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# Gehring-Hayman Theorem for conformal deformations

Pekka Koskela and Päivi Lammi\*

**Abstract.** We study conformal deformations of a uniform space that satisfies the Ahlfors Q-regularity condition on balls of Whitney type. We verify the Gehring-Hayman Theorem by using a Whitney covering of the space.

Mathematics Subject Classification (2010). 30C65.

Keywords. Conformal deformations, uniform space, Whitney covering.

## 1. Introduction

Given  $x, y \in B^2(0, 1)$ , the hyperbolic geodesic [x, y] is essentially the shortest curve joining x to y in  $B^2(0, 1)$ . More precisely

$$\ell([x,y]) \le \frac{\pi}{2}\ell(\gamma)$$

whenever  $\gamma$  is a path that joins x to y in  $B^2(0,1)$ . This simple fact is an instance of a theorem of Gehring and Hayman in [GH]: If  $f: B^2(0,1) \to \Omega \subset \mathbb{C}$  is a conformal mapping and  $\gamma$  is a path joining points x and y, then

$$\int_{[x,y]} |f'(z)| \, ds \le C \int_{\gamma} |f'(z)| \, ds, \tag{1.1}$$

where  $C \ge 1$  is an absolute constant. The density  $\rho(z) = |f'(z)|$  satisfies a Harnack inequality

$$\frac{\rho(z)}{A} \le \rho(w) \le A\rho(z)$$

whenever  $z \in B^2(0,1)$  and  $w \in B(z,(1-|z|)/2)$ . It also satisfies the area growth estimate

$$\int_{B\rho(z,r)} \rho^2 \, dA \le \pi r^2,$$

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where  $B_{\rho}(z, r)$  refers to the ball with centre z and radius r in the path metric

$$d_{\rho}(x, y) = \inf \int_{\gamma} \rho \, ds,$$

where the infimum is taken over all curves  $\gamma$  joining points x and y.

In [BKR] the Gehring-Hayman inequality (1.1) was extended to  $B^n(0,1)$ ,  $n \ge 2$ , for conformal deformations of the Euclidean metric. By a conformal deformation (a conformal density)  $\rho$  we mean a continuous function  $\rho \colon B^n(0,1) \to (0,\infty)$  that satisfies a Harnack inequality with a constant A > 1,

$$\frac{\rho(z)}{A} \le \rho(w) \le A\rho(z) \quad \text{for all } w \in B(z, (1-|z|)/2) \text{ and all } z \in B^n(0,1),$$

and a volume growth condition with a constant B > 0,

$$\int_{B_{\rho}(z,r)} \rho^n dm_n \le Br^n \quad \text{for all } z \in B^n(0,1) \text{ and all } r > 0,$$

with respect to n-dimensional Lebesgue measure  $m_n$ .

Subsequently, Herron showed in [H1] that  $B^n(0,1)$  can be replaced by any uniform space  $(\Omega,d)$  of bounded geometry. In this setting conformal densities are defined by conditions analogous to those given above – see Section 2 for details. Here uniformity is a substitute for the "roundness" of  $B^n(0,1)$ . The assumption of bounded geometry includes two conditions. First, it requires that  $\Omega$  carries a Borel regular measure  $\mu$  that satisfies the (Ahlfors) Q-regularity condition on balls of Whitney type for some Q > 1. That is, there is a constant  $C_1 \ge 1$  such that if  $r \le d(z, \partial\Omega)/2$ , then

$$C_1^{-1}r^Q \le \mu(B(z,r)) \le C_1r^Q$$
.

Secondly, it requires that balls  $B(z, d(z, \partial\Omega)/2)$  allow for nice lower bounds for the Q-modulus (see e.g. [HK], [BHK]). In fact, the Q-regularity condition on balls of Whitney type is not explicitly stated in [H1] but it follows from the other assumptions. The precise definition of a uniform space is given in Section 2 below. This concept, introduced in [BHK], generalizes the notion of a uniform domain introduced by Jones [Jo] and Martio and Sarvas [MaSa], see also [GO]. The volume growth condition for  $\rho$  then refers to integrals of  $\rho^Q$  with respect to the measure  $\mu$ . For predecessors of the results in [H1], see [HN], [HR]. For connections to Gromov hyperbolicity, see [Gr], [BHK] and [BB].

In this paper we show that, surprisingly, lower bounds on the Q-modulus are not needed to prove the Gehring–Hayman inequality.

**Theorem 1.1** (Gehring–Hayman Theorem). Let Q > 1 and let  $(\Omega, d, \mu)$  be a non-complete uniform space equipped with a measure that is Q-regular on balls

of Whitney type. If  $\rho: \Omega \to (0, \infty)$  is a conformal density on  $\Omega$ , then there is a constant  $C \geq 1$  that depends only on the data associated with  $\Omega$  and  $\rho$  such that

$$\ell_{\rho}([x,y]) \leq C\ell_{\rho}(\gamma),$$

whenever [x, y] is a quasihyperbolic geodesic and y is a curve joining x to y in  $\Omega$ .

The definition of a quasihyperbolic geodesic is given in Section 2 and the proof of the theorem is in Section 4. Especially Subcase D of the proof is the novelty, that allows us to avoid the use of lower bounds for the Q-modulus. The previous arguments [BKR], [H1], [HN] and [HR] rely on modulus estimates.

The Gehring-Hayman Theorem was a central tool in [BHR], [BKR], [H1] and [H2]. We expect that Theorem 1.1 will allow one to remove the use of modulus bounds in [BHR], [BKR], [H1] and [H2] and thus extend large parts of those papers to a much more general setting. A very simple example of a space that satisfies the assumptions of Theorem 1.1 but does not support lower bounds for the *Q*-modulus is

$$\Omega = \{(x, y) \in \mathbb{R}^2 : |y| \le |x|, \ -1 < x < 1\}$$

equipped with the path metric and Lebesgue measure.

