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## Entropy on Riemann surfaces and the Jacobians of finite covers

Curtis T. McMullen\*

**Abstract.** This paper characterizes those pseudo-Anosov mappings whose entropy can be detected homologically by taking a limit over finite covers. The proof is via complex-analytic methods. The same methods show the natural map  $\mathcal{M}_g \rightarrow \prod \mathcal{A}_h$ , which sends a Riemann surface to the Jacobians of all of its finite covers, is a contraction in most directions.

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### 1. Introduction

Let  $f : S \rightarrow S$  be a pseudo-Anosov mapping on a surface of genus  $g$  with  $n$  punctures. It is well-known that the topological entropy  $h(f)$  is bounded below in terms of the spectral radius of  $f^* : H^1(S, \mathbb{C}) \rightarrow H^1(S, \mathbb{C})$ ; we have

$$\log \rho(f^*) \leq h(f).$$

If we lift  $f$  to a map  $\tilde{f} : \tilde{S} \rightarrow \tilde{S}$  on a finite cover of  $S$ , then its entropy stays the same but the spectral radius of the action on homology can increase. We say the entropy of  $f$  can be *detected homologically* if

$$h(f) = \sup \log \rho(\tilde{f}^* : H^1(\tilde{S}) \rightarrow H^1(\tilde{S})),$$

where the supremum is taken over all finite covers to which  $f$  lifts.

In this paper we will show:

**Theorem 1.1.** *The entropy of a pseudo-Anosov mapping  $f$  can be detected homologically if and only if the invariant foliations of  $f$  have no odd-order singularities in the interior of  $S$ .*

The proof is via complex analysis. Hodge theory provides a natural embedding  $\mathcal{M}_g \rightarrow \mathcal{A}_g$  from the moduli space of Riemann surfaces into the moduli space of

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Abelian varieties, sending  $X$  to its Jacobian. Any characteristic covering map from a surface of genus  $h$  to a surface of genus  $g$ , branched over  $n$  points, provides a similar map

$$\mathcal{M}_{g,n} \rightarrow \mathcal{M}_h \rightarrow \mathcal{A}_h. \tag{1.1}$$

It is known that the hyperbolic metric on a Riemann surface  $X$  can be reconstructed using the metrics induced from the Jacobians of its finite covers ([Kaz]; see the Appendix). Similarly, it is natural to ask if the Teichmüller metric on  $\mathcal{M}_{g,n}$  can be recovered from the Kobayashi metric on  $\mathcal{A}_h$ , by taking the limit over all characteristic covers  $\mathcal{C}_{g,n}$ . We will show such a construction is impossible.

**Theorem 1.2.** *The natural map  $\mathcal{M}_{g,n} \rightarrow \prod_{\mathcal{C}_{g,n}} \mathcal{A}_h$  is not an isometry for the Kobayashi metric, unless  $\dim \mathcal{M}_{g,n} = 1$ .*

It is an open problem to determine if the Kobayashi and Carathéodory metrics on moduli space coincide when  $\dim \mathcal{M}_{g,n} > 1$  (see e.g. [FM], Problem 5.1). An equivalent problem is to determine if Teichmüller space embeds holomorphically and isometrically into a (possibly infinite) product of bounded symmetric domains. Theorem 1.2 provides some support for a negative answer to this question.

Here is a more precise version of Theorem 1.2, stated in terms of the lifted map

$$\mathcal{T}_{g,n} \rightarrow \mathcal{T}_h \xrightarrow{J} \mathcal{S}_h$$

from Teichmüller space to Siegel space determined by a finite cover.

**Theorem 1.3.** *Suppose the Teichmüller mapping between a pair of distinct points  $X, Y \in \mathcal{T}_{g,n}$  comes from a quadratic differential with an odd order zero. Then*

$$\sup d(J(\tilde{X}), J(\tilde{Y})) < d(X, Y),$$

where the supremum is taken over all compatible finite covers of  $X$  and  $Y$ .

