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Rigidity of pseudo-Anosov flows transverse to $\mathbb{R}$-covered foliations

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Abstract. A foliation is $\mathbb{R}$-covered if the leaf space of the lifted foliation to the universal cover is homeomorphic to the set of real numbers. We show that, up to topological conjugacy, there are at most two pseudo-Anosov flows transverse to a fixed $\mathbb{R}$-covered foliation. If there are two transverse pseudo-Anosov flows, then the foliation is weakly conjugate to the stable foliation of an $\mathbb{R}$-covered Anosov flow. The proof uses the universal circle for $\mathbb{R}$-covered foliations.

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1. Introduction

Pseudo-Anosov flows are extremely common amongst 3-manifolds [GK1], [Mo2], [Fe2], [Cal2], [Cal3] and they yield important topological and geometrical information about the manifold. For example they imply that the manifold is irreducible and the universal cover is homeomorphic to $\mathbb{R}^3$ [Ga-Oe], [Fe-Mo]. There are also relations with the atoroidal property [Fe3]. Finally there are consequences for the large scale geometry of the universal cover when the manifold is atoroidal: In that case it follows that the fundamental group is Gromov hyperbolic [GK2] and in certain cases the dynamics structure of the flow produces a flow ideal boundary to the universal cover which is equivariantly homeomorphic to the Gromov boundary and yields many geometric results [Fe7].

As for the existence of pseudo-Anosov flows, it turns out that many classes of Reebless foliations in atoroidal 3-manifolds admit transverse or almost transverse pseudo-Anosov flows which are constructed using the structure of the foliation. For example this occurs for: 1) fibrations over the circle [Th1], 2) finite depth foliations [Mo2], 3) $\mathbb{R}$-covered foliations [Cal2], [Fe2] and 4) Foliations with one sided branching [Cal3]. Pseudo-Anosov flows also survive under the majority of Dehn surgeries on closed orbits [Fr], [GK1], which makes them extremely common. On the other

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hand there are some examples of non existence of pseudo-Anosov flows in certain specific manifolds: see [Br] for examples in Seifert fibered spaces and [Ca-Du], [Fe5] for examples in hyperbolic manifolds.

In this article we consider the uniqueness question for such flows: Up to topological conjugacy, how many pseudo-Anosov flows are there in a closed 3-manifold? *Topological conjugacy* means that there is a homeomorphism of the manifold sending orbits to orbits. The less flows there are, the more rigid these flows are and consequently more likely to give information about the manifold. In this generality the question is, at this point, very hard to tackle. Here we start the study of this question and we consider how many pseudo-Anosov flows are there transverse to a given foliation. This is very natural, since as explained above, many pseudo-Anosov flows are constructed from the foliation and are transverse to it. We will consider a class of foliations called $\mathbb{R}$-*covered*: this means that the leaf space of the lifted foliation to the universal cover is homeomorphic to the set of real numbers [Fe1]. This is the simplest situation with respect to this question. The uniqueness analysis involves a detailed understanding of the topology and geometry of the foliation and flow in this case.

There are many examples of $\mathbb{R}$-covered foliations: 1) Fibrations over the circle. 2) Many stable and unstable foliations of Anosov flows, which are then called $\mathbb{R}$-covered Anosov flows. These include geodesic flows of hyperbolic surfaces and many examples in hyperbolic 3-manifolds [Fe1]; 3) *Uniform* foliations [Th2]: this means that given any two leaves of the lifted foliation in the universal cover, they are a bounded distance from each other. Obviously the bound depends on the pair of leaves. This is associated with slitherings over the circle [Th2]. 4) Many examples foliations which are $\mathbb{R}$-covered but not uniform in hyperbolic 3-manifolds [Cal1].

We should stress that in this article pseudo-Anosov flows include flows without singularities, that is (topological) Anosov flows. On the other hand, we do not allow 1-prong singularities. With 1-prongs almost all control is lost, for example $\mathbf{S}^2 \times \mathbf{S}^1$ has a pseudo-Anosov flow with 1-prongs and the manifold is not even irreducible.

A flow transverse to a foliation is *regulating* if an arbitrary orbit in the universal cover intersects every leaf of the lifted foliation. In particular this implies that the foliation is $\mathbb{R}$-covered. This is strongly connected with the atoroidal property: Given an $\mathbb{R}$-covered foliation with a transverse, regulating pseudo-Anosov flow, it follows that either the manifold is atoroidal or it fibers over the circle with fiber a torus and Anosov monodromy [Fe3]. Conversely if the manifold is atoroidal and acylindrical and the foliation is transversely orientable, then there is a regulating, pseudo-Anosov flow transverse to the $\mathbb{R}$-covered foliation [Fe2], [Cal2]. So transverse pseudo-Anosov flows are as general as possible in this situation and the uniqueness question is a very natural one in this setting.

There is one case where the uniqueness question for transverse flows is known, which is the simplest case of foliations: a fibration over the circle. It is easy to see that any transverse flow is regulating. Any two transverse flows induce homotopic

and hence isotopic monodromies of the fiber S. This works even if the flow is not pseudo-Anosov. If the flow is pseudo-Anosov, then the associated monodromy is a pseudo-Anosov homeomorphism of S [Th1]. In particular the fiber cannot be the sphere or the projective plane. If the fiber is Euclidean, then the flow has no singularities and is a topological Anosov flow. In this case it is not hard to prove that there is at most one transverse pseudo-Anosov flow up to conjugacy. Suppose then that the fiber is hyperbolic and therefore the monodromy is pseudo-Anosov with singularities. It is proved in [FLP], exposé 12, that any two homotopic pseudo-Anosov homeomorphisms are in fact conjugate. This implies that the corresponding flows are also topologically conjugate and consequently in this case there is only one transverse pseudo-Anosov flow up to conjugacy.

This result turns out to be very close to what happens in general for $\mathbb{R}$-covered foliations:

Main theorem. *Let $\mathcal{G}$ be an $\mathbb{R}$-covered foliation in M^3 closed. Then up to topological conjugacy there is at most one transverse pseudo-Anosov flow which is regulating for $\mathcal{G}$. In addition, up to conjugacy, there is also at most one non regulating transverse pseudo-Anosov flow to $\mathcal{G}$. If there is a transverse pseudo-Anosov flow which is non regulating for $\mathcal{G}$, then this flow has no singular orbits and is a topological Anosov flow which is $\mathbb{R}$-covered. In addition in this case, after a blow down of foliated I-bundles of $\mathcal{G}$, then the resulting foliation $\mathcal{G}'$ is conjugate to either the stable or the unstable foliation of the Anosov flow.*

Consequently if $\mathcal{G}$ is not a blow up of the stable/unstable foliation of an $\mathbb{R}$-covered Anosov flow then up to topological conjugacy, there is at most one pseudo-Anosov flow transverse to $\mathcal{G}$. Furthermore there is one such flow if M is atoroidal.

A *foliated I-bundle* of $\mathcal{G}$ is an I-bundle V embedded in M so that V is a union of leaves of $\mathcal{G}$, which are transverse to the I-fibers in V. In particular the boundary of V is an union of leaves of $\mathcal{G}$. In general the base of the bundle is not a compact surface. The blow down operation collapses a foliated I-bundle onto a single leaf, by collapsing I-fibers to points. In the theorem above one may need to do this blow down operation a countable number of times. With reference to the abstract of this article, the phrase $\mathcal{G}$ is weakly conjugate to a foliation $\mathcal{F}$, means that some blow down $\mathcal{G}'$ of $\mathcal{G}$ is topologically conjugate to $\mathcal{F}$.

This theorem generalizes the result for fibrations, because as explained above in that case any transverse flow is regulating.

In order to prove the theorem we split into two cases: the regulating and non regulating situations. The non regulating case was studied in [Fe4] where all of the statements concerning non regulating flows were proved except for the uniqueness of the transverse pseudo-Anosov flow. In the last section of this article we use the constructions and results of [Fe4] to finish the proof of uniqueness in the non regulating case. For completeness here is an outline of the proof of the other statements in the

non regulating case. In the universal cover $\widetilde{M}$ of M, the lifted flow has stable and unstable foliations. Since $\mathcal{G}$ is $\mathbb{R}$-covered there is only one transverse direction to the lift $\widetilde{\mathcal{G}}$ of the foliation $\mathcal{G}$ to $\widetilde{M}$. After a considerable analysis, using the topological theory of pseudo-Anosov flows [Fe4], [Fe6], this implies that there is only one transverse direction to the stable and unstable foliations of the flow in the universal cover. In particular we show that there are no singularities of the flow – it is a (topological) Anosov flow. In addition we prove that the stable and unstable foliations of the flow – which now are non singular foliations – are $\mathbb{R}$-covered foliations. Therefore the flow is an $\mathbb{R}$-covered Anosov flow.

The next step is to show that for each leaf of $\widetilde{\mathcal{G}}$ there is a well defined stable (or unstable) leaf in the universal cover associated to it and these two leaves (one stable/unstable and the other a leaf of $\widetilde{\mathcal{G}}$) are a bounded Hausdorff distance from each other. For simplicity assume they are stable leaves. After collapsing foliated I-bundles of $\mathcal{G}$, this correspondence between leaves of the stable foliation in the universal cover and leaves of $\widetilde{\mathcal{G}}$ is a bijection. Since the leaf of $\widetilde{\mathcal{G}}$ and the corresponding stable leaf are a bounded Hausdorff distance from each other, there is a map between them which sends a point in one leaf to a point at a bounded distance in the other leaf. As both foliations are $\mathbb{R}$-covered then this map is a quasi-isometry. Since leaves of the stable foliation are Gromov hyperbolic [Pl], [Su] and any leaf of $\widetilde{\mathcal{G}}$ is quasi-isometric to a stable leaf, it follows that the leaves of $\widetilde{\mathcal{G}}$ are also Gromov hyperbolic. In particular in the non regulating case, there are no parabolic leaves in $\mathcal{G}$. In [Fe4] the analysis was done under the assumption that leaves of $\mathcal{G}$ are Gromov hyperbolic. The argument above shows that this assumption is not necessary. Using a result of Candel [Can], we can assume that the leaves of $\mathcal{G}$ are hyperbolic leaves.

The next step is to show that for each flow line in a fixed leaf of the stable foliation in the universal cover there is a unique geodesic in the corresponding leaf of $\widetilde{\mathcal{G}}$, so that they are a bounded Hausdorff distance from each other. These geodesics in leaves of $\widetilde{\mathcal{G}}$ jointly produce a flow, which projects to a flow in M whose flow lines are contained in leaves of $\mathcal{G}$. In [Fe4] we show that this new flow is conjugate to the original Anosov flow and therefore $\mathcal{G}$ is topologically conjugate to the stable foliation of the original Anosov flow. Essentially what is left to prove is the uniqueness of the new flow.

We remark that it is very easy to construct non regulating examples for certain foliations: let $\mathcal{G}$ be the stable foliation of a smooth $\mathbb{R}$-covered Anosov flow Ψ, so that $\mathcal{G}$ is transversely orientable. Perturb the flow Ψ slightly along the unstable leaves, to produce a new Anosov flow Φ which is transverse to $\mathcal{G}$ and non regulating for $\mathcal{G}$ – see details in [Fe4].

The bulk of this article concerns the regulating situation, whose analysis is completely different from the non regulating case: in that case the proof was internal to $\widetilde{M}$ – we only used the topology of the pseudo-Anosov flow and showed that stable/unstable leaves in $\widetilde{M}$ and leaves of $\widetilde{\mathcal{G}}$ are basically parallel to each other. Clearly this cannot happen in the regulating situation. In the regulating case we use the asymptotics of the foliation, contracting directions between leaves, the universal cir-

cle for foliations and relations of these with the flow. We show that the universal circle of the foliation can be thought of as an ideal boundary for the orbit space of a regulating pseudo-Anosov flow and this can be used to completely determine the flow from outside in – from the universal circle ideal boundary to the universal cover of the manifold in an equivariant way.

The proof of the theorem goes as follows. Let Φ be a transverse flow which is regulating for the foliation $\mathcal{G}$. Suppose first that there is a parabolic leaf in $\mathcal{G}$. Then we show that there has to be a compact leaf which is parabolic. Hence the manifold fibers over the circle with fiber this leaf and the flow is topologically conjugate to a suspension Anosov flow. In this case there is at most one pseudo-Anosov flow transverse to $\mathcal{G}$, since there cannot be a non regulating transverse pseudo-Anosov flow. This is done in Section 2.

In the case that all leaves are Gromov hyperbolic, we use Candel's theorem [Can] and assume the leaves are hyperbolic. The orbit space of a pseudo-Anosov flow is the space of orbits in the universal cover. It is always homeomorphic to the plane [Fe-Mo] and the fundamental group of the manifold acts naturally on this orbit space. Given two regulating pseudo-Anosov flows transverse to $\mathcal{G}$ we produce a homeomorphism between the corresponding orbit spaces, which is group equivariant. This is the main step here. Using the foliation $\widetilde{\mathcal{G}}$ which is transverse to each lifted flow, this produces a homeomorphism of the universal cover of the manifold, which takes orbits of one flow to orbits of the other flow and is group equivariant. This produces the conjugacy.

In order to produce the homeomorphism between the orbit spaces, we use in an essential way the universal circle for foliations as introduced by Thurston [Th2], [Th3], [Th4]. For $\mathbb{R}$-covered foliations, the universal circle is canonically identified to the circle at infinity of any leaf of $\widetilde{\mathcal{G}}$ [Fe2], [Cal2]. Notice that the universal circle depends only on the foliation and not on the particular the transverse pseudo-Anosov flow. We first consider only one pseudo-Anosov flow transverse to $\mathcal{G}$. We show that the orbit space of the flow in $\widetilde{M}$ can be compactified with the universal circle of the foliation to produce a closed disk. This is canonically identified with the standard compactification of any hyperbolic leaf of $\widetilde{\mathcal{G}}$. Here one has to show that the topology of the orbit space of the flow in $\widetilde{M}$ union the universal circle of the foliation is compatible with the topology of the compactification of the leaves of $\widetilde{\mathcal{G}}$ and also that this topology is independent of the particular leaf of $\widetilde{\mathcal{G}}$. To prove this fact, one has to distinguish between uniform and non uniform foliations. Recall that uniform means that any two leaves of $\widetilde{\mathcal{G}}$ are a finite Hausdorff distance from each other – for example fibrations over the circle. The uniform case is simple. The non uniform case requires arguments involving the denseness of contracting directions between leaves, after a possible blow down of foliated I-bundles. Using the same ideas we analyse how stable/unstable leaves in the universal cover intersect leaves of $\widetilde{\mathcal{G}}$, particularly with relation to the universal circle. We proved in [Fe6] that for any pseudo-Anosov flow transverse to a foliation with hyperbolic leaves the following happens: given any ray in the intersection of a stable/unstable leaf (in the universal cover) with a leaf

of $\widetilde{\mathcal{G}}$, then this ray limits to a single point in the circle at infinity of this leaf of $\widetilde{\mathcal{G}}$. In this article we show if $\mathcal{G}$ is $\mathbb{R}$-covered then given a fixed stable (or unstable) leaf and varying the leaves of $\widetilde{\mathcal{G}}$, then the ideal points of these intersections in different leaves of $\widetilde{\mathcal{G}}$ follow the identifications prescribed by the universal circle. So clearly the universal circle is intrinsically connected with any regulating, transverse pseudo-Anosov flow. This is done in Section 4. These two results are the key tools used in the analysis of the theorem.

