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Exponential mixing for the Teichmüller flow in the space of quadratic differentials

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Abstract. We consider the Teichmüller flow on the unit cotangent bundle of the moduli space of compact Riemann surfaces with punctures. We show that it is exponentially mixing for the Ratner class of observables. More generally, this result holds for the restriction of the Teichmüller flow to an arbitrary connected component of stratum. This result generalizes [AGY] which considered the case of strata of squares.

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1. Introduction

Let $g, n \geq 0$ be integers such that $3g - 3 + n > 0$ and let $\mathcal{T}_{g,n}$ be the Teichmüller space of marked Riemann surfaces of genus g with n punctures. There is a natural $\mathrm{SL}(2,\mathbb{R})$ action on the unit cotangent bundle $\mathcal{Q}^1_{g,n}$ to $\mathcal{T}_{g,n}$, which preserves the natural (infinite) Liouville measure. The orbits of the diagonal flow project to the geodesics of the Teichmüller metric on $\mathcal{T}_{g,n}$.

Let $\mathcal{Q}^* = \mathcal{Q}^*_{g,n}$ be the quotient of $\mathcal{Q}^1_{g,n}$ by the modular group $\mathrm{Mod}(g,n)$. The $\mathrm{SL}(2,\mathbb{R})$ action descends to $\mathcal{Q}^*_{g,n}$. The Liouville measure descends to a *finite* measure $\mu = \mu_{g,n}$ on $\mathcal{Q}^*_{g,n}$. The diagonal flow $T_t \colon \mathcal{Q}^*_{g,n} \to \mathcal{Q}^*_{g,n}$ is called the *Teichmüller geodesic flow*.

Veech showed that T_t is mixing with respect to μ: if ϕ and ψ are observables (L^2 functions) with zero mean then

$$\lim_{t\to\infty} \int \phi(\psi \circ T_t) d\mu = \frac{1}{\mu(\mathcal{Q}^*)} \int \phi d\mu \int \psi d\mu. \tag{1}$$

Here we are interested in the speed of mixing, that is, the rate of convergence of (1). As usual, it is necessary to specify a class of "regular" observables. The class

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for which our results apply is the *Ratner class* H of observables which are Hölder with respect to the $\mathrm{SO}(2,\mathbb{R})$ action. More precisely, letting R_θ denote the rotation of angle $2\pi\theta$, H is the set of all $\phi \in L^2(\mu)$ such that $\theta \mapsto R_\theta\phi \in L^2(\mu)$ is a Hölder function (this includes all functions which are Hölder with respect to the metric of the fiber). This is a natural class to consider, since exponential mixing for observables in the Ratner class is known to be equivalent to the "spectral gap" property for the $\mathrm{SL}(2,\mathbb{R})$ action, (the hard direction of this equivalence being due to Ratner, see the Appendix B of [AGY] for a discussion).

Theorem 1.1. *The Teichmüller flow is exponentially mixing with respect to μ for observables in the Ratner class.*

In the sequel we will see that Theorem 1.1 is a special case of a more general result on the restriction of Teichmüller flow to "strata" and discuss some of the ingredients in the proof. First we will discuss in more detail the main notions involved in this statement.

1.1. Quadratic differentials and half-translation surfaces. A *quadratic differential* q on a Riemann surface S (compact, with punctures) assigns to each point of the surface a complex quadratic form on the corresponding tangent space, depending holomorphically on the point. Given any local coordinate z on S, the quadratic differential may be written as $q_z = \phi(z)dz^2$ where the coefficient $\phi(z)$ is a holomorphic function; then the expression $q_w = \phi'(w)dw^2$ with respect to any other local coordinate w is determined by

$$\phi'(w) = \phi(z)\left(\frac{dz}{dw}\right)^2$$

on the intersection of the domains. The *norm* of a quadratic differential is defined by $\|q\| = \int |\phi|\, dz\, d\bar{z}$ (the integral does not depend on the choice of the local coordinates). Quadratic differentials with finite norm are called *integrable*: in this case the quadratic differential naturally extends to a *meromorphic* quadratic differential on the completion of S, with at worse simple poles at the punctures. Below we will restrict considerations to integrable quadratic differentials.

