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Commentarii Mathematici Helvetici

2-block Springer fibers: convolution algebras and coherent sheaves

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Abstract. For a fixed 2-block Springer fiber, we describe the structure of its irreducible components and their relation to the Białynicki-Birula paving, following work of Fung. That is, we consider the space of complete flags in $\mathbb{C}^n$ preserved by a fixed nilpotent matrix with 2 Jordan blocks, and study the action of diagonal matrices commuting with our fixed nilpotent. In particular, we describe the structure of each component, its set of torus fixed points, and prove a conjecture of Fung describing the intersection of any pair.

Then we define a convolution algebra structure on the direct sum of the cohomologies of pairwise intersections of irreducible components and closures of $\mathbb{C}^*$-attracting sets (that is Białynicki-Birula cells), and show this is isomorphic to a generalization of the arc algebra of Khovanov defined by the first author. We investigate the connection of this algebra to Cautis and Kamnitzer's recent work on link homology via coherent sheaves and suggest directions for future research.

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Keywords. Springer fiber, convolution algebra, coherent sheaves, Khovanov homology, Fukaya category, category $\mathcal{O}$, torus fixed points.

Contents

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Introduction

Many important algebras arising in representation theory (Hecke algebras, universal enveloping algebras, etc.) have a geometric description based on convolution products.

Besides their intrinsic interest, realizing an algebra in terms of convolution allows for a geometric understanding of the representation theory of that algebra, in particular, the construction of collections of standard and costandard modules, indicating the existence of an interesting representation theory along the lines of highest weight categories or quasi-hereditary algebras. This approach has been applied with great success to the representation theory of Weyl groups, Hecke algebras of various flavors and universal enveloping algebras, as is ably documented in the book of Chriss and Ginzburg [CG97].

Using 2-block Springer fibers we present a construction of a family of convolution algebras with a somewhat different nature than the above examples (see Section 4 for a precise description). With a certain specific choice of parameters and the two Jordan blocks of the same size, the algebra is related to the Ext-algebra of certain *coherent sheaves* on a resolution of the corresponding Slodowy slice and to a graphically defined algebra, called the *arc algebra* $\mathcal{H}^\bullet$, introduced by Khovanov [Kho00]. For the general 2-Jordan-block case (not necessarily equally sized), we establish an isomorphism to the more general version of the arc algebra as introduced in [Str09] and [CK06].

Our construction is built on a careful explicit geometric and combinatorial analysis of the geometry of the Springer fiber and its components. Apart from the 2-Jordan-block case, the structure of irreducible components of Springer fibers is not sufficiently well understood to generalize this construction, though significant progress on the structure of components and their intersections has been achieved in the square-zero (i.e. two column) case studied in [MP06], in addition to the 2-Jordan-block case studied here.

Khovanov used his arc algebra to define a categorification of the Jones polynomial ([Kho00]), followed by a representation theoretic categorification of the Jones polynomial and the Reshetikhin–Turaev $U(\mathfrak{sl}_2)$-tangle invariant obtained by the first author in [Str05]. The choice of Jordan block sizes corresponds there to a choice of a specific weight space in a tensor product of many copies of the natural $U(\mathfrak{sl}_2)$-module, hence naturally extends the case of two blocks of the same size. It is known that after restriction to a suitable subcategory, this categorification of the Jones polynomial agrees with Khovanov's ([Str09], [BS08a], [BS08b]).

On the other hand, Cautis and Kamnitzer ([CK08]) used the geometry of spaces connected with two-row Springer fibers to define a related knot homology theory using certain categories of coherent sheaves, whereas Seidel, Smith [SS06] and Manolescu [Man07] constructed a symplectic version of Khovanov homology using a certain Fukaya category connected to the Springer fibres of interest to us. Our convolution algebra construction is motivated by both of these constructions, though perhaps more strongly the latter. More precisely, the cohomology of the intersection of two components in our picture should be seen in analogy to the morphism space between two (compact) Lagrangian submanifolds in [SS06] and one of our main results will be the definition of a convolution product structure on the direct sum of all these morphism spaces mimicking the composition of morphisms in the Fukaya category.

We hope that our description of the convolution algebra will ultimately shed some light on the connection between the algebraic-representation theoretic categorification and the geometric ones. In particular, we expect that, with the correct identifications, the algebras appearing in all three contexts are isomorphic, establishing some rather surprising equivalences of categories (see Section 5 for results in this direction).

An analogous construction associating an algebra to a hypertoric variety has been developed by the second author with Braden, Licata and Proudfoot ([BLPW08]). Like the algebra we define, this hypertoric algebra is quasi-hereditary and moreover Koszul (which is known to be true for our algebra as well, [BS10]). The Koszul dual of this hypertoric algebra is the algebra associated via this convolution construction to the Gale dual hypertoric variety.

Let us outline the content of the paper in more detail. For any nilpotent endomorphism N of $\mathbb{C}^n$, we have the following (in general, not smooth) subvariety of the full flag variety, which only depends (up to isomorphism) on the conjugacy class of N:

Definition. The *Springer fiber* of a nilpotent map $N: \mathbb{C}^n \to \mathbb{C}^n$ is the variety of all complete flags $\mathcal{F}$ in $\mathbb{C}^n$ fixed under N (i.e. for any space F_i of the flag $\mathcal{F}$, we have that $NF_i \subset F_{i-1}$).

We can always naturally associate a Springer fiber with any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{sl}_n$ containing the standard Borel of all upper triangular matrices: given $\mathfrak{p}$, we have a composition of n which, in turn, determines a Jordan type, hence a nilpotent conjugacy class in $M(n \times n, \mathbb{C})$. More canonically, this is a regular nilpotent in the Levi of $\mathfrak{p}$. In the present paper, we restrict to the case where N is nilpotent with two Jordan blocks (i.e. where $\mathfrak{p}$ is maximal or, equivalently, $\dim \ker N = 2$).

In Sections 1–3, we will concentrate on combinatorial and geometric preliminaries. We first recall the description of irreducible components of these Springer fibers (following [Fun03]), and more generally consider the closure of cells in the Białynicki-Birula paving of the Springer fiber. For all such closures, we verify Fung's conjecture that pairwise intersections of such are smooth, iterated $\mathbb{P}^1$-bundles and explicitly determine their cohomology rings as quotients of the cohomology ring of the full flag variety.

Then, in Section 4, we equip the direct sum of all these cohomologies (with appropriate grading shifts) with a non-commutative convolution product which turns it into a finite dimensional graded algebra $H^\bullet$. In the case where the two Jordan blocks have the same size, the underlying vector space is isomorphic to the one underlying

Khovanov's arc algebra. For all block lengths, we obtain the vector spaces underlying the generalized versions of Khovanov algebras.

The generalized versions of Khovanov's algebra have a quasi-hereditary cover $\widetilde{\mathcal{H}}^\bullet$ described by the first author in [Str09] and now sometimes called KS-algebras. For a detailed description and further properties of these generalized Khovanov algebras and their representation theory, we refer to [BS08a] and [BS10]. We construct a quasi-hereditary cover $\widetilde{H}^\bullet$ of our first convolution algebra using a Białynicki-Birula paving of the Springer fiber, with respect to a generic cocharacter of the maximal torus commuting with N. The set of fixed points for this torus action are in natural bijection with the idempotents in the algebra $\mathcal{H}^\bullet$ (and hence with indecomposable projective modules in the parabolic category $\mathcal{O}_0^{\mathfrak{p}}$ or in the quasi-hereditary cover of the generalized arc algebra). We denote by $\mathcal{Y}w$ the closures of Białynicki-Birula cells. (In our special case these can also be viewed as the stable manifolds under the Morse flow of the moment map of this cocharacter or as the closure of a fixed point attracting set). Taking cohomology over $\mathbb{C}$, we show that

$$\widetilde{H}^\bullet := \bigoplus_{w,w'} H^\bullet(\mathcal{Y}w \cap \mathcal{Y}w')\langle d(w,w')\rangle$$

can be equipped with a convolution algebra structure which is a (non-negatively) $\mathbb{Z}$-graded algebra after appropriate grading shifts $\langle d(w,w')\rangle$ indicated on the right hand side.

We then show the main result of our paper.

Theorem. *The algebra $H^\bullet$ (resp. the extended version $\widetilde{H}^\bullet$) and the generalized arc algebra $\mathcal{H}^\bullet$ (resp. its quasi-hereditary cover $\widetilde{\mathcal{H}}^\bullet$) are isomorphic as graded algebras.*

Note that by Theorem 3 of [Str09] the category of $\mathcal{H}^\bullet$-modules is equivalent to the category of *perverse sheaves* on a Grassmannian (constructible with respect to the Schubert stratification). Hence this algebra actually has two geometric realizations, one arising from constructible sheaves, and one which seems to be related to the Fukaya category and coherent sheaves.

Since the KS-algebras are the endomorphism rings of certain projectives in parabolic category $\mathcal{O}_0^{\mathfrak{p}}$, by Koszul duality ([BGS96], Theorem 1.1.3), this is isomorphic to an Ext-algebra of simple modules in a singular block of category $\mathcal{O}$ corresponding to a weight precisely fixed by $W_{\mathfrak{p}}$ (in the so-called dot-action). This theorem then suggests that we have an embedding of this singular category $\mathcal{O}$ into the Fukaya category of the Slodowy slice $\mathcal{S}_{n-k,k}$. In this way one might hope for a direct connection with the construction in [SS06].

Finally in Section 5, we consider how our model (and thus, indirectly, the KS-algebra and category $\mathcal{O}_0^{\mathfrak{p}}$) is related to the sheaf-theoretic model of Khovanov homology given by Cautis and Kamnitzer [CK08]. To each crossingless matching

$a \in \mathrm{Cup(n)}$ (that means to each primitive idempotent in Khovanov's arc algebra) their model associates a certain coherent sheaf $i_*\Omega(a)^{1/2}$ on a certain compact smooth variety related with Slodowy slices. The variety naturally contains the Springer fiber we had considered previously, and the sheaves in question are supported on the component we associated with a. As our notation suggest, these sheaves arise from square roots of canonical bundles (Theorem 39).

We show that, as a vector space, the Ext-algebra of these sheaves can be identified with our algebra $H^\bullet$ (and thus also with Khovanov's algebra):

Theorem. *With the notation in Section 5 there is an isomorphism of graded vector spaces*

$$\mathrm{Ext}^\bullet_{\mathsf{Coh}(\mathcal{S}_{n-k,k})}(i_*\Omega(a)^{1/2}, j_*\Omega(b)^{1/2}) \cong H^\bullet(a \cap b)\langle d(a,b)\rangle,$$

We have not been able to determine whether the Yoneda product on this space is isomorphic to the arc algebra $\mathcal{H}^\bullet$. Obviously, this would be a very interesting question to resolve. It might be a first step to solve the question of whether the functorial tangle invariants of Cautis and Kamnitzer ([CK08]) can be identified with the functorial tangle invariants of Khovanov ([Kho02]) and (equivalently) of the second author ([Str05]).

The half-densities $\Omega(a)^{1/2}$ are simple objects in the heart $\mathcal{C}$ of a certain t-structure on the category of coherent sheaves on a smooth compact space Z_n. This space is a certain compactification of the pre-image under the Springer resolution of a normal slice to the nilpotent orbit through N at N.

We describe the other simple objects in this heart, and show that it carries a highest-weight structure with the same Kazhdan–Lusztig polynomials as the corresponding highest weight KS-algebra, say A.

Conjecture. There is an isomorphism between $\mathcal{C}$ and the category of finite dimensional modules over the algebra A.

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Preliminaries

In the following, all vector spaces and cohomologies are defined over $\mathbb{C}$. We abbreviate $\otimes = \otimes_{\mathbb{C}}$. An *algebra* will always be a unitary associative $\mathbb{C}$-algebra. A *graded vector space* will always be $\mathbb{Z}$-graded. For a graded vector space M and $i \in \mathbb{Z}$ we denote by $M\langle i\rangle$ the graded vector space with homogeneous components $(M\langle i\rangle)_j = M_{j-i}$.

Let V be an n-dimensional complex vector space and $N\colon V \to V$ be a nilpotent endomorphism of Jordan type $(n-k,k)$. For ease, we assume $2k < n$. Explicitly, we equip V with an ordered basis $\{p_1,\dots,p_{n-k},q_1,\dots,q_k\}$ with the action of N defined by

$$N(p_i) = p_{i-1}, \quad N(q_i) = q_{i-1}$$

where, by convention, $p_0 = q_0 = 0$ and often write $\mathbb{C}^n$ instead of V. We let $P = \langle p_1,\dots,p_{n-k}\rangle$ and $Q = \langle q_1,\dots,q_k\rangle$.

Let X be the variety of complete flags in V, and let Y be the fixed points of $\exp(N)$ acting on X. So, Y consists of all complete flags $F_0 \subset F_1 \subset \cdots \subset V$ such that $N(F_i) \subseteq F_{i-1}$.

The ordering on the basis equips V with a *standard flag*

$$\{0\} \subset \langle p_1\rangle \subset \langle p_1,p_2\rangle \subset \cdots \subset \langle p_1,\dots,p_{n-k},q_1,\dots,q_{k-1}\rangle \subset V,$$

which is invariant under N.

1. Irreducible components and their cohomology

1.1. Matchings and tableaux. In order to describe the irreducible components of Y, we will first have to define some combinatorial machinery. This section will cover a number of results from the article of Fung [Fun03], which will be necessary for later.

Definition 1. A *standard tableau* is a filling of the Young diagram of a partition such that the rows and columns are strictly decreasing (read from the top left corner).

Definition 2. A *crossingless matching* is a planar diagram consisting of n points, k cups, and $n-2k$ rays pointing directly downward such that each point is attached to exactly one cup or ray, cups only pass below points, not above them, and no cup or ray crosses any other. We say that a point at the end of a cup is *matched* and one at the end of a ray is *orphaned*.

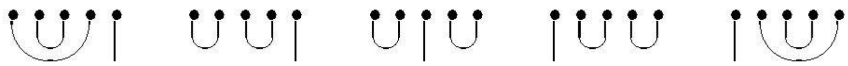
Given any standard tableau S of shape $(n-k,k)$, we can associate a crossingless matching $\mathbf{m}(S)$ of n points, numbered from left to the right, such that the bottom row of the tableau contains all the numbers which are at the left end of a cup, and the top row of the diagram contains all the numbers which are at the right endpoint of a cup, or are the endpoint of a ray.

Proposition 3. *This assignment gives in fact a bijection between standard tableaux of shape $(n-k, k)$ and crossingless matchings/cup diagrams of n points with k cups and $n-2k$ rays.*

Example 4. Let $k = 2$, $n = 5$. Then we have the following five standard tableaux

$$\begin{array}{|c|c|c|}\hline 5&4&3\\ \hline 2&1\\ \cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline 5&4&2\\ \hline 3&1\\ \cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline 5&3&2\\ \hline 4&1\\ \cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline 5&3&1\\ \hline 4&2\\ \cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline 5&4&1\\ \hline 3&2\\ \cline{1-2}\end{array}$$

and the associated cup diagrams (with one orphaned point in each case):



Example 5. The following will be our running example, (and the notation should be kept in mind): Let $k = 2$, $n = 4$. Then we have two standard tableaux

$$S(\Cup) := \begin{array}{|c|c|}\hline 4&3\\ \hline 2&1\\ \hline\end{array}\qquad S(\cup\cup) := \begin{array}{|c|c|}\hline 4&2\\ \hline 3&1\\ \hline\end{array}$$

where the first corresponds to the cup diagram $\mathrm{Cup}(\Cup)$ with two nested cups, the second to the cup diagram $\mathrm{Cup}(\cup\cup)$ with two cups next to each other. There are no orphaned points.

Given a tableau S of shape $(n-k, k)$ let $S_\vee$ be set of numbers in the lower row of the tableau, and $S_\wedge$ the set of numbers in the top row. If S is standard, the cup diagram $\mathbf{m}(S)$ defines a map $\sigma\colon S_\vee \to S_\wedge$ sending the beginning of a cup to its end.

