Separatrices for $\mathbb{C}^2$ actions on 3-manifolds

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Abstract. We prove the existence of a separatrix for the codimension 1 singular foliation spanned by two commuting holomorphic vector fields on a neighborhood of the origin in $\mathbb{C}^3$.

1. Introduction

The problem of the existence of separatrices is a central theme in the local theory of singular holomorphic foliations. On a neighborhood of a singularity in dimension 2, the existence of separatrices was settled in [C-S] completing a classical work of Briot and Bouquet. Here a separatrix for a singular holomorphic foliation is by definition an (germ of) analytic curve passing through the singularity and invariant under the foliation in question. However, as we increase the dimension and consider foliations on manifolds having dimension equal to 3, it becomes necessary to distinguish between foliations of dimension 1 and foliations of dimension 2 (or of codimension 1). By using local coordinates, we can place ourselves on a neighborhood of the origin in $\mathbb{C}^3$. Then, in the case of a 1-dimensional foliation, a separatrix still is an (germ of) analytic curve passing through the origin and invariant by the foliation. As to codimension 1 foliations, a separatrix in this context should be understood as a germ of surface (i.e. 2-dimensional analytic set) passing through the origin and invariant by the foliation. Unfortunately, the existence of separatrices is no longer valid for all foliations as above. In [GM-L] the reader will find examples of 1-dimensional foliations without separatrices. For codimension 1 foliations the existence of counterexamples goes back to Jouanolou [J-1]. By studying the existence of invariant curves for foliations in the complex projective plane, it is easy to produce examples of codimension 1 foliations without separatrices as it will be discussed below. The existence of separatrices for codimension 1 foliations was also the object of the remarkable papers [Ca],

[C-C] where it is proved, in particular, that a non-dicritical codimension 1 foliation on a neighborhood of $\mathbb{C}^3$ always possesses a separatrix (for the definition of non-dicritical foliation see Section 2). As it follows from the preceding discussion, the set of foliations that fail to be *non-dicritical* is not negligible. For completeness, let us also mention the work of Sancho de Salas [S] concerning invariant sets for a vector field having a singularity of high codimension. Similarly Stolovitch has investigated normal forms for certain families of commuting vector fields having rank at least 2 and their applications to the existence of invariant sets [St].

In the present paper we consider codimension 1 foliations that are generated by an action of $\mathbb{C}^2$ of rank 2 on a complex manifold of dimension 3. More precisely we shall work on a neighborhood of the origin in $\mathbb{C}^3$. In this local setting, we consider the foliation spanned by two commuting holomorphic vector fields that are linearly independent at generic points. The main result of this paper is as follows.

Main Theorem. *Consider holomorphic vector fields X, Y defined on a neighborhood of the origin of $\mathbb{C}^3$. Suppose that they commute and are linearly independent at generic points so that they span a codimension 1 foliation denoted by $\mathcal{D}$. Then $\mathcal{D}$ possesses a separatrix.*

Note that the foliation $\mathcal{D}$ can be much more degenerate than the vector fields X, Y themselves since their k-jets may coincide to an order higher than the first non-trivial homogeneous component of X, Y. This is a considerable source of difficulty in the proof of our theorem. Also, if $\mathcal{D}$ is defined by the differential 1-form induced by the vector product of X, Y, this form may have a singular set of codimension 1 even though the singular sets of X, Y are of codimension ≥ 2.

An interesting application of the above theorem concerns the case of a $\mathbb{C}^2$-action having rank 2 on a complex manifold of dimension 3. By a rank 2 action, it is simply meant an action of $\mathbb{C}^2$ whose orbits have dimension 2 at generic points. On a neighborhood of a singular point for this action, the vector fields X, Y can, in addition, be chosen semi-complete. A good deal of information about these vector fields can then be derived from their restrictions to the separatrix of $\mathcal{D}$ since semi-complete vector fields in dimension 2 are well understood. A particularly remarkable example of this situation can be found in [G]. This example was first discovered by Lins-Neto [LN] in connection with the so-called Painlevé problem, whereas the corresponding geometry and dynamics was described in [G]. It consists of choosing X, Y respectively as the vector fields

$$Z_\infty = 2z_2(-z_1 + z_3)\frac{\partial}{\partial z_1} + (3z_1^2 - z_2^2)\frac{\partial}{\partial z_2} + 2z_3z_2\frac{\partial}{\partial z_3},$$

$$Z_0 = (-3z_1^2 + z_2^2 + 2z_1z_3)\frac{\partial}{\partial z_1} + 2z_2(-3z_1 + 2z_3)\frac{\partial}{\partial z_2} + 2z_3(3z_1 - z_3)\frac{\partial}{\partial z_3}.$$

These vector fields correspond to an action of $\mathbb{C}^2$ on a suitable 3-manifold. The family spanned by them is such that a generic element has an isolated singularity at the origin. Yet some elements, such as Z_∞, possesses a singular set with codimension 2.

Let us now indicate how to construct numerous examples of codimension 1 foliations on a neighborhood of $(0, 0, 0) \in \mathbb{C}^3$ without separatrices. Consider a homogeneous polynomial vector field Z defined on $\mathbb{C}^3$ and having an isolated singularity at $(0, 0, 0) \in \mathbb{C}^3$. Unless Z is a multiple of the radial vector field $R = x\partial/\partial x + y\partial/\partial y + z\partial/\partial z$, it induces a 1-dimensional holomorphic foliation $\mathcal{F}_{\mathbb{C}P(2)}$ of dimension 1 on $\mathbb{C}P(2)$. Conversely every 1-dimensional foliation on $\mathbb{C}P(2)$ is induced by a homogeneous vector field on $\mathbb{C}^3$. Next we consider the 2-dimensional distribution of planes on $\mathbb{C}^3$ which is spanned by Z and by R. The Euler relation (Equation 3) shows that Z, R generates a Lie algebra isomorphic to the Lie algebra of the affine group. The corresponding distribution is therefore integrable and hence yields a codimension 1 foliation that is going to be denoted by $\mathcal{D}$. Clearly the punctual blow-up $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ does not leave the exceptional divisor $\pi^{-1}(0)$ invariant (for details see Lemma 1). In fact, the intersections of the leaves of $\widetilde{\mathcal{D}}$ with $\pi^{-1}(0)$ coincide with the leaves of $\mathcal{F}_{\mathbb{C}P(2)}$.

As far as the existence of separatrices for $\mathcal{D}$ is concerned, the upshot of the preceding construction is as follows: if $\mathcal{D}$ possesses a separatrix, the tangent cone of this separatrix yields an algebraic curve in $\pi^{-1}(0)$ which must be invariant under $\mathcal{F}_{\mathbb{C}P(2)}$. Nonetheless it is known that, in a very strong sense, most choices of Z leads to a foliation $\mathcal{F}_{\mathbb{C}P(2)}$ that does not leave any proper analytic set invariant (cf. for example [LN-S], [L-R]). As a result the codimension 1 foliation obtained by means of Z, R, for a generic choice of Z, does not have separatrices. We also note that, for these examples, no separatrix can be produced by adding "higher order terms" to $\mathcal{D}$.

To show that this phenomenon cannot take place in our context, we shall consider the intersection of our codimension 1 foliation with a given component of the exceptional divisor. Unless this component is invariant by the codimension 1 foliation, this intersection defines a foliation of dimension 1 on it. Except for some rather special situations that are already "linear" in a suitable sense, we are going to show that all the leaves of the latter foliation are properly embedded (in particular they are compact provided that the mentioned component of the exceptional divisor is so). This statement is, indeed, equivalent to saying that the corresponding foliation admits a non-constant meromorphic first integral as it follows from [J-2]. In general we shall directly work with the existence of a non-constant meromorphic first integral for foliations as above. Next we will be led to consider the special situations of "linear" foliations that may not possess any non-constant meromorphic first integral. Fortunately in these cases the existence of a separatrix can be established by more direct methods. An example of a "linear case" would consist of a pair of vector fields X, Y with X linear and Y equal to the radial vector field $x\partial/\partial x + y\partial/\partial y + z\partial/\partial z$. These two vector fields commute and span a codimension 1 foliation whose (punctual) blow-up at the origin does not leave the corresponding exceptional divisor invariant.

Furthermore the foliation induced on the corresponding exceptional divisor by the mentioned blown-up foliation coincides with the foliation induced on $\mathbb{C}P(2)$ by X. In particular X can be chosen so that the "generic" leaf is not compact. However, in this situation the foliation induced by X on $\mathbb{C}P(2)$ still has a compact leaf which "immediately" leads to the existence of the desired separatrix. Apart from these so-called "linear situations", the fact that the above mentioned leaves are all compact will be obtained by exploiting the mutual symmetries of X, Y yielded by their commutativity and by the fact that their proper transform should vanish identically over the whole exceptional divisor.

A similar example concerning blow-ups centered at a curve (as opposed to a single point) was pointed out to us by D. Cerveau. It serves to illustrate both the problem about the existence of first integrals as above and the contents of Lemmas 2 and 5 which are crucial for establishing the existence of these first integrals. This example goes as follows. Consider the pair of vector fields X, Y given by

$$X = zy\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z} \quad \text{and} \quad Y = x^2\frac{\partial}{\partial x} + axy\frac{\partial}{\partial y}.$$

These two vector fields commute and span a codimension 2 foliation denoted by $\mathcal{D}$. They also leave the axis $\{y = z = 0\}$ invariant. Consider the blow-up of $\mathcal{D}$, X, Y centered over $\{y = z = 0\}$. The transform $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ does not leave the exceptional divisor invariant. Furthermore the leaves of the foliation induced on the non-compact exceptional divisor by intersecting it with the leaves of $\widetilde{\mathcal{D}}$ are themselves non-compact. The explanation for this phenomenon is that the blow-up of X is regular at generic points of the exceptional divisor. Indeed, X is already regular at generic points of the axis $\{y = z = 0\}$. Hence this case must be considered as "linear" (indeed even "regular"). As it will be clear in Section 3, the appropriate notion of order of a vector field relative to a curve is such that the resulting order for X as above is *zero*. This is totally coherent with the fact that X is regular at generic points of this axis. With these definitions the situations that were called "linear" in the above discussion essentially coincides with the cases where the vector fields X, Y have order strictly less than 2 at the center of some blow-up map.

The organization of the paper is as follows. In Section 2 we consider the case of a single punctual blow-up. The condition for the proper transforms of X, Y to vanish over the corresponding exceptional divisor amounts to saying that the linear parts at the origin of X, Y are zero (i.e. the Taylor series of X, Y at the center of the blow-up must begin with terms whose order is at least 2). Under this condition we prove that, if the codimension 1 foliation spanned by X, Y does not leave the exceptional divisor invariant, then all (1-dimensional) leaves induced by it on the exceptional divisor are compact (Proposition 1). Section 3 is devoted to obtaining an analogue of Proposition 1 for the case of blow-ups centered at a (smooth, irreducible) curve. In particular, this will require a suitable analogue of the "linear parts" of X, Y which is adapted to the curve in question. This is going to raise some minor additional

difficulties as indicated by the above example. After introducing the appropriate setting, the main result of Section 3 will be Proposition 2 which is a faithful analogue of Proposition 1.

Since it is hard to imagine a theorem about arbitrarily degenerate singularities being proved without resorting to a suitable "desingularization" theorem, the fundamental results of [C-C], [Ca] about reduction of singularities of codimension 1 foliations will play a role in this paper. They will be brought to bear in Section 4. First we shall prove that the desired separatrix must exist provided that the 1-dimensional foliations induced on the non-invariant compact components of the (total) exceptional divisor have only compact leaves (in other words, provided that each of these foliations possesses a non-constant meromorphic first integral). In these cases, the existence of the separatrix will follow from the combination of the compactness of the mentioned (1-dimensional) leaves with the fact that the "reduced singularities" are simple enough to allow for a total understanding of their (local) separatrices. To prove our main result, we are then led to discuss the effect of the blow-up procedure of [C-C], [Ca] on the initial vector fields X, Y. The outcome of this discussion is that, to a large extent, Propositions 1 and 2 can be applied to guarantee that the (1-dimensional) leaves in question are compact. Thus, at this point, we shall have the existence of the separatrix established except for some "special" cases in which the assumptions of Propositions 1 and 2 are not fulfilled. These remaining cases are however simple enough to be amenable to more direct integration methods.

2. On the dicritical character of $\mathcal{D}$, Part I: Blowing-up a point

Consider two commuting holomorphic vector fields, X, Y, defined on $(\mathbb{C}^3, 0)$. Throughout this section, the vector fields X, Y are supposed to satisfy the following conditions:

(1) X, Y are linearly independent at generic points.

(2) The linear parts of X, Y at the origin are zero.

Next, let $X = \sum_{i=1}^{3} X_i \partial/\partial x_i$ and $Y = \sum_{i=1}^{3} Y_i \partial/\partial x_i$. Because X and Y commute, they define a codimension 1 singular foliation $\mathcal{D}$ which is represented by

the holomorphic 1-form

$$(X_2Y_3 - X_3Y_2)dx_1 + (X_3Y_1 - X_1Y_3)dx_2 + (X_1Y_2 - X_2Y_1)dx_3. \qquad (1)$$

As usual we can assume that Sing $(\mathcal{D})$ has codimension greater than or equal to 2. In other words, we can eliminate all non-trivial common factors from the components of the 1-form considered above.

In this work we shall deal with the codimension 1 foliation $\mathcal{D}$ as well as with the foliations $\mathcal{F}_X$, $\mathcal{F}_Y$ associated respectively to the vector fields X, Y. Unlike $\mathcal{D}$, the foliations $\mathcal{F}_X$, $\mathcal{F}_Y$ have *dimension* 1. Nonetheless, their singular sets can also be supposed to have codimension 2 or greater.