The next step is to analyse how an element of the fundamental group acts on the universal circle. If an element of the fundamental group is associated with a closed orbit of the flow, then we show that some power of it acts on the universal circle with a finite even number ≥ 4 of fixed points and vice versa. This key result depends on the analysis in Section 4 and on further properties of the intersections of leaves of $\widetilde{\mathcal{G}}$ and stable/unstable leaves, which is done in Section 5.

Finally in Section 6 we consider two pseudo-Anosov flows transverse and regulating for $\mathcal{G}$. We first prove that for each lift of a periodic orbit of the first flow, there is a unique periodic orbit of the second flow associated to it. This depends essentially on the study of group actions in Section 5. This produces a map between the orbit spaces of the two flows restricted to lifts of closed orbits. The final step is to show that this can be extended to an equivariant homeomorphism between the orbit spaces. This finishes the proof of the main theorem.

At the end of the article we also study the following two questions: 1) Given Φ, Ψ pseudo-Anosov flows transverse to a foliation $\mathcal{G}$ which is $\mathbb{R}$-covered, when is there a topological conjugacy between Φ and Ψ which also *preserves direction* along flowlines? Given the analysis of the main theorem, if this happens, then either both Φ and Ψ are regulating or they are both non regulating. By the main theorem again, this question reduces to asking whether there is a topological conjugacy between Φ and its inverse Φ^{-1} which preserves the direction along flow lines. Here Φ^{-1} is the same flow Φ traversed in the opposite direction. We show that all possibilities can occur. 2) The other question we analyse is whether the conjugating homeomorphism can be chosen to be isotopic to the identity. We show that this is always the case.

We thank the reviewer for several useful suggestions, in particular the simplification of the proof of Theorem 2.1 and also the implied suggestion of the isotopy result. We also thank Lee Mosher who asked the same isotopy question and also helped with conjugacy issues in the mapping class group of surfaces, which was useful to address question 1) above.

2. The case of parabolic leaves

Leaves of the foliation $\mathcal{G}$ are conformally either spherical, Euclidean or hyperbolic. In this section we quickly rule out the first option and prove the main theorem in the second option. We say that a leaf is *parabolic* if it is conformally Euclidean.

The stable and unstable foliations of Φ induce 1-dimensional perhaps singular foliations in any leaf of $\mathcal{G}$. There are no 1-prongs in the stable foliation and no centers, so Euler characteristic arguments disallow the existence of spherical leaves.

Theorem 2.1. *Let $\mathcal{G}$ be an $\mathbb{R}$-covered foliation transverse to a pseudo-Anosov flow Φ. If $\mathcal{G}$ has a parabolic leaf, then there is a compact leaf C which is parabolic and M fibers over the circle with fiber C. In this case the flow is an Anosov flow and is conjugate to a suspension flow with fiber C. Therefore if an $\mathbb{R}$-covered foliation $\mathcal{G}$ has a parabolic leaf, then up to topological conjugacy, there is at most one pseudo-Anosov flow transverse to $\mathcal{G}$.*

Proof. If the pseudo-Anosov flow Φ is not regulating for $\mathcal{G}$ then as explained in the introduction, the leaves of $\mathcal{G}$ are Gromov hyperbolic and therefore not conformally Euclidean. Therefore Φ has to be regulating.

We assume first that M is orientable.

Let L be a parabolic leaf of $\mathcal{G}$.

Suppose first that $\mathcal{G}$ has a compact leaf. Since $\mathcal{G}$ is $\mathbb{R}$-covered, it was shown by Goodman and Shields [Go-Sh] that any compact leaf is a fiber of M over the circle. We show that there is a compact leaf which is parabolic. This is not true in general, but it holds for $\mathbb{R}$-covered foliations. We may assume that L is not compact. Using the $\mathbb{R}$-covered hypothesis we show that L limits on a compact leaf. Consider the component of the complement of the compact leaves which contains L and let O be the closure of this component. Then O is homeomorphic to $C \times [0, 1]$ and in addition we can assume that $\mathcal{G}$ is transverse to the I-fibration in O (see [Fe2]). Identify C with the lower boundary of O. Look at the points that L hits in a fixed I-fiber J. Let x be the infimum of these points. If x is in the boundary of O we are done. The foliation in O is determined by its holonomy which is a homomorphism from $\pi_1(C)$ into the group of orientation preserving homeomorphisms of J. This holonomy has to fix x for otherwise some element would bring x closer to C and hence L would have a point in J lower than x. Since the holonomy fixes x then the leaf through x is compact, contrary to assumption that there are no compact leaves in the interior of O.

We conclude that L limits on a compact leaf C and since L is parabolic, then so is C. The flow Φ is regulating for $\mathcal{G}$ and so every orbit through C intersects C again, in other words Φ is conjugate to a suspension flow and the cross section is an Euclidean surface. In particular Φ is an Anosov flow. Any two pseudo-Anosov flows transverse to $\mathcal{G}$ will generate suspension flows in M transverse to C. As explained in the introduction, any two such flows are topologically conjugate. This finishes the analysis (in the orientable case) when there is a compact leaf.

Suppose now that there is no compact leaf. Our goal is to show that this cannot happen. As proved in Proposition 2.6 of [Fe2] there is a unique minimal set $\mathcal{Z}$ in $\mathcal{G}$. Since L must limit on leaves in a minimal set, then there are parabolic leaves in

the minimal set, and hence all leaves in the minimal set are parabolic. There are at most countably many components in $M - \mathcal{Z}$ each of which has a closure which is an I-bundle over a non compact surface. In addition the flow can be taken to be the I-fibration in this closure [Fe2]. Therefore these I-bundles can be blown down to single leaves and this yields a foliation which is still transverse to Φ and now is a minimal foliation. Clearly this happens for any pseudo-Anosov flow transverse to $\mathcal{G}$. Hence if there are no compact leaves, we may assume that $\mathcal{G}$ is minimal.

Let now L be an arbitrary leaf of $\mathcal{G}$, which is parabolic. Since L has polynomial growth, then Plante [Pl] showed that there is a holonomy invariant transverse measure supported in the closure of L. Since L is dense, this shows that the support of the measure is all of M.

The lift of this transverse measure to $\tilde{M}$ identifies the leaf space of $\mathcal{G}$ to $\mathbb{R}$ in a way that covering translations preserve the measure, that is, act by translations. Therefore there is a single subgroup G of $\pi_1(M)$ which is the stabilizer of any leaf E of $\tilde{\mathcal{G}}$. In particular all leaves of $\mathcal{G}$ are homeomorphic. This group G is the kernel of the holonomy homomorphism $\pi_1(M) \to \mathrm{Hom}(\mathbb{R})$ and therefore it is a normal subgroup of $\pi_1(M)$. In addition G is the fundamental group of a parabolic leaf L, which is not compact and which is orientable, hence G is either trivial or isomorphic to $\mathbb{Z}$.

If $G = 1$ then all leaves of $\mathcal{G}$ are planes. In this case Rosenberg [Ros] proved that M is homeomorphic to the 3-torus and hence $\pi_1(M)$ has polynomial growth of degree 3. On the other hand a manifold with a pseudo-Anosov flow has fundamental group with exponential growth [Pl-Th]. Therefore this case cannot happen.

If $G = \mathbb{Z}$ then since $\mathcal{G}$ is transversely orientable and M is orientable, the leaves of $\mathcal{G}$ are orientable and hence all annuli. We show that M is a nilmanifold. Start with a simple closed curve γ in L which is not null homotopic in L and let B a small closed annulus transverse to $\mathcal{G}$ and with one boundary γ. Since there is no holonomy in $\mathcal{G}$ the foliation induced by $\mathcal{G}$ in B is a foliation by circles near γ and we may assume the other boundary β is also a closed curve in L as L is dense. Then β is not null homotopic in L, for otherwise γ would be null homotopic in M contradicting Novikov's theorem [No]. Hence γ and β bound an annulus A in L and the union $A \cup B$ can be perturbed to a surface S transverse to $\mathcal{G}$ and foliated by circles (again by the no holonomy condition). In addition it is easy to see that S is transverse to the flow, hence double sided and therefore has to be a torus as M is orientable.

Cut M along S to produce a manifold M_1 with an induced 2-dimensional foliation $\mathcal{G}_1$ transverse to ∂M_1. If a leaf $\mathcal{G}_1$ is non compact then there is a leaf of $\mathcal{G}$ not intersecting S, contradiction. Hence every leaf of $\mathcal{G}_1$ is compact and as every leaf of $\mathcal{G}$ is an annulus, it now follows that every leaf of $\mathcal{G}_1$ is a compact annulus. We deduce that M_1 is $S \times [0, 1]$ and M is obtained from M_1 by a glueing which preserves circle foliations. Hence M is a nilpotent 3-manifold. It follows that $\pi_1(M)$ has polynomial growth, again contradicting the fact that $\pi_1(M)$ has exponential growth [Pl-Th]. So again we conclude that this cannot happen.

We conclude that if M is orientable, then $\mathcal{G}$ has to have a compact leaf C, which is a fiber of a fibration of M over S^1 and Φ is topologically conjugate to a suspension. The result is proved in this case.

If M is non orientable then it is doubly covered by an orientable manifold and the result applies to the double cover. The fiber C' in the double cover projects to a leaf C of $\mathcal{G}$ in M which intersects every orbit of the flow. Hence C is a fiber in M and has to be a torus as there are no pseudo-Anosov homeomorphisms of the Klein bottle. The rest of the proof is the same. This finishes the proof the theorem. □

3. General facts about $\mathbb{R}$-covered foliations

Remark. Unless otherwise stated, from now on we assume that $\mathcal{G}$ has only Gromov hyperbolic leaves.

A theorem of Candel [Can] then shows that there is a metric in M so that leaves of $\mathcal{G}$ are hyperbolic surfaces. We assume this is the metric we are using. Let $\pi : \tilde{M} \to M$ bee the universal covering space of M. The following facts concerning $\mathbb{R}$-covered foliations are proved in [Fe2], [Cal2]. There are two possibilities for $\mathcal{G}$:

- $\mathcal{G}$ is *uniform*. Given any two leaves L, E of $\tilde{\mathcal{G}}$, then they are a finite Hausdorff distance from each other. This was defined by Thurston [Th2]. If a is the Hausdorff distance between the leaves L, E (which depends on the pair L, E), then for any x in L choose $f(x)$ in E so that $d(x, f(x)) \leq a$. Note that f in general may not even be continuous. However, given the $\mathbb{R}$-covered hypothesis, then f is boundedly well defined: any two choices of $f(x)$ are a bounded distance from each other. The bound depends on the pair of leaves. The map f is a quasi-isometry between L and E and hence induces a homeomorphism between the corresponding circles at infinity still denoted by $f : \partial_\infty L \to \partial_\infty E$. Clearly these identifications between circles at infinity are group equivariant under the action by $\pi_1(M)$. In addition they satisfy a cocycle property: given 3 leaves L, E, S of $\tilde{\mathcal{G}}$, then the identifications between $\partial_\infty L$ and $\partial_\infty E$ composed with those between $\partial_\infty E$ and $\partial_\infty S$, induce the direct identifications between $\partial_\infty L$ and $\partial_\infty S$. Hence all circles at infinity are identified to a single circle, which is called the *universal circle* of $\mathcal{G}$ or $\tilde{\mathcal{G}}$ and is denoted by $\mathcal{U}$. By the equivariance property, $\pi_1(M)$ acts on $\mathcal{U}$. The fact to remember here is that given x in $\partial_\infty L$ and q in $\partial_\infty E$, then x, q are associated to the same point of $\mathcal{U}$ if and only if a geodesic ray r in L defining x is a finite Hausdorff distance in $\tilde{M}$ from a geodesic ray r' in E defining q.
- $\mathcal{G}$ is not uniform. If $\mathcal{G}$ is not a minimal foliation, then it has up to countably many foliated I-bundles. One can collapse the I-bundles to produce a foliation which is minimal (notice this does not work in the uniform case, for instance when $\mathcal{G}$

is a fibration). If a pseudo-Anosov flow is transverse to $\mathcal{G}$, then one can do the blow down so that the flow is still transverse to the blow down foliation [Fe2]. Sometimes we will assume in this case that $\mathcal{G}$ is minimal. If $\mathcal{G}$ is minimal then the following important fact is proved in [Fe2], [Cal2]: for any L, E leaves of $\widetilde{\mathcal{G}}$, then there is a dense set of contracting directions between them. A *contracting direction* is given by a geodesic r in L so that the distance between r and E converges to 0 as one escapes in r. Notice this only depends on the ideal point of r in $\partial_\infty L$ as all such rays are asymptotic because L is the hyperbolic plane. Any such direction produces a *marker* m. This is an embedding

$$m\colon [0,\infty)\times[0,1]\to\widetilde{M}$$

so that for each s in $[0,1]$ there is a leaf F_s of $\widetilde{\mathcal{G}}$ so that

$$m([0,\infty)\times\{s\})\subset F_s$$

is a parametrized geodesic ray in F_s. In addition, $m(\{t\}\times I)$ is a transversal to $\widetilde{\mathcal{G}}$ for each t in $[0,+\infty)$, and for all $s_1, s_2\in I$,

$$d(m(t,s_1),m(t,s_2))\to 0 \quad \text{as } t\to\infty.$$

Hence these geodesics of F_{s_1}, F_{s_2} are asymptotic in $\widetilde{M}$. The contracting directions between L, E induce an identification between dense sets in $\partial_\infty L$, $\partial_\infty E$ which preserves the circular ordering. This extends to a homemorphism between $\partial_\infty L$ and $\partial_\infty E$. These homeomorphisms are clearly $\pi_1(M)$ equivariant and in addition they satisfy the cocycle property as in the uniform case. Hence as before each circle at infinity is canonically identified to a fixed circle $\mathcal{U}$, the universal circle of $\mathcal{G}$ or $\widetilde{\mathcal{G}}$. Finally $\pi_1(M)$ acts on $\mathcal{U}$.