Each quadratic differential q induces a special geometric structure on the completion of S, as follows. Near any non-singular point (puncture or zero) one can choose *adapted coordinates* ζ for which the local expression of q reduces to $q_\zeta = d\zeta^2$. Given any pair ζ_1 and ζ_2 of such adapted coordinates,

$$(d\zeta_1)^2 = (d\zeta_2)^2 \quad \text{or, equivalently,} \quad \zeta_1 = \pm\zeta_2 + \text{const}. \tag{2}$$

Thus, we say that the set of adapted coordinates is a *half-translation atlas* on the complement of the singularities and S is a *half-translation surface*. In particular,

$S\backslash\{\text{singularities}\}$ is endowed with a flat Riemannian metric imported from the plane via the half-translation atlas. The total area of this metric coincides with the norm $\|q\|$. Adapted coordinates ζ may also be constructed in the neighborhood of each singularity z_i such that

$$q_\zeta = \zeta^{l_i}\, d\zeta^2$$

with $l_i \geq -1$. Through them, the flat metric can be completed with a conical singularity of angle equal to $\pi(l_i + 2)$ at z_i (thus $l_i = 0$ corresponds to removable singularities).

A quadratic differential q is *orientable* if it is the square of some *Abelian differential*, that is, some holomorphic complex 1-form ω. Notice that square roots can always be chosen locally, at least far from the singularities, so that orientability has mostly to do with having a globally consistent choice. In the orientable case adapted coordinates may be chosen so that $\omega_\zeta = d\zeta$. Changes between such coordinates are given by

$$d\zeta_1 = d\zeta_2 \quad \text{or, equivalently,} \quad \zeta_1 = \zeta_2 + \text{const} \tag{3}$$

instead of (2). One speaks of *translation atlas* and *translation surface* in this case. We shall be particularly interested in the case when q is *not* orientable.

1.2. Strata. Each element of $\mathcal{Q}^*_{g,n}$ admits a representation as a meromorphic quadratic differential q on a compact Riemann surface of genus g with at most n simple poles and with $\|q\| = 1$. To each $q \in \mathcal{Q}^*_{g,n}$ we can associate a symbol $\sigma = (k, \nu, \varepsilon)$ where

(1) k is the number of poles,

(2) $\nu = (\nu_j)_{j\geq 1}$ and ν_j is the number of zeros of order j,

(3) $\varepsilon \in \{-1, 1\}$ is equal to 1 if q is the square of an Abelian differential and to -1 otherwise.

We denote by $\mathcal{Q}^*_g(\sigma)$ the *stratum* of all q with symbol σ. A non-empty stratum is an analytic orbifold of real dimension $4g + 2k + 2\sum \nu_j + \varepsilon - 3$ which is invariant under the Teichmüller flow. Each non-empty stratum carries a natural volume form and the corresponding measure, $\mu_g(\sigma)$ has finite mass and is invariant under the Teichmüller flow.

A stratum $\mathcal{Q}^*_{g,n}(\sigma)$ is not necessarily connected, but it is finitely connected, and the connected components are obviously $\mathrm{SL}(2, \mathbb{R})$ invariant (see [KZ], [L1], [L3]). Veech showed that the restriction of the Teichmüller flow restricted to any connected component of $\mathcal{Q}^*_{g,n}(\sigma)$ is ergodic with respect to the restriction of $\mu_{g,n}(\sigma)$. In [AGY] it was shown that in the case of strata of squares (that is, with $\varepsilon = 1$) the Teichmüller flow is exponentially mixing (for a class of Hölder observables) with respect to μ. Their approach is followed here and generalized to yield:

Theorem 1.2. *The Teichmüller flow is exponentially mixing with respect to each ergodic component of $\mu_g(\sigma)$ for observables in the Ratner class.*

There is a single stratum $\mathcal{Q}^*_{g,n}(\sigma)$ with maximum dimension, which is open and connected and has full $\mu_{g,n}$ measure: for this stratum, $\mu_{g,n}$ coincides with $\mu_{g,n}(\sigma)$. Thus Theorem 1.1 is a particular case of this one.

1.3. Outline of the proof. Our approach to exponential mixing follows [AGY] which develops around a combinatorial description of the moduli space of Abelian differentials.