Let $\delta(i) = (\sigma(i) - i + 1)/2$ be the number of cups nested inside the one connecting i and $\sigma(i)$ for any $i \in S_\vee$. Note that $\delta(i)$ encodes the size of the cup starting at i. For instance, the diagram $\Cup$ has $\delta(2) = 1$ and $\delta(1) = 2$. We let $c(i)$ be the column number of i, i.e. the number of columns to the left (inclusive) of the one which i lies in.

1.2. Components and matchings. Spaltenstein [Spa76] and Vargas [Var79] established a bijection between the irreducible components of Y and the standard tableaux of shape $(n-k, k)$ which allowed them to describe the components as closures of explicitly given locally closed subspaces:

Definition 6. Let S be a standard tableau of shape $(n-k, k)$. The associated irreducible component Y_S is the closure of the set of complete flags $F_0 \subset \cdots \subset F_n = V$ in Y such that for all $i \in S_\vee$, we have $F_i \subseteq F_{i-1} + \operatorname{im} N^{c(i)-1}$.

Alternatively, (see [Fun03]) one can use the following much more handy definition: let t_i be the number of indices smaller than or equal to i in the top row, and similarly for b_i and the bottom row, then we have

Proposition 7. *A complete flag* $\{0\} = F_0 \subset \cdots \subset F_n = V$ *lies in* Y_S *if and only if for all* $i \in S_\vee$, *we have* $N^{\delta(i)}(F_{\sigma(i)}) = F_{i-1}$, *and for each* $i \in S_\wedge \setminus \sigma(S_\vee)$, *we have* $F_i = N^{-b_i}(\operatorname{im} N^{n-k-t_i+b_i})$.

Note that the condition of being in a component associated to S means that the spaces F_i where i labels either on orphaned point or the right end of a cup in $\mathbf{m}(S)$ (i.e. $i \in S_\wedge$) as a set, are completely determined by the spaces F_{i-1} corresponding to the point at the left endpoint of a cup as a set, but this is not true for the ends of an individual cup.

Example 8. For our running example we have

$$\begin{aligned} Y_{\cup\cup} &= \{F_0 \subset F_1 \subset N^{-1}(F_0) = \langle p_1, q_1\rangle \subset F_3 \subset N^{-1}(F_2) = \mathbb{C}^4\} \subset Y, \\ Y_{\Cup} &= \{F_0 \subset F_1 \subset F_2 \subset N^{-1}(F_1) \subset N^{-2}(F_0) = \mathbb{C}^4\} \subset Y. \end{aligned}$$

Hence $Y_{\cup\cup} \cong \mathbb{P}^1 \times \mathbb{P}^1$, whereas $Y_{\Cup}$ is a non-trivial $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ (a Hirzebruch surface).

By [Fun03], Proposition 5.1, all irreducible components are iterated $\mathbb{P}^1$-bundles; in particular, they are smooth.

1.3. Cohomology of components. The variety X carries n tautological line bundles of the form $V_i = F_i/F_{i-1}$ where we use F_i to denote the corresponding tautological vector bundle on X, and its restriction to Y and Y_S. These line bundles generate $\operatorname{Pic}(X)$, and their first Chern classes $x_i = c_1(V_i)$ generate the cohomology ring $H^\bullet(X;\mathbb{C})$. This presentation is due to Borel and gives an isomorphism of $H^\bullet(X;\mathbb{C})$ with the algebra of coinvariants for the obvious action of the symmetric group S_n on $\mathbb{C}[x_1, \cdots, x_n]$, that is,

$$H^\bullet(X;\mathbb{C}) \cong \mathbb{C}[x_1, \cdots x_n]/(\epsilon_1(\mathbf{x}), \cdots, \epsilon_n(\mathbf{x}))$$

where ϵ_i is the i-th elementary symmetric polynomial in the variables x_i (see e.g. [Ful97]).

Theorem 9. *The cohomology ring of* Y_S *has a natural presentation of the form*

$$H^\bullet(Y_S;\mathbb{C}) \cong \mathbb{C}[\{x_i\}_{i\in S_\vee}]/(\{x_i^2\}_{i\in S_\vee}).$$

The pullback map $i_S^*\colon H^\bullet(X) \to H^\bullet(Y_S)$ *is surjective, and given in this presentation by*

$$i_S^*(x_i) = \begin{cases} x_i, & i \in S_\vee, \\ -x_{\sigma^{-1}(i)}, & i \in \sigma(S_\vee) \subset S_\wedge, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. By [Fun03], Proposition 5.1, Y_S an iterated $\mathbb{P}^1$-bundle, with the maps to $\mathbb{P}^1$ given by the line bundles V_i for $i \in S_\vee$. Hence, the cohomology ring $H^\bullet(Y_S;\mathbb{C})$ is generated by their first Chern classes (since these give a generating set in the associated graded with respect to the filtration coming from the Leray–Serre spectral sequence). Since these line bundles are pullbacks from X, the map i_S^* is surjective.

We will find relations between these using the Chern classes of related bundles. First, note that by the definition of Y_S, we have exact sequences of vector bundles for each $i \in S_\vee$:

$$0 \longrightarrow \ker N^{\delta(i)} \longrightarrow F_{\sigma(i)} \xrightarrow{N^{\delta(i)}} F_{\sigma(i)} \longrightarrow F_{\sigma(i)}/F_{i-1} \longrightarrow 0,$$
$$0 \longrightarrow \ker N^{\delta(i)-1} \longrightarrow F_{\sigma(i)-1} \xrightarrow{N^{\delta(i)-1}} F_{\sigma(i)-1} \longrightarrow F_{\sigma(i)-1}/F_i \longrightarrow 0.$$

It may happen at $\delta(i) = 1$, in which case we interpret N^0 as the identity map, and the lower exact sequence becomes trivial. Since $\ker N^{\delta(i)}$ is a trivial subbundle of F_n, we obtain in K-theory

$$[F_{\sigma(i)}/F_{i-1}] - [F_{\sigma(i)-1}/F_i] = [V_{\sigma(i)} \oplus V_i] = 0.$$

The Chern classes of a bundle only depend on its class in K-theory, so that the following equalities hold in $H^\bullet(Y_S;\mathbb{C})$:

$$c_1(V_{\sigma(i)} \oplus V_i) = x_{\sigma(i)} + x_i = 0,$$
$$c_2(V_{\sigma(i)} \oplus V_i) = x_{\sigma(i)} x_i = 0.$$

If $i \in S_\wedge \setminus \sigma(S_\vee)$, then the bundles F_i and F_{i-1} are both trivial, so $x_i = 0$.

Thus, the Chern classes x_i for $i \in S_\vee$ generate the cohomology of Y_S, and the relations which we claimed hold. These must be sufficient, since the quotient by the relations we have proven above and $H^\bullet(Y_S)$ both have dimension 2^k, the latter by Theorem 5.3 of [Fun03]. $\square$

Example 10. Let $R \cong \mathbb{C}[X]/(X^2)$. We have isomorphisms of graded rings

$$H^\bullet(Y_{C(\Cup)}) \cong \mathbb{C}[x_1, x_2]/(x_1^2, x_2^2) \cong R \otimes R,$$

and

$$H^\bullet(Y_{C(\cup\cup)}) \cong \mathbb{C}[x_1, x_3]/(x_1^2, x_3^2) \cong R \otimes R.$$

2. The Białynicki-Birula paving and stable manifolds

2.1. The torus action and fixed points. The torus $(\mathbb{C}^*)^n$ of diagonal matrices in the basis given by the p_i's and q_i's acts on the flag variety X in the natural way and induces on the Springer fiber Y an action of a maximal torus of $Z_G(N)$. This torus is

2-dimensional, and its action is explicitly given by $(r,s)\cdot p_i = rp_i$, $(r,s)\cdot q_i = sq_i$ for $(r,s)\in(\mathbb{C}^*)^2$.

This action has isolated fixed points which we want to label by row strict tableaux of $(n-k,k)$-shape (i.e. tableaux which are decreasing in the rows, but with no condition on the columns). To any arbitrary row strict tableau w of shape $(n-k,k)$ we associate the full flag $\mathcal{F}_\bullet(w)$ such that

$$\mathcal{F}_i(w) = \langle\{p_j, q_r \mid j \le t_i, r \le b_i\}\rangle,$$

where t_i is the number of indices smaller than or equal to i in the top row, and similarly for b_i and the bottom row. Note that the standard flag is of the form $\mathcal{F}_\bullet(w_{\text{dom}}^{n-k,k})$, where $w_{\text{dom}}^{n-k,k}$ is the row strict tableau with $1, 2, \ldots, n-k$ in the first row; for example $w_{\text{dom}}^{3,2} = \begin{array}{|c|c|c|}\hline 3&2&1\\\hline 5&4\\\cline{1-2}\end{array}$ and $w_{\text{dom}}^{2,2} = \begin{array}{|c|c|}\hline 2&1\\\hline 4&3\\\hline\end{array}$

To any row strict tableau w of shape $(n-k,k)$ we will later associate a crossingless matching $\mathbf{m}(w)$ of n points by the same rule as before for standard tableaux (but the resulting matching might have in the extreme case only rays and no cups at all); see the paragraph before Theorem 15 for a precise definition. There are $\binom{n}{n-k}$ row strict tableaux, which is also the same as the number of fixed points and Φ defines an explicit bijection:

Lemma 11. *The map $\Phi : w \mapsto \mathcal{F}_\bullet(w)$ defines a bijection between row strict tableaux of shape $(n-k,k)$ and torus fixed points of Y.*

Proof. It is easy to check that $\mathcal{F}_\bullet(w)$ is in fact a point in Y, and obviously a fixed point, since all its component subspaces are spanned by weight vectors. The map is Φ injective by construction.

On the other hand, if $\mathcal{F}$ is a T-fixed flag, then each of its constituent subspaces F_i is spanned by the intersections $F_i \cap P$ and $F_i \cap Q$. These, in turn are invariant subspaces for $N|_P$ and $N|_Q$. But these restrictions are regular nilpotents, so there is a unique invariant subspace of any possible dimension, which is of the form $\langle p_1, \ldots, p_i\rangle$ (and similarly for q_j). Thus, $\mathcal{F}$ is of the form $\mathcal{F}_\bullet(w)$ for some row-strict tableau, and Φ is surjective. $\square$

Let w, S be tableaux of shape $(n-k,k)$, where w is row strict and S is standard with associated cup diagram $\mathbf{m}(S)$. We consider the sequences $\mathbf{a} = a_1a_2a_3\ldots a_n$, where $a_i = \wedge$ if $i \in w_\wedge$ and $a_i = \vee$ if $i \in w_\vee$ and call it the *weight sequence* of w. For instance $w_{\text{dom}}^{3,2}$ has weight sequence $\wedge\wedge\wedge\vee\vee$. (We refer to Example 16 for more examples of weight sequences with their cup diagrams.) We can put the weight sequence on top of the diagram $\mathbf{m}(S)$ and obtain a diagram $w\mathbf{m}(S)$ where the upper ends of each cup or line are decorated with an orientation. We call $w\mathbf{m}(S)$ *oriented* if these decorations induce a well-defined orientation on $\mathbf{m}(S)$. For instance, if $\mathbf{m}(S)$

is one of the cup diagrams from Example 4, then $w_{\mathrm{dom}}^{3,2}\mathbf{m}(S)$ is only oriented if $\mathbf{m}(S)$ is the last diagram in the list. Note that the number of cups in a cup diagram $\mathbf{m}(S)$, where S is a standard tableau, is always k, hence for any orientation $w\mathbf{m}(S)$, the decoration at each orphaned vertex will be a $\wedge$.

Lemma 12. *A fixed point $\mathcal{F}_\bullet(w)$ is in an irreducible component Y_S associated with a cup diagram C if and only if wC is an oriented cup diagram. In particular, every component contains exactly 2^k fixed points.*

Proof. Let first C be oriented with the orientation on all cups pointing (counter-clockwise) from left to right (and all lines pointing up). This is exactly the case when $w = S$ is the standard tableau associated with C. We claim $\mathcal{F}_\bullet(w)$ satisfies the conditions of Proposition 7. If i is on the top row, then there are exactly $n-k-c(i)+1$ numbers smaller than or equal to i on the top row, and so F_i/F_{i-1} is spanned by $p_{t_i} = p_{n-k-c(i)+1} \in \operatorname{im} N^{c(i)-1}$. If i is on the bottom row, then F_i/F_{i-1} is spanned by $p_{n-c(i)+1} \in \operatorname{im} N^{c(i)-1}$. The claim follows.

Consider now the general case. Let first i and $\sigma(i)$ be the labels for the two endpoints of a cup. The condition $N^{\delta(i)}(F_{\sigma(i)}) = F_{i-1}$ is equivalent to exactly half of the indices between i and $\sigma(i)$ (inclusive) are contained in $S_\wedge$ (or $S_\vee$ respectively). For cups connecting two points next to each other this is directly equivalent to being oriented. By induction on the length of the cup, we may assume that each cup between i and $\sigma(i)$ is oriented. Since there are no orphaned points below a cup, getting exactly half $\wedge$'s and half $\vee$, means the labels i and $\sigma(i)$ must carry the opposite orientations, i.e. the cup is oriented.

Now $i \in S_\wedge \backslash \sigma(S_\vee)$ is the same as saying the point with label i is orphaned. The necessary condition for $\mathcal{F}_\bullet(w)$ only depends on $c(i)$, which is the same for all w where wC is oriented, because it only depends on the number of cups and lines to the left of the point i. Hence the argument at the beginning of the proof implies the lemma. $\square$

Example 13. There are six row strict tableaux in case $n = 4$, $k = 2$, hence six fixed points $w_1, w_2, \ldots, w_6$, corresponding to the six weight sequences

$$\wedge\wedge\vee\vee, \quad \wedge\vee\wedge\vee, \quad \vee\wedge\wedge\vee, \quad \wedge\vee\vee\wedge, \quad \vee\wedge\vee\wedge, \quad \vee\vee\wedge\wedge.$$

The fixed point w_1 is the standard flag. Now the component $Y_{S(\cup\cup)}$ contains w_i, $i \in \{2,3,4,5\}$, whereas $Y_{S(\Cup)}$ contains the w_i, $i \in \{1,2,5,6\}$.

2.2. The paving. If we choose a cocharacter $\mathbb{C}^* \hookrightarrow T$ which has the same fixed points as the whole torus, then we can consider the behavior of points as t approaches infinity. We will fix the choice of $t \mapsto (t^{-1}, t)$, that is, subspaces are attracted toward the q_i's as t approaches ∞ and towards the p_i's as t approaches 0.

Definition. If $\mathcal{F}_\bullet(w) \in Y^T$ is a torus fixed point, then we denote the *stable manifold* or *attracting set* by

$$\mathcal{Y}w^0 = \{y \in Y \mid \lim_{t\to\infty} t \cdot y = \mathcal{F}_\bullet(w)\},$$

and its closure $\mathcal{Y}w = \overline{\mathcal{Y}w^0}$.

For each flag $\mathcal{F}$ in Y, we can obtain a flag $\mathcal{F}'$ (with no longer necessarily distinct spaces) in P by taking the intersections $\mathcal{P}_i = F_i \cap P$, and similarly in $V/P \cong Q$ given by $\mathcal{Q}_i = F_i/(F_i \cap P)$. We can define the new flag $\mathcal{F}'$ by putting $F_i' := \mathcal{P}_i + \mathcal{Q}_i \subset P \oplus Q = V$, which is obviously T-equivariant.