Recall that a *separatrix* for a *foliation of dimension* 1 (such as $\mathcal{F}_X$, $\mathcal{F}_Y$) is a germ of analytic curve passing through the origin and invariant by the foliation in question. Note that this definition does not exclude the possibility of having a separatrix entirely contained in the singular set of the corresponding foliation.

On the other hand, if we have a *codimension* 1 *foliation* (such as $\mathcal{D}$), then a separatrix is a germ of *analytic surface* passing through the origin and invariant by the corresponding foliation. We can also say that, in the latter case, a separatrix is given by an irreducible germ of analytic function f that divides $\Omega \wedge df$, where Ω stands for a 1-form defining the foliation. Since the singular set of any foliation has codimension at least 2, a separatrix for a codimension 1 foliation is always obtained from a regular leaf that accumulates on the singular set (unless it coincides locally with the leaf of the foliation through a regular point).

As mentioned in the Introduction, in Sections 2 and 3 we shall consider the case of a single blowing-up map leading to an exceptional divisor that is not invariant by the corresponding blown-up foliation. Our purpose will be to find sufficient conditions to ensure that the (1-dimensional) foliation induced on the exceptional divisor in question must admit a non-constant meromorphic first integral. To motivate this discussion, let us indicate how the existence of separatrices is related to this type of behavior on dicritical divisors. Indeed, the proof of the main result in this paper depends on a "topological" theorem for codimension 1 foliations, not necessarily spanned by commuting vector fields, that reads as follows:

Theorem 1. *Let $\mathcal{D}$ be a codimension* 1 *foliation defined on a neighborhood of the origin in $\mathbb{C}^3$ and consider a reduction of singularities (cf. Section* 4*)*

$$\mathcal{D} = \mathcal{D}^0 \xleftarrow{\ \pi_1\ } \mathcal{D}^1 \xleftarrow{\ \pi_2\ } \cdots \xleftarrow{\ \pi_k\ } \mathcal{D}^k$$

for $\mathcal{D}$. Consider those irreducible components E_j of the total exceptional divisor that are compact but that are not invariant by $\mathcal{D}^k$. Suppose that for every component E_j as above the (1-dimensional) foliation induced on E_j by intersecting E_j with the leaves of $\mathcal{D}^k$ possesses a non-constant meromorphic first integral. Then $\mathcal{D}$ possesses a separatrix.

Theorem 1 provides a sufficient condition to guarantee the existence of separatrices for a codimension 1 foliation which will be used to establish the theorem stated in the Introduction. Also Theorem 1 points out a kind of "universal character" in the previously discussed examples of codimension 1 foliations without separatrices.

Naturally the statement of the preceding theorem makes an implicit use of the existence of a "reduction of singularities" procedure for codimension 1 foliations defined on a neighborhood of $(0,0,0) \in \mathbb{C}^3$. The existence of this procedure is a fundamental theorem appearing in [C-C] and [Ca]. The accurate form of these results will be recalled in Section 4 where the proof of Theorem 1 will be supplied. We point out in particular that the mentioned proof can be read independently of the material discussed in this section and in Section 3.

Let us however mention once and for all that the procedure of reducing the singularities of $\mathcal{D}$ given in [C-C], [Ca] requires us to use two types of blow-up, namely those centered at a single point and those centered at a smooth (irreducible) curve. In either case the transform $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ is well-defined and consists of a singular holomorphic foliation defined on the new (blown-up) manifold. Here is a good point to recall some terminology that will be used throughout this paper. First a blow-up map with center C is called a *dicritical blow-up for a codimension* 1 *singular foliation* $\mathcal{D}$ if and only if C is invariant by $\mathcal{D}$ and the exceptional divisor introduced by this blow-up map is not invariant by the transform $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ (equivalently the exceptional divisor is transverse to $\widetilde{\mathcal{D}}$ at generic points). A foliation of codimension 1 is called *dicritical* if and only if there is a finite sequence of blow-ups with invariant centers such that the last blow-up is dicritical for the corresponding transform of the initial foliation. If such a sequence of blow-ups does not exist, the foliation is said to be *non-dicritical*. Concerning deciding whether or not a given codimension 1 foliation is dicritical, the reduction theorem of [Ca] yields a suitable condition: a foliation is dicritical if and only if there is a dicritical blow-up in the sequence of blowing ups leading to the reduction of singularities (more details can be found in Section 4). Finally an irreducible component of a divisor in an ambient space where a codimension 1 foliation $\mathcal{D}$ is defined is said to be *dicritical for* $\mathcal{D}$ if and only if this component is not invariant by $\mathcal{D}$ (and thus it is transverse to $\mathcal{D}$ at generic points).

Consider then the transform $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ under a dicritical blow-up as above. To implement the strategy outlined in the Introduction and, in particular, to be able to make use of Theorem 1, our next task is to find conditions ensuring that the foliation induced on the corresponding exceptional divisor by $\widetilde{\mathcal{D}}$ admits a non-constant meromorphic first integral. The remainder of this section is devoted to this question in the case of a blow-up centered at a point. The case of blow-ups centered over curves is going to be treated in Section 3. Naturally in the course of this discussion we shall assume that $\mathcal{D}$ is spanned by two commuting vector fields X, Y that are linearly independent at generic points. The desired conditions are summarized by Propositions 1 and 2. Proposition 1 concerning punctual blow-ups is the main result of this section. Proposition 2 is the analogue of Proposition 1 for the case of blow-

ups centered over a curve and it will be the object of Section 3. Finally, though the statement of Propositions 1 and 2 do not cover all foliations we are going to deal with, it will turn out that the class of foliations failing to satisfy their conditions is rather small: it essentially consists of foliations that are "almost linear" in an appropriate sense. In particular the existence of separatrices for these "almost linear" foliations can directly be checked.

From now on we fix a holomorphic 1-form Ω defining $\mathcal{D}$ and having singular set of codimension at least 2. In particular the singular set Sing $(\mathcal{D})$ of $\mathcal{D}$ consists of isolated points and analytic curves. In the remainder of this section we shall exclusively deal with the case of punctual blow-ups.

Suppose then that the blow-up centered at the origin is dicritical for $\mathcal{D}$. We are going to prove that the foliation induced on the resulting exceptional divisor by the transform of $\mathcal{D}$ possesses a non-constant meromorphic first integral provided that $\mathcal{D}$ is spanned by vector fields X, Y satisfying Conditions 1 and 2 in the beginning of the section (cf. Proposition 1). In particular observe that Condition 2 already gives some hint on the "linear nature" of foliations failing to satisfy the assumption of Theorem 1 (or of Propositions 1 and 2).

To begin our approach to Main Theorem and, more precisely, to Proposition 1, we are going to give a characterization of foliations for which the blow-up centered at the origin is dicritical. With this purpose, let us denote by R the radial vector field (Euler vector field)

$$R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

Given a holomorphic vector field defined about $(0,0,0) \in \mathbb{C}^3$, its first non-trivial homogeneous component at the origin is the homogeneous component of lowest degree (and not identically zero) of the Taylor series of this vector field at the origin.

Lemma 1. *Suppose that $\mathcal{D}$ is a singular codimension 1 foliation and suppose that the punctual blow-up π of the origin is dicritical for $\mathcal{D}$. Then there is a holomorphic vector field Z tangent to $\mathcal{D}$ whose first non-trivial homogeneous component at the origin is a multiple of the radial vector field R. Conversely the existence of a vector field Z as above ensures that the blow-up of the origin is dicritical for $\mathcal{D}$.*

Proof. Let $\mathcal{D}$ be given by the holomorphic 1-form $\Omega = F\,dx + G\,dy + H\,dz$ whose singular set has codimension 2 or greater. Denote by Ω_d the first non-trivial homogeneous component of Ω at $(0,0,0)$. Here d stands for the degree of Ω_d. Furthermore set $\Omega_d = F^d\,dx + G^d\,dy + H^d\,dz$. A direct inspection shows that π is dicritical for $\mathcal{D}$ if and only if

$$xF^d + yG^d + zH^d = 0. \tag{2}$$

The triple (x, y, z) is therefore a solution of $\Omega_d = 0$. Hence it can be expressed as a linear combination over the rational functions of the two independent solutions

$(G^d, -F^d, 0)$ and $(H^d, 0, -F^d)$. Thus we have

$$(x, y, z) = A(G^d, -F^d, 0) + B(H^d, 0, -F^d)$$

for certain rational functions A, B. Now a rational multiple Z of the vector field $A(G, -F, 0) + B(H, 0, -F)$ satisfies the requirements. For the converse, just note that the first non-trivial homogeneous component of a vector field tangent to $\mathcal{D}$ must yield a solution for $\{\Omega_d = 0\}$. This applies in particular to the vector field Z. It then follows that Equation (2) is satisfied and thus that π is dicritical for $\mathcal{D}$.   □

Let us now return to our original setting where $\mathcal{D}$ is spanned by vector fields X, Y satisfying conditions 1 and 2.

Consider a homogeneous polynomial vector field Z of degree $d \geq 2$. The Euler relation then provides

$$[R, Z] = (d - 1)Z. \tag{3}$$

In particular R and Z do not commute. Furthermore, if Z is not a multiple of R, then the same holds when R is replaced by a multiple hR. Indeed, we have

$$[hR, Z] = h[R, Z] - (Z.h)\, R = (d - 1)hZ - \left(\frac{\partial h}{\partial Z}\right) R. \tag{4}$$

It is then clear that $[hR, Z] \neq 0$ provided that R, Z are linearly independent.

Next we have a simple lemma concerning the vanishing of the Lie bracket for a special type of vector fields.

Lemma 2. *Let Z_1, Z_2 be commuting vector fields defined about $(0, 0, 0) \in \mathbb{C}^3$. Suppose that the first non-trivial homogeneous component of Z_1 at $(0, 0, 0)$ is a multiple of R. Suppose also that the linear part of Z_2 at the origin is zero. Then Z_1, Z_2 are linearly dependent at every point.*

Proof. Consider the punctual blow-up $\widetilde{Z}_1$ (resp. $\widetilde{Z}_2$) of Z_1 (resp. Z_2) at the origin. Since the first non-trivial homogeneous component of Z_1 is a multiple of R, we set $Z_1^d = hR$ where h is a homogeneous polynomial of degree $d - 1$. Then it follows that the foliation associated to $\widetilde{Z}_1$ is transverse to $\pi^{-1}(0)$ away from the transform of $\{h = 0\}$. Thus we can fix local coordinates (x, t, u), $\pi^{-1}(0) \subset \{x = 0\}$, about a generic point of $\pi^{-1}(0)$ in which the vector field $\widetilde{Z}_1$ becomes $f(x, t, u)\partial/\partial x$. In these coordinates, we set $\widetilde{Z}_2 = \widetilde{Z}_{2,1}\partial/\partial x + \widetilde{Z}_{2,2}\partial/\partial t + \widetilde{Z}_{2,3}\partial/\partial u$. The equation $[\widetilde{Z}_1, \widetilde{Z}_2] = 0$ then yields

$$\frac{\partial \widetilde{Z}_{2,2}}{\partial x} = \frac{\partial \widetilde{Z}_{2,3}}{\partial x} = 0.$$

Therefore $\widetilde{Z}_{2,2}$ and $\widetilde{Z}_{2,3}$ do not depend on the variable x. However, since the linear part of Z_2 equals zero at the origin, $\widetilde{Z}_2$ vanishes identically on $\pi^{-1}(0)$. Thus its components $\widetilde{Z}_{2,2}$, $\widetilde{Z}_{2,3}$ must vanish everywhere since they do not depend on x. It

follows that Z_1, Z_2 are linearly dependent on an open set and therefore they are so everywhere. The lemma is proved.   □

Remark 1. Considering the vector fields X, Y introduced in the beginning of the section, let X^H (resp. Y^H) denote the first non-trivial homogeneous component of X (resp. Y) at the origin. As a consequence of the above lemma, we conclude that neither X^H nor Y^H is a multiple of the radial vector field R.

Also note that the assumption concerning the vanishing of the linear part of Z_2 at the origin was used only to ensure that the blown-up vector field $\tilde{Z}_2$ vanishes identically on the corresponding exceptional divisor.

Let us go back to the vector fields X, Y considered in the beginning of the section. Recall that $\mathcal{D}$ stands for the codimension 1 foliation spanned by X, Y. Suppose that the punctual blow-up π of the origin is dicritical for $\mathcal{D}$. Then, according to Lemma 1, there are holomorphic functions f, g and h such that

$$fX + gY = hZ, \tag{5}$$

where Z is a holomorphic vector field whose first non-trivial homogeneous component at the origin is a multiple of R. Let ord (fX) (resp. ord (gY), ord (hZ)) denote the order of the vector field fX (resp. gY, hZ) at the origin, i.e. the degree of the first jet of fX (resp. gY, hZ) that is not zero at the origin or, equivalently, the degree of the component of lowest degree effectively appearing in the Taylor expansion of fX (resp. gY, hZ) at the origin.

Lemma 3. *With the above notations we have the following alternative*

(1) ord $(hZ) > \min\{$ord (fX), ord $(gY)\}$.

