, [J], p. 60, as follows.

5.2 Lemma ([H]). *Let A be an amenable Banach algebra. Then all Banach (left, right, or bi-) modules over A are flat.*

Proof. We shall show only that the dual right module, X^* , of a left A -module X is injective. Replacing X with X^* in the diagram defining injectivity on p. 1023, and taking ι and λ_0 as morphisms of right A -modules, we consider $\lambda_0 \circ \ell$ as element of the Banach space, $L(Y, X^*)$, of bounded linear maps from Y into X^* . Turning it into an A -bimodule by $(aT)(y) = T(ya)$ and $(Ta)(y) = (Ty)a$, for $a \in A, T \in L(Y, X^*)$, $y \in Y$, we obtain a bounded linear map $D: A \rightarrow L(Y, X^*)$, $Da = a(\lambda_0 \circ \ell) - (\lambda_0 \circ \ell)a$, $a \in A$, whose values vanish on the closed submodule ιY_0 of Y , thus defining a new map, $D_0: A \rightarrow L(Y/\iota Y_0, X^*)$, by the formula $(D_0a)(\pi y) = (Da)(y)$, $a \in A$, $y \in Y$, π denoting the canonical morphism from Y onto $Y/\iota Y_0$. Endowing the projective tensor product $Y/\iota Y_0 \hat{\otimes} X$ with A -actions $a(\pi y \otimes x) = \pi y \otimes ax$ and $(\pi y \otimes x)a = \pi ya \otimes x$, the Banach space $L(Y/\iota Y_0, X^*) = (Y/\iota Y_0 \hat{\otimes} X)^*$, cf. [CLM], II.1.7, p. 54, becomes a dual A -bimodule and D_0 a derivation so that, by the amenability of A , $D_0a = aS - Sa$, $a \in A$, for some $S \in L(Y/\iota Y_0, X^*)$. Comparing with the definition of D_0 yields

$$a(\lambda_0 \circ \ell - S \circ \pi) = (\lambda_0 \circ \ell - S \circ \pi)a \quad (a \in A),$$

such that $\lambda = \lambda_0 \circ \ell - S \circ \pi$ is a morphism extending λ_0 along ι . Hence X^* is injective and X flat. \square

5.3. Proof of Corollary b. Let X be a non-zero flat left $L^1(G)$ -module, reflexive as a Banach space. Then X^* is a non-zero injective right $L^1(G)$ -module and equally reflexive, implying the amenability of G by the Proposition. If, conversely, the group G is amenable, then the Banach algebra $L^1(G)$ is amenable, [J], Theorem 2.5, p. 32, so that every left $L^1(G)$ -module is flat by the lemma above. \square

6. An open problem

Let \mathcal{M} be a von Neumann algebra admitting a non-zero injective normal Banach left module, reflexive as a Banach space. Does this entail the injectivity of \mathcal{M} ? Cf. [R2], in particular Corollary 2.6, p. 2533.

Acknowledgements I am obliged to Dr Paul Ramsden (Surrey, formerly at Leeds) for providing me with a copy of his unpublished Będlewo notes, [Ra]. The work for this paper was done at The Erwin Schrödinger International Institute for Mathematical Physics, Vienna, during the workshop “Bialgebras in Free Probability”. The author’s indebtedness to the referee is acknowledged in the text.

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Received April 22, 2011

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