We now explain what happens if $\mathcal{G}$ is not uniform and not minimal. This was not discussed in [Fe2] but it is a simple consequence of the analysis of the minimal case as follows: Let $\mathcal{Z}$ be the unique minimal set of $\mathcal{G}$ [Fe2]. Blow down $\mathcal{G}$ to a minimal foliation $\mathcal{G}'$. The analysis above produces the universal circle $\mathcal{U}'$ for $\mathcal{G}'$. Let $\delta\colon M\to M$ be the blow down map sending leaves of $\mathcal{G}$ to leaves of $\mathcal{G}'$ and homotopic to the identity. Lift the homotopy to produce a lift $\tilde{\delta}$ of δ, which is a homeomorphism of $\widetilde{M}$. For any A, B leaves of $\widetilde{\mathcal{G}}'$, there are F, E leaves in $\widetilde{\mathcal{Z}}$ so that A, B are between F, E. Let $F'=\tilde{\delta}(F)$, $E'=\tilde{\delta}(E)$. Then in $\widetilde{\mathcal{G}}'$ there is a dense set of contracting directions between F' and E'. For any such there is a ray r' in F' asymptotic to a ray l' in E'. Under the blow up map, this produces corresponding rays in F, E: a ray r in F which is a bounded distance from a ray l in E. By the $\mathbb{R}$-covered property, the ideal point of the ray l is the unique direction for which there is a ray a bounded distance from r in $\widetilde{M}$. This provides an identification between dense sets in $\partial_\infty F$ and $\partial_\infty E$. This is equivariant and satisfies the cocycle property. This can be extended

to a group equivariant homeomorphism between $\partial_\infty F$ and $\partial_\infty E$. This produces the universal circle in this case.

Calegari [Cal1] produced many examples of $\mathbb{R}$-covered, non uniform foliations in closed, hyperbolic 3-manifolds.

4. Intersections between leaves of $\widetilde{\mathcal{G}}$ and pseudo-Anosov foliations

The main goal of this section is to show that a pseudo-Anosov flow transverse to an $\mathbb{R}$-covered foliation interacts very well with the universal circle. We show that leaves of $\widetilde{\Lambda}^s$ are essentially vertical products with respect to the universal circle. Let Φ be a pseudo-Anosov flow in M^3 closed. Background on pseudo-Anosov flows can be found in [Mo1], [Fe6]. We stress that pseudo-Anosov flows do not have 1-prong singular orbits. Let Λ^s, Λ^u be the stable/unstable foliations of Φ and $\widetilde{\Phi}$, $\widetilde{\Lambda}^s$, $\widetilde{\Lambda}^u$ the lifts to the universal cover of Φ, Λ^s, Λ^u respectively. Given z in $\widetilde{M}$ let $\widetilde{W}^s(z)$ be the stable leaf containing z and similarly define $\widetilde{W}^u(z)$. Our assumption is that Φ is transverse to the foliation $\mathcal{G}$ and is regulating for $\mathcal{G}$. Therefore given any leaf L of $\widetilde{\mathcal{G}}$, the foliations $\widetilde{\Lambda}^s$, $\widetilde{\Lambda}^u$ are transverse to L and they induce 1-dimensional singular foliations $\widetilde{\Lambda}^s_L$, $\widetilde{\Lambda}^u_L$ in L. We are in the case that leaves of $\widetilde{\mathcal{G}}$ are isometric to the hyperbolic plane. The orbit space of $\widetilde{\Phi}$ is $\mathcal{O} = \widetilde{M}/\widetilde{\Phi}$ with the quotient topology and it is homeomorphic to $\mathbb{R}^2$ [Fe-Mo]. The foliations $\widetilde{\Lambda}^s$, $\widetilde{\Lambda}^u$ induce 1-dim foliations $\mathcal{O}^s$, $\mathcal{O}^u$ in $\mathcal{O}$. If x is in $\mathcal{O}$, then $\mathcal{O}^s(x)$ is the leaf of $\mathcal{O}^s$ through x and similarly for $\mathcal{O}^u$.

One fundamental fact used here is that we proved in [Fe6] that each ray of a leaf of $\widetilde{\Lambda}^s_L$ or $\widetilde{\Lambda}^u_L$ accumulates in a single point of $\partial_\infty L$. This works even if $\mathcal{G}$ is not $\mathbb{R}$-covered.

A convention that will be used throughout the article is the following: the group $\pi_1(M)$ acts on several objects: the universal cover $\widetilde{M}$, the orbit space $\mathcal{O}$, the universal circle $\mathcal{U}$, the foliations $\widetilde{\Lambda}^s$, $\widetilde{\Lambda}^u$, $\mathcal{O}^s$, $\mathcal{O}^u$, etc. If g is an element of $\pi_1(M)$ we still use the same g to denote the induced actions on all these spaces $\widetilde{M}$, $\mathcal{O}$, $\mathcal{U}$, $\widetilde{\Lambda}^s$, $\widetilde{\Lambda}^u$, $\mathcal{O}^s$, $\mathcal{O}^u$, etc.

Lemma 4.1. *Suppose that a pseudo-Anosov flow Φ is regulating for an $\mathbb{R}$-covered foliation $\mathcal{G}$. Then the stable and unstable foliations $\widetilde{\Lambda}^s$, $\widetilde{\Lambda}^u$ have Hausdorff leaf space. It follows that for any leaf L of $\widetilde{\mathcal{G}}$, the leaves of the one dimensional foliations $\widetilde{\Lambda}^s_L$, $\widetilde{\Lambda}^u_L$ are uniform quasigeodesics in L.*

Proof. This is stronger than the fact that rays in these leaves limit to single points in $\partial_\infty L$. If we suppose on the contrary that (say) $\widetilde{\Lambda}^s$ does not have Hausdorff leaf space, then there are closed orbits α, β of Φ (maybe with multiplicity), so that they are freely homotopic to the inverse of each other, see [Fe6]. Lift them coherently to

orbits $\tilde{\alpha}, \tilde{\beta}$ of $\widetilde{\Phi}$. Since Φ is regulating for $\mathcal{G}$, then both $\tilde{\alpha}$ and $\tilde{\beta}$ intersect every leaf of $\widetilde{\mathcal{G}}$.

Let g in $\pi_1(M)$ non trivial with g leaving $\tilde{\alpha}$ invariant and sending points in $\tilde{\alpha}$ forward (in terms of the flow parameter). Therefore g acts in an increasing way in the leaf space of $\widetilde{\mathcal{G}}$. By the free homotopy, g also leaves $\tilde{\beta}$ invariant and g acts decreasingly in $\tilde{\beta}$, hence also in the leaf space of $\widetilde{\mathcal{G}}$. This is a contradiction.

Hence the leaf spaces of $\widetilde{\Lambda}^s$, $\widetilde{\Lambda}^u$ are Hausdorff. As proved in Proposition 6.11 of [Fe6] this implies that for any L in $\widetilde{\mathcal{G}}$, then all leaves of $\widetilde{\Lambda}^s_L$, $\widetilde{\Lambda}^u_L$ are uniform quasigeodesics in L. The bounds are independent of the leaf of $\widetilde{\Lambda}^s_L$, $\widetilde{\Lambda}^u_L$ in L and also of the leaf L of $\widetilde{\mathcal{G}}$. For non singular leaves, this implies that any such leaf is a bounded distance (in the hyperbolic metric of L) from a minimal geodesic in L. For singular p-prong leaves of $\widetilde{\Lambda}^s_L$, $\widetilde{\Lambda}^u_L$ the same is true for any properly embedded copy of $\mathbb{R}$ in such leaves. □

In this section we want to show that the asymptotic behavior of leaves of $\widetilde{\Lambda}^s_L$, $\widetilde{\Lambda}^u_L$ is coherent with the identifications prescribed by the universal circle.

Let $\mathcal{H}$ be the leaf space of $\widetilde{\mathcal{G}}$, which is homeomorphic to the set of real numbers.

Let A be a leaf of $\widetilde{\Lambda}^s$ (or $\widetilde{\Lambda}^u$). We will show that each half leaf of A has a single point of the universal circle associated to it. In order to do that choose an arbitrary leaf L of $\widetilde{\mathcal{G}}$ to start with and let r be a ray of $A \cap L$ – this is a ray of $\widetilde{\Lambda}^s_L$. Let now E be an arbitrary leaf of $\widetilde{\mathcal{G}}$ or an element of $\mathcal{H}$. Since Φ is regulating, then $\widetilde{\Phi}_{\mathbb{R}}(q)$ intersects E for any q in r. The intersection of $\widetilde{\Phi}_{\mathbb{R}}(r)$ and E is a ray of $\widetilde{\Lambda}^s_E$ – again because of the regulating condition. This ray also defines an unique ideal point in $\partial_\infty E$. Since $\partial_\infty E$ is canonically identified with the universal circle $\mathcal{U}$ this defines a map

$$f_r : \mathcal{H} \to \mathcal{U},$$

$$f_r(E) = \{\text{equivalence class in } \mathcal{U} \text{ of the ideal point in } \partial_\infty E \text{ of the ray } (\widetilde{\Phi}_{\mathbb{R}}(r) \cap E)\}.$$

The set $\widetilde{\Phi}_{\mathbb{R}}(r)$ is a half leaf of A. Clearly the map f_r only depends on the equivalence class of half leaves of A, where two half leaves are equivalent if they both contain a half leaf of A.

Proposition 4.2. *Any leaf A of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ is a product with respect to the universal circle, that is, the ideal points of $A \cap L$ for L leaves of $\widetilde{\mathcal{G}}$ are constant in the universal circle. More specifically given a ray r of $A \cap L$, where L is in $\widetilde{\mathcal{G}}$, then the corresponding map $f_r : \mathcal{H} \to \mathcal{U}$ defined above is a constant map.*

Proof. The proof depends on whether $\mathcal{G}$ is uniform or not.

Case 1. $\mathcal{G}$ is uniform.

Claim. *If $\mathcal{G}$ is uniform and Φ is transverse and regulating for $\mathcal{G}$, then for any S, E leaves of $\widetilde{\mathcal{G}}$, there is a bound on the length of flow lines from S to E. The bound depends on the pair S, E.*

Otherwise we find p_i in S with $\widetilde{\Phi}_{t_i}(p_i)$ in E and t_i converging to (say) infinity. Up to subsequence assume that $\pi(p_i)$ converges to a point p in M. Take covering translations g_i in $\pi_1(M)$ with $g_i(p_i)$ converging to p_0. For each i take q_i in $g_i(E)$ with $d(q_i, g_i(p_i)) < a$ for fixed a. This uses the uniform property. Up to subsequence assume that q_i converges and hence $g_i(E)$ converges to a leaf E_0. The orbit of $\widetilde{\Phi}$ through p_0 intersects E_0, since the flow is regulating. Hence there is t_0 with $\widetilde{\Phi}_{t_0}(p_0)$ in E_0. By continuity of flow lines of $\widetilde{\Phi}$, then for any z in $\widetilde{M}$ near p_0 and G leaf of $\widetilde{\mathcal{G}}$ near E_0, then there is t near t_0 so that $\widetilde{\Phi}_t(z)$ is in G. But $\widetilde{\Phi}_{t_i}(g_i(p_i))$ is in $g_i(E)$, which is a leaf near E_0 and t_i converges to infinity, contradiction. This proves the claim. Notice that it is not necessary for Φ to be pseudo-Anosov in this claim, just that it is regulating.

Since r is a quasigeodesic in L, let l be the geodesic ray in L with starting point p and a finite Hausdorff distance (in L) from r. By the above $\widetilde{\Phi}_{\mathbb{R}}(r)$ intersects E in a ray r' of $\widetilde{\Lambda}^s_E$ which is a bounded distance from r in $\widetilde{M}$. The ray r' is also a uniform quasigeodesic ray in E, hence r' is a bounded distance in E from a geodesic ray l'. Then l, l' are a finite distance from each other in $\widetilde{M}$. The definition of the universal circle in the uniform case implies that r, r' define the same point in $\mathcal{U}$. This establishes this case.

Case 2. $\mathcal{G}$ is not uniform.

In this case, first assume that $\mathcal{G}$ is minimal. Therefore between any two leaves of $\widetilde{\mathcal{G}}$, there is a dense set of contracting directions. The proof essentially uses that flow lines cannot cross these contracting directions. The proof will be done by contradiction. Let r be a ray of a leaf of $\widetilde{\Lambda}^s_L$ for some L in $\widetilde{\mathcal{G}}$ with initial point p. Let a be the ideal point of r in $\partial_\infty L$. Suppose that for some E leaf of $\widetilde{\mathcal{G}}$, then

$$r' = \widetilde{\Phi}_{\mathbb{R}}(r) \cap E \text{ defines a distinct point in } \mathcal{U}.$$

Let b be the point in $\partial_\infty L$ identified to the ideal point of r' in $\partial_\infty E$, by the universal circle identification. Hence a, b are different. By density of contracting directions between L and E, there are points c, d in $\partial_\infty L$ which separate a from b in $\partial_\infty L$ and so that c, d correspond to contracting directions between L and E. Let m_1, m_2 be markers between L and E associated to the contracting directions c, d respectively. Let $B_i = \text{Image}(m_i)$ and let C be the union of the points in $\widetilde{M}$ contained in leaves intersecting the markers m_1, m_2. Removing initial pieces if necessary we may assume that B_1, B_2 are disjoint. Since $m_i(\{t\} \times I)$ is a very small transverse arc if t is big enough, we can also assume the following: if z is in B_1 or B_2 then $\widetilde{\Phi}_{\mathbb{R}}(z)$ will intersect any leaf S in C near z, producing a small transversal from L to E passing

through z. For each leaf S of $\widetilde{\mathcal{G}}$ intersected by the markers, let

$$r_S = \text{ geodesic arc in } S \text{ joining the endpoints of } \text{Image}(m_1) \cap S \text{ and } \text{Image}(m_2) \cap S.$$

Let A be the union of the r_S for such S. This is topologically a rectangle with the bottom in L the top in E and the sides transversals from L to E. Then $A \cup B_1 \cup B_2$ separates C into 2 components C_1, C_2. Since $\{a, b\}$ is disjoint from $\{c, d\}$ the ray r does not accumulate on c or d in $\partial_\infty L$. Hence starting with a smaller ray r if necessary we may assume also that r, r' are disjoint from B_i and far away from it. In particular the flow line through any point of r will not intersect B_i, since points in B_i are in very short transversals from L to E.