(2) *the first non-trivial homogeneous component X^H of X (resp. Y^H of Y) admits a non-constant meromorphic first integral.*

Proof. Suppose that ord $(hZ) = \min\{$ord (fX), ord $(gY)\}$. All we have to do is to show that X^H possesses a non-constant first integral. Denote respectively by f^H, g^H, h^H the first non-trivial homogeneous components of f, g, h. Analogously Z^H is the first non-trivial homogeneous component of Z. Without loss of generality, we can assume that ord $(fX) \leq$ ord (gY). We claim that, indeed, these two orders must be equal, i.e. ord $(fX) =$ ord (gY). To check this claim first recall that X^H is not a multiple of the radial vector field R, cf. Remark 1. Then just note that the existence of a strict inequality ord $(fX) <$ ord (gY) would imply that $f^H X^H$, and thus X^H itself, is a multiple of R as it can be seen by considering the first non-trivial homogeneous component in Equation (5). From this we conclude that ord $(fX) =$ ord (gY) as desired.

Therefore, by considering again the first non-trivial homogeneous component in Equation (5), it follows that

$$f^H X^H + g^H Y^H = h^H q^H R, \tag{6}$$

where, by assumption, none of the two sides vanishes identically. Furthermore q^H is also a homogeneous polynomial.

Because X, Y commute, so do X^H, Y^H. Thus we have

$$
\begin{aligned}
[X^H, Y^H] &= \left[X^H, \frac{h^H q^H}{g^H} R - \frac{f^H}{g^H} X^H \right] \\
&= \left[X^H . \left(\frac{h^H q^H}{g^H} \right) \right] R - \frac{h^H q^H}{g^H} [R, X^H] - \left[X^H . \left(\frac{f^H}{g^H} \right) \right] X^H \\
&= \left[X^H . \left(\frac{h^H q^H}{g^H} \right) \right] R - \left[(d-1) \frac{h^H q^H}{g^H} - X^H . \left(\frac{f^H}{g^H} \right) \right] X^H \\
&= 0,
\end{aligned}
$$

where d denotes de degree of X^H. In particular

$$
\left[X^H . \left(\frac{h^H q^H}{g^H} \right) \right] R = \left[(d-1) \frac{h^H q^H}{g^H} - X^H . \left(\frac{f^H}{g^H} \right) \right] X^H
$$

The expression between brackets on the left hand side (i.e. the expression multiplying R) must vanish identically for otherwise X^H would be a multiple of the radial vector field. It then follows that

$$
X^H . \left(\frac{h^H q^H}{g^H} \right) = 0.
$$

In other words, $h^H q^H / g^H$ is a meromorphic first integral for X^H.

It only remains to prove that $h^H q^H / g^H$ is not constant. However, if this function is constant, then we can assume $h^H q^H / g^H = 1$ without loss of generality. Hence dividing (6) by g^H, it would follow

$$
\frac{f^H}{g^H} X^H + Y^H = R.
$$

This last equation is however impossible since Y^H has degree at least 2 and the expression $f^H X^H / g^H$ is homogeneous. Therefore $h^H q^H / g^H$ cannot be constant. Since the argument is symmetric in the vector fields X, Y, the last assertion completes our proof. $\qquad \square$

We are now able to prove the main proposition of this section. It summarizes the preceding results and clarifies the nature of the foliation induced by $\mathscr{D}$ on the exceptional divisor in the case where the blow-up of the origin is dicritical for $\mathscr{D}$.

Proposition 1. *Let X, Y be two commuting vector fields satisfying Conditions 1 and 2 at the beginning of the section. Denote by $\mathfrak{D}$ the codimension 1 foliation spanned by X, Y and suppose that the punctual blow-up π of the origin is dicritical for $\mathfrak{D}$. Then one has:*

(1) *The foliations $\widetilde{\mathcal{F}}_X$ and $\widetilde{\mathcal{F}}_Y$ coincide in their restriction to the exceptional divisor $\pi^{-1}(0)$, where $\widetilde{\mathcal{F}}_X$ (resp. $\widetilde{\mathcal{F}}_Y$) stands for the transform of $\mathcal{F}_X$ (resp. $\mathcal{F}_Y$) by the blow-up map π in question.*

(2) *The restrictions to $\pi^{-1}(0)$ of $\widetilde{\mathcal{F}}_X$ and $\widetilde{\mathcal{F}}_Y$ also coincide with the foliation induced on $\pi^{-1}(0)$ by intersecting $\pi^{-1}(0)$ with the leaves of $\widetilde{\mathfrak{D}}$, where $\widetilde{\mathfrak{D}}$ stands for the transform of $\mathfrak{D}$.*

(3) *The restrictions of $\widetilde{\mathcal{F}}_X$, $\widetilde{\mathcal{F}}_Y$ to $\pi^{-1}(0)$ possess a meromorphic first integral. In other words, the foliation induced by $\widetilde{\mathfrak{D}}$ on $\pi^{-1}(0)$ defines a pencil on $\pi^{-1}(0) \simeq \mathbb{C}P(2)$.*

Proof. Recall that X^H, Y^H denote the first non-trivial homogeneous components of respectively X, Y at the origin. We know that none of the vector fields X^H, Y^H is a multiple of the radial vector field R. In particular each of them induces a non-trivial foliation on the exceptional divisor $\pi^{-1}(0) \simeq \mathbb{C}P(2)$. Let us first check that these two foliations actually coincide. The condition for these two foliations to coincide is that either the vector fields X^H, Y^H are parallel or they span a 2-dimensional plane containing the radial direction (away from a proper analytic set). If none of these possibilities hold, then a direct inspection in the coefficients of the 1-form defining $\mathfrak{D}$, cf. Equation 1, will show that Equation 2 cannot be satisfied. In fact, the coefficients of the mentioned 1-form are given by the vectorial product between X^H, Y^H so that they cannot satisfy Equation 2 unless one of the two possibilities above is verified. In turn the fact that Equation 2 is not satisfied leads to a contradiction with the dicritical character of π for $\mathfrak{D}$. Therefore it follows that the foliations induced by X^H, Y^H on the exceptional divisor are the same. Finally it becomes equally clear that these two foliations coincide also with the foliation obtained by intersecting $\pi^{-1}(0)$ with the leaves of $\mathfrak{D}$. This proves the first two conclusions in the statement of our proposition.

To complete the proof it suffices to show that X^H admits non-constant meromorphic first integral. Indeed, since X^H is not a multiple of the radial vector field the mentioned first integral also yields a non-constant first integral for the foliation induced by the projection of X^H on $\mathbb{C}P(2)$.

To show the existence of the desired first integral for X^H, recall that Lemma 1 ensures the existence of a holomorphic vector field Z satisfying Equation 5 for suitable holomorphic functions f, g, h. Furthermore the first non-trivial homogeneous component of Z at the origin is a multiple of the radial vector field R. In turn Lemma 3 allows us to suppose that ord $(hZ) > \min\{\text{ord}\,(fX),\ \text{ord}\,(gY)\}$ (strictly) without loss of generality. In particular, we must have ord $(fX) = \text{ord}\,(gY)$ and, in fact,

$$f^H X^H + g^H Y^H = 0.$$

Alternatively we write

$$\frac{f^H}{g^H} X^H + Y^H = 0.$$

Therefore

$$\left[X^H, \frac{f^H}{g^H} X^H + Y^H \right] = [X^H, 0] = 0.$$

However, since $[X^H, Y^H] = 0$, the above equation amounts to

$$\left[X^H \cdot \left(\frac{f^H}{g^H} \right) \right] . X^H = 0$$

so that $X^H .(f^H/g^H)$ must vanish identically. In other words f^H/g^H is a first integral for X^H. The statement is then proved unless f^H/g^H is constant. Therefore we just need to consider this latter possibility. This means that X^H and Y^H differ by a multiplicative constant. Set $X^H = cY^H$ for some $c \in \mathbb{C}^*$. Now note that the order of the new vector field $Y' = X - cY$ must be strictly larger than the order of X. Besides Y' is not constant equal to *zero* since X, Y are linearly independent at generic points. In fact, Y' is itself linearly independent with X at generic points. Furthermore the vector fields X, Y' still commute and they span the same foliation $\mathcal{D}$ as the initial pair X, Y. Therefore we can repeat the argument using X, Y' instead of X, Y. By construction the first non-trivial homogeneous component of Y' cannot differ from X^H by a multiplicative constant. Therefore X^H must admit a non-constant meromorphic first integral. The proposition is proved. $\square$

3. On the dicritical character of $\mathcal{D}$, Part II: Blowing-up a curve

The next step towards the proof of the existence of separatrices consists of obtaining an analogue of Proposition 1 for the case of blow-ups centered over smooth (irreducible) curves contained in Sing $(\mathcal{D})$. Indeed this section is entirely devoted to discussing the effect of blowing-up a smooth curve contained in the singular set of $\mathcal{D}$.

Consider a point p belonging to a smooth curve contained in Sing $(\mathcal{D})$. On a neighborhood of p, there are local coordinates (x, y, z) in which the curve in question coincides with the z-axis, i.e. it is given by $\{x = y = 0\}$. For the blow-up centered at $\{x = y = 0\}$, we can introduce affine coordinates (x, t, z) and (u, y, z) such that the resulting blow-up map π_z is given by $\pi_z(x, t, z) = (x, tx, z)$ (resp. $\pi_z(u, y, z) = (uy, y, z)$). In the context of blow-ups centered over a fixed curve, the expression "a generic point of $\{x = y = 0\}$" is a synonymous of "except for a finite set of points". If the neighborhood of $(0, 0, 0) \in \mathbb{C}^3$ can be reduced without affecting the generality of the discussion then "a generic point" becomes an expression

equivalent to "for every point in $\{x = y = 0\}$ with possible exception of the origin". Finally let R_z denote the vector field defined by

$$R_z = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

In the case of punctual blow-ups, the first non-trivial homogeneous component of X, Y played the main role in establishing the existence of a non constant meromorphic first integral for the foliation induced over $\mathbb{C}\mathbb{P}(2)$ under appropriate conditions. We shall begin this section by introducing an analogue of these "first non-trivial homogeneous components" which are adapted to deal with the blow-up over the curve $\{x = y = 0\}$. In particular, the corresponding "homogeneous" vector fields will satisfy the same properties as those satisfied by the first components of X, Y in the case of punctual blow-ups. Most notably, they will still commute and they will also encode the information determining the dicritical/non-dicritical nature of the corresponding blow-up for $\mathcal{D}$.

To motivate the definition, note that the first non-trivial homogeneous component of X can be recovered as

$$\lim_{\lambda \to 0} \frac{1}{\lambda^{d-1}}\Gamma_\lambda^* X,$$

where $\Gamma_\lambda^* X$ denotes the pull-back of X by the homothety

$$\Gamma_\lambda \colon (x, y, z) \mapsto (\lambda x, \lambda y, \lambda z)$$

and d stands for the degree of the first non-trivial homogeneous component of X. In the present case we shall consider an adapted notion of homothety, namely the one obtained through the family of automorphisms given by

$$\Lambda_\lambda \colon (x, y, z) \mapsto (\lambda x, \lambda y, z).$$

In this case, the pull-back of X by Λ_λ becomes

$$\Lambda_\lambda^* X = \frac{1}{\lambda}\left[X_1(\lambda x, \lambda y, z)\frac{\partial}{\partial x} + X_2(\lambda x, \lambda y, z)\frac{\partial}{\partial y}\right] + X_3(\lambda x, \lambda y, z)\frac{\partial}{\partial z}$$

Denote by k (resp. l) the maximal power of λ that divides $X_1(\lambda x, \lambda y, z)\partial/\partial x + X_2(\lambda x, \lambda y, z)\partial/\partial y$ (resp. $X_3(\lambda x, \lambda y, z)\partial/\partial z$). It corresponds to the degree of the first non-trivial homogeneous components relative to the variables x, y in each expression. Then

$$\Lambda_\lambda^* X = \lambda^{k-1}\left[(X_1^k(x, y, z) + \lambda\overline{X}_{1,\lambda}(x, y, z))\frac{\partial}{\partial x}\right.$$

$$+ (X_2^k(x, y, z) + \lambda\overline{X}_{2,\lambda}(x, y, z))\frac{\partial}{\partial y}\Big]$$

$$+ \lambda^l(X_3^l(x, y, z) + \lambda\overline{X}_{3,\lambda}(x, y, z))\frac{\partial}{\partial z},$$

where X_i^j stands for the homogeneous component of degree j, of X_i, relative to the variables x, y. Since the powers of λ in the different components do not coincide, three cases must be considered according to the possibilities $l > k - 1, l = k - 1$ or $l < k - 1$. The expression for the limit vector field $\bar{X}$ will change accordingly. Naturally, when it comes to blow-ups centered over a smooth curve, the vector field $\bar{X}$ is going to represent the desired analogue of the "first non-trivial homogeneous component" as considered in the preceding section.

a) Suppose $l > k - 1$. In this case we consider the vector fields

$$X_\lambda = \frac{1}{\lambda^{k-1}} \Lambda_\lambda^* X$$

and the limit

$$\bar{X} = \lim_{\lambda \to 0} X_\lambda.$$

The vector field $\bar{X}$ has the form

$$\bar{X} = X_1^k(x, y, z) \frac{\partial}{\partial x} + X_2^k(x, y, z) \frac{\partial}{\partial y}$$

and it can be thought of as the "first non-trivial homogeneous component in the variables x, y".

b) If $l = k - 1$ we still consider the vector field $X_\lambda = \Lambda_\lambda^* X / \lambda^{k-1}$ and define $\bar{X}$ as above. A similar argument shows that $\bar{X}$ has the form

$$\bar{X} = X_1^k(x, y, z) \frac{\partial}{\partial x} + X_2^k(x, y, z) \frac{\partial}{\partial y} + X_3^{k-1}(x, y, z) \frac{\partial}{\partial z}.$$

c) Finally, if $l < k - 1$ we let

$$X_\lambda = \frac{1}{\lambda^l} \Lambda_\lambda^* X$$

and consider $\bar{X} = \lim_{\lambda \to 0} X_\lambda$ which, in this case, is a "vertical" vector field

$$\bar{X} = X_3^l(x, y, z) \frac{\partial}{\partial z}.$$

As mentioned, the vector field $\bar{X}$ represents the analogue of the first non-trivial homogeneous component of X under the adapted homothety Λ_λ. Given a vector field X as above, throughout this section the *first non-trivial homogeneous component of X in the variables x, y* is, by definition, the vector field $\bar{X}$ constructed above. The *degree w.r.t. the variables x, y* (or simply *degree* when no misunderstanding is possible) of $\bar{X}$ is by definition the minimum between k and $l + 1$. Similarly the *order of X in the*

variables x, y will be degree w.r.t. the variables x, y of $\overline{X}$. Finally note that $\overline{X}$ may also be viewed as a homogeneous polynomial vector field in the variables x, y with coefficients in $\mathbb{C}[z]$ (except that the degree of the third component may differ from the degree of the other two cf. below).

