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## Orbit closures and rank schemes

Christine Riedtmann and Grzegorz Zwara

*Dedicated to Andrzej Skowroński on the occasion of his 60th birthday*

**Abstract.** Let  $A$  be a finitely generated associative algebra over an algebraically closed field  $k$ , and consider the variety  $\text{mod}_A^d(k)$  of  $A$ -module structures on  $k^d$ . In case  $A$  is of finite representation type, equations defining the closure  $\overline{\mathcal{O}}_M$  are known for  $M \in \text{mod}_A^d(k)$ ; they are given by rank conditions on suitable matrices associated with  $M$ . We study the schemes  $\mathcal{C}_M$  defined by such rank conditions for modules over arbitrary  $A$ , comparing them with similar schemes defined for representations of quivers and obtaining results on singularities. One of our main theorems is a description of the ideal of  $\overline{\mathcal{O}}_M$  for a representation  $M$  of a quiver of type  $\mathbb{A}_n$ , a result Lakshmibai and Magyar established for the equioriented quiver of type  $\mathbb{A}_n$  in [12].

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### 1. Introduction

Throughout the paper,  $k$  denotes an algebraically closed field of arbitrary characteristic. By abuse of notation, a  $k$ -scheme  $\mathcal{X}$  and its functor of points, i.e. the functor from the category of commutative  $k$ -algebras to the category of sets sending  $\mathcal{X}$  to the set of morphisms  $\text{Spec}(R) \rightarrow \mathcal{X}$ , will be denoted by the same symbol. Any scheme  $\mathcal{X}$  considered in the paper will be of finite type over  $k$ . In fact,  $\mathcal{X}(k)$  can be viewed as the set of closed points of the scheme  $\mathcal{X}$ .

Let  $d \in \mathbb{N}$ . We denote by  $\mathbb{M}_d$  the  $k$ -scheme of  $d \times d$ -matrices and by  $\text{GL}_d$  the group  $k$ -scheme of invertible  $d \times d$ -matrices. Let  $A$  be a finitely generated associative  $k$ -algebra with a unit. The module scheme  $\text{mod}_A^d$  can be easily described in terms of its functor of points

$$\text{mod}_A^d(R) = \text{Hom}_{k\text{-alg.}}(A, \mathbb{M}_d(R)).$$



$v_{\underline{a}'}$  have isomorphic cokernels. In Section 4, we define analogous rank schemes for quiver representations and use them in Section 5 to extend Bongartz' results on a geometric version of the Morita equivalence to rank schemes.

In [12], Lakshmibai and Magyar proved a result which turns out to be equivalent to the following (see Section 4): If  $M$  is a representation of an equioriented Dynkin quiver of type  $\mathbb{A}$ , then  $\mathcal{C}_M = (\mathcal{C}_M)_{\text{red}} = \overline{\mathcal{O}}_M$ . In [17], the second author introduced so-called hom-controlled exact functors. This tool allowed him to show that some types of singularities in orbit closures of modules over two different algebras coincide. In Section 6 we study hom-controlled exact functors for rank schemes, and we obtain one of our main results, a generalization of the result above to representations of Dynkin quivers of type  $\mathbb{A}$ , not necessarily equioriented (see Theorem 6.4). Thus the ideal defining  $\overline{\mathcal{O}}_M$  is now known for Dynkin quivers of type  $\mathbb{A}$ ; it is an open question whether this result can be generalized to representations of arbitrary Dynkin quivers.

The main advantage of the scheme  $\mathcal{C}_M$  over  $\overline{\mathcal{O}}_M$  is that its tangent space at some  $N \in \overline{\mathcal{O}}_M$  has a module theoretic interpretation; we will explain this in Section 7 and use it in Section 8 to study the regularity of  $\mathcal{C}_M$  at  $N$ . Under the assumption that the algebra is representation-finite, we will characterize the singular locus of  $\mathcal{C}_M$ . The motivation is that the knowledge of the singular locus for  $\mathcal{C}_M$  helps to describe the singular locus for the orbit closure  $\overline{\mathcal{O}}_M$ . We will show in a forthcoming paper that in fact both loci coincide if  $M$  is a nilpotent representation of an oriented cycle.

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## 2. Rank ideals

Throughout the section  $R$  denotes a commutative ring. Let  $\mathbb{M}_{p \times q}$  be the affine scheme of  $p \times q$ -matrices, and fix  $U \in \mathbb{M}_{p \times q}(R)$  and  $t \in \{1, \dots, \min(p, q)\}$ . Following [7], 1.B, we denote by  $I_t(U)$  the ideal in  $R$  generated by the minors of  $U$  of size  $t$ . It will be convenient to define  $I_0(U) = R$  and  $I_t(U) = 0$  for  $t > \min(p, q)$ . Thus we have

$$R = I_0(U) \supseteq I_1(U) \supseteq I_2(U) \supseteq \dots$$

We first collect a few properties of  $I_t(U)$ .

**Lemma 2.1.** *Let  $U \in \mathbb{M}_{p \times q}(R)$  and  $V \in \mathbb{M}_{q \times r}(R)$ . Then*

$$I_t(UV) \subseteq I_t(U) \cap I_t(V).$$

*Proof.* Recall that, given a matrix  $W \in \mathbb{M}_{p' \times q'}(R)$  and the corresponding  $R$ -homomorphism  $U: R^{q'} \rightarrow R^{p'}$ , the entries of the matrix of the  $R$ -homomorphism

$\Lambda^t(U): \Lambda^t(R^{q'}) \rightarrow \Lambda^t(R^{p'})$  with respect to the standard bases are just the  $t \times t$ -minors of  $W$ . For two subsets  $K \subseteq \{1, \dots, p'\}$ ,  $L \subseteq \{1, \dots, q'\}$  of the same cardinality, we denote the minor of  $W$  corresponding to rows in  $K$  and columns in  $L$  by  $W_{K,L}$ . Using the functoriality of  $\Lambda^t$ , we obtain

$$\det(UV)_{K,N} = \sum_L \det U_{K,L} \det V_{L,N}$$

for any two subsets  $K \subseteq \{1, \dots, p\}$ ,  $N \subseteq \{1, \dots, r\}$  of cardinality  $t$ , where  $L$  ranges over all subsets of  $\{1, \dots, q\}$  of cardinality  $t$ . Our claim follows.  $\square$

**Lemma 2.2.** *Let  $U, V$  and  $W$  be matrices with coefficients in  $R$ , where  $U$  and  $W$  are invertible and of a size that the product  $UVW$  exists. Then  $I_t(UVW) = I_t(V)$ .*

*Proof.* Apply Lemma 2.1 to  $V' = UVW$  and to  $V = U^{-1}V'W^{-1}$ .  $\square$

We leave the proof of the next lemma to the reader. Given two matrices  $U$  and  $V$  we set  $U \oplus V = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ .

**Lemma 2.3.** *We have  $I_t(U \oplus V) = \sum_{i=0}^t I_i(U)I_{t-i}(V)$ . In particular, if  $V$  is the identity matrix of size  $s \leq t$ , then  $I_t(U \oplus V) = I_{t-s}(U)$ .*

### 3. Definition and first properties of $\mathfrak{C}_M$

Let  $A$  be a finitely generated associative  $k$ -algebra and  $d \in \mathbb{N}$ . The coordinate algebra  $k[\text{mod}_A^d]$  can be constructed as follows: Choosing generators  $a_1, \dots, a_r$  of  $A$  we obtain an isomorphism of  $A$  with the quotient of a free  $k$ -algebra  $k\langle x_1, \dots, x_r \rangle$  by a two-sided ideal  $J$ . We consider  $rd^2$  independent variables  $x_{i,j}^l, l \leq r, i, j \leq d$ , arranged into  $r$  matrices  $X_l = (x_{i,j}^l)$  of size  $d \times d$ . Then  $k[\text{mod}_A^d]$  is the quotient of the polynomial algebra  $k[x_{i,j}^l]$  by the ideal generated by the entries of the  $d \times d$ -matrices  $\rho(X_1, \dots, X_r), \rho \in J$ . Let  $X_A^d$  be the element in  $\text{mod}_A^d(k[\text{mod}_A^d])$  defined by the equalities  $X(a_l) = \bar{X}_l$  for  $l \leq r$ . We call  $X_A^d$  a universal module in  $\text{mod}_A^d$ , as it satisfies the following universal property: For any commutative  $k$ -algebra  $R$  and any element  $N \in \text{mod}_A^d(R)$  there is a unique algebra homomorphism  $\varphi: k[\text{mod}_A^d] \rightarrow R$  such that  $N = \text{mod}_A^d(\varphi)(X_A^d)$ .

The coordinate algebra  $k[\mathbb{M}_{p \times q}]$  is the polynomial algebra  $k[y_{i,j}]$  with  $i \leq p, j \leq q$ . We denote by  $\mathcal{V}_{p \times q}^r \subseteq \mathbb{M}_{p \times q}$  the closed subscheme of “matrices of rank at most  $r$ ” defined by the ideal  $I_{r+1}(Y) \subseteq k[\mathbb{M}_{p \times q}]$  for  $Y = (y_{i,j}) \in \mathbb{M}_{p \times q}(k[\mathbb{M}_{p \times q}])$ .

Let  $\underline{a} = (a_{i,j})$  be a  $p \times q$  matrix with coefficients in  $A$ . The assignment

$$N \mapsto N(\underline{a}) = \begin{pmatrix} N(a_{1,1}) & \cdots & N(a_{1,q}) \\ \dots & \dots & \dots \\ N(a_{p,1}) & \cdots & N(a_{p,q}) \end{pmatrix}$$

leads to a regular morphism  $\Theta_{\underline{a}}: \text{mod}_A^d \rightarrow \mathbb{M}_{pd \times qd}$ . For an  $A$ -module  $M \in \text{mod}_A^d(k)$ , we set

$$\mathcal{C}_{M,\underline{a}} = \Theta_{\underline{a}}^{-1}(\mathcal{V}_{pd \times qd}^{\text{rk } M(\underline{a})}).$$

Note that  $\mathcal{C}_{M,\underline{a}} = \text{Spec}(k[\text{mod}_A^d]/\mathcal{I}_{M,\underline{a}})$ , where  $\mathcal{I}_{M,\underline{a}} = I_{1+\text{rk } M(\underline{a})}(X_A^d(\underline{a})) \subseteq k[\text{mod}_A^d]$ .

**Lemma 3.1.** *The subscheme  $\mathcal{C}_{M,\underline{a}} \subseteq \text{mod}_A^d$  is stable under  $\text{GL}_d$ .*

*Proof.* Fix a commutative  $k$ -algebra  $R$ . We need to show that  $N \in \mathcal{C}_{M,\underline{a}}(R)$  and  $g \in \text{GL}_d(R)$  implies  $g * N \in \mathcal{C}_{M,\underline{a}}(R)$ , or equivalently that all  $r \times r$ -minors of  $(g * N)(\underline{a})$  vanish, where  $r = 1 + \text{rk } M(\underline{a})$ . But  $I_r(g * N(\underline{a})) = I_r(N(\underline{a}))$  by Lemma 2.2 as

$$(g * N)(\underline{a}) = \begin{pmatrix} g & 0 & \cdots & 0 \\ 0 & g & \cdots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & g \end{pmatrix} \cdot N(\underline{a}) \cdot \begin{pmatrix} g^{-1} & 0 & \cdots & 0 \\ 0 & g^{-1} & \cdots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & g^{-1} \end{pmatrix}. \quad \square$$

We define the rank scheme associated to  $M$  as

$$\mathcal{C}_M = \bigcap \mathcal{C}_{M,\underline{a}},$$

where  $\underline{a}$  ranges over all  $p \times q$ -matrices with coefficients in  $A$  for all  $p$  and  $q$ . Thus  $\mathcal{C}_M$  is the closed  $\text{GL}_d$ -subscheme of  $\text{mod}_A^d$  defined by  $\mathcal{I}_M = \sum \mathcal{I}_{M,\underline{a}}$ . Note that isomorphic modules define the same rank scheme.