The combinatorial description which we will use in the treatment of quadratic differentials, essentially equivalent the one of [BL], builds from the observation that the space of (non-orientable) quadratic differentials can be viewed as a subset of the space of Abelian differentials with involution. Indeed, it is well known that given any quadratic differential q on a Riemann surface S of genus g there exists a double covering $\pi\colon \widetilde{S} \to S$, branched over the singularities of odd order, and there is an Abelian differential ω on the surface $\widetilde{S}$ such that $\pi_*(\omega^2) = q$. In other words, q lifts to an orientable quadratic differential on $\widetilde{S}$. In this construction,

- to each zero of q with even multiplicity $l_i \geq 1$ corresponds a pair of zeros of ω with multiplicity $m_j = l_i/2$;
- to each zero of q with odd multiplicity $l_i \geq 1$ corresponds a zero of ω with multiplicity $m_j = l_i + 1$;
- to each pole of q with $l_i = -1$ corresponds a removable (that is, order 0) singularity of ω.

The surface $\widetilde{S}$ is connected if and only if q is non-orientable. Notice $i_*(\omega) = \pm\omega$, where $i\colon \widetilde{S} \to \widetilde{S}$ is the involution permuting the points in each fiber of the double cover π.

An Abelian differential induces a translation structure on the surface. In particular we can speak of the horizontal flow to the "east" and the vertical flow to the "north" (the involution exchanges north with south and east with west).

Thus, we consider moduli spaces of Abelian differentials with involution and a certain combinatorial marking. The combinatorial marking includes the order of the zeros at the singularities, but also a distinguished singularity with a fixed eastbound separatrix. This moduli space $\mathcal{M}$ is a finite cover of $\mathcal{Q}_{g,n}(\sigma)$ where $\mathrm{SL}(2,\mathbb{R})$ is still acting, and thus it is enough to prove the result on this space.

We parametrize the moduli space as a moduli space of zippered rectangles with involution as follows. Choosing a convenient segment I inside the separatrix, we look at the first return map under the northbound flow to the union of the I and its image under the involution. This map is an *interval exchange transformation with involution*, and the original northbound flow becomes a suspension flow, living in the union of some rectangles. The original surface can be obtained from the rectangles

by gluing appropriately. This construction can be carried out in a large open set of $\mathcal{M}$ (with complement of codimension 2, see [Ve3]).

Once this combinatorial model is setup, one can view the Teichmüller flow on $\mathcal{M}$ as a suspension flow over a (weakly) hyperbolic transformation, which is itself a skew-product over a (weakly) expanding transformation, the Rauzy algorithm with involution.

We then consider some appropriate compact subset of the domain of the Rauzy algorithm with involution: the induced transformation is automatically expanding and the Teichmüller flow is thus modelled on an "excellent hyperbolic flow" in the language of [AGY]. Two properties need to be verified to deduce exponential mixing: the return time should not be cohomologous to locally constant, and it should have exponential tails. The first property is an essentially algebraic consequence of the zippered rectangle construction. The second depends essentially on proving some distortion estimate. Both proofs of the distortion estimate in [AGY] depend heavily on certain properties of the usual Rauzy induction (simple description of transition probabilities for a random walk) which seem difficult to generalize to our setting. We provide here an alternative proof which is less dependent on precise estimates for the random walk.

Remark 1.3. Since the moduli space of zippered rectangles with involution can be regarded as a (Teichmüller flow-invariant) subspace of the larger moduli space of all zippered rectangles, it would seem natural to carry out the analysis around an appropriate restriction of the usual Rauzy algorithm. However, while the Rauzy algorithm can be modelled as a random walk on a finite graph, this property does not persist after restriction (though there is still a natural random walk model, it takes place in an infinite graph).

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2. Excellent hyperbolic semi-flows

In [AGY], an abstract result for exponential mixing was proved for the class of so-called *excellent hyperbolic semi-flows*, following the work of Baladi–Vallée [BV] based on the foundational work of Dolgopyat [D]. This result can be directly used in our work. In this section we state precisely this result, which will need several definitions.

By definition, a Finsler manifold is a smooth manifold endowed with a norm on each tangent space, which varies continuously with the base point.

Definition 2.1. A *John domain* Δ is a finite dimensional connected Finsler manifold, together with a measure Leb on Δ, with the following properties.

(1) For $x, x' \in \Delta$, let $d(x, x')$ be the infimum of the length of a C^1 path contained in Δ and joining x and x'. For this distance, Δ is bounded and there exist constants C_0 and ϵ_0 such that, for all $\epsilon < \epsilon_0$, for all $x \in \Delta$, there exists $x' \in \Delta$ such that $d(x, x') \leq C_0\epsilon$ and such that the ball $B(x', \epsilon)$ is compactly contained in Δ.

