Small points on rational subvarieties of tori

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Abstract. Let V be a subvariety of a torus defined over the rational numbers. We study the distribution of points of small Weil's height on V. We simplify the proof and we improve previous results by the first author and S. David. We obtain a totally explicit version of a generalized Dobrowolski result on the Lehmer problem.

1. Introduction

In this article we study the distribution of the small points of proper subvarieties of the torus $\mathbb{G}_m^n$ defined over $\mathbb{Q}$. For $n = 1$, the problem corresponds to finding lower bounds for the Weil height of an algebraic number. Let α be a non-zero algebraic number of degree D which is not a root of unity. Lehmer (see [Leh]) asked whether there exists an absolute constant $c > 0$ such that $h(\alpha) \geq \frac{c}{D}$. The best known result in this direction is Dobrowolski's result ([Dob]): if $D > 1$,

$$ h(\alpha) \geq \frac{c}{D} \left(\frac{\log D}{\log \log D} \right)^{-3} $$

for some absolute constant $c > 0$. Dobrowolski's theorem was generalized to $\mathbb{Q}$-irreducible subvarieties $V \subseteq \mathbb{G}_m^n$ in a series of articles by David and the first author. They prove the Generalized Dobrowolski Bound stated below. Their proofs are long and involved. Mainly, they need an intricate descent argument, hard to read by non specialists. This descent has been used in several occasions by other authors. Our first achievement in this paper is a simple and short proof of an explicit and improved version of the Generalized Dobrowolski Bound. More precisely, we generalize this statement describing the distribution of small points for different invariants. In addition we improve some bounds in the applications.

We fix the usual embedding of $\mathbb{G}_m^n$ in $\mathbb{P}^n$ given by $x = (x_1, \ldots, x_n) \mapsto (1 : x_1 : \cdots : x_n)$. For a set $S \subseteq \mathbb{G}_m^n$, we denote by $\overline{S}$ the Zariski closure of S in $\mathbb{G}_m^n$. On $\mathbb{P}^n$

we consider the Weil logarithmic absolute height, denoted by $h(\cdot)$. Given $\varepsilon > 0$ we denote by $S(\varepsilon)$ the set of $\boldsymbol{\alpha} \in S \cap \mathbb{G}_m(\bar{\mathbb{Q}})$ of height $\leq \varepsilon$. A variety $V \subseteq \mathbb{G}_m^n$ is the intersection of $\mathbb{G}_m^n$ with a variety of $\mathbb{P}^n$ defined over $\bar{\mathbb{Q}}$. Note that the varieties which appear in this paper are not necessarily irreducible or equidimensional. However we consider only proper subvarieties of $\mathbb{G}_m^n$, therefore we say subvariety of $\mathbb{G}_m^n$ for proper subvariety of $\mathbb{G}_m^n$. We define the essential minimum $\hat{\mu}^{\mathrm{ess}}(V)$ of V as the infimum of the set of $\varepsilon > 0$ such that $V(\varepsilon)$ is Zariski-dense in V. We say that $B \subset \mathbb{G}_m^n$ is torsion if it is a translate of a subtorus by a torsion point. The Kronecker theorem for points and the Bogomolov conjecture (Zhang [Zha]) for varieties of positive dimension yield

$$\hat{\mu}^{\mathrm{ess}}(V) > 0 \text{ if and only if } V \text{ is not a union of torsion varieties.} \tag{1.1}$$

According to different geometric and arithmetic assumptions, we relate $\hat{\mu}^{\mathrm{ess}}(V)$ to different invariants of V, proving essentially sharp effective versions of (1.1). Lehmer's conjecture can be seen as a sharp effective version of (1.1) for points. The Generalized Dobrowolski Bound is a quasi optimal effective version of (1.1) for varieties *defined over* $\mathbb{Q}$ of arbitrary dimension. For varieties over arbitrary number fields which are not union of translates of subtori we speak of Effective Bogomolov. This case has been treated in our previous work [Amo-Via]. Note that there are intersections between the two problems, namely for varieties over $\mathbb{Q}$ which are not translates. Therefore an interesting new case treated in this work, is the one of translates defined over $\mathbb{Q}$ and specially the case of 0-dimensional varieties consisting of the conjugates of a non-torsion point $\boldsymbol{\alpha} \in \mathbb{G}_m^n(\bar{\mathbb{Q}})$. Naturally the Galois group plays a key role in this work.

Let us introduce relevant invariants of a proper projective subvariety $V \subseteq \mathbb{P}^n$. The obstruction index $\omega(V)$ is the minimum degree of a hypersurface Z containing V. Define $\delta(V)$ as the minimal degree δ such that V is, as a set, the intersection of hypersurfaces of degree $\leq \delta$. Finally, define $\delta_0(V)$ as the minimal degree δ_0 such that there exists an intersection $X = Z_1 \cap \cdots \cap Z_r$ of hypersurfaces Z_j of degree $\leq \delta_0$ such that any $\bar{\mathbb{Q}}$-irreducible component of V is a $\bar{\mathbb{Q}}$-irreducible component of X. In Corollary 2.3 we prove that if V is defined over $\mathbb{Q}$, we can choose the above hypersurfaces $Z, Z_1, \ldots, Z_r$ also defined over $\mathbb{Q}$.

The following effective version of (1.1) is proved in [Amo-Dav] for $\dim V = 0$, in [Amo-Dav] for $\mathrm{codim}\, V = 1$ and in [Amo-Dav] for varieties of arbitrary dimension.

Generalized Dobrowolski Bound. *Let V be a subvariety of $\mathbb{G}_m^n$ defined over $\mathbb{Q}$ of codimension k. Let us assume that V is not contained in any union of proper torsion varieties.*

Then, there exist two positive constants $c(n)$ and $\kappa(k) = (k+1)(k+1)!^k - k$ such that

$$\hat{\mu}^{\mathrm{ess}}(V) \geq \frac{c(n)}{\omega(V)} \left(\log 3\omega(V)\right)^{-\kappa(k)}. \tag{1.2}$$

To recover a slightly weaker version of Dobrowolski's theorem it is sufficient to take V equal to the set of conjugates of the algebraic number α.

For a subvariety V of $\mathbb{G}_{\mathrm{m}}^n$, we denote by V^* the complement in V of the union of the torsion varieties $B \subseteq V$. By (1.1) the minimum of the height on $V^*(\overline{\mathbb{Q}})$ is > 0. In [Amo-Dav] is proved that for a $\mathbb{Q}$-irreducible V and $\boldsymbol{\alpha} \in V^*(\overline{\mathbb{Q}})$

$$h(\boldsymbol{\alpha}) \geq \frac{c(n)}{\delta(V)}(\log 3\delta(V))^{-\kappa(n)}. \tag{1.3}$$

where $c(n) > 0$ is not computed and where $\kappa(n) \approx n^{n^2}$ is as above. Notice that this lower bound implies (1.2), with a possible worse exponent on the remainder term. To see that, apply (1.3) to a hypersurface $Z \supseteq V$ defined over $\mathbb{Q}$ and of degree $\omega(V)$.

For $n = 1$ Dobrowolski's result remains the best known. In order to simplify the exposition and the computation of the constants we prefer to assume $n \geq 2$. Our first achievement is a simple and short proof of an explicit and improved version of (1.3):

Theorem 1.1. *Let* $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ *be a* $\mathbb{Q}$*-irreducible variety of dimension* d. *Then, for any* $\boldsymbol{\alpha} \in V^*(\overline{\mathbb{Q}})$

$$h(\boldsymbol{\alpha}) \geq \delta(V)^{-1}\big(935n^5 \log(n^2\delta(V))\big)^{-(d+1)(n+1)^2}.$$

In short, the exponent $\kappa(n)$ on the remainder term is improved by one exponential. In addition the constant $c(n)$ is computed. This could be useful in possible applications. However, the most interesting aspect remains the simplicity of the new method. We avoid the technical descent argument and the generalization of Philippon zero's estimate used in [Amo-Dav]. This new method could find other applications, as for instance in the context of the Relative Lehmer Problem, where methods similar to the ones of David and the first author are used (see [Del]).

To be able to use a conclusive geometric induction similar to the one presented in [Amo-Via] we first need to produce a new sharp lower bound for $\hat{\mu}^{\mathrm{ess}}(V)$ in terms of $\delta_0(V)$ for varieties which are not union of torsion varieties.

Theorem 1.2. *Let* V *be a subvariety of* $\mathbb{G}_{\mathrm{m}}^n$ *of codimension* k, *defined and irreducible over* $\mathbb{Q}$. *Assume that* V *is not a union of torsion varieties. Let*

$$\theta_0 = \delta_0(V)(52n^2 \log(n^2\delta_0(V)))^{(n+1)(k+1)}.$$

Then there exists a hypersurface Z *defined over* $\mathbb{Q}$ *of degree at most* θ_0 *which does not contain* V *and such that*

$$V\big(\theta_0^{-1}\big) \subseteq V \cap Z.$$

This theorem is the arithmetic counterpart to [Amo-Via], Theorem 2.1. On one side, V has to be defined over $\mathbb{Q}$, assumption not necessary in [Amo-Via]. On the

other side V can be a union of translates of torsion varieties by non-torsion points, situation to avoid in [Amo-Via]. Despite some similarity, the methods used in other works are not sufficient to prove this theorem. As in [Amo-Via], we first produce an inequality involving some parameters, $\hat{\mu}^{\text{ess}}(V)$ and the Hilbert functions of two varieties related to V (Theorem 3.1). Some ingredients of the proof of Theorem 3.1 come from [Amo-Dav]. The main difference is the following. In the quoted paper, using Siegel's lemma, the authors construct one auxiliary function vanishing on V and then they extrapolate to show that the obstruction index of $[p]V$ is small. Here we use Siegel's lemma in its full power and we find a family of linearly independent auxiliary functions vanishing on V. Then, we extrapolate at $[p]V$ for each auxiliary function. We don't use an interpolation determinant, as in [Amo-Via], because the problem is not symmetric. Another important difference is that, to decode the diophantine information in Theorem 3.1 it is not sufficient to use the estimates for the Hilbert function due to M. Chardin and P. Philippon [Cha-Phi], like we do in [Amo-Via]. In the present situation we need a refinement of their results which is proved in the appendix of this article by M. Chardin and P. Philippon. A further subtle point is to control the behavior of δ_0 under the action of groups (Proposition 2.7). The final geometric induction allows us to prove the main result of this article:

Theorem 1.3. *Let $V_0 \subseteq V_1$ be subvarieties of $\mathbb{G}_{\mathrm{m}}^n$, defined over $\mathbb{Q}$, of codimensions k_0 and k_1 respectively. Assume that V_0 is $\mathbb{Q}$-irreducible. Let*

$$\theta = \delta(V_1)\big(935n^5 \log(n^2\delta(V_1))\big)^{(k_0-k_1+1)(k_0+1)(n+1)}.$$

Then,

- *either there exists a $\mathbb{Q}$-irreducible B union of torsion varieties such that $V_0 \subseteq B \subseteq V_1$ and $\delta_0(B) \le \theta$,*

- *or there exists a hypersurface Z defined over $\mathbb{Q}$ of degree at most θ such that $V_0 \nsubseteq Z$ and $V_0(\theta^{-1}) \subseteq Z$.*

In Section 5, we show how to deduce Theorem 1.1. In addition we prove some corollaries. Combining Theorem 1.1 with the estimate on the sum of the degrees of the maximal torsion varieties of V ([Amo-Via], Corollary 5.3), we can give the following complete description of the small points of V.