Sending  $a \in A$  to the  $A$ -homomorphism  $v_a: A \rightarrow A$ ,  $v_a(b) = b \cdot a$  defines a bijection from  $A$  to  $\text{Hom}_A(A, A)$ . Given a  $p \times q$ -matrix  $\underline{a} = (a_{i,j})$  with coefficients in  $A$ , we define an  $A$ -homomorphism

$$v_{\underline{a}}: A^p \xrightarrow{(v_{a_{j,i}})} A^q.$$

This gives a bijection between the set of  $p \times q$ -matrices with coefficients in  $A$  and the space  $\text{Hom}_A(A^p, A^q)$ . Moreover, for a  $q \times s$ -matrix  $\underline{b}$  with coefficients in  $A$  we get

$$v_{\underline{a} \cdot \underline{b}} = v_{\underline{b}} \circ v_{\underline{a}}.$$

**Lemma 3.2.** *Let  $\underline{a}'$  and  $\underline{a}''$  be two matrices with coefficients in  $A$ . If the cokernels  $\text{Coker}(v_{\underline{a}'})$  and  $\text{Coker}(v_{\underline{a}''})$  are  $A$ -isomorphic then  $\mathcal{I}_{M,\underline{a}'} = \mathcal{I}_{M,\underline{a}''}$ .*

As a consequence, we obtain a well defined scheme  $\mathcal{C}_{M,L}$  for any finitely presented  $A$ -module  $L$  by choosing a presentation

$$A^p \xrightarrow{v_{\underline{a}}} A^q \rightarrow L \rightarrow 0$$

and setting  $\mathcal{C}_{M,L} = \mathcal{C}_{M,\underline{a}}$ . Note that  $\mathcal{C}_M = \bigcap \mathcal{C}_{M,L}$ , where  $L$  ranges over representatives of all isomorphism classes of finitely presented  $A$ -modules.

*Proof of the lemma.* Let  $\underline{a}'$  and  $\underline{a}''$  be two matrices with coefficients in  $A$  of sizes  $p_1 \times q_1$  and  $p_2 \times q_2$ , respectively. Setting  $f_1 = v_{\underline{a}'}$  and  $f_2 = v_{\underline{a}''}$  we obtain two  $A$ -homomorphisms

$$A^{p_1} \xrightarrow{f_1} A^{q_1} \quad \text{and} \quad A^{p_2} \xrightarrow{f_2} A^{q_2}.$$

We assume that there is an  $A$ -isomorphism  $\xi: \text{Coker}(f_1) \rightarrow \text{Coker}(f_2)$ . We claim that there are matrices  $\underline{b}$  and  $\underline{c}$  with coefficients in  $A$  such that

$$\begin{pmatrix} \underline{a}' & 0 \\ 0 & 1_{dq_2} \end{pmatrix} = \underline{b} \cdot \begin{pmatrix} 1_{dq_1} & 0 \\ 0 & \underline{a}'' \end{pmatrix} \cdot \underline{c}.$$

Using the property that free  $A$ -modules are projective we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccc} A^{p_1} & \xrightarrow{f_1} & A^{q_1} & \xrightarrow{g_1} & \text{Coker}(f_1) & \longrightarrow & 0 \\ \downarrow h & & \downarrow h' & & \simeq \downarrow \xi & & \\ A^{p_2} & \xrightarrow{f_2} & A^{q_2} & \xrightarrow{g_2} & \text{Coker}(f_2) & \longrightarrow & 0 \\ \downarrow l & & \downarrow l' & & \simeq \downarrow \xi^{-1} & & \\ A^{p_1} & \xrightarrow{f_1} & A^{q_1} & \xrightarrow{g_1} & \text{Coker}(f_1) & \longrightarrow & 0. \end{array}$$

In particular,

$$h'f_1 = f_2h, \quad l'f_2 = f_1l, \tag{3.1}$$

and  $g_1l'h' = g_1$ . The latter implies that  $\text{Im}(1 - l'h')$  is contained in  $\text{Im}(f_1)$ , and consequently  $1 - l'h'$  factors through  $f_1$ . From this, and by symmetry, we get two  $A$ -homomorphisms  $\varphi_i: A^{q_i} \rightarrow A^{p_i}$ ,  $i = 1, 2$ , such that

$$l'h' + f_1\varphi_1 = 1_{A^{q_1}} \quad \text{and} \quad h'l' + f_2\varphi_2 = 1_{A^{q_2}}. \tag{3.2}$$

We conclude from (3.1) and (3.2) that

$$\begin{pmatrix} f_1 & 0 \\ 0 & 1_{A^{q_2}} \end{pmatrix} = \begin{pmatrix} f_1\varphi_1 & -l' \\ h' & 1_{A^{q_2}} \end{pmatrix} \cdot \begin{pmatrix} 1_{A^{q_1}} & 0 \\ 0 & f_2 \end{pmatrix} \cdot \begin{pmatrix} f_1 & l' \\ -h & \varphi_2 \end{pmatrix}.$$

We get the claim by choosing matrices  $\underline{b}$  and  $\underline{c}$  such that

$$v_{\underline{b}} = \begin{pmatrix} f_1 & l' \\ -h & \varphi_2 \end{pmatrix} \quad \text{and} \quad v_{\underline{c}} = \begin{pmatrix} f_1\varphi_1 & -l' \\ h' & 1_{A^{q_2}} \end{pmatrix}.$$

Let  $X = X_A^d$  be a universal module for  $\text{mod}_A^d$ . The claim implies that

$$\begin{pmatrix} X(\underline{a}') & 0 \\ 0 & 1_{dq_2} \end{pmatrix} = X(\underline{b}) \cdot \begin{pmatrix} 1_{dq_1} & 0 \\ 0 & X(\underline{a}'') \end{pmatrix} \cdot X(\underline{c}).$$

By Lemmas 2.1 and 2.3, we know that

$$I_{t-dq_2}(X(\underline{a}')) = I_t \begin{pmatrix} X(\underline{a}') & 0 \\ 0 & 1_{dq_2} \end{pmatrix} \subseteq I_t \begin{pmatrix} 1_{dq_1} & 0 \\ 0 & X(\underline{a}'') \end{pmatrix} = I_{t-dq_1}(X(\underline{a}'')).$$

for any  $t \geq q_1d, q_2d$ . Applying the functor  $\text{Hom}_A(-, M)$  we obtain the exact sequences

$$\begin{aligned} 0 \rightarrow \text{Hom}_A(\text{Coker}(f_1), M) &\rightarrow \text{Hom}_A(A^{q_1}, M) \xrightarrow{\text{Hom}_A(v_{\underline{a}'}, M)} \text{Hom}_A(A^{p_1}, M), \\ 0 \rightarrow \text{Hom}_A(\text{Coker}(f_2), M) &\rightarrow \text{Hom}_A(A^{q_2}, M) \xrightarrow{\text{Hom}_A(v_{\underline{a}''}, M)} \text{Hom}_A(A^{p_2}, M). \end{aligned}$$

Let  $w = \dim_k \text{Hom}_A(\text{Coker}(f_1), M) = \dim_k \text{Hom}_A(\text{Coker}(f_2), M)$ . Identifying the space  $\text{Hom}_A(A^s, M)$  with  $k^{ds}$ ,  $s \in \mathbb{N}$ , we get

$$\text{Hom}_A(v_{\underline{a}'}, M) = M(\underline{a}') \quad \text{and} \quad \text{Hom}_A(v_{\underline{a}''}, M) = M(\underline{a}'').$$

Consequently,

$$\text{rk}(M(\underline{a}')) = dq_1 - w, \quad \text{rk}(M(\underline{a}'')) = dq_2 - w$$

and

$$\mathcal{I}_{M, \underline{a}'} = I_{1+dq_1-w}(X(\underline{a}')) \subseteq I_{1+dq_2-w}(X(\underline{a}'')) = \mathcal{I}_{M, \underline{a}''}.$$

In a similar way we prove the reverse inclusion, which finishes the proof. □

**Lemma 3.3.** *For finitely presented  $A$ -modules  $L_1$  and  $L_2$ , we have that  $\mathcal{I}_{M, L_1 \oplus L_2} \subseteq \mathcal{I}_{M, L_1} + \mathcal{I}_{M, L_2}$ .*

*Proof.* We fix matrices  $\underline{a}'$  and  $\underline{a}''$  with coefficients in  $A$  such that the cokernels of  $v_{\underline{a}'}$  and  $v_{\underline{a}''}$  are isomorphic to  $L_1$  and  $L_2$ , respectively. Let  $r_1 = \text{rk}(M(\underline{a}'))$  and  $r_2 = \text{rk}(M(\underline{a}''))$ , and set  $X = X_A^d$ . Using the fact that

$$R = I_0(U) \supseteq I_1(U) \supseteq I_2(U) \supseteq \dots$$

for any matrix  $U$  with coefficients in a commutative ring  $R$ , we get from Lemma 2.3

$$\begin{aligned}
\mathcal{I}_{M,L_1 \oplus L_2} &= \mathcal{I}_{M,\underline{a}' \oplus \underline{a}''} = I_{1+r_1+r_2}(X(\underline{a}') \oplus X(\underline{a}'')) \\
&= \sum_{t=0}^{1+r_1+r_2} I_t(X(\underline{a}')) \cdot I_{1+r_1+r_2-t}(X(\underline{a}'')) \\
&\subseteq \sum_{t=0}^{r_1} I_{1+r_1+r_2-t}(X(\underline{a}'')) + \sum_{t=1+r_1}^{1+r_1+r_2} I_t(X(\underline{a}')) \\
&= I_{1+r_2}(X(\underline{a}'')) + I_{1+r_1}(X(\underline{a}')) = \mathcal{I}_{M,\underline{a}''} + \mathcal{I}_{M,\underline{a}'} \\
&= \mathcal{I}_{M,L_1} + \mathcal{I}_{M,L_2}. \quad \square
\end{aligned}$$

Let  $L$  be a finitely presented  $A$ -module. Then the space  $\text{Hom}_A(L, M)$  is finite dimensional, and we choose a basis  $f_1, \dots, f_s$ . We denote by  $L_M$  the kernel of the map

$$L \xrightarrow{(f_1, \dots, f_s)^t} M^s.$$

Note that  $L_M$  does not depend on the choice of the basis  $f_1, \dots, f_s$ .

**Lemma 3.4.** *Using the above notation, we have  $\mathcal{I}_{M,L} \subseteq \mathcal{I}_{M,L/L_M}$ .*

*Proof.* As there is an injective  $A$ -homomorphism from  $L/L_M$  to  $M^s$ , the module  $L/L_M$  is finite dimensional and thus finitely presented, as  $A$  is finitely generated. We may choose presentations of  $L$  and of  $L/L_M$  for which there is a commutative diagram

$$\begin{array}{ccccccc}
A^p & \xrightarrow{v_a} & A^q & \longrightarrow & L & \longrightarrow & 0 \\
v_b \downarrow & & \parallel & & \downarrow & & \\
A^t & \xrightarrow{v_c} & A^q & \longrightarrow & L/L_M & \longrightarrow & 0
\end{array}$$

with exact rows.

From  $v_a = v_c \circ v_b$  we see that  $\underline{a} = \underline{b} \circ \underline{c}$ . Note that the injection from  $L/L_M$  to  $M^s$  induces an isomorphism from  $\text{Hom}_A(L/L_M, M)$  to  $\text{Hom}_A(L, M)$  and thus  $\text{rk}(M(\underline{a})) = \text{rk}(M(\underline{c}))$ . By Lemma 2.1 we conclude that

$$\begin{aligned}
\mathcal{I}_{M,L} &= \mathcal{I}_{M,\underline{a}} = I_{1+\text{rk } M(\underline{a})}(X_A^d(\underline{a})) \\
&= I_{1+\text{rk } M(\underline{a})}(X_A^d(\underline{b}) \circ X_A^d(\underline{c})) \\
&\subseteq I_{1+\text{rk } M(\underline{c})}(X_A^d(\underline{c})) = \mathcal{I}_{M,\underline{c}} \\
&= \mathcal{I}_{M,L/L_M}. \quad \square
\end{aligned}$$

Next we study the behavior of rank schemes under an algebra homomorphism  $\varphi: A \rightarrow B$ . For a  $p \times q$ -matrix  $\underline{a} = (a_{i,j})$  with coefficients in  $A$ , we denote

the corresponding  $p \times q$ -matrix with coefficients in  $B$  by  $\varphi(\underline{a}) = (\varphi(a_{i,j}))$ . Any  $B$ -module can be considered as an  $A$ -module via  $\varphi$ ; we will write  ${}_A M$  for the  $A$ -module corresponding to the  $B$ -module  ${}_B M$ . In addition,  $\varphi$  induces a regular  $\text{GL}_d$ -morphism  $\varphi^d: \text{mod}_B^d \rightarrow \text{mod}_A^d$  defined by  $[\varphi^d(N)](a) = N(\varphi(a))$ , which is a closed immersion if  $\varphi$  is surjective. If  $d = \dim_k M$ , then  $\varphi^d(\mathcal{O}_{B M}) = \mathcal{O}_{A M}$ , and consequently  $\bar{\mathcal{O}}_{B M} \subseteq (\varphi^d)^{-1}(\bar{\mathcal{O}}_{A M})$ . A similar result holds for rank schemes.