Conversely, if the Teichmüller map from  $X$  to  $Y$  has only even order singularities, then there is a double cover such that  $d(J(\tilde{X}), J(\tilde{Y})) = d(X, Y)$  (cf. [Kra]). In particular, the complex geodesics generated by squares of holomorphic 1-forms map isometrically into  $\mathcal{A}_g$ . The only directions contracted by the map  $\mathcal{M}_g \rightarrow \prod \mathcal{A}_h$  are those identified by Theorem 1.3.

Theorem 1.1 follows from Theorem 1.3 by taking  $X$  and  $Y$  to be points on the Teichmüller geodesic stabilized by the mapping-class  $f$ . It would be interesting to find a direct topological proof of Theorem 1.1.

As a sample application, let  $\beta \in B_n$  be a pseudo-Anosov braid whose monodromy map  $f : S \rightarrow S$  (on the  $n$ -times punctured plane) has an odd order singularity. Then Theorem 1.1 implies the image of  $\beta$  under the Burau representation satisfies

$$\log \sup_{|q|=1} \rho(B(q)) < h(f).$$

Indeed,  $\rho(B(q))$  at any  $d$ -th root of unit is bounded by  $\rho(\tilde{f}^*)$  on a  $\mathbb{Z}/d$  cover  $S$  [Mc2]. This improves a result in [BB]. Similar statements hold for other homological representations of the mapping–class group.

**Notes and references.** For  $C^\infty$  diffeomorphisms of a compact smooth manifold, one has  $h(f) \geq \log \sup_i \rho(f^*|H^i(X))$  [Ym], and equality holds for holomorphic maps on Kähler manifolds [Gr]. The lower bound  $h(f) \geq \log \rho(f^*|H^1(X))$  also holds for homeomorphisms [Mn]. For more on pseudo-Anosov mappings, see e.g. [FLP], [Bers] and [Th].

A proof that the inclusion of  $\mathcal{T}_{g,n}$  into universal Teichmüller space is a contraction, based on related ideas, appears in [Mc1].

## 2. Odd order zeros

We begin with an analytic result, which describes how well a monomial  $z^k$  of odd order can be approximated by the square of an analytic function.

**Theorem 2.1.** *Let  $k \geq 1$  be odd, and let  $f(z)$  be a holomorphic function on the unit disk  $\Delta$  such that  $\int |f(z)|^2 = 1$ . Then*

$$\left| \int_{\Delta} f(z)^2 \left( \frac{\bar{z}}{|z|} \right)^k \right| \leq C_k = \frac{\sqrt{k+1}\sqrt{k+3}}{k+2} < 1.$$

Here the integral is taken with respect to Lebesgue measure on the unit disk.

*Proof.* Consider the orthonormal basis  $e_n(z) = a_n z^n, n \geq 0, a_n = \sqrt{n+1}/\sqrt{\pi}$ , for the Bergman space  $L^2_{\alpha}(\Delta)$  of analytic functions on the disk with  $\|f\|_2^2 = \int |f(z)|^2 < \infty$ . With respect to this basis, the nonzero entries in the matrix of the symmetric bilinear form  $Z(f, g) = \int f(z)g(z)\bar{z}^k/|z|^k$  are given by

$$Z(e_n, e_{k-n}) = a_n a_{k-n} \int_{\Delta} |z|^k = \frac{2\sqrt{n+1}\sqrt{k-n+1}}{k+2}.$$

In particular,  $Z(e_i, e_i) = 0$  for all  $i$  (since  $k$  is odd), and  $Z(e_i, e_j) = 0$  for all  $i, j > k$ .