The vector field $\overline{Y}$ is analogously defined. The commutativity of $\overline{X}$ and $\overline{Y}$ follows easily from the commutativity between X and Y. In fact, $\Lambda_\lambda^* X$ commutes with $\Lambda_\lambda^* Y$ for every λ. The same being true when these two vector fields are multiplied by arbitrary constants. Hence, by taking a suitable limit, we conclude that the vector fields $\overline{X}$ and $\overline{Y}$ must commute as well.

A similar notion of "first non-trivial homogeneous component" is also natural for a codimension 1 foliation given by a holomorphic 1-form $F dx + G dy + H dz$. Again this "first non-trivial homogeneous component in the variables x, y" is obtained as an appropriate limit of pull-backs of $F dx + G dy + H dz$ by the automorphisms $\Lambda_\lambda : (x, y, z) \mapsto (\lambda x, \lambda y, z)$. Details are left to the reader. We point out, however, that the resulting component can equally well be seen as a polynomial 1-form in the variables x, y with coefficients in $\mathbb{C}[z]$. Nonetheless the case analogous to the case "b)" (where $l = k - 1$) of vector fields in which this "first non-trivial homogeneous component in the variables x, y" may have non-trivial components in all the directions dx, dy, dz is now associated to the possibility $l = k + 1$.

Consider now the codimension 1 foliation $\mathcal{D}$ spanned by X, Y. The following lemma will play, in the context of blow-ups centered over curves, a role analogous to the role played by Lemma 1 in the preceding section.

Lemma 4. *Suppose that $\mathcal{D}$ is singular over $\{x = y = 0\}$ and denote by π_z the blow-up map centered over this curve. Suppose that π_z is dicritical for $\mathcal{D}$. Then there exists a holomorphic vector field Z tangent to $\mathcal{D}$ whose first non-trivial homogeneous component $\overline{Z}$ in the variables x, y is a multiple of R_z having the form $P_z(x, y) R_z$, where P_z stands for a homogeneous polynomial in the variables x, y with coefficients in $\mathbb{C}[z]$.*

Proof. To prove the statement let us consider a holomorphic 1-form $\omega = F dx + G dy + H dz$ defining $\mathcal{D}$ and having singular set of codimension at least 2. Denote by $\bar{\omega} = f_{d,z} dx + g_{d,z} dy + h_{d,z} dz$ the first non-trivial homogeneous component of ω relative to the variables x, y. Recall that $f_{d,z}$, $g_{d,z}$ (resp. $h_{d,z}$) are either identically zero or homogeneous polynomials of degree d (resp. $d - 1$) in the variables x, y with coefficients in $\mathbb{C}[z]$. The dicritical nature of π_z for $\mathcal{D}$ is equivalent to saying that the vector field R_z is tangent to the leaves of the foliation defined by $\bar{\omega}$ i.e. π_z is dicritical for $\mathcal{D}$ if and only if

$$x f_{d,z} + y g_{d,z} = 0.$$

Though this statement is slightly less known than its analogous in the case of punctual blow-ups, it can explicitly be checked. Recalling the existence of affine coordinates

(x, t, z) for the blow-up of $\mathbb{C}^3$ over $\{x = y = 0\}$ in which the blow-up map π_z becomes $\pi_z(x, t, z) = (x, tx, z)$, the transform $\pi_z^*(\bar{\omega})$ of $\bar{\omega}$ is

$$\pi_z^*(\bar{\omega}) = [f_{d,z}(1, t) + t g_{d,z}(1, t)]dx + x g_{d,z}(1, t)dt + x h_{d,z}(1, t)dz. \qquad (7)$$

Thanks to the above formula, it becomes immediate to check that the exceptional divisor (locally given by $\{x = 0\}$) will be invariant by the transform of $\mathcal{D}$ if and only if $f_{d,z}(1, t) + t g_{d,z}(1, t)$ vanishes identically. In turn this establishes our claim.

To complete the proof of the lemma, we still need to construct the vector field Z. This however goes as in Lemma 1. The triple $(x, y, 0)$ is a solution for $\bar{\omega} = f_{d,z}dx + g_{d,z}dy + h_{d,z}dz = 0$ and therefore it must be given as a linear combination over rational functions of the two independent solutions that can be obtained by means of the expression for $\bar{\omega} = 0$ (assuming for example that none of the polynomials $f_{d,z}$, $g_{d,z}$, $h_{d,z}$ vanishes identically, these solutions may be chosen as $(g_{d,z}, -f_{d,z}, 0)$ and $(h_{d,z}, 0, -f_{d,z})$). The rest of the proof is analogous to the proof of Lemma 1. $\qquad \square$

Next let $\mathcal{F}_X$ be the foliation associated to X. We assume that the axis $\{x = y = 0\}$ is invariant by X. This is a good point to note the difference between being invariant by $\mathcal{F}_X$ and being invariant by X. To be invariant by $\mathcal{F}_X$, the axis $\{x = y = 0\}$ must be contained in the singular set of $\mathcal{F}_X$ or contained in a regular leaf of $\mathcal{F}_X$ or yet contained in the union of a regular leaf with the singular set. If $\{x = y = 0\}$ is invariant by $\mathcal{F}_X$ then it is automatically invariant by X as well. The converse however is not true: for example, we might have $X = x\partial/\partial x$. The axis $\{x = y = 0\}$ is invariant by X since it is contained in the zero-set of X, however it is not invariant by $\mathcal{F}_X$ which is induced by $\partial/\partial x$. Let k and l be as defined above (in connection with the first non-trivial homogeneous component $\bar{X}$ in the variables x, y of X).

The notion of dicritical blow-up (resp. dicritical component of a divisor in the ambient space) for foliations of dimension 1 is analogous to the corresponding notion for foliations of codimension 1. Thus a blow-up with center C is said to be dicritical for a 1-dimensional foliation $\mathcal{F}$ if and only if C is invariant by $\mathcal{F}$ and the transform of $\mathcal{F}$ is transverse to the corresponding exceptional divisor at generic points. A component of a divisor in the ambient space is dicritical for $\mathcal{F}$ if and only if $\mathcal{F}$ is transverse to this component at generic points.

Going back to X, $\mathcal{F}_X$, a direct inspection in the formulas related to the preceding possibilities a), b) and c) for the nature of $\bar{X}$ makes it clear that the blow-up π_z is dicritical for $\mathcal{F}_X$ if and only if $\bar{X}$ is a multiple of R_z at points in $\{x = y = 0\}$, i.e. if and only if $\bar{X}$ has the form $P_z(x, y)(x\partial/\partial x + y\partial/\partial y)$ for some homogeneous polynomial P_z in the variables x, y with coefficients in $\mathbb{C}[z]$.

Next we state:

Lemma 5. *Suppose that Z_1, Z_2 are two commuting vector fields defined on $(\mathbb{C}^3, 0)$. Suppose that the first non-trivial homogeneous component of Z_1 in the variables x, y at points in $\{x = y = 0\}$ has the form $P_z(x, y)(x\partial/\partial x + y\partial/\partial y)$ for a homogeneous*

polynomial P in the variables x, y with coefficients in $\mathbb{C}[z]$. Suppose also that the order of Z_2 relative to the variables x, y at points in $\{x = y = 0\}$ is at least 2. Then Z_1, Z_2 are linearly dependent everywhere.

Proof. Keeping the preceding notations, consider the blow-up map π_z. The transforms $\widetilde{Z}_1$, $\widetilde{Z}_2$ of Z_1, Z_2 under π_z are holomorphic on a neighborhood of the corresponding exceptional divisor $\pi_z^{-1}(0)$. Besides the foliation associated to $\widetilde{Z}_1$ is transverse to $\pi_z^{-1}(0)$ at generic points. Thus, on a neighborhood of a generic point of $\pi_z^{-1}(0)$, we can introduce coordinates (x, t, u) such that:

(1) The foliation associated to $\widetilde{Z}_1$ is given by $\partial/\partial x$.

(2) $\{x = 0\} \subset \pi_z^{-1}(0)$ is contained in the set of zeros of $\widetilde{Z}_2$.

It then follows that $\widetilde{Z}_1$ is given in these coordinates by $f\,\partial/\partial x$ for some holomorphic function f. The rest of the proof goes exactly as in the proof of Lemma 2. More precisely, the condition on the vanishing of the Lie bracket of $\widetilde{Z}_1$, $\widetilde{Z}_2$ ensures that the components of $\widetilde{Z}_2$ in the coordinates t, z do not depend on the variable x. Since $\widetilde{Z}_2$ equals zero over the exceptional divisor, locally given by $\{x = 0\}$, it follows that these components must be identically zero. In other words, $\widetilde{Z}_1$, $\widetilde{Z}_2$ must be parallel on an open set and hence everywhere. $\qquad\square$

Remark 2. Note that the assumptions concerning the order of Z_2 and the first nontrivial homogeneous component of Z_1 in the variables x, y were used only in the items 1 and 2 in the above proof. The assumption on the order of Z_2 relative to the variables x, y at points in $\{x = y = 0\}$ was necessary to guarantee that $\widetilde{Z}_2$ must be equal to zero on all of the exceptional divisor $\pi_z^{-1}(0)$. In view of the examples discussed in the Introduction, it is clear that this assumption cannot be removed from the statement. Here it might be a good point to remind the reader that the order of the vector field $X = zy\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}$ in the variables y, z over the axis $\{y = z = 0\}$ equals indeed *zero*. Similarly the blow-up of X over $\{y = z = 0\}$ yields a holomorphic vector field which does not vanish at generic points of the resulting exceptional divisor.

On the other hand the assumption on Z_1 simply means that the blow-up π_z is dicritical for the foliation associated to Z_1.

Before being able to state and prove Proposition 2, we are going to need the corresponding analogue of Lemma 3.

Again we go back to the vector fields X, Y that span the codimension 1 foliation $\mathcal{D}$. However now we suppose that $\{x = y = 0\}$ is contained in Sing $(\mathcal{D})$ and that the blow-up map π_z centered over $\{x = y = 0\}$ is dicritical for $\mathcal{D}$. Then, according to Lemma 4, there are holomorphic functions f, g and h such that

$$fX + gY = hZ, \tag{8}$$

where Z is a holomorphic vector field whose first non-trivial homogeneous component $\overline{Z}$ in the variables x, y at points in $\{x = y = 0\}$ is a multiple of R_z. Let

ord (fX) (resp. ord (gY), ord (hZ)) denote the order of the vector field fX (resp. gY, hZ) in the variables x, y at a generic point in $\{x = y = 0\}$.

Lemma 6. *With the above notations, we have the following alternative:*

(1) ord (hZ) > min{ord (fX), ord (gY)}.