#### 2. Preliminaries

Let  $(\Omega, d)$  be a metric space. A *curve* means a continuous map  $\gamma : [a, b] \to \Omega$  from an interval  $[a, b] \subset \mathbb{R}$  to  $\Omega$ . We also denote the image set  $\gamma([a, b])$  of  $\gamma$  by  $\gamma$ . The *length*  $\ell_d(\gamma)$  of  $\gamma$  with respect to the metric d is defined as

$$\ell_d(\gamma) = \sup_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all partitions  $a = t_0 < t_1 < \cdots < t_m = b$  of the interval [a, b]. If  $\ell_d(\gamma) < \infty$ , then  $\gamma$  is said to be a *rectifiable curve*. When the parameter interval is open or half-open, we set

$$\ell_d(\gamma) = \sup \ell_d(\gamma|_{[c,d]}),$$

where the supremum is taken over all compact subintervals [c, d]. For a rectifiable curve  $\gamma$  we define the *arc length*  $s: [a, b] \to [0, \infty)$  along  $\gamma$  by

$$s(t) = \ell_d(\gamma|_{[a,t]}).$$

Next, let us assume that  $\rho \colon \Omega \to [0, \infty]$  is a Borel function. For each rectifiable curve  $\gamma \colon [a, b] \to \Omega$  we define the  $\rho$ -length  $\ell_{\rho}(\gamma)$  of  $\gamma$  by

$$\ell_{\rho}(\gamma) = \int_{\gamma} \rho \, ds = \int_{a}^{b} \rho(\gamma(t)) \, ds(t).$$

If  $\Omega$  is rectifiably connected – that is, every pair of points in  $\Omega$  can be joined by a rectifiable curve – then  $\rho$  determines a distance function

$$d_{\rho}(x, y) = \inf \ell_{\rho}(y),$$

where the infimum is taken over all rectifiable curves  $\gamma$  joining  $x, y \in \Omega$ . In general, the distance function  $d_{\rho}$  need not be a metric. However, it is a metric – called a  $\rho$ -metric – if  $\rho$  is positive and continuous. If  $\rho \equiv 1$ , then  $\ell_{\rho}(\gamma) = \ell_{d}(\gamma)$  is the length of the curve  $\gamma$  with respect to the metric d. Furthermore, if  $\ell_{d}(\gamma) = d(x, y)$  for some curve  $\gamma$  joining points  $x, y \in \Omega$ , then  $\gamma$  is said to be a geodesic. If every pair of points in  $\Omega$  can be joined by a geodesic, then  $(\Omega, d)$  is called a geodesic space.

Let  $(\Omega,d)$  be a locally compact, rectifiably connected and non-complete metric space and denote by  $\bar{\Omega}$  its metric completion. Then the *boundary*  $\partial\Omega:=\bar{\Omega}\setminus\Omega$  is nonempty. We write

$$d(z) = \operatorname{dist}_d(z, \partial \Omega) = \inf\{d(z, x) : x \in \partial \Omega\}$$

for  $z \in \Omega$ . If we choose

$$\rho(z) = \frac{1}{d(z)},$$

we obtain the *quasihyperbolic metric* k in  $\Omega$ . In this special case we denote the metric  $d_{\rho}$  by k and the quasihyperbolic length of the curve  $\gamma$  by  $\ell_k(\gamma)$ . That  $\ell_k(\gamma) = \ell_{\rho}(\gamma)$  is shown in [BHK], Appendix. Moreover, [x, y] refers to a quasihyperbolic geodesic joining points x and y in  $\Omega$ .

Given a real number  $D \ge 1$ , a curve  $\gamma : [a,b] \to (\Omega,d)$  is called a *D-uniform* curve if it is quasiconvex:

$$\ell_d(\gamma) \le Dd(\gamma(a), \gamma(b)),$$
 (2.1)

and

$$\min\{\ell_d(\gamma|_{[a,t]}), \ell_d(\gamma|_{[t,b]})\} \le Dd(\gamma(t)) \tag{2.2}$$

for every  $t \in [a, b]$ . A metric space  $(\Omega, d)$  is called a *D-uniform space* if every pair of points in it can be joined by a *D*-uniform curve.

If  $(\Omega, d)$  is a uniform space, then by Proposition 2.8 and Theorem 2.10 of [BHK] the quasihyperbolic space  $(\Omega, k)$  is complete, proper (closed balls are compact), and geodesic. Furthermore, each quasihyperbolic geodesic [x, y] is a D'-uniform curve for every  $x, y \in \Omega$ , where  $D' = D'(D) \ge 1$ . Quasihyperbolic geodesics are also locally D'-uniform curves – that is, every subcurve  $[u, v] \subset [x, y]$  is a D'-uniform curve – because [u, v] is a quasihyperbolic geodesic as well. We also have an estimate for a quasihyperbolic distance of every pair of points x and y in the D-uniform space  $(\Omega, d)$  (see [BHK], Lemma 2.13):

$$k(x, y) \le 4D^2 \log \left(1 + \frac{d(x, y)}{\min\{d(x), d(y)\}}\right).$$
 (2.3)

Let us consider a continuous function  $\rho: \Omega \to (0, \infty)$ , called a *density*. The metric  $d_{\rho}$  is then well defined. We use the subscript  $\rho$  for metric notations which refer to  $d_{\rho}$ , and similarly for k and d. For example,  $B_{\rho}(a,r)$ ,  $B_{k}(a,r)$  and  $B_{d}(a,r)$  are open balls with centre a and radius r in metrics  $d_{\rho}$ , k and d. Furthermore, we abbreviate the "Whitney ball"  $B_{d}(z,\frac{1}{2}d(z))$  to  $B_{z}$ .

Let  $\mu$  be a Borel regular measure on  $(\Omega, d)$  with dense support. We call  $\rho$  a conformal density provided it satisfies both a Harnack type inequality, HI(A), for some constant A > 1:

$$\frac{1}{A} \le \frac{\rho(x)}{\rho(y)} \le A$$
 for all  $x, y \in B_z$  and all  $z \in \Omega$ ,  $HI(A)$ 

and a volume growth condition, VG(B), for some constant B > 0:

$$\mu_{\rho}(B_{\rho}(z,r)) \leq Br^{Q}$$
 for all  $z \in \Omega$  and  $r > 0$ .  $VG(B)$ 

Here  $\mu_{\rho}$  is the Borel measure on  $\Omega$  defined by

$$\mu_{\rho}(E) = \int_{E} \rho^{Q} d\mu$$
 for a Borel set  $E \subset \Omega$ ,

and Q is a positive real number. Generally Q will be the Hausdorff dimension of our space  $(\Omega, d)$ .

We defined in the introduction the concept of Q-regularity on balls of Whitney type. The immediate consequence is that the measure  $\mu$  is also doubling on balls of Whitney type: there exists a constant  $C_2 \ge 1$  such that

$$\mu(B_d(z,2r)) \le C_2 \mu(B_d(z,r))$$
 (2.4)

for every  $z \in \Omega$  and every  $0 < r \le \frac{1}{4}d(z)$ .