Corollary 1.4. *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be a $\mathbb{Q}$-irreducible variety of dimension d. Let*

$$\theta = \delta(V)\big(935n^5 \log(n^2\delta(V))\big)^{(d+1)(n+1)^2}.$$

Then

$$\overline{V(\theta^{-1})} = B_1 \cup \cdots \cup B_t,$$

where $B_1, \ldots, B_t$ are the maximal torsion varieties of V. In addition, $\delta_0(B_j) \leq \theta$ and

$$\sum_{j=1}^{t} \theta^{\dim(B_j)} \deg(B_j) \leq \theta^n.$$

A direct application of Theorem 1.3 allows us to show

Corollary 1.5. *Let $V \subseteq \mathbb{G}_m^n$ be a $\mathbb{Q}$-irreducible subvariety of codimension k which is not contained in any union of proper torsion varieties. Then*

$$\hat{\mu}^{\mathrm{ess}}(V) \geq \omega(V)^{-1} \big(935 n^5 \log(n^2 \omega(V)) \big)^{-k(k+1)(n+1)}.$$

As mentioned, also Theorem 1.1 implies a similar but less sharp lower bound for the essential minimum, where the exponent on the remainder term is $n(n+1)^2$ instead of the better $k(k+1)(n+1)$.

An important application of Corollary 1.5 is a lower bound for the product of the heights of multiplicatively independent algebraic numbers. For instance, this kind of result is used by Bombieri, Masser and Zannier to show the finiteness of the intersection of a transverse curve with the union of all subtori of codimension two [Bom-Mas-Zan]. From Corollary 1.5 we deduce the following refined version of [Amo-Dav], Theorem 1.6:

Corollary 1.6. *Let $\alpha_1, \ldots, \alpha_n$ be multiplicatively independent algebraic numbers in a number field K of degree $D = [K : \mathbb{Q}]$. Then*

$$h(\alpha_1) \ldots h(\alpha_n) \geq D^{-1} \big(1050 n^5 \log(3D) \big)^{-n^2(n+1)^2}.$$

The dependence on δ (or ω) of our results is essentially sharp. However, the dependence in the dimension n of the ambient variety remains mysterious. One could conjecture that for all $\mathbb{Q}$-irreducible linear subvarieties $V \subseteq \mathbb{G}_m^n$ and for all $\boldsymbol{\alpha} \in V^*(\overline{\mathbb{Q}})$ we had $h(\boldsymbol{\alpha}) \geq c$ for some positive absolute constant c (not depending on n). This is false, as the following example shows. Let $V_n \subseteq \mathbb{G}_m^n$ be the hypersurface defined by the equation

$$x_1 + \cdots + x_{n-1} + x_n = 0.$$

We claim that, as n tends to ∞,

$$\min_{\boldsymbol{\alpha} \in V_n^*} h(\boldsymbol{\alpha}) \to 0.$$

Indeed, let $n \geq 3$. Consider for instance the point $\boldsymbol{\alpha} \in \mathbb{G}_m^n(\overline{\mathbb{Q}})$ whose coordinates are the roots $\alpha_1, \ldots, \alpha_n$ of the polynomial $f(x) = x^n - 2x - 6$. Observe that f is

irreducible by Eisenstein's criterion. Moreover $\boldsymbol{\alpha} \in V_n$, because the coefficient of x^{n-1} in f is zero. We now show that $\boldsymbol{\alpha}$ has small height. For a non-zero integer a, let $\mathbf{a} = (a, \ldots, a) \in \mathbb{G}_m^n$. Since $\boldsymbol{\alpha}^n = \mathbf{2} \cdot \boldsymbol{\alpha} + \mathbf{6}$ we obtain

$$n\,h(\boldsymbol{\alpha}) = h(\boldsymbol{\alpha}^n) = h(\mathbf{2} \cdot \boldsymbol{\alpha} + \mathbf{6}) \leq h(\mathbf{2} \cdot \boldsymbol{\alpha}) + h(\mathbf{6}) + \log 2 \leq h(\boldsymbol{\alpha}) + \log 24.$$

Thus

$$h(\boldsymbol{\alpha}) \leq \frac{\log 24}{n-1}.$$

We claim that $\boldsymbol{\alpha} \in V_n^*$. Assume on the contrary that $\boldsymbol{\alpha}$ is in a torsion variety contained in V_n. From the description of [Sch], p. 163, of the torsion varieties contained in a linear variety, we see that there exist $i < j$ such that $u = \alpha_i / \alpha_j$ is a root of unity. Note that $u \neq 1$ because f has distinct roots. Thus

$$0 = f(\alpha_j) - f(u\alpha_j) = (1 - u^n)\alpha_j^n - 2(1-u)\alpha_j.$$

Let $\gamma = (1 - u^n)/(1 - u)$. Then γ is an algebraic integer and $\gamma\alpha_j^{n-1} = 2$. Passing to norms, we infer that $\pm 6 = \mathrm{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\alpha_j)}(\alpha_j)$ divides a power of 2. This is a contradiction. Thus $\boldsymbol{\alpha} \in V^*(\overline{\mathbb{Q}})$ and $h(\boldsymbol{\alpha}) \leq \frac{\log 24}{n-1}$.

2. Geometry

2.1. Algebraic interpolation. In the introduction, we have already mentioned the definitions of $\omega(V)$ and $\delta_0(V)$ for a projective variety $V \subseteq \mathbb{P}^n$. Let us be more precise and give some further details and useful relations.

Definition 2.1. *Let $V \subseteq \mathbb{P}^n$ be a projective variety and let K be a subfield of $\overline{\mathbb{Q}}$.*

  i) *The obstruction index $\omega_K(V)$ is the minimum degree of a hypersurface defined over K containing V.*

  ii) *We define $\delta_{K,0}(V)$ as the minimal degree δ such that there exists an intersection X of hypersurfaces defined over K of degree $\leq \delta$ such that every $\overline{\mathbb{Q}}$-irreducible component of V is a $\overline{\mathbb{Q}}$-irreducible component of X.*

  iii) *Suppose that V is defined over K. We define $\delta_K(V)$ as the minimal degree δ such that V is, as a set, the intersection of hypersurfaces defined over K of degree $\leq \delta$.*

If $K = \overline{\mathbb{Q}}$ we shall omit the index $\overline{\mathbb{Q}}$.

Note that the definition of $\delta_{K,0}$ makes sense for every number field K, independently of the field of definition L of V. Indeed, $V' = \bigcup_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)} \sigma(V)$ is defined over K and the $\overline{\mathbb{Q}}$-irreducible components of V are components of V'. On the contrary, δ_K can only be defined for extensions of the field of definition of V. Indeed if V

is the intersection of hypersurfaces over K then it is also defined over K. In addition, if V is defined over K, then in the above definition ii), it is equivalent to require that every K-irreducible component of V is a K-irreducible component of X.

Clearly, for L a field extension of K, $\omega_K \geq \omega_L$, $\delta_{K,0} \geq \delta_{L,0}$ and $\delta_K \geq \delta_L$. We are now going to show that these are equalities for extensions L of the field of definition K of V.

Let G be a group acting on $\mathbb{G}_m^n$. For any subset S of $\mathbb{G}_m^n$ we define

$$S^G = \bigcap_{g \in G} g(S),$$

$$G \cdot S = \bigcup_{g \in G} g(S).$$

In what follows we provide relations between the obstruction indices of V and V^G in two special cases, namely for G the Galois group (Lemma 2.2 below) and for G the kernel of the "multiplication by l" (Lemma 2.4).

Lemma 2.2. *Let K be a number field and let Z be a hypersurface defined over some extension L of K. Then there exist $D \leq [L : K]$ and hypersurfaces $Z_1, \ldots, Z_D$ defined over K and of degree $\leq \deg Z$ such that*

$$Z^{\mathrm{Gal}(\overline{\mathbb{Q}}/K)} = Z_1 \cap \cdots \cap Z_D.$$

Proof. Let $F(\boldsymbol{x}) \in L[\boldsymbol{x}]$ be an equation defining Z. We fix a basis $\{e_j\}$ of L/K and we write $F(\boldsymbol{x}) = \sum e_j F_j(\boldsymbol{x})$ with $F_j(\boldsymbol{x}) \in K[\boldsymbol{x}]$. Up to order, we can suppose $F_j(\boldsymbol{x}) \neq 0$ for $j = 1, \ldots, D$ and $F_j(\boldsymbol{x}) = 0$ for $j > D$. Define Z_j to be the zero set of $F_j(\boldsymbol{x})$, for $j \leq D$. Clearly $Z^{\mathrm{Gal}(\overline{\mathbb{Q}}/K)} \supseteq Z_1 \cap \cdots \cap Z_D$. We now show the reverse inclusion. Let $\boldsymbol{\alpha} \in Z^{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$. Let each $\sigma_1, \ldots, \sigma_{[L:K]}$ be an extension to $\overline{\mathbb{Q}}$ of each of the $[L : K]$ embeddings of L in $\overline{\mathbb{Q}}$ fixing K. Then, for every i, also $\sigma_i^{-1}(\boldsymbol{\alpha}) \in Z^{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$. Since the F_j are invariant under the action of any such σ_i, we obtain that for every $i \leq [L : K]$

$$0 = \sigma_i(F(\sigma_i^{-1}(\boldsymbol{\alpha})) = \sigma_i \left(\sum e_j F_j(\sigma_i^{-1}(\boldsymbol{\alpha})) \right)$$
$$= \sigma_i \left(\sum e_j(\sigma_i^{-1} F_j(\boldsymbol{\alpha})) \right) = \sum \sigma_i(e_j) F_j(\boldsymbol{\alpha}).$$

The matrix $(\sigma_i e_j)_{i,j}$ is non singular. This implies that $F_j(\boldsymbol{\alpha}) = 0$ for all $1 \leq j \leq [L : K]$. This shows the inclusion $Z^{\mathrm{Gal}(\overline{\mathbb{Q}}/K)} \subseteq Z_1 \cap \cdots \cap Z_D$. $\square$

Corollary 2.3. *Let V be a variety defined over a number field K. Then $\delta_K(V) = \delta(V)$, $\omega_K(V) = \omega(V)$ and $\delta_{K,0}(V) = \delta_0(V)$.*

Proof. We already mentioned that such invariants decrease by fields extensions. Then we have only to show that $\delta_K(V) \le \delta(V)$, $\omega_K(V) \le \omega(V)$ and $\delta_{K,0}(V) \le \delta_0(V)$.