(2) $\overline{X}$, *the first homogeneous component in the variables x, y of X, admits a non-constant meromorphic first integral.*

Proof. Suppose for a contradiction that the above estimate does not hold. For fX, gY and hZ we are going to consider their first homogeneous components in the variables x, y. Denote by $\overline{X}$, $\overline{Y}$ the respective non-trivial homogeneous components of X, Y and by $\overline{f}$, $\overline{g}$, $\overline{h}$ the homogeneous components of f, g, h (all these homogeneous components are to be understood as relative to the variables x, y). With these notations, one has:

$$\overline{f}\,\overline{X} + \overline{g}\,\overline{Y} = \overline{h}\overline{q}R_z, \tag{9}$$

where $\overline{q}$ is a homogeneous polynomial in the variables x, y with coefficients in $\mathbb{C}[z]$. Since X, Y commute, it follows that $\overline{X}$, $\overline{Y}$ commute as well. Therefore

$$[\overline{X}, \overline{Y}] = \left[\overline{X}, \frac{\overline{h}\overline{q}}{\overline{g}}R_z - \frac{\overline{f}}{\overline{g}}\overline{X}\right]$$

$$= \left[\overline{X}.\left(\frac{\overline{h}\overline{q}}{\overline{g}}\right)\right]R_z - \frac{\overline{h}\overline{q}}{\overline{g}}[R_z, \overline{X}] - \left[\overline{X}.\left(\frac{\overline{f}}{\overline{g}}\right)\right]\overline{X}$$

$$= 0.$$

The commutator $[R_z, \overline{X}]$ is given by

$$[R_z, \overline{X}] = \left(x\frac{\partial \overline{X}_1}{\partial x} + y\frac{\partial \overline{X}_1}{\partial y} - X_1\right)\frac{\partial}{\partial x}$$

$$+ \left(x\frac{\partial \overline{X}_2}{\partial x} + y\frac{\partial \overline{X}_2}{\partial y} - X_2\right)\frac{\partial}{\partial y} + \left(x\frac{\partial \overline{X}_3}{\partial x} + y\frac{\partial \overline{X}_3}{\partial y}\right)\frac{\partial}{\partial z}.$$

As previously seen, the components $\overline{X}_1$, $\overline{X}_2$ are homogeneous of degree k in the variables x, y while $\overline{X}_3$ is homogeneous of degree $k - 1$, if not identically zero. In fact, the remaining case in which $\overline{X}$ has only a component in the direction of $\partial/\partial z$ obviously admits a non-constant first integral so that it can be ignored in this discussion. Therefore

$$x\frac{\partial \overline{X}_i}{\partial x} + y\frac{\partial \overline{X}_i}{\partial y} = kX_i$$

for $i = 1, 2$, while

$$x\frac{\partial \overline{X}_3}{\partial x} + y\frac{\partial \overline{X}_3}{\partial y} = (k - 1)X_3$$

unless $\bar{X}_3$ is identically zero. In all cases, the equation $[R_z, \bar{X}] = (k-1)\bar{X}$ holds. Combined to the above equations, it follows that

$$\left[\bar{X}.\left(\frac{\bar{h}\bar{q}}{\bar{g}}\right)\right]R_z = \left[(k-1)\frac{\bar{h}\bar{q}}{\bar{g}} + \bar{X}.\left(\frac{\bar{f}}{\bar{g}}\right)\right]\bar{X}.$$

If the expression multiplying $\bar{X}$ on the right-hand side does not vanish identically, then $\bar{X}$ is a multiple of R_z. This is however impossible since it would imply that X and Y are linearly dependent everywhere, cf. Lemma 5. Therefore the mentioned expression is constant equal to *zero* and hence $\bar{X}.\left(\frac{\bar{h}\bar{q}}{\bar{g}}\right)$ is identically *zero* as well. This implies that $\bar{h}\bar{q}/\bar{g}$ is a meromorphic first integral for $\bar{X}$. The same argument of Lemma 3 proves that this first integral is not reduced to a constant. $\square$

Thanks to the preceding lemmas, the desired analogue of Proposition 1 can finally be stated.

Proposition 2. *Let X, Y be two commuting vector fields that are linearly independent at generic points. Denote by $\mathcal{D}$ the codimension 1 foliation spanned by X, Y. The foliation $\mathcal{D}$ is supposed to be singular over the axis $\{x = y = 0\}$ and, besides, the blow-up map π_z centered over this axis is supposed to be dicritical for $\mathcal{D}$. Finally we also suppose that the order of X, Y in the variables x, y at points in $\{x = y = 0\}$ is greater than equal to 2. Then one has:*

(1) *The transforms $\widetilde{\mathcal{F}}_X$, $\widetilde{\mathcal{F}}_Y$ of the foliations $\mathcal{F}_X$, $\mathcal{F}_Y$ coincide in their restriction to the exceptional divisor $\pi_z^{-1}(0)$.*

(2) *The restrictions to $\pi_z^{-1}(0)$ of $\widetilde{\mathcal{F}}_X$ and $\widetilde{\mathcal{F}}_Y$ also coincide with the foliation induced on $\pi_z^{-1}(0)$ by intersecting $\pi_z^{-1}(0)$ with the leaves of $\widetilde{\mathcal{D}}$, where $\widetilde{\mathcal{D}}$ stands for the transform of $\mathcal{D}$.*

(3) *The restrictions of $\widetilde{\mathcal{F}}_X$, $\widetilde{\mathcal{F}}_Y$ to $\pi_z^{-1}(0)$ possess a non-constant meromorphic first integral.*

Proof. All the material was prepared so that the proof of Proposition 1 applies word-by-word in the present setting. It suffices to replace "first homogeneous component" (at a point) by "first homogeneous components in the variables x, y" (over the curve $\{x = y = 0\}$). $\square$

4. Reduction of singularities and proper transforms of vector fields

In this section we shall first give a simplified statement of the reduction procedure of [Ca] which will be sufficient to establish Theorem 1. Then we shall consider the transforms of the (initial) vector fields X, Y, defined on a neighborhood of $(0,0,0) \in \mathbb{C}^3$,

under the corresponding sequence of blow-up maps. The discussion of these transforms of X, Y is obviously important since Propositions 1 and 2 can only be applied to those components of the (total) exceptional divisor at which the corresponding transformed vector fields vanish identically. The section closes with a much more detailed account of the results in [Ca] so as to prepare the way for the proof of our Main Theorem.

Let us however remind the reader that, by means of the construction explained in the Introduction, "almost all homogeneous polynomial vector fields in three variables" yield a codimension 1 foliation without separatrix at the origin of $\mathbb{C}^3$. Not surprisingly the proof of our Main Theorem relies heavily on the fact that the corresponding foliation is spanned by two commuting vector fields, i.e. by two vector fields generating an abelian Lie algebra. The examples provided by the foliations spanned by a polynomial homogeneous vector field and by the radial vector field R also show that the statement of Main Theorem does not generalize to the case of vector fields generating an affine Lie algebra without further conditions.

4.1. Reduction of singularities of codimension 1 foliations and proof of Theorem 1.

Recall that Cano and Cerveau have proved a theorem of reduction of singularities for codimension 1 foliations on $(\mathbb{C}^3, 0)$ that are *non-dicritical* [C-C]. More recently, Cano obtained a general reduction theorem for singularities of codimension 1 foliations on $(\mathbb{C}^3, 0)$ [Ca]. The latter theorem asserts the existence of a finite sequence of blowing-up maps along with the corresponding transforms of $\mathcal{D}$,

$$\mathcal{D} = \mathcal{D}^0 \xleftarrow{\pi_1} \mathcal{D}^1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_k} \mathcal{D}^k \qquad (10)$$

such that:

(1) The center of each blow-up map is invariant by the corresponding transform of $\mathcal{D}$ and it has normal crossings with the corresponding exceptional divisor.

(2) $\mathcal{D}^k$ has only simple singularities.

The total exceptional divisor arising from the above procedure is going to be denoted by $(\pi_1 \circ \cdots \circ \pi_k)^{-1}(0)$ even though the center of π_1 may, in principle, be a curve rather than the origin itself. As mentioned a more accurate version of this theorem will be stated later. For the time being, it suffices to remind the reader that the above theorem also yields a test to check whether or not a foliation is dicritical. Namely $\mathcal{D}$ is dicritical if and only if in the above procedure there is at least one irreducible component of $(\pi_1 \circ \cdots \circ \pi_k)^{-1}(0)$ that is not invariant by $\mathcal{D}^k$ (i.e. the exceptional divisor contains a component that is dicritical for $\mathcal{D}^k$). In other words, if all the irreducible components of $(\pi_1 \circ \cdots \circ \pi_k)^{-1}(0)$ are invariant by $\mathcal{D}^k$, then $\mathcal{D}$ is non-dicritical what means that for every sequence of blow-ups satisfying Condition 1 above, the resulting foliation will leave the exceptional divisor fully invariant.

The notion of simple singularities for a codimension 1 foliation defined on a complex 3-manifold was introduced in [C-C]. For the convenience of the reader, we briefly recall the possible models below referring to [C-C], [Ca] for further information.

Models of Type A. in this case, the foliation is locally given by a pair of commuting vector fields Z_1, Z_2, having the general form: $Z_1 = \partial/\partial z$, $Z_2 = x\partial/\partial x + a(x, y)\partial/\partial y, a(0, 0) = 0$.

In addition, the eigenvalues of the linear part of Z_2 at the origin are such that their quotient is not positive rational. The restriction of Z_2 to the invariant plane $\{z = 0\}$ is then a simple singularity in the usual sense of vector fields in dimension 2. In particular, if this singularity is not a saddle-node (i.e. if both eigenvalues are different from zero), then the vector field obtained as restriction of Z_2 to $\{z = 0\}$ has exactly 2 separatrices. Considering the special form of the vector field Z_1, it becomes obvious that these separatrices give rise to (codimension 1) separatrices for the corresponding (codimension 1) foliation. Actually the foliation spanned by Z_1, Z_2 is nothing but the cylinder over the foliation induced by Z_2 on the plane $\{z = 0\}$.

Models of Type B. Here there are still locally defined commuting vector fields Z_1, Z_2 spanning the foliation and given by $Z_1 = x\partial/\partial x + a(x, y, z)\partial/\partial z$ and $Z_2 = y\partial/\partial y + b(x, y, z)\partial/\partial z$, with $a(0, 0, 0) = b(0, 0, 0) = 0$.

In the present case, we can assume that the eigenvalues of both Z_1, Z_2 possess a quotient lying in the complement of $\mathbb{Q}_+$.

Remark 3. As mentioned the statement of Cano's reduction theorem will further be detailed in the next paragraph, however we can already mention that, in the final statement of his theorem, only singularities of Type A may intersect a dicritical component of the total exceptional divisor.

Having explained what is meant by "simple singularities" in Condition 2, we need to say a few additional words about the nature of the centers of the above mentioned blow-ups. In particular, to obtain Conditions 1 and 2 stated above, it may be necessary to blow-up points lying away from the singular set of the foliation in question. The centers of a blow-up map that are not contained in the singular set of the corresponding foliation are however invariant by the foliations. In other words, the part of this center lying away from the singular set of the mentioned foliations is still contained in a single leaf of this foliation. This fact has an important consequence that will be used in our proofs.

In the reduction procedure (10), those blow-up maps whose center are not invariant by the (corresponding transforms of the) vector fields X, Y have a special role in our discussion. Their special character is related to the fact that, in this case, the (new) transforms of X, Y will be *meromorphic* over the component of the exceptional divisor added by the blow-up map in question. In particular, *a priori*, blow-ups centered at regular points of (the transforms of) $\mathcal{D}$ are natural candidates to yield

meromorphic vector fields. This issue will be detailed discussed later. For the time being, let us just make some easy remarks concerning blow-ups that are not centered at the singular set of our codimension 1 foliations. Hence suppose that π_i is a blow-up whose center in not fully contained in the singular set of $\mathcal{D}^{i-1}$. Suppose also that the center of π_i is a curve C_{i-1} (necessarily irreducible and smooth). This center may also intersect non-trivially the singular set of $\mathcal{D}^{i-1}$. This intersection however consists of finitely many points $\{q_1, \ldots, q_r\}$. Since $\mathcal{D}^{i-1}$ is regular on $C_{i-1} \setminus \{q_1, \ldots, q_r\}$ we have:

Lemma 7. *With the preceding notations, the blow-up map π_i is non-dicritical for the foliation $\mathcal{D}^{i-1}$. Furthermore all the singularities of $\mathcal{D}^i$ lying over $\pi_i^{-1}(C_{i-1} \setminus \{q_1, \ldots, q_r\})$ are simple and, in fact, non-dicritical.*

Proof. Note that on a neighborhood of a regular point of $\mathcal{D}^{i-1}$ this foliation admits a holomorphic first integral and this property is clearly stable by all types of blow-up maps. In particular the singularity is non-dicritical. To check that singularities of $\mathcal{D}^i$ lying over $\pi_i^{-1}(C_{i-1} \setminus \{q_1, \ldots, q_r\})$ are simple and non-dicritical, just note that the blow-up map π_i is essentially two-dimensional and the two-dimensional situation is question is such that the corresponding blow-up possesses a unique (2-dimensional) singularity over the exceptional divisor which has eigenvalues $1, -1$. The statement follows at once. $\square$

Remark 4. Note that the above argument applies equally well to the case of a punctual blow-up at a regular point of $\mathcal{D}^{i-1}$.

To finish this brief review on reduction of singularities, let us point out a minor issue concerning the compactness of the exceptional divisor arising from proceeding the reduction of singularities of $\mathcal{D} = \mathcal{D}^0$. In general this divisor is not compact due to the fact that the singularities of $\mathcal{D} = \mathcal{D}^0$ need not be isolated. In particular, in the local context of a neighborhood of the origin in $\mathbb{C}^3$, the exceptional divisor arising from blowing-up a curve of singularities of $\mathcal{D} = \mathcal{D}^0$ is already non-compact. In turn, this non-compact component may lead to further non-compact components and this constitutes the source for the lack of compactness for the total exceptional divisor.

To abridge notations let $\Pi = \pi_1 \circ \cdots \circ \pi_k$ and observe that in any event the preimage $\Pi^{-1}(0)$ of the origin under the total blow-up map Π is necessarily compact and constituted by strata of dimension either 1 or 2 depending on whether the center of the corresponding blow-up was a point or a curve. Moreover whenever E is a non-compact component of the total exceptional divisor the intersection $E \cap \Pi^{-1}(0)$ (if not empty) yields a stratum of dimension 1 of $\Pi^{-1}(0)$. In particular all components of the exceptional divisor that are collapsed to the origin by Π are necessarily compact.

Let us close this paragraph with the proof of Theorem 1.

Proof of Theorem 1. We need to prove the existence of a separatrix for $\mathcal{D}$ containing the origin of $\mathbb{C}^3$. Since it was proved in [C-C] that a non-dicritical foliation has a separatrix, we can assume that the total exceptional divisor appearing in (10) is not fully invariant by $\mathcal{D}^k$. Nonetheless we are going to show that the strategy of proof used in [C-C] can still be carried out in our case.

Let us then consider an irreducible component E of this total exceptional divisor associated to the procedure (10) which is transverse to the leaves of $\mathcal{D}^k$ at generic points. Next we denote by $\mathcal{D}^k_{|E}$ the 1-*dimensional foliation* induced on E by intersecting E with the transverse leaves of $\mathcal{D}^k$. Let us first consider the case in which E can be chosen compact. Since E is compact, our assumption says that $\mathcal{D}^k_{|E}$ has a non constant meromorphic integral.