## 3. Whitney covering

In this section we assume that  $(\Omega, d, \mu)$  is a locally compact, rectifiably connected, and non-complete metric measure space such that the measure  $\mu$  is doubling on balls of Whitney type. Let r(z) = d(z)/50. From the family of balls  $\{B_d(z, r(z))\}_{z \in \Omega}$  we select a maximal (countable) subfamily  $\{B_d(z_i, r(z_i)/5)\}_{i \in I}$  of pairwise disjoint balls. Let  $\mathcal{B} = \{B_i\}_{i \in I}$ , where  $B_i = B_d(z_i, r_i)$  and  $r_i = r(z_i)$ . We call the family  $\mathcal{B}$  the Whitney covering of  $\Omega$ . Let us list a few facts concerning the Whitney covering. The last property is a consequence of the doubling on balls of Whitney type property of the measure  $\mu$ . For more properties of the Whitney covering, see e.g. Theorem III.1.3 of [CW], Lemma 2.9 of [MaSe], Lemma 7 of [HKT], and [BS], Theorem 5.3 and Lemma 5.5.

**Lemma 3.1.** There is  $N \in \mathbb{N}$  such that

- (i) the balls  $B_d(z_i, r_i/5)$  are pairwise disjoint,
- (ii)  $\Omega = \bigcup_{i \in I} B_d(z_i, r_i),$
- (iii)  $B_d(z_i, 5r_i) \subset \Omega$ ,
- (iv)  $\sum_{i=1}^{\infty} \chi_{B_d(z_i,5r_i)}(x) \leq N \text{ for all } x \in \Omega.$

The family  $\mathcal{B}$  has the same kind of properties as the usual Whitney decomposition  $\mathcal{W}$  of a domain  $\Omega \subset \mathbb{R}^n$  and next we prove a couple of them. In addition to the assumptions above, we assume that for each pair of points in  $B \in \mathcal{B}$  for every  $B \in \mathcal{B}$  can be joined by a D-uniform curve in  $\Omega$ .

**Lemma 3.2.** Let  $x, y \in (\Omega, d, \mu)$  and  $d(x, y) \ge d(x)/2$ . There is a constant  $C = C(C_2, D) > 0$  such that

$$C^{-1}N(x, y) \le k(x, y) \le CN(x, y),$$

where N(x, y) is the number of balls  $B \in \mathbb{B}$  intersecting a quasihyperbolic geodesic [x, y].

*Proof.* Let  $x, y \in \Omega$  be points so that  $d(x, y) \ge d(x)/2$ . Since 24 diam<sub>d</sub>(B)  $\le d(z)$  for every  $B \in \mathcal{B}$  and for every  $z \in B$ , then the basic estimate (2.3) implies

$$\operatorname{diam}_{k}(B) \leq 4D^{2} \log \left( 1 + \frac{\operatorname{diam}_{d}(B)}{24 \operatorname{diam}_{d}(B)} \right) = 4D^{2} \log \frac{25}{24}.$$

Thus

$$N(x, y) \ge \frac{k(x, y)}{4D^2 \log \frac{25}{24}}$$

Lemma 3.1 (iv) says that there are only N balls  $B \in \mathbb{B}$  that contain x. Fix one of them and denote it by  $B_1$ . A *neighbour* of the ball  $B_1$  is a ball  $B \in \mathbb{B}$  which intersects the ball  $5B_1 = B_d(z_1, 5r_1) = B_d(z_1, d(z_1)/10)$ . Because the measure  $\mu$  is doubling in every ball  $B_d(z, r)$  with radius  $0 < r \le d(z)/4$ , the ball  $B_1$  has a uniformly bounded number of neighbours. Let this number be  $N' \in \mathbb{N}$  and let  $y_1 \in [x, y]$  be the first point such that  $y_1$  does not belong to any neighbour of  $B_1$ . This choice is possible because  $d(x, y) \ge d(x)/2$ . The geodesic  $[x, y_1]$  intersects at most N' balls  $B \in \mathbb{B}$  and

$$k(x, y_1) = \int_{[x, y_1]} \frac{1}{d(z)} ds \ge \int_{5B_1 \cap [x, y_1]} \frac{10}{11d(z_1)} ds$$
  
 
$$\ge \frac{10}{11d(z_1)} \left(\frac{d(z_1)}{10} - \frac{d(z_1)}{50}\right) = \frac{4}{55}.$$
(3.1)

Let  $B_2 \in \mathcal{B}$  be a ball such that  $y_1 \in B_2$  and  $B_2 \cap B \neq \emptyset$  for some neighbour  $B \in \mathcal{B}$  of  $B_1$ . Again there are only N' balls  $B \in \mathcal{B}$  which are neighbours of  $B_2$ . Let  $y_2 \in [x, y]$  be the first point so that  $y_2$  does not belong to any neighbour of  $B_2$ . Then the geodesic  $[y_1, y_2]$  intersects at most N' balls  $B \in \mathcal{B}$  and  $k(y_1, y_2) \geq \frac{4}{55}$ , by the same way than in inequality (3.1). We continue this process until we end up with a ball  $B_m$  whose neighbours contain  $[y_{m-1}, y]$ . This process really ends and  $m < \infty$ , because [x, y] is compact. We may start doing this process from every ball B that contains x. Thus we obtain the upper bound to the number of balls that intersects the quasihyperbolic geodesic [x, y]:

$$N(x,y) \le \frac{55}{4} N N' k(x,y). \qquad \Box$$

Fix a ball  $B_0$  from the Whitney covering  $\mathcal{B}$  and let  $z_0$  be its centre point. For each  $B_i \in \mathcal{B}$  we fix a geodesic  $[z_0, z_i]$ . Furthermore, for each  $B_i \in \mathcal{B}$  we set  $P(B_i) = \{B \in \mathcal{B} : B \cap [z_0, z_i] \neq \emptyset\}$  and define the *shadow* S(B) of a ball  $B \in \mathcal{B}$  by

$$S(B) = \bigcup_{\substack{B_i \in \mathcal{B} \\ B \in P(B_i)}} B_i.$$

For  $n \in \mathbb{N}$  we set

$$\mathcal{B}_n = \{ B_i \in \mathcal{B} : n \le k(z_0, z_i) < n+1 \}.$$

The next two lemmas are metric space analogues of [KL], Lemma 2.1 and Lemma 2.2.