Proposition 14. *A flag $\mathcal{F}$ in Y is contained in $\mathcal{Y}w^0$ if and only if the new flag $\mathcal{F}'$ obtained from it by the procedure above is $\mathcal{F}_\bullet(w)$.*

Proof. Obviously, the new flag $\mathcal{F}'$ does only depend on the orbit of $\mathcal{F}$, in the sense that it does not change if we move inside the torus orbit O containing $\mathcal{F}$. Thus, for any point $\mathcal{G}$ in the closure of O we have $(\mathcal{F}_i \cap P) \subseteq (\mathcal{G}_i \cap P)$ and hence $(\mathcal{F}_i' \cap P) \subseteq (\mathcal{G}_i' \cap P)$ for any $1 \leq i \leq n$, since containing a vector is a closed condition on a subspace.

On the other hand, since P has minimal weight under $\mathbb{C}^*$, no vector not in P is attracted to P as $t \to \infty$, so the size of the intersection with P can only stay the same or decrease in that limit. Thus intersection with P must be fixed under the limit. Since the image in Q has complementary dimension, it must also be fixed. □

This makes it clear that $\mathcal{Y}w^0$ is algebraically isomorphic to an affine space (in particular, diffeomorphic to a disk), since the set of vector spaces projecting to a given one under a linear map is an affine space.

The structure of these stable manifolds can be understood in terms of cup diagrams, in much the same way as the structure of the components. To w we attach two (in general different) cup diagrams, $\mathbf{m}(w)$ and $C(w)$ as follows:

For each fixed point $\mathcal{F}_\bullet(w)$, there is the diagram $\mathbf{m}(w)$ with the property that $w\mathbf{m}(w)$ has the maximal number of cups amongst all cup diagrams C such that wC is oriented and contains only counter-clockwise cups. This diagram will have $k_w \leq k$ cups, with equality $k_w = k$ if and only if w is standard. One can build this diagram inductively by adding an arc between any adjacent pair $\vee\wedge$, and then continuing the process for the sequence with these points excluded. We then add lines to the remaining points. We call $\mathbf{m}(w)$ *the cup diagram associated with* w.

Rather than adding these lines, we could complete to an oriented cup diagram $C(w)$ with k cups, by matching all the $\vee$'s in the only possible way. Call the corresponding standard tableau $S(w)$.

Theorem 15. *Let w be a row strict tableau. Then $\mathcal{Y}w$ is the subset of $Y_{S(w)}$ containing exactly the flags which satisfy the additional property: if $i \in w_\wedge \cap S(w)_\vee$, then F_i coincides with the i-th subspace of the fixed point $\mathcal{F}_\bullet(w)$.*

In particular, for any standard tableau S, we have $\mathcal{Y}S = Y_S$.

Proof. First we confirm that these relations hold on $\mathcal{Y}w^0$ (and thus on $\mathcal{Y}w$, since they are closed conditions).

Let $\mathcal{F}$ be a point in $\mathcal{Y}w^0$. Let us first assume there is at least one cup in $\mathbf{m}(w)$, so in particular a minimal one not containing any other cup. This means there is some index $i \in w_\vee$ with $\delta(i) = 1$. The result we desire is that $F_{i+1} = N^{-1}(F_{i-1})$.

First, note that since the index $i+1$ is marked with an $\wedge$ in w, then we must have

$$F_{i+1} \supset F_i + N_P^{-1}(F_i \cap P).$$

On the other hand, since i is marked with an $\wedge$, we must have $F_i \cap P = F_{i-1} \cap P$ and it follows that

$$N^{-1}(F_{i-1}) \supset F_i + N_P^{-1}(F_{i-1} \cap P) = F_i + N_P^{-1}(F_i \cap P).$$

All of these spaces are of dimension $i+1$, so we must have $F_{i+1} = N^{-1}(F_{i-1})$.

Let w' denote w with $i, i+1$ removed. Applying N to all spaces of dimension bigger than $i+1$ provides a map $q_i : \mathcal{Y}w^0 \to \mathcal{Y}w'^0$ which extends to a map $q_i : \mathcal{Y}w \to \mathcal{Y}w'$ between the closures. The relation for a cup in $S(w')$ pulls back to that for the corresponding cup of $S(w)$.

Thus, by induction, we may reduce to the case where there are no cups in $\mathbf{m}(w)$ (that is, w is a series of $\wedge$'s followed by $\vee's$). In this case, our claim simply reduces to the claim that $\mathcal{Y}w = \{\mathcal{F}_\bullet(w)\}$. This is indeed the case, since for any index in $w_\wedge$, we must have $F_i \subset P$, and N acts regularly on P so all N-invariant subspaces are also T-equivariant. Similarly, for any $i \in w_\vee$, we must have $F_i \supset P$, and N acts regularly on $V/P \cong Q$. Therefore $\mathcal{F}$ satisfies the required relations.

On the other hand $\mathcal{F}_\bullet(w)$ obviously satisfies the conditions coming from cups in $C(w)$, and our requirement on elements of $w_\wedge \cap S(w)_\vee$, and any flag satisfying these relations is in the closure of $\mathcal{Y}w^0$. ☐

Example 16. The cup diagrams $\mathbf{m}(w_i)$ associated to the weights w_i, $1 \le i \le 6$, from Example 13 are as follows:

On the other hand, the cup diagrams $C(w_i)$ for the weights w_i, $1 \le i \le 6$, are as follows:

There are the two irreducible components from Example 8,

$$\mathcal{Y}w_5 = \{F_0 \subset F_1 \subset N^{-1}(F_0) = \langle p_1, q_1\rangle \subset F_3 \subset N^{-1}(F_2) = \mathbb{C}^4\} \subset Y,$$
$$\mathcal{Y}w_6 = \{F_0 \subset F_1 \subset F_2 \subset N^{-1}(F_1) \subset N^{-2}(F_0) = \mathbb{C}^4\} \subset Y$$

and the additional stable manifolds

$$\mathcal{Y}w_4 = \{F_0 \subset \langle p_1\rangle \subset \langle p_1, q_1\rangle \subset F_3 \subset \mathbb{C}^4\} \subset \mathcal{Y}w_5,$$
$$\mathcal{Y}w_3 = \{F_0 \subset F_1 \subset \langle p_1, q_1\rangle \subset \langle p_1, p_2, q_1\rangle \subset \mathbb{C}^4\} \subset \mathcal{Y}w_5,$$
$$\mathcal{Y}w_2 = \{F_0 \subset \langle p_1\rangle \subset F_2 \subset \langle p_1, p_2, q_1\rangle \subset \mathbb{C}^4\} \subset \mathcal{Y}w_6,$$
$$\mathcal{Y}w_1 = \{F_0 \subset \langle p_1\rangle \subset \langle p_1, p_2\rangle \subset \langle p_1, p_2, q_1\rangle \subset \mathbb{C}^4\} = \{w_1\} \subset \mathcal{Y}w_6.$$

2.3. The cohomology of stable manifolds. The proof of Theorem 15 with the result of Theorem 9 enables us to calculate the cohomology of the stable manifolds $\mathcal{Y}w$. Let $\mathbf{m}_\vee(w)$ (resp. $\sigma(\mathbf{m}_\vee(w))$) be the set of indices of vertices which are at the left (resp, right) end of a cup in $\mathbf{m}(w)$ (i.e. those at which we have a free choice, and are not constrained to match the fixed point).

Theorem 17. *The cohomology ring of $\mathcal{Y}w$ has a natural presentation of the form*

$$H^\bullet(\mathcal{Y}w;\mathbb{C}) \cong \mathbb{C}[\{x_i\}_{i\in\mathbf{m}_\vee(w)}]/(\{x_i^2\}_{i\in\mathbf{m}_\vee(w)})$$

with the surjective pullback map $i_w^*\colon H^\bullet(X) \to H^\bullet(\mathcal{Y}w)$ given in this presentation by

$$i_S^*(x_i) = \begin{cases} x_i, & i \in \mathbf{m}_\vee(w),\\ -x_{\sigma^{-1}(i)}, & i \in \sigma(\mathbf{m}_\vee(w)),\\ 0, & \textit{otherwise}.\end{cases} \tag{2.1}$$

Proof. Theorem 15 implies that the map in question is surjective and gives the last case in (2.1). The second relation has to hold because of Theorem 15 together with Theorem 9. Finally, the proof of Theorem 15 implies that the dimension of $H^\bullet(\mathcal{Y}w;\mathbb{C})$ equals 2^a, where a is the number of cups in $\mathbf{m}(w)$, hence there are not more relations and the statement follows. □

Example 18. In the situation of Example 13 we have isomorphisms as follows:

$$
\begin{aligned}
H^\bullet(\mathcal{Y}w_1) &\cong \mathbb{C},\\
H^\bullet(\mathcal{Y}w_2) &\cong \mathbb{C}[x_2]/(x_2^2) \cong R,\\
H^\bullet(\mathcal{Y}w_3) &\cong \mathbb{C}[x_1]/(x_1^2) \cong R,\\
H^\bullet(\mathcal{Y}w_4) &\cong \mathbb{C}[x_3]/(x_3^2) \cong R,\\
H^\bullet(\mathcal{Y}w_5) &\cong \mathbb{C}[x_1, x_3]/(x_1^2, x_3^2) \cong R \otimes R,\\
H^\bullet(\mathcal{Y}w_6) &\cong \mathbb{C}[x_1, x_2]/(x_1^2, x_2^2) \cong R \otimes R.
\end{aligned}
$$

3. Pairwise intersections of stable manifolds

3.1. Fixed points of intersections. The first step in understanding the structure of the intersection of stable manifolds is to calculate the torus fixed points which lie in the intersection.

Let w and w' be two row-strict tableaux of shape $(n-k, k)$ with associated cup diagrams $C = \mathbf{m}(w)$ and $D = \mathbf{m}(w')$. Let $\overline{D}C$ be the diagram obtained by taking D, reflecting it in the horizontal line containing the dots and putting it on top of the diagram C, identifying the points with the same label. The result will be (up to homotopy) a collection of lines and circles.

Definition. An *orientation* of $\overline{D}C$ or $\overline{C}D$ is a row strict tableau v such that vD and vC are oriented. In particular, this requires the weight sequence for v to match the one for w at any unmatched points in C, and the one for w' at any unmatched points in D.

Lemma 19. *Let $\mathcal{Y}w, \mathcal{Y}w'$ be stable manifolds in Y with associated cup diagrams C and D. Then the number of fixed points contained in $\mathcal{Y}w \cap \mathcal{Y}w'$ equals the number of orientations of the diagram $\overline{D}C$. In particular, the number of fixed points is either*

- *zero (if there is at least one line where the orientations required by orphaned points are incompatible),*
- *one (if all lines are oriented and there are no circles),*
- *or 2^c (otherwise), where c is the number of circles in $\overline{D}C$.*

Proof. By Lemma 12 the number of fixed points in the intersection of two irreducible components is the number of weight sequences which give rise to an orientation of C and D at the same time, and hence to an orientation of $\overline{D}C$. By Theorem 15 this is true more generally for intersections of two stable manifolds. For each circle there are exactly two such choices of an orientation and for each line there is a unique orientation. There is no orientation if the endpoints of some line are contained in the same cup diagram. The statement follows. $\square$

Corollary 20. *The intersection $\mathcal{Y}w \cap \mathcal{Y}w'$ is*

- *non-empty if and only if $\bar{D}C$ has an orientation,*
- *a single point if and only if there is a unique such orientation.*

Proof. The intersection $\mathcal{Y}w \cap \mathcal{Y}w'$ is projective, and so it is either empty or has a fixed point by Borel's fixed point theorem. Moreover, if $\mathcal{Y}w \cap \mathcal{Y}w'$ contains a point x which is not a fixed point, then the limits $\lim_{t\to 0} t \cdot x$ and $\lim_{t\to\infty} t \cdot x$ exist and are different torus fixed points, since they have different moment map images. □

Example 21. Using the cup diagram in Example 16 one easily obtains the following three sets telling when the intersection $\mathcal{Y}w_i \cap \mathcal{Y}w_j$ is empty, contains exactly one fixed point, or contains exactly two fixed points respectively:

$$\begin{aligned}
&(i, j) \in \{(1,3),(1,4),(1,5)\},\\
&(i, j) \in \{(1,2),(1,6),(2,3),(2,4),(2,5),(3,4),(3,6),(4,6)\},\\
&(i, j) \in \{(2,6),(3,5),(4,5),(5,6)\}.
\end{aligned}$$

3.2. Structure of intersections. To fully describe the structure of the intersections, we will require a bit more machinery. We first restate once more the condition for a flag $\mathcal{F}_\bullet \in Y$ being contained in an irreducible component Y_S. We introduce an equivalence relation on the subspaces F_i of $\mathcal{F}$ by grouping them into classes such that fixing one element of a given class determines our choice of all the others. Consider the cup diagram C associated to S, and let $\epsilon(a) = \sigma(a+1)$.

Definition 22. Let $i \sim j$ (or more precisely $i \sim_C j$) be the equivalence relation on the set $\{1, 2, \dots, n\}$ obtained by taking the transitive closure of the reflexive and symmetric relations $i = j$, or $\epsilon(i) = j$ or $\epsilon(j) = i$ (when ϵ is defined). Note that the set of minimal representatives of the equivalence classes equals $S_\vee$.

For all i, j such that $\epsilon(i) = j$ or vice versa, we have $F_i = N^{(j-i)/2}(F_j)$ for any flag $\mathcal{F} \in Y_S$. Since this condition is transitive, we obtain that whenever $i \sim j$, we have $F_i = N^{(j-i)/2}(F_j)$, and along with attaching a fixed subspace to each orphaned vertex, this is a full set of relations for Y_S.

Proposition 23. *If $b = \sigma(a)$, then $b \sim a - 1$ and $a \sim b - 1$.*

Proof. The first relation is by definition. To get the second, note that $\epsilon(a) < b$ (by the non-crossing condition), and either $\epsilon(a) = b - 1$ (in which case we obtain the desired equivalence), or $a < \epsilon(a) < \epsilon^2(a) < b$ (again, by non-crossing). Since there are finitely many indices between a and b, we must have $b - 1 = \epsilon^\ell(a)$ for some ℓ, and so $a \sim b - 1$. □

Now we introduce distinguished representatives for each equivalence class:

Definition 24. Given two row strict tableaux w and w' with associated cup diagrams C and D, we let $i \approx j$ (or more precisely $i \approx_{C,D} j$ or $i \approx_{w,w'} j$) be the transitive closure of the relations of the form $i \sim_C j$ *and* those of the form $i \sim_D j$. We let $\mathcal{E}(C,D)$ or $\mathcal{E}(w,w')$ denote the set of minimal representatives for $\approx_{C,D}$ with the subset $\mathcal{E}_c(C,D) = \mathcal{E}_c(w,w')$ given by all points lying on a circle in $\bar{D}C$.

Example 25. The equivalence classes for our running example are

$$\sim_{S(\cup\cup)}\colon\ \{0,2,4\},\ \{1\},\{3\},\qquad \sim_{S(\Cup)}\colon\ \{0,4\},\ \{1,3\},\{2\}.$$

There are two equivalence classes for $\approx_{S(\cup\cup),S(\Cup)}$, namely $\{0,2,4\}$ and $\{1,3\}$. The set of minimal representatives are

$$\mathcal{E}(S(\cup\cup),S(\Cup)) = \{0,1\} \quad \text{and} \quad \mathcal{E}_c(S(\cup\cup),S(\Cup)) = \{1\}.$$

Example 26. We denote by $S_1, S_2, \dots, S_5$ the five standard tableaux of Example 4. The set of equivalence classes of $\sim_{S_i}$ are the following:

$$\begin{aligned}
&\sim_{S_1}\colon\ \{\{0,4\},\{1,3\},\{2\},\{5\}\}, && \sim_{S_2}\colon\ \{\{0,2,4\},\{1\},\{3\},\{5\}\},\\
&\sim_{S_3}\colon\ \{\{0,2\},\{1\},\{3,5\},\{4\}\}, && \sim_{S_4}\colon\ \{\{0\},\{1,3,5\},\{2\},\{4\}\},\\
&\sim_{S_5}\colon\ \{\{0\},\{1,5\},\{2,4\},\{3\}\}.
\end{aligned}$$

Now the equivalence classes for $\approx_{S_1,S_4}$ are for instance

$$\{\{0,4\},\{1,3,5\},\{2\}\}$$

with $\mathcal{E}(S_1,S_4) = \{0,1,2\}$ and $\mathcal{E}_c(S_1,S_4) = \{2\}$, since 1 labels a point on a line, whereas 2 labels a point on a circle. The flags contained in $Y_{S_1} \cap Y_{S_4}$ are exactly the flags in Y of the form

$$\{0\} \subset \operatorname{im} N^2 \subset F_2 \subset N^{-1}(F_1) \subset N^{-2}(\{0\}) \subset N^{-2}(F_1) = \mathbb{C}^5.$$

Theorem 27. *The set $\mathcal{E}(D,C)$ of minimal representatives of the equivalence classes contains, apart from zero, exactly the left most points in either a circle or line of $\bar{D}C$.*

Proof. Indeed, let a, b, c be the labels of three points in $\bar{D}C$ such that a and b are connected via a cup and b and c via a cap. According to Proposition 23, we have $c \sim_D b-1 \sim_C a$, so $c \approx_{C,D} a$.