As in [C-C] consider a local analytic curve Γ transverse to the total exceptional divisor and satisfying the following conditions:

- Γ is not contained in the singular set of $\mathcal{D}^k$.

- Γ is an invariant analytic space for $\mathcal{D}^k$.

The existence of Γ is obvious since there is an irreducible component E of the total exceptional divisor which is not invariant by $\mathcal{D}^k$. Naturally Γ can be chosen so that its intersection with the total exceptional divisor is a point $p \in E$. In addition, since E is supposed to be compact, it follows that $\Pi(p) = (0, 0, 0)$ (recall that $\Pi = \pi_1 \circ \cdots \circ \pi_k$).

Now let us consider the (germ of) invariant surface S obtaining by "sliding" Γ over the leaf through p of the foliation $\mathcal{D}^k_{|E}$. The statement amounts to proving that this germ can be extended to an analytic set invariant by $\mathcal{D}^k$ and defined on a neighborhood of the total exceptional divisor. First it is clear that the surface S can be extended over the regular part of $\mathcal{D}^k$. By using the compactness of the leaves of the foliations induced by $\mathcal{D}^k$ on the *compact dicritical components* of the total exceptional divisor, the argument employed in [C-C] works word-by-word in our context modulo checking that S can be extended over a neighborhood of singularities of *Type A*. In fact, since the foliations considered in [C-C] are all non-dicritical, only singularities of Type B were discussed in the paper in question.

Thus consider that S accumulates on a point q belonging to a dicritical component E_1 of the total exceptional divisor. Though this is not important for the argument, let us point out that this extension is needed only if $\Pi(q) = (0, 0, 0)$. Let then $\mathcal{D}^k_{|E_1}$ denote the foliation induced on E_1 by $\mathcal{D}^k$. We need to show that S can be extended over q. Clearly we can assume that q is a singular point for $\mathcal{D}^k$ and therefore, it is a singularity of Type A (recall that singularities of Type B do not intersect dicritical components of the exceptional divisor, cf. Remark 3). This means that $\mathcal{D}^k$ is locally spanned by a pair of vector fields Z_1, Z_2 given in local coordinates (x, y, z) by

$$Z_1 = \partial/\partial z \quad \text{and} \quad Z_2 = x\partial/\partial x + a(x, y)\partial/\partial y,$$

where $a(0,0) = 0$ and $\{z = 0\} \subset E_1$. Denoting by Sing $(\mathcal{D}^k)$ the singular set of $\mathcal{D}^k$, it follows that Sing $(\mathcal{D}^k) \cap E_1$ has an isolated point at q. In particular the intersection $S \cap E_1$ locally coincides with a separatrix of $\mathcal{D}^k_{|E_1}$ so that the intersection $S \cap E_1$ can be continued through q. To finish the proof it suffices to check that the resulting extension of S through q over the mentioned separatrix is not "pinched at q" and, indeed, that it defines a local analytic surface containing q. This is however clear since the vector field Z_1 shows that $\mathcal{D}^k$ is (locally) a cylinder over the (1-dimensional) foliation induced on E_1. This establishes the existence of a separatrix.

To finish the proof of the theorem, we just need to consider the case in which none of the components of the exceptional divisor that are dicritical for $\mathcal{D}^k$ is compact. First we need a local analytic curve Γ satisfying the previous conditions and, in particular, intersecting the exceptional divisor at a point p satisfying $\Pi(p) = (0,0,0)$. The existence of this curve Γ can be obtained as in [C-C], by resorting to a 2-dimensional consideration relying in [C-S]. The above discussion can then be repeated to yield a germ of invariant surface S. Here it is convenient to point out that, if S happens to intersect a component of the exceptional divisor that is not compact, we only need to extend S locally over the singularities that are, indeed, projected to the origin by Π (i.e. no information concerning the global character of the foliation induced by $\mathcal{D}^k$ on a dicritical non-compact component of the exceptional divisor is needed). Now it is clear that the above constructed (germ of) invariant surface S is projected by Π onto a separatrix for $\mathcal{D}$ containing the origin. The theorem is proved. $\square$

4.2. The blowing-up strategy.

Consider a reduction procedure for the singularities of $\mathcal{D} = \mathcal{D}^0$ as in (10). In addition to obtaining a foliation $\mathcal{D}^k$ all of whose singularities are simple, to prove the existence of separatrices for $\mathcal{D}$ it will also be important to keep track of the choice of the centers for each of the blow-up maps π_i appearing in (10). This paragraph is essentially devoted to further detail the results of [Ca] by explaining the strategy for choosing the above mentioned centers as well as their role in the proof of existence of separatrices for $\mathcal{D}$.

To begin with, consider the vector fields X, Y spanning $\mathcal{D}$ and denote by $\widetilde{X}^i$, $\widetilde{Y}^i$ their respective transforms under the composition $\pi_1 \circ \cdots \circ \pi_i$. The general principle to ensure the existence of a separatrix for $\mathcal{D}$ is to reduce the statement *as much as possible* to Theorem 1. This requires us to show that the foliation induced by $\mathcal{D}^k$ on every (compact, irreducible) dicritical component of the total exceptional divisor $(\pi_1 \circ \cdots \circ \pi_k)^{-1}(0)$ has a non-trivial meromorphic first integral. In turn, to prove the existence of these first integrals we may resort to Propositions 1 and 2 what, finally, leads us to consider the vector fields $\widetilde{X}^i$, $\widetilde{Y}^i$ and their zero-sets. Ultimately we shall look for conditions ensuring that $\widetilde{X}^k$, $\widetilde{Y}^k$ must vanish identically on every compact, dicritical component of $(\pi_1 \circ \cdots \circ \pi_k)^{-1}(0)$.

So far all that has been mentioned about the center C_{i-1} of the blow-up map π_i is its invariance under $\mathcal{D}^{i-1}$ and the fact that it is smooth. In particular it may happen that C_{i-1} is not contained in the singular set of $\mathcal{D}^{i-1}$. This already poses a

difficulty for us since the blow-up of a holomorphic vector field Z centered at a set that *is not invariant* under Z yields a *meromorphic* vector field having poles over the corresponding exceptional divisor. Conversely we have:

Lemma 8. *Let Z be a holomorphic vector field and denote by $\widetilde{Z}$ the transform of Z with respect to a blow-up π of center C. If C is invariant by Z, then $\widetilde{Z}$ is holomorphic as well.*

Proof. C being invariant by Z, the local flow generated by Z naturally acts on the tangent bundle of C. It then follows that $\widetilde{Z}$ is holomorphic. $\square$

Next denote by $\mathcal{F}_Z$ the foliation associated to the above mentioned vector field Z. Recall that the singular set of $\mathcal{F}_Z$ has codimension at least 2 though the zero-set of Z may have components with codimension equal to 1. A useful refinement of Lemma 8 reads as follows.

Lemma 9. *Let Z, $\widetilde{Z}$, π and C be as in Lemma 8. Suppose that*

(1) *C is invariant under the foliation $\mathcal{F}_Z$.*

(2) *C is contained in a codimension 1 component of the zero-set of Z.*

Then $\widetilde{Z}$ is holomorphic and it vanishes identically over the corresponding exceptional divisor.

Proof. Consider a point $p \in C$. Our assumptions imply the existence of local coordinates (x, y, z) about p in which Z is given by $f(x, y, z)Z_1$ where Z_1 is a holomorphic vector field having singular set of codimension 2 or greater that leaves C invariant. Besides f is a holomorphic function satisfying $f(p) \simeq f(0, 0, 0) = 0$. The vector field $\widetilde{Z}$ is therefore given by the product between the transform of f and the blow-up of Z_1. Since the blow-up of Z_1 is itself holomorphic thanks to Lemma 8, the statement follows at once. $\square$

Going back to the reduction procedure (10), it is clear that blow-up maps π centered at regular points of $\mathcal{D}^{i-1}$ will be hard to handle since there is *a priori* no reason why these centers should be invariant by either $\widetilde{X}^{i-1}$ or $\widetilde{Y}^{i-1}$. As far as the foliation $\mathcal{D}^{i-1}$ is concerned, one such blow-up is never dicritical. Besides the singularities of $\mathcal{D}^i$ lying over $\pi_i^{-1}(C_{i-1} \setminus \{q_1, \dots, q_r\})$ are simple and do not give rise to any dicritical component of $(\pi_1 \circ \cdots \circ \pi_k)^{-1}(0)$, cf. Lemma 7. Analogous statements hold also for the case of a punctual blow-up at a regular point of $\mathcal{D}^{i-1}$, cf. Remark 4. Yet, as pointed out by the referee, there is still one difficulty that might possibly arise: using the notations of Lemma 7, let π_i be as above and suppose that π_{i+1} happens to be a dicritical blow-up for $\mathcal{D}^i$ which is centered at a singular point Q_1 of $\mathcal{D}_i$ lying over $\pi_i^{-1}(q_1)$. Since the vector fields $\widetilde{X}^i$, $\widetilde{Y}^i$ have divisors of poles containing $\pi_i^{-1}(C_{i-1})$ and thus passing through Q_1, the above lemmas cannot

be used to ensure that $\widetilde{X}^{i+1}$, $\widetilde{Y}^{i+1}$ vanish identically over $\pi_{i+1}^{-1}(Q_1)$. A similar problem arises, more generally, if a center that is not invariant by the corresponding transforms of X or of Y is blown-up. Here it should be noted that, in principle, even a center contained in the singular set of $\mathcal{D}$ may fail to be invariant by X or by Y.

Strictly speaking, the blowing-up of regular points of proper transforms of $\mathcal{D}$ are only performed at the last steps of the reduction procedure (10). More precisely, there is a reduction subprocedure

$$\mathcal{D} = \mathcal{D}^0 \xleftarrow{\pi_1} \mathcal{D}^1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} \mathcal{D}^n \tag{11}$$

leading to a foliation $\mathcal{D}^n$ with simple singularities but for which the total exceptional divisor may not be a *normal crossing divisor*. Blow-ups centered at regular points are only required to turn the preceding exceptional divisor into a divisor with normal crossings, cf. [Ca], pages 914 and 1007. In particular no dicritical component arise after performing the first blow-up π_i with a regular center. Summarizing, for our purposes it suffices to consider a reduction procedure as in (11) where all the centers $C_0, \ldots, C_{n-1}$ are contained in the singular sets of the corresponding foliations $\mathcal{D} = \mathcal{D}^0, \ldots, \mathcal{D}^{n-1}$. Therefore in the rest of this paper we shall focus exclusively on the procedure (11) rather than in the procedure (10).

To close this section, let us collect below some further information on the choice of the centers $C_0, \ldots, C_{n-1}$ according to the general procedure of [Ca]. To keep the technical details to a minimum, we are not going to reproduce all the definitions and invariants involved in the procedure, for which the reader is referred to [Ca]. In fact, our primary purpose is to split the procedure in a few steps carried out according to certain specific rules so that the amount of information provided here will be enough for the applications in the next section.

First, we shall restrict the procedure (11) to the case of *presimple* singularities. The passage from presimple singularities to simple singularities is subsequently done in a rather explicit way, cf. pages 998–1008 of [Ca]. This part of the proof can independently be read and do not add new difficulties to our discussion.

Let Sing $(\mathcal{D})$ the singular set of $\mathcal{D}$ and consider the subset $\text{Sing}^*(\mathcal{D})$ of Sing $(\mathcal{D})$ consisting of those singularities that are not presimple. Naturally the aim of the reduction procedure (11) is to eliminate the latter set modulo modifying $\mathcal{D}$ by means of suitable blow-ups. The reduction procedure (11) can now be summarized as follows.

(1) The set $\text{Sing}^*(\mathcal{D})$ is split into the disjoint union of the set of *good points* and the set of *bad points*. The definition of a good point can be found on page 938 of [Ca]. A bad point is a point that is not good.

(2) The set of bad points is finite (page 947 of [Ca]).

(3) The first step of the procedure (11), to reach presimple singularities, is therefore the destruction of the set of bad points. Let us mention here that this step actually

corresponds to the core of the paper [Ca]. The elimination of bad points can further be divided as follows.

- Blow-up the foliation to obtain "weak normal crossing" and to destroy the "cycles" (Proposition 23, Proposition 25 and "proof of Theorem 1" on page 955 of [Ca]). These blow-ups are centered at "good centers" whose definition is given on page 948 of [Ca]. In particular all punctual blow-ups performed at this stage are centered at bad points since "good points are not good centers".

- Continue the reduction procedure following the "global criteria of blowing-up" described on pages 954 and 955 until all the bad points are eliminated. Once again it follows from the mentioned criteria that during this step all punctual blow-ups performed are centered at bad points.

(4) We have now obtained a foliation whose singular set is entirely constituted by good points or by presimple singularities. The next step is to obtain only presimple singularities. This is done by means of the equi-reduction sequence (Proposition 19). The reduction of good points to presimple singularities is essentially 2-dimensional and it essentially amounts to blowing-up "permissible" centers.

(5) Finally, the passage from presimple to simple singularities is easy and carried out explicitly on pages 998–1008. In particular the above rules for the equi-reduction sequence are still fulfilled.