**Lemma 3.5.** *Let  $\varphi: A \rightarrow B$  be an algebra homomorphism and let  $M$  belong to  $\text{mod}_B^d(k)$ . Then*

$$\mathcal{C}_{B M} \subseteq (\varphi^d)^{-1}(\mathcal{C}_{A M}).$$

*If  $\varphi$  is surjective, the above inclusion is an equality.*

*Proof.* Note that  $\Theta_{\varphi(\underline{a})} = \Theta_{\underline{a}} \circ \varphi^d$  and  ${}_B M(\varphi(\underline{a})) = {}_A M(\underline{a})$  for a  $B$ -module  ${}_B M$ , and thus  $\mathcal{C}_{B M, \varphi(\underline{a})} = (\varphi^d)^{-1}(\mathcal{C}_{A M, (\underline{a})})$ . □

The algebra  $B = A / \text{Ann } M$  is finite dimensional, being a subalgebra of  $\text{End}_k(M)$ . By the above lemma, we can work over the finite dimensional algebra  $B = A / \text{Ann } M$  instead of  $A$  and consider  $M$  as a  $B$ -module. For a finite dimensional algebra, any finitely presented module is isomorphic to a direct sum of indecomposables, and we obtain the following consequence.

**Corollary 3.6.** *Let  $A$  be finite dimensional, and let  $\mathcal{L}$  be a complete set of pairwise non-isomorphic indecomposable  $A$ -modules which can be embedded into finite powers of  $M$ . Then*

$$\mathcal{I}_M = \sum_{L \in \mathcal{L}} \mathcal{I}_{M, L}.$$

We construct  $\mathcal{C}_M$  on a simple but instructive example.

**Example 3.7.** Let  $A = k[\varepsilon] \simeq k[x]/(x^2)$  be the algebra of dual numbers and  $M: A \rightarrow \mathbb{M}_2(k)$  be the unique algebra homomorphism satisfying  $M(\varepsilon) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ , so that the corresponding module is isomorphic to  ${}_A A$ . Choosing  $\varepsilon$  as a generator of  $A$ , we identify the coordinate algebra  $k[\text{mod}_A^2]$  with

$$k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}] / (\text{entries of } \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}^2).$$

The set  $\mathcal{L}$  considered in the previous corollary consists of two modules:  $M$  and its one-dimensional simple submodule denoted by  $S$ . In fact, any indecomposable  $A$ -module is isomorphic to either  $M$  or  $S$ , so the algebra  $A$  is representation finite and therefore  $(\mathcal{C}_M)_{\text{red}} = \bar{\mathcal{O}}_M$ , as mentioned in the introduction. Since  $M$  is free as an  $A$ -module, we have a free presentation  $0 = A^0 \rightarrow A^1 \rightarrow M \rightarrow 0$  giving us no condition, i.e.  $\mathcal{I}_{M, M} = 0$ . Thus choosing a free presentation  $A^1 \xrightarrow{\nu(\varepsilon)} A^1 \rightarrow S \rightarrow 0$

and denoting by  $\bar{x}_{i,j}$  the residue class of  $x_{i,j}$  in the coordinate algebra  $k[\text{mod}_A^2]$ , we get

$$\mathcal{I}_M = \mathcal{I}_{M,S} = I_{1+\text{rk } M(\varepsilon)} \begin{pmatrix} \bar{x}_{1,1} & \bar{x}_{1,2} \\ \bar{x}_{2,1} & \bar{x}_{2,2} \end{pmatrix} = I_2 \begin{pmatrix} \bar{x}_{1,1} & \bar{x}_{1,2} \\ \bar{x}_{2,1} & \bar{x}_{2,2} \end{pmatrix} = \left( \det \begin{pmatrix} \bar{x}_{1,1} & \bar{x}_{1,2} \\ \bar{x}_{2,1} & \bar{x}_{2,2} \end{pmatrix} \right).$$

Obviously the trace of  $\begin{pmatrix} \bar{x}_{1,1} & \bar{x}_{1,2} \\ \bar{x}_{2,1} & \bar{x}_{2,2} \end{pmatrix}$  does not belong to  $\mathcal{I}_M$  (but its third power does), so the ideal  $\mathcal{I}_M$  is not radical and  $\mathcal{C}_M$  is not reduced.

#### 4. Rank schemes for representations of quivers

We first recall the classical definition of the representation space of a quiver with relations for a given dimension vector, acted upon by a product of general linear groups, and then view this space as the  $k$ -points of an affine scheme with the action of a group scheme.

Let  $Q = (Q_0, Q_1, s, t)$  be a finite quiver, i.e. a finite set  $Q_0$  of vertices and a finite set  $Q_1$  of arrows  $\alpha: s\alpha \rightarrow t\alpha$ , where  $s\alpha$  and  $t\alpha$  denote the starting and the terminating vertex of  $\alpha$ , respectively. A representation of  $Q$  over  $k$  is a collection  $(X(i); i \in Q_0)$  of finite dimensional  $k$ -vector spaces together with a collection  $(X(\alpha): X(s\alpha) \rightarrow X(t\alpha); \alpha \in Q_1)$  of  $k$ -linear maps. A morphism  $f: X \rightarrow Y$  between two representations is a collection  $(f(i): X(i) \rightarrow Y(i))$  of  $k$ -linear maps such that

$$f(t\alpha) \circ X(\alpha) = Y(\alpha) \circ f(s\alpha) \quad \text{for all } \alpha \in Q_1.$$

The dimension vector of a representation  $X$  of  $Q$  is the vector

$$\mathbf{dim} X = (\dim X(i)) \in \mathbb{N}^{Q_0}.$$

We denote the category of representations of  $Q$  by  $\text{rep}(Q)$ , and for any vector  $\mathbf{d} = (d_i) \in \mathbb{N}^{Q_0}$ ,

$$\text{rep}_Q^{\mathbf{d}}(k) = \prod_{\alpha \in Q_1} \mathbb{M}_{d_{t\alpha} \times d_{s\alpha}}(k)$$

is the affine space of representations  $X$  of  $Q$  with  $X(i) = k^{d_i}$ ,  $i \in Q_0$ . The group

$$\text{GL}_{\mathbf{d}}(k) = \prod_{i \in Q_0} \text{GL}_{d_i}(k)$$

acts on  $\text{rep}_Q^{\mathbf{d}}(k)$  by

$$((g_i) \star X)(\alpha) = g_{t\alpha} \circ X(\alpha) \circ g_{s\alpha}^{-1}.$$

Note that the  $\text{GL}_{\mathbf{d}}(k)$ -orbit of  $X$ , denoted by  $\mathcal{O}_X$ , consists of the representations  $Y$  in  $\text{rep}_Q^{\mathbf{d}}(k)$  which are isomorphic to  $X$ .



where  $p' = \sum d_{u_i}$  and  $q' = \sum d_{v_j}$ . We keep track in  $\omega_{i,j}$  of the vertices  $v_j$  and  $u_i$  even if  $\omega_{i,j} = 0$ . For  $M \in \text{rep}_Q^d(k)$ ,  $\Theta_{\underline{\omega}}(M)$  is a  $p' \times q'$ -matrix with coefficients in  $k$ . We set

$$\mathcal{C}_{M,\underline{\omega}} = \Theta_{\underline{\omega}}^{-1}(\mathcal{V}_{p' \times q'}^{\text{rk } \Theta_{\underline{\omega}}(M)}), \quad \mathcal{C}_M = \bigcap \mathcal{C}_{M,\underline{\omega}},$$

where  $\underline{\omega}$  ranges over all possible matrices of paths with all possible sets of starting and terminating vertices. Note that  $\mathcal{C}_{M,\underline{\omega}} = \text{Spec}(k[\text{rep}_Q^d]/\mathcal{I}_{M,\underline{\omega}})$ , where  $\mathcal{I}_{M,\underline{\omega}} \subseteq k[\text{rep}_Q^d]$  is the ideal generated by all minors of size  $1 + \text{rk } \Theta_{\underline{\omega}}(M)$  of the matrix  $\Theta_{\underline{\omega}}(X_Q^d)$ , and that  $\mathcal{C}_M = \text{Spec}(k[\text{rep}_Q^d]/\mathcal{I}_M)$ , where  $\mathcal{I}_M = \sum \mathcal{I}_{M,\underline{\omega}}$ . We leave the necessary adjustments for quivers with relations to the reader.

All results presented before in the context of module scheme have a corresponding version in terms of representations of bound quivers  $(Q, J)$ . The main difference is that instead of finitely generated free presentations of modules we consider projective presentations of representations using the projectives  $(kQ/J) \cdot \varepsilon_i$ ,  $i \in Q_0$ . In particular, if  $Q$  is an equioriented Dynkin quiver of type  $\mathbb{A}_n$ ,

$$Q: 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n,$$

$J = 0$ ,  $\omega_{j,i} = \alpha_{j-1}\alpha_{j-2}\cdots\alpha_i$  and  $M$  is a representation in  $\text{rep}_Q^d(k)$ , then

$$\mathcal{I}_M = \sum_{1 \leq i < j \leq n} \mathcal{I}_{M,(\omega_{j,i})} = \sum_{1 \leq i < j \leq n} I_{1+\text{rk}(M(\omega_{j,i}))}(X_Q^d(\omega_{j,i})).$$

Thus  $\mathcal{I}_M$  is exactly the ideal generated by determinantal conditions as considered by Lakshmibai and Magyar in [12]. Therefore we can reformulate their main result as follows:

**Theorem 4.1.** *Let  $M$  be a representation in  $\text{rep}_Q^d(k)$ , where  $Q$  is an equioriented Dynkin quiver of type  $\mathbb{A}$ . Then the ideal  $\mathcal{I}_M$  is radical and  $\mathcal{C}_M = \bar{\Theta}_M$ .*

## 5. A geometric version of Morita equivalence for rank schemes

The purpose of this section is to relate rank schemes for quiver representations to rank schemes for modules over algebras.

Let  $A$  be a finite dimensional algebra, and let  $S_1, \dots, S_s$  be representatives for the isomorphism classes of simple  $A$ -modules. The Grothendieck group  $K_0(A)$  can be identified with  $\mathbb{Z}^s$ , and the dimension vector  $\mathbf{dim} N \in \mathbb{Z}^s$  of a finite dimensional  $A$ -module  $N$  is the vector

$$\mathbf{dim} N = (d_1, \dots, d_s) \in K_0(A),$$

where  $d_l$  is the multiplicity of  $S_l$  in any composition series for  $N$ . If  $e^l \in A$  is a primitive idempotent such that  $Ae^l$  is a projective cover for  $S_l$ , we have

$$d_l = \dim_k \operatorname{Hom}_A(Ae^l, N) = \operatorname{rk} N(e^l).$$

By [9] or Lemma 1 of [3], there is a connected component  $\operatorname{mod}_A^{\mathbf{d}}$  of the scheme  $\operatorname{mod}_A^{\mathbf{d}}$ , characterized by the fact that

$$\operatorname{mod}_A^{\mathbf{d}}(k) = \{N \in \operatorname{mod}_A^{\mathbf{d}}(k) : \mathbf{dim} N = \mathbf{d}\}$$

for any vector  $\mathbf{d} = (d_1, \dots, d_s) \in \mathbb{N}^s$ .