Note that the ratio above is less than one, by the inequality between the arithmetic and geometric means, and it is maximized when  $n < k/2 < n+1$ . Thus the maximum of  $|Z(f, f)|/\|f\|_2^2$  over  $L^2_{\alpha}(\Delta)$  is achieved when  $f = e_n + e_{n+1}, n = (k-1)/2$ , at which point it is given by  $C_k$ . □

### 3. Siegel space

In this section we describe the Siegel space of Hodge structures on a surface  $S$ , and its Kobayashi metric.

**Hodge structures.** Let  $S$  be a closed, smooth, oriented surface of genus  $g$ . Then  $H^1(S) = H^1(S, \mathbb{C})$  carries a natural involution  $C(\alpha) = \bar{\alpha}$  fixing  $H^1(S, \mathbb{R})$ , and a natural Hermitian form

$$\langle \alpha, \beta \rangle = \frac{\sqrt{-1}}{2} \int_S \alpha \wedge \bar{\beta}$$

of signature  $(g, g)$ . A *Hodge structure* on  $H^1(S)$  is given by an orthogonal splitting

$$H^1(S) = V^{1,0} \oplus V^{0,1}$$

such that  $V^{1,0}$  is positive-definite and  $V^{0,1} = C(V^{1,0})$ . We have a natural norm on  $V^{1,0}$  given by  $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$ .

The set of all possible Hodge structures forms the *Siegel space*  $\mathfrak{S}(S)$ . To describe this complex symmetric space in more detail, fix a splitting  $H^1(S) = W^{1,0} \oplus W^{0,1}$ . Then for any other Hodge structure  $V^{1,0} \oplus V^{0,1}$ , there is a unique operator

$$Z: W^{1,0} \rightarrow W^{0,1}$$

such that  $V^{1,0} = (I + Z)(W^{1,0})$ . This means  $V^{1,0}$  coincides with the graph of  $Z$  in  $W^{1,0} \oplus W^{0,1}$ .

The operator  $Z$  is determined uniquely by the associated bilinear form

$$Z(\alpha, \beta) = \langle \alpha, CZ(\beta) \rangle$$

on  $W^{1,0}$ , and the condition that  $V^{1,0} \oplus V^{0,1}$  is a Hodge structure translates into the conditions

$$Z(\alpha, \beta) = Z(\beta, \alpha) \quad \text{and} \quad |Z(\alpha, \alpha)| < 1 \text{ if } \|\alpha\| = 1. \quad (3.1)$$

Since the second inequality above is an open condition, the tangent space at the base point  $p \sim W^{1,0} \oplus W^{0,1}$  is given by

$$T_p \mathfrak{S}(S) = \{\text{symmetric bilinear maps } Z: W^{1,0} \times W^{1,0} \rightarrow \mathbb{C}\}.$$

**Comparison maps.** Any Hodge structure on  $H^1(S)$  determines an isomorphism

$$V^{1,0} \cong H^1(S, \mathbb{R}) \quad (3.2)$$

sending  $\alpha$  to  $\Re(\alpha) = (\alpha + C(\alpha))/2$ . Thus  $H^1(S, \mathbb{R})$  inherits a norm and a complex structure from  $V^{1,0}$ .

Put differently, (3.2) gives a *marking* of  $V^{1,0}$  by  $H^1(S, \mathbb{R})$ . By composing one marking with the inverse of another, we obtain the real-linear *comparison map*

$$T = (I + Z)(I + CZ)^{-1} : W^{1,0} \rightarrow V^{1,0} \tag{3.3}$$

between any pair of Hodge structures. It is characterized by  $\Re(\alpha) = \Re(T(\alpha))$ .

**Symmetric matrices.** The classical Siegel domain is given by

$$\mathfrak{S}_g = \{Z \in M_g(\mathbb{C}) : Z_{ij} = Z_{ji} \text{ and } I - Z\bar{Z} \gg 0\}.$$

(cf. [Sat], Chapter II.7). It is a convex, bounded symmetric domain in  $\mathbb{C}^N$ ,  $N = g(g + 1)/2$ . The choice of an orthonormal basis for  $W^{1,0}$  gives an isomorphism  $Z \mapsto Z(\omega_i, \omega_j)$  between  $\mathfrak{S}(S)$  and  $\mathfrak{S}_g$ , sending the basepoint  $p$  to zero.