**Lemma 3.3.** Let  $\gamma$  be a quasihyperbolic geodesic in  $\Omega$  starting at the point  $z_0$ . Then there is a constant  $C = C(C_2, D) > 0$  such that, for each  $n \in \mathbb{N}$ ,

$$\#\{B \in \mathcal{B}_n : B \cap \gamma \neq \emptyset\} \leq C.$$

Proof. Put

$$a_n := \#\{B \in \mathcal{B}_n : B \cap \gamma \neq \emptyset\} < \infty.$$

Let  $B_1, \ldots, B_{a_n} \in \mathcal{B}_n$  be the balls intersecting  $\gamma$ , ordered so that if k < l, then there exists  $x_k \in B_k \cap \gamma$  such that for every  $z \in B_l \cap \gamma$ , we have  $k(z_0, x_k) \le k(z_0, z)$ . We may assume that  $d(x_1, x_{a_n}) \ge d(x_1)/2$ , otherwise  $x_{a_n} \in B_{x_1}$  and we get the result by doubling on balls of Whitney type. Thus by Lemma 3.2,  $k(x_1, x_{a_n}) \ge \frac{a_n}{C}$ . Since  $k(z_i, x_i) \le \frac{1}{49} < 1$  for all  $i = 1, \ldots, a_n$ , we may compute

$$\frac{a_n}{C} \le k(x_1, x_{a_n}) = k(z_0, x_{a_n}) - k(z_0, x_1) 
\le k(z_0, z_{a_n}) + k(z_{a_n}, x_{a_n}) - (k(z_0, z_1) - k(x_1, z_1)) 
\le (n+1) + 1 - n + 1 = 3.$$

Hence  $a_n \leq 3C$ .

**Lemma 3.4.** There is a constant  $C = C(C_2, D) > 0$  such that, for each  $n \in \mathbb{N}$ ,

$$\sum_{B\in\mathcal{B}_n}\chi_{S(B)}(x)\leq C$$

whenever  $x \in \Omega$ .

*Proof.* Let  $x \in \Omega$ . The number of balls  $B \in \mathcal{B}$  containing x is bounded, so we may assume that there is a unique ball, denote it by  $B_1$ , in  $\mathcal{B}$  such that  $x \in B_1$ . Let  $[z_0, z_1]$  be the fixed geodesic joining  $z_0$  to  $z_1$ . Then  $x \in S(B)$  for  $B \in \mathcal{B}_n$  if and only if  $[z_0, z_1] \cap B \neq \emptyset$ . By Lemma 3.3, the number of balls  $B \in \mathcal{B}_n$  is bounded by a constant that is independent of n.

# 4. Gehring-Hayman Theorem

We begin with *Frostman's Lemma*. First we recall the definitions of the Hausdorff measure and the weighted Hausdorff measure.

Let (X, d) be a compact metric space. Let  $0 \le s < \infty$  and  $0 < \delta \le \infty$ . We set

$$\lambda_{\delta}^{s}(X) = \inf \left\{ \sum_{i=1}^{\infty} c_{i} \operatorname{diam}_{d}(E_{i})^{s} : \chi_{X} \leq \sum_{i} c_{i} \chi_{E_{i}}, c_{i} > 0, \operatorname{diam}_{d}(E_{i}) \leq \delta \right\}.$$

The weighted Hausdorff s-measure of X is

$$\lambda^{s}(X) = \lim_{\delta \to 0} \lambda^{s}_{\delta}(X).$$

In the special case, where  $c_i = 1$  for every i = 1, 2, ..., we set  $\mathcal{H}^s_{\delta}(X) = \lambda^s_{\delta}(X)$ , and we obtain the *Hausdorff s-measure* 

$$\mathcal{H}^{s}(X) = \lim_{\delta \to 0} \mathcal{H}^{s}_{\delta}(X).$$

The Hausdorff s-content of X is

$$\mathcal{H}^{s}_{\infty}(X) = \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}_{d}(E_{i})^{s} : X \subset \bigcup_{i=1}^{\infty} E_{i} \right\}.$$

By Lemma 8.16 of [Ma] we know that  $\mathcal{H}^s(X) \leq 30^s \lambda^s(X)$ , but in fact from the proof of that lemma one obtains that

$$\mathcal{H}_{30\delta}^{s}(X) \leq 30^{s} \lambda_{\delta}^{s}(X)$$
 for every  $0 < \delta \leq \infty$ .

In particular

$$\mathcal{H}^s_{\infty}(X) \leq 30^s \lambda^s_{\infty}(X).$$

The following formulation of Frostman's Lemma (cf. [Ma], Theorem 8.17, and [BO], Theorem 2) is suitable for our purposes.

**Theorem 4.1** (Frostman's Lemma). For any  $s \ge 0$  there is a Radon measure  $\omega$  on X such that

$$\omega(X) = \lambda_{\infty}^{s}(X)$$

and

$$\omega(E) \leq \operatorname{diam}_d(E)^s$$
 for all  $E \subset X$ .

In particular, when s = 1 and X is connected, we obtain

$$\omega(X) \ge \frac{1}{30} \mathcal{H}^1_{\infty}(X) \ge \frac{\operatorname{diam}_d(X)}{60}.$$

In this paper we apply the version of Frostman's Lemma, where X is connected and s = 1.

For the rest of the paper we assume that  $(\Omega, d, \mu)$  is a locally compact, non-complete and D-uniform metric measure space such that the measure  $\mu$  is Q-regular on balls of Whitney type for some Q > 1. Let  $\rho$  be a conformal density such that the number Q in the definition VG(B) coincides with the previous Q > 1.

Proof of Theorem 1.1. Let x and y be points in  $\overline{\Omega}$  and let [x, y] be a quasihyperbolic geodesic in  $\Omega$  joining points x and y. Because quasihyperbolic geodesics are D'-uniform curves, [x, y] is rectifiable in the metric d.

Let  $\gamma$  be another rectifiable curve in  $\Omega$  joining points x and y. Let  $a \in [x, y]$  be the point such that  $\ell_d([x, a]) = \ell_d([a, y])$ , and write p = d(x, a). Moreover, for each  $j = 0, 1, 2, \ldots$ , write  $A_j = (\overline{B}_d(x, 2^{-j}p) \setminus B_d(x, 2^{-(j+1)}p)) \cap \Omega$ . Let  $[x_{j+1}, x_j] \subset [x, a] \subset [x, y]$  be a subcurve, where  $x_{j+1}$  is the last point of [x, y] in  $\overline{B}(x, 2^{-(j+1)}p)$  and  $x_j$  is the last point of [x, y] in  $\overline{B}(x, 2^{-j}p)$ , and set  $\gamma_j = \gamma \cap A_j$ . We may clearly assume that  $\gamma_j$  is connected. By summing and symmetry it suffices to prove that

$$\ell_{\rho}([x_{j+1}, x_j]) \le C\ell_{\rho}(\gamma_j) \tag{4.1}$$

for every j = 0, 1, 2, ...