Let $X \supseteq V$ be an intersection of hypersurfaces of degree $\le \delta$, for $\delta \in \mathbb{N}$. By Lemma 2.2 $X^{\mathrm{Gal}(\bar{\mathbb{Q}}/K)}$ is an intersection of hypersurfaces defined over K, of degree $\le \delta$. Since V is defined over K, $V = V^{\mathrm{Gal}(\bar{\mathbb{Q}}/K)} \subseteq X^{\mathrm{Gal}(\bar{\mathbb{Q}}/K)}$.

Choosing $\delta = \delta(V)$ and $X = V$ we see that $\delta_K(V) \le \delta(V)$. Choosing $\delta = \omega(V)$ and $X \supseteq V$ a hypersurface defined over $\bar{\mathbb{Q}}$ of minimal degree δ we see that $\omega_K(V) \le \omega(V)$. Choose at last $\delta = \delta_0(V)$ and $X \supseteq V$ such that every $\bar{\mathbb{Q}}$-irreducible component of V is a $\bar{\mathbb{Q}}$-irreducible component of X. Let W be a $\bar{\mathbb{Q}}$-irreducible component of V. Then W is a $\bar{\mathbb{Q}}$-irreducible component of X. Since $V \subseteq X^{\mathrm{Gal}(\bar{\mathbb{Q}}/K)} \subseteq X$, we see that W is a $\bar{\mathbb{Q}}$-irreducible component of $X^{\mathrm{Gal}(\bar{\mathbb{Q}}/K)}$, too. Thus $\delta_{K,0}(V) \le \delta_0(V)$. $\square$

We shall recall some important relations between the obstruction indices. If V is equidimensional of codimension k, then, by a result of M. Chardin ([Cha]),

$$\omega(V) \le n \deg(V)^{1/k}. \tag{2.4}$$

Moreover,

$$\omega(V) \le \delta_0(V) \le \delta(V) \le \deg(V) \le \delta_0(V)^k. \tag{2.5}$$

The first three inequalities are immediate. The last one follows from [Phi], Corollary 5, p. 357 (with $m = n$, $S = \mathbb{P}^n$ and $\delta = \delta_0(V)$).

2.2. An upper bound for $\delta_0([l]V)$. Let V be an equidimensional variety and let $l \ne 0$ be an integer. We need a bound for $\delta_0([l]V)$. We denote by $[l] : \mathbb{G}_m^n \to \mathbb{G}_m^n$, $\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha}^l = (\alpha_1^l, \dots, \alpha_n^l)$ the "multiplication by l" and by $\ker[l]$ its kernel. The following lemma is analogue to Lemma 2.2. Here we consider the action of $\ker[l]$, whereas in Lemma 2.2 we considered the Galois action.

Lemma 2.4. *Let $Z \subset \mathbb{G}_m^n$ be a hypersurface. Then, there exist $D \le l^n$ and hypersurfaces $Z_1, \dots, Z_D$ of degree $\le \deg Z$ such that $\ker[l] \cdot Z_j = Z_j$ and*

$$Z^{\ker[l]} = Z_1 \cap \cdots \cap Z_D.$$

Proof. Let $F(\boldsymbol{x}) \in \bar{\mathbb{Q}}[\boldsymbol{x}]$ be an equation for Z. Performing the euclidean divisions by l on the exponents of each monomial, we can write

$$F(\boldsymbol{x}) = \sum_{\boldsymbol{\lambda} \in \Lambda} \boldsymbol{x}^{\boldsymbol{\lambda}} F_{\boldsymbol{\lambda}}(\boldsymbol{x}^l)$$

where $\boldsymbol{x}^l = (x_1^l, \dots, x_n^l)$ and $\boldsymbol{\lambda}$ runs over the set Λ of integral multi-indices $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ with $0 \le \lambda_i < l$. Let Z_j be the hypersurfaces defined by the non-trivial

$F_\lambda(x^l)$. Clearly $\ker[l] \cdot Z_j = Z_j$. Moreover $Z^{\ker[l]} \supseteq Z_1 \cap \cdots \cap Z_D$. We now show the reverse inclusion. Let $\alpha \in Z^{\ker[l]}$. Then, for every $\zeta \in \ker[l]$,

$$0 = F(\zeta\alpha) = \sum_{\lambda \in \Lambda} (\zeta\alpha)^\lambda F_\lambda((\zeta\alpha)^l) = \sum_{\lambda \in \Lambda} \zeta^\lambda \alpha^\lambda F_\lambda(\alpha^l).$$

Let ζ_i varying over all elements of $\ker[l]$ and λ_j varying over all elements of Λ. Then we can write the following homogenous linear system

$$(\zeta_i^{\lambda_j})_{i,j} (\alpha^{\lambda_j} F_{\lambda_j}(\alpha^l))_j = \mathbf{0}.$$

Since the matrix $(\zeta_i^{\lambda_j})_{i,j}$ is non singular, $(\alpha^{\lambda_j} F_{\lambda_j}(\alpha))_j$ must be the zero vector. We remark that no monomial vanishes on $\mathbb{G}_m^n$. Then we have $\alpha \in Z_1 \cap \cdots \cap Z_D$. This shows that $Z^{\ker[l]} \subseteq Z_1 \cap \cdots \cap Z_D$. $\square$

To estimate δ_0, we need a generalization of Lemma 3.7 of [Amo-Via], which holds for $\overline{\mathbb{Q}}$-irreducible varieties. Here the variety is not necessarily $\overline{\mathbb{Q}}$-irreducible. In general, the lemma does not extend to all equidimensional varieties, however it extends under some additional assumptions.

Lemma 2.5. *Let V be a $\mathbb{Q}$-irreducible subvariety of $\mathbb{G}_m^n$ and let l be a positive integer. Let K be the field of definition of one of the $\overline{\mathbb{Q}}$-irreducible components of V. Assume that $K \cap \mathbb{Q}(\zeta_l) = \mathbb{Q}$, for a primitive l-th root of unity ζ_l. Then*

$$\delta_0(\ker[l] \cdot V) \le l^n \delta_0(V).$$

Proof. The first step is to prove the following remark. By definition of $\delta_0(V)$, there exists a variety X defined by rational equations of degree $\le \delta_0(V)$ such that V is a $\mathbb{Q}$-irreducible component of X. Let $W_1, \ldots, W_t$ be the $\overline{\mathbb{Q}}$-irreducible components of V.

Remark 2.6. Let $\zeta \in \ker[l]$. Assume that for some i the variety $\zeta W_i \subseteq X$. Then $\zeta W_j \subseteq X$ for any index j.

Proof. We remark that the Galois group permutes transitively $W_1, \ldots, W_t$. Let K_i be the field of definition of W_i. By assumption $K_i \cap \mathbb{Q}(\zeta) = \mathbb{Q}$. Thus $[K_i(\zeta) : K_i] = [\mathbb{Q}(\zeta) : \mathbb{Q}]$. Hence, for any $j = 1, \ldots, t$ there exists $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\tau(W_i) = W_j$ and $\tau(\zeta) = \zeta$. We infer that $\zeta W_j = \tau(\zeta W_i)$ is included in $\tau(X) = X$. $\square$

In what follows we say that a $\overline{\mathbb{Q}}$-irreducible variety $W \subseteq \mathbb{G}_m^n$ is imbedded in a variety $X \subseteq \mathbb{G}_m^n$ if V is a subset of X but not an irreducible component of X. Let

us denote $W = W_1$. Let S be the set of $\zeta \in \ker[l]$ such that ζW is imbedded in X. Then, by the remark above, $V \subseteq \zeta^{-1} X$. We define

$$X' = X \cap \bigcap_{\zeta \in S} \zeta^{-1} X.$$

Note that $V \subseteq X'$. Furthermore, the varieties X and $\zeta^{-1} X$ are intersections of hypersurfaces of degree $\leq \delta_0(V)$. Thus $\delta(X') \leq \delta_0(V)$.

We shall show that no translate ζW_j is imbedded in X'. Assume by contradiction that ζW_j was imbedded in X' for some $\zeta \in \ker[l]$ and for some $j \in \{1, \ldots, n\}$. We will prove that $\mathbf{1} \in S$. Then W would be imbedded in X, which contradicts the fact that W is a component of X. Since ζ has finite order, to prove $\mathbf{1} \in S$ it is sufficient to prove that $\zeta^n \in S$, for all positive integers n. We proceed by induction. Since $X' \subseteq X$, ζW_j is imbedded in X and so $\zeta \in S$. We now assume $\zeta^n \in S$ for some $n \geq 1$ and we prove that $\zeta^{n+1} \in S$. Since $X' \subseteq \zeta^{-n} X$, ζW_j is imbedded in $\zeta^{-n} X$. Thus $\zeta^{n+1} W_j$ is imbedded in X and $\zeta^{n+1} \in S$.

We now define

$$Y = \ker[l] \cdot X'.$$

Clearly $\ker[l] \cdot V \subseteq Y$ and $\delta(Y) \leq l^n \delta(X') \leq l^n \delta_0(V)$. Let ζW_j ($\zeta \in \ker[l]$, $j \in \{1, \ldots, t\}$) be a $\overline{\mathbb{Q}}$-irreducible component of $\ker[l] \cdot V$. Assume by contradiction ζW_j imbedded in Y. Then ζW_j is imbedded in $\eta X'$ for some $\eta \in \ker[l]$. Thus $\eta^{-1} \zeta W_j$ is imbedded in X', which contradicts the construction of X'. $\qquad \square$

At last we provide the necessary upper bound for $\delta_0([l]V)$.