**The Kobayashi metric.** Let  $\Delta \subset \mathbb{C}$  denote the unit disk, equipped with the metric  $|dz|/(1 - |z|^2)$  of constant curvature  $-4$ . The *Kobayashi metric* on  $\mathfrak{S}(S)$  is the largest metric such that every holomorphic map  $f : \Delta \rightarrow \mathfrak{S}(S)$  satisfies  $\|Df(0)\| \leq 1$ . It determines both a norm on the tangent bundle and a distance function on pairs of points [Ko].

**Proposition 3.1.** *The Kobayashi norm on  $T_p\mathfrak{S}(S)$  is given by*

$$\|Z\|_K = \sup\{|Z(\alpha, \alpha)| : \|\alpha\| = 1\},$$

and the Kobayashi distance is given in terms of the comparison map (3.3) by

$$d(V^{1,0}, W^{1,0}) = \log \|T\|.$$

*Proof.* Choosing a suitable orthonormal basis for  $W^{1,0}$ , we can assume that

$$Z(\omega_i, \omega_j) = \lambda_i \delta_{ij}$$

with  $\lambda_1 \geq \lambda_2 \geq \dots \lambda_g \geq 0$ . Since  $\mathfrak{S}_g$  is a convex symmetric domain, the Kobayashi norm at the origin and the Kobayashi distance satisfy

$$\|Z\|_K = r \quad \text{and} \quad d(0, Z) = \frac{1}{2} \log \frac{1+r}{1-r},$$

where  $r = \inf\{s > 0 : Z \in s\mathfrak{S}_g\}$  (see [Ku]). Clearly  $r = \lambda_1 = \sup |Z(\alpha, \alpha)|/\|\alpha\|^2$ , and by (3.3), we have

$$\|T\|^2 = \|T(\sqrt{-1}\omega_1)\|^2 = \left\| \frac{\omega_1}{1-\lambda_1} + \frac{\lambda_1\bar{\omega}_1}{1-\lambda_1} \right\|^2 = \frac{1+\lambda_1}{1-\lambda_1},$$

which gives the expressions above. □

#### 4. Teichmüller space

This section gives a functorial description of the derivative of the map from Teichmüller space to Siegel space.

**Markings.** Let  $\bar{S}$  be a compact oriented surface of genus  $g$ , and let  $S \subset \bar{S}$  be a subsurface obtained by removing  $n$  points.

Let  $\text{Teich}(S) \cong \mathcal{T}_{g,n}$  denote the Teichmüller space of Riemann surfaces marked by  $S$ . A point in  $\text{Teich}(S)$  is specified by a homeomorphism  $f: S \rightarrow X$  to a Riemann surface of finite type. This means there is a compact Riemann surface  $\bar{X} \supset X$  and an extension of  $f$  to a homeomorphism  $\bar{f}: \bar{S} \rightarrow \bar{X}$ .

**Metrics.** Let  $Q(X)$  denote the space of holomorphic quadratic differentials on  $X$  such that

$$\|q\|_X = \int_X |q| < \infty.$$

There is a natural pairing  $(q, \mu) \mapsto \int_X q\mu$  between the space  $Q(X)$  and the space  $M(X)$  of  $L^\infty$ -measurable Beltrami differentials  $\mu$ . The tangent and cotangent spaces to Teichmüller space at  $X$  are isomorphic to  $M(X)/Q(X)^\perp$  and  $Q(X)$  respectively.