By renaming C_1, C_2 we may assume that r is contained in C_1 and r' is contained in C_2. For each z in r it is in C_1, then the flow line through z intersects E in r' which is in C_2. Therefore this flow line has to intersect $A \cup B_1 \cup B_2$. The above remarks imply that this flow line cannot intersect either B_1 or B_2. Hence this flow line must intersect A. Since A is compact we can choose z_i in r escaping in r so that $\widetilde{\Phi}_{\mathbb{R}}(z_i)$ intersects A in

$$q_i = \widetilde{\Phi}_{t_i}(z_i) \quad \text{and} \quad q_i \to q \in A.$$

Since z_i escapes in r, it follows that t_i converges to infinity. By the regulating property of Φ, the orbit through q intersects L. Hence nearby orbits intersect L in bounded time, contradicting that t_i converges to infinity.

This contradiction shows that r' has to define the same point in $\mathcal{U}$ that r does. This finishes the proof when $\mathcal{G}$ is minimal.

If $\mathcal{G}$ is not minimal, then first blow down $\mathcal{G}$ to a minimal foliation $\mathcal{G}'$. We can assume that Φ is still transverse to $\mathcal{G}'$. Now use the proof for $\mathcal{G}'$ as above. The walls $A \cup B_1 \cup B_2$ for $\widetilde{\mathcal{G}}'$ pull back to walls for $\widetilde{\mathcal{G}}$. Because the foliation $\mathcal{G}$ is a blow up of $\mathcal{G}'$ and Φ is transverse to both of them, it follows that flowlines of $\widetilde{\Phi}$ cannot cross the two ends of the pullback walls and if necessary can only cross the compact part of these walls. Therefore the same arguments as above prove the result in this case. This finishes the proof of Proposition 4.2. □

A leaf F of $\widetilde{\mathcal{G}}$ is isometric to the hyperbolic plane, so we consider the canonical compactification $F \cup \partial_\infty F$ with a circle at infinity. Given any two leaves F, E in $\widetilde{\mathcal{G}}$, then using the universal circle analysis there is a homeomorphism between $\partial_\infty F$ and $\partial_\infty E$. In addition if a flow Φ is regulating for $\mathcal{G}$ then there is also a homeomorphism between F, E by moving along flow lines. We next show that these two homeomorphisms are compatible:

Proposition 4.3. *Given F, E in $\widetilde{\mathcal{G}}$ consider the map g from $F \cup \partial_\infty F$ to $E \cup \partial_\infty E$ defined by: if x is in F then move along the flow line of $\widetilde{\Phi}$ through x until it hits E. The intersection point is $\zeta(x)$. If x is in $\partial_\infty F$, let $\zeta(x)$ be the point in $\partial_\infty E$ associated*

to x by the universal circle identification. Then ζ is a homeomorphism. In addition these homeomorphisms are group equivariant and satisfy the cocycle condition.

Proof. The map ζ is a bijection. We only need to show that it is continuous, since the inverse is a map of the same type. The equivariance and cocycle properties follow immediately from the same properties for flowlines and identifications induced by the universal circle.

We now prove continuity of ζ: This is very similar to the previous proposition and we will use the setup of that proposition. The first possibility is that $\mathcal{G}$ is uniform. The claim in Proposition 4.2 shows that the map $\zeta\colon F \to E$ is a quasi-isometry and it induces a homeomorphism ζ^* from $F \cup \partial_\infty F$ to $E \cup \partial_\infty E$. The image of an ideal point p in $\partial_\infty F$ is determined by the ideal point of $\zeta(r)$ where r is a geodesic ray in F with ideal point p. But $\zeta(r)$ is a bounded distance from r in $\tilde{M}$ and this is exactly the identification associated with the universal circle.

Suppose now that $\mathcal{G}$ is not uniform. Assume first that $\mathcal{G}$ is minimal. We know that ζ restricted to both F and $\partial_\infty F$ are homeomorphisms. Since F is open in $F \cup \partial_\infty F$ all we need to do is to show that ζ is continuous in $\partial_\infty F$. Let a in $\partial_\infty F$ and (a_i) converging to a in $F \cup \partial_\infty F$, so we may assume that a_i is in F. Suppose by way of contradiction that $\zeta(a_i)$ converges to $\zeta(b)$ where b is not a. Choose c, d in $\partial_\infty F$ which separate a, b in $\partial_\infty F$. Then construct the wall $A \cup B_1 \cup B_2$ as in Proposition 4.2. The flow lines from a_i to $g(a_i)$ have to intersect this wall in a compact set, contradiction as in Proposition 4.2. This finishes the proof if $\mathcal{G}$ is minimal.

If $\mathcal{G}$ is not minimal, then use the same arguments as in the end of the previous proposition to deal with this case. □

Topology in $\mathcal{O} \cup \mathcal{U}$. Proposition 4.3 allows us to put a topology in $\mathcal{O} \cup \mathcal{U}$ as follows: Consider any leaf L of $\tilde{\mathcal{G}}$. There are homeomorphisms between L and $\mathcal{O}$ and $\partial_\infty L$ and $\mathcal{U}$. The combined map induces a topology in $\mathcal{O} \cup \mathcal{U}$ from the topology in $L \cup \partial_\infty L$. Proposition 4.3 shows that this topology is independent of the leaf L we start with. In addition covering translations induce homeomorphisms of $\mathcal{O} \cup \mathcal{U}$ – this is because if L is in $\tilde{\mathcal{G}}$ and f in $\pi_1(M)$ then f is a homeomorphism from $L \cup \partial_\infty L$ to $(f(L) \cup \partial_\infty f(L))$, both of which are homeomorphic to $\mathcal{O} \cup \mathcal{U}$. We think of this as an action on $\mathcal{O} \cup \mathcal{U}$. Given f in $\pi_1(M)$, then the notation f will also denote the induced map in $\mathcal{O} \cup \mathcal{U}$. The analysis above makes it clear that f in $\pi_1(M)$ acts as an orientation preserving way on $\mathcal{O}$ if and only if it acts as an orientation preserving way on $\mathcal{U}$.

5. Action of elements of $\pi_1(M)$

The main purpose of this section is to analyse how elements of $\pi_1(M)$ act on $\mathcal{U}$ for an $\mathbb{R}$-covered foliation $\mathcal{G}$, particularly with respect to a transverse pseudo-Anosov

flow. We first need a couple of auxiliary results. Let

$$\Theta\colon\ \tilde{M} \ \to\ \mathcal{O} \ \text{ be the projection map.}$$

A point x in $\mathcal{O}$ is called *periodic* if there is $g \neq id$ in $\pi_1(M)$ with $g(x) = x$ and an orbit α of $\tilde{\Phi}$ is periodic if $\Theta(\alpha)$ is periodic. A *line leaf* of $\tilde{\Lambda}^s_L$ is a properly embedded copy l of $\mathbb{R}$ in a leaf of $\tilde{\Lambda}^s_L$ of a leaf L of $\tilde{\mathcal{G}}$ so that: if l is in a singular leaf r of $\tilde{\Lambda}^s_L$, then r does not have prongs on both sides of l in L. A singular leaf with a p-prong singularity has p lines leaves. Consecutive line leaves intersect in a ray of $\tilde{\Lambda}^s_L$. Non singular leaves are line leaves themselves. Similarly one defines line leaves for $\tilde{\Lambda}^u_L, \mathcal{O}^s, \mathcal{O}^u, \tilde{\Lambda}^s, \tilde{\Lambda}^u$ (the last two are pullbacks to $\tilde{M}$ of line leaves of $\mathcal{O}^s, \mathcal{O}^u$). Given z in $\tilde{M}$ let $\widetilde{W}^s(z)$ be the stable leaf containing z. The *sectors* of $\widetilde{W}^s(z)$ are the connected components of $\tilde{M} - \widetilde{W}^s(z)$.

Lemma 5.1. *Let Φ be a pseudo-Anosov regulating for a foliation $\mathcal{G}$ which is $\mathbb{R}$-covered with hyperbolic leaves. Let l_i be line leaves of $\tilde{\Lambda}^s_{L_i}$ where L_i are leaves of $\tilde{\mathcal{G}}$. Suppose that there are x_i in l_i so that x_i converges in $\tilde{M}$ to a point x in a leaf L of $\tilde{\mathcal{G}}$. If $\widetilde{W}^s(x)$ is singular assume that all x_i are in the closure of a sector of $\widetilde{W}^s(x)$. Then there is a line leaf l of $\tilde{\Lambda}^s_L$ with x in l and l_i converging to l in the geometric topology of $\tilde{M}$. In addition if s_i are the geodesics in L_i a bounded distance from l_i in L_i and s is the geodesic a bounded distance from l in L then s_i converges to s in the geometric topology of $\tilde{M}$.*

Proof. We first prove the statement about l_i and l. Geometric convergence means that if z is in l then there are z_i in l_i with the sequence (z_i) converging to z and in addition if z_{i_k} is in l_{i_k} and (z_{i_k}) converges to w in $\tilde{M}$ then w is in l.

Since the flow Φ is regulating for $\mathcal{G}$, then l_i flows into line leaves r_i of $\tilde{\Lambda}^s_L$. The points x_i flow to q_i in L and clearly q_i converges to x. Hence there is a line leaf l of $\tilde{\Lambda}^s_L$ through x, so that any point z in l is the limit of a sequence (z'_i) with z'_i in r_i. If $\widetilde{W}^s(x)$ is singular, this uses the fact that the x_i are all in the closure of a sector of $\widetilde{W}^s(x)$. Otherwise it could easily be that different subsequences of r_i converge to distinct line leaves of $\tilde{\Lambda}^s_L$. Let $z_i = \tilde{\Phi}_{\mathbb{R}}(z'_i) \cap l_i$. Then z_i converges to z. This shows that any z in l is the limit of a sequence in l_i.

Now suppose that (z_{i_k}) is a sequence converging to z with z_{i_k} in L_{i_k}. Here x_{i_k} is in L_{i_k} and x is in L and hence L_{i_k} converges to L in the leaf space $\mathcal{H}$ of $\tilde{\mathcal{G}}$. Since $\mathcal{H}$ is Hausdorff then no sequence of points in L_{i_k} converges to a point in another leaf of $\tilde{\mathcal{G}}$. If follows that z is in L. Let

$$V_k = \widetilde{W}^s(x_{i_k}), \quad V = \widetilde{W}^s(x).$$

Then V_k converges to V. By Lemma 4.1 the leaf space of $\tilde{\Lambda}^s$ is also Hausdorff. It follows that z is in V. Hence z is in $L \cap V = \tau$. It was also proved in [Fe6] that $L \cap V$ is connected and hence τ is exactly the leaf of $\tilde{\Lambda}^s_L$ containing x.

If τ is non singular this finishes the proof of the first statement. Suppose then that $\widetilde{W}^s(x)$ is singular. Since the x_i are in the closure of a sector of $\widetilde{W}^s(x)$ then so are the l_{i_k} and hence the z_{i_k}. Consequently the same is true of z. The boundary of this sector is a line leaf of $\widetilde{W}^s(x)$ and so z is in the corresponding line leaf of $\widetilde{\Lambda}^s_L$, which is l. This finishes the proof of the first statement of Lemma 5.1.

We now consider the second part of Lemma 5.1. By Lemma 4.1 the leaves of $\widetilde{\Lambda}^s_E$ are uniform quasigeodesics in E for any E leaf of $\widetilde{\mathcal{G}}$. Let then $b > 0$ so that any line leaf of $\widetilde{\Lambda}^s_E$ is $\leq b$ from the corresponding geodesic in E and likewise for arcs in such leaves. Let l_i be line leaves of $\widetilde{\Lambda}^s_{L_i}$, l its limit in a leaf L of $\widetilde{\mathcal{G}}$ as in the first part of the lemma. Let s_i be the geodesics in L_i corresponding to l_i and let s be the geodesic in L corresponding to l.

For any $\epsilon > 0$ there is fixed $\mu(\epsilon) > 0$ so that if two geodesic segments in the hyperbolic plane have length bigger than $3\mu(\epsilon)$ and the corresponding endpoints are less than $2b + 2$ from each other, then except for segments of length $\mu(\epsilon)$ adjacent to the endpoints, then the rest of the segments are less than $\epsilon/3$ from each other.

Let then z in s. Given $\epsilon > 0$, find w', u' in s which are exactly $(3\mu(\epsilon) + 2b + 1)$ distant from z. There are w, u in l with

$$d_L(w, w') < b + \frac{1}{2}, \quad d_L(u, u') < b + \frac{1}{2}.$$

Let τ be the segment of l between w, u. There is a corresponding segment of τ_i of l_i between points w_i, u_i so that the Hausdorff distance in $\widetilde{M}$ from τ to τ_i is $<< 1$. The corresponding geodesic segment m_i from w_i to u_i in L_i is less than b from τ_i and by choice of w', u' then the midpoint of m_i is less than $\epsilon/3$ from a point v_i in s_i. Hence v_i is less than ϵ from z. By adjusting the ϵ to converge to 0 and the i to increase, one finds v_i in s_i with v_i converging to z.

Suppose now that z_{i_k} are in s_{i_k} with s_{i_k} contained in L_{i_k}. Suppose that the sequence z_{i_k} converges to z in $\widetilde{M}$. The proof is very similar to the above: Fix $\epsilon > 0$. Choose big segments in s_{i_k} centered in z_{i_k}. The length is fixed and depends on ϵ. There are geodesic arcs of L_{i_k} with endpoints in the leaves l_{i_k} whose midpoints are very close to z_{i_k}. Very close depends on ϵ and the length above. There are arcs in l_i with these endpoints so that the above arcs converge up to a subsequence to a segment in l by the first part of the lemma. The geodesic arcs above converge to a geodesic arc with endpoints in l. Up to subsequence the midpoints of the geodesic arcs (which are ϵ close to the z_{i_k}) converge to a point (this point is z) which is close to a point in s, closeness depending on ϵ. Now make ϵ converge to 0 and prove that z is in s. This finishes the proof of Lemma 5.1. □

At this point it is convenient to do the following: for the remainder of this section we fix a leaf L of $\widetilde{\mathcal{G}}$. The bijection $L \cup \partial_\infty L \to \mathcal{O} \cup \mathcal{U}$ is a homeomorphism. Therefore the action of $\pi_1(M)$ on $\mathcal{O} \cup \mathcal{U}$ induces an action by homeomorphisms on

$L \cup \partial_\infty L$ under this identification. This action leaves invariant the foliations $\widetilde{\Lambda}^s_L$, $\widetilde{\Lambda}^u_L$, because $\mathcal{O}^s, \mathcal{O}^u$ are $\pi_1(M)$ invariant and $\mathcal{O}^s, \mathcal{O}^u$ are identified with $\widetilde{\Lambda}^s_L$, $\widetilde{\Lambda}^u_L$ by the bijection above.