Proposition 2.7. *Let V be a $\mathbb{Q}$-irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$ and let l be a positive integer. Let K be the field of definition of one of the $\overline{\mathbb{Q}}$-irreducible component of V. Assume that $K \cap \mathbb{Q}(\zeta_l) = \mathbb{Q}$. Then*

$$\delta_0([l]V) \leq l^{n-1} \delta_0(V).$$

Proof. By Lemma 2.5 there exist hypersurfaces $Z_1, \ldots, Z_r$ of degree $\leq l^n \delta_0(V)$ such that every $\overline{\mathbb{Q}}$-irreducible component of $\ker[l] \cdot V$ is a component of $Z_1 \cap \cdots \cap Z_r$. By Lemma 2.4 we can assume $\ker[l] \cdot Z_i = Z_i$. Thus

$$[l]V \subseteq [l]Z_1 \cap \cdots \cap [l]Z_r$$

and $\deg([l]Z_i) = l^{-1} \deg(Z_i)$. We now show that each component of $[l]V$ is isolated in such an intersection. Suppose on the contrary that U is a $\overline{\mathbb{Q}}$-irreducible component of V such that

$$[l]U \subsetneq Y \subseteq [l]Z_1 \cap \cdots \cap [l]Z_r$$

for some $\overline{\mathbb{Q}}$-irreducible Y. Then there exists a $\overline{\mathbb{Q}}$-irreducible component Y' of $[l]^{-1}Y$ such that

$$U \subsetneq Y' \subseteq (\ker[l] \cdot Z_1) \cap \cdots \cap (\ker[l] \cdot Z_r) = Z_1 \cap \cdots \cap Z_r.$$

This contradicts the fact that each component of V is isolated in $Z_1 \cap \cdots \cap Z_r$. $\quad\square$

2.3. Exceptional primes. Let $V \subseteq \mathbb{G}_m^n$ be a $\mathbb{Q}$-irreducible variety and let $\wp$ be a finite set of primes. In what follows, we need a lower bound for the degree of $\bigcup_{p \in \wp} [p]V$ and an upper bound for $\delta_0([p]V)$ for $p \in \wp$. This holds outside a set of "bad" primes. One has to ensure that there are few bad primes. This is the object of the next proposition. Part of the proof was already in [Amo-Dav], Section 2. We prefer to reproduce the integral argument.

Proposition 2.8. *Let $V \subseteq \mathbb{G}_m^n$ be a $\mathbb{Q}$-irreducible variety of dimension d. Assume that V is not a union of torsion varieties. Then there exists a set of prime numbers $E(V)$ of cardinality*

$$|E(V)| \le \frac{d+1}{\log 2} \log \deg(V)$$

such that for all prime numbers $p \notin E(V)$,

$$\delta_0([p]V) \le p^{n-1} \delta_0(V) \tag{2.6}$$

and, for all finite subsets $\wp$ of primes lying outside $E(V)$,

$$\deg \left(\bigcup_{p \in \wp} [p]V \right) \ge |\wp| \deg(V). \tag{2.7}$$

Proof. We remark that the Galois group permutes transitively the $\overline{\mathbb{Q}}$-irreducible components $W = W_1, \ldots, W_k$ of V. We recall the definition of stabilizer:

$$\mathrm{Stab}(W) = \{\boldsymbol{\alpha} \in \mathbb{G}_m^n \text{ such that } \boldsymbol{\alpha} W = W\}.$$

Define $H = \mathrm{Stab}(W)/\mathrm{Stab}(W)^0$ where $\mathrm{Stab}(W)^0$ is the connected component of $\mathrm{Stab}(W)$ through the neutral element. Then, H is a finite group of cardinality

$$|H| \le \deg(\mathrm{Stab}\, W) \le \deg(W)^{d+1}. \tag{2.8}$$

We denote $d_0 = \dim \mathrm{Stab}(W) \le d$. We remark that for any natural number l, it holds that

$$|\ker[l] \cap \mathrm{Stab}(W)| = |\ker[l] \cap \mathrm{Stab}(W)^0| \cdot |\ker[l] \cap H| = l^{d_0} |\ker[l] \cap H|,$$

where we identify $[l]$ with the "multiplication" by l in the quotient $\mathbb{G}_m^n / \mathrm{Stab}(W)^0$. Furthermore, denote by K the field of definition of W. Then $[K : \mathbb{Q}] = k$.

Let E_1 be the set of prime numbers p such that p divides $|H|$. Let E_2 be the set of primes p such that $[p]W = [p]W_i$ for some $1 < i \le k$. Let E_3 be the set of primes p such that $K \cap \mathbb{Q}(\zeta_p) \ne \mathbb{Q}$, where as usual ζ_p is a primitive p-th root of unity. We define

$$E(V) = E_1 \cup E_2 \cup E_3.$$

Since $E_3 \subseteq E(V)$, Proposition 2.7 shows that the upper bound (2.6) holds. We now prove (2.7). First we show that

$$p \nmid |H| \implies \deg([p]W_i) \geq \deg(W_i), \qquad \text{for } i = 1, \ldots, k. \qquad (2.9)$$

If $p \nmid |H|$, then $|\ker[p] \cap H| = 1$. By the degree formula for the image of the multiplication by p (see for instance [Dav-Phi], Proposition 2.1 (i)),

$$\deg([p]W) = p^{d-d_0}|\ker[p] \cap H|^{-1} \deg(W) = p^{d-d_0} \deg(W) \geq \deg(W).$$

This shows (2.9).

We now show that, for l_1, l_2 natural integers,

$$\begin{aligned} V \neq \text{ union of torsion varieties and } l_1 \neq l_2 \\ \implies [l_1]W_i \neq [l_2]W_j \text{ for } i, j = 1, \ldots, k. \end{aligned} \qquad (2.10)$$

Assume on the contrary that $[l_1]W$ is a Galois conjugate to $[l_2]W$. Since the multiplication by natural numbers commute with the Galois action, the same holds replacing l_i by l_i^r for $r \in \mathbb{N}$, as well. We can suppose $l_1 < l_2$. Let $\hat{h}$ be the normalised height for subvarieties of $\mathbb{G}_m^n$ (see for instance [Dav-Phi]). Then $\hat{h}([l_1]W) = \hat{h}([l_2]W)$. By the height formula for the image of the multiplication by an integer ([Dav-Phi], Proposition 2.1 (i)), we obtain

$$\begin{aligned} l_1^{d-d_0+1}|\ker[l_1] \cap H|^{-1}\hat{h}(W) = \hat{h}([l_1]W) = \hat{h}([l_2]W) \\ = l_2^{d-d_0+1}|\ker[l_2] \cap H|^{-1}\hat{h}(W). \end{aligned}$$

Since V is not a union of torsion varieties, W is not torsion. Then $\hat{h}(W) > 0$. Thus

$$l_2/l_1 \leq (l_2/l_1)^{d-d_0+1} \leq \frac{|\ker[l_2] \cap H|}{|\ker[l_1] \cap H|} \leq |H|.$$

Replacing l_1 and l_2 with l_1^r and l_2^r and letting $r \to +\infty$ we get a contradiction.

Let $\wp$ be a set of primes lying outside $E(V)$ and assume that V is not a union of torsion varieties. The statements (2.9) and (2.10) and the definition of $E(V)$ show that

$$\begin{aligned} \deg\Big(\bigcup_{p \in \wp}[p]V\Big) &= \deg\Big(\bigcup_{j=1}^{k}\bigcup_{p \in \wp}[p]W_j\Big) = \sum_{j=1}^{k}\sum_{p \in \wp}\deg\big([p]W_j\big) \\ &\geq \sum_{j=1}^{k}\sum_{p \in \wp}\deg(W_j) = |\wp|\deg(V). \end{aligned}$$

To conclude the proof, we need to provide an upper bound for the cardinality of $E(V) = E_1 \cup E_2 \cup E_3$. First we remark that by (2.8) the set E_1 of primes p dividing $|H|$ has cardinality

$$\leq \frac{\log |H|}{\log 2} \leq \frac{d+1}{\log 2} \log \deg(W) = \frac{d+1}{\log 2} \log(\deg(V)/k).$$

Below we detail the proof that the set E_2 has cardinality

$$|E_2| \leq \frac{\log k}{\log 2}. \tag{2.11}$$

We have still to estimate the cardinality of the set E_3 of primes p such that $K \cap \mathbb{Q}(\zeta_p) \neq \mathbb{Q}$. It holds that

$$|E_3| \leq \frac{\log k}{\log 2}. \tag{2.12}$$

Indeed, for $l \in \mathbb{N}$, define $K_l = K \cap \mathbb{Q}(\zeta_l)$. Thus, $K_l/\mathbb{Q}$ is Galois. We note that for $n, m \in \mathbb{N}$ coprime, $K_n \cap K_m = \mathbb{Q}$ and $K_n K_m \subseteq K_{nm}$. By induction we easily see that

$$k = [K : \mathbb{Q}] \geq \left[\prod_{p \in E_3} K_p : \mathbb{Q} \right] = \prod_{p \in E_3} [K_p : \mathbb{Q}] \geq 2^{|E_3|}.$$

This is equivalent to (2.12). We conclude that

$$\begin{aligned}
|E(V)| \leq |E_1| + |E_2| + |E_3| &\leq \frac{d+1}{\log 2} \log(\deg(V)/k) + \frac{2\log k}{\log 2} \\
&\leq \frac{d+1}{\log 2} \log(\deg(V)) + \frac{1-d}{\log 2} \log k \\
&\leq \frac{d+1}{\log 2} \log \deg(V)
\end{aligned}$$

as required.

The upper bound for $|E_2|$ is a variant of the corresponding lemma of Dobrowolski ([Dob], Lemma 3). For a natural integer l and for $i \in \{1, \ldots, k\}$, let

$$\mathcal{I}(l,i) = \{j, \, [l]W_i = [l]W_j\}.$$

Thus, for a fixed l, these sets have the same cardinality. Moreover, $p \in E_2$ if and only if $\mathcal{I}(p,1) \geq 2$.

Let l_1, l_2 be coprime integers. Then, by the definition of the sets $\mathcal{I}$,

$$\mathcal{I}(l_1 l_2, i) \supseteq \bigcup_{j \in \mathcal{I}(l_1, i)} \mathcal{I}(l_2, j). \tag{2.13}$$

Indeed, if $m \in \mathcal{I}(l_2, j)$ for some $j \in \mathcal{I}(l_1, i)$, we have $[l_2]W_j = [l_2]W_m$ and $[l_1]W_i = [l_1]W_j$ which implies $[l_1 l_2]W_i = [l_1 l_2]W_j = [l_1 l_2]W_m$. This immediately

gives the inclusion. Moreover, for $j \in \mathcal{I}(l_1, i)$ the sets $\mathcal{I}(l_2, j)$ are pairwise distinct. Indeed, let $j_1, j_2 \in \mathcal{I}(l_1, i)$ such that $\mathcal{I}(l_2, j_1) \cap \mathcal{I}(l_2, j_2) \neq \emptyset$. Then $[l_1]W_{j_1} = [l_1]W_{j_2}$ and $[l_2]W_{j_1} = [l_2]W_{j_2}$. Thus, there exist $x_1 \in \ker[l_1]$ and $x_2 \in \ker[l_2]$ such that $W_{j_2} = x_1 W_{j_1} = x_2 W_{j_1}$. This implies that $x_2^{-1} x_1 \in \mathrm{Stab}(W_{j_1})$. Since l_1, l_2 are coprime, by the Bézout identity, there exist integers u_1, u_2 such that $u_1 l_1 + u_2 l_2 = 1$. Thus

$$x_1 = x_1^{1-u_1 l_1} = x_1^{u_2 l_2} = (x_2^{-1} x_1)^{u_2 l_2} \in \mathrm{Stab}(W_{j_1}).$$

Hence $W_{j_2} = x_1 W_{j_1} = W_{j_1}$, and $j_1 = j_2$. This proves that (2.13) is a disjoint union. We infer

$$|\mathcal{I}(l_1 l_2, i)| \geq \sum_{j \in \mathcal{I}(l_1, i)} |\mathcal{I}(l_2, j)| = |\mathcal{I}(l_1, 1)||\mathcal{I}(l_2, 1)|.$$

Iterating this process, we see that

$$k \geq \left| \mathcal{I}\left(\prod_{p \in E_2} p, 1 \right) \right| \geq \prod_{p \in E_2} |\mathcal{I}(p, 1)| \geq 2^{|E_2|},$$

which proves (2.11) and concludes the proof of the proposition. $\square$

We remark that the inequalities (2.9) and (2.11) in the proof of the previous proposition hold even for a $\mathbb{Q}$-irreducible variety which is the union of torsion varieties.