The Teichmüller and Kobayashi metrics on  $\text{Teich}(S)$  coincide [Roy1], [Hub], Chapter 6. They are given by the norm

$$\|\mu\|_T = \sup \{ |\int q\mu| : \|q\|_X = 1 \}$$

on the tangent space at  $X$ ; the corresponding distance function

$$d(X, Y) = \inf \frac{1}{2} \log K(\phi)$$

measures the minimal dilatation  $K(\phi)$  of a quasiconformal map  $\phi: X \rightarrow Y$  respecting their markings.

**Hodge structure.** The periods of holomorphic 1-forms on  $X$  serve as classical moduli for  $X$ . From a modern perspective, these periods give a map

$$J: \text{Teich}(S) \rightarrow \mathfrak{S}(\bar{S}) \cong \mathfrak{S}_g,$$

sending  $X$  to the Hodge structure

$$H^1(\bar{S}) \cong H^1(\bar{X}) \cong H^{1,0}(\bar{X}) \oplus H^{0,1}(\bar{X}).$$

Here the first isomorphism is provided by the marking  $\bar{f}: \bar{S} \rightarrow \bar{X}$ . We also have a natural isomorphism between  $H^{1,0}(\bar{X})$  and the space of holomorphic 1-forms  $\Omega(\bar{X})$ . The image  $J(X)$  encodes the complex analytic structure of the Jacobian variety  $\text{Jac}(\bar{X}) = \Omega(\bar{X})^*/H_1(\bar{X}, \mathbb{Z})$ . (It does not depend on the location of the punctures of  $X$ .)

**Proposition 4.1.** *The derivative of the period map sends  $\mu \in M(X)$  to the quadratic form  $Z = DJ(\mu)$  on  $\Omega(\bar{X})$  given by*

$$Z(\alpha, \beta) = \int_{\bar{X}} \alpha\beta\mu.$$

This is a basis-free reformulation of Ahlfors’ variational formula [Ah], §5; see also [Ra], [Roy2] and Proposition 1 of [Kra]. Note that  $\alpha\beta \in Q(X)$ .

### 5. Contraction

This section brings finite covers into play, and establishes a uniform estimate for contraction of the mapping  $\mathcal{T}_{g,n} \rightarrow \mathcal{T}_h \rightarrow \mathfrak{S}_h$ .

**Jacobians of finite covers.** A finite connected covering space  $S_1 \rightarrow S_0$  determines a natural map

$$P : \text{Teich}(S_0) \rightarrow \text{Teich}(S_1)$$

sending each Riemann surface to the corresponding covering space  $X_1 \rightarrow X_0$ . By taking the Jacobian of  $X_1$ , we obtain a map  $J \circ P : \text{Teich}(S_0) \rightarrow \mathfrak{S}(\bar{S}_1)$ .

Let  $q_0 \in Q(X_0)$  be a holomorphic quadratic differential with a zero of odd order  $k$ , say at  $p \in X_0$ . Let  $\mu = \bar{q}_0/|q_0| \in M(X_0)$ ; then  $\|\mu\|_T = 1$ . Let  $\pi : X_1 \rightarrow X_0$  denote the natural covering map, and let  $q_1 = \pi^*(q_0)$ .

We will show that  $J(X_1)$  cannot change too rapidly under the unit deformation  $\mu$  of  $X_0$ . Indeed, if  $J(X_1)$  were to move at nearly unit speed, then  $\pi^*(\mu) = \bar{q}_1/|q_1|$  would pair efficiently with  $\alpha^2$  for some unit-norm  $\alpha \in \Omega(\bar{X}_1)$ , which is impossible because of the many odd-order zeros of  $q_1$ .

To make a quantitative estimate, choose a holomorphic chart  $\phi : (\Delta, 0) \rightarrow (X_0, p)$  such that  $\phi^*(\mu) = z^k/|z|^k d\bar{z}/dz$ . Let  $U = \phi(\Delta)$ , and let

$$m(U) = \inf\{\|q\|_U : q \in Q(X_0), \|q\|_X = 1\}.$$

(Here  $\|q\|_U = \int_U |q|$ .) Since  $Q(X_0)$  is finite-dimensional, we have  $m(U) > 0$ .