**Lemma 5.1.** *For  $M \in \operatorname{mod}_A^{\mathbf{d}}(k)$  we have  $\mathcal{C}_M \subseteq \operatorname{mod}_A^{\mathbf{d}}$ .*

*Proof.* As  $\operatorname{mod}_A^{\mathbf{d}}$  is a connected component in  $\operatorname{mod}_A^{\mathbf{d}}$ , it suffices to show that  $\mathcal{C}_M(k) \subseteq \operatorname{mod}_A^{\mathbf{d}}(k)$ . Let  $N \in \mathcal{C}_M(k)$ , and set  $\mathbf{dim} N = \mathbf{d}' = (d'_1, \dots, d'_s)$ . Considering the ideal  $I_{M, (e^l)}$ , where  $(e^l)$  is the  $1 \times 1$ -matrix having the idempotent defined above as its entry, we get that

$$d'_l = \operatorname{rk}(N(e^l)) \leq d_l = \operatorname{rk} M(e^l),$$

for  $l = 1, \dots, s$ . But

$$d = \dim_k M = \sum_{l=1}^s d_l \dim_k S_l = \dim_k N = \sum_{l=1}^s d'_l \dim_k S_l,$$

and thus  $\mathbf{d}' = \mathbf{d}$ . □

Let  $B$  be a maximal semisimple subalgebra of  $A$ . We know that

$$B \simeq \prod_{l=1}^s \mathbb{M}_{n_l}(k),$$

where we set  $n_l = \dim_k S_l$ . Denote by  $e^l_{i,j}$ ,  $l = 1, \dots, s$ ,  $i, j = 1, \dots, n_l$  the canonical basis of  $B$ , and set  $e = \sum_{l=1}^s e^l_{1,1}$ . Then  $eAe$  is a basic algebra Morita equivalent to  $A$ . There is a quiver  $Q$  with the set of vertices  $\{1, \dots, s\}$  together with an admissible ideal  $J$  in  $kQ$  and an algebra isomorphism  $\Phi: eAe \rightarrow kQ/J$  such that  $\Phi(e^l_{1,1}) = \varepsilon_l + J$ .

The inclusion  $\varphi: B \rightarrow A$  of  $k$ -algebras induces a regular morphism  $\varphi^{\mathbf{d}}: \operatorname{mod}_A^{\mathbf{d}} \rightarrow \operatorname{mod}_B^{\mathbf{d}}$ , which restricts to a regular  $\operatorname{GL}_d$ -equivariant morphism  $p: \operatorname{mod}_A^{\mathbf{d}} \rightarrow \operatorname{mod}_B^{\mathbf{d}}$ . Bongartz showed in [3] that the fiber of some special element  $E \in \operatorname{mod}_B^{\mathbf{d}}$  is isomorphic to  $\operatorname{rep}_{Q,J}^{\mathbf{d}}$ . In fact, he proved that  $p$  is a fiber bundle with fiber  $p^{-1}E$ . We now recall

his construction and describe explicitly a closed immersion  $\eta: \text{rep}_{Q,J}^d \rightarrow \text{mod}_A^d$  which is an isomorphism onto  $p^{-1}E$ . First we need some more notation.

Recall that  $d = \sum_{l=1}^s n_l d_l$ . According to the decomposition

$$1_A = \sum_{l \leq s} \sum_{i \leq n_l} e_{i,i}^l$$

into a sum of primitive orthogonal idempotents, we subdivide a  $d \times d$ -matrix  $W$  first into  $s^2$  “large” blocks, the block  $W^{l',l''}$  being of size  $n_{l'} d_{l'} \times n_{l''} d_{l''}$ ,  $l', l'' \leq s$ , and then we subdivide each block  $W^{l',l''}$  into  $n_{l'} n_{l''}$  blocks, the block  $W_{i,j}^{l',l''}$  being of size  $d_{l'} \times d_{l''}$ ,  $i \leq n_{l'}$ ,  $j \leq n_{l''}$ . In order to handle these blocks we introduce the obvious injective scheme morphisms

$$t_{i,j}^{l',l''}: \mathbb{M}_{d_{l'} \times d_{l''}} \rightarrow \mathbb{M}_d, \quad l', l'' \leq s, \quad i \leq n_{l'}, \quad j \leq n_{l''}.$$

We define a subfunctor  $E$  of  $\text{mod}_B^d$  by  $E(R)(e_{i,j}^l) = t_{i,j}^{l,l}(1_{n_l})$  for a commutative  $k$ -algebra  $R$ , where  $1_n$  denotes the identity matrix in  $\mathbb{M}_n(R)$ . So  $E$  is a closed point of the scheme  $\text{mod}_B^d$ . Using the decomposition of an element  $a \in A$ ,

$$a = \left( \sum_{l' \leq s} \sum_{i \leq n_{l'}} e_{i,i}^{l'} \right) \cdot a \cdot \left( \sum_{l'' \leq s} \sum_{j \leq n_{l''}} e_{j,j}^{l''} \right) = \sum_{l', l'' \leq s} \sum_{i \leq n_{l'}} \sum_{j \leq n_{l''}} e_{i,1}^{l'} \cdot (e_{1,i}^{l'} \cdot a \cdot e_{j,1}^{l''}) \cdot e_{1,j}^{l''},$$

and the fact that  $e_{1,i}^{l'} \cdot a \cdot e_{j,1}^{l''}$  belongs to  $eAe$ , we define the scheme morphism

$$\eta: \text{rep}_{Q,J}^d \rightarrow \text{mod}_A^d, \quad (\eta N)(a) = \sum_{l', l'' \leq s} \sum_{i \leq n_{l'}} \sum_{j \leq n_{l''}} t_{i,j}^{l',l''} (N(\Phi(e_{1,i}^{l'} \cdot a \cdot e_{j,1}^{l''}))).$$

Then  $\eta$  is an isomorphism onto  $p^{-1}(E)$ . Note that if we view elements of  $\text{rep}_{Q,J}^d(k)$  and  $\text{mod}_A^d(k)$  as representations and modules, respectively, then the map

$$\eta(k): \text{rep}_{Q,J}^d(k) \rightarrow \text{mod}_A^d(k)$$

is in accordance with an equivalence between the category of representations of  $(Q, J)$  and the category of  $A$ -modules.

**Proposition 5.2.** *With the above notations we have*

$$\eta^{-1}(\mathcal{C}_{\eta M}) = \mathcal{C}_M \subseteq \text{rep}_{Q,J}^d$$

for any  $M \in \text{rep}_{Q,J}^d(k)$ .

*Proof.* The result is a consequence of the following two facts.

- (1) For any  $p \times q$ -matrix  $\underline{a}$  with coefficients in  $A$  there are  $p' + q'$  vertices  $u_1, \dots, u_{p'}, v_1, \dots, v_{q'}$  of  $Q$  and elements  $\omega_{i',j'} \in \varepsilon_{u_{i'}} \cdot kQ/J \cdot \varepsilon_{v_{j'}}$  yielding  $\underline{\omega}(\underline{a}) = (\omega_{i',j'})$  such that

$$\mathcal{C}_{M, \underline{\omega}(\underline{a})} = \eta^{-1} \mathcal{C}_{\eta M, \underline{a}}.$$

- (2) For any  $p', q'$ , any vertices  $u_1, \dots, u_{p'}, v_1, \dots, v_{q'}$  of  $Q$  and any elements  $\omega_{i',j'} \in \varepsilon_{u_{i'}} \cdot kQ/J \cdot \varepsilon_{v_{j'}}$  with  $\underline{\omega} = (\omega_{i',j'})$ , there is a  $p' \times q'$ -matrix  $\underline{a}(\underline{\omega})$  with coefficients in  $A$  such that

$$\mathcal{C}_{M, \underline{\omega}} = \eta^{-1} \mathcal{C}_{\eta M, \underline{a}(\underline{\omega})}.$$

In order to prove (1) we first construct  $\underline{\omega}(a)$  for  $a \in A$  such that  $N(\underline{\omega}(a)) = (\eta N)(a)$  for  $N \in \text{rep}_{Q,J}^d$ : We set  $p' = q' = n = \sum_{l=1}^s n_l$ , choose  $u_{(l,i)} = v_{(l,i)} = l \in Q_0$  for  $l = 1, \dots, s, i = 1, \dots, n_s$ , and set  $\underline{\omega} = (\omega_{(l',i),(l'',j)})$  with

$$\omega_{(l',i),(l'',j)}(a) = \Phi(e_{1,i}^{l'} \cdot a \cdot e_{j,1}^{l''}) \in \varepsilon_{l'} \cdot kQ/J \cdot \varepsilon_{l''}.$$

By the definition of  $\eta$ , we have

$$(\eta N)(a) = \sum_{l', l'' \leq s} \sum_{i \leq n_{l'}} \sum_{j \leq n_{l''}} t_{i,j}^{l', l''} (N(\Phi(e_{1,i}^{l'} \cdot a \cdot e_{j,1}^{l''}))) = N(\underline{\omega}(a)).$$

as desired. For a  $p \times q$ -matrix  $\underline{a} = (a_{i',j'})$  with coefficients in  $A$ , we set  $\underline{\omega}(\underline{a}) = (\underline{\omega}(a_{i',j'}))$ . As above, we conclude that

$$(\eta N)(\underline{a}) = N(\underline{\omega}(\underline{a})).$$

In particular,  $r = 1 + \text{rk}(\eta M)(\underline{a}) = 1 + \text{rk} M(\underline{\omega}(\underline{a}))$ . As a consequence we obtain that, for any commutative  $k$ -algebra  $R$ ,  $I_r(N(\underline{\omega}(\underline{a}))) = 0$  if and only if  $I_r((\eta N)(\underline{a})) = 0$ , for any  $N \in \text{rep}_{Q,J}^d(R)$ , which is equivalent to  $\mathcal{C}_{M, \underline{\omega}(\underline{a})}(R) = (\eta^{-1} \mathcal{C}_{\eta M, \underline{a}})(R)$ .

For the proof of (2), we set  $\underline{a}(\underline{\omega}) = (\Phi^{-1} \omega_{i',j'})$  for  $\underline{\omega} = (\omega_{i',j'})$ . For any commutative  $k$ -algebra  $R$  and any  $N \in \text{rep}_{Q,J}^d(R)$ , the only possibly non-zero entries of the  $d \times d$ -matrix  $(\eta N)(\Phi^{-1} \omega_{i',j'}) = t_{1,1}^{u_{i'}, v_{j'}} (N(\omega_{u_{i'}, v_{j'}}))$  sit in the small block in the upper left corner of the big block corresponding to  $l' = u_{i'}, l'' = v_{j'}$ . Therefore the rows and the columns of the  $p'd \times q'd$ -matrix  $(\eta N)(\underline{a}(\underline{\omega}))$  can be permuted in such a way that the upper left corner becomes  $N(\underline{\omega})$  and all other entries are zero. In other words, there are invertible matrices, in fact permutation matrices,  $U$  and  $V$  such that

$$U \cdot (\eta N)(\underline{a}(\underline{\omega})) \cdot V = \begin{pmatrix} N(\underline{\omega}) & 0 \\ 0 & 0 \end{pmatrix}.$$

Then clearly  $r = 1 + \text{rk}(\eta M)(\underline{a}(\underline{\omega})) = 1 + \text{rk} M(\underline{\omega})$  and by Lemma 2.2 we have

$$I_r((\eta N)(\underline{a}(\underline{\omega}))) = I_r \begin{pmatrix} N(\underline{\omega}) & 0 \\ 0 & 0 \end{pmatrix} = I_r(N(\underline{\omega})).$$

Therefore  $I_r(N(\underline{\omega})) = 0$  if and only if  $I_r((\eta N)(\underline{a}(\underline{\omega}))) = 0$ , and we conclude that  $\mathcal{C}_{M,\underline{\omega}}(R) = (\eta^{-1}\mathcal{C}_{\eta M,\underline{a}(\underline{\omega})})(R)$ .  $\square$

Following Hesselink (see (1.7) in [11]) we call two pointed schemes  $(\mathcal{X}, x_0)$  and  $(\mathcal{Y}, y_0)$  smoothly equivalent if there are smooth morphisms  $f: \mathcal{Z} \rightarrow \mathcal{X}$ ,  $g: \mathcal{Z} \rightarrow \mathcal{Y}$  sending a point  $z_0 \in \mathcal{Z}$  to  $x_0$  and  $y_0$ , respectively. This is an equivalence relation and an equivalence class will be denoted by  $\text{Sing}(\mathcal{X}, x_0)$  and called the type of singularity of  $\mathcal{X}$  at  $x_0$ . Assuming  $\text{Sing}(\mathcal{X}, x_0) = \text{Sing}(\mathcal{Y}, y_0)$ , the scheme  $\mathcal{X}$  is regular (or reduced, normal, Cohen–Macaulay, respectively) at  $x_0$  if and only if the same is true for the scheme  $\mathcal{Y}$  at  $y_0$  (see [10], Section 17, for more information about smooth morphisms).