5. Proof of the existence of a separatrix

In this last section we are going to complete the proof of the existence of a separatrix for the foliation $\mathcal{D}$. To begin with, and in view of the discussion carried out in the preceding section, we fix once and for all a procedure of reduction of singularities

$$\mathcal{D} = \mathcal{D}^0 \xleftarrow{\pi_1} \mathcal{D}^1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} \mathcal{D}^n \tag{12}$$

such that all the singularities of $\mathcal{D}^n$ are simple. Furthermore the centers $C_0, \ldots, C_{n-1}$ of the above indicated blow-up maps are supposed to be contained in the singular sets of the corresponding transforms of $\mathcal{D}$. Finally the choice of these centers is made in accordance with the strategy detailed in Paragraph 4.2. In particular each C_j $(j = 0, \ldots, n-1)$ is either a single point or a smooth curve. Furthermore when the center C_j is reduced to a single point there cannot exist a holomorphic vector field tangent to the foliation $(\mathcal{D}^j)$ and regular at C_j, otherwise C_j would not be a bad point. In fact, since C_j is a singular point its orbit for the local flow Φ^T of a vector field tangent to the foliation and regular at C_j would entirely be constituted by singular points of $\mathcal{D}^j$. Nonetheless Φ induces a family of diffeomorphisms preserving the

foliation $\mathcal{D}^j$ since the vector field is tangent to $\mathcal{D}^j$. Hence, if C_j were a bad point, all the points in its orbit by Φ would also be bad points what is not possible since there are only finitely many of these.

Recall that $\widetilde{X}^i$, $\widetilde{Y}^i$ stand for the transforms of X, Y under the composition $\pi_1 \circ \cdots \circ \pi_i$. Their associated foliations are denoted by $\mathcal{F}_X^i$, $\mathcal{F}_Y^i$. Finally the singular set of $\mathcal{D}^i$ will be denoted by $\mathrm{Sing}\,(\mathcal{D}^i)$.

It is clear that the singular set $\mathrm{Sing}\,(\mathcal{D}^i)$ of $\mathcal{D}^i$ is *globally* invariant by the vector fields $\widetilde{X}^i$, $\widetilde{Y}^i$. Let us begin with a sharper statement.

Lemma 10. *If Z is a vector field tangent to $\mathcal{D}^i$, then the center C_i is invariant by Z. In particular C_i is invariant by the vector fields $\widetilde{X}^i$, $\widetilde{Y}^i$ and also for the (1-dimensional) foliations $\mathcal{F}_X^i$, $\mathcal{F}_Y^i$ associated to them, $i = 0, \ldots, n-1$.*

Proof. Recall that $C_i \subseteq \mathrm{Sing}\,(\mathcal{D}^i)$ and that C_i is either a single point or a smooth irreducible curve. Suppose first that C_j is a (smooth) curve. Consider a holomorphic vector field Z tangent to $\mathcal{D}^i$. We are going to show that Z must leave C_j invariant what will clearly imply that C_j is invariant by both $\widetilde{X}^i$ and $\mathcal{F}_X^i$ (since the vector field $\widetilde{X}^i$ can be divided by a suitable holomorphic function in the case where the zero-set of $\widetilde{X}^i$ has components with codimension equal to 1). To check this, note that at a generic point of C_j the analytic set $\mathrm{Sing}\,(\mathcal{D}^i)$ is smooth as well. Therefore there are local coordinates (x, y, z) where this singular set is given by $\{y = z = 0\}$ and, in particular, it locally coincides with C_j. Now a vector field Z as above must preserve the singular set $\mathrm{Sing}\,(\mathcal{D}^i)$ of $\mathcal{D}^i$ so that it (locally) preserves C_j in the (x, y, z)-coordinates. It then follows that Z must globally preserve C_j as claimed.

If C_j is reduced to a single point, then we have seen that there is no holomorphic vector field tangent to $\mathcal{D}^i$ and regular at C_j. It then follows that both the vector field $\widetilde{X}^i$ and the foliation $\mathcal{F}_X^i$ must be singular at C_j. The statement follows. $\qquad\square$

The following corollary will be useful in the sequel.

Corollary 1. *If π_{i+1} is a punctual blow-up belonging to the procedure (12), then its center C_i is a singular point for both foliations $\mathcal{F}_X^i$, $\mathcal{F}_Y^i$.* $\qquad\square$

In view of Lemma 8, we conclude that both $\widetilde{X}^n$, $\widetilde{Y}^n$ are holomorphic vector fields. Also Lemma 9 combined with the preceding discussion implies the following.

Lemma 11. *Suppose that C_i is contained in a codimension 1 component of the zero-set of $\widetilde{X}^i$ (resp. $\widetilde{Y}^i$). Then $\widetilde{X}^n$ vanishes identically over the divisor*

$$(\pi_{i+1} \circ \cdots \circ \pi_n)^{-1}(C_i). \qquad\square$$

Before starting the proof of the existence of separatrices for $\mathcal{D}$, let us summarize the contents of the preceding lemmas. Recall that our aim is to reduce as far as possible

the proof of the Main Theorem to the statement of Theorem 1. Consider again the resolution procedure (12). To conclude the existence of a separatrix, it would be sufficient to show that for every irreducible component E of the total exceptional that is simultaneously compact and dicritical for $\mathcal{D}^n$, the foliation induced on E by $\mathcal{D}^n$ possesses a non-constant meromorphic first integral. As a further step towards the existence of separatrices in full generality, let us prove:

Theorem 2. *Let X, Y and $\mathcal{D}$ be as in the statement of Main Theorem. Suppose that the linear parts of both X, Y at the origin are zero. Then $\mathcal{D}$ possesses a separatrix passing through the origin.*

Proof. Let us begin with the reduction procedure (12). The first observation concerning procedure (12) is that the origin $(0, 0, 0) \in \mathbb{C}^3$ can be chosen as the center C_0 of the first blow-up map. In fact, if a punctual blow-up centered at $(0, 0, 0) \in \mathbb{C}^3$ is not compatible with the strategy of [Ca], then $(0, 0, 0)$ is a good point for $\mathcal{D} = \mathcal{D}^0$. Thus, modulo reducing the neighborhood of $(0, 0, 0) \in \mathbb{C}^3$, we can assume that all singular points of $\mathcal{D} = \mathcal{D}^0$ are good points since there are only finitely many bad points. This means that the reduction of the singularities of $\mathcal{D} = \mathcal{D}^0$ can be obtained by means of the equi-reduction procedure. This is a 2-dimensional situation (in fact, there is a regular vector field tangent to $\mathcal{D} = \mathcal{D}^0$) so that the existence of the desired separatrix is already known. Thus we suppose henceforth that π_1 is the (punctual) blow-up of $(0, 0, 0) \in \mathbb{C}^3$.

Next recall that the existence of a separatrix will be established provided that the foliation induced by $\mathcal{D}^n$ on every compact, dicritical component of the exceptional divisor has a non-constant meromorphic first integral. Because they are compact, these components either coincide with $\pi_1^{-1}(0, 0, 0) \simeq \mathbb{C}P(2)$ or are projected in $\pi_1^{-1}(0, 0, 0)$ by the sub-procedure $\mathcal{D}^1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} \mathcal{D}^n$. Because the linear parts of X, Y at the origin are zero, the transformed vector fields $\widetilde{X}^1$, $\widetilde{Y}^1$ vanish identically over $\pi_1^{-1}(0, 0, 0)$. If, in addition, π_1 is dicritical for $\mathcal{D} = \mathcal{D}^0$, then it follows from Proposition 1 that the foliation induced on this component by $\mathcal{D}^1$ possesses a non-constant first integral. Naturally this latter foliation coincides with the foliation induced by $\mathcal{D}^n$ on the same component. Now a simple induction argument based on Lemma 11 shows that both vector fields $\widetilde{X}^n$, $\widetilde{Y}^n$ vanish identically over all compact components of the total exceptional divisor. If E is one of these components that happen to be dicritical for $\mathcal{D}^n$, then Propositions 1 and 2 ensure that the foliation induced on E by $\mathcal{D}^n$ admits a non-constant first integral. The existence of the separatrix then follows from Theorem 1. $\square$

Let us now consider the cases in which one of the vector fields X, Y have a non-zero linear part at the origin of $\mathbb{C}^3$. The existence of a separatrix for $\mathcal{D}$ can directly be established in many cases. For example if the linear part of X at the origin is hyperbolic and belongs to the Poincaré domain then X is linearizable or conjugate to

its Poincaré–Dulac normal form. In any event it follows that X is tangent to finitely many smooth surfaces passing through the origin. Since Y commutes with X, these invariant surfaces must be invariant by Y as well so that they constitute separatrices for $\mathcal{D}$. However a systematic analysis of the vector field X is rather involved since there are well-known subtleties concerning, for example, singularities in the Siegel domain.

To establish the existence of a separatrix for $\mathcal{D}$ in the above case, we shall adapt the proof of Theorem 2. The difficulty in doing so lies in the fact that the transforms of X (resp. Y) may not vanish identically over (dicritical) compact components of the total exceptional divisor what prevents us from applying Propositions 1 and 2. We are then led to understanding dicritical blow-ups giving rise to (1-dimensional) foliations without non-constant meromorphic integrals. Our first aim will be to characterize these situations. The discussion can naturally be split in two cases according to whether the center of the blow-up map in question is a point or a curve.

The case of a punctual blow-up. Let us begin with a simple lemma.

Lemma 12. *Let Z_1, Z_2 be vector fields defined about $(0,0,0) \in \mathbb{C}^3$. Suppose that the linear part of Z_1 at $(0,0,0)$ is a (constant, non-zero) multiple of R. Suppose, in addition, that $[Z_1, Z_2] = 0$. Then in suitable coordinates Z_1 coincides with R and Z_2 becomes a linear vector field. In particular, if the first non-trivial homogeneous component of Z_2 is itself a multiple of R, then Z_1, Z_2 are parallel everywhere.*

Proof. Since the linear part of Z_1 is a *non-zero* multiple of the radial vector field, there exists coordinates in which $X = R$. The assumption on the commutativity of Z_1, Z_2 together with the Euler relation (3) implies that the terms of order greater than or equal to 2 of Z_2 must vanish. Hence Z_2 is linear. The lemma is proved.   □

So we are in the following situation: Z_1, Z_2 are two commuting holomorphic vector fields defined on a neighborhood of $(0,0,0) \in \mathbb{C}^3$ and satisfying the following conditions:

(1) The linear part of Z_1 at $(0,0,0) \in \mathbb{C}^3$ is not zero and is not a (constant) multiple of the radial vector field.

(2) Z_1, Z_2 span a codimension 1 foliation $\mathcal{D}$.

(3) The blow-up π of $(0,0,0) \in \mathbb{C}^3$ is dicritical for $\mathcal{D}$. Besides the foliation induced by the transform $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ on $\pi^{-1}(0,0,0)$ does not admit a non-constant meromorphic first integral.

We shall prove:

Lemma 13. *Under the preceding assumptions, the foliation induced on $\pi^{-1}(0,0,0)$ by $\widetilde{\mathcal{D}}$ coincides with the foliation induced on $\pi^{-1}(0,0,0)$ by the blow-up of Z_1.*

Proof. Lemma 12 allows us to suppose that the linear part of Z_1 at the origin is not a (constant) multiple of R. In particular the blow-up of Z_1 does induce a foliation on $\pi^{-1}(0, 0, 0)$. The lemma then follows from the fact that π is dicritical for $\mathcal{D}$: the foliation associated to the blow-up of Z_1 is tangent to the leaves of $\widetilde{\mathcal{D}}$. In turn the intersection of the leaves of $\widetilde{\mathcal{D}}$ with $\pi^{-1}(0, 0, 0)$ are of dimension 1 (at generic points). □

Whereas $\widetilde{\mathcal{D}}$ may not possess a non-constant meromorphic first integral, it possesses the following property:

Corollary 2. *The foliation $\widetilde{\mathcal{D}}_{|\pi^{-1}(0)}$ induced on $\pi^{-1}(0, 0, 0)$ by $\widetilde{\mathcal{D}}$ is such that for each singularity $p \in \pi^{-1}(0, 0, 0)$ of $\widetilde{\mathcal{D}}_{|\pi^{-1}(0)}$ and each (irreducible) separatrix S for $\widetilde{\mathcal{D}}_{|\pi^{-1}(0)}$ at p, the separatrix S is contained in an algebraic curve invariant by $\widetilde{\mathcal{D}}_{|\pi^{-1}(0)}$.*

We are going to see that Corollary 2 will be enough to prove our Main Theorem. The last step before starting its proof consists of obtaining a version of this corollary valid for the case of (dicritical) blow-ups centered over curves. In the sequel the blown-up singularity leading to the situation described above will be called a singularity of type SING-1 for the corresponding foliation.

The case of a blow-up centered over a curve C. Let us now consider the case of a blow-up π_z centered over a compact curve C and dicritical for $\mathcal{D}$. Again $\mathcal{D}$ is spanned by a pair of commuting holomorphic vector fields Z_1, Z_2 leaving C invariant. We suppose that the foliation induced on the exceptional divisor $\pi_z^{-1}(C)$ by the transform $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ does not admit a non-constant meromorphic first integral.

Next consider local coordinates (x, y, z) about a generic point of C and such that C is locally given by $\{x = y = 0\}$. In particular the first homogeneous component Z_1 (resp. Z_2) in the variables (x, y) can be considered as in Section 3. Here the order of Z_1 with respect to the variables x, y is supposed to be either 1 or *zero*. First we need an analogue of Lemma 12.