Let  $j = 0, 1, 2, \ldots$  From the definition of the curve  $\gamma_j$  it follows that

$$\ell_d(\gamma_j) \ge 2^{-(j+1)} p. \tag{4.2}$$

From the definition of the quasihyperbolic geodesic  $[x_{j+1}, x_j]$  and from the local D'-uniformity of the curve [x, y], we have that

$$\ell_d([x_{j+1}, x_j]) \le D'd(x_{j+1}, x_j) \le D'2^{-j+1}p, \tag{4.3}$$

$$2^{-(j+1)}p \le \ell_d([x,z]) \le D'd(z) \quad \text{for every } z \in [x_{j+1}, x_j], \tag{4.4}$$

and

$$k(x_{j+1}, x_j) = \int_{[x_{j+1}, x_j]} \frac{1}{d(z)} ds \le \frac{D'}{p} 2^{j+1} \ell_d([x_{j+1}, x_j]) \le 4D'^2.$$
 (4.5)

The proof consists of two parts: the "easy part", Case A, and the "hard part", Case B. Furthermore, Case B is divided into two parts, Subcase C and Subcase D. Here Subcase D is the hardest part and the novelty of our proof.

Case A. We first prove that inequality (4.1) holds when the curves  $[x_{j+1}, x_j]$  and  $\gamma_j$  are "close" to each other in the quasihyperbolic metric k. Let

$$M > \max \left\{ 4D^2 \frac{\log(4D'^2)}{\log 2} + 1, 4D^2 \frac{\log(B(2 + A^2/6)^Q/c_1)}{\log 2} \right\},$$

where  $c_1 > 0$  is a sufficiently small constant depending on  $A, C_1, D$  and Q, and let us assume that  $\operatorname{dist}_k([x_{j+1}, x_j], \gamma_j) \leq M$ . Let  $y_j \in [x_{j+1}, x_j]$  and  $\tilde{y}_j \in \gamma_j$  be points such that  $k(y_j, \tilde{y}_j) \leq M$ . Let us show that we may estimate the  $\rho$ -length of the quasihyperbolic geodesic  $[x_{j+1}, x_j]$  from above by  $2^{-j} p\rho(y_j)$  in the following way

$$\ell_{\rho}([x_{j+1}, x_j]) \le A^b D' \rho(y_j) 2^{-j+1} p, \tag{4.6}$$

where  $b = 4c_2D^2$  and  $c_2 = c_2(C_1, D) > 0$  is the constant from Lemma 3.2.

If there exists  $z \in [x_{j+1}, x_j]$  such that  $[x_{j+1}, x_j] \subset B_z = B_d(z, d(z)/2)$ , we obtain from HI(A) and (4.3)

$$\ell_{\rho}([x_{j+1}, x_j]) \le A\rho(y_j)\ell_d([x_{j+1}, x_j]) \le AD'\rho(y_j)2^{-j+1}p.$$

Otherwise we may assume that  $d(x_{j+1}, x_j) \ge d(x_{j+1})/2$ . From Lemma 3.2 and inequality (4.5), it follows that

$$N(x_{j+1}, x_j) \le 4c_2 D'^2 =: b,$$

where the constant  $c_2 = c_2(C_1, D) > 0$  is the constant from Lemma 3.2. Then by HI(A), every  $z \in [x_{j+1}, x_j]$  satisfies

$$\rho(z) \le A^b \rho(y_i).$$

This with (4.3) gives us inequality (4.6)

$$\ell_{\rho}([x_{j+1}, x_j]) \le A^b \rho(y_j) \ell_d([x_{j+1}, x_j])$$
  
 
$$\le A^b D' \rho(y_j) 2^{-j+1} p.$$

Next we estimate the  $\rho$ -length of the curve  $\gamma_j$  from below by  $2^{-j} p \rho(y_j)$ . If  $[x_{j+1}, x_j] \cap B_{\tilde{y}_j} \neq \emptyset$ , we easily get from HI(A) an estimate for  $\ell_{\rho}(\gamma_j)$ :

$$\ell_{\rho}(\gamma_j) \ge \frac{1}{A^{b+1}} \rho(y_j) \ell_d(\gamma_j \cap B_{\tilde{y}_j}). \tag{4.7}$$

Furthermore, for every  $z \in [x_{j+1}, x_j] \cap B_{\tilde{y}_j}$ , using inequalities (4.2) and (4.4) it holds that

$$\ell_{d}(\gamma_{j} \cap B_{\tilde{y}_{j}}) \geq \begin{cases} 2^{-(j+1)}p & \text{if } \gamma_{j} \subset B_{\tilde{y}_{j}}, \\ \frac{1}{2}d(\tilde{y}_{j}) \geq \frac{1}{2}(\frac{3}{2}d(z)) \geq \frac{3}{4D'}2^{-(j+1)}p & \text{if } \gamma_{j} \not\subset B_{\tilde{y}_{j}}. \end{cases}$$
(4.8)

In this case, combining (4.6), (4.7) and (4.8) we obtain the desired result (4.1)

$$\ell_{\rho}([x_{j+1}, x_j]) \le \frac{16}{3} A^{2b+1} D^{2} \ell_{\rho}(\gamma_j).$$

Therefore we may assume that  $[x_{j+1}, x_j] \cap B_{\tilde{y}_j} = \emptyset$ . This implies that  $d(y_j, \tilde{y}_j) \ge d(\tilde{y}_j)/2$ . By Lemma 3.2 there are at most  $h := Mc_2$  balls in the Whitney covering  $\mathbb{B}$  that intersect  $[y_j, \tilde{y}_j]$  and hence, by HI(A),

$$\rho(y_j) \le A^h \rho(\tilde{y}_j). \tag{4.9}$$

On the other hand, by HI(A) and (4.2),

$$\ell_{\rho}(\gamma_{j}) \geq \frac{1}{A} \rho(\tilde{y}_{j}) \ell_{d}(\gamma_{j} \cap B_{\tilde{y}_{j}}) \geq \begin{cases} \frac{1}{A} \rho(\tilde{y}_{j}) 2^{-(j+1)} p & \text{if } \gamma_{j} \subset B_{\tilde{y}_{j}}, \\ \frac{1}{2A} \rho(\tilde{y}_{j}) d(\tilde{y}_{j}) & \text{if } \gamma_{j} \not\subset B_{\tilde{y}_{j}}. \end{cases}$$
(4.10)

If  $\gamma_j \subset B_{\tilde{y}_j}$ , again we obtain the desired inequality (4.1) by combining inequalities (4.6), (4.9) and (4.10). If  $\gamma_j \not\subset B_{\tilde{y}_j}$ , then (4.10) with (4.9) gives

$$\rho(y_j) \le A^{h+1} \frac{2}{d(\tilde{y}_j)} \ell_{\rho}(\gamma_j). \tag{4.11}$$