**Theorem 5.1.** *The image  $Z$  of the vector  $[\mu]$  under the derivative of  $J \circ P$  satisfies*

$$\|Z\|_K \leq \delta < 1 = \|\mu\|_T,$$

where  $\delta = \max(1/2, 1 - (1 - C_k)m(U)/2)$  does not depend on the finite cover  $S_1 \rightarrow S_0$ .

*Proof.* The derivative of  $P$  sends  $\mu$  to  $\pi^*(\mu)$ . By Proposition 3.1, to show  $\|Z\|_K \leq \delta$  it suffices to show that

$$|Z(\alpha, \alpha)| = \left| \int_{X_1} \alpha^2 \pi^* \mu \right| \leq \delta$$

for all  $\alpha \in \Omega(\bar{X}_1)$  with  $\|\alpha^2\|_{X_1} = 1$ . Setting  $q = \pi_*(\alpha^2)$ , we also have

$$|Z(\alpha, \alpha)| = \left| \int_{X_0} q \mu \right| \leq \|q\|_{X_0},$$

so the proof is complete if  $\|q\|_{X_0} \leq 1/2$ . Thus we may assume that

$$\|\alpha^2\|_V \geq \|q\|_U \geq m(U)\|q\|_{X_0} \geq m(U)/2,$$

where  $V = \pi^{-1}(U) = \bigcup_1^d V_i$  is a finite union of disjoint disks. Using the coordinate charts  $V_i \cong U \cong \Delta$  and Theorem 2.1, we find that on each of these disks we have

$$\left| \int_{V_i} \alpha^2 \pi^*(\mu) \right| = \left| \int_{\Delta} \alpha(z)^2 \left( \frac{z}{|z|} \right)^k \right| \leq C_k \|\alpha^2\|_{V_i}.$$

Summing these bounds and using the fact that  $\|\alpha^2\|_{(X_1-V)} + \|\alpha^2\|_V = 1$ , we obtain

$$\left| \int_{X_1} \alpha^2 \pi^*(\mu) \right| \leq \|\alpha^2\|_{(X_1-V)} + C_k \|\alpha^2\|_V \leq 1 - \frac{(1 - C_k)m(U)}{2} \leq \delta. \quad \square$$

### 6. Conclusion

It is now straightforward to establish the results stated in the Introduction.

*Proof of Theorem 1.3.* Assume the Beltrami coefficient of the Teichmüller mapping between  $X, Y \in \mathcal{T}_{g,n}$  has the form  $\mu = k\bar{q}/q$ , where  $q \in Q(X)$  has an odd order zero. Then the same is true for the tangent vectors to the Teichmüller geodesic  $\gamma$  joining  $X$  to  $Y$ . Theorem 5.1 then implies that  $D(J \circ P)|_\gamma$  is contracting by a factor  $\delta < 1$  independent of  $P$ , and therefore

$$d(J \circ P(X), J \circ P(Y)) = d(J(\tilde{X}), J(\tilde{Y})) < \delta \cdot d(X, Y). \quad \square$$

*Proof of Theorem 1.2.* The contraction of  $\mathcal{M}_{g,n} \rightarrow \prod \mathcal{E}_{g,n} \mathcal{A}_h$  in some directions is immediate from the uniformity of the bound in Theorem 1.3, using the fact that the Kobayashi metric on a product is the sup of the Kobayashi metrics on each term, and that there exist  $q \in Q(X)$  with simple zeros whenever  $X \in \mathcal{M}_{g,n}$  and  $\dim \mathcal{M}_{g,n} > 1$ .  $\square$

*Proof of Theorem 1.1.* Let  $f : S_0 \rightarrow S_0$  be a pseudo-Anosov mapping. If  $f$  has only even order singularities, then its expanding foliation is locally orientable, and hence there is a double cover  $\tilde{S} \rightarrow \tilde{S}$  such that  $\log \rho(\tilde{f}^*) = h(f)$ .