3. Diophantine analysis

3.1. Coding the information. Let $I \subset \overline{\mathbb{Q}}[x]$ be a homogeneous radical ideal, where $x = (x_1, \ldots, x_n)$. For $\nu \in \mathbb{N}$ we denote by $H(\overline{\mathbb{Q}}[x]/I; \nu)$ the Hilbert function $\dim[\overline{\mathbb{Q}}[x]/I]_\nu$. Let T be a positive integer. We denote by $I^{(T)}$ the T-symbolic power of I, *i.e.* the ideal of polynomials vanishing on the variety defined by I with multiplicity at least T. Let V be a variety of $\mathbb{G}_m^n$. Let I be the radical homogeneous ideal in $\overline{\mathbb{Q}}[x]$ defining the Zariski closure of V in $\mathbb{P}^n$. By abuse of notation, we set $H(V; \nu) = H(\overline{\mathbb{Q}}[x]/I; \nu)$ and $H(V, T; \nu) = H(\overline{\mathbb{Q}}[x]/I^{(T)}; \nu)$.

Proposition 3.1. *Let ν, T be positive integers and let $\wp$ be a finite set of prime numbers. Let V be a subvariety of $\mathbb{G}_m^n$ defined over $\mathbb{Q}$. Define $V' = \bigcup[p]V$ for p running over $\wp$. Then, for some $p \in \wp$,*

$$\hat{\mu}^{\mathrm{ess}}(V) \geq \frac{1}{p\nu}\left(T \log p - \frac{T H(V, T; \nu)}{H(V'; \nu)}\Big(\log(\nu + 1) + \log p \Big) - n \log(\nu + 1) \right).$$

Proof. Denote for simplicity $H = H(V, T; \nu)$ and $H' = H(V'; \nu)$ and choose a real ε such that $\varepsilon > \hat{\mu}^{\mathrm{ess}}(V)$. We remark that the lower bound for $\hat{\mu}^{\mathrm{ess}}(V)$ of the proposition is obviously negative if $H \geq H'$. Hence we assume $H' > H$.

As usual in diophantine approximation, we first construct the auxiliary function. We are going to show that there exists an homogeneous polynomial $F \in \mathbb{Q}[x]_\nu$ vanishing on V with multiplicity $\geq T$ but not vanishing identically on V' and such that the Weil height of the vector of its coefficients satisfies

$$(H' - H)h(F) \leq H((T + n)\log(\nu + 1) + \nu\varepsilon). \tag{3.14}$$

Consider the vector space E of homogeneous polynomials $F \in \mathbb{Q}[x]_\nu$ vanishing on V with multiplicity $\geq T$. Let

$$L = \binom{\nu + n}{n}.$$

Then $\dim(E) = L - H$. Note that $L - H > L - H' \geq 0$. Thus $\dim(E) \geq 1$. Then there exists a basis $F_1, \ldots, F_{L-H}$ of E such that

$$\sum_{j=1}^{L-H} h(F_j) \leq H((T + n)\log(\nu + 1) + \nu\varepsilon). \tag{3.15}$$

This is a standard application of Bombieri and Vaaler's version of Siegel's lemma. The proof can be found in [Amo-Dav], Theorem 4.1. We briefly give a sketch. Theorem 8 of [Bom-Vaa] shows that there exists a basis $\{F_1, \ldots, F_{L-H}\}$ of E such that $\sum_{j=1}^{L-H} h(F_j)$ is bounded by the logarithmic L_2-height (defined choosing the L_2-norm at the infinite places) $h_2(E)$. By the duality principle (see the proof of Theorem 9 of [Bom-Vaa]) $h_2(E)$ is equal to the L_2-height of the vector space $E^\perp$ of dimension H. Given $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{G}_m^n(\overline{\mathbb{Q}})$ and a multi-index $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$ we define $\boldsymbol{\alpha}^{\boldsymbol{\lambda}} = \alpha_1^{\lambda_1} \ldots \alpha_n^{\lambda_n}$. Given two multi-indices $\boldsymbol{\lambda}, \boldsymbol{\mu}$ we write $\binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}}$ for the product over j of $\binom{\lambda_j}{\mu_j}$. Since $V(\varepsilon)$ is Zariski-dense in V, the space $E^\perp$ is spanned by the vectors

$$\left(\binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}}\boldsymbol{\alpha}^{\boldsymbol{\lambda}-\boldsymbol{\mu}}\right)_{|\boldsymbol{\lambda}| \leq \nu} \qquad (\boldsymbol{\alpha} \in V(\varepsilon),\ |\boldsymbol{\mu}| \leq T) \tag{3.16}$$

of L_2-height $\leq (T + n)\log(\nu + 1) + \nu\varepsilon$ (use $\sum_{|\boldsymbol{\lambda}| \leq \nu} \binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} \leq (\nu + 1)^{T+n}$). Since the L_2-height of a vector space is bounded by the sum of the L_2-height of a basis (by an application of Hadamard's inequality, [Bom-Vaa], equation (2.6)) we find that $h_2(E) \leq H((T + n)\log(\nu + 1) + \nu\varepsilon)$. Then equation (3.15) is proved.

We can assume $F_1, \ldots, F_{L-H} \in \mathbb{Z}[x]$ and $h(F_1) \leq \cdots \leq h(F_{L-H})$. We claim that there exists $j_0 \leq L - H' + 1$ such that F_{j_0} does not vanish on V'. Indeed, if all $F_1, \ldots, F_{L-H'+1}$ vanish on V', then $H' \leq L - (L - H' + 1) = H' - 1$. Let $F = F_{j_0}$. Then

$$\sum_{j=1}^{L-H} h(F_j) \geq (L - H - j_0 + 1)h(F) \geq (H' - H)h(F).$$

Using (3.15) we deduce that $h(F)$ satisfy (3.14).

The extrapolation step is based on a generalization of Dobrowolski's main lemma ([Dob], lemme 1). We recall that F does not vanish on V' and $\varepsilon > \hat{\mu}^{\mathrm{ess}}(V)$. Then there exists $\boldsymbol{\alpha} \in V(\varepsilon)$ such that $F(\boldsymbol{\alpha}^p) \neq 0$ for some prime $p \in \wp$. Let v be a place dividing p. By [Amo-Dav], Theorem 3.1,

$$|F(\boldsymbol{\alpha}^p)|_v \leq p^{-T} |\boldsymbol{\alpha}|_v^{pv}$$

where $|\boldsymbol{\alpha}| = \max\{1, |\alpha_1|_v, \ldots, |\alpha_n|_v\}$. Moreover, for an arbitrary place v,

$$|F(\boldsymbol{\alpha}^p)|_v \leq \begin{cases} |\boldsymbol{\alpha}|_v^{pv} & \text{if } v \nmid \infty, \\ L|F|_v |\boldsymbol{\alpha}|_v^{pv} & \text{if } v \mid \infty. \end{cases}$$

Note that $L \leq (v+1)^n$ and $h(\boldsymbol{\alpha}) \leq \varepsilon$. The product formula gives

$$0 \leq -T \log p + n \log(v+1) + h(F) + pv\varepsilon.$$

Comparing with (3.14) we get

$$(H' - H)(T \log p - n \log(v+1) - pv\varepsilon) \leq H((T+n)\log(v+1) + v\varepsilon)$$
$$\leq H((T+n)\log(v+1) + pv\varepsilon),$$

which easily implies our claim.   □

3.2. Decoding the information.

To decode the information of Proposition 3.1 we need an upper bound for the Hilbert function. The proposition below follows from a result of M. Chardin [Cha]. It is proved in Lemma 2.5 of [Amo-Dav].

Proposition 3.2. *Let* $V \subseteq \mathbb{P}_n$ *be an equidimensional variety of dimension* d *and codimension* $k = n - d$. *Let* v, T *be positive integers. Then*

$$H(V, T; v) \leq \binom{T-1+k}{k}\binom{v+d}{d} \deg(V).$$

We also need a sharp lower bound for the Hilbert function. This is a deep result of M. Chardin and P. Philippon. Let K be a subfield of $\overline{\mathbb{Q}}$ and let V be a K-irreducible variety. They prove ([Cha-Phi], Corollary 3) that for an equidimensional V,

$$H(V; v) \geq \binom{v+d-m}{d} \deg(V)$$

for $v > m$ and $m = k(\delta_0(V) - 1)$.

We need a generalization of this result. Consider finitely many equidimensional varieties V_j of the same dimension d. Let $k = n - d$,

$$m = -1 + \sum_j \big(k(\delta_0(V_j) - 1) + 1\big) < k \sum_j \delta_0(V_j).$$

Let us consider the equidimensional variety $V' = \bigcup V_j$. In the appendix of this article, M. Chardin and P. Philippon prove (see Subsection 6.1)

$$H(V'; \nu) \geq \binom{\nu + d - m}{d} \deg(V') \qquad (3.17)$$

for $\nu > m$.

Let $\wp$ be a set of prime numbers. We apply the previous result to $V' = \bigcup_{p \in \wp} [p]V$. Using the upper bound (2.6) of Proposition 2.8 and (3.17) we get:

Proposition 3.3. *Let $V \subseteq \mathbb{G}_m^n$ be a $\mathbb{Q}$-irreducible variety of dimension d and codimension $k = n - d$ which is not a union of torsion varieties. Let N be a positive real number and let $\wp$ be a set of prime numbers with $p \leq N$ lying outside the set $E(V)$ of Proposition 2.8. Define*

$$V' = \bigcup_{p \in \wp} [p]V$$

and

$$m = [kN^n \delta_0(V)].$$

Then for any $\nu \geq m$ we have

$$H(V'; \nu) \geq \binom{\nu + d - m}{d} \deg(V').$$

We are now ready to prove the main result of this section, Theorem 1.2. Let us recall the statement.