**Theorem 5.3.** *Let  $\eta: \text{rep}_{Q,J}^d \rightarrow \text{mod}_A^d$  be the morphism defined above. Suppose  $M$  and  $M'$  in  $\text{rep}_{Q,J}^d(k)$  are such that  $M'$  belongs to  $\mathcal{C}_M(k)$ . Then  $\eta M'$  belongs to  $\mathcal{C}_{\eta M}(k)$  and*

$$\text{Sing}(\mathcal{C}_{\eta M}, \eta M') = \text{Sing}(\mathcal{C}_M, M').$$

*Proof.* The orbit map  $\psi: \text{GL}_d \rightarrow \text{mod}_B^d$  defined by  $\psi(g) = g * E$  is smooth and induces an isomorphism of schemes  $\text{GL}_d / \text{GL}_d \rightarrow \text{mod}_B^d$ , where  $\text{GL}_d = \prod_{l=1}^s \text{GL}_{d_l}$  is embedded into  $\text{GL}_d$  via

$$(g_1, \dots, g_s) \mapsto \sum_{l=1}^s \sum_{i=1}^{n_l} t_{i,i}^{l,l}(g_l).$$

It is not hard to see (compare e.g. [6]) that the diagram

$$\begin{array}{ccc} \text{GL}_d \times \text{rep}_{Q,J}^d & \xrightarrow{\lambda} & \text{mod}_A^d \\ \downarrow \pi & & \downarrow p \\ \text{GL}_d & \xrightarrow{\psi} & \text{mod}_B^d \xrightarrow{\simeq} \text{GL}_d / \text{GL}_d \end{array}$$

is a pullback, where  $\pi$  is the projection to the first factor and  $\lambda(g, N) = g * \eta N$ . Note that  $\lambda$  is smooth as smoothness is preserved under base change. As  $\lambda(k)$  is surjective and thus contains  $\eta M'$  in its image, it is enough to show that  $\lambda^{-1}\mathcal{C}_{\eta M} = \text{GL}_d \times \mathcal{C}_M$ .

A pair  $(g, N) \in \text{GL}_d(R) \times \mathcal{C}_{\eta M}(R)$  belongs to  $(\lambda^{-1}\mathcal{C}_{\eta M})(R)$ , for a commutative  $k$ -algebra  $R$ , if and only if  $g * \eta N \in \mathcal{C}_{\eta M}(R)$ . As by Lemma 2.2  $\mathcal{C}_{\eta M}(R)$  is stable under  $\text{GL}_d(R)$ , this is equivalent to  $\eta N \in \mathcal{C}_{\eta M}(R)$ , which is in turn equivalent to  $(g, N) \in \text{GL}_d(R) \times \mathcal{C}_M(R)$  by Proposition 5.2.  $\square$

**6. Hom-controlled exact functors**

Let  $\varphi: A \rightarrow B$  be a homomorphism of finite dimensional algebras and  $\varphi^*: \text{mod } B \rightarrow \text{mod } A$  the induced change of scalars functor. For a  $B$ -module  $M$  we will use the notation  $M = {}_B M$  and  $\varphi^*(M) = {}_A M$ . Thus  $\varphi^d(\mathcal{O}_{BM}) = \mathcal{O}_{AM}$  for any module  $M$  in  $\text{mod}_B^d(k)$ .

Following [17], we call an exact functor  $\mathcal{F}: \text{mod } B \rightarrow \text{mod } A$  hom-controlled, if there is a bilinear form  $\xi: K_0(B) \times K_0(B) \rightarrow \mathbb{Z}$  such that

$$[\mathcal{F}U, \mathcal{F}V]_A - [U, V]_B = \xi(\mathbf{dim } U, \mathbf{dim } V)$$

for any  $U, V \in \text{mod } B$ . Here and later on, we abbreviate  $\dim_k \text{Hom}_B(U, V)$  by  $[U, V]_B$ , for any  $U, V \in \text{mod } B$  and similarly for  $A$ -modules.

Assume now that the functor  $\varphi^*$  is hom-controlled. It follows from Theorem 1.1 of [17] that the restriction of  $\varphi^d$

$$\bar{\mathcal{O}}_{BM} \rightarrow \bar{\mathcal{O}}_{AM}$$

is a smooth morphism. The aim of this section is to show this is still true if we replace the orbit closures by the rank schemes  $\mathcal{C}_{BM}$  and  $\mathcal{C}_{AM}$ .

Let  $L$  be a finite dimensional  $A$ -module and  $t \in \mathbb{N}$ . We choose a  $p \times q$ -matrix  $\underline{a}$  such that  $\text{Coker}(v_{\underline{a}})$  is  $A$ -isomorphic to  $L$ . Let  $\text{mod}_{A,L,t}^d$  be the closed subscheme of  $\text{mod}_A^d$  defined by the ideal  $I_{1+qd-t}(X_A^d(\underline{a}))$  in  $k[\text{mod}_A^d]$ . The proof of Lemma 3.2 can easily be generalized to show that this ideal is determined uniquely by  $t$  and the isomorphism class of  $L$ . By  $(\text{mod}_{A,L,t}^d)^0$  we denote the open subscheme of  $\text{mod}_{A,L,t}^d$  whose  $k$ -points are the modules  $N$  with  $[L, N]_A = t$ .

It has been proved in Section 4 of [17] that  $\varphi^d$  restricts to a smooth morphism from  $\text{mod}_B^d$  to  $(\text{mod}_{A,AB,t}^d)^0$  for any  $\mathbf{d} \in K_0(B)$ , where  $d$  is the common dimension of all modules in  $\text{mod}_B^d$  and  $t = d + \xi(\mathbf{dim } B, \mathbf{d})$ . We denote by

$$\psi: \text{mod}_B^d \rightarrow \text{mod}_{A,AB,t}^d$$

the composition of this morphism with the open immersion into  $\text{mod}_{A,AB,t}^d$ , which is still smooth.

**Theorem 6.1.** *Let  $\varphi: A \rightarrow B$  be an algebra homomorphism such that  $\varphi^*$  is a hom-controlled exact functor and fix  $M \in \text{mod}_B^d(k)$ . Then the morphism  $\psi$  restricts to a morphism*

$$\mathcal{C}_{BM} \rightarrow \mathcal{C}_{AM},$$

*which is smooth.*

*Proof.* We know that  $\mathcal{C}_{BM}$  is a subscheme of  $\text{mod}_B^d$ . Thus the claim will be proved if we can show that  $\mathcal{C}_{AM}$  is a subscheme of  $\text{mod}_{A,AB,t}^d$ , where  $t = d + \xi(\mathbf{dim } B, \mathbf{d})$ ,

and that  $\psi^{-1}(\mathcal{C}_{AM}) = \mathcal{C}_{BM}$ , or equivalently that

$$(\varphi^d)^{-1}\mathcal{C}_{AM} \cap \text{mod}_B^d = \mathcal{C}_{BM}.$$

The first part is easy, because

$$[{}_AB, {}_AM]_A = [{}_BB, {}_BM]_B + \xi(\mathbf{dim} B, \mathbf{dim} M) = t$$

and consequently,  $I_{1+qd-t}(X_A^d(\underline{a})) = \mathcal{I}_{AM, AB}$ , where  $\underline{a}$  is a  $p \times q$ -matrix with coefficients in  $A$  and with  ${}_AB = \text{Coker } v_{\underline{a}}$ . The inclusion  $\mathcal{C}_{BM} \subseteq (\varphi^d)^{-1}\mathcal{C}_{AM} \cap \text{mod}_B^d$  follows from Lemma 3.5 and Lemma 5.1. In order to prove the reverse inclusion we will show that, for any  $B$ -module  ${}_BL$ , we have

$$\mathcal{I}_{BM, BL} \subseteq k[\text{mod}_B^d] \cdot (\varphi^d)^* \mathcal{I}_{AM, AL} + \mathcal{I}(\text{mod}_B^d) \quad (6.1)$$

in  $k[\text{mod}_B^d]$ , where  $\mathcal{I}(\text{mod}_B^d)$  is the ideal defining  $\text{mod}_B^d$ .

Let  $L$  be a finite dimensional  $B$ -module. Choosing a finite free presentation of  ${}_AL$  we obtain the exact sequence of  $A$ -modules

$$A^p \xrightarrow{v_{\underline{a}}} A^q \rightarrow {}_AL \rightarrow 0$$

for some  $p, q \geq 1$  and a  $p \times q$ -matrix  $\underline{a}$  with coefficients in  $A$ . We apply the tensor functor  $B \otimes_A (-)$  to get another exact sequence

$$B^p \xrightarrow{v_{\varphi(\underline{a})}} B^q \rightarrow B \otimes_A L \rightarrow 0.$$

Using the homomorphism  $\varphi$  we have a left and a right  $A$ -module structure on  $B$ , and the functor  $\varphi^*$  can be identified with the functor  ${}_AB \otimes_B (-)$  as well as with  $\text{Hom}_B({}_B B_A, -)$ . Observe that

$$B \otimes_A L = B \otimes_A \varphi^*({}_BL) \simeq B \otimes_A B \otimes_B L = \Omega \otimes_B L,$$

where  $\Omega$  is the  $B$ - $B$ -bimodule  $B \otimes_A B$ , and that for any  $B$ -module  $Y$  we have

$$\begin{aligned} \text{Hom}_A({}_AL, {}_AY) &= \text{Hom}_A({}_AB \otimes_B L, \text{Hom}_B({}_B B_A, {}_BY)) \\ &\simeq \text{Hom}_B({}_B B \otimes_A B \otimes_B L, {}_BY) \\ &= \text{Hom}_B(\Omega \otimes_B L, {}_BY). \end{aligned}$$

As  $\varphi^*$  is hom-controlled, we obtain

$$[\Omega \otimes_B L, Y]_B - [L, Y]_B = \xi(\mathbf{dim} L, \mathbf{dim} Y)$$

for any  $B$ -module  $Y$ .

Let  $\{P_1, \dots, P_n\}$  be a complete set of pairwise non-isomorphic indecomposable projective  $B$ -modules and  $S_i = P_i / \text{rad}(P_i)$  for  $i \leq n$ . Note that  $\{S_1, \dots, S_n\}$  is a complete set of pairwise non-isomorphic simple  $B$ -modules.

Let  $s_i = \xi(\mathbf{dim} L, \mathbf{dim} S_i)$  for  $i \leq n$  and  $P_L = \bigoplus_i P_i^{s_i}$ , and let  $y_i$  denote the  $i$ -th coordinate of  $\mathbf{dim} Y$ . Then

$$\begin{aligned} [P_L, Y] &= \sum_{i=1}^n s_i \cdot [P_i, Y] = \sum_{i=1}^n \xi(\mathbf{dim} L, \mathbf{dim} S_i) \cdot y_i \\ &= \xi(\mathbf{dim} L, \mathbf{dim}(\bigoplus_{i=1}^n S_i^{y_i})) = \xi(\mathbf{dim} L, \mathbf{dim} Y), \end{aligned}$$

and consequently,

$$[\Omega \otimes_B L, Y] = [L \oplus P_L, Y].$$

The latter holds for any finite dimensional  $B$ -module  $Y$ , hence  $\Omega \otimes_B L \simeq L \oplus P_L$ , by Auslander’s theorem. This implies that the ideal generated by  $(\varphi^d)^* \mathcal{I}_{AM,AL}$  equals  $\mathcal{I}_{BM,B(L \oplus P_L)}$ .