Now suppose  $f$  has an odd-order singularity. Let  $X_0 \in \text{Teich}(S_0)$  be a point on the Teichmüller geodesic stabilized by the action of  $f$  on  $\text{Teich}(S_0)$ . Then  $h(f) = d(f \cdot X_0, X_0) > 0$  (see e.g. [FLP] and [Bers]).

Let  $\tilde{f}: S_1 \rightarrow S_1$  be a lift of  $f$  to a finite covering of  $S_0$ , and let  $X_1 = P(X_0) \in \text{Teich}(S_1)$ . Using the marking of  $X_1$  and the isomorphism  $H^1(X_1, \mathbb{R}) \cong H^{1,0}(X_1)$ , we obtain a commutative diagram

$$\begin{CD} H^1(S_1, \mathbb{R}) @>\tilde{f}^*>> H^1(S_1, \mathbb{R}) \\ @VVV @VVV \\ H^{1,0}(\bar{X}_1) @>T>> H^{1,0}(\bar{X}_1) \end{CD}$$

where  $T$  is the comparison map between  $J(X_1)$  and  $J(\tilde{f} \cdot X_1)$  (see equation (3.3)). Then Theorem 1.3 and Proposition 3.1 yield the bound

$$\log \rho(\tilde{f}^*) \leq \log \|T\| = d(J(X_1), \tilde{f} \cdot J(X_1)) \leq \delta d(X_0, f \cdot X_0) = \delta h(f),$$

where  $\delta < 1$  does not depend on the finite covering  $S_1 \rightarrow S_0$ . Consequently,  $\sup \log \rho(\tilde{f}^*) < h(f)$ . □

**Appendix. The hyperbolic metric via Jacobians of finite covers**

Let  $X = \Delta/\Gamma$  be a compact Riemann surface, presented as a quotient of the unit disk by a Fuchsian group  $\Gamma$ . Let  $Y_n \rightarrow X$  be an ascending sequence of finite Galois covers which converge to the universal cover, in the sense that

$$Y_n = \Delta/\Gamma_n, \quad \Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \cdots, \quad \text{and} \quad \bigcap \Gamma_i = \{e\}. \quad (\text{A.1})$$

The Bergman metric on  $Y_n$  (defined below) is invariant under automorphisms, so it descends to a metric  $\beta_n$  on  $X$ . This appendix gives a short proof of:

**Theorem A.1** (Kazhdan). *The Bergman metrics inherited from the finite Galois covers  $Y_n \rightarrow X$  converge to a multiple of the hyperbolic metric; more precisely, we have*

$$\beta_n \rightarrow \frac{\lambda_X}{2\sqrt{\pi}}$$

*uniformly on  $X$ .*

The argument below is based on [Kaz], §3; for another, somewhat more technical approach, see [Rh].

**Metrics.** We begin with some definitions. Let  $\Omega(X)$  denote the Hilbert space of holomorphic 1-forms on a Riemann surface  $X$  such that

$$\|\omega\|_X^2 = \int_X |\omega|^2 < \infty.$$

The area form of the *Bergman metric* on  $X$  is given by

$$\beta_X^2 = \sum |\omega_i|^2, \tag{A.2}$$

where  $(\omega_i)$  is any orthonormal basis of  $\Omega(X)$ . Equivalently, the Bergman length of a tangent vector  $v \in TX$  is given by

$$\langle \beta_X, v \rangle = \sup_{\omega \neq 0} \frac{|\omega(v)|}{\|\omega\|_X}. \tag{A.3}$$

This formula shows that inclusions are contracting: if  $Y$  is a subdomain of  $X$ , then  $\beta_Y \geq \beta_X$ .