Repeating this argument implies the following: If two points on a circle in $\bar{D}C$ are joined by a path with an even number of arcs, then they are equivalent. Thus all indices on any circle are equivalent either to its leftmost point p, or to a point adjacent to p by a single arc. Applying Proposition 23 again, this shows that each point in the circle is equivalent to p or $p-1$, the latter of which must be equivalent to the leftmost point in another circle or to 0, by induction. $\square$

We note that the set $\mathcal{E}(D, C)$ can be equipped with a partial order defined by $a \geq b$ if the circle a lies on is nested inside that b lies on. Hence outer circles are minimal in this ordering. This poset has a natural rank function $r\colon \mathcal{E}(D, C) \to \mathbb{Z}$ given by 0 on all lines, 1 on all circles not nested inside any other, and thereafter increasing with the depth of nesting. Recall that a flag indexed by a ranked poset is a map of ranked posets from that poset to the ranked poset of subspaces of a given vector space.

The equivalence relation $\approx$ allows us to prove Conjecture 7.1 of [Fun03]:

Theorem 28. *The variety $\mathcal{Y}w \cap \mathcal{Y}w'$ is canonically isomorphic to the space of flags indexed by the ranked poset $\mathcal{E}(D, C)$ invariant under N. In particular,*

(1) *$\mathcal{Y}w \cap \mathcal{Y}w'$ is an iterated fiber bundle of base type $(\mathbb{P}^1, \mathbb{P}^1, \dots, \mathbb{P}^1)$ where the numbers of terms is the number c of closed circles in $\bar{D}C$ (if $c = 0$, the intersection is a point),*

(2) *$\mathcal{Y}w \cap \mathcal{Y}w'$ is smooth, and*

(3) *$H^\bullet(\mathcal{Y}w \cap \mathcal{Y}w') \cong R^{\otimes c}$ as graded vector spaces.*

Proof. The consequences are clear, hence we only prove the first statement. Since any comparable circles are on the same side of each line, we can divide our poset into subsets consisting of the circles between any adjacent lines. The space of flags indexed by this sub-poset in V is isomorphic to space of such flags in $V/F_{\ell(a)}$, where for $a \in \mathcal{E}(D, C)$, we let $\ell(a)$ be the left-most point on the right-most line that a lies on the right side of, and thus our claim is that our intersection is isomorphic to the product of these spaces of flags.

Consider the subspaces $G_a = N^{(a-r(a)+\ell(a))/2}(F_a)/F_{\ell(a)}$. This is a subspace of $V/F_{\ell(a)}$ of dimension $r(a)$.

If $a \geq b$, and $r(a) = r(b) + 1$, then we have $a - 1 \approx b$, since either $a - 1 = b$, or $a - 1$ lies on a circle with leftmost point a'. Since $a' \geq b$, we have $r(a') > r(b)$, so $a \not\geq a'$. Thus, we have $a - 1 \approx a' - 1$, and by induction, our claim follows. Thus $G_a \supset N^{(a-r(a)+\ell(a))/2}(F_{a-1})/F_{\ell(a)} = G_b$, since

$$a - r(a) = ((a-1) - b) + (b - r(b))$$

and $\ell(a) = \ell(b)$.

By induction, this establishes that G_a is indeed a flag over our poset.

Conversely, we can define an element of our intersection, given such a flag, by defining F_i by $N^{-(i-r(i'))/2}(G_{i'} + F_{\ell(i)})$ where i' is the representative of i in $\mathcal{E}(D, C)$.

This variety is an iterated $\mathbb{P}^1$-bundle, since forgetting the vector space attached to a maximal element a obviously defines a map to the set of flags indexed by a poset with this point removed. This map is surjective, since the interval below a is a chain, so the space attached to it can be chosen in increasing order. On the other hand, the fiber of this map is $\mathbb{P}(N^{-1}(G_{a'})/G_{a'})$ for a' the unique element that a covers in this poset (the circle immediately containing it). This is a $\mathbb{P}^1$, since $G_{a'} \subset \operatorname{im} N$, for any $a' \neq a$ for simple dimensional reasons (we must have $r(a') < r(a) \leq k$ since no diagram can have more than k circles, and thus no more than 1 of rank k). This is thus a general result for flags indexed by any poset where all intervals are chains, and the rank is bounded by k. □

This theorem has a natural generalization to intersections of arbitrary finitely many numbers of $\mathcal{Y}w_i$, given by a rank function on the set of equivalence classes of the relation generated by $\sim_{w_i}$ for all i. Let $\mathcal{E}(w_1, \cdots, w_n)$ be the set of minimal elements of these equivalence classes. This can be defined inductively by the following rule:

- If an equivalence class contains a line, $r(i) = 0$.
- If $i \in \mathcal{E}(w_1, \ldots, w_n)$, and $j \in \mathcal{E}(w_1, \ldots, w_n)$ is the minimal representative of $i - 1$, then

$$r(i) = \begin{cases} r(i-1) + 1 & \text{if } i - 1 \equiv j \pmod 2, \\ r(i-1) & \text{if } i - 1 \not\equiv j \pmod 2. \end{cases}$$

This rank function will be of great importance in the next section.

3.3. The cohomology of pairwise intersections as bimodules. Theorem 28 enables us to calculate the cohomology $H^\bullet(\mathcal{Y}w \cap \mathcal{Y}w')$ of the intersection of two stable manifolds as a module over the cohomology of $H^\bullet(X)$, and thus as a $(H^\bullet(\mathcal{Y}w), H^\bullet(\mathcal{Y}w'))$-bimodule.

For any $1 \leq i, j \leq n$ we set $\epsilon(i, j) = 0$ if i and j are not on the same circle in $\overline{\mathbf{m}(w)}\mathbf{m}(w')$, and $\epsilon(i, j)_{w,w'} = \epsilon(i, j) = (-1)^a$ if i and j lie on the same circle with a being the number of arcs in a path between them. Note that, although a depends on the chosen path, the number $(-1)^a$ does not.

Theorem 29. *Assume the intersection $\mathcal{Y}w \cap \mathcal{Y}w'$ is non-empty. Then the cohomology ring $H^\bullet(\mathcal{Y}w \cap \mathcal{Y}w')$ has the presentation*

$$H^\bullet(\mathcal{Y}w \cap \mathcal{Y}w') = \mathbb{C}\left[\{x_i\}\right] / (\{x_i^2\}), \tag{3.1}$$

where the index i runs through $\mathcal{E}_c(\mathbf{m}(w), \mathbf{m}(w'))$. The pullback map

$$i^*_{w,w'} : H^\bullet(X) \to H^\bullet(\mathcal{Y}w \cap \mathcal{Y}w')$$

is surjective and given by

$$i^*_{w,w'}(x_i) = \sum_{j \in \mathcal{E}_c(w,w')} \epsilon(i, j) x_j.$$

In particular, the image of x_i is zero if and only if i does not lie on a closed circle.

Proof. By Theorem 17 we know in particular $\ker i^*_{w,w'} \supseteq \ker i^*_w + \ker i^*_{w'}$. Hence there is a well-defined map

$$\psi\colon H^\bullet(X)/(\ker i^*_w + \ker i^*_{w'}) \to H^\bullet(\mathcal{Y}w \cap \mathcal{Y}w').$$

By Theorem 28, ψ is surjective since the cohomology of the intersection is generated in degree two. Comparing dimensions (Theorem 17 provides the dimension of the left hand side whereas Corollary 28 gives the dimension of the right hand side), we see ψ must be an isomorphism. □

Example 30. The only interesting cases for $\mathcal{Y}w_i \cap \mathcal{Y}w_j$ where $i \neq j$ (notation as in Example 13) are

$$H^\bullet(\mathcal{Y}w_2 \cap \mathcal{Y}w_6) \cong \mathbb{C}[x_2]/(x_2^2), \quad H^\bullet(\mathcal{Y}w_3 \cap \mathcal{Y}w_5) \cong \mathbb{C}[x_1]/(x_1^2),$$
$$H^\bullet(\mathcal{Y}w_4 \cap \mathcal{Y}w_5) \cong \mathbb{C}[x_3]/(x_3^2), \quad H^\bullet(\mathcal{Y}w_5 \cap \mathcal{Y}w_6) \cong \mathbb{C}[x_1]/(x_1^2),$$

since in all other cases where the intersection is non-trivial we get $\mathbb{C}$.

Similar bimodules have appeared previously: first in work of Khovanov [Kho00], [Kho02] in the case $2k = n$, for pairs of standard tableaux; then in the general case in work of the first author [Str09] and [BS08a]. Our construction agrees with the latter two, and so the cohomology rings of stable manifolds $\mathcal{Y}w$ are naturally isomorphic to the endomorphism ring of the indecomposable projective module corresponding to $\mathbf{m}(w)$ for the algebra denoted $\mathcal{K}^{n-k,k}$ of [Str09], [BS08a]. The category of modules over this algebra is equivalent to the category of perverse sheaves on the Grassmannian of k-planes in $\mathbb{C}^n$ (see [Str09]) and related to the representation theory (the so-called category $\mathcal{O}$) of the general Lie algebra $\mathfrak{gl}(n,\mathbb{C})$.

3.4. Background from category $\mathcal{O}$. Let us briefly recall the construction of [Str09] and the connection to (parabolic) category $\mathcal{O}$. For details on and properties of category $\mathcal{O}$ and its parabolic version see for example [BGG76] and [Car80], or Chapter 9 in the recent book [Hum08]).

The symmetric group S_n acts (from the right) on the set $W(n-k,k)$ of weight diagrams with $n-k$ $\wedge$'s and k $\vee$'s by permutation. The stabilizer of the weight $w_{\mathrm{dom}} = \wedge \wedge \cdots \wedge \vee \cdots \vee$ is the Young subgroup $S_{n-k} \times S_k$ of S_n. Hence we get a bijection between the set $W^{n-k,k}$ of shortest coset representatives $S_{n-k} \times S_k \backslash S_n$ and the set $W(n-k,k)$ under which w_{dom} corresponds to the identity element in S_n. On the other hand, the set $W^{n-k,k}$ labels in a natural way also the simple modules in the principal block $\mathcal{O}_0^{n-k,k}$ of the parabolic category $\mathcal{O}^{n-k,k}$ for the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$.

These simple modules are exactly the simple highest weight modules $L(x \cdot 0)$ in the principal block of $\mathcal{O}$ for $\mathfrak{gl}(n,\mathbb{C})$) which are locally finite with respect to the parabolic $\mathfrak{p} = \mathfrak{b} + \mathfrak{l}$, where $\mathfrak{b}$ is the standard Borel given by upper triangular matrices and $\mathfrak{l} \cong \mathfrak{gl}(n-k, \mathbb{C}) \times \mathfrak{gl}(k, \mathbb{C})$ is the subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ given by all $(n-k, k)$-block matrices. Let $P(x \cdot 0) \in \mathcal{O}_0^{n-k,k}$ be the (indecomposable) projective cover of $L(x \cdot 0)$. We have now set up a bijection between the indecomposable modules $P(x \cdot 0)$ and the stable manifolds $\mathcal{Y}w$ by mapping $P(x \cdot 0)$ to its weight diagram in $W(n-k, k)$ which then in turn is associated with some row strict tableau $w = w(x)$ determining the stable manifold $\mathcal{Y}w = \mathcal{Y}w(x)$. Let $\mathrm{Cup}(x)$ be the corresponding cup diagram.

The endomorphism algebra $\mathcal{K}^{n-k,k}$ of a minimal projective generator $\bigoplus_x P(x \cdot 0)$ in $\mathcal{O}_0^{n-k,k}$ has the following description: Let $x, y \in W^{n-k,k}$. Then $\mathrm{Hom}_{\mathcal{O}}(P(x \cdot 0), P(y \cdot 0)) = \{0\}$ in case the diagram $\overline{\mathrm{Cup}(x)}\,\mathrm{Cup}(y)$ cannot be oriented. Otherwise there is an isomorphism of vector spaces

$$\mathrm{Hom}_{\mathcal{O}}(P(x \cdot 0), P(y \cdot 0)) = R^{\otimes c(x,y)},$$

where $c(x, y)$ is the number of circles in the diagram $\overline{\mathrm{Cup}(x)}\,\mathrm{Cup}(y)$ (with $R^{\otimes 0} = \mathbb{C}$ by definition). In particular, thanks to Theorem 28,

$$\mathrm{Hom}_{\mathcal{O}}(P(x \cdot 0), P(y \cdot 0)) \cong H^\bullet(\mathcal{Y}w(x) \cap \mathcal{Y}w(y))$$

as vector spaces.

The endomorphism algebra $\mathcal{K}^{n-k,k}$ can be equipped with a Koszul grading ([BGS96] or [BS10] (Theorem 5.6) and [BS08b] (Theorem 1.1) for a more elementary proof). Let $\widetilde{P}(x \cdot 0)$ be the standard graded lift of $P(x \cdot 0)$. This is a graded $\mathcal{K}^{n-k,k}$-module whose head is concentrated in degree zero and which is isomorphic to $P(x \cdot 0)$ after forgetting the grading. Since $P(x \cdot 0)$ is indecomposable, such a standard graded lift is unique up to isomorphism ([BGS96], Lemma 2.5.3). Then the space $\mathrm{Hom}_{\mathcal{K}^{n-k,k}}(\widetilde{P}(x \cdot 0), \widetilde{P}(y \cdot 0))$ is a graded vector space isomorphic to

$$H^\bullet(\mathcal{Y}w(x) \cap \mathcal{Y}w(y))\langle d(x, y)\rangle, \tag{3.2}$$

where $d(x, y) = k - c(x, y)$. (Since $c(x, y)$ is the dimension of $\mathcal{Y}w(x) \cap \mathcal{Y}w(y)$, the shift encodes its codimension in a Lagrangian in which it is contained.) In particular, $\mathrm{End}_{\mathcal{K}^{n-k,k}}(\widetilde{P}(x \cdot 0)) \cong H^\bullet(\mathcal{Y}w(x))$. The multiplication in $\mathcal{K}^{n-k,k}$ was defined using a TQFT-procedure generalizing Khovanov's (see [Str09], [BS08a], [Kho00]). From the definitions it follows in particular,

$$\mathrm{End}_{\mathcal{K}^{n-k,k}}(\widetilde{P}(x \cdot 0)) \cong H^\bullet(\mathcal{Y}w(x))$$

as graded algebras.

Remark 31. There is an alternative description of the connection between category $\mathcal{O}$ and the geometry of the Slodowy slice. By work of the second author in [Web] and the localization theorem of Ginzburg [Gin08] and Dodd–Kremnitzer [DK09], Theorem 7.4, a singular block $\mathcal{O}_{n-k,k}$ of category $\mathcal{O}$ with singularity type $(n-k,k)$ is equivalent to a subcategory of a category of sheaves for a quantization of the structure sheaf on the Slodowy slice. These sheaves are easily seen to be supported inside the closure of the set of points attracted to the Springer fiber under a particular $\mathbb{C}^*$-action.