We need one more auxiliary fact. This is a technical result concerning ideal points of leaves of $\widetilde{\Lambda}^s_L$, $\widetilde{\Lambda}^u_L$.

Lemma 5.2. *Let E be a leaf of $\widetilde{\mathcal{G}}$ and l_1, l_2 distinct leaves of $\widetilde{\Lambda}^s_E$ or $\widetilde{\Lambda}^u_E$. Then l_1, l_2 do not share an ideal point in $\partial_\infty E$.*

Proof. Roughly the proof goes like this: rays in $\widetilde{\Lambda}^s_E$ with same ideal points are a bounded distance from each other. Zoom in to the ideal point and use covering translations to bring it back to a compact region and produce line leaves of say $\widetilde{\Lambda}^s_L$ (for appropriate L) with *both* ideal points identified. Then use the transitive property, pseudo-Anosov dynamics and the regulating property to derive a contradiction. Here are the details:

Suppose first by way of contradiction that there are l_1, l_2 rays in leaves of $\widetilde{\Lambda}^s_E$ for some E in $\widetilde{\mathcal{G}}$ with the same ideal point a in $\partial_\infty E$ and so that l_1, l_2 do not share a subray. We can assume that l_1, l_2 do not have singularities. Let u_j, $j = 1, 2$ be the starting points of l_j. Let r_j, $j = 1, 2$ be a line leaf of $\widetilde{\Lambda}^s_E$ containing l_j. Choose points x_i in l_1 escaping in l_1. As explained before the leaves of $\widetilde{\Lambda}^s_E$ are uniform quasigeodesics in E and hence they are at a bounded distance in E from geodesics in E. This implies that there are q_i in l_2 so that q_i are a bounded distance from x_i in E. Up to taking a subsequence we may assume that $\pi(x_i)$ converges in M. Let then g_i in $\pi_1(M)$ with $g_i(x_i)$ converging to x_0. For simplicity of explanation we assume that the leaf of $\widetilde{\mathcal{G}}$ containing x_0 is the fixed leaf L as above. Let v_1 be the line leaf of $\widetilde{\Lambda}^s_L$ containing x_0 and which is the limit of the $g_i(r_1)$ as proved in the previous lemma. If $\widetilde{W}^s(x_0)$ is singular then, up to taking a subsequence, we may assume that the $g_i(x_i), g_i(r_i)$ satisfy the requirements of the previous lemma.

Since the distance in $g_i(E)$ from $g_i(x_i)$ to $g_i(q_i)$ is bounded we may assume up to subsequence that $g_i(q_i)$ also converges and let q_0 be its limit. It follows that q_0 is also in L and let v_2 be the line leaf of $\widetilde{\Lambda}^s_L$ containing q_0 which is the limit of $g_i(r_2)$. Here the rays

$$g_i(l_1),\ g_i(l_2) \text{ in } E \text{ have the same ideal point } g_i(a) \text{ in } \partial_\infty(g_i(E)).$$

The line leaves r_j are uniform quasigeodesics in E and a bounded distance from a geodesic s_j in E. Hence the geodesics $g_i(s_1), g_i(s_2)$ share an ideal point in $\partial_\infty g_i(E)$. By the second part of the previous lemma $g_i(s_j)$ converges to a geodesic t_j in L with same ideal points as v_j for both $j = 1, 2$. By continuity of geodesics in leaves of $\widetilde{\mathcal{G}}$, it follows that t_1 and t_2 share an ideal point. Therefore v_1, v_2 share an ideal point in $\partial_\infty L$.

We claim that v_1, v_2 also share the other ideal point. The line leaves $g_i(r_1), g_i(r_2)$ have big segments from

$$g_i(u_1) \text{ to } g_i(x_k) \quad \text{and} \quad g_i(u_2) \text{ to } g_i(q_k)$$

which are boundedly close to each other. Here $k \gg i$ and so $g_i(x_i)$ is in these segments. Also $g_i(x_i)$ converges to x_0. The corresponding geodesic arcs between the points above have endpoints which are boundedly close to each other. As explained in the proof of the previous lemma they have middle thirds which are arbitrarily close to each other. The limits of the geodesic arcs are contained in t_1 and t_2. This shows that t_1 and t_2 have infinitely many points in common and therefore are the same geodesic.

Suppose first that v_1, v_2 are distinct. The two line leaves v_1, v_2 of $\widetilde{\Lambda}^s_L$ have the same two ideal points, which we denote by a_1, a_2. The line leaves

$$v_1, v_2 \text{ bound a region } R \text{ in } L.$$

For any stable leaf l of $\widetilde{\Lambda}^s_L$ in R then l has ideal points which can only be a_1, a_2. But l is a quasigeodesic in L. Therefore this leaf is non singular and has ideal points exactly a_1, a_2. Now consider a periodic orbit α of $\widetilde{\Phi}$ intersecting L in R very close to v_1 so that the unstable leaf $\widetilde{W}^u(\alpha)$ intersects v_1. Notice that the set of periodic orbits of Φ is dense in M when Φ is transitive as proved by Mosher [Mo1]. In addition if M is atoroidal then Φ is transitive [Mo1]. In the situation here, Φ is regulating and $\mathcal{G}$ has hyperbolic leaves, which implies that M is atoroidal as mentioned in the introduction.

We now use that $L \cup \partial_\infty L$ is identified with $\mathcal{O} \cup \mathcal{U}$. Let g in $\pi_1(M)$ non trivial so that $g(\alpha) = \alpha$ and in addition g leaves invariant all components of $\widetilde{W}^s(\alpha) - \alpha$. Under the identifications above then

$$g \text{ fixes } a_1 \text{ and } a_2 \text{ in } \partial_\infty L.$$

Notice that a_1, a_2 are the ideal points of $\widetilde{W}^s(\alpha) \cap L$ in $\partial_\infty L$. Assume that $g^n(\widetilde{W}^s(v_1))$ moves away from $\widetilde{W}^s(\alpha)$ when n converges to infinity. Since v_1 (line leaf of $\widetilde{\Lambda}^s_L$) has ideal points a_1, a_2, it follows that the same happens for all leaves $g^n(\widetilde{W}^s(v_1)) \cap L$. These line leaves are nested in L and they are uniform quasigeodesics in L, so they cannot escape compact sets in L. Hence they have to limit in a line leaf v of $\widetilde{\Lambda}^s_L$. Since the leaf space of $\widetilde{\Lambda}^s_L$ is Hausdorff, the limit is unique, which implies that $g(v) = v$. The leaf z of $\mathcal{O}^s$ corresponding to v is also invariant under g. This produces a point y of $\mathcal{O}$ in z which is invariant under g. Let β be the orbit of $\widetilde{\Phi}$ with $\Theta(\beta) = y$. But g also leaves invariant the point $w = \Theta(\alpha)$. This shows that there are 2 fixed points in $\mathcal{O}$ under g. Then $\pi(\alpha), \pi(\beta)$ are closed orbits of $\widetilde{\Phi}$ which up to powers are freely homotopic to the inverse of each other. Since Φ is regulating, this is impossible: notice that g is associated to the negative flow direction in α – as it acts

as an expansion in the set of orbits of $\widetilde{W}^u(\alpha)$. The regulating property applied to α implies that g acts freely and in an decreasing fashion on the leaf space $\mathcal{H}$ of $\widetilde{\mathcal{G}}$. The property that $\pi(\beta)$ is freely homotopic to the inverse of $\pi(\alpha)$ implies that g would have to act in an increasing way on $\mathcal{H}$, contradiction. Notice that the last argument is about the leaf space of $\widetilde{\mathcal{G}}$ and not of $\widetilde{\Lambda}^s$. This contradiction shows that l_1, l_2 cannot have the same ideal point in E. This finishes the analysis if v_1, v_2 are distinct.

If $v_1 = v_2$, then for i big enough we may assume that x_i is very closed to q_i. Then one can choose α periodic with $\widetilde{W}^u(\alpha)$ intersecting both l_1 and l_2. It follows that $\widetilde{W}^s(\alpha) \cap L$ has one endpoint a. Then one applies the same arguments as in the case v_1, v_2 distinct to produce a contradiction. This finishes the first part of the lemma.

We now prove that if l_1 is a ray in a leaf of $\widetilde{\Lambda}^s_L$ and l_2 is ray in a leaf of $\widetilde{\Lambda}^u_L$ then they cannot share an ideal point in $\partial_\infty L$. Suppose this is not the case. Apply the same limiting procedure as above to produce a stable line leaf s_1 in $\widetilde{\Lambda}^s_L$ and an unstable line leaf s_2 in $\widetilde{\Lambda}^u_L$ which share two ideal points. Clearly in this case they cannot be the same leaf and they bound a region R in L with ideal points a_1, a_2. Consider a non singular stable leaf l intersecting s_2. Then it enters R and cannot intersect the boundary of R (in L) again. Therefore it has to limit in either a_1 or a_2 and share an ideal point with a ray of s_1. This is disallowed by the first part of the proof. □

Given these facts the following happens: For any L in $\widetilde{\mathcal{G}}$ and leaf l in $\widetilde{\Lambda}^s_L$ if l is non singular let l^* be the geodesic in L with same ideal points as l. If l is a p-prong leaf, let $\delta_1, \ldots, \delta_p$ be the line leaves of l and let δ_i^* be the corresponding geodesics. In this case let l^* be the union of the δ_i^*, which is a p-sided ideal polygon in L. Let $\widetilde{\mathcal{L}}^s_L$ be the union of such l^* for l in $\widetilde{\Lambda}^s_L$ and similarly define $\widetilde{\mathcal{L}}^u_L$.

Lemma 5.1 implies that $\widetilde{\mathcal{L}}^s_L$, $\widetilde{\mathcal{L}}^u_L$ are closed subsets of L and so are geodesic laminations in L. Lemma 5.1 also implies that the complementary regions of $\widetilde{\mathcal{L}}^s_L$ are exactly those associated to p-prong leaves of $\widetilde{\Lambda}^s_L$, and so these complementary regions are finite sided ideal polygons. As leaves of $\widetilde{\Lambda}^s_L$ are uniform quasigeodesics (Lemma 4.1), then $\widetilde{\mathcal{L}}^s_L$ varies continuously if L varies in $\widetilde{\mathcal{G}}$. This produces a lamination in M which intersects leaves of $\mathcal{G}$ in geodesic laminations. As $\widetilde{\Lambda}^s_L$, $\widetilde{\Lambda}^u_L$ have no rays which share an ideal point, it follows that $\widetilde{\mathcal{L}}^s_L$ is transverse to $\widetilde{\mathcal{L}}^u_L$. It now follows that for any y in $\partial_\infty L$, then y has a neighborhood system in $L \cup \partial_\infty L$ defined by a sequence of leaves in either $\widetilde{\mathcal{L}}^s_L$ or $\widetilde{\mathcal{L}}^u_L$. Therefore the same holds for $\widetilde{\Lambda}^s_L$, $\widetilde{\Lambda}^u_L$ as these are uniform quasigeodesics.

We are now ready to analyse the properties of the action of $\pi_1(M)$ on $\mathcal{U}$.

Proposition 5.3. *Let $\mathcal{G}$ be an $\mathbb{R}$-covered foliation with a transverse regulating pseudo-Anosov flow Φ. Let g in $\pi_1(M)$ be a non trivial element. Then one of the following options must happen:*

I. *If g fixes 3 or more points in $\mathcal{U}$, then g does not act freely on $\mathcal{O}$ and has a unique fixed point x in $\mathcal{O}$. Here g is associated to a closed orbit of Φ. In addition g acts by an orientation preserving homeomorphism of $\mathcal{O}$ and g leaves invariant each prong of $\mathcal{O}^s(x)$, $\mathcal{O}^u(x)$ when acting on $\mathcal{O}$. Hence g fixes the ideal points of $\mathcal{O}^s(x)$, $\mathcal{O}^u(x)$ in $\mathcal{U}$ which are even in number. These are the only fixed points of g in $\mathcal{U}$ and they are alternatively repelling and attracting;*

II. *g fixes exactly two points in $\mathcal{U}$. Then, either* 1) *g acts freely on $\mathcal{O}$ and there is one attracting and one repelling fixed point in $\mathcal{U}$; or* 2) *g fixes a point x in $\mathcal{O}$ and leaves invariant exactly two prongs of (say) $\mathcal{O}^s(x)$ but not those of $\mathcal{O}^u(x)$ or any other possible prongs of $\mathcal{O}^s(x)$ (or vice versa). Here g reverses orientation in $\mathcal{O}$. The orbit associated to x may be non singular in which case all prongs of $\mathcal{O}^s(x)$ are left invariant and there are 4 fixed points in $\mathcal{U}$ under the square of g. The orbit associated to x may be singular. Then the square of g has more than 4 fixed points in $\mathcal{U}$.*

III. *g has no fixed point in $\mathcal{U}$. Then g fixes a single point x in $\mathcal{O}$ and a power of g fixes an even number ≥ 4 of points in $\mathcal{U}$.*

Consequently, g always fixes a finite even number of points in $\mathcal{U}$ (it may be zero).

Proof. Since g acts on $\mathcal{O}$ and leaves invariant the foliation $\mathcal{O}^s$, then it acts on the leaf space $\mathcal{H}^s$ of $\mathcal{O}^s$. This is the same as the leaf space of $\widetilde{\Lambda}^s_L$ (under the identification of $\mathcal{O}$ with L), and is also the same as the leaf space of $\widetilde{\Lambda}^s$. Recall that in our situation the leaf space of $\mathcal{O}^s$ is Hausdorff. Therefore the leaf space $\mathcal{H}^s$ of $\mathcal{O}^s$ (same as the leaf space of $\widetilde{\Lambda}^s$) is a topological tree [Fe3]. The same happens for the leaf space of $\mathcal{O}^u$.

Given any g in $\pi_1(M)$ it induces a homeomorphism of this topological tree $\mathcal{H}^s$. $\mathbb{Z}$ actions on such trees are well understood [Ba3], [Fe3], [Ro-St]. There are two options:

– g acts freely and has an axis v. Elements in the axis are those z in $\mathcal{H}^s$ for which $g(z)$ separates z from $g^2(z)$, or

– g fixes a point in $\mathcal{H}^s$.