Now suppose  $X$  is a compact surface of genus  $g > 0$ . Then (A.2) shows its Bergman area is given by

$$\int_X \beta_X^2 = \dim \Omega(X) = g. \tag{A.4}$$

In this case  $\beta_X$  is also the pullback, via the Abel–Jacobi map, of the natural Kähler metric on the Jacobian of  $X$ .

Finally suppose  $X = \Delta/\Gamma$ . Then the hyperbolic metric of constant curvature  $-1$ ,

$$\lambda_\Delta = \frac{2|dz|}{1 - |z|^2},$$

descends to give the *hyperbolic metric*  $\lambda_X$  on  $X$ . Using the fact that  $\|dz\|_\Delta = \pi$ , it is easy to check that  $4\pi\beta_\Delta^2 = \lambda_\Delta^2$ .

*Proof of Theorem A.1.* We will regard the Bergman metric  $\beta_n$  on  $Y_n$  as a  $\Gamma_n$ -invariant metric on  $\Delta$ . It suffices to show that  $\beta_n/\beta_\Delta \rightarrow 1$  uniformly on  $\Delta$ .

Let  $g$  and  $g_n$  denote the genus of  $X$  and  $Y_n$  respectively, and let  $d_n$  denote the degree of  $Y_n/X$ ; then  $g_n - 1 = d_n(g - 1)$ . By (A.1), the injectivity radius of  $Y_n$  tends to infinity. In particular, there is a sequence  $r_n \rightarrow 1$  such that  $\gamma(r_n\Delta)$  injects into  $Y_n$  for any  $\gamma \in \Gamma$ . Since inclusions are contracting, this shows

$$\beta_n \leq (1 + \epsilon_n)\beta_\Delta \tag{A.5}$$

where  $\epsilon_n \rightarrow 0$ .

Next, note that both  $\beta_n$  and  $\beta_\Delta$  are  $\Gamma$ -invariant, so they determine metrics on  $X$ . By (A.4), we have

$$\int_X \beta_n^2 = \frac{1}{d_n} \int_{Y_n} \beta_n^2 = \frac{g_n}{d_n} \rightarrow (g - 1) = \int_X \beta_\Delta^2$$

(since  $\int_X \lambda_X^2 = 2\pi(2g - 2)$  by Gauss–Bonnet). Together with (A.5), this implies

$$\int_X |\beta_n - \beta_\Delta|^2 \rightarrow 0. \tag{A.6}$$

To show  $\beta_n \rightarrow \beta_\Delta$  uniformly, consider any sequence  $p_n \in \Delta$  and let  $x \in [0, 1]$  be a limit point of  $(\beta_n/\beta_\Delta)(p_n)$ . It suffices to show  $x = 1$ .

Passing to a subsequence and using compactness of  $X$ , we can assume that  $p_n \rightarrow p \in \Delta$  and that  $\beta_n(p_n) \rightarrow x\beta_\Delta(p)$ . By changing coordinates on  $\Delta$ , we can also assume  $p = 0$ . By (A.6) we can find  $q_n \rightarrow 0$  such that  $\beta_n(q_n) \rightarrow \beta_\Delta(0)$ . Then by (A.3), there exist  $\Gamma_n$ -invariant holomorphic 1-forms  $\omega_n(z) dz$  on  $\Delta$  such that  $\int_{Y_n} |\omega_n|^2 = 1$  and

$$|\omega_n(q_n)| = \beta_n(q_n) \rightarrow \beta_\Delta(0) = \frac{|dz|}{\pi}.$$

Since  $\omega_n$  is holomorphic and  $\int_{r_n\Delta} |\omega_n|^2 < 1$ , the equation above easily implies that  $|\omega_n| \rightarrow |dz|/\pi$  uniformly on compact subsets of  $\Delta$ . But we also have

$$\beta_n(p_n) \geq |\omega_n(p_n)| \rightarrow \beta_\Delta(0),$$

and thus  $\beta_n(p_n) \rightarrow \beta_\Delta(0)$  and hence  $x = 1$ . □

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