If  $P_L = 0$ , the inclusion (6.1) clearly holds. Otherwise, choose matrices  $\underline{b}'$  and  $\underline{b}''$  with coefficients in  $B$  such that the Coker( $v_{\underline{b}'}$ )  $\simeq L$  and Coker( $v_{\underline{b}''}$ )  $\simeq P_L$ . Let  $r_1 = \text{rk } M(\underline{b}')$ ,  $r_2 = \text{rk } M(\underline{b}'')$ , and set  $X = X_B^d$ . Then

$$\mathcal{I}_{BM,B(L \oplus P_L)} = I_{1+r_1+r_2} \begin{pmatrix} X(\underline{b}') & 0 \\ 0 & X(\underline{b}'') \end{pmatrix} \supseteq I_{1+r_1}(X(\underline{b}')) \cdot I_{r_2}(X(\underline{b}')),$$

by Lemma 2.3. Obviously  $I_{1+r_1}(X(\underline{b}')) = \mathcal{I}_{BM,BL}$  and therefore

$$\mathcal{I}_{BM,B(L \oplus P_L)} + \mathcal{I}(\text{mod}_B^d) \supseteq \mathcal{I}_{BM,BL} \cdot (I_{r_2}(X(\underline{b}'')) + \mathcal{I}(\text{mod}_B^d)).$$

Thus it suffices to show that

$$I_{r_2}(X(\underline{b}'')) + \mathcal{I}(\text{mod}_B^d) = k[\text{mod}_B^d].$$

Let  $N$  be a module in  $\text{mod}_B^d(k)$ . The condition that  $I_{r_2}(X(\underline{b}''))$  vanishes on  $N$  means  $[P_L, N] > [P_L, M]$  while  $\mathcal{I}(\text{mod}_B^d)$  vanishes on  $N$  if and only if  $\mathbf{dim} N = \mathbf{dim} M$ . Since  $\mathbf{dim} N = \mathbf{dim} M$  implies  $[P_L, N] = [P_L, M]$ , there exists no point  $N$  on which the ideal  $I_{r_2}(X(\underline{b}'')) + \mathcal{I}(\text{mod}_B^d)$  vanishes.  $\square$

Theorem 1.2 in [17] says that

$$\text{Sing}(\overline{\mathcal{O}}_{\mathcal{F}M}, \mathcal{F}M') = \text{Sing}(\overline{\mathcal{O}}_M, M')$$

for a hom-controlled exact functor  $\mathcal{F}$ ,  $M \in \text{mod}_A^d(k)$  and  $M' \in \overline{\mathcal{O}}_M$ . The proof can be adapted to rank schemes, yielding the following result. The only slight difficulty is taken care of by the lemma following the theorem. Recall that if  $M'$  is a point of  $\mathcal{C}_M(k)$ , the modules  $M$  and  $M'$  have the same dimension vector, and so do their images  $\mathcal{F}(M)$  and  $\mathcal{F}(M')$  under an exact functor.

**Theorem 6.2.** *Let  $\mathcal{F} : \text{mod } B \rightarrow \text{mod } A$  be a hom-controlled exact functor and fix  $M, M' \in \text{mod}_B^d(k)$  with  $M' \in \bar{\mathcal{O}}_M$ . Let  $e$  be the common dimension vector of  $\mathcal{F}M$  and  $\mathcal{F}M'$ . Identifying  $\mathcal{F}M$  and  $\mathcal{F}M'$  with the corresponding elements in  $\text{mod}_A^e(k)$  we obtain that  $\mathcal{F}M' \in \mathcal{C}_{\mathcal{F}M}$  and*

$$\text{Sing}(\mathcal{C}_{\mathcal{F}M}, \mathcal{F}M') = \text{Sing}(\mathcal{C}_M, M').$$

**Lemma 6.3.** *Let  $B = C \times D$  be the product of an algebra  $C$  with a semisimple algebra  $D$ , both finite dimensional, fix a  $B$ -module  $M = (M_1, M_2)$ , and choose  $M' = (M'_1, M'_2) \in \mathcal{C}_M(k)$ . Then we have*

$$\text{Sing}(\mathcal{C}_M, M') = \text{Sing}(\mathcal{C}_{M_1}, M'_1).$$

*Proof.* The easiest way to see this is to replace the algebras by quivers and relations using Theorem 5.3. Then we have

$$\mathcal{C}_M = \mathcal{C}_{M_1} \times \mathcal{C}_{M_2}, \quad \mathcal{C}_{M'} = \mathcal{C}_{M'_1} \times \mathcal{C}_{M'_2}.$$

As  $D$  is semisimple, its quiver consists of some vertices but no arrows, and thus  $\mathcal{C}_{M_2} = \mathcal{C}_{M'_2} = \{M_2\} = \{M'_2\}$ .  $\square$

The above theorem remains true if modules are replaced by representations of quivers, by Theorem 5.3. In particular, applying the theorem to the exact functors constructed in [1] and [2] we may generalize Theorem 4.1 as follows.

**Theorem 6.4.** *Let  $M$  be a representation in  $\text{rep}_Q^d(k)$ , where  $Q$  is a Dynkin quiver of type  $\mathbb{A}$ . Then the ideal  $\mathcal{I}_M$  is radical and  $\mathcal{C}_M = \bar{\mathcal{O}}_M$ .*

Let us describe the ideal  $\mathcal{I}_M$  explicitly. We know from Corollary 3.6 that  $\mathcal{I}_M = \sum \mathcal{I}_{M,L}$ , where  $L$  ranges over all indecomposable representations of  $Q$ . Suppose the underlying graph of  $Q$  is

$$1 \text{ --- } 2 \text{ --- } \cdots \text{ --- } n.$$

Independently of the orientations of the arrows, an indecomposable  $L$  is given by an interval  $L = [l, l']$  in  $[1, n]$ , for some  $l \leq l'$ : Each vertex in  $[l, l']$  is represented by  $k$ , each arrow between such vertices by the matrix (1). Denote the full subquiver of  $Q$  with vertex set  $[l, l']$  by  $Q_{[l, l']}$ . We associate with  $L$  the sequence  $l \leq v_1 < \cdots < v_q \leq l'$  of all sources of  $Q_{[l, l']}$  and the sequence  $l - 1 \leq u_1 < \cdots < u_p \leq l' + 1$  consisting of all sinks in  $Q_{[l, l']}$  distinct from  $l, l'$  in addition to

$$\begin{cases} l - 1 & \text{if } 1 < l \text{ and there is an arrow } l - 1 \leftarrow l \in Q_1, \\ l' + 1 & \text{if } l' < n \text{ and there is an arrow } l' \rightarrow l' + 1 \in Q_1. \end{cases}$$

For any  $u_i$  there is either some  $v_{j'} < u_i$  and a path  $\omega_{i,j'}: v_{j'} \rightarrow u_i$  in  $Q$  or some  $v_{j''} > u_i$  and a path  $\omega_{i,j''}: v_{j''} \rightarrow u_i$  in  $Q$  or both, in which case we must have  $j'' = j' + 1$ . The  $p \times q$ -matrix  $\underline{\omega}$  corresponding to  $L = [l, l']$  has all its entries 0, except for those just described.

In the special case

$$Q = 1 \xrightarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 3 \xleftarrow{\alpha_3} 4 \xrightarrow{\alpha_4} 5,$$

the matrices to be considered are

$$\begin{aligned} &(\alpha_1), \quad (\alpha_2), \quad (\alpha_3), \quad (\alpha_4), \\ &(\alpha_1 \ \alpha_2), \quad (\alpha_2 \circ \alpha_3), \quad \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix}, \\ &(\alpha_1 \ \alpha_2 \circ \alpha_3), \quad \begin{pmatrix} \alpha_2 \circ \alpha_3 \\ \alpha_4 \end{pmatrix}, \quad \begin{pmatrix} \alpha_1 & \alpha_2 \circ \alpha_3 \\ 0 & \alpha_4 \end{pmatrix}. \end{aligned}$$

### 7. Tangent spaces

Let  $N$  belong to  $\mathcal{C}_M(k)$ . The main aim of this section is to describe the tangent space  $\mathcal{T}_{\mathcal{C}_M, N}$  in terms of selfextensions of the module  $N$ .

The tangent space  $\mathcal{T}_{\text{mod}_A^d, N}$  can be identified with the space of 1-cocycles  $\mathbb{Z}_A^1(N, N)$ , that is, with the set of  $k$ -linear maps  $Z: A \rightarrow \mathbb{M}_d(k)$  with the property that  $Z(a_1 a_2) = N(a_1)Z(a_2) + Z(a_1)N(a_2)$  for any  $a_1, a_2 \in A$ . Note that from a 1-cocycle  $Z$  we obtain a module structure on  $k^d \oplus k^d$  given by

$$\begin{pmatrix} N & Z \\ 0 & N \end{pmatrix} (a) = \begin{pmatrix} N(a) & Z(a) \\ 0 & N(a) \end{pmatrix}$$

and that the sequence

$$\varphi(Z): 0 \rightarrow N \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \begin{pmatrix} N & Z \\ 0 & N \end{pmatrix} \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} N \rightarrow 0$$

is exact.

The tangent space  $\mathcal{T}_{\mathcal{O}_N, N}$  can be identified with the space of 1-coboundaries  $\mathbb{B}_A^1(N, N) = \{h \cdot N - N \cdot h; h \in \mathbb{M}_d(k)\}$ . By [9], Proposition 1.1, the map  $\varphi$  induces an isomorphism, called Voigt's isomorphism,

$$\mathcal{T}_{\text{mod}_A^d, N} / \mathcal{T}_{\mathcal{O}_N, N} \simeq \mathbb{Z}_A^1(N, N) / \mathbb{B}_A^1(N, N) = \text{Ext}_A^1(N, N).$$

Since  $\mathcal{T}_{\mathcal{O}_N, N} \subseteq \mathcal{T}_{\mathcal{C}_M, N} \subseteq \mathcal{T}_{\text{mod}_A^d, N}$ , the tangent space  $\mathcal{T}_{\mathcal{C}_M, N}$  corresponds to a subspace of  $\mathbb{Z}_A^1(N, N)$  containing  $\mathbb{B}_A^1(N, N)$ , which we now describe.

Let  $\mathcal{F}$  and  $\mathcal{F}'$  be complete sets of pairwise non-isomorphic indecomposable modules  $X$  and  $X'$  such that  $[N, X] = [M, X]$  and  $[X', N] = [X', M]$ , respectively. Set

$$\begin{aligned} \mathcal{E}(Y, Z) &= \{[\sigma: 0 \rightarrow Z \rightarrow W \rightarrow Y \rightarrow 0]_{\sim} \in \text{Ext}_A^1(Y, Z); \delta_{\sigma}(X) = 0 \text{ for all } X \in \mathcal{F}\} \\ &= \{[\sigma: 0 \rightarrow Z \rightarrow W \rightarrow Y \rightarrow 0]_{\sim} \in \text{Ext}_A^1(Y, Z); \delta'_{\sigma}(X') = 0 \text{ for all } X' \in \mathcal{F}'\}. \end{aligned}$$

for two  $A$ -modules  $Y, Z$ , where

$$\begin{aligned} \delta_{\sigma}(X) &= \dim_k \text{Hom}_A(Z \oplus Y, X) - \dim_k \text{Hom}_A(W, X), \\ \delta'_{\sigma}(X') &= \dim_k \text{Hom}_A(X', Y \oplus Z) - \dim_k \text{Hom}_A(X', W). \end{aligned}$$

Note that the pushout or pullback of an exact sequence in  $\mathcal{E}$  belongs to  $\mathcal{E}$  again. As a consequence,  $\mathcal{E}(-, -)$  is a  $k$ -subfunctor of  $\text{Ext}_A^1(-, -)$ .

**Proposition 7.1.** *For  $N \in \mathcal{C}_M(k)$ , Voigt's isomorphism restricts to an isomorphism*

$$\mathcal{T}_{\mathcal{C}_M, N} / \mathcal{T}_{\mathcal{O}_N, N} \simeq \mathcal{E}(N, N).$$

The following corollary is an immediate consequence.

**Corollary 7.2.** *Let  $N$  be a point of  $\bar{\mathcal{O}}_M$ . Then  $\text{codim}(M, N) \leq \dim_k \mathcal{E}(N, N)$ , and equality holds if and only if  $N$  is a regular point of  $\mathcal{C}_M$ .*

By definition,  $\text{codim}(M, N) = \dim \mathcal{O}_M - \dim \mathcal{O}_N$ .