Suppose first that g acts freely on $\mathcal{H}^s$. Then g has an axis v for its action on $\mathcal{H}^s$ and consequently an axis for its action on the leaf space of $\widetilde{\Lambda}^s_L$. Because $\mathcal{H}^s$ is Hausdorff it follows that the axis v is properly embedded in $\mathcal{H}^s$ [Fe3]. Let l be a leaf of $\widetilde{\Lambda}^s_L$ in the axis and we may assume that l is non singular again because $\mathcal{H}^s$ is Hausdorff [Fe3]. By the axis properties it follows that the leaves

$$\{g^n(l),\ \ n \in \mathbb{Z}\}$$

are nested in L and they are uniform quasigeodesics. Since they escape when viewed in the leaf space of $\widetilde{\Lambda}^s_L$, the same is true in L. As they are uniform quasigeodesics and nested, then there are unique points y, z in $\partial_\infty L$ so that $g^n(l)$ converges to y if n converges to infinity and to z if n converges to minus infinity. Hence under the

identification of $\mathcal{U}$ with $\partial_\infty L$, then y, z are the unique fixed points of (any power of) g in $\mathcal{U}$, where y is attracting and z repelling. In this case the action of g in $\mathcal{O}$ could be orientation preserving or not. This is case II, 1).

From now on in the proof we assume that g has a fixed point in $\mathcal{H}^s$, so there is a leaf C of $\mathcal{H}^s$ with $g(C) = C$. Then the leaf

$$\Theta(C) \text{ of } \mathcal{O}^s \text{ contains unique } x \text{ in } \mathcal{O} \text{ with } g(x) = x.$$

If g has no fixed points in $\mathcal{U}$ then it acts as an orientation preserving homeomorphism on $\mathcal{U}$ and hence the same happens for the action on $\mathcal{O}$.

There is a smallest positive integer i_0 so that $h = g^{i_0}$ leaves invariant all prongs of $\mathcal{O}^s(x), \mathcal{O}^u(x)$. If there are $2n$ such prongs, each generates an ideal point of L and also a point of $\mathcal{U}$. By Lemma 5.2 any two distinct prongs have different ideal points in $\mathcal{U}$. Hence h has at least $2n$ fixed points in $\mathcal{U}$. Let α be the flow line of $\tilde{\Phi}$ with $\Theta(\alpha) = x$. Without loss of generality assume that the prongs above are circularly ordered with corresponding ideal points

$$a_1, b_1, \ldots, a_n, b_n \quad \text{in } \mathcal{U}$$

where

$$\partial\mathcal{O}^s(x) = \{a_1, a_2, \ldots, a_n\}, \quad \text{and} \quad \partial\mathcal{O}^u(x) = \{b_1, b_2, \ldots, b_n\}.$$

Suppose that g is associated to the positive flow direction in α. Fix a prong τ of $\mathcal{O}^s(x)$ and let I be the maximal interval of $\mathcal{U} - \partial\mathcal{O}^u(x)$ containing the ideal point of τ. Let now μ be an arbitrary unstable leaf of $\mathcal{O}^u$ intersecting τ. Then as μ gets closer to prongs of $\mathcal{O}^u(x)$, the ideal points of μ approach the endpoints of I. The action of h on τ is as follows: h fixes x and for a leaf μ as above then h takes it to a leaf farther away from x. This is because in $\tilde{M}$ the flow lines along stable leaves move closer in forward time. It follows that h acts as an expansion in τ with a single fixed point in x. Given μ as above then $h^n(\mu) \cap \tau$ escapes in τ as n converges to infinity. These also form a nested collection of leaves. If the sequence $h^n(\mu)$ does not escape compact sets in $\mathcal{O}$, then it limits in a collection

$$\mathcal{W} = \{W_i,\ i \in J\}$$

of leaves of $\mathcal{O}^u$, where J is an interval in $\mathbb{Z}$ either finite or all of $\mathbb{Z}$ [Fe6]. In addition h leaves invariant $\mathcal{W}$. If $\mathcal{W}$ is not finite, then in particular it is not a single point and then the leaf space of $\mathcal{O}^u$ is not Hausdorff, contrary to our situation. If on the other hand $\mathcal{W}$ is a single leaf W, then $h(W) = W$ and there is a single periodic point z in W with $h(z) = z$. Then h fixes x and z and this is also impossible as seen previously.

It follows that $h^n(\mu)$ escapes compact sets in $\mathcal{O}$ and as seen in the free action case, they can only limit in a single point of $\mathcal{U}$, which corresponds to the ideal point t of τ. This shows that h acts as a contraction in I with fixed point t. Hence the

points $a_i, 1 \leq i \leq n$ are attracting fixed points of h in $\mathcal{U}$. Using h^{-1} one shows that the $b_i, 1 \leq i \leq n$ are repelling fixed points and these are the only fixed points of h in $\mathcal{U}$. Hence h fixes exactly $2n$ points in $\mathcal{U}$, where $n \geq 2$.

We now return to g. If g is orientation reversing on $\mathcal{U}$, then so is the action on $\mathcal{O}$. In this case there are exactly 2 fixed points of g in $\mathcal{U}$. The square of g is now orientation preserving on $\mathcal{U}$ and it has fixed points. In particular any fixed point of g^{2i} is a fixed point of g^2. It follows that h is equal to g^2 and this is case II, 2).

Suppose finally that g is orientation preserving on $\mathcal{U}$. Since $h = g^{i_0}$ has fixed points in $\mathcal{U}$, then either g has no fixed points in $\mathcal{U}$ or g has exactly the same fixed points in $\mathcal{U}$ as h does. In the second case h is equal to g and g has exactly $2n$ fixed points in $\mathcal{U}$, which are alternatively attracting and contracting. This is case I). In the first case g acts essentially as a rotation in $\mathcal{U}$ and $\mathcal{O}$. This is case III).

This finishes the proof of Proposition 5.3. □

Notice that cases I, II and III are mutually exclusive.

6. Construction of the conjugacy

We are now ready to prove the main theorem. Let then Φ, Ψ be two pseudo-Anosov flows transverse to the $\mathbb{R}$-covered foliation $\mathcal{G}$ and both regulating for $\mathcal{G}$. Fix a transverse orientation to $\mathcal{G}$ and we assume that both Φ, Ψ are positively transverse to $\mathcal{G}$. We may assume that because as we defined conjugation, the identity is a topological conjugacy between a flow and its inverse. We want to show that Φ and Ψ are topologically conjugate. Let $\mathcal{O}$ be the orbit space of $\widetilde{\Phi}$ and $\mathcal{T}$ be the orbit space of $\widetilde{\Psi}$. The first and main step is to construct a $\pi_1(M)$-equivariant homeomorphism from $\mathcal{O}$ to $\mathcal{T}$. Let

$$\Theta_1 : \widetilde{M} \to \mathcal{O} \quad \text{and} \quad \Theta_2 : \widetilde{M} \to \mathcal{T}$$

be the corresponding orbit space projection maps. Let $\mathcal{O}^s$, $\mathcal{O}^u$ be the projections of the stable and unstable foliations of $\widetilde{\Phi}$ to $\mathcal{O}$ and $\mathcal{T}^s$, $\mathcal{T}^u$ the corresponding objects for $\widetilde{\Psi}$. Recall that $\pi : \widetilde{M} \to M$ is the universal covering map.

The main property to note here is that the universal circle $\mathcal{U}$ depends only on $\mathcal{G}$ and not on Φ or Ψ. The same is true for the action of $\pi_1(M)$ on $\mathcal{U}$. This will allow us to go from Φ to $\mathcal{U}$ and then back to Ψ, using Proposition 5.3. Before we prove the theorem, we first construct an identification between closed orbits of Φ and Ψ.

Lemma 6.1. *Let α be an orbit of $\widetilde{\Phi}$ so that $\pi(\alpha)$ is a closed orbit of Φ. Let g be the element of $\pi_1(M)$ associated to the closed orbit $\pi(\alpha)$. Then there is a unique orbit β of $\widetilde{\Psi}$ so that $\pi(\beta)$ is a closed orbit of Ψ and associated to g, that is, $\pi(\beta)$ is freely homotopic to $\pi(\alpha)$.*

Proof. Let $x = \Theta_1(\alpha)$ and g non trivial in $\pi_1(M)$ with $g(x) = x$ and indivisible with respect to this property, hence g is associated with α. Suppose that g is associated to the forward flow direction of $\pi(\alpha)$. Let h be the smallest power of g so that h leaves invariant all prongs of $\mathcal{O}^s(x), \mathcal{O}^u(x)$. Proposition 5.3, Case I shows that h has $2n$ fixed points in $\mathcal{U}$, with $n \geq 2$. Now apply this proposition to h and Ψ. Since h has $2n$ fixed points in $\mathcal{U}$ and $n \geq 2$, Proposition 5.3 implies that there is a unique y in $\mathcal{T}$ with $h(y) = y$. Let

$$\beta \text{ be the orbit of } \tilde{\Psi} \text{ with } \Theta_2(\beta) = y, \text{ so } h(\beta) = \beta.$$

If g acts freely on $\mathcal{T}$ then the analysis of Proposition 5.3 shows that h can have only 2 fixed points in $\mathcal{U}$, impossible (this is case II.1 of Proposition 5.3). It follows that g cannot act freely on $\mathcal{T}$ and therefore the only fixed point of g in $\mathcal{T}$ is y – as it is fixed by a power of g. This implies that $g(\beta) = \beta$ and consequently $\pi(\alpha)$ is freely homotopic to a power of $\pi(\beta)$. Reversing the roles of α and β implies that $\pi(\alpha)$ and $\pi(\beta)$ are freely homotopic to each other or their inverses. The action of h on $\mathcal{U}$ shows that the first option is the one that happens – this is because they both have attracting fixed points in $\mathcal{U}$ in the same points. This finishes the proof of the lemma. □

This defines a map from the periodic points of $\mathcal{O}$ to the periodic points of $\mathcal{T}$. Notice that in the lemma above $\partial\mathcal{O}^s(x) = \partial\mathcal{T}^s(y)$ as points in $\mathcal{U}$ and similarly for $\mathcal{O}^u(x), \mathcal{T}^u(y)$. This is the key property which will characterize the map between orbit spaces as shown in the next result.

Theorem 6.2. *Let Φ, Ψ be pseudo-Anosov flows, which are transverse and regulating for an $\mathbb{R}$-covered foliation $\mathcal{G}$. Then Φ, Ψ are topologically conjugate.*

Proof. Given a transversal orientation to $\mathcal{G}$ we may suppose that both Φ, Ψ are positively transverse to $\mathcal{G}$. Fix a leaf L of $\tilde{\mathcal{G}}$. We first define a map η from $\mathcal{O}$ to $\mathcal{T}$ which extends the correspondence between periodic points obtained previously. The map η will assign to any point in the orbit space $\mathcal{O}$ a corresponding point in $\mathcal{T}$ so that corresponding stable and unstable leaves in $\mathcal{O}$ and $\mathcal{T}$ have the same ideal points in $\mathcal{U}$. More specifically, given x in $\mathcal{O}$, we will let y be the unique point of $\mathcal{T}$ with

$$\partial\mathcal{T}^s(y) = \partial\mathcal{O}^s(x), \quad \partial\mathcal{T}^u(y) = \partial\mathcal{O}^u(x). \tag{1}$$

If x is periodic the previous lemma shows that there is an unique such a y.

Now consider x not periodic and let x_n in $\mathcal{O}$ which are periodic and converging to x. We want to show that the y_n associated to x_n converge to a single point y. We may assume that no x_n is singular since the singular orbits form a discrete subset of $\mathcal{O}$. We can also assume that $(\mathcal{O}^s(x_n))$ forms a nested sequence, and so does $(\mathcal{O}^u(x_n))$. Let z_n, q_n points in $\mathcal{U}$ with

$$\partial\mathcal{O}^s(x_n) = \{z_n, q_n\} \text{ and let } \{z, q\} = \partial\mathcal{O}^s(x).$$

Then up to renaming we can assume that z_n converges to z in $\mathcal{U}$ and q_n converges to q in $\mathcal{U}$. Let y_n in $\mathcal{T}$ periodic with $\partial\mathcal{T}^s(y_n) = \{z_n, q_n\}$. Notice that the $l_n = \mathcal{T}^s(y_n)$ are leaves of $\mathcal{T}^s$, which are nested in $\mathcal{T}$ because their ideal points are nested in $\mathcal{U}$. By the identification of L with $\mathcal{O}$, then the l_n are associated to uniform quasigeodesics in L which have ideal points which converge to distinct points in $\partial_\infty L$ (associated to z, q in $\mathcal{U}$). Therefore these quasigeodesics converge to a single quasigeodesic in L and consequently

$$\mathcal{T}^s(y_n) \text{ converges to a leaf } l \text{ of } \mathcal{T}^s.$$

Similarly $\mathcal{T}^u(y_n)$ converges to a leaf s of $\mathcal{T}^u$. For all n, the pairs $\partial\mathcal{O}^s(x_n)$, $\partial\mathcal{O}^u(x_n)$ link each other in $\mathcal{U}$, so the same happens for $\partial\mathcal{T}^s(y_n)$, $\partial\mathcal{T}^u(y_n)$. It follows that the ideal points of l, s link each other in $\mathcal{U}$, for otherwise we would have a leaf of $\mathcal{T}^s$ sharing an ideal point with a leaf of $\mathcal{T}^u$ – which is disallowed by Lemma 5.2. Therefore

$$y_n = \mathcal{T}^s(y_n) \cap \mathcal{T}^u(y_n)$$

converges to a point y in $\mathcal{T}$. Clearly $\partial\mathcal{T}^s(y)$ contains $\partial\mathcal{O}^s(x)$ and similarly $\partial\mathcal{T}^u(y)$ contains $\partial\mathcal{O}^u(x)$. If y is a singular orbit, one could apply the inverse process to produce x' in $\mathcal{O}$, x' singular so that $\partial\mathcal{O}^s(x')$ contains $\partial\mathcal{T}^s(y)$. But then $\partial\mathcal{O}^s(x')$ contains $\partial\mathcal{O}^s(x)$ and x is non singular. This is disallowed by Lemma 5.1. Therefore y is non singular and hence equation (1) holds for y and x. In addition y is well defined, that is, given x in $\mathcal{O}$ there is a unique y in $\mathcal{T}$ satisfying equation (1): If y_1 and y_2 satisfy (1), then $\partial\mathcal{T}^s(y_1) = \partial\mathcal{T}^s(y_2)$ and $\partial\mathcal{T}^u(y_1) = \partial\mathcal{T}^u(y_2)$. By Lemma 5.2 the first fact implies that $\mathcal{T}^s(y_1) = \mathcal{T}^s(y_2)$ and the second fact implies that $\mathcal{T}^u(y_1) = \mathcal{T}^u(y_2)$. Therefore their intersection is $y_1 = y_2$.