The connection to our picture is related to the already mentioned Koszul duality. Recall that the endomorphism algebra of a minimal projective generator of the parabolic category $\mathcal{O}^{n-k,k}$, turns into the Ext-algebra of simples in the singular block $\mathcal{O}_{n-k,k}$. Thus, the two-fold appearance (in the context of the geometry of the Springer) of the algebra studied in our paper is not surprising. However, the precise connection between this representation theoretic quantization construction and the constructions of this paper has not been worked out yet.

3.5. An isomorphism of bimodules. In Conjecture 5.9.2 of [Str09], it was conjectured that for any two standard tableaux S and S', the cohomology $H^\bullet(Y_S \cap Y_{S'})$ is isomorphic, as a bimodule (with the above identifications), to the Hom-space between the corresponding indecomposable projective modules over the algebra $\mathcal{K}^{n-k,k}$. We have the following more general result:

Theorem 32. *There are isomorphisms of graded algebras*

$$\Psi_x : \mathrm{End}_{\mathcal{K}^{n-k,k}}(\tilde{P}(x\cdot 0)) \cong H^\bullet(\mathcal{Y}w(x)), \quad x \in W^{n-k,k}, \tag{3.3}$$

such that under these identifications one can find isomorphism of graded bimodules

$$\Psi_{x,y} : \mathrm{Hom}_{\mathcal{K}^{n-k,k}}(\tilde{P}(x\cdot 0), \tilde{P}(y\cdot 0)) \cong H^\bullet(\mathcal{Y}w(x) \cap \mathcal{Y}w(y))\langle d(x,y)\rangle$$

for any $x, y \in W^{n-k,k}$.

Proof. Let $x \in W^{n-k,k}$. Consider the circle diagram $\overline{\mathrm{Cup}(x)}\,\mathrm{Cup}(x)$ and pick some odd vertex in each circle. If $I(x)$ denotes the set of these vertices, then $H^\bullet(\mathcal{Y}w(x)) \cong \mathbb{C}[\{x_i\}_{i\in I(x)}]/(\{x_i^2\}_{i\in I(x)})$. This follows from Theorem 17 by mapping x_j for $j \in w_\vee$ to $a_j x_i$, where i lies on the same circle as j and $a_j = 1$ if j is odd, whereas $a_j = -1$ if j is even. On the other hand

$$\mathbb{C}[\{x_i\}_{i\in I(x)}]/(\{x_i^2\}_{i\in I(x)}) \cong R^{c(x,x)} \cong \mathrm{End}_{\mathcal{K}^{n-k,k}}(\tilde{P}(x\cdot 0))$$

by mapping the x_i to the X associated with the circle where i lies on. These isomorphisms define graded algebra isomorphisms Ψ_x of the form (3.3). Similarly we define an isomorphism of vector spaces

$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}^{n-k,k}}(\tilde{P}(x\cdot 0), \tilde{P}(y\cdot 0)) &\cong R^{\otimes c(x,y)} \cong \mathbb{C}[\{x_i\}_{i\in I(x,y)}]/(\{x_i^2\}_{i\in I(x,y)}) \\ &= H^\bullet(\mathcal{Y}w(x) \cap \mathcal{Y}w(y))\langle d(x,y)\rangle\end{aligned}$$

by choosing a set $I(x, y)$ of odd vertices, one for each circle in the diagram $\overline{\mathrm{Cup}(x)}\,\mathrm{Cup}(y)$. Hence we have the family $\Psi_{x,y}$ of isomorphisms of vector spaces, which we claim are isomorphisms of bimodules.

To see this let $1 \otimes \cdots \otimes 1 \otimes X \otimes 1 \otimes \cdots \otimes 1$ be the element in $R^{\otimes c(x,x)}$ where the X-factor corresponds to a circle C with leftmost vertex labeled by say m. It acts on $R^{\otimes c(x,y)}$ by multiplication with X on the factor corresponding to the circle containing the vertex m. Under the isomorphism $\Psi_{x,y}$ it corresponds to multiplication with $a_r x_r$, where $r \in I(x, y)$ is on the same circle as m.

Under the isomorphism Ψ_x the element $1 \otimes \cdots \otimes 1 \otimes X \otimes 1 \otimes \cdots \otimes 1$ is mapped to $a_i x_i$, where $i \in I(x)$ is the chosen vertex on the circle C, hence acts by multiplication with $a_i x_i$ on

$$H^\bullet(\mathcal{Y}w(x) \cap \mathcal{Y}w(y))\langle d(x, y)\rangle = \mathbb{C}[\{x_i\}_{i \in I(x,y)}]/(\{x_i^2\}_{i \in I(x,y)}).$$

Since all the elements in $I(x)$ and $I(x, y)$ are odd, this is the same (thanks to the relations in $H^\bullet(\mathcal{Y}w(x) \cap \mathcal{Y}w(y))$ as multiplication with $a_r x_r$ where $r \in I(x, y)$ lies on the same circle as i.

Hence the isomorphisms $\Psi_{x,y}$ are equivariant with respect to the left action. The arguments for the right action are completely analogous. $\square$

4. Convolution algebras

4.1. Definition of convolution. The purpose of this section is to introduce an algebra structure on the direct sum of all bimodules $H^\bullet(\mathcal{Y}w \cap \mathcal{Y}w')$ via a convolution product and compare it with the algebra $\mathcal{K}^{n-k,k}$.

Let $\widetilde{\mathcal{Y}}$ be the disjoint union of the stable manifolds $\mathcal{Y}w$ over all weights w, and let $\widetilde{Y}$ be the disjoint union of the components Y_S over all standard tableaux S, equipped with the obvious maps $\widetilde{\mathcal{Y}} \to Y$ and $\widetilde{Y} \to Y$, so

$$\widetilde{\mathcal{Y}} := \bigsqcup_w \mathcal{Y}w \to Y, \quad \widetilde{Y} := \bigsqcup_S Y_S \to Y.$$

Choose an element $f \in H^*(\widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}})$, that is, a cohomology class of the intersection of each ordered triple of stable manifolds.

Both the cohomology groups

$$H^\bullet(\widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}}) = \bigoplus_{w,w'} H^\bullet(\mathcal{Y}w \cap \mathcal{Y}w'), \tag{4.1}$$

$$H^\bullet(\widetilde{Y} \times_Y \widetilde{Y}) = \bigoplus_{S,S'} H^\bullet(Y_S \cap Y_{S'}) \tag{4.2}$$

have a natural product structure defined by a convolution product we denote $*_f$, given by pulling back, cupping together with f and pushing forward on the diagram.

$$\begin{tikzcd}
\widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}} & & \\
& \widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}} \arrow[ul, "p_{12}"] \arrow[dl, "p_{23}"'] \arrow[r, "p_{13}"] & \widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}} \\
\widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}} & &
\end{tikzcd} \tag{4.3}$$

More explicitly, the product of two classes $\alpha \in H^\bullet(\mathcal{Y}w \cap \mathcal{Y}w')$ and $\beta \in H^\bullet(\mathcal{Y}w' \cap \mathcal{Y}w'')$ is

$$\alpha *_f \beta = (p_{13})_*(p_{12}^*\alpha \cup p_{23}^*\beta \cup f) \in H^\bullet(\mathcal{Y}w \cap \mathcal{Y}w'').$$

In essence, we do the usual convolution after replacing the usual fundamental class on $\widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}}$ with its cap product with f, which one can think of as a "virtual fundamental class." Since any change of orientation can be absorbed into f, we will always give these manifolds the complex orientation.

Obviously, if one is not very careful in one's choice of f, this algebra will lack many desirable properties; in particular, it will not be associative or graded. However, for certain choices of f, it will have these properties. In the next section we will describe a good choice for f which gives us our desired associative graded algebra. In the section following that, we will describe the case $f = 1$, which gives an interesting non-associative graded algebra.

A toric analogue of our situation yielding to interesting associative graded algebras is studied in §4 of [BLPW08]. In that case, the choice of f was simply a careful choice of orientations (so in that case, f is degree 0, but not the identity).

Although our set up algebro-geometric, all the mentioned examples can be interpreted as collections of Lagrangians in ambient symplectic manifolds and hence the following appears natural:

Question 33. What conditions on f must be satisfied to induce an associative product? Can this be expressed in terms of symplectic geometry as a general property of collections of Lagrangian submanifolds (as the components of the Springer fiber are inside the resolved Slodowy slice), maybe together with the data of a certain distinguished bundle on them?

Remark 34. While superficially similar, the convolution constructions in this paper are quite different in flavor from those of Chriss–Ginzburg [CG97]. Our algebra is modeled on the behavior of coherent sheaves (as we discuss later in this paper) or the Fukaya category of a symplectic space in which the Springer fiber lies, not on those of constructible sheaves. If one took the constant constructible sheaves on the components and looked at their Ext-algebra, one could also interpret this in terms of a convolution algebra on the same underlying vector space, but with a different product structure and *different grading*.

We wish to emphasize that these techniques from [CG97] have no bearing on the structure of coherent sheaves, and thus could not be applied to prove Conjecture 42 below; it seems to be something of a remarkable coincidence that cohomology of these varieties describe Ext-algebras in different categories.

4.2. An isomorphism

Theorem 35. *There exists a class $f \in H^\bullet(\widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}})$ such that the bimodule isomorphisms $\Psi_{x,y}$ from Theorem 32 define in fact an isomorphism of algebras*

$$\mathcal{K}^{n-k,k} \cong H^\bullet(\widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}}; \mathbb{C}),$$

where the latter is given the multiplication $*_f$.

With the additional grading shifts as in (3.2) this isomorphism is compatible with the grading. In the case $n = 2k$, it induces an isomorphism of (graded) subalgebras $\mathcal{H}^{k,k} \cong H^\bullet(\widetilde{Y} \times_Y \widetilde{Y})$.

Proof. For purposes of the proof, it will be convenient to use cohomology classes $z_i = (-1)^i x_i$ as our generators, rather than x_i.

Let w', w, w'' be row strict tableaux, with corresponding cup diagrams $C' = \mathbf{m}(w')$, $C = \mathbf{m}(w)$, $C'' = \mathbf{m}(w'')$. The multiplication map

$$\mathcal{K}_{w',w} \otimes \mathcal{K}_{w,w''} \to \mathcal{K}_{w',w''}$$

is described by a cobordism from $\overline{C'}C \sqcup \overline{C}C''$ given by saddle moves on the pairs of cups in C and connecting the "loose ends" of C, with the result given by $\overline{C'}C''$ (see [BS08a]). We view this cobordism as a movie of length $n - k_w$ where we do one saddle move or connection at a time (see (4.4) for an easy example). Of course, this requires choosing a total order on the set of cups and rays in C compatible with the nesting partial order on cups.

Then we construct a sequence $Z_0, Z_1, \dots, Z_{n-k_w}$ of varieties $Z_i \subseteq \mathcal{Y}w' \times \mathcal{Y}w''$, one to each stage in the movie, with equations corresponding to the state of the circle diagram at that point in the cobordism. By definition, Z_i consists of all tuples $(F_\bullet, F'_\bullet)$ of flags such that for any index $j \in \mathcal{E}(w)$

- $F_j = F'_j$ and $F_{\sigma(j)} = F'_{\sigma(j)}$ if j lies on a cup and the corresponding saddle has been done already,
- $F_{\sigma(j)} = N^{-\delta(j)}(F_{j-1})$ and $F'_{\sigma(j)} = N^{-\delta(j)}(F'_{j-1})$ if j lies on a cup and the corresponding saddle has not been done already,

- F_j and F'_j coincide with the space for the fixed point if instead j lies on a line segment and j has not been connected yet, and just
- $F_j = F'_j$ if instead j lies on a line segment and j has been connected already.

Obviously, $Z_0 \cong (\mathcal{Y}w' \cap \mathcal{Y}w) \times (\mathcal{Y}w \cap \mathcal{Y}w'')$ and $Z_{n-k_w} = (\mathcal{Y}w' \cap \mathcal{Y}w)_\Delta$, the diagonal embedding of that variety in $\mathcal{Y}w' \times \mathcal{Y}w''$.

Furthermore, one can easily check that at the i-th step of the cobordism, the dimension of Z_i is the number of circles in the diagram, and that there are seven possibilities (analogous to [BS08a], Section 6), for the next move:

- two circles become one: the dimension drops by 1.
- one circle becomes two: the dimension jumps by 1.
- a circle and line segment become a line segment: the dimension drops by 1.
- a line segment "births" a circle: the dimension jumps by 1.
- there is a saddle between two line segments: the variety is unchanged.
- a line segment becomes a circle: the dimension jumps by 1.
- two line segments become one: the variety is unchanged.

In particular, for any i, we have either $Z_i \subseteq Z_{i+1}$ or $Z_i \supseteq Z_{i+1}$ as smooth subvarieties of $\mathcal{Y}w' \times \mathcal{Y}w''$. The most important claim of this proof is that at each step of the sequence of varieties, pushing forward or pulling back under this inclusion induces (the first author's reinterpretation of) Khovanov's action of the cobordism up to that point on the cohomology of Z_0. when we inductively identify the cohomology of Z_i with the corresponding vector space in Khovanov's construction (or rather the extension of [Str09]) associated to the picture at the i-th place in the movie.

Throughout, we will implicitly use the fact that $H^\bullet(Z_i)$ is a quotient of $H^\bullet(\mathcal{Y}w' \times \mathcal{Y}w'')$, and so use $z_{j,i}$ and $z'_{j,i}$ to denote the image of z_j in $H^\bullet(Z_i)$ coming from the first and second factor respectively. We then identify $H^\bullet(Z_i)$ with the vector space associated to the diagram of Z_i by identifying the degree 2 class on a circle with $z_{j,i}$ (resp. $z'_{j,i}$) if the circle passes through the j-th on the bottom (resp. top) row (see again (4.4) for the obvious definition of first and second row). This class does not depend on which point on the circle we choose (see Section 3.3).

In each case, the pullback or push-forward map is a map of $H^\bullet(\mathcal{Y}w' \times \mathcal{Y}w'')$-modules, so we only have to calculate the image of the identity of $H^*(Z_i)$ at each step of the cobordism.

The claim is true for Z_0 by definition. So assume it to be true for Z_{i-1} and consider the next move.

If the variety is unchanged, the asserted statement is trivial, since the combinatorial multiplication also does nothing in this case ([Str09], §5.4, or Theorem 6.1 (i) in [BS08a]).

Similarly when merging two circles, the pullback of 1 is sent to 1 and $z_{j,i}$ and $z'_{j,i}$ are sent to the same class (since the two corresponding line bundles are isomorphic) which by our convention is $z_{j,i+1} = z'_{j,i+1}$. This is precisely Khovanov's multiplication rule.

In the merging of a circle and line segment, the variable for the circle ($z_{j,i}$ or $z'_{j,i}$) is sent to 0, since the line bundle become trivialized. The same happens in the combinatorial multiplication, [BS08a], (6.1).

Thus, the only tricky case is push-forward. If we do a saddle move on the cup from j to $\sigma(j)$, then 1 is sent to $x_j - x_{\sigma(j)} = (-1)^j(z_{j,i+1} + z_{\sigma(j),i+1})$, since Z_i is the vanishing set in Z_{i+1} of $N^{\delta(j)}\colon V_{\sigma(j)} \to V_j$ and this is the first Chern class of $\mathrm{Hom}(V_{\sigma(j)}, V_j)$. This is precisely the extension of Khovanov's rule (recall that $z_{j,i}$ is 0 if j lies on a line segment, as in [Str09]), (5.4.2), except for the difference of sign, which we will deal with momentarily.