We will prove Proposition 7.1 in several steps. We begin by characterizing the tangent space to the scheme  $\mathcal{V}_{p \times q}^r$  at some matrix  $N \in \mathcal{V}_{p \times q}^r(k)$  as a subspace of the tangent space of  $\mathbb{M}_{p \times q}$  at  $N$ , which we identify with  $\mathbb{M}_{p \times q}(k)$ .

**Lemma 7.3.** *Fix  $r \leq p, q$ , and choose a matrix  $N \in \mathcal{V}_{p \times q}^r(k)$ . Then*

$$\mathcal{T}_{\mathcal{V}_{p \times q}^r, N} = \begin{cases} \mathbb{M}_{p \times q}(k) & \text{if } \text{rk } N < r, \\ \{D \in \mathbb{M}_{p \times q}(k); \text{rk} \begin{pmatrix} N & D \\ 0 & N \end{pmatrix} = 2r\} & \text{if } \text{rk } N = r. \end{cases}$$

*Proof.* The algebraic group scheme  $\text{GL}_p \times \text{GL}_q$  acts on  $\mathbb{M}_{p \times q}$  via  $(g, h) * N' = g \cdot N' \cdot h^{-1}$ , and we know that  $N = g \cdot \begin{pmatrix} 1_s & 0 \\ 0 & 0 \end{pmatrix} \cdot h^{-1}$  for some  $g \in \text{GL}_p(k), h \in \text{GL}_q(k)$ , where  $s = \text{rk } N$ . As the tangent space to  $\mathcal{V}_{p \times q}^r$  at  $g \cdot N' \cdot h^{-1}$  is  $g \cdot \mathcal{T}_{\mathcal{V}_{p \times q}^r, N'} \cdot h^{-1}$ , it suffices to prove the claim for  $N = \begin{pmatrix} 1_s & 0 \\ 0 & 0 \end{pmatrix}$ .

It is obviously true for  $s < r$ . In case  $s = r$ , we decompose

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$$

into blocks; the size of  $D_{11}$  is  $s \times s$ . A straightforward computation yields that

$$\mathcal{T}_{\mathcal{V}_{p \times q}, N}^r = \left\{ D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}; D_{22} = 0 \right\}.$$

But note that

$$\text{rk} \begin{pmatrix} N & D \\ 0 & N \end{pmatrix} = \text{rk} \begin{pmatrix} 1_r & 0 & D_{11} & D_{12} \\ 0 & 0 & D_{21} & D_{22} \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2r + \text{rk } D_{22}.$$

So  $\text{rk} \begin{pmatrix} N & D \\ 0 & N \end{pmatrix} = 2r$  if and only if  $D_{22} = 0$ , and the lemma is established.  $\square$

Let  $\underline{a}$  be a  $p \times q$ -matrix with coefficients in  $A$ , set  $L = \text{Coker } v_{\underline{a}}$ , and fix  $N \in \mathcal{C}_{M, \underline{a}}(k)$ . Mapping the exact sequence

$$A^p \xrightarrow{v_{\underline{a}}} A^q \longrightarrow L \longrightarrow 0$$

to a module  $N' \in \text{mod}_A^{d'}(k)$  and identifying  $\text{Hom}_A(A, N')$  with  $N'$ , we see that

$$\text{rk } N'(\underline{a}) + \dim_k \text{Hom}_A(L, N') = qd'. \tag{7.1}$$

**Corollary 7.4.** *Using the notions just introduced, we have that the tangent space  $\mathcal{T}_{\mathcal{C}_{M, L, N}} = \mathcal{T}_{\mathcal{C}_{M, \underline{a}, N}}$  equals*

- (1)  $\mathcal{T}_{\text{mod}_A^d, N}$  provided that  $\dim_k \text{Hom}_A(L, M) < \dim_k \text{Hom}_A(L, N)$ ;
- (2)  $\{Z \in \mathcal{T}_{\text{mod}_A^d, N}; \dim_k \text{Hom}_A(L, \begin{pmatrix} N & Z \\ 0 & N \end{pmatrix}) = 2 \dim_k \text{Hom}_A(L, N)\}$  provided that  $\dim_k \text{Hom}_A(L, M) = \dim_k \text{Hom}_A(L, N)$ .

*Proof.* Remember that

$$\mathcal{C}_{M, \underline{a}} = \Theta_{\underline{a}}^{-1}(\mathcal{V}_{pd \times qd}^{\text{rk } M(\underline{a})}).$$

Using (7.1), the corollary is a direct consequence of Lemma 7.3 and the fact that

$$\text{rk} \begin{pmatrix} N & Z \\ 0 & N \end{pmatrix}(\underline{a}) = \text{rk} \begin{pmatrix} N(\underline{a}) & Z(\underline{a}) \\ 0 & N(\underline{a}) \end{pmatrix}. \quad \square$$

Now Proposition 7.1 is easy to prove. Indeed, we have

$$\mathcal{T}_{\mathcal{C}_{M, N}} = \bigcap \mathcal{T}_{\mathcal{C}_{M, L, N}},$$

where the intersection is taken over representatives  $L$  of all isomorphism classes of  $A$ -modules which are finitely presented, or, by Lemma 3.4, even finite dimensional. In case  $\dim_k \text{Hom}_A(L, M) < \dim_k \text{Hom}_A(L, N)$ , this gives no restriction. The condition  $\dim_k \text{Hom}_A(L, M) = \dim_k \text{Hom}_A(L, N)$  is equivalent to  $L \in \text{add } \mathcal{F}'$  by definition, and having the equality

$$\dim_k \text{Hom}_A \left( L, \begin{pmatrix} N & Z \\ 0 & N \end{pmatrix} \right) = 2 \dim_k \text{Hom}_A(L, N)$$

for all  $L \in \text{add } \mathcal{F}'$  is equivalent to  $\varphi(Z) \in \mathcal{E}(N, N)$ .

## 8. Singular loci

In this last section, we assume  $A$  to be representation finite, except for the final remark and example. All  $A$ -modules considered will be finite dimensional, and we fix  $M \in \text{mod}_A^d(k)$ ,  $N \in \overline{\mathcal{O}}_M$ . We denote the Auslander–Reiten quiver of  $A$  by  $\Gamma_A$ . In order to study the singularity of  $\mathcal{C}_M$  at  $N$ , we need some definitions and some preliminary results on source and sink maps, also called approximations by some authors.

We define the shadow  $\mathcal{S}$  of the degeneration from  $M$  to  $N$  to be the set of all meshes in  $\Gamma_A$  which start in a vertex  $X \notin \mathcal{F}$ , or equivalently which stop in a vertex  $X' \notin \mathcal{F}'$ ; the shadow  $\mathcal{S}_\sigma$  of an exact sequence  $\sigma$  of  $A$ -modules consists of all meshes of  $\Gamma_A$  with starting vertex  $Y$  with  $\delta_\sigma(Y) > 0$  or equivalently with ending vertex  $Y'$  with  $\delta'_\sigma(Y') > 0$ . We call an exact sequence

$$\sigma: 0 \longrightarrow Z \longrightarrow W \longrightarrow Y \longrightarrow 0$$

fit for  $(M, N)$  if its class  $[\sigma]$  belongs to  $\mathcal{E}(Y, Z)$ , or equivalently if  $\mathcal{S}_\sigma \subseteq \mathcal{S}$  or  $\delta_\sigma(X) = 0$  for all  $X \in \mathcal{F}$ .

For an  $A$ -module  $Z$ , we call a morphism  $f: Z \rightarrow W$  a universal morphism from  $Z$  to  $\text{add } \mathcal{F}$  if  $W \in \text{add } \mathcal{F}$  and any morphism from  $Z$  to some  $W' \in \text{add } \mathcal{F}$  factors through  $f$ . It is easy to see that universal morphisms from  $Z$  to  $\text{add } \mathcal{F}$  exist. Such a morphism is necessarily injective as all injective indecomposables belong to  $\mathcal{F}$ . A universal morphism  $f: Z \rightarrow W$  is called a source map if any endomorphism  $\varphi$  of  $W$  for which  $\varphi \circ f$  is still universal is invertible. A source map  $f_Z: Z \rightarrow W_Z$  is unique up to isomorphism, and it is characterized by the fact that the morphism  $W_Z \rightarrow V_Z$  in the exact sequence

$$\sigma_Z: 0 \longrightarrow Z \xrightarrow{f_Z} W_Z \longrightarrow V_Z \longrightarrow 0$$

is radical.

Sink maps from  $\text{add } \mathcal{F}'$  to some module  $Y$  are defined dually. We will denote the exact sequence obtained from a sink map  $W'_Y \rightarrow Y$  from  $\text{add } \mathcal{F}'$  to  $Y$  by

$$\sigma'_Y: 0 \longrightarrow U_Y \longrightarrow W'_Y \longrightarrow Y \longrightarrow 0.$$

**Lemma 8.1.** *Let  $Y, Z$  be  $A$ -modules.*

- (1) *The sequences  $\sigma_Z$  and  $\sigma'_Y$  are fit for  $(M, N)$ .*
- (2) *Mapping  $Y$  to  $\sigma_Z$ , we obtain an exact sequence*

$$0 \rightarrow \text{Hom}(Y, Z) \rightarrow \text{Hom}(Y, W_Z) \rightarrow \text{Hom}(Y, V_Z) \rightarrow \mathcal{E}(Y, Z) \rightarrow 0.$$

- (3) *Mapping  $\sigma'_Y$  to  $Z$ , we obtain an exact sequence*

$$0 \rightarrow \text{Hom}(Y, Z) \rightarrow \text{Hom}(W'_Y, Z) \rightarrow \text{Hom}(U_Y, Z) \rightarrow \mathcal{E}(Y, Z) \rightarrow 0.$$

(4) We have  $\mathcal{S}_\sigma \subseteq \mathcal{S}_{\sigma_Z}$  and  $\mathcal{S}_\sigma \subseteq \mathcal{S}_{\sigma'_Y}$  for any exact sequence  $\sigma$  with  $[\sigma] \in \mathcal{E}(Y, Z)$ .

*Proof.* Statement (1) holds by definition. For (2), note that the pullback of  $\sigma_Z$  under any morphism in  $\text{Hom}(Y, V_Z)$  will still have a splitting pushout under any morphism from  $Z$  to  $X \in \text{add } \mathcal{F}$  and thus belongs to  $\mathcal{E}(Y, Z)$ . By the definition of  $\sigma_Z$ , any exact sequence  $\sigma$  with  $[\sigma] \in \mathcal{E}(Y, Z)$  is a pullback of  $\sigma_Z$ . The proof of (3) is dual, and (4) follows from (2) and (3) as shadows cannot grow under pushouts nor under pullbacks.  $\square$

As an immediate consequence we obtain the following corollary.

**Corollary 8.2.**  $\dim_k \mathcal{E}(Y, Z) = \delta_{\sigma'_Y}(Z) = \delta'_{\sigma_Z}(Y)$ .

**Lemma 8.3.** For an  $A$ -module  $X$  the following properties are equivalent:

- (1)  $X \in \text{add } \mathcal{F}$ ,
- (2)  $\mathcal{E}(-, X) = 0$ ,
- (3)  $\mathcal{E}(N, X) = 0$ .

There is a dual statement characterizing  $X' \in \text{add } \mathcal{F}'$ .

*Proof.* The implications from (1) to (2) and from (2) to (3) are immediate. In order to show that (3) implies (1), it is enough to prove the inclusion  $\mathcal{S} \subseteq \mathcal{S}_{\sigma'_N}$ ; in fact then both shadows coincide as  $\sigma'_N$  is fit for  $(M, N)$ . By [16] there is a short exact sequence

$$\sigma: 0 \longrightarrow Z' \longrightarrow Z' \oplus M \longrightarrow N \longrightarrow 0.$$

By the definition of  $\mathcal{F}$ , we know that  $\mathcal{S}_\sigma = \mathcal{S}$ . Therefore  $[\sigma] \in \mathcal{E}(N, Z')$ , which implies that  $\mathcal{S} = \mathcal{S}_\sigma \subseteq \mathcal{S}_{\sigma'_N} \subseteq \mathcal{S}$  by Lemma 8.1 (4).  $\square$

**Lemma 8.4.** The following conditions are equivalent:

- (1)  $U_N$  belongs to  $\text{add}(\mathcal{F}')$ ,
- (2)  $U_Y$  belongs to  $\text{add}(\mathcal{F}')$  for any module  $Y$ ,
- (3)  $V_N$  belongs to  $\text{add}(\mathcal{F})$ ,
- (4)  $V_Z$  belongs to  $\text{add}(\mathcal{F})$  for any module  $Z$ .