This defines a map $\eta\colon \mathcal{O} \to \mathcal{T}$, given by $\eta(x) = y$, if x, y satisfy equation (1). The same argument as above that shows that η is well defined, also shows that η is injective – when one applies the argument to the domain rather than to the range. In addition, the map η clearly has an inverse by applying the same procedure from Ψ to Φ. Therefore η is a bijection.

We claim that η is continuous and by symmetry, then the inverse will also be continuous. Let then x in $\mathcal{O}$ and (x_n) a sequence in $\mathcal{O}$ converging to x. Assume first that x is non singular. Then

$$\mathcal{O}^s(x_n) \text{ converges to } \mathcal{O}^s(x) \text{ in } \mathcal{O} \text{ and } \partial\mathcal{O}^s(x_n) \text{ converges to } \partial\mathcal{O}^s(x) \text{ in } \mathcal{U}.$$

Hence $\partial\mathcal{T}^s(\eta(x_n)) = \partial\mathcal{O}^s(x_n)$ converges to $\partial\mathcal{T}^s(\eta(x)) = \partial\mathcal{O}^s(x)$ and similarly for $\partial\mathcal{T}^u(\eta(x_n))$. This shows that $\eta(x_n)$ converges to $\eta(x)$ in $\mathcal{T}$.

Suppose finally that x is singular. Up to subsequence we may assume that (x_n) are all in a sector of $\mathcal{O}^s(x)$ bounded by the line leaf l (contained in $\mathcal{O}^s(x)$). Then $\mathcal{O}^s(x_n)$ converges to l and $\partial\mathcal{O}^s(x_n)$ converges to ∂l in $\mathcal{U}$. It follows that

$$\partial\mathcal{T}^s(\eta(x_n)) \text{ converges to } \partial l - \text{a subset of } \mathcal{U}$$

which is contained in $\partial\mathcal{T}^s(\eta(x))$. The same happens if x_n are in $\mathcal{O}^s(x)$, that is, if $\partial\mathcal{T}^s(\eta(x_n))$ is contained in $\partial\mathcal{T}^s(\eta(x))$. This shows that $\partial\mathcal{T}^s(\eta(x_n))$ only accumulates in $\partial\mathcal{T}^s(\eta(x))$. The same is true for $\partial\mathcal{T}^u(\eta(x_n))$, which only accumulates in $\partial\mathcal{T}^u(\eta(x))$. Then

$$\eta(x_n) = \mathcal{T}^s(\eta(x_n)) \cap \mathcal{T}^u(\eta(x_n))$$

only accumulates in

$$\eta(x) = \mathcal{T}^s(\eta(x)) \cap \mathcal{T}^u(\eta(x))$$

This shows that $\eta(x_n)$ has to converge to $\eta(x)$. This shows that η is a homeomorphism from $\mathcal{O}$ to $\mathcal{T}$.

In addition η is $\pi_1(M)$ equivariant, and in fact it commutes with the action of $\pi_1(M)$ on $\mathcal{O}$. Again this is because of property (1) above. Here is a detailed explanation: If g is in $\pi_1(M)$ and x is in $\mathcal{O}$, then the stable and unstable leaves

$$g(\mathcal{O}^s(x)) = \mathcal{O}^s(g(x)), \quad g(\mathcal{O}^u(x)) = \mathcal{O}^u(g(x))$$

have ideal points in $\mathcal{U}$

$$\partial\mathcal{O}^s(g(x)) \quad \text{and} \quad \partial\mathcal{O}^u(g(x))$$

respectively. Hence these are also the ideal points of

$$\mathcal{T}^s(\eta(g(x))), \quad \mathcal{T}^u(\eta(g(x))).$$

In addition

$$\partial\mathcal{T}^s(\eta(x)) = \partial\mathcal{O}^s(x) \quad \text{and} \quad \partial(g(\mathcal{T}^s(\eta(x)))) = \partial\mathcal{T}^s(g(\eta(x))).$$

Hence they are the same as $\partial\mathcal{T}^s(\eta(g(x))$. Since this is also true for the unstable foliations, it follows that

$$\eta(g(x)) = g(\eta(x)), \quad \text{commutation with } \pi_1(M) \text{ action (2).}$$

In other words, equation (1) says that η is defined by having the same ideal points in the universal circle $\mathcal{U}$. Since the action of $\pi_1(M)$ on $\mathcal{U}$ is independent from the flow, then one expects the commuting relation above.

We now finish the proof of topological conjugacy between Φ and Ψ. We define a map $\tilde{h}\colon \tilde{M} \to \tilde{M}$ as follows. Given z in $\tilde{M}$, then z is in a leaf L of $\tilde{\mathcal{G}}$. Define

$$\tilde{h}(z) = \tilde{\Psi}_{\mathbb{R}}(\eta(\Theta_1(z))) \cap L,$$

here $\Theta_1(z)$ is in $\mathcal{O}$ and $\eta(\Theta_1(z))$ is in $\mathcal{T}$. Essentially we look at the orbit $\alpha = \tilde{\Phi}_{\mathbb{R}}(z)$ of the flow $\tilde{\Phi}$ through z and consider the corresponding orbit of $\tilde{\Psi}$ under the map η: that is the orbit $\tilde{\Psi}_{\mathbb{R}}(\eta(\Theta_1(z)))$ of $\tilde{\Psi}$. Then we intersect this orbit of $\tilde{\Psi}$ with L. This map $\tilde{h}$ preserves the leaves of $\tilde{\mathcal{G}}$ – not just the foliation $\tilde{\mathcal{G}}$, but the leaves themselves.

In addition $\tilde{h}$ sends orbits of $\tilde{\Phi}$ to orbits of $\tilde{\Psi}$. By the first part of the proof the map $\tilde{h}$ is clearly continuous and hence defines a homeomorphism of $\tilde{M}$. From the commuting property (2) of η the same follows for $\tilde{h}$, that is, for any g and z in $\tilde{M}$, then $\tilde{h}(g(z)) = g(\tilde{h}(z))$. Therefore h induces a homeomorphism of M, which sends orbits of Φ to orbits of Ψ. Hence Φ and Ψ are topologically conjugate. This finishes the proof of Theorem 6.2. □

We now improve this result and prove that the conjugating homeomorphism is actually isotopic to the identity. Here we also must consider the case with parabolic leaves.

Proposition 6.3. *Let Φ and Ψ be pseudo-Anosov flows transverse to an $\mathbb{R}$-covered foliation $\mathcal{G}$ and assume they are both regulating for $\mathcal{G}$ and both (say) positively transverse to $\mathcal{G}$. Then there is a topological conjugacy h between Φ and Ψ which is a homeomorphism isotopic to the identity.*

Proof. Suppose first that $\mathcal{G}$ has parabolic leaves. In the proof of Theorem 2.1 we showed the following facts: 1) $\mathcal{G}$ has a compact leaf C, 2) Any two pseudo-Anosov flows Φ and Ψ transverse to $\mathcal{G}$ are Anosov flows and their monodromies are maps of C which are homotopic. Homotopic homeomorphisms of surfaces are isotopic, and this implies that the conjugating homemorphism between Φ and Ψ is isotopic to the identity.

From now on assume that the leaves of $\mathcal{G}$ are hyperbolic. We start with the topological conjugacy h defined in Theorem 6.2 and we let $\tilde{h}$ be the lift to $\tilde{M}$ as in the proof of Theorem 6.2. For every z in $\tilde{M}$ then z and $\tilde{h}(z)$ are in the same leaf L which is isometric to the hyperbolic plane. Hence there is an unique geodesic γ_z in L from z to $\tilde{h}(z)$, parametrized with constant speed passing through z at time 0 and through $\tilde{h}(z)$ at time 1. Define $\tilde{h}_t(z)$ to the $\gamma_z(t)$. The map $\tilde{h}$ is continuous and geodesics in leaves of $\tilde{\mathcal{G}}$ vary continuously, because the hyperbolic metrics in leaves of $\mathcal{G}$ vary continuously [Can]. It now follows that $\tilde{h}_t$ is a homotopy in $\tilde{M}$, preserving leaves of $\tilde{\mathcal{G}}$. Clearly the homotopy $\tilde{h}_t, 0 \leq t \leq 1$ is $\pi_1(M)$ equivariant and so induces a homotopy $h_t, 0 \leq t \leq 1$ in M between the identity and h.

Since Φ is a regulating pseudo-Anosov flow for $\mathcal{G}$ and the leaves of $\mathcal{G}$ are hyperbolic, then the flow Φ has singularities [Fe2], [Cal2]. Blowing up the singularities produces an essential lamination [Ga-Oe], which is *genuine*, that is, the complementary regions are not all I-bundles. In addition since there is a regulating pseudo-Anosov flow for $\mathcal{G}$ and $\mathcal{G}$ does not have parabolic leaves, then M is atoroidal [Fe2], [Fe3]. Given these conditions Gabai and Kazez [GK3] proved that if a homemorphism h of M is homotopic to the identity, then h is in fact isotopic to the identity. This finishes the proof of Proposition 6.3. □

Remarks (The question of preserving the flow direction). 1) Notice again that by definition the conjugating homeomorphism h is not required to preserve flow direction

along flowlines. In particular the identity is a topological conjugacy between a flow Φ and its inverse Φ^{-1}.

2) If in addition one requires the conjugating homeomorphism to preserve flow direction along orbits, then there may be two transverse regulating flows for $\mathcal{G}$, that is, Φ and its inverse Φ^{-1}. For example if $\mathcal{G}$ is a fibration, the question of whether Φ is direction preserving conjugate to its inverse boils down to a question about the holonomy g of the fibration $\mathcal{G}$. In particular it depends on whether g is conjugate to its inverse in the mapping class group of the fiber. This question has been analysed by Mosher and others. In the case of torus fiber this question has a well known and fairly simple characterization [Mo3]: Given a matrix representative A of g, the conjugacy invariant has two parts: A cyclic word W in the letters R (for right) and L (for left) and the sign of the trace. First find an element in the conjugacy class in $\mathrm{SL}(2,\mathbb{Z})$ of the form $M \cdot (\pm I)$, where M is a positive matrix, that is, all entries of M are positive. This is possible if and only if the conjugacy class is Anosov. The sign of the trace is the $\pm$ sign in this expression. Then one factors the positive matrix M as a product of matrices

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The word W obtained in R's and L's is unique up to cyclic permutation [Mo3]. The cyclic word for the inverse conjugacy class is obtained from W by writing it backwards and replacing each R with an L and each L with an R. The sign of the trace is invariant under inverse. Using this characterization it is easy to see that both possibilities occur. For example given representative matrices for g below

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

Then the conjugacy class of A has cyclic word $W = RL$ and the inverse A^{-1} has the same cyclic word. In this case the corresponding suspension flow Φ is conjugate to Φ^{-1} by a conjugacy which preserves direction along orbits. As for B, its conjugacy class has cyclic word $W = RLR$ and the one for B^{-1} is $W' = LLR$. This shows that the associated monodromies g and g^{-1} are not in the same conjugacy class. In this case the resulting suspension Anosov flow is not direction preserving conjugate to its inverse. We remark that the higher genus case is much more unclear because the conjugacy invariants in the higher genus case are much more complicated [Mo3].

7. The non regulating case

In order to finish the analysis of the non regulating case we first need some information about the structure of $\mathbb{R}$-covered Anosov flows. Let $\mathcal{G}$ be an $\mathbb{R}$-covered foliation and let Φ a pseudo-Anosov flow transverse to $\mathcal{G}$ and non regulating. We also need to

understand the projection of leaves of $\widetilde{\mathcal{G}}$ to the orbit space $\mathcal{O}$ of $\widetilde{\Phi}$. Since the flow is not regulating, this projection is not the whole orbit space, in particular the boundary of this projection is relevant to us here. Recall that $\Theta\colon \widetilde{M} \to \mathcal{O}$ is the projection map to the orbit space $\mathcal{O}$. Details of the results here are in [Fe1], [Fe4]. As proved in [Fe4] the non regulating hypothesis implies that Φ is an $\mathbb{R}$-covered Anosov flow. Therefore there are 2 options for the flow Φ. In both cases we describe the structure of the foliations in the orbit space $\mathcal{O}$.

- *Skewed type.* The orbit space is homeomorphic to the strip U in the plane bounded by $x = 0$ and $x = 1$. Stable leaves are horizontal segments and unstable leaves are segments making oriented angle $\pi/4$ with the positive x axis. A stable leaf and an unstable leaf which have a common "ideal point" z in ∂U are said to form a perfect fit [Fe3], [Fe4]. They do not intersect, but just barely.

  In this case given a leaf L of $\widetilde{\mathcal{G}}$, then its projection $\Theta(L)$ to the orbit space is an open subset whose boundary is an union of *exactly two leaves*: one stable leaf E and another unstable leaf S and the leaves E, S form a perfect fit. The leaf E is denoted by $\tau_s(L)$ and if the foliation $\mathcal{G}$ is minimal then the map $\tau_s\colon \mathcal{H} \to \mathcal{H}^s$ is a homeomorphism [Fe4]. Similarly $L \to S$ defines $\tau_u\colon \mathcal{H} \to \mathcal{H}^u$, another homeomorphism.

- *Product type.* Here the orbit space is also homeomorphic to U as above. Stable leaves are horizontal segments and unstable leaves are vertical lines. Notice that any stable leaf intersects every unstable leaf and vice versa. This does not occur in the skewed case. In this case the flow is topologically conjugate to a suspension Anosov flow [Ba1].

  In this case given a leaf L of $\widetilde{\mathcal{G}}$, its projection $\Theta(L)$ is an open subset of the orbit space $\mathcal{O}$, whose boundary is a *single leaf* which is either stable or unstable.

Theorem 7.1. *Let $\mathcal{G}$ be an $\mathbb{R}$-covered foliation.*

1) *If there is a pseudo-Anosov flow Φ transverse to $\mathcal{G}$, but non regulating for $\mathcal{G}$ then Φ is an $\mathbb{R}$-covered Anosov flow and $\mathcal{G}$ is weakly conjugate to either the stable or the unstable foliation of Φ. In addition,*

2) *up to topological conjugacy there is at most one such flow Φ. If Φ has skewed type then there is only one such flow up to direction preserving conjugacy, whereas if Φ is product there may be two such flows, that is, Φ and its inverse.*

Proof. Part 1) of the theorem was proved in [Fe4] and most of part 2) as well. As explained in the introduction one can blow down $\mathcal{G}$ to a minimal foliation, still transverse to Φ. Also the leaves of $\mathcal{G}$ can be assumed to be hyperbolic.