For closing a segment to a circle, we obtain $(-1)^{j+1}z_{j,i+1}$, which again matches the rule from §5.4 of [Str09] (up to sign), since this subvariety is given by the zero set of a section of $\mathrm{Hom}(F_j/F_{j-1}, \mathbb{C})$.

Altogether, this proves that our geometric and the Khovanov–Stroppel algebraic multiplications agree up to signs (not necessarily an overall sign); however, we know exactly how the signs are off from the algebraic multiplication, and thus can correct for them by choosing a different orientation of Z_i. We change the orientation by $(-1)^{\sum_{\ell=1}^{i} j_\ell}$ where

$$j_i = \begin{cases} j & \text{if the } i\text{-th step is a saddle creating a circle on the } (i,\sigma(i)) \text{ cup,} \\ j+1 & \text{if the } i\text{-th step closes a circle at } j, \\ 0 & \text{otherwise.} \end{cases}$$

This precisely corrects for the signs which appeared in the description of the multiplication and we arrive at the Khovanov–Stroppel formulas.

In order to describe this in terms of cohomology classes on $\widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}}$, we need to use base change for clean intersections: if we have a diagram of cleanly intersecting submanifolds

$$\begin{tikzcd} & X & \\ A \arrow[ur, hook, "i_A"] & & B \arrow[ul, hook', "i_B"'] \\ & A \cap B \arrow[ul, hook', "j_A"] \arrow[ur, hook, "j_B"'] & \end{tikzcd}$$

then by standard algebraic topology (analogous to [CG97], Proposition 2.6.47), we have that

$$i_B^*(i_A)_* g = (j_A)_*(e(E) \cup j_B^* g)$$

where $E = i_A^* j_A^* T_X/(j_A^* T_A + j_B^* T_B)$ is the excess bundle of the intersection.

This shows that if we have any chain of manifolds $A_1 \supseteq A_2 \subseteq A_3 \supseteq \cdots \subseteq A_\ell$, such that the intersection of any subset of the A_i's is clean, we can always shorten the chain by doing base change, at the cost of multiplying by the Euler class of a bundle on one of the A_h's. By induction, the iterated push-pull can be described as a pulling back to $\bigcap_i A_i$, multiplying the Euler class of a bundle on $\bigcap_i A_i$, and pushing forward to A_ℓ.

Applying this to the Z_i's, we get a bundle on $\bigcap_i Z_i = \mathcal{Y}w' \cap \mathcal{Y}w \cap \mathcal{Y}w''$ whose Euler class is the desired f on that given component of $\widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}} \times_Y \widetilde{\mathcal{Y}}$; the result follows. □

Since this proof describes the cohomology class f in a somewhat implicit manner, let us attempt to give a more intuitive description at the cost of some loss of precision. We can write Khovanov's cobordism in normal form, that means as union of 3 pieces: one where circles just join together, one where we add some handles to the cobordism, and one where the cobordism branches out to meet the circles of $\overline{C'}C''$.

Geometrically, each of these portions of the cobordism match up with parts of the convolution procedure:

- The merging portion of the cobordism corresponds to pull-back to the triple intersection $\mathcal{Y}w' \cap \mathcal{Y}w \cap \mathcal{Y}w''$.
- The handles portion corresponds to multiplying by f on the triple intersection; in particular, the degree of f is equal to twice the number of handles, i.e. the genus of the cobordism.
- The branching portion corresponds to the push-forward to $\mathcal{Y}w' \cap \mathcal{Y}w''$.

In particular, if the cobordism has genus zero, than it only consists of merges and branches. In this case it follows from the proof of Theorem 35 that the multiplication map

$$H^\bullet(\mathcal{Y}w' \cap \mathcal{Y}w) \otimes H^\bullet(\mathcal{Y}w \cap \mathcal{Y}w'') \to H^\bullet(\mathcal{Y}w' \cap \mathcal{Y}w'')$$

giving rise to our desired algebra is just pulling back to the triple intersection and pushing forward, but with appropriate choices of orientations of the involved manifolds.

4.3. Comparison with the natural choice of orientation. For the sake of completeness we would like to indicate (without proof) in which sense the convolution algebra with our choice of orientation differs from the convolution algebra obtained when we choose the natural complex orientation. The difference will depend on a parameter α, where we set $\alpha = 1$ in case we chose the natural complex orientation, and $\alpha = -1$ for our choice of orientation.

Theorem 36. *Let w', w, w'' be standard tableaux, with the corresponding cup diagrams $C' = \mathbf{m}(w')$, $C = \mathbf{m}(w)$, $C'' = \mathbf{m}(w'')$. The image of*

$$_{w'}1_w \otimes {}_w1_{w''} \in H^\bullet(\mathcal{Y}w' \cap \mathcal{Y}w) \otimes H^\bullet(\mathcal{Y}w \cap \mathcal{Y}w'')$$

in $H^\bullet(\mathcal{Y}w' \cap \mathcal{Y}w'')$ *under the convolution product (in either case) can be calculated as follows: Place* $\overline{C'}C$ *over* $\overline{C}C''$ *and consider the minimal cobordism* $\mathfrak{C}'$ *from this collection of circles to the collection of circles given by* $\overline{C'}C''$ *(see* [Kho00], [BS08a]*).*

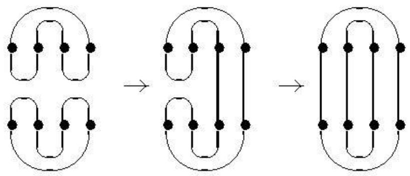
If we consider this cobordism as a union of saddle moves corresponding to the set $S_\vee$ *with respect to* w *(with some fixed order compatible with the nesting) then* ${}_{w'}1_w \otimes {}_w1_{w''}$ *goes to the product* $\prod_{i \in S_\vee} \varphi(i)$ *where*

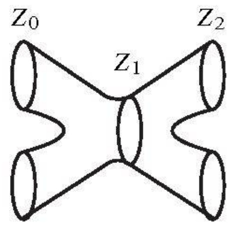
$$\varphi(i) = \begin{cases} 1 & \textit{if the saddle of } i \textit{ joins two circles,} \\ \alpha x_i + x_{\sigma(i)} & \textit{if the saddle of } i \textit{ creates two circles, and } \gamma_{\sigma(i)} \textit{ contains } \gamma_i, \\ x_i + \alpha x_{\sigma(i)} & \textit{if the saddle of } i \textit{ creates two circles, and } \gamma_i \textit{ contains } \gamma_{\sigma(i)}, \\ \alpha x_i + \alpha x_{\sigma(i)} & \textit{otherwise,} \end{cases}$$

where γ_j *denotes the created circle containing the vertex labeled by* j *for any* j.

Proof. Left to the reader. □

Example 37. If, for instance, $C' = \mathrm{Cup}(⋓) = C''$, $C = \mathrm{Cup}(\cup\cup)$ then we have the following possible sequence of diagrams describing $\mathfrak{C}'$ (which is in this case a pair of pants joining two circles to one circle followed by a pair of pants which splits this one circle into two).





The element ${}_{w'}1_w \otimes {}_w1_{w''}$ is then mapped to $(x_1 + \alpha x_2)_{w'}1_{w''}$, since the only place where a circle is split into two is at the cup/cap pair attached to the vertices 1 and 2 (from the left). Alternatively we could have chosen the sequence where we first remove the cup/cap pair attached to the vertices 1 and 2, so that ${}_{w'}1_w \otimes {}_w1_{w''}$ is then mapped to $(x_3 + \alpha x_4)_{w'}1_{w''}$ which equals $(x_1 + \alpha x_2)_{w'}1_{w''}$ in $H^\bullet(\mathcal{Y}w' \cap \mathcal{Y}w'')$. The result will always be independent from the chosen sequence, since any such sequence describes the convolution product. If we swap the roles of C' and C'' then ${}_{w'}1_w \otimes {}_w1_{w''}$ would be mapped to $(\alpha x_1 + \alpha x_3)_{w'}1_{w''}$ in $H^\bullet(\mathcal{Y}w' \cap \mathcal{Y}w'')$.

If $\alpha = -1$, then the resulting algebra is not associative, if $\alpha = 1$ then this is exactly Khovanov's arc algebra (with the extension from [Str09]). It seems natural to search for a topological construction making transparent the difference between these two algebra structures on the same vector space. Our suggestion is to use a

TQFT-like procedure like Khovanov's, but one which is sensitive to the embedding of cobordisms in 3-space. This is what we propose to call an *embedded* 2*-dimensional TQFT*.

Equivalently, one can say that our cobordisms keep track of the nestedness of the circles. In particular, there will be two types of pair of pants cobordisms, namely one which connects one circle with two disjoint, not nested circles in the usual embedding for trousers and a second "unusual" one which connects one circle with two disjoint, but nested circles, with one of the trouser legs pushed down the middle of the other, see Figure 1.

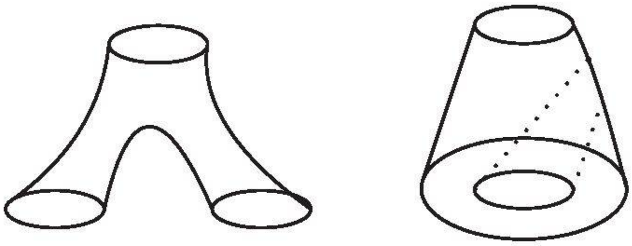


Figure 1. Embedded pair of pants: The usual one for not nested circles, the unusual one for nested circles.

For instance, the minimal cobordism displayed in the previous example would be a composition of a usual pair of pants connecting two circles to one followed by a generalized pair of pants splitting one circle into two nested circles. We now define an embedded version of Khovanov's algebra by assigning the following maps to the pair of pants morphisms:

- To a usual pair of pants joining two (not-nested) circles to one circle, we associate the multiplication $m\colon R \otimes R \to R$, $1 \otimes 1 \mapsto 1$, $X \otimes 1 \mapsto X$ $1 \otimes X \mapsto X$, $X \otimes X \mapsto 0$.
- To the reverse cobordism, splitting one circle into two (not-nested) circles, we associate the comultiplication $\Delta\colon R \to R \otimes R$, $1 \mapsto -X \otimes 1 - 1 \otimes X$, $X \mapsto -X \otimes X$.
  (So far it is exactly the setup of [Kho00], except that our $-X$ is X there.)
- To the "unusual" pair of pants joining two nested circles to one circle, we associate the map $m'\colon R \otimes R \to R$, $1 \otimes 1 \mapsto 1$, $X \otimes 1 \mapsto X$, $1 \otimes X \mapsto -X$, $X \otimes X \mapsto 0$, where the first tensor factor is associated with the outer circle and the second with the inner circle.
- To the reverse cobordism, we associate the linear map $\Delta'\colon R \to R \otimes R$, $1 \mapsto X \otimes 1 - 1 \otimes X$, $X \mapsto -X \otimes X$, where again the first tensor factor is outer and the second is inner.

Keeping track of the nestedness using the rules above describes exactly the (non-associative) multiplication on the convolution algebra with the ordinary complex orientation.

5. Coherent sheaves and cup functors

In this section, we want to connect our approach with the one of [CK08], where an alternative (geometric) categorification of the Jones polynomial was obtained. It agrees on the K_0-group level with the Reshetikhin–Turaev tangle invariant [RT90] associated with $\mathcal{U}_q(\mathfrak{sl}_2)$, hence also with the decategorification of [Str05] which in turn restricts to Khovanov's functorial invariant. The precise categorical or functorial connection between the geometric and algebraic-representation theoretic picture is however open at the moment. In the following, we give some partial results which indicate that the geometric picture might differ slightly from the algebraic one. We note that our results partially overlap with those obtained independently in [Ann].

5.1. Geometric background. Now, we consider the Springer fiber as a Lagrangian subvariety inside a larger smooth space. This ambient space is best defined as the pre-image under the Springer resolution of a normal slice to the nilpotent orbit through N at N. We denote this space by $\mathcal{S}_{n-k,k}$. Our Springer fiber is included as the fiber over N. The interested reader can consult [MV07] for details. For our purposes, the only important fact about the varieties is that they are smooth, and each component of the Springer fiber is a Lagrangian subvariety inside $\mathcal{S}_{n-k,k}$. These spaces were for instance used in the geometric construction of knot invariants via Floer homology in the work of [SS06] and [Man07].

In the case where $n = 2k$, this variety has a more convenient description, which played an important role in the work of Cautis and Kamnitzer [CK08], who used a compactification of it to define homological knot invariants. So from now on let $n = 2k$. Let M be the nilpotent endomorphism of $\mathbb{C}^{2n}$ with two equally sized Jordan blocks. Let $\{p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n\}$ be the basis of $\mathbb{C}^{2n}$ such that M has Jordan Normal Form (with $Mp_i = p_{i-1}$ and $Mq_i = q_{i-1}$) with the $\mathbb{C}^*$-action as before on V. Now define the space of flags

$$Z_n = \{F_0 \subset F_i \subset \cdots \subset F_{n-1} \subset F_n \subset \mathbb{C}^{2n} \mid \dim_{\mathbb{C}} F_i = i, MF_i \subset F_{i-1}\}.$$

We can identify our original vector space V with the span of the p_i, q_i for $1 \le i \le n$, with the endomorphism M restricting to the nilpotent endomorphism N. Thus, we can identify Y with the subset of Z_n where $F_n = V$. Furthermore, $\mathcal{S}_{k,k}$ can be identified with the subset of Z_n where the projection of F_n onto V (by forgetting the coordinates with higher indices) is an isomorphism.

In Section 4 of [CK08], the authors define functors between the bounded derived categories $\mathcal{D}(Z_{n'})$ of ($\mathbb{C}^*$-equivariant) coherent sheaves on $Z_{n'}$ (for varying n') which provide a categorified tangle/knot invariant, in the following sense: to each (n_1, n_2)-tangle, there is an associated functor from $\mathcal{D}(Z_{n_1})$ to $\mathcal{D}(Z_{n_2})$ which is a tangle invariant, up to isomorphism, and decategorifies to the Reshetikhin–Turaev tangle invariant associated with (the quantum group of) $\mathfrak{sl}_2$ on the level of the K_0-group.

In fact, for all k, the space $\mathcal{S}_{n-k,k}$ is embedded in Z_n matching the obvious inclusion of the Springer fiber (see [MV07]). The compactification obtained by closing this embedding seems to be a likely candidate for extending [CK08] beyond the case of blocks of equal size. However, we will not pursue this idea further in this paper.

Let $\mathsf{Coh}(Z_n)$ be the category of coherent sheaves on Z_n with its bounded derived category $\mathcal{D}^b(Z_n)$.

For our purposes, the $\mathbb{C}^*$-action carefully tracked in [CK08] is unnecessary, so we will ignore it. Since all the functors of concern are defined by Fourier–Mukai transforms, they have non-equivariant analogues.

Note that, Z_0 is just a point, and so $\mathsf{Coh}(Z_0)$ is the category of vector spaces over $\mathbb{C}$.

If C is a cup diagram corresponding to a standard tableau S with two rows of size k, we can view it as a $(0, 2k)$-tangle and consider the associated functor

$$\varphi_C : \mathcal{D}^b(Z_0) = \mathcal{D}^b(\mathsf{Vect}) \to \mathcal{D}^b(Z_n)$$

as defined in [CK08] (the interested reader may note Equation (5.2) below serves as an inductive definition of this functor). In general, the functors associated with crossingless tangles are not exact in the standard t-structure on $\mathsf{Coh}(Z_n)$ (though of course, they are exact in the triangulated sense). In the special case of a $(0, 2k)$-tangle, the situation is much easier: First of all, the functor maps a vector space to an actual sheaf (i.e. is exact in the usual t-structure), hence defines (or comes from) a functor

$$\varphi_C : \mathsf{Coh}(Z_0) = \mathsf{Vect} \to \mathsf{Coh}(Z_n). \tag{5.1}$$

Secondly, as with any exact functor from vector spaces to any abelian category, φ_C is already determined by its value on $\mathbb{C}$.