Theorem 1.2. *Let V be a variety of $\mathbb{G}_m^n$ of codimension k, defined and irreducible over $\mathbb{Q}$. Assume that V is not a union of torsion varieties. Let*

$$\theta_0 = \delta_0(V)(52n^2 \log(n^2 \delta_0(V)))^{(n+1)(k+1)}.$$

Then there exists a hypersurface Z defined over $\mathbb{Q}$ of degree at most θ_0 which does not contain V and such that

$$V\left(\theta_0^{-1}\right) \subseteq V \cap Z.$$

Proof. For simplicity, denote $\delta_0 = \delta_0(V)$. We prove a slightly more precise result. Namely that

$$V\left(\delta_0^{-1} n^{-2} (39n^2 \log(n^2 \delta_0))^{-(n+1)(k+1)+1}\right)$$

is contained in a hypersurface Z defined over $\mathbb{Q}$, such that $V \nsubseteq Z$ and

$$\deg Z \leq \delta_0 n^2 (39n^2 \log(n^2 \delta_0))^{(n+1)(k+1)}.$$

Since $39n^{2/((n+1)(k+1))} \leq 39n^{1/(n+1)} \leq 39 \cdot 4^{1/5} \leq 52$ this statement implies the statement of Theorem 1.2. Let

$$N = (39n^2 \log(n^2\delta_0))^{k+1}.$$

We need a lower and an upper bound for $\log N$. We have

$$\log N \geq 2 \log(39 \cdot 4 \log 4) \geq 10.75 \tag{3.18}$$

and (using $\log x < \sqrt{x}$ for $x > 0$, $k + 1 \leq 1.5n$ and $39 \leq 2^{5.29} \leq n^{5.29}$)

$$\begin{aligned}
\log N &\leq (k+1) \log \left(39n^2 \cdot \sqrt{n^2\delta_0}\right) \\
&\leq 1.5n \log(n^{8.29}\delta_0) \leq 6.22n \log(n^2\delta_0).
\end{aligned} \tag{3.19}$$

We define $\wp$ as the set of prime numbers p such that $N^{3/4} \leq p \leq N$ and $p \notin E(V)$ where $E(V)$ is as in Proposition 2.8. Thus

$$|\wp| \geq \pi(N) - \pi(N^{3/4}) - |E(V)|,$$

where, as usual, $\pi(t)$ is the cardinality of the set of prime numbers $\leq t$. By Theorem 1 of [Ros-Sch] we have, for $t \geq 59$,

$$\frac{t}{\log t} + \frac{t}{2(\log t)^2} < \pi(t) \leq \frac{t}{\log t} + \frac{3t}{2(\log t)^2}.$$

By Proposition 2.8 and by the last inequality in (2.5),

$$\begin{aligned}
|E(V)|/\sqrt{N} &\leq \frac{d+1}{\log 2} \log \deg(V) \cdot \frac{1}{(39n^2 \log(n^2\delta_0))^{(k+1)/2}} \\
&\leq \frac{nk \log \delta_0}{\log 2 \cdot 39n^2 \log(n^2\delta_0)} \leq \frac{1}{39 \log 2}.
\end{aligned}$$

Thus $|\wp| \geq \frac{f(N)N}{\log N}$, where

$$f(t) = 1 + \frac{1}{2 \log t} - \frac{1}{t^{1/4} \cdot 3/4} - \frac{3}{2t^{1/4}(3/4)^2 \log t} - \frac{\log t}{39(\log 2)t^{1/2}}.$$

Since $f(t) \geq 0.937$ for $\log t \geq 10.75$, we obtain,

$$|\wp| \geq \frac{0.937N}{\log N}.$$

As in Proposition 3.1, we set

$$V' = \bigcup_{p \in \wp} [p]V.$$

We constructed $\wp$ such that $\wp \cap E(V) = \emptyset$. Then, by Proposition 2.8,

$$\deg(V') \geq |\wp| \deg(V) \geq \frac{0.937N}{\log N} \deg(V). \tag{3.20}$$

As in the statement of Proposition 3.3, let $m = [kN^n \delta_0]$. Choose

$$\nu = md + m \qquad \text{and} \qquad T = [39n^2 \log(n^2 \delta_0)].$$

We remark that

$$\nu + 1 \leq n^2 N^n \delta_0. \tag{3.21}$$

Let

$$\theta := \delta_0 n^2 (39n^2 \log(n^2 \delta_0))^{(n+1)(k+1)-1}.$$

Let W be the Zariski closure of the set $V(\theta^{-1})$ and let $W' = \bigcup_{p \in \wp} [p]W$. We remark that W is defined over $\mathbb{Q}$ because the small points of V are invariant under the Galois action. Then

$$\hat{\mu}^{\mathrm{ess}}(W) \leq \theta^{-1}. \tag{3.22}$$

Furthermore

$$H(W, T; \nu) \leq H(V, T; \nu) \quad \text{and} \quad H(W'; \nu) \leq H(V'; \nu).$$

We are going to prove that the last inequality is strict. Assume on the contrary that

$$H(W'; \nu) = H(V'; \nu). \tag{3.23}$$

Apply Proposition 3.2 to V and Proposition 3.3 to V'. Then, by (3.20),

$$\frac{H(W, T; \nu)}{H(W'; \nu)} \leq \frac{H(V, T; \nu)}{H(V'; \nu)} \leq \frac{\binom{T-1+k}{k} \binom{\nu+d}{d}}{\binom{\nu+d-m}{d}} \times \frac{\log N}{0.937N}.$$

We remark that $\binom{T-1+k}{k} \leq T^k$. Moreover, by the choice $\nu = md + m$,

$$\binom{\nu+d}{d} \binom{\nu+d-m}{d}^{-1} = \prod_{j=1}^{d} \frac{\nu+j}{\nu-m+j} \leq \left(1 + \frac{m}{\nu-m}\right)^d = \left(1 + \frac{1}{d}\right)^d \leq e.$$

Thus,

$$\lambda := \frac{TH(W, T; \nu)}{H(W'; \nu)} \leq \frac{e(\log N)T^{k+1}}{0.937N} \leq 2.91 \log N. \tag{3.24}$$

By Proposition 3.1 (with V replaced by W) there exists a prime $p \in \wp$ such that

$$\theta^{-1} \geq \frac{1}{p\nu} \left((T+1)\log p - \lambda(\log(\nu+1) + \log N) - n\log(\nu+1) - \log N\right).$$

By the choice of T, we have $T + 1 \geq 39n^2 \log(n^2\delta_0)$. By (3.24), (3.21) and (3.19),

$$\lambda(\log(v + 1) + \log N) + n \log(v + 1) + \log N$$
$$\leq 2.91(\log N)\big(\log(n^2\delta_0) + (n + 1)\log N\big) + n \log(n^2\delta_0) + (n^2 + 1)\log N$$
$$\leq 2.91(6.22n(n + 1) + 1)\log(n^2\delta_0)\log N + n \log(n^2\delta_0) + (n^2 + 1)\log N$$
$$\leq c \cdot 39n^2 \log(n^2\delta_0)\log N$$

with

$$c = \frac{2.91(6.22 \cdot 1.5 + 0.25) + 0.5/10.75 + (1 + 0.25)/\log 4}{39} \leq 0.74$$

(use $n \geq 2$ and (3.18)). Let

$$f(t) = \frac{N}{t}\left(\frac{\log t}{\log N} - 0.74\right)\log N.$$

Then

$$\theta^{-1} > \frac{39n^2 f(p) \log(n^2\delta_0)}{Nv}.$$

We remark that $f(t)$ has a single stationary point on $[0, +\infty]$ which is a local maximum. Since $p \in [N^{3/4}, N]$, we have $f(p) \geq \min\{f(N^{3/4}), f(N)\}$. Moreover, by (3.18),

$$f(N^{3/4}) \geq e^{10.75/4}(3/4 - 0.74) \cdot 10.75 > 1$$

and $f(N) \geq (1 - 0.74) \cdot 10.75 > 1$. Thus $f(p) > 1$. Using (3.21), we finally obtain

$$\theta < \frac{Nv}{39n^2 \log(n^2\delta_0)} \leq \frac{n^2 N^{n+1}\delta_0}{39n^2 \log(n^2\delta_0)} = \delta_0 n^2 (39n^2 \log(n^2\delta_0))^{(n+1)(k+1)-1} = \theta.$$

This contradiction shows that the assumption (3.23) cannot hold. Thus we have:

$$H(W'; v) < H(V'; v).$$

Equivalently, there exists a homogeneous polynomial F of degree v which vanishes on W' but not on V'. The varieties are defined over the rationals, so we can assume $F \in \mathbb{Q}[x]$. Since F does not vanish on V', there exists a prime number $p \in \wp$ such that F does not vanish on $[p]V$. Let Z be the zero set of $F(x^p) = 0$. Then $V \not\subseteq Z$ and $V(\theta^{-1}) \subseteq W \subseteq Z$. We have

$$\deg(Z) \leq N \deg F \leq Nv \leq n^2 N^{n+1}\delta_0 = \delta_0 n^2 (39n^2 \log(n^2\delta_0))^{(n+1)(k+1)}$$

as required. $\square$

4. Distribution of the small points

A geometric reduction process, close to that of [Amo-Via], applied to each variety involved, allows us to prove the main result of this article using Theorem 1.2.

Theorem 1.3. *Let $V_0 \subseteq V_1$ be subvarieties of $\mathbb{G}_m^n$, defined over $\mathbb{Q}$, of codimensions k_0 and k_1 respectively. Assume that V_0 is $\mathbb{Q}$-irreducible. Let*

$$\theta = \delta(V_1)\big(935n^5 \log(n^2\delta(V_1))\big)^{(k_0-k_1+1)(k_0+1)(n+1)}.$$

Then,

- *either there exists a $\mathbb{Q}$-irreducible B union of torsion varieties such that $V_0 \subseteq B \subseteq V_1$ and $\delta_0(B) \leq \theta$,*

- *or there exists a hypersurface Z defined over $\mathbb{Q}$ of degree at most θ such that $V_0 \nsubseteq Z$ and $V_0(\theta^{-1}) \subseteq Z$.*

Proof. Theorem 1.3 is analogue to Theorem 2.2 of [Amo-Via]. The proof is similar. Let us give the details.