(2) The measure Leb is a fully supported finite measure on Δ, satisfying the following inequality: for all $C > 0$, there exists $A > 0$ such that, whenever a ball $B(x, r)$ is compactly contained in Δ, $\mathrm{Leb}(B(x, Cr)) \leq A\mathrm{Leb}(B(x, r))$.

Definition 2.2. Let L be a finite or countable set, let Δ be a John domain, and let $\{\Delta^{(l)}\}_{l \in L}$ be a partition into open sets of a full measure subset of Δ. A map $T\colon \bigcup_l \Delta^{(l)} \to \Delta$ is a *uniformly expanding Markov map* if the following holds.

(1) For each l, T is a C^1 diffeomorphism between $\Delta^{(l)}$ and Δ, and there exist constants $\kappa > 1$ (independent of l) and $C_{(l)}$ such that, for all $x \in \Delta^{(l)}$ and all $v \in T_x\Delta$, $\kappa \|v\| \leq \|DT(x) \cdot v\| \leq C_{(l)} \|v\|$.

(2) Let $J(x)$ be the inverse of the Jacobian of T with respect to Leb. Denote by $\mathcal{H}$ the set of inverse branches of T. The function $\log J$ is C^1 on each set $\Delta^{(l)}$ and there exists $C > 0$ such that, for all $h \in \mathcal{H}$, $\|D((\log J) \circ h)\|_{C^0(\Delta)} \leq C$.

Such a map T preserves a unique absolutely continuous measure μ. Its density is bounded from above and from below and is C^1.

Definition 2.3. Let $T\colon \bigcup_l \Delta^{(l)} \to \Delta$ be a uniformly expanding Markov map on a John domain. A function $r\colon \bigcup_l \Delta^{(l)} \to \mathbb{R}_+$ is a *good roof function* if

(1) there exists $\epsilon_1 > 0$ such that $r \geq \epsilon_1$;

(2) there exists $C > 0$ such that, for all $h \in \mathcal{H}$, $\|D(r \circ h)\|_{C^0} \leq C$;

(3) it is not possible to write $r = \psi + \phi \circ T - \phi$ on $\bigcup_l \Delta^{(l)}$, where $\psi\colon \Delta \to \mathbb{R}$ is constant on each set $\Delta^{(l)}$ and $\phi\colon \Delta \to \mathbb{R}$ is C^1.

If r is a good roof function for T, we will write $r^{(n)}(x) = \sum_{k=0}^{n-1} r(T^k x)$.

Definition 2.4. A good roof function r as above has *exponential tails* if there exists $\sigma_0 > 0$ such that $\int_\Delta e^{\sigma_0 r}\, \mathrm{dLeb} < \infty$.

If $\widehat{\Delta}$ is a Finsler manifold, we will denote by $C^1(\widehat{\Delta})$ the set of functions $u\colon \widehat{\Delta} \to \mathbb{R}$ which are bounded, continuously differentiable, and such that $\sup_{x \in \widehat{\Delta}} \|Du(x)\| < \infty$. Let

$$\|u\|_{C^1(\widehat{\Delta})} = \sup_{x \in \widehat{\Delta}} |u(x)| + \sup_{x \in \widehat{\Delta}} \|Du(x)\| \tag{4}$$

be the corresponding norm.

Definition 2.5. Let $T\colon \bigcup_l \Delta^{(l)} \to \Delta$ be a uniformly expanding Markov map, preserving an absolutely continuous measure μ. A *hyperbolic skew-product* over T is a map $\widehat{T}$ from a dense open subset of a bounded connected Finsler manifold $\widehat{\Delta}$, to $\widehat{\Delta}$, satisfying the following properties.

(1) There exists a continuous map $\pi\colon \widehat{\Delta} \to \Delta$ such that $T \circ \pi = \pi \circ \widehat{T}$ whenever both members of this equality are defined.

(2) There exists a probability measure ν on $\widehat{\Delta}$, giving full mass to the domain of definition of $\widehat{T}$, which is invariant under $\widehat{T}$.