By elementary inequalities in [GP], Lemma 2.1, and [BHK], Inequality (2.4), we obtain

$$\log\left(1 + \frac{d(y_j, \tilde{y}_j)}{\min\{d(y_i), d(\tilde{y}_i)\}}\right) \le k(y_j, \tilde{y}_j) \le M$$

and further,

$$\frac{1}{d(\tilde{y}_i)} \le \frac{e^M - 1}{d(y_i, \tilde{y}_i)}.\tag{4.12}$$

Moreover, the assumption  $d(y_i, \tilde{y}_i) \ge d(\tilde{y}_i)/2$  gives us

$$d(y_j) \le d(y_j, \tilde{y}_j) + d(\tilde{y}_j) \le 3d(y_j, \tilde{y}_j).$$

This, along with inequalities (4.11), (4.12) and (4.4), yields an estimate for the  $\rho$ -length of  $\gamma_j$ :

$$\rho(y_j) \le 2A^{h+1} \frac{e^M - 1}{d(y_j, \tilde{y}_j)} \ell_{\rho}(\gamma_j) \le 6A^{h+1} \frac{e^M - 1}{d(y_j)} \ell_{\rho}(\gamma_j) 
\le 6A^{h+1} (e^M - 1) \frac{D'}{p} 2^{j+1} \ell_{\rho}(\gamma_j).$$
(4.13)

Now combining (4.6) and (4.13) we obtain

$$\ell_{\rho}([x_{i+1}, x_i]) \le 24(e^M - 1)A^{b+h+1}D^{\prime 2}\ell_{\rho}(\gamma_i).$$

Thus (4.1) is proven when the curves  $[x_{j+1}, x_j]$  and  $\gamma_j$  are "close" to each other in the quasihyperbolic metric.

Case B. By Case A we may assume that  $\operatorname{dist}_k([x_{j+1},x_j],\gamma_j)>M$ . Let  $w_j\in [x_{j+1},x_j]$  satisfy  $d(x,w_j)=3\cdot 2^{-(j+2)}p$ . Let  $r:=\ell_\rho(\gamma_j)$  and let  $w\in\gamma_j$ . Let us consider the  $\rho$ -ball  $B_\rho(w,2r)$ .

**Subcase** C. If  $\operatorname{dist}_k(w_j, B_\rho(w, 2r)) < M$ , there exists  $u \in B_\rho(w, 2r)$  such that  $k(w_j, u) \leq M$  and hence  $\rho(w_j) \leq A^h \rho(u)$  (cf. inequality (4.9)). We may assume that  $\gamma_j \cap B_u = \emptyset$ . Otherwise  $\operatorname{dist}_k([x_{j+1}, x_j], \gamma_j) \leq M + 1$  and replacing M with M+1 we obtain the result by the case A. As we have assumed  $\gamma_j \cap B_u = \emptyset$ ,

$$2\ell_{\rho}(\gamma_{j}) = 2r > \operatorname{dist}_{\rho}(u, \gamma_{j})$$

$$\stackrel{\text{HI}(A)}{\geq} \frac{1}{2A} \rho(u) d(u)$$

$$\stackrel{(4.9)}{\geq} \frac{1}{2A^{h+1}} \rho(w_{j}) d(u)$$

$$\stackrel{(*)}{\geq} \frac{1}{2A^{h+1}e^{M}} \rho(w_{j}) d(w_{j})$$

$$\stackrel{(4.4)}{\geq} \frac{2^{-(j+1)}p}{2A^{h+1}D'e^{M}} \rho(w_{j})$$

$$\stackrel{(4.6)}{\geq} \frac{1}{8A^{b+h+1}D'^{2}e^{M}} \ell_{\rho}([x_{j+1}, x_{j}]).$$

The inequality (\*) above follows from the elementary estimate ([GP], Lemma 2.1, [BHK], Inequality (2.3))

$$\left|\log \frac{d(w_j)}{d(u)}\right| \le k(w_j, u) \le M.$$

Again we find a constant  $C \ge 1$  such that  $\ell_{\rho}([x_{j+1}, x_j]) \le C\ell_{\rho}(\gamma_j)$ . So (4.1) is satisfied.

**Subcase D.** By Subcase C we may assume that the  $\rho$ -ball  $B_{\rho}(w, 2r)$  is "far away" from the quasihyperbolic geodesic  $[x_{j+1}, x_j]$ . More precisely, we may assume that  $\operatorname{dist}_k(w_j, B_{\rho}(w, 2r)) \geq M$ . Our plan is to prove that the volume growth condition  $\operatorname{VG}(B)$  does not hold for such a  $\rho$ -ball. This is done by considering subcurves of  $\rho$ -length r of quasihyperbolic geodesics  $[z, w_j]$  with  $z \in \gamma_j$  and "averaging over  $\gamma_j$ " with respect to a suitable Frostman measure.

Let for every  $z \in \gamma_j$ ,  $[z, w_j]$  be a quasihyperbolic geodesic which joins z and  $w_j$ . Cover  $[z, w_j]$  with balls  $\{B_1, \ldots, B_{n(z)}\}\subset \mathcal{B}$  ordered so that if m < n, then

there exists  $z_m \in B_m \cap [z, w_j]$  such that for every  $\tilde{z} \in B_n \cap [z, w_j]$ , we have  $k(z, z_m) \leq k(z, \tilde{z})$ . Recall that  $n(z) < \infty$ .

Let  $[z, w_z] \subset [z, w_j]$ , where  $w_z$  is the first point which does not belong to  $B_\rho(w, 2r)$ . Thus  $\ell_\rho([z, w_z]) \geq r$ . Let  $\{B_1, \ldots, B_{n_r(z)}\} \subset \{B_1, \ldots, B_{n(z)}\}$  be those balls which cover  $[z, w_z]$ . So by  $\operatorname{HI}(A)$  and by the local D'-uniformity (quasiconvexity) of quasihyperbolic geodesics we obtain

$$r \leq \ell_{\rho}([z, w_{z}]) \leq \sum_{i=1}^{n_{r}(z)} A\rho(z_{i})\ell_{d}([z, w_{z}] \cap B_{i})$$

$$\leq AD' \sum_{i=1}^{n_{r}(z)} \rho(z_{i}) \operatorname{diam}_{d}(B_{i}).$$

$$(4.14)$$

We next provide a tool that will be used to estimate the  $\mu_{\rho}$ -measure of the  $\rho$ -ball  $B_{\rho}(w, 2r)$ . We claim that if  $B \in \mathcal{B}$  intersects  $B_{\rho}(w, 2r)$ , then  $B \subset B_{\rho}(w, (2 + \frac{A^2}{6})r)$ . To show this, it suffices to prove that if  $B \in \mathcal{B}$  intersects  $B_{\rho}(w, 2r)$  then