Lemma 14. *Let Z_1, Z_2 be as above. Suppose that Z_1 vanishes identically over C and that the linear part of Z_1 w.r.t. the variables x, y is a (constant, non-zero) multiple of R_z. Suppose, in addition, that $[Z_1, Z_2] = 0$ and that they span a codimension 1 foliation $\mathcal{D}$. Then the linear part of Z_2 w.r.t. the variables x, y is not zero. Besides this linear part is not a (constant) multiple of R_z.*

Proof. The assumption concerning the linear part (w.r.t. the variables x, y) of Z_1 ensures that the transform of the foliation associated to Z_1 itself is transverse to the exceptional divisor $\pi_z^{-1}(C)$. In particular, since Z_1, Z_2 span a codimension 1 foliation $\mathcal{D}$, the argument of Lemma 5 implies that the linear part of Z_2 (w.r.t.

the variables x, y) cannot be zero. Besides, since the transforms of the higher order homogeneous components of Z_2 (w.r.t. the variables x, y) vanish identically over $\pi_z^{-1}(C)$, the same argument also shows that the linear parts of Z_1, Z_2 (w.r.t. the variables x, y) cannot coincide up to a multiplicative factor. The lemma is proved. $\qquad\square$

Lemma 14 allows us to suppose that the linear part of Z_1 (w.r.t. the variables x, y) is neither zero nor a multiple of R_z. In particular the transform of Z_1 induces a foliation on $\pi_z^{-1}(C)$. In turn this foliation coincides with the foliation induced on $\pi_z^{-1}(C)$ by the transform $\widetilde{\mathcal{D}}$ of $\mathcal{D}$. Now we have the following:

Lemma 15. *If the foliation induced by $\widetilde{\mathcal{D}}$ on $\pi_z^{-1}(C)$ does not admit a non-constant meromorphic first integral, then, modulo permuting the names of Z_1, Z_2, it follows that Z_1 does not vanish identically over C.*

Proof. As mentioned above, we can assume that the foliation induced by $\widetilde{\mathcal{D}}$ on $\pi_z^{-1}(C)$ coincides with the foliation induced on $\pi_z^{-1}(C)$ by the transform of Z_1. Next assume that Z_1 vanishes identically over C. Then in the coordinates (x, t, z) for the blow-up π_z, it becomes clear that the leaves of the foliation induced on $\pi_z^{-1}(C)$ by the transform of Z_1 are contained in the curves $\{x = 0\,;\,z = \text{cte}\}$. This contradicts the fact that this foliation does not admit a non-constant meromorphic first integral. $\qquad\square$

Summarizing what precedes, we just need to consider the cases in which Z_1 does not vanish identically over C. Since C is compact and Z_1 is holomorphic, it follows that C is either a rational curve or an elliptic curve. Let us separately consider each possibility.

a) *Case of a rational curve*: The exceptional divisor $\pi_z^{-1}(C)$ is a Hirzebruch surface (i.e. a fibration over $\mathbb{C}P(1)$ with fiber isomorphic to $\mathbb{C}P(1)$). The vector field Z_1 has singularities on C. The foliation induced on $\pi_z^{-1}(C)$ by the transform of Z_1 is transverse to the natural fibration of $\pi_z^{-1}(C)$ over C. In other words, this foliation can be viewed as a Riccati foliation on $\pi_z^{-1}(C)$. In fact, it is tangent to a non-trivial holomorphic vector field that projects on Z_1. The nature of the generic leaf of this foliation is therefore determined by the holonomy representation of the fundamental group of $C \setminus \text{Sing}\,(Z_1, C)$ in PSL $(2, \mathbb{C})$, where $\text{Sing}\,(Z_1, C)$ stands for the zero-set of Z_1 restricted to C. In particular, if $\text{Sing}\,(Z_1, C)$ is reduced to a single point, then $C \setminus \text{Sing}\,(Z_1, C)$ is simply connected so that the holonomy representation in question is trivial. It then follows that all the leaves of the mentioned Riccati foliation on $\pi_z^{-1}(C)$ are compact and hence that this foliation has a meromorphic first integral.

It remains to discuss the case where $\text{Sing}\,(Z_1, C)$ consists of two points q_1, q_2. In this case the image of the holonomy representation is the cyclic group generated by an element γ of PSL $(2, \mathbb{C})$. Besides the singularities of the Riccati foliation in question

are contained in the invariant fibers sitting over q_1, q_2. Each of these invariant fibers contains one or two singularities of the Riccati foliation. Now we have the following possibilities:

(1) If γ has finite order in PSL $(2, \mathbb{C})$. Then all leaves of the Riccati foliation are compact and this foliation possesses a non-constant meromorphic first integral.

(2) If γ has infinite order and fixes two points in $\mathbb{C}P(1)$. A direct inspection using blow-up coordinates (x, t, z) shows that, in this case, the vector field projecting on Z_1 and tangent to the Riccati foliation in question admits a non-trivial component tangent to the fibers of the fibration. For a fixed fiber, this component is naturally a linear vector field with two singular points.

It then follows that, apart from the invariant fibers, the resulting Riccati foliation possesses exactly two compact curves (associated to the two fixed points of γ). Each of these compact curves intersect each invariant fiber at a singular point for the foliation in question. In particular the Riccati foliation has exactly four singular points. Furthermore the foliation has two eigenvalues different from zero at each of these singular points. Finally through each singular points, there pass two compact curves invariant by the foliation itself, namely the invariant fiber and one of the two compact leaves that are not fibers.

(3) If γ has infinite order and fixes a single point in $\mathbb{C}P(1)$. Again a direct inspection shows that the vector field projecting on Z_1 and tangent to the Riccati foliation in question admits a non-trivial component tangent to the fibers of the fibration. For a fixed fiber, this component is naturally a linear vector field with a unique singular point.

In this case, each invariant fiber of the fibration contains a unique singular point of the Riccati foliation. This singularity is a 2-dimensional saddle-node at it possesses exactly two separatrices (one of them being contained in the invariant fiber). Besides, the fixed point of γ in $\mathbb{C}P(1)$ represents a third compact leaf L of the Riccati foliation (the other two being the invariant fibers). In total the Riccati foliation possesses two singularities (contained in the invariant fibers) and three compact leaves. Besides L intersect each invariant fiber at its singular point so that L defines a separatrix for each of these singular points.

In the cases 2 and 3 above, we are going to say that C represents a singularity of type SING-2 for the corresponding foliation. The corollary below summarizes the above discussion by retaining the information needed for the proof of Main Theorem.

Corollary 3. *Suppose that C represents a singularity of type SING-2 for the foliation $\mathfrak{D}$ (for a fixed dicritical blow-up π_z). Then the 2-dimensional foliation $\widetilde{\mathfrak{D}}_{|\pi^{-1}(C)}$ induced on $\pi_z^{-1}(C)$ by the transform of $\mathfrak{D}$ satisfies the following condition: given a singularity $p \in \pi^{-1}(C)$ of $\widetilde{\mathfrak{D}}_{|\pi^{-1}(C)}$ and a local (irreducible) separatrix S for $\widetilde{\mathfrak{D}}_{|\pi^{-1}(C)}$ at p, there exists an compact curve invariant by $\widetilde{\mathfrak{D}}_{|\pi^{-1}(C)}$ that contains S.*

Remark 5. Here is a minor complement to the statements of Corollaries 2 and 3. Both statements concern certain "linear" (1-dimensional) foliations on a compact surface E. In turn E is embedded in a 3-dimensional ambient equipped with a codimension 1 foliation $\mathcal{D}$ whose intersection with E induces the initial "linear" foliation on E. Because these "linear" foliations on E do not admit non-constant meromorphic first integrals, their singularities cannot have both eigenvalues in $\mathbb{Z}_+$. In other words, these singularities are not dicritical (in dimension 2). This implies that 3-dimensional blow-ups over centers containing the mentioned singularities will lead to an exceptional divisor whose intersection with the transform of E is a curve simultaneously invariant by the foliations induced on E and (if any) by the foliation induced on the created exceptional divisor (in case the latter is dicritical for the transform of $\mathcal{D}$). In particular, if a leaf of the transformed foliation is followed from E to this exceptional, it must pass through a singularity of the mentioned transform lying over the intersection curve and there are only finitely many of such singularities.

b) *Case of an elliptic curve*: In this case the restriction of Z_1 to C must be a nowhere zero holomorphic vector field. The terminology corresponding to this case will consist of saying that C represents a singularity of type SING-3 for the corresponding foliation. The only thing we need to know about this case is that the foliation induced on $\pi_z^{-1}(C)$ by the transform of $\mathcal{D}$ has no singular points.

We are finally able to prove the main result of this paper.

Proof of Main Theorem. We resume the context used in the proof of Theorem 1 and of Theorem 2. Let the reduction procedure (12) be fixed. In particular π_1 is supposed to be the blow-up of the origin of $\mathbb{C}^3$. The compact irreducible components E_j of the total exceptional divisor satisfy one of the following conditions:

(1) E_j is invariant by $\mathcal{D}^n$

(2) E_j is dicritical (i.e. non-invariant) for $\mathcal{D}^n$ but the 1-dimensional foliation induced on E_j by $\mathcal{D}^n$ possesses a non-constant meromorphic first integral.

(3) E_j is dicritical (i.e. non-invariant) for $\mathcal{D}^n$ and the 1-dimensional foliation induced on E_j by $\mathcal{D}^n$ does not admit any non-constant meromorphic first integral.

As previously seen, the components E_j belonging to the third group above arise from blowing-up a singularity of type SING-1, SING-2 or SING-3 (for the corresponding foliation) in the reduction procedure (12). Besides we can assume the existence of irreducible components in the third group since otherwise Theorem 1 ensures the existence of the desired separatrix.

Next consider a local analytic curve Γ transverse to the total exceptional divisor and satisfying the same conditions stated in the proof of Theorem 1. We can choose Γ such that its intersection with the exceptional divisor lies in a component E_j that is invariant by $\mathcal{D}^n$ (for example add to the procedure (12) a "superfluous" non-dicritical blow-up and resort to a 2-dimensional consideration as in [C-C] to guarantee the

existence of Γ). As in the proof of Theorem 1, let us consider the (germ of) invariant surface S obtaining by "sliding" Γ over the leaf of $\mathcal{D}^n$. We need to prove that S can be globalized in an analytic set invariant by $\mathcal{D}^n$. This has already been done unless S intersects a component E_j of the total exceptional divisor belonging to the third group above. This intersection however can only happen at a singularity of $\mathcal{D}^n$ of type A (since type B singularities do not intersect dicritical components). In this case, the "regular direction $\partial/\partial z$" of a type A singularity is transverse to the component E_j in question. The local continuation of S is always possible and it is performed by continuing S as one of the two separatrices of $\mathcal{D}^n$ containing the direction $\partial/\partial z$. In particular the local intersection of S with E_j is contained in a separatrix of a singularity of the foliation $\mathcal{D}^n_{|E_j}$ induced by $\mathcal{D}^n$ on E_j. To deduce the statement, it suffices to check that the (local) separatrix $S \cap E_j$ of the foliation $\mathcal{D}^n_{|E_j}$ is actually contained in a (global) compact leaf of $\mathcal{D}^n_{|E_j}$. For this, note that E_j must arise from a singularity of type SING-1 or SING-2 (since the type SING-3 gives rise to regular foliations on the exceptional divisor). Now Corollaries 2 and 3 ensure that the mentioned local separatrix is contained in a compact curve invariant by $\mathcal{D}^n_{|E_j}$. The theorem is proved.   □

Motivated by the classical situation of vector fields in dimension 2, it is natural to ask whether there must exist infinitely many separatrices for a foliation having dicritical components. Simple linear examples involving the radial vector field R and another linear vector field X shows that this is not true in general.

Suppose however that we are in the context of Theorem 1, i.e. we begin with "sufficiently non-linear" vector fields X, Y so as to be able to ensure that the condition of the theorem in question is satisfied. For example, we assume that both X, Y have zero linear parts at a chosen singular point of the foliation $\mathcal{D}$ spanned by X, Y (this singular point will be identified with $(0, 0, 0) \in \mathbb{C}^3$). Consider the reduction procedure (10). Then a careful reading of the proof of Theorem 1 makes it clear that infinitely many separatrices must always exist provided that there is a dicritical component for $\mathcal{D}^k$ that, in addition, is compact.

If none of the components of the total exceptional divisor that are dicritical for $\mathcal{D}^k$ is compact, then the existence of infinitely many separatrices may fail in accordance with the following specific situation: suppose that the intersection of $\Pi^{-1}(0)$ with a dicritical component E (of the total exceptional divisor) is reduced to a curve that is invariant by the restriction of $\mathcal{D}^k$ to E. Indeed, this curve should be contained in the singular set of $\mathcal{D}^k$. Here the existence of a separatrix for $\mathcal{D}$ may be obtained from the argument in [C-C] if there is no other dicritical component. In some sense this situation means that, although the exceptional divisor may contain dicritical components, its intersection with $\Pi^{-1}(0)$ is "essentially non-dicritical".

An interesting remark concerning the case where this situation actually takes place, so that in particular the separatrices for $\mathcal{D}$ at the origin are obtained with the

help of the method used in [C-C], is as follows: the separatrices obtained through [C-C] do not pass through a "generic" point of a singular curve of $\mathcal{D}$ (note that this curve has to exist otherwise $\Pi^{-1}(0)$ will contain only 2-dimensional components). In particular, these "generic" singular points of $\mathcal{D}$ will themselves have separatrices due to the preceding result even though X, Y have non-trivial linear parts at these latter singularities. With little extra effort, one can show the existence of *infinitely many (germs of) surfaces invariant by $\mathcal{D}$ and passing through "generic" singular points of $\mathcal{D}$*. Naturally, in the above situation, the origin is not "generic" among the singularities of $\mathcal{D}$.

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