(3) There exists a family of probability measures $\{\nu_x\}_{x\in\Delta}$ on $\widehat{\Delta}$ which is a disintegration of ν over μ in the following sense: $x \mapsto \nu_x$ is measurable, ν_x is supported on $\pi^{-1}(x)$ and, for every measurable set $A \subset \widehat{\Delta}$, $\nu(A) = \int \nu_x(A)\, \mathrm{d}\mu(x)$.

Moreover, this disintegration satisfies the following property: there exists a constant $C > 0$ such that, for any open subset $O \subset \bigcup \Delta^{(l)}$, for any $u \in C^1(\pi^{-1}(O))$, the function $\bar{u}\colon O \to \mathbb{R}$ given by $\bar{u}(x) = \int u(y)\, \mathrm{d}\nu_x(y)$ belongs to $C^1(O)$ and satisfies the inequality

$$\sup_{x\in O} \|D\bar{u}(x)\| \le C \sup_{y\in\pi^{-1}(O)} \|Du(y)\|.$$

(4) There exists $\kappa > 1$ such that, for all $y_1, y_2 \in \widehat{\Delta}$ with $\pi(y_1) = \pi(y_2)$, holds

$$d(\widehat{T}y_1, \widehat{T}y_2) \le \kappa^{-1} d(y_1, y_2).$$

Let $\widehat{T}$ be an hyperbolic skew-product over a uniformly expanding Markov map T. Let r be a good roof function for T, with exponential tails. It is then possible to define a space $\widehat{\Delta}_r$ and a semi-flow $\widehat{T}_t$ over $\widehat{T}$ on $\widehat{\Delta}$, using the roof function $r \circ \pi$, in the following way. Let $\widehat{\Delta}_r = \{(y, s) : y \in \bigcup_l \widehat{\Delta}_l, 0 \le s < r(\pi y)\}$. For almost all $y \in \widehat{\Delta}$, all $0 \le s < r(\pi y)$ and all $t \ge 0$, there exists a unique $n \in \mathbb{N}$ such that $r^{(n)}(\pi y) \le t + s < r^{(n+1)}(\pi y)$. Set $\widehat{T}_t(y, s) = (\widehat{T}^n y, s + t - r^{(n)}(\pi y))$. This is a semi-flow defined almost everywhere on $\widehat{\Delta}_r$. It preserves the probability measure $\nu_r = \nu \otimes \mathrm{Leb}/(\nu \otimes \mathrm{Leb})(\widehat{\Delta}_r)$. Using the canonical Finsler metric on $\widehat{\Delta}_r$, namely the product metric given by $\|(u, v)\| := \|u\| + \|v\|$, we define the space $C^1(\widehat{\Delta}_r)$ as in (4). Notice that $\widehat{\Delta}_r$ is not connected, and the distance between points in different connected components is infinite.

Definition 2.6. A semi-flow $\widehat{T}_t$ as above is called an *excellent hyperbolic semi-flow*.

Theorem 2.7 ([AGY]). *Let $\widehat{T}_t$ be an excellent hyperbolic semi-flow on a space $\widehat{\Delta}_r$, preserving the probability measure ν_r. There exist constants $C > 0$ and $\delta > 0$ such that, for all functions $U, V \in C^1(\widehat{\Delta}_r)$ and for all $t \ge 0$,*

$$\left| \int U \cdot V \circ \widehat{T}_t \, \mathrm{d}\nu_r - \left(\int U \, \mathrm{d}\nu_r \right) \left(\int V \, \mathrm{d}\nu_r \right) \right| \le C \, \|U\|_{C^1} \, \|V\|_{C^1} \, e^{-\delta t}.$$

3. The Veech flow with involution

3.1. Rauzy classes and interval exchange transformations with involution

3.1.1. Interval exchange transformations with involution. Let $\mathcal{A}$ be an alphabet on $2d \geq 4$ letters with an involution $i\colon \mathcal{A} \to \mathcal{A}$ and let $* \notin \mathcal{A}$. When considering objects modulo involution, we will use underline: for instance the involution class of an element $\alpha \in \mathcal{A}$ will be denoted by $\underline{\alpha} \in \underline{\mathcal{A}} = \mathcal{A}/i$. An *interval exchange transformation with involution* of type $(\mathcal{A}, i, *)$ depends on the specification of the following data:

Combinatorial data: Let $\pi\colon \mathcal{A} \cup