We simply denote $\delta = \delta(V_1)$. By contradiction, we suppose that the conclusion of Theorem 1.3 does not hold. Thus

V_0 is not contained in any union $B \subseteq V_1$ of proper torsion varieties with $\delta_0(B) \leq \theta$
$$(4.25)$$

and

Each hypersurface Z defined over $\mathbb{Q}$, of degree $\leq \theta$, with $V_0(\theta^{-1}) \subseteq Z$ contains V_0.
$$(4.26)$$

For $r \in \{0, \ldots, k_0 - k_1 + 1\}$ we define

$$D_r = \delta\big(935n^5 \log(n^2\delta)\big)^{r(k_0+1)(n+1)}.$$

Since $r \leq k_0 - k_1 + 1$, we have $D_r \leq \theta$. Using an inductive process on r, we are going to construct a chain of varieties

$$X_0 \supseteq \cdots \supseteq X_r \supseteq X_{r+1} \supseteq \cdots \supseteq X_{k_0-k_1+1}$$

defined over $\mathbb{Q}$ which satisfy:

Claim.

i) $V_0 \subseteq X_r$.

ii) *Each $\mathbb{Q}$-irreducible component of X_r containing V_0 has codimension $\geq r + k_1$.*

iii) $\delta(X_r) \leq D_r$.

Theorem 1.3 is proved if we show the claim for $r = k_0 - k_1 + 1$. Indeed, by i) there exists a $\mathbb{Q}$-irreducible component W of $X_{k_0-k_1+1}$ which contains V_0. By ii) codim $W \geq k_0 + 1$. This gives a contradiction.

We now define X_r and prove our claim by induction on r.

• For $r = 0$, we simply choose $X_0 = V_1$.

• We assume that our claim holds for some $r \in \{0, \ldots, k_0 - k_1\}$ and we prove that it holds for $r + 1$, as well. Since $V_0 \subseteq X_r$, there exists at least one $\mathbb{Q}$-irreducible component of X_r which contains V_0. Let $1 \leq s \leq t$ be integers and let $W_1, \ldots, W_s, W_{s+1}, \ldots, W_t$ be the $\mathbb{Q}$-irreducible components of X_r. We enumerate these components so that

$$V_0 \subseteq W_j \quad \text{if and only if} \quad j = 1, \ldots, s.$$

Assertion ii) of our claim for r implies that $r + k_1 \leq \mathrm{codim}(W_j) \leq k_0$, for $j = 1, \ldots, s$.

Let $j \in \{1, \ldots, s\}$. Since $\delta(X_r) \leq D_r$, the variety W_j is a $\mathbb{Q}$-irreducible component of an intersection of hypersurfaces defined over $\mathbb{Q}$ of degree $\leq D_r$. Thus $\delta_0(W_j) \leq D_r \leq \theta$. Moreover

$$V_0 \subseteq W_j \subseteq X_r \subseteq X_0 = V_1.$$

By assumption (4.25), W_j is not a union of torsion varieties.

Let

$$\theta_0 = D_r \big(52n^2 \log(n^2 D_r)\big)^{(n+1)(k_0+1)}.$$

In view of Theorem 1.2, the set $W_j(\theta_0^{-1})$ is contained in a hypersurface Z_j defined over $\mathbb{Q}$ which does not contain W_j and such that $\deg Z_j \leq \theta_0$. We show that $\theta_0 \leq D_{r+1}$. For this we need an upper bound for $\log(n^2 D_r)$. Using $\log x < \sqrt{x}$ for $x > 0$, we obtain

$$D_r = \delta\big(935n^5 \log(n^2\delta)\big)^{r(k_0+1)(n+1)} \leq \delta(935n^5 \cdot n\delta)^{r(k_0+1)(n+1)}$$

$$\leq \delta(935n^6\delta)^{n(n+1)^2}.$$

We have $n^2 \leq n^{n^3/4}$, $n(n+1)^2 \leq (9/4)n^3$ and $935 \leq n^{(\log 935)/\log 2}$. Thus $n^2 D_r \leq (n^2\delta)^{cn^3}$ with

$$c = \frac{1}{8} + \frac{9}{4} \cdot \frac{1}{2}\left(\frac{\log 935}{\log 2} + 6\right) < 17.98.$$

We deduce

$$\theta_0 \leq D_r \big(52n^2 \cdot 17.98n^3 \log(n^2\delta)\big)^{(n+1)(k_0+1)}$$

$$\leq D_r \big(935n^5 \log(n^2\delta)\big)^{(n+1)(k_0+1)}$$

$$= D_{r+1}.$$

Since $V_0 \subseteq W_j$

$$V_0(\theta_0^{-1}) \subseteq W_j(\theta_0^{-1}) \subseteq Z_j.$$

As $\deg Z_j \leq \theta_0 \leq D_{r+1} \leq \theta$, relation (4.26) implies that $V_0 \subseteq Z_j$. Thus, for $j = 1, \ldots, s$ we have $V_0 \subseteq Z_j$ and

$$V_0 \subseteq \bigcap_{j=1}^{s} Z_j.$$

Let

$$X_{r+1} = X_r \cap Z_1 \cap \cdots \cap Z_s.$$

Then $V_0 \subseteq X_{r+1} \subseteq X_r$.

Recall that $\deg Z_j \leq \theta_0 \leq D_{r+1}$. Then

$$\delta(X_{r+1}) \leq \max\{\delta(X_r), D_{r+1}\} \leq \max\{D_r, D_{r+1}\} = D_{r+1}.$$

We decompose

$$X_{r+1} = W_1' \cup \cdots \cup W_s' \cup W_{s+1}' \cup \cdots \cup W_t',$$

where $W_j' = W_j \cap Z_1 \cap \cdots \cap Z_s$.

Let $j \in \{1, \ldots, s\}$. Since $W_j \not\subseteq Z_j$, every $\mathbb{Q}$-irreducible component of W_j' has codimension $\geq \mathrm{codim}(W_j) + 1 \geq r + 1 + k_1$.

Let $j \in \{s + 1, \ldots, t\}$. Since $V_0 \not\subseteq W_j$, the variety V_0 is not contained in any $\mathbb{Q}$-irreducible component of W_j'.

We conclude that X_{r+1} satisfies our claim for $r + 1$. $\qquad\square$

5. Proofs of Theorem 1.1 and of the corollaries

Theorem 1.1 becomes a corollary of Theorem 1.3:

Proof of Theorem 1.1. Let

$$\theta = \delta(V)\big(935n^5 \log(n^2 \delta(V))\big)^{(d+1)(n+1)^2}.$$

We have to show that $V^*(\theta^{-1}) = \emptyset$. Let V_0 be one of the finitely many $\mathbb{Q}$-irreducible components of $\overline{V(\theta^{-1})}$. Then $\overline{V_0(\theta^{-1})} = V_0$. Apply Theorem 1.3 to V_0 and $V_1 = V$. We have $k_0 \leq n$ and $k_1 = n - d$. Thus

$$(k_0 - k_1 + 1)(k_0 + 1)(n + 1) \leq (d + 1)(n + 1)^2.$$

Since $V(\theta^{-1})$ is dense in V_0, the first assertion of Theorem 1.3 must hold. So $\overline{V_0(\theta^{-1})}$ is contained in a union of torsion varieties $B \subseteq V$. Varying V_0 over all components of $\overline{V(\theta^{-1})}$, we conclude that $\overline{V(\theta^{-1})} \subseteq B$ where $B \subseteq V$ is a union of torsion varieties. Thus $V^*(\theta^{-1}) = \emptyset$. $\qquad\square$

On the one hand, Theorem 1.1 tells us that the small points of V are contained in the union V^u of torsion varieties included in V. On the other hand, the torsion is dense in a torsion varieties and V^u is a finite union of the maximal torsion varieties of V. Thus, the closure of the small points must be V^u. In [Amo-Via], Corollary 5.3, we estimate the sum of the degrees of these maximal torsion varieties. This is the line of

Proof of Corollary 1.4. Let $V^u = B_1 \cup \cdots \cup B_t$ where B_j are the maximal torsion varieties of V. By [Amo-Via], Corollary 5.3, $\delta_0(B_j) \le \theta'$ and

$$\sum_{j=1}^t \theta'^{\dim(B_j)} \deg(B_j) \le \theta'^m$$

where $\theta' \le \theta$. Since $V^* = V \backslash V^u$, Theorem 1.1 shows that

$$V(\theta^{-1}) \subseteq V^u = B_1 \cup \cdots \cup B_t.$$

In addition

$$V^u = \overline{V(0)} \subseteq \overline{V(\theta^{-1})}. \qquad \Box$$

Let $V \subseteq \mathbb{G}_m^n$ be a $\mathbb{Q}$-irreducible subvariety which is not contained in any union of proper torsion varieties. As remarked in the introduction, Theorem 1.1 implies a lower bound for the essential minimum. The slightly better lower bound of Corollary 1.5 is obtained directly from Theorem 1.3.

Proof of Corollary 1.5. Choose a hypersurface Z defined over $\mathbb{Q}$ containing V of minimal degree $\omega(V)$. The result follows choosing $V_0 = V$, $V_1 = Z$, $k_0 = k$ and $k_1 = 1$ in Theorem 1.3. $\qquad \Box$

Finally, we prove the lower bound for the product of the heights of multiplicatively independent algebraic numbers announced in the introduction in Corollary 1.6.

Proof of Corollary 1.6. We reorder $\alpha_1, \ldots, \alpha_n$ in such a way that $h(\alpha_1) \le \cdots \le h(\alpha_n)$. Let $A_i = [2h(\alpha_i)/h(\alpha_1)]$ and choose algebraic numbers $\beta_1, \ldots, \beta_n$ such that $\beta_i^{A_i} = \alpha_i$. We apply Corollary 1.5 to the 0-dimensional variety V of degree $[\mathbb{Q}(\boldsymbol{\beta}) : \mathbb{Q}]$, consisting of the conjugates of $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$. We have

$$\hat{\mu}^{\text{ess}}(V) = h(\boldsymbol{\beta}) \le \sum_i A_i^{-1} h(\alpha_i) \le n h(\alpha_1).$$

By the bound (2.4) of Chardin, we deduce

$$\begin{aligned} \omega(V) &\le n[\mathbb{Q}(\boldsymbol{\beta}) : \mathbb{Q}]^{1/n} \\ &\le n(DA_1 \ldots A_n)^{1/n} \\ &\le 2n(h(\alpha_1) \ldots h(\alpha_n))^{1/n} h(\alpha_1)^{-1} D^{1/n}. \end{aligned}$$

In view of the upper bound for the essential minimum and in view of Corollary 1.5 we obtain

$$nh(\alpha_1) \geq \hat{\mu}^{\mathrm{ess}}(V)$$
$$\geq (2n)^{-1}(h(\alpha_1)\dots h(\alpha_n))^{-1/n}h(\alpha_1)D^{-1/n}\big(935n^5\log(n^2\omega(V))\big)^{-n(n+1)^2}$$

or equivalently

$$h(\alpha_1)\dots h(\alpha_n) \geq D^{-1}(2n^2)^{-n}\big(935n^5\log(n^2\omega(V))\big)^{-n^2(n+1)^2}.$$