Suppose first that Φ has skewed type. As described above we proved in [Fe4] that for each leaf L of $\widetilde{\mathcal{G}}$ there is a unique leaf $E = \tau_s(L)$ of $\widetilde{\Lambda}^s$ associated to it, producing a homeomorphism between the leaf spaces of $\widetilde{\mathcal{G}}$ and $\widetilde{\Lambda}^s$. This uses the

skewed hypothesis. The leaf E projects to the stable boundary of $\Theta(L)$. In addition the orbits of $\widetilde{\Phi}$ in E are in one to one correspondence with a pencil of geodesics of L with ideal point c_E in $\partial_\infty L$ and also each orbit of $\widetilde{\Phi}$ in E is a fixed bounded distance (in $\widetilde{M}$) from a single geodesic with ideal point c_E in L. One constructs a flow $\widetilde{\Phi}'$ in $\widetilde{M}$ where the flowlines are tangent to the pencil of geodesics in L with forward ideal point c_E. This induces a flow Φ' in M by geodesics in leaves of $\mathcal{G}$. The above describes a homeomorphism between the orbit spaces of $\widetilde{\Phi}$ and $\widetilde{\Phi}'$.

Unlike in the regulating case this does not easily produce a homeomorphism to $\widetilde{M}$ sending orbits of $\widetilde{\Phi}$ to orbits of $\widetilde{\Phi}'$. Still using the homeomorphism between the orbit spaces one can produce a homotopy equivalence of M which sends orbits of Φ to orbits of Φ' but may not be injective along orbits [Gh], [Ba2]. Using averaging techniques along orbits of Φ, Φ', this homotopy can be deformed into a homeomorphism h which sends orbits of Φ to those of Φ' [Gh], [Ba2]. The topological conjugacy h preserves the flow direction along orbits [Fe4].

Claim. *There is only one flow Φ' no matter what flow Φ we start with.*

Clearly the flow Φ' is completely determined by the ideal points c_E in leaves L of $\widetilde{\mathcal{G}}$. We claim that these points depend only on L and not on Φ or E. It was proved in [Fe4] that the flow Φ' in M which is by geodesics in leaves of $\mathcal{G}$, is an Anosov flow. For every point q in $\partial_\infty L$ which is not c_E then q is the endpoint of a geodesic l of L with q as the negative ideal point of the associated flow line of $\widetilde{\Phi}'$. Since l is contained in an unstable leaf of the flow $\widetilde{\Phi}'$ then this direction is a contracting direction for the foliation $\widetilde{\mathcal{G}}$. Therefore there is a single non contracting direction in L, which must be c_E. Hence c_E is uniquely determined by L and so is the flow Φ'.

This shows that any non regulating flow Φ is topologically conjugate to Φ' by a conjugacy preserving the flow direction. This finishes the proof of 2) in the skewed case. In particular this shows that Φ is direction preserving conjugate to its inverse Φ^{-1}, unlike the situation in the regulating case.

Now consider the case that Φ is product. The difference here is that the corresponding projection $\Theta(L)$ has boundary which is a single leaf and can be *either* a stable or unstable leaf. In [Fe4] the analysis was done assuming that the boundary of $\Theta(L)$ is a stable leaf. If now this boundary is an unstable leaf, then the analysis in [Fe4] would consider Φ^{-1} instead of $\Phi -$ which then produces stable boundary for $\Theta(L)$. Once this is done, the analysis proceeds as above. Therefore the same arguments as in the skewed case above show that either Φ or Φ^{-1} is direction preserving conjugate to Φ' and this proves 2) in the product case. However as explained in the previous section there are examples where Φ is not direction preserving conjugate to Φ^{-1} in this case. In particular there are such examples when Φ is a suspension Anosov flow. This finishes the proof of Theorem 7.1. □

Finally we again consider the question as to whether the conjugacy between the flows is isotopic to the identity, now in the non regulating situation.

Proposition 7.2. *Suppose that Φ, Ψ are non regulating pseudo-Anosov flows transverse to an $\mathbb{R}$-covered foliation $\mathcal{G}$, which are direction preserving conjugated by a homeomorphism h'. Then h' is isotopic to the identity.*

Proof. Let Φ' be the flow by geodesics in leaves of $\mathcal{G}$ constructed in [Fe4] and described above. Let h be the conjugacy between Φ and Φ' described in the previous theorem. First we need some additional information about $\tilde{h}$: by its definition, the homeomorphism τ_s between the leaf spaces $\mathcal{H}$ of $\widetilde{\mathcal{G}}$ and $\mathcal{H}^s$ of $\widetilde{\Lambda}^s$ is group equivariant: $g(\tau_s(L)) = \tau_s(g(L))$ for any L in $\widetilde{\mathcal{G}}$ and any g in $\pi_1(M)$. Notice that here we are assuming that $\Theta(L)$ has stable boundary − but exactly the same proof works when $\Theta(L)$ has only unstable boundary. Given L in $\widetilde{\mathcal{G}}$ the identification between orbits of $\widetilde{\Phi}$ in $\tau_s(L)$ and a fixed pencil of geodesics in L is by bounded distance, so this is also group equivariant. Therefore the homeomorphism between the orbit spaces of $\widetilde{\Phi}$ and $\widetilde{\Phi}'$ commutes with the action of $\pi_1(M)$. By doing the averaging steps carefully it follows that the lift of the conjugacy $\tilde{h}$ also commutes with the action of $\pi_1(M)$ [Gh], [Ba2], [Fe4].

Under appropriate identifications of $\pi_1(M)$ with $\pi_1(M, y)$ and $\pi_1(M, h(y))$ for a given y in M, this implies that h induces the identity in the fundamental group level. Because M has a pseudo-Anosov flow it follows that M is a $K(\pi, 1)$ and this implies that h is homotopic to the identity.

If M is toroidal then Waldhausen's theorem shows that h is isotopic to the identity [He]. If M is atoroidal, then since $\mathcal{G}$ has hyperbolic leaves it was proved in [Fe2], [Cal2] that $\mathcal{G}$ has a transverse pseudo-Anosov flow Δ which is regulating for $\mathcal{G}$. Notice that Δ is completely different from the non regulating transverse flows Φ, Ψ. In particular this pseudo-Anosov flow Δ has singularities [Fe2], [Cal2]. Then exactly as proved in Proposition 7.2, the result of Gabai and Kazez [GK3] implies that h is isotopic to the identity. In the same way there is another homeomorphism h_*, isotopic to the identity, so that h_* conjugates Φ' to Ψ preserving flow direction. This finishes the proof of the proposition. □

Remark. In some cases it is very easy to see that the conjugating homeomorphism between Φ and Φ^{-1} is isotopic to the identity. For example let Φ be the geodesic flow in the unit tangent bundle M of a closed, orientable hyperbolic surface S. We first construct the isotopy: let h_t be the homeomorphism in M that corresponds to turning each vector in S by an angle $t\pi$. This uses the fact that S is orientable. Then h_t is an isotopy in M. Projecting to S, the homeomorphism h_1 sends geodesics to the same geodesics, but with the opposite direction. However they are going backwards, and hence they are going forwards for the inverse flow Φ^{-1}. In this case h_1 is a conjugacy between Φ and Φ^{-1} which preserves direction along flow lines and h_1 is isotopic to the identity.

References

[Ba1] T. Barbot, Caractérization des flots d'Anosov pour les feuilletages. *Ergodic Theory Dynam. Systems* **15** (1995), 247–270. Zbl 0826.58025 MR 1332403

[Ba2] T. Barbot, Flots d'Anosov sur les variétés graphées au sens de Waldhausen. *Ann. Inst. Fourier Grenoble* **46** (1996), 1451–1517. Zbl 0861.58028 MR 1427133

[Ba3] T. Barbot, Actions de groupes sur les 1-variétés non séparées et feuilletages de codimension un. *Ann. Fac. Sci. Toulouse* **7** (1998), 559–597. Zbl 0932.57027 MR 1693597

[Br] M. Brittenham, Essential laminations in Seifert fibered spaces. *Topology* **32** (1993), 61–85. Zbl 0791.57013 MR 1204407

[Cal1] D. Calegari, **R** covered foliations of hyperbolic 3-manifolds. *Geom. Topol.* **3** (1999), 137–153. Zbl 0924.57014 MR 1695533

[Cal2] D. Calegari, The geometry of **R**-covered foliations. *Geom. Topol.* **4** (2000), 457–515. Zbl 0964.57014 MR 1800151

[Cal3] D. Calegari, Foliations with one-sided branching. *Geom. Ded.* **96** (2003), 1–53. Zbl 1064.57015 MR 1956833

[Ca-Du] D. Calegari and N. Dunfield, Laminations and groups of homeomorphisms of the circle. *Invent. Math.* **152** (2003), 149–204. Zbl 1025.57018 MR 1965363

[Can] A. Candel, Uniformization of surface laminations. *Ann. Sci. Ecole Norm. Sup.* **26** (1993), 489–516. Zbl 0785.57009 MR 1235439

[FLP] A. Fathi, F. Laudenbach, and V. Poénaru (eds.), *Travaux de Thurston sur les surfaces*. Astérisque 66–67, Soc. Math. France, Paris 1979. MR 0568308Zbl 0406.00016

[Fe1] S. Fenley, Anosov flows in 3-manifolds. *Ann. of Math.* **139** (1994), 79–115. Zbl 0796.58039 MR 1259365

[Fe2] S. Fenley, Foliations, topology and geometry of 3-manifolds: $\mathbb{R}$-covered foliations and transverse pseudo-Anosov flows. *Comment. Math. Helv.* **77** (2002), 415–490. Zbl 1010.57008 MR 1933786

[Fe3] S. Fenley, Pseudo-Anosov flows and incompressible tori. *Geom. Dedicata* **99** (2003), 61–102. Zbl 1047.37017 MR 1998929

[Fe4] S. Fenley, Regulating flows, topology of foliations and rigidity. *Trans. Amer. Math. Soc.* **357** (2005), 4957–5000. Zbl 1076.37017 MR 2165394

[Fe5] S. Fenley, Laminar free hyperbolic 3-manifolds. *Comment. Math. Helv.* **82** (2007), 247–321. Zbl 1136.57015 MR 2319930

[Fe6] S. Fenley, Geometry of foliations and flows I: Almost transverse pseudo-Anosov flows and asymptotic behavior of foliations. *J. Differential Geom.* **81** (2009), no. 1, 1-–89. Zbl 1160.57026 MR 2477891

[Fe7] S. Fenley, Ideal boundaries of pseudo-Anosov flows and uniform convergence groups, with connections and applications to large scale geometry. *Geom. Topol.* **16** (2012), no. 1, 1–110. Zbl 06035978 MR 2872578

[Fe-Mo] S. Fenley and L. Mosher, Quasigeodesic flows in hyperbolic 3-manifolds. *Topology* **40** (2001), 503–537. Zbl 0990.53040 MR 1838993

[Fr] D. Fried, Transitive Anosov flows and pseudo-Anosov maps. *Topology* **22** (1983), 299–303. Zbl 0516.58035 MR 0710103

[GK1] D. Gabai and W. Kazez, Pseudo-Anosov maps and surgery on fibred 2-bridge knots. *Topology. Appl.* **37** (1990), 93–100. Zbl 0714.57004 MR 1075377

[GK2] D. Gabai and W. Kazez, Group negative curvature for 3-manifolds with genuine laminations. *Geom. Topol.* **2** (1998), 65–77. Zbl 0905.57011 MR 1619168

[GK3] D. Gabai and W. Kazez, Homotopy, isotopy and genuine laminations of 3-manifolds. In *Geometric topology* (Athens, GA 1993), AMS/IP Stud. Adv. Math. 2.1, Amer. Math. Soc., Providence, RI, 1997, 123–138. Zbl 0893.57012 MR 1470725

[Ga-Oe] D. Gabai and U. Oertel, Essential laminations and 3-manifolds. *Ann. of Math.* **130** (1989), 41–73. Zbl 0685.57007 MR 1005607

[Gh] E. Ghys, Flots d'Anosov sur les variétés fibres en cercles. *Ergodic Theory Dynam. Systems* **4** (1984), 67–80. Zbl 0527.58030 MR 0758894

[Go-Sh] S. Goodman and S. Shields, A condition for the stability of $\mathbb{R}$-covered foliations. *Trans. Amer. Math. Soc.* **352** (2000), 4051–4065. Zbl 0965.57026 MR 1624178

[He] J. Hempel, *3-manifolds*. Ann. of Math. Studies 86, Princeton University Press, Princeton, NJ, 1976. Zbl 0345.57001 MR 0415619

[Mo1] L. Mosher, Dynamical systems and the homology norm of a 3-manifold II. *Invent. Math.* **107** (1992), 243–281. Zbl 0758.58027 MR 1144424

[Mo2] L. Mosher, *Laminations and flows transverse to finite depth foliations*. Part I: Branched surfaces and dynamics, available from http://newark.rutgers.edu:80/~mosher/; Part II in preparation.

[Mo3] L. Mosher, Conjugacy classes of pseudo-Anosov homeomorphisms, private communication.

[No] S. P. Novikov, Topology of foliations. *Trans. Moscow Math. Soc.* **14** (1965), 268–305. Zbl 0247.57006 MR 200938

[Pl] J. Plante, Foliations with measure preserving holonomy. *Ann. of Math.* **107** (1975), 327–361. Zbl 0314.57018 MR 0391125

[Pl-Th] J. Plante and W. Thurston, Anosov flows and the fundamental group. *Topology* **11** (1972), 147–150. Zbl 0246.58014 MR 0295389

[Ro-St] R. Roberts and M. Stein, Group actions on order trees. *Topol. Appl.* **115** (2001), 175–201. Zbl 0985.57007 MR 1847462

[Ros] Rosenberg, Foliations by planes. *Topology* **7** (1968), 131–138. Zbl 0157.30504 MR 0228011

[Su] D. Sullivan, Cycles for the dynamical study of foliated manifolds and complex manifolds. *Invent. Math.* **36** (1976), 225–255. Zbl 0335.57015 MR 0433464

[Th1] W. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc.* **19**, (1988) 417–431. Zbl 0674.57008 MR 0956596

[Th2] W. Thurston, 3-manifolds, foliations and circles I. Preprint 1997; arXiv:math/9712268 [math.GT].

[Th3] W. Thurston, 3-manifolds, foliations and circles II: Transverse asymptotic geometry of foliations. Preprint.

[Th4] W. Thurston, *The universal circle for foliations and transverse laminations.* Lectures in M.S.R.I., 1997.

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