*Proof.* Obviously (2) implies (1) and (4) implies (3). Thus, up to duality, it suffices to show that (1) implies (4). Let  $Z$  be a module. As  $W_Z \in \text{add } \mathcal{F}$  and  $W'_N \in \text{add } \mathcal{F}'$ ,

the exact sequences  $\sigma'_N$  and  $\sigma_Z$  induce the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
\mathrm{Hom}_A(W'_N, W_Z) & \longrightarrow & \mathrm{Hom}_A(U_N, W_Z) & \longrightarrow & 0 & & \\
\downarrow & & \downarrow \beta & & & & \\
\mathrm{Hom}_A(W'_N, V_Z) & \xrightarrow{\alpha} & \mathrm{Hom}_A(U_N, V_Z) & \longrightarrow & \mathcal{E}(N, V_Z) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
0 & & \mathcal{E}(U_N, Z) & & & & 
\end{array}$$

As  $U_N$  belongs to  $\mathrm{add} \mathcal{F}'$  by our hypothesis,  $\beta$  is surjective. Then  $\alpha$  is surjective as well, thus  $\mathcal{E}(N, V_Z) = 0$ , which implies  $V_Z \in \mathrm{add} \mathcal{F}$  by Lemma 8.3.  $\square$

We are now ready to give a first characterization of the regularity  $\mathcal{C}_M$  at  $N$ .

**Proposition 8.5.** *The scheme  $\mathcal{C}_M$  is regular at  $N$  if and only if  $\mathcal{E}(M, M) = \{0\}$  and one of the equivalent conditions in Lemma 8.4 holds.*

*Proof.* We compute the difference  $\dim_k \mathcal{E}(N, N) - \mathrm{codim}(M, N)$ . Observe that

$$\mathrm{codim}(M, N) = \delta'_{M,N}(N) + \delta_{M,N}(M).$$

By Corollary 8.2,

$$\begin{aligned}
\dim_k \mathcal{E}(N, N) - \dim_k \mathcal{E}(N, M) &= \delta_{\sigma'_N}(N) - \delta_{\sigma'_N}(M) \\
&= \delta'_{M,N}(U_N \oplus N) - \delta'_{M,N}(W'_N) \\
&= \delta'_{M,N}(U_N \oplus N),
\end{aligned}$$

$$\begin{aligned}
\dim_k \mathcal{E}(N, M) - \dim_k \mathcal{E}(M, M) &= \delta'_{\sigma_M}(N) - \delta'_{\sigma_M}(M) \\
&= \delta_{M,N}(M \oplus V_M) - \delta_{M,N}(W_M) \\
&= \delta_{M,N}(M \oplus V_M).
\end{aligned}$$

Thus

$$\dim_k \mathcal{E}(N, N) - \mathrm{codim}(M, N) = \dim_k \mathcal{E}(M, M) + \delta'_{M,N}(U_N) + \delta_{M,N}(V_M).$$

By Corollary 7.2, the scheme  $\mathcal{C}_M$  is regular at  $N$  if and only if  $\mathcal{E}(M, M) = 0$  and  $\delta'_{M,N}(U_N) = \delta_{M,N}(V_M) = 0$ . As by Lemma 8.4  $\delta'_{M,N}(U_N) = 0$  forces  $\delta_{M,N}(V_M) = 0$ , our claim follows.  $\square$

**Lemma 8.6.** *Assume that  $\mathcal{E}(M, M) = 0$ . Then  $\mathcal{C}_M$  is singular at  $N$  if and only if there exists an indecomposable  $U$  such that the sequence*

$$\sigma_U: 0 \rightarrow U \xrightarrow{f_U} W_U \xrightarrow{g} V_U \rightarrow 0$$

*satisfies the following conditions:*

- (1) The representation  $V_U$  is indecomposable.
- (2) The morphism  $g: W_U \rightarrow V_U$  is a sink map from  $\text{add } \mathcal{F}$  to  $V_U$ .
- (3) If the mesh stopping at some indecomposable  $Y$  belongs to  $\mathcal{S}_{\sigma_U}$ , then  $Y \notin \mathcal{F}$ .

*Proof.* We first prove that  $\mathcal{C}_M$  is singular at  $N$  if these conditions hold. Note that  $\sigma_U$  does not split, as  $U \in \mathcal{F}$  would imply  $V_U = 0$ , but  $V_U$  is indecomposable. Therefore the mesh stopping at  $V_U$  belongs to  $\mathcal{S}_{\sigma_U}$ , and condition (3) implies  $V_U \notin \mathcal{F}$ . The claim then follows from Proposition 8.5 as condition (4) in Lemma 8.4 is violated.

In order to show the converse implication, observe that the surjection  $W_N \rightarrow V_N$  factors through the sink map  $\theta: C \rightarrow V_N$  from  $\text{add } \mathcal{F}$  to  $V_N$ . In particular  $\theta$  is surjective and we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} \sigma_N: & 0 & \longrightarrow & N & \longrightarrow & W_N & \longrightarrow & V_N & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \parallel & & \\ \varphi: & 0 & \longrightarrow & B & \longrightarrow & C & \xrightarrow{\theta} & V_N & \longrightarrow & 0. \end{array}$$

Thus  $\varphi$  is fit for  $(M, N)$ , being a pushout of  $\sigma_N$ . A decomposition of  $V_N$  into a direct sum of submodules yields a corresponding decomposition of  $\varphi$  as a direct sum. We choose a direct summand of  $\varphi$ :

$$\eta: 0 \longrightarrow U \xrightarrow{f} W \xrightarrow{g} V \longrightarrow 0$$

such that  $V$  is indecomposable and does not belong to  $\text{add}(\mathcal{F})$ . As  $g$  is radical,  $f$  is a source map from  $U$  to  $\text{add } \mathcal{F}$ . But a source map from a decomposable module has a decomposable cokernel, and therefore  $U$  must be indecomposable and  $\eta$  is isomorphic to  $\sigma_U$ .

Finally, suppose the mesh stopping at some indecomposable  $Y$  belongs to  $\mathcal{S}_\eta$ . Equivalently, we have  $\delta'_\eta(Y) \neq 0$ . If  $Y$  belongs to  $\mathcal{F}$ , any morphism from  $Y$  to  $V$  factors through  $g$ , and thus  $\delta'_\eta(Y) = 0$ , because  $g$  is a sink map from  $\text{add } \mathcal{F}$  to  $V$ . Our last claim follows. □

**Lemma 8.7.** *Assume that the algebra  $A$  is directed and consider the exact sequence  $\eta$  from Lemma 8.6. Then  $\text{codim}(W, U \oplus V) = 1$ .*

*Proof.* Since  $\eta$  is fit for  $(M, N)$  and  $W$  belongs to  $\text{add}(\mathcal{F})$ , we have  $\delta_\eta(W) = 0$ . Since  $U$  and  $V$  are indecomposable and  $A$  is directed,  $\text{Ext}_A^1(U, U) = \{0\}$  and  $\text{End}_A(V) \simeq k$ . We conclude from the long exact sequences

$$\begin{aligned} 0 &\rightarrow \text{Hom}_A(U, U) \rightarrow \text{Hom}_A(U, W) \rightarrow \text{Hom}_A(U, V) \rightarrow \text{Ext}_A^1(U, U), \\ 0 &\rightarrow \text{Hom}_A(V, U) \rightarrow \text{Hom}_A(V, W) \rightarrow \text{Hom}_A(V, V) \rightarrow \mathcal{E}(V, U) \rightarrow 0 \end{aligned}$$

induced by  $\eta$  that  $\delta'_\eta(U) = 0$  and  $\delta'_\eta(V) = 1$ . Thus

$$\text{codim}(W, U \oplus V) = \delta'_\eta(U \oplus V) + \delta_\eta(W) = 1. \quad \square$$

We end this section with a few remarks on singularities.

**Remark 8.8.** For a finitely generated algebra  $A$  and  $M \in \text{mod}_A^d(k)$ , we have a chain of inclusions of schemes  $\bar{\mathcal{O}}_M \subseteq (\mathcal{C}_M)_{\text{red}} \subseteq \mathcal{C}_M$ . Fix  $N \in \bar{\mathcal{O}}_M$ , and let us compare regularity at  $N$  for  $\bar{\mathcal{O}}_M$ ,  $(\mathcal{C}_M)_{\text{red}}$ , and  $\mathcal{C}_M$ . Clearly, if  $\mathcal{C}_M$  is regular at  $N$ , the subscheme  $(\mathcal{C}_M)_{\text{red}}$  will be as well. Remember that in Example 3.7 we have that  $(\mathcal{C}_M)_{\text{red}} = \bar{\mathcal{O}}_M$  and that the tangent space of  $\bar{\mathcal{O}}_M$  (and thus of  $(\mathcal{C}_M)_{\text{red}}$ ) at  $N$  is a proper subspace of  $\mathcal{T}_{\mathcal{C}_M, N}$ . So there might be cases where  $\mathcal{C}_M$  is singular at  $N$  while  $(\mathcal{C}_M)_{\text{red}}$  is regular at that point.

As  $\bar{\mathcal{O}}_M$  is an irreducible component of  $(\mathcal{C}_M)_{\text{red}}$  by Proposition 1 of [4], we have that regularity of  $(\mathcal{C}_M)_{\text{red}}$  at  $N$  implies regularity of  $\bar{\mathcal{O}}_M$  at  $N$ . The following simplified version of Carlson's example shows that the reverse implication is false.

**Example 8.9.** Let  $Q$  be the quiver

$$1 \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2 \begin{array}{c} \xrightarrow{\gamma} \\ \xrightarrow{\delta} \end{array} 3$$

and let  $J$  be the ideal generated by  $\delta\alpha$ ,  $\gamma\beta$  and  $\gamma\alpha - \delta\beta$ . Consider the representations

$$M = k \begin{array}{c} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} k^2 \begin{array}{c} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \xrightarrow{\quad} \end{array} k, \quad U_\lambda = 0 \begin{array}{c} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \xrightarrow{\quad} \end{array} k \begin{array}{c} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \\ \xrightarrow{\quad} \end{array} k,$$

and

$$V_\mu = k \begin{array}{c} \begin{pmatrix} \mu \\ 1 \end{pmatrix} \\ \xrightarrow{\quad} \end{array} k \begin{array}{c} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \xrightarrow{\quad} \end{array} 0,$$

for  $\lambda, \mu \in k$ . It is not difficult to see and can be found in [14] that  $\bar{\mathcal{O}}_M$  and the closure of  $\bigcup_{\lambda, \mu \in k} \text{GL}_d(k) * (U_\lambda \oplus V_\mu)$  are irreducible components of  $\mathcal{C}_M$  and that they intersect in the closure of  $\bigcup_{\lambda \in k} \text{GL}_d(k) * (U_\lambda \oplus V_{-\lambda})$ , where  $d = (1, 2, 1)$ . So  $\mathcal{C}_M$  is singular at  $N = U_1 \oplus V_{-1}$ .

On the other hand, a computation shows that the morphism given by

$$k \begin{array}{c} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \\ \xrightarrow{\quad} \end{array} k^2 \begin{array}{c} \begin{pmatrix} z_1 & z_2 \\ t_1 & t_2 \end{pmatrix} \\ \xrightarrow{\quad} \end{array} k \mapsto \begin{pmatrix} x_1 & x_2 & y_1 & y_2 \\ -t_2 & t_1 & z_2 & -z_1 \end{pmatrix}$$

is an isomorphism from  $\bar{\mathcal{O}}_M$  to the variety  $\mathcal{V}_{2 \times 4}^1(k)$  of  $2 \times 4$ -matrices of rank at most 1, which has a single singularity at 0, the image of the semisimple representation. Hence  $\bar{\mathcal{O}}_M$  is regular at  $N$ .

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