5.2. Half-densities. We let $\Omega^{1/2}(Y_S)$ denote a square-root of the canonical bundle on the component Y_S. This sheaf exists by the theorem below (but more generally, it exists at least as a twisted sheaf) and is unique, since the Picard group of any iterated $\mathbb{P}^1$-bundle is torsion-free.

Lemma 38. *Each component Y_S carries a unique square-root of the canonical bundle. In fact, $\Omega^{1/2}(Y_S) \cong \bigotimes_{i \in S_\vee} V_i$.*

Proof. Abbreviate $A = Y_S$. As in any bundle, one can always compute the canonical bundle on the total space as the product of the canonical bundle on the base and the relative canonical bundle. Since each component is fibered over one for a smaller diagram, to show the result by induction, we need only show that the relative canonical bundle of that fibration has a square root.

Let i be an index such that $\sigma(i) = i + 1$. In this case, our fibration is

$$\mathbf{q}_i : A \to A',$$

where A' is the component for our cup diagram with the cup from i to $i + 1$ deleted. Since V_i^{-1} is isomorphic to $\mathcal{O}(1)$ on the fibers, we have that our fibration is the projectivization of the bundle $\mathbf{q}_{i*}V_i^{-1} \cong V_j \oplus V_j^{-1}$ where j is the left end of the cup immediately nested over i. Thus, we have an exact sequence

$$0 \to \Omega_{A/A'} \to \mathrm{Hom}(\mathbf{q}_i^*(V_j \oplus V_j^{-1}), V_i) \to \mathrm{Hom}(V_i, V_i) \to 0.$$

The multiplicativity of determinants in exact sequences shows that

$$\Omega_{A/A'} \cong \det(\Omega_{A/A'}) \cong \det\big(\mathrm{Hom}(\mathbf{q}_i^*(V_j \oplus V_j^{-1}), V_i)\big) \cong V_i^2.$$

Thus, $\Omega_{A/A'}^{1/2} \cong V_i$. On the other hand, $\mathbf{q}_i^*\Omega_{A'} \cong \bigotimes_{j \in S_\vee \setminus \{i\}} V_i$. Thus, the result follows by induction. □

In fact, these square roots are exactly the images of the 1-dimensional vector space under the functors φ_C associated to cup diagrams:

Theorem 39. *Let W be any finite dimensional vector space. Then*

$$\varphi_C(W) \cong W \otimes_{\mathbb{C}} \Omega^{1/2}(Y_S).$$

Proof. Our proof is by induction. Assume that the result is true for all smaller n, in particular for the corresponding cup diagrams with less than n points. This set of diagram include for instance the diagram C' which is C with one of its minimal cups removed. Denote by S' the corresponding standard tableau and let j and $j + 1$ be the endpoints of this cup.

Then if $\mathbf{i} = \mathbf{i}_j$ is the inclusion of the locus where $N(F_{j+1}) = F_{j-1}$ holds, and $\mathbf{q} = \mathbf{q}^j$ is the projection defined on this locus to Z_{n-2} given by forgetting F_j and F_{j+1} as well as applying N to all subspaces larger than F_{j+1}, we have ([CK08], 4.2.1) the equation

$$\varphi_C(W) = \mathbf{i}_*(V_j \otimes \mathbf{q}^*(\varphi_{C'}(W))). \tag{5.2}$$

By induction, our proposition holds for C', so this equation becomes

$$\varphi_C(W) = \mathbf{i}_*(V_j \otimes \mathbf{q}^*(W \otimes_{\mathbb{C}} \Omega^{1/2}(Y_{S'})).$$

On the other hand, we have the usual exact sequence of normal bundles

$$0 \to \mathbf{q}^*\mathcal{N}_{Y_{S'}/Y_{n-2}} \to \mathcal{N}_{Y_S/Y_n} \to V_{j+1}^* \otimes V_j\big|_{Y_S} \to 0.$$

Since $V_{j+1}^*|_{X_n^j} \cong V_j|_{X_n^j}$, we see that $\Omega(Y_S) \cong \mathbf{q}^*\Omega(Y_{S'}) \otimes V_j^{\otimes 2}$, so $\Omega^{1/2}(Y_S) \cong \mathbf{q}^*\Omega^{1/2}(Y_{S'}) \otimes V_j$. Applying this in equation (5.2), we obtain the desired result. □

On the way of trying to connect the different categorifications of the Reshetikhin–Turaev tangle invariants one could hope for an isomorphism of rings

$$\mathrm{Ext}^\bullet_{\mathrm{Coh}(\mathcal{S}_{n-k,k})}(i_*\Omega^{1/2}(A), i_*\Omega^{1/2}(A)) \cong \mathrm{End}(\tilde{P}(x \cdot 0))$$

where $P(x \cdot 0)$ is the indecomposable projective module associated with a component A under the isomorphisms of (3.3), or more generally a formula like

$$\mathrm{Ext}^\bullet_{\mathrm{Coh}(\mathcal{S}_{n-k,k})}(i_*\Omega^{1/2}(A), i_*\Omega^{1/2}(B)) \cong \mathrm{End}(\tilde{P}(x \cdot 0), P(y \cdot 0)) \tag{5.3}$$

as graded vector spaces (up to our usual shifts). On the other hand, based on work such that of Leung ([Leu02]), one might expect that

$$\mathrm{Ext}^\bullet_{\mathrm{Coh}(\mathcal{S}_{n-k,k})}(i_*\mathcal{O}_A, i_*\mathcal{O}_A) \cong H^*(A), \tag{5.4}$$

or, more generally,

$$\mathrm{Ext}^\bullet_{\mathrm{Coh}(\mathcal{S}_{n-k,k})}(i_*\mathcal{O}_A, i_*\mathcal{O}_B) \cong H^\bullet(A \cap B) \tag{5.5}$$

as graded vector spaces (up to our usual shifts), where $\mathcal{O}_A$ denotes the structure sheaf on A. In the following we will show that, in fact, all of the above statements are true, except the last one (which might appear as a surprise).

The importance of these square roots of canonical bundles (the so-called *half-densities*) in connection with derived categories of coherent sheaves and the failure of (5.5) have previously been noticed by physicists in connection with the so-called *Freed–Witten anomaly*, (see [FW99]).

A mathematical manifestation of this phenomenon appears when considering the spectral sequences computing the $\mathrm{Ext}^\bullet$-groups of the square roots of the canonical sheaves in contrast to the ones computing the $\mathrm{Ext}^\bullet$-groups of the structure sheaves of these varieties, as carefully explained for instance in papers such as [KS02], [Sha04].

The crucial point hereby is that by the adjunction formula for the canonical bundle on a subvariety ([Huy05], Proposition 2.2.17), using half-densities instead of structure sheaves compensates for the appearance of the normal bundle in the E_2-term of the spectral sequence of [KS02] which we use below.

Let now $n = (n-k)+k$ as usual. Let A, B be components in the corresponding Springer fiber Y included in the resolution to the Slodowy slice $\mathcal{S}_{n-k,k}$. Let $i: A \hookrightarrow \mathcal{S}_{n-k,k}$, and $j: B \hookrightarrow \mathcal{S}_{n-k,k}$ be the natural inclusions. The formula (5.3) is by Theorem 32 equivalent to the following result:

Theorem 40. *There is an isomorphism of graded vector spaces*

$$\mathrm{Ext}^\bullet_{\mathrm{Coh}(\mathcal{S}_{n-k,k})}(i_*\Omega(A)^{1/2}, j_*\Omega(B)^{1/2}) \cong H^\bullet(A \cap B)\langle d(A,B)\rangle,$$

Proof. First, note that since $\mathcal{S}_{n-k,k}$ is holomorphic symplectic and the components of the Springer fiber are Lagrangian, so the symplectic form induces an isomorphism between the normal bundle and cotangent bundle. Further, this shows that on an intersection, the quotient

$$T_{\mathcal{S}_{n-k,k}}|_{A\cap B}/(T_A|_{A\cap B} + T_B|_{A\cap B})$$

will be the cotangent bundle $T^*_{A\cap B}$.

Given these facts, the result follows almost immediately from Theorem A.1 of [CKS03] (though the theorem appeared with a less complete proof in [KS02]). In our case, this gives a spectral sequence

$$H^p(A\cap B, \wedge^q T^*_{A\cap B}) \cong H^{p,q}(A\cap B) \Rightarrow \mathrm{Ext}^{p+q+d(A,B)}_{\mathsf{Coh}(\mathcal{S}_{n-k,k})}(i_*\Omega(A)^{1/2}, j_*\Omega(B)^{1/2})$$

where $H^{p,q}$ denotes the usual Dolbeault cohomology. The first Chern classes of line bundles (which lie in $H^{1,1}(A\cap B)$) generate $H^{p,q}(A\cap B)$, so it has only (p,p) Dolbeault cohomology. Thus, this spectral sequence has no non-trivial differentials, and we obtain the desired isomorphism. □

Corollary 41. *There is an isomorphism of graded vector spaces*

$$\mathrm{Ext}^\bullet_{\mathsf{Coh}(\mathcal{S}_{n-k,k})}\Big(\bigoplus_A i_{A*}\Omega(A)^{1/2}, \bigoplus_A i_{A*}\Omega(A)^{1/2}\Big) \cong H(\widetilde{Y}\times_Y \widetilde{Y}),$$

where the sum runs over all irreducible components A.

Of course, both the left and right side of this isomorphism have natural ring structures given by Yoneda product and by convolution. The statement of the following conjecture would give a very explicit description of the Ext-algebra of half-densities:

Conjecture 42. There is an isomorphism of algebras

$$\mathrm{Ext}^\bullet_{\mathsf{Coh}(\mathcal{S}_{n-k,k})}\Big(\bigoplus_A i_{A*}\Omega(A)^{1/2}, \bigoplus_A i_{A*}\Omega(A)^{1/2}\Big) \cong H(\widetilde{Y}\times_Y \widetilde{Y}).$$

Remark 43. Of course, this Ext-algebra is, as a vector space, *also* isomorphic to Khovanov's arc algebra, and at the moment, the authors are unsure as to which product on this vector space corresponds to Yoneda's. Having clarified this conjecture it would not be too difficult to extend it to (4.1).

An affirmative or negative answer to this conjecture would direct us toward further questions on the correct geometric perspective on knot homology:

Question 44. Is it possible to construct a functorial tangle invariant and categorification of the Jones polynomial using our new convolution algebras? If so what is the relation to previous geometrical ones ([CK08], [SS06], [Man07]) and to algebraic/representation theoretic approaches ([Kho02], [Str05])?

As was noted in [Ann], these half-densities are so-called *exotic sheaves* as introduced by Bezrukavnikov [Bez06]. This suggests that the conjecture and questions above could be investigated using the noncommutative Springer resolution and related techniques of algebraic geometry.

We can perform a partial verification of Conjecture 42, considering only a single component at a time.

Theorem 45. *Let A be an irreducible component of Y and $i\colon A \hookrightarrow \mathcal{S}_{n-k,k}$ the inclusion. Let $\mathcal{O}_A$ be the structure sheaf on A. Then there are isomorphisms of graded rings*

$$\operatorname{Ext}^{\bullet}_{\mathsf{Coh}(\mathcal{S}_{n-k,k})}(i_*\Omega^{1/2}(A), i_*\Omega^{1/2}(A)) \cong \operatorname{Ext}^{\bullet}_{\mathsf{Coh}(\mathcal{S}_{n-k,k})}(i_*\mathcal{O}_A, i_*\mathcal{O}_A) \cong H^{\bullet}(A).$$

Remark 46. Note that thanks to (3.3) the rings appearing in the theorem can also be identified with the endomorphism rings of indecomposable projective and at the same time injective modules in the associated parabolic category $\mathcal{O}$ for $\mathfrak{sl}_n$. Based on the results of this paper, the slight generalization from components to arbitrary stable manifolds should not be too difficult by mimicking [BS10], Section 6.

Proof of Theorem 45. The first isomorphism follows from the fact that $\Omega^{1/2}(A)$ deforms to a global line bundle on $\mathcal{S}_{n-k,k}$, the pullback of $\prod_{i\in S} V_i$ from Z_n. (It's worth noting, this isomorphism does *not* hold in general.)

To compute the Ext-algebra on the left hand side we first compute the Ext-sheaves $\mathcal{E}xt^{\bullet}(i_*\mathcal{O}_A, i_*\mathcal{O}_A)$. The irreducible component A is smooth, hence a local complete intersection ([Har77], Example 8.22.1). Since we can work locally, we might assume that A is the zero locus of a regular section $s \in H^0(E)$ for some bundle E on Z.

Then we have the Koszul resolution

$$0 \to \textstyle\bigwedge^n E^* \to \bigwedge^{n-1} E^* \to \cdots \to \bigwedge^1 E^* \to E^* \to \mathcal{O}_Z \to i_*\mathcal{O}_C \to 0, \qquad (5.6)$$

where the differential maps $f_1 \wedge f_2 \wedge \cdots \wedge f_r \in \bigwedge^r E^*$ to

$$\sum_{i=1^r} (-1)^{i-1} f_i(s) f_1 \wedge f_2 \wedge \cdots \wedge f_{i-1} \wedge f_{i+1} \wedge \cdots \wedge f_r.$$

The Koszul complex is exact, since s is a regular section (see [GH78], page 688).

The beginning of the resolution (5.6) defines a surjection

$$E^* \to \mathcal{I} \to 0 \qquad (5.7)$$

where $\mathcal{I}$ is the ideal sheaf of A in Z. Tensoring with $i_*\mathcal{O}_A$, we get a surjection $i_*E^* \to \mathcal{I}/\mathcal{I}^2 = \mathcal{N}^*_{A/Z}$. This map is an isomorphism for dimension reasons.

Now $\mathcal{E}xt^\bullet(i_*\mathcal{O}_A, i_*\mathcal{O}_A)$ can be calculated as the cohomology sheaves of the complex $i_* \wedge^\bullet E$. Since $i_* s = 0$, the differentials in this complex are all zero, hence

$$\mathcal{E}xt^\bullet(i_*\mathcal{O}_A, i_*\mathcal{O}_A) \cong \wedge^* \mathcal{N}_{A/Z} \tag{5.8}$$

as graded vector spaces.

We have to compare the ring structure. We first claim that there is a map of differential graded algebras

$$c\colon \bigwedge^\bullet E \to \mathcal{E}xt^\bullet(i_*\mathcal{O}_A, i_*\mathcal{O}_A)$$

sending $\xi \in \wedge^r E$ to the contraction with ξ, denoted c_ξ. The differentials in the Koszul complex (5.6) are given by contraction c_s with the section s, and c_ξ and c_s super commute. Therefore $c(\xi)$ is a chain map of degree k. Since contraction satisfies $c_\xi \circ c_\zeta = c_{\xi\wedge\zeta}$, the map c intertwines the wedge product on the source space with the composition in the target space. Passing to cohomology, we obtain that (5.8) is an isomorphism of algebras.

Since the component A is Lagrangian inside Z, we have a canonical isomorphism between the normal bundle of A in Z and the cotangent bundle of A, in formulas $\mathcal{N}_{A/Z} \cong T_A^*$.

It thus follows from the isomorphism of (5.8) that the cohomology of the Ext-sheaf $\mathcal{E}xt^\bullet(i_*\mathcal{O}_A, i_*\mathcal{O}_A)$ is canonically isomorphic to the Dolbeault cohomology $H^\bullet(A; \wedge^\bullet T_A^*)$ (here we abuse notation, and identify the vector bundle $\wedge^q T_A^*$ with its sheaf of holomorphic sections) with its usual product induced by $\wedge$. By the Hodge theorem, this is isomorphic to the de Rham cohomology $H^\bullet(A;\mathbb{C})$ equipped with the cup product.