$$\operatorname{diam}_{\rho}(B) \le \frac{A^2}{6}r. \tag{4.15}$$

Consider such a ball  $B \in \mathcal{B}$ . It follows from HI(A) that

$$\operatorname{diam}_{\rho}(B) \le A\rho(z_B)\operatorname{diam}_{d}(B) = \frac{A}{25}\rho(z_B)d(z_B)$$

for each  $B \in \mathcal{B}$ , where  $z_B$  is the centre of B. Hence it actually suffices to prove that

$$\rho(z_B)d(z_B) \le \frac{25}{6}Ar. \tag{4.16}$$

Let  $y \in B \cap B_{\rho}(w, 2r)$ . If  $w \notin B_{z_B}$ , then there exists a curve  $\gamma$ , which joins points w and y and

$$2r \ge \int_{\gamma} \rho(z) \, ds \ge \frac{1}{A} \rho(z_B) \ell_d(\gamma \cap B_{z_B})$$
  
 
$$\ge \left(\frac{1}{2} - \frac{1}{50}\right) \frac{1}{A} \rho(z_B) d(z_B) = \frac{12}{25A} \rho(z_B) d(z_B),$$

and the inequality (4.16) is proven.

Let us assume that  $w \in B_{z_B}$ . The elementary estimate (2.3) implies

$$M \le k(w_j, w) \le 4D^2 \log \left(1 + \frac{d(w_j, w)}{\min\{d(w_i), d(w)\}}\right).$$

Along with the assumption that  $M > 4D^2 \frac{\log(4D'^2)}{\log 2} + 1$ , we see that

$$\min\{d(w_j), d(w)\} \le \frac{d(w_j, w)}{e^{M/4D^2} - 1} \le 2^{-j + 1 - (M - 1)/4D^2} p. \tag{4.17}$$

The assumption  $M > 4D^2 \frac{\log(4D'^2)}{\log 2} + 1$  and (4.4) give us

$$d(w_j) \ge \frac{p}{D'} 2^{-(j+1)} = 2^{-j+1-(M-1)/4D^2} p \frac{2^{(M-1)/4D^2}}{2^2 D'}$$

$$\ge 2^{-j+1-(M-1)/4D^2} p. \tag{4.18}$$

Thus it follows from inequality (4.17) that

$$d(w) \le 2^{-j+1-(M-1)/4D^2} p \le 2^{-(j+1)} p.$$

Hence, from the definition of the curve  $\gamma_j$  and inequality (4.2) we know that  $\gamma_j$  cannot be a subset of  $B_w$ . Then by  $\mathrm{HI}(A)$ 

$$r = \int_{\gamma_i} \rho(z) \, ds \ge \frac{1}{2A} \rho(z_B) d(w) \ge \frac{1}{4A} \rho(z_B) d(z_B),$$

and (4.16) is proven.

Now we know that if  $B \in \mathbb{B}$  intersects  $B_{\rho}(w, 2r)$ , then  $B \subset B_{\rho}(w, (2 + \frac{1}{6}A^2)r)$ . Then by HI(A), Lemma 3.1 (iv) and Q-regularity on balls of Whitney type, we have

$$\mu_{\rho}(B_{\rho}(w, (2 + \frac{1}{6}A^{2})r)) = \int_{B\rho(w, (2 + \frac{1}{6}A^{2})r)} \rho^{Q} d\mu$$

$$\geq \sum_{\substack{B \in \mathbb{B} \\ B \cap B_{\rho}(w, 2r) \neq \emptyset}} \frac{1}{NA^{Q}} \rho(z_{B})^{Q} \mu(B)$$

$$\geq \sum_{\substack{B \in \mathbb{B} \\ B \cap B_{\rho}(w, 2r) \neq \emptyset}} c_{3}\rho(z_{B})^{Q} \left(\frac{\operatorname{diam}_{d}(B)}{2}\right)^{Q},$$

$$(4.19)$$

where 
$$c_3 = \frac{1}{NC_1AQ}$$
.

Let us choose the basepoint  $z_0$  to be  $w_j$ . According to Frostman's Lemma (Theorem 4.1) there is a Radon measure  $\omega$  supported on  $\gamma_j$  such that  $\omega(\gamma_j) \geq \frac{\operatorname{diam}_d(\gamma_j)}{60}$  and  $\omega(E) \leq \operatorname{diam}_d(E)$  for every  $E \subset \gamma_j$ . Then with (4.14) we obtain (a version of Fubini's theorem)

$$\omega(\gamma_{j})r \leq AD' \int_{\gamma_{j}}^{n_{r}(z)} \sum_{i=1}^{n_{r}(z)} \rho(z_{i}) \operatorname{diam}_{d}(B_{i}) d\omega(z)$$

$$\leq AD' \sum_{n=M-1}^{\infty} \sum_{\substack{B \in \mathcal{B}_{n} \\ B \cap [z, w_{z}] \neq \emptyset \\ z \in \gamma_{j}}} \rho(z_{B}) \operatorname{diam}_{d}(B)\omega(S(B) \cap \gamma_{j}). \tag{4.20}$$

By Hölder's inequality we obtain that

$$\sum_{n=M-1}^{\infty} \sum_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \rho(z_B) \operatorname{diam}_d(B) \omega(S(B) \cap \gamma_j)$$

$$\leq \left(\sum_{n=M-1}^{\infty} \sum_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \rho(z_B)^{\mathcal{Q}} \operatorname{diam}_d(B)^{\mathcal{Q}}\right)^{\frac{1}{\mathcal{Q}}}$$

$$\left(\sum_{n=M-1}^{\infty} \sum_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)^{\frac{\mathcal{Q}}{\mathcal{Q}-1}}\right)^{\frac{\mathcal{Q}-1}{\mathcal{Q}}}.$$

Combining this with (4.20), (4.19) and the assumption  $\operatorname{dist}_k(w_j, B_\rho(w, 2r)) \geq M$  we obtain the estimate

$$\omega(\gamma_{j})r \leq AD' \left(\frac{2^{Q}}{c_{3}}\mu_{\rho}\left(B_{\rho}\left(w,\left(2+\frac{1}{6}A^{2}\right)r\right)\right)\right)^{\frac{1}{Q}}$$

$$\left(\sum_{n=M-1}^{\infty}\sum_{\substack{B\in\mathcal{B}_{n}\\B\cap\left[z,w_{z}\right]\neq\emptyset\\z\in\gamma_{j}}}\omega(S(B)\cap\gamma_{j})^{\frac{Q}{Q-1}}\right)^{\frac{Q-1}{Q}}$$

$$= c_{4}\left(\mu_{\rho}\left(B_{\rho}\left(w,\left(2+\frac{1}{6}A^{2}\right)r\right)\right)\right)^{\frac{1}{Q}}\left(\sum_{n=M-1}^{\infty}\sum_{\substack{B\in\mathcal{B}_{n}\\B\cap\left[z,w_{z}\right]\neq\emptyset\\z\in\gamma_{j}}}\omega(S(B)\cap\gamma_{j})^{\frac{Q}{Q-1}}\right)^{\frac{Q-1}{Q}},$$

$$(4.21)$$

where  $c_4 = 2AD'c_3^{-\frac{1}{Q}} = 2(NC_1)^{\frac{1}{Q}}A^2D'$ .