To conclude the proof, we use an effective lower bound for the height due to P. Voutier. Note that α_1 is not a root of unity. By [Vou], Corollary 2, $h(\alpha_1) \geq 2D^{-1}\log(3D)^{-3}$. Moreover we can clearly assume $D \geq 2$ and

$$h(\alpha_1)\dots h(\alpha_n) \leq D^{-1}(n\log(3D))^{-3n}.$$

Thus,

$$\omega(V) \leq 2n \cdot D^{-1/n}(n\log(3D))^{-3} \cdot \frac{1}{2}D\log(3D)^3 \cdot D^{1/n} = n^{-2}D$$

and (using $(2n^2)^{1/n(n+1)^2} \cdot 935 \leq 8^{1/18} \cdot 935 \leq 1050$ for $n \geq 2$)

$$(2n^2)^n\big(935n^5\log(n^2\omega(V))\big)^{n^2(n+1)^2} \leq (2n^2)^n\big(935n^5\log D\big)^{n^2(n+1)^2}$$
$$\leq \big(1050n^5\log(3D)\big)^{n^2(n+1)^2}. \qquad \square$$

6. Appendix

The following appendix by M. Chardin and P. Philippon contains two results. The first one is an extension of the lower bound for the Hilbert function proved in [Cha-Phi]. This result is crucial in the proof of Proposition 3.3. The second result in this appendix deals with a filtration of invariants starting with ω and ending with δ_0. Let $V \subset \mathbb{P}^n$ be a K-irreducible variety of codimension k defined by a homogeneous prime ideal $I \subseteq A = \mathbb{K}[x_0,\dots,x_n]$. Let $1 \leq r \leq k$. Philippon (see [Phi], Corollary 6) defines $\delta'_r(I)$ as the minimal degree δ such that there exist homogeneous polynomials $f_1,\dots,f_r \in A$ of degree δ which form a regular sequence in IA_I. Thus $\delta'_r(I)$ is the minimal degree δ such that there exists an intersection X of hypersurfaces defined over K of degree $\leq \delta$ containing V and of local codimension $\geq r$ at V. The proof of Corollary 2.3 shows that it is not restrictive to require also that all hypersurfaces are defined over $\overline{\mathbb{Q}}$. Thus $\delta'_1(I) = \omega(V)$ and $\delta'_k(I) = \delta_0(V)$. In addition, one can show that V is an isolated component of an intersection of k hypersurfaces of

degree $\delta'_1(I), \ldots, \delta'_k(I)$. Thus, by Bézout's theorem, $\deg(V) \leq \delta'_1(I) \ldots \delta'_k(I)$. In the second part of the appendix, M. Chardin and P. Philippon prove that there exist hypersurfaces $Z_1, \ldots, Z_k$ of degree $d_1, \ldots, d_k$ such that V is an isolated component of $Z_1 \cap \cdots \cap Z_k$ and

$$(n^{k(k+1)/2} 2^{nk(k-1)})^{-1} d_1 \ldots d_k \leq \deg(V) \leq d_1 \ldots d_k.$$

Obviously, by definition $\delta'_r \leq d_r$. In addition, since $\deg(V) \leq \delta'_1(I) \ldots \delta'_k(I)$, we deduce

$$(n^{k(k+1)/2} 2^{nk(k-1)})^{-1} \delta'_1(I) \ldots \delta'_k(I) \leq \deg(I) \leq \delta'_1(I) \ldots \delta'_k(I). \qquad (6.27)$$

Even if these inequalities are not needed here, we believe that they will be useful.

Complément à [Cha-Phi]

Par M. Chardin et P. Philippon

6.1. Extension de la minoration de fonction de Hilbert.

Dans l'énoncé suivant, nous utilisons la notion de modules et schémas $(m, \mathfrak{b})$-parfaits telle qu'introduite dans [Cha-Phi]. Rappelons que dans cette propriété m est un entier et $\mathfrak{b}$ est un idéal homogène de l'anneau de base (supposé gradué). En particulier, l'espace projectif $\mathbb{P}^n$ est 0-régulier et son anneau de coordonnées $A = \mathbf{k}[x_0, \ldots, x_n]$ est $(0, A)$-régulier (en tant que A-module).

Théorème 6.1. *Soient $V_1, \ldots, V_s$ des sous-schémas de $\mathbb{P}^n$, équi-dimensionnels de même dimension D et de supports deux à deux distincts. Notons $\mathfrak{b}_1, \ldots, \mathfrak{b}_s$ des idéaux homogènes de l'anneau de coordonnées $A = \mathbf{k}[x_0, \ldots, x_n]$. On suppose que V_i est $(m_i, \mathfrak{b}_i)$-parfait pour $i = 1, \ldots, s$ et on note V un sous-schéma de dimension D contenu dans $V_1 \cup \cdots \cup V_s$. Alors on a*

$$\mathcal{H}(V, \nu) \geq \deg(V) \binom{\nu + D - m}{D}$$

dès que $\nu > m := m_1 + \cdots + m_s + s - 1$.

Nota Bene – Posons $\delta_0(V)$ le plus petit entier tel que V soit composante d'une intersection de $n - D$ formes de degré au plus δ_0. On sait que $m_i \leq (n-D)(\delta_0(V_i)-1)$ et on a donc dans l'énoncé ci-dessus :

$$m \leq (n - D)(\delta_0(V_1) + \cdots + \delta_0(V_s) - s) + s - 1.$$

Démonstration. On procède par récurrence sur D, on note $A = \mathbf{k}[x_0, \ldots, x_n]$ et $I_1, \ldots, I_s$ les idéaux des V_i. Pour $D = 0$ on sait que le A-module A/I_i est $(m_i, \mathfrak{b}_i)$-parfait et donc m_i-régulier d'après [Cha-Phi], proposition 3. D'après le théorème 2.4

de [Con-Her] (appliqué avec $M = A/(I_1 \cap \cdots \cap I_{i-1})$ et A/I_i qui est de dimension 1) on sait que la régularité de $I_i/(I_1 \cap \cdots \cap I_i)$ est majorée par la somme de la régularité de $A/(I_1 \cap \cdots \cap I_{i-1})$ et de celle de I_i (qui est égale à celle de A/I_i plus 1). De plus, la régularité de $A/(I_1 \cap \cdots \cap I_i)$ est le maximum de celle de $I_i/(I_1 \cap \cdots \cap I_i)$ et de celle de A/I_i, d'où les inégalités

$$\mathrm{reg}(A/(I_1 \cap \cdots \cap I_i)) \leq \max\left(\mathrm{reg}(I_i/(I_1 \cap \cdots \cap I_i)); \mathrm{reg}(A/I_i)\right)$$
$$\leq \mathrm{reg}(A/(I_1 \cap \cdots \cap I_{i-1})) + \mathrm{reg}(A/I_i) + 1.$$

Comme la régularité de A/I_i est majorée par m_i on obtient par téléscopage que la régularité de $A/(I_1 \cap \cdots \cap I_s)$ est majorée par $m_1 + \cdots + m_s + s - 1$. L'idéal J de V contient $I_1 \cap \cdots \cap I_s$ et A/J a même dimension D, la minoration cherchée résulte alors de [Cha-Phi], proposition 4, dans ce cas.

Pour passer de $D - 1$ à D on intersecte, comme dans *loc. cit.*, V par une forme linéaire x assez générale de sorte que pour tout $i, j \in \{1, \ldots, s\}$ on ait $\dim(V_i \cap V_j \cap Z(x)) < D - 1$ et $\dim(V_i \cap Z(\mathfrak{b}_i + xA)) < D - 1$. On note W_i la partie de dimension $D-1$ de $V_i \cap Z(x)$ et on vérifie que W_i est $(m_i, b_i\mathfrak{b}_i)$-parfait pour un $b_i \in A$ convenable. De plus les W_i sont deux à deux distincts, en posant $W = W_1 \cup \cdots \cup W_s$ on a $\deg(W) = \deg(V)$ et

$$\mathcal{H}(V, \nu) - \mathcal{H}(V, \nu - 1) = \mathcal{H}(V \cap Z(x), \nu) \geq \mathcal{H}(W, \nu).$$

L'hypothèse de récurrence entraîne donc

$$\mathcal{H}(V, \nu) - \mathcal{H}(V, \nu - 1) \geq \deg(V) \binom{\nu + D - 1 - m}{D - 1}$$

puis la minoration voulue par intégration finie.   □

6.2. Complément à l'interpolation : estimations du degré.

Dans le théorème 2 de [Cha-Phi], on vérifie de plus :

$$(n^{r(r+1)/2} 2^{nr(r-1)})^{-1} d_1 \ldots d_r \leq \deg(X) \leq d_1 \ldots d_r.$$

La majoration $\deg(X) \leq d_1 \ldots d_r$ est une conséquence du théorème de Bézout. Pour l'autre inégalité, on peut en fait établir les propriétés supplémentaires suivantes, à annexer à celles $(1)_i$, $(2)_i$ et $(3)_i$ du théorème 2 de [Cha-Phi]. Pour $i = 1, \ldots, r$ on pose $c_i = (n^{i(i+1)/2} 2^{ni(i-1)})^{-1}$ et cette propriété s'énonce :
$(4)_i$ pour toute composante Y de X_i on a $\mathcal{H}(Y, d_i - 1) \geq c_i d_1 \ldots d_i \binom{d_i + n - i}{n - i}$. Et en particulier $\deg(X_i) \geq \deg(Y) \geq c_i d_1 \ldots d_i \geq c_i \deg(X_i)$.

La démonstration se fait dans la récurrence sur $i = 1, \ldots, r$ et pour $i = r$ on a bien $\deg(X) \geq c_r d_1 \ldots d_r \geq c_r \deg(X)$ car $X_r = X$. Le cas $i = 1$ résulte déjà de $(2)_1$ $(c_1 = 1/n)$ et pour la récurrence l'argument à ajouter est le suivant $(1 < i \leq r)$:

Comme $X \subset Y$ on a, par $(2)_i$ et $(4)_{i-1}$,

$$\mathcal{H}(Y, d_i - 1) \geq \mathcal{H}(X, d_i - 1)$$

$$\geq c(n, i)^{-1} \deg(X_{i-1}) d_i \binom{d_i + n - i}{n - i}$$

$$\geq c(n, i)^{-1} c_{i-1} d_1 \dots d_i \binom{d_i + n - i}{n - i}.$$

Ce qui conclut car $c_i \leq c_{i-1} c(n, i)^{-1}$, vu que $c(n, i) = \frac{n!}{(n-i)!} 2^{(i-1)(2n-i)} \leq n^i 4^{n(i-1)}$.

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