Thus, we have the local-global spectral sequence

$$\begin{aligned} E_2^{p,q} : H^p(A; \wedge^q T^*A) &= H^p(A; \wedge^q \mathcal{N}_{A/Z}) = H^{p+q}(A; \mathcal{E}xt^\bullet(i_*\mathcal{O}_A, i_*\mathcal{O}_A)) \\ &\Longrightarrow \mathrm{Ext}^{p+q}_{\mathsf{Coh}(\mathcal{S}_{n-k,k})}(i_*\mathcal{O}_A, i_*\mathcal{O}_A). \end{aligned}$$

This sequence collapses due to the Hodge diamond only having diagonal support, as in the proof of Theorem 40, and thus induces a ring isomorphism from $H^\bullet(A;\mathbb{C})$ to the ring $\mathrm{Ext}^\bullet_{\mathsf{Coh}(\mathcal{S}_{n-k,k})}(i_*\mathcal{O}_A, i_*\mathcal{O}_A)$. □

6. Exotic sheaves and highest weight categories

In fact, we would like to propose a correspondence between weight sequences and certain sheaves on Z_n, which extends that sending a full crossingless matching on n points to half-densities on the corresponding component of the Springer fiber.

Let w be a weight sequence of length n. We denote by $r(w)$ be the number of cups in $C(w)$. Let

$$Z_w = \{F_* \in Z_n \mid F_{i-1} = N^{\delta(i)} F_{\sigma(i)} \text{ for } i \text{ and } \sigma(i) \text{ connected in } C(w)\}$$

with its embedding $j = j_w \colon Z_w \to Z_n$. If $r(w) = 1$ we have the map $q \colon Z_w \to Z_{n-2}$ as in (5.1), and in general a map $p \colon Z_w \to Z_{n-2r(w)}$ by taking compositions of such maps, one for each cup.

Consider the line bundle $V_w = \bigotimes_{i \in w_\vee} V_i$ on Z_w and set $\mathcal{S}_w = j_* V_w$. In the setup of [CK08], the latter has the following description: to the cup diagram $C(w)$, Cautis and Kamnitzer associated a functor $F \colon D^b(Z_{n-2r(w)}) \to D^b(Z_n)$ and (by comparing the definitions) we have $j_* V_w = F(V_{\tilde{w}})$, where $\tilde{w}$ is the induced weight sequence on the orphaned points of $C(w)$.

We have the following two extreme cases:

- If $r(w) = 0$, then F is just the identity functor and we have $\mathcal{S}_w = V_w$.
- If $r(w) = k$, then Z_w is just a point and in fact, $\mathcal{S}_w = \varphi_{C(w)}(\mathbb{C})$ as in Theorem 39.

Let Θ_w be the set of weight diagrams which differ from w by switching the signs on opposite ends of any number of cups in $\mathrm{Cup}(w)$. For an object $M \in D^b(Z_n)$ we denote by $[M]$ its class in $K_0(D^b(Z_n))$. Then the following holds.

Proposition 47.

$$[\mathcal{S}_w] = \sum_{w' \in \Theta_w} (-1)^{\ell(w)-\ell(w')} [V_{w'}]. \tag{6.1}$$

In particular, the classes of $[\mathcal{S}_w]$ and $[V_w]$ span the same sublattice of the Grothendieck group.

Proof. We induct on the number of cups in $C(w)$. If this is 0, we have reduced to the fact that $\mathcal{S}_w \cong V_w$ in this case. Otherwise, we can write $\mathcal{S}_w = \varphi_i(\mathcal{S}_v)$ where i is on the left end of a minimal cup in $C(w)$ and v is the induced weight sequence on $S - \{i, i+1\}$. Now, we can assume that $[\mathcal{S}_v] = \sum_{v' \in \Theta_v} (-1)^{\ell(v)-\ell(v')} [V_{v'}]$. Let v^+ be v with the cup at $i, i+1$ added and marked with $\vee\wedge$, and v^- be the same, but with $\wedge\vee$ at $i, i+1$ instead. Then, as we noted previously, we have an exact sequence

$$0 \to V_{v^-} \to V_{v^+} \to \varphi_i(V_v) \to 0 \tag{6.2}$$

and thus in the Grothendieck group, $[\varphi_i(V_v)] = [V_{v^+}] - [V_{v^-}]$.

Note that $\Theta_w = \Theta_v^+ \sqcup \Theta_v^-$, and $\ell(v) \equiv \ell(v^+) \equiv \ell(v^-) + 1 \pmod 2$ so

$$\begin{aligned} [\mathcal{S}_w] = [\varphi_i(\mathcal{S}_v)] &= \sum_{v' \in \Theta_v} (-1)^{\ell(v)-\ell(v')} \left([V_{(v')^+}] - [V_{(v')^-}]\right) \\ &= \sum_{w' \in \Theta_w} (-1)^{\ell(w)-\ell(w')} [V_{w'}]. \end{aligned}$$

$\square$

Remark 48. Proposition 47 should be compared with [BS08a], (5.12), which implies that $[\mathcal{S}_w] = \sum_{w' \in \Theta_w} d_{w,w'}(-1)[V_{w'}]$, where $d_{w,w'}$ is a Kazhdan–Lusztig polynomial (arising from perverse sheaves on Grassmannians).

By the Cellular Fibration Lemma ([CG97], Lemma 5.5.1) and Theorem 6.2 of [CK08], the V_w's generate $D^b(Z_n)$, and in fact are a basis of the Grothendieck group. As a consequence of Theorem 47 we have the following:

Corollary 49. *The objects $\mathcal{S}_w$ generate the category $D^b(Z_n)$ and are a basis of its Grothendieck group.*

By Remark 48, the transformation matrix between the two bases is given by Kazhdan–Lusztig polynomials.

Following ideas of Bezrukavnikov, we now define a t-structure on $D^b(Z_n)$ (not the standard one) for which the $\mathcal{S}_w$ form a complete set of simple objects in the heart. This heart will then be equivalent to the category of finite dimensional modules over our convolution algebra K_n. The algebra K_n is quasi-hereditary with the standard modules given by the line bundles V_w's.

First, we define the necessary ordering on the set of weights. This is the standard ordering on weights which can be explicitly given in this case by saying that $w \leq v$ if for each i, there are more $\vee$'s in the last i indices for v than w. Alternatively, it's the partial ordering generated by the basic relation that changing $\wedge\vee$ to $\vee\wedge$ is getting smaller in the ordering.

Lemma 50. *The full additive category generated by the $\mathcal{S}_w$'s is semisimple. Let still be $n = 2k$. Let w, w' weights. Then $\operatorname{Hom}_{D^b(\mathrm{Coh}(Z_n))}(\mathcal{S}_w, \mathcal{S}_{w'}) = \{0\}$ for $w \neq w'$ and $\operatorname{Hom}_{D^b(\mathrm{Coh}(Z_n))}(\mathcal{S}_w, \mathcal{S}_{w'}) = \mathbb{C}$ otherwise.*

Proof. Claim: let $d(w, w')$ be as in (3.2) then either $\operatorname{Ext}^i_{\mathrm{Coh}(Z_n)}(V_w, V_w)$ is trivial or its minimal nonzero degree is $i = d(w, w')$, and so the lemma follows directly. Note that in case w, w' correspond to standard tableaux, then the claim is clear by Theorem 40. It of course also holds for $n = 2$.

Assume first w is minimal in the partial order $\leq$ and w' is arbitrary. If $w = w'$, then $\operatorname{Ext}^i_{\mathrm{Coh}(Z_n)}(\mathcal{S}_w, \mathcal{S}_w) \cong \operatorname{Ext}^i_{\mathrm{Coh}(Z_n)}(V_w, V_w) \cong H^i(V_w^* \otimes V_w) \cong H^i(\mathcal{O}_{Z_n}) = \mathbb{C}$ and the statement follows. If $w \neq w'$ then $C(w')$ has at least one minimal cup connecting say i and $i + 1$. Using the adjunctions [CK08], Lemma 4.4, for cup and cap functors in we can remove this cup in expense of applying a cap functor $F_i[1]$ to V_w. Let a, b be the i-th and $i + 1$-st labels of w and denote by v the weight which is obtained from w by removing these two points. Then by [CK08], 6.3, we have the following four cases: $F_i V_w = 0$ if $ab = \vee\vee$ or $ab = \wedge\wedge$, and then of course $\operatorname{Ext}^i_{\mathrm{Coh}(Z_n)}(V_w, V_w) = \{0\}$. We have $F_i V_w \cong V_v[1]$ if $ab = \wedge\vee$, in which case the claim follows by induction (note that we removed a clockwise cup/cap). We have $F_i V_w \cong V_v$ if $ab = \vee\wedge$, in which case the claim follows by induction noting that we removed a counter-clockwise cup/cap. Hence the statement is true for minimal w. Assume w is not minimal. Choose a minimal cup in $C(w)$ say at the vertices $i, i+1$. Applying again adjunction properties, we can remove this cup by the expense of a cap functor $F_i[-1]$. If this cap creates a circle, we have $\mathrm{Ext}^{\bullet}_{\mathsf{Coh}(Z_n)}(V_w, V_{w'}) \cong \mathrm{Ext}^{\bullet}_{\mathsf{Coh}(Z_n)}(V_v, V_{v'}) \oplus \mathrm{Ext}^{\bullet+2}_{\mathsf{Coh}(Z_n)}(V_v, LB_{v'})$ by [CK08], Corollary 5.10. Since $d(w,w') = d(v,v')$, the statement follows. If this cap does not create a circle, and $\mathrm{Ext}^{\bullet}_{\mathsf{Coh}(Z_n)}(V_w, V'_w) \neq \{0\}$, then using again Corollary 5.10 of [CK08] and adjointness properties we get

$$\begin{aligned}
&\mathrm{Ext}^{\bullet+1}_{\mathsf{Coh}(Z_n)}(V_w, V'_w) \oplus \mathrm{Ext}^{\bullet-1}_{\mathsf{Coh}(Z_n)}(V_w, V'_w)\\
&\quad\cong \mathrm{Ext}^{\bullet}_{\mathsf{Coh}(Z_n)}(G_i F_i V_w, V'_w)\\
&\quad\cong \mathrm{Ext}^{\bullet}_{\mathsf{Coh}(Z_n)}(G_i F_i V_x, V'_w)\\
&\quad\cong \mathrm{Ext}^{\bullet}_{\mathsf{Coh}(Z_n)}(V_x, F_i G_i V'_w)\\
&\quad\cong \mathrm{Ext}^{\bullet+1}_{\mathsf{Coh}(Z_n)}(V_w, V_z) \oplus \mathrm{Ext}^{\bullet-1}_{\mathsf{Coh}(Z_n)}(V_w, V_z),
\end{aligned}$$

where z is obtained from w', and x is obtained from w, by swapping the labels at the vertices i and $i+1$. In particular,

$$\mathrm{Ext}^{\bullet}_{\mathsf{Coh}(Z_n)}(V_x, V'_w) \cong \mathrm{Ext}^{\bullet}_{\mathsf{Coh}(Z_n)}(V_w, V_z).$$

On the other hand $d(x,w') = d(w,z)$. (To see this assume first vertex l and k are connected to the vertices i and $i+1$ via a cup diagram in $C(w')$). $\square$

The following is now a direct consequence of Lemma 3 in [Bez03]:

Theorem 51. *There exists a unique t-structure of $D^b(\mathsf{Coh}(Z_n))$, such that the $\mathcal{S}_w$'s form the simple objects.*

Proof. We only have to verify the assumptions of [Bez03], Lemma 3. These are however just Lemma 50 together with the observation that the $\mathcal{S}_w$'s are sheaves (so that $\mathrm{Hom}_{D^b(\mathsf{Coh}(Z_n))}(\mathcal{S}_w, \mathcal{S}_{w'}[l]) = \{0\}$ for any positive l). $\square$

Following Bezrukavnikov, we call this the *exotic* t-structure. We call the heart of this t-structure the category of exotic sheaves $\mathfrak{Ex}_n$. The main result of this section is the following:

Theorem 52. *There is a highest weight structure on $\mathfrak{Ex}_n$ such that the sheaves V_w are standard.*

Lemma 53. *The sheaf V_w is exotic, and its composition factors are all of the form $\mathcal{S}_{w'}$ for $w' \leq w$, with $\mathcal{S}_w$ appearing exactly once.*

Proof. We induct on both the number of points, and the ordering given above. Our base case is still that where $C(w)$ is empty, where this is obvious.

As we noted before, we can write w as v^+ for some sequence v on fewer points. Recall the exact sequence (6.2). Now, by induction on the number of points $\varphi_i(V_v)$ is exotic, and has the desired composition series (since $\mathcal{S}_v$ appears once in V_v, we have $\mathcal{S}_w = \varphi_i(\mathcal{S}_v)$ appearing once), and by induction on the partial order, V_{v^-} is exotic, and all its composition factors are strictly smaller than w. □

Lemma 54. *The line bundles V_v form an exceptional sequence, that is, we have*

$$\mathrm{Ext}^\bullet_{\mathrm{Coh}(Z_n)}(V_w, V_v) = 0,$$

for all $v \not\geq w$.

Proof. As usual, we have $\mathrm{Ext}^i(V_w, V_v) \cong H^i(V_w^* \otimes V_v)$. Thus our problem reduces to computing the cohomology of certain line bundles.

Consider the map $\pi\colon Z_n \to Z_{n-1}$ given by forgetting the top space. We note that if $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_{n-1})$ is a vector valued in $\{1, 0, -1\}$, then

$$\pi_*(V_{\boldsymbol{\epsilon}} \otimes V_n^j) \cong \begin{cases} V_{\boldsymbol{\epsilon}} \otimes \mathrm{Sym}^{-j}(W), & j \leq 0, \\ 0, & j = 1, \\ V_{\boldsymbol{\epsilon}} \otimes \mathrm{Sym}^{j-2}(W)[-1], & j \geq 2, \end{cases}$$

where $W \cong \pi_* V_n$ is a rank 2 vector bundle which is an extension

$$0 \to V_{n-1}^{-1} \to W \to V_{n-1} \to 0.$$

Thus, if a vector bundle is an extension of line bundles of the form $V_{\boldsymbol{\epsilon}} \otimes V_n^j[m]$, for $|j-1| \leq k$, then its push-forward is an extension of ones of the form $V_1^{\epsilon_1} \otimes \cdots \otimes V_{n-2}^{\epsilon_{n-2}} \otimes V_{n-1}^{j'}[m']$ where $|j'-1| \leq k + \epsilon_{n-1}$.

Applying this inductively, we see that the $\ell - n$-fold push-forward $\pi_*^{\ell,n} V_w^* \otimes V_v$ is an extension of line bundles of the form $V_{\boldsymbol{\epsilon}} \otimes V_n^j[m]$ where $|j-1| \leq g_n + 1$ where g_n is the difference between the number of $\vee$'s in the last $\ell - n$ places of w and those in those places in v. If this number is ever negative, then $j = -1$, so the $\ell - n + 1$-fold push-forward is trivial. Thus, if this push-forward is non-trivial, we must have this number always non-negative, that is, we must have $v \geq w$. □

Proof of Theorem 52. Lemmata 53 and 54 show the line bundles V_w's are standard covers of the simple modules $\mathcal{S}_w$. This shows that an object has negative Ext vanishing with all V_w if and only if it does with $\mathcal{S}_w$ (since $\mathrm{Ext}^i(\mathcal{S}_w, X) = \mathrm{Ext}^i(V_w, X)$ for $i < 0$ if $\mathrm{Ext}^i(\mathcal{S}_v, X) = 0$ for all $i < 0$ and $v < w$), and the Serre subcategory generated by $\{V_v[i]\}_{i \geq 0}$ is the same as that generated by $\{\mathcal{S}_v[i]\}_{i \geq 0}$. That is, the exotic t-structure is exactly the one which Bezrukavnikov calls the t-structure of the exceptional sequence $\{V_v\}$. By [Bez03], Proposition 2, the heart of this t-structure is highest weight, with $\{V_w\}$ as its standards. □

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