In order to estimate the measure of the shadow of the ball  $B \in \mathcal{B}_n$ , let us make a couple of preliminary estimates. For every  $v \in B \cap [z, w_j]$ , where  $B \in \mathcal{B}$  and  $z \in \gamma_j$ , we have by uniformity (quasiconvexity) and inequality (4.3) that

$$d(w_j, v) \le \ell_d([w_j, v]) \le \ell_d([w_j, z]) \le D'd(w_j, z) \le 2^{-j+1} pD'.$$

In the same way as in inequalities (4.17) and (4.18), we obtain from inequality (4.4) and the assumption  $n \ge M - 1 \ge 4D^2 \frac{\log(4D'^2)}{\log 2}$  that for every  $v \in B \cap [z, w_j]$ , where  $B \in \mathcal{B}_n$  and  $z \in \gamma_j$ , it holds that

$$d(v) \le 2^{-j+1-n/4D^2} pD'.$$

Furthermore, for every centre point  $z_B \in B \in \mathcal{B}_n$ , such that  $B \cap [z, w_j] \neq \emptyset$  for some  $z \in \gamma_j$ , it holds that

$$d(z_B) \le \frac{50}{49}d(v) \le 2^{-j+1-n/4D^2}p\frac{50D'}{49}.$$
 (4.22)

Also from the uniformity of the space  $(\Omega, d)$  and inequality (4.22) it follows that there exist a constant  $c_5 = c_5(C_1, D) \ge 1$  such that for every  $B \in \mathcal{B}_n$ , so that  $B \cap [z, w_i] \ne \emptyset$  for some  $z \in \gamma_i$ , it holds

$$\operatorname{diam}_{d}(S(B)) \le c_{5} \operatorname{diam}_{d}(B) \le 2^{-j+2-n/4D^{2}} p c_{5} \frac{50D'}{49}. \tag{4.23}$$

Now for every  $n \ge M - 1$  it holds by Lemma 3.4, Frostman's Lemma and inequality (4.23) that

$$\sum_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)^{\frac{Q}{Q-1}}$$

$$\leq \max_{\substack{B \in \mathcal{B}_n \\ S \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)^{\frac{1}{Q-1}} \sum_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)$$

$$\leq c_6 \omega(\gamma_j) \max_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)^{\frac{1}{Q-1}}$$

$$\leq c_6 \omega(\gamma_j) \max_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \operatorname{diam}_d(S(B) \cap \gamma_j)^{\frac{1}{Q-1}}$$

$$\leq c_6 \left(\frac{200D'c_5}{49}\right)^{\frac{1}{Q-1}} \omega(\gamma_j)(2^{-j-n/4D^2}p)^{\frac{1}{Q-1}},$$

where  $c_6 = c_6(C_1, D)$  is from Lemma 3.4. Furthermore, using this we may compute that

$$\sum_{n=M-1}^{\infty} \sum_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)^{\frac{Q}{Q-1}}$$

$$\leq c_6 \left(\frac{200D'c_5}{49}\right)^{\frac{1}{Q-1}} \omega(\gamma_j) \sum_{n=M-1}^{\infty} (2^{-j-n/4D^2}p)^{\frac{1}{Q-1}}$$

$$\leq c_7 \omega(\gamma_j) p^{\frac{1}{Q-1}} 2^{\frac{-j}{Q-1}} 2^{\frac{-M}{4D^2(Q-1)}},$$

where 
$$c_7 = c_6 \left(\frac{200D'c_5}{49}\right)^{\frac{1}{Q-1}} \frac{2^{\frac{2}{4D^2(Q-1)}}}{2^{\frac{1}{4D^2(Q-1)}} - 1}$$
. Thus with (4.21) we have

$$\omega(\gamma_j)^{Q} r^{Q} \le c_4^{Q} c_7^{Q-1} \mu_{\rho}(B_{\rho}(w, (2 + \frac{1}{6}A^2)r)) \omega(\gamma_j)^{Q-1} 2^{-j - \frac{M}{4D^2}} p.$$

Furthermore  $\omega(\gamma_j) \ge \frac{\operatorname{diam}_d(\gamma_j)}{60}$ , and this gives us

$$\mu_{\rho}(B_{\rho}(w, (2 + \frac{1}{6}A^{2})r)) \geq \omega(\gamma_{j}) \frac{1}{c_{4}^{Q} c_{7}^{Q-1}} \frac{2^{j + \frac{M}{4D^{2}}}}{p} r^{Q}$$

$$\geq \frac{2^{-j-1}p}{60} \frac{1}{c_{4}^{Q} c_{7}^{Q-1}} \frac{2^{j + \frac{M}{4D^{2}}}}{p} r^{Q}$$

$$= 2^{\frac{M}{4D^{2}}} c_{1} r^{Q},$$

where 
$$c_1 = \frac{49 \cdot 2^{\frac{-2}{4D^2} - 1} \left(2^{\frac{1}{4D^2(Q-1)}} - 1\right)^{Q-1}}{12000c_5 NC_1 (2A^2)^Q D'^{Q+1} c_6^{Q-1}}.$$

This is a contradiction because when M is sufficiently big, the volume growth condition VG(B) will not hold. Consequently, if  $k([x_{j+1}, x_j], \gamma_j) > M$  then our  $\rho$ -ball is in the quasihyperbolic metric k so big that  $\operatorname{dist}_k(w_j, B_\rho(w, 2r)) \leq M$ . Thus the conclusion is that  $\ell_\rho([x_{j+1}, x_j]) \leq C\ell_\rho(\gamma_j)$ , where  $C = C(A, B, C_1, D, Q)$ .

There is nothing special about the constant  $\frac{1}{2}$  in condition HI(A) and the constants  $\frac{1}{50}$  and 5 in Whitney covering. The only restriction in the Whitney covering is that if  $\lambda_1 B_d(z_1, d(z_1)/\lambda_2) \cap \lambda_1 B_d(z_2, d(z_2)/\lambda_2) \neq \emptyset$ , then  $\lambda_1 B_d(z_1, d(z_1)/\lambda_2)$  must be included in some ball  $B_d(z_2, d(z_2)/\lambda_3)$  on which the measure  $\mu$  is doubling. Otherwise one can choose the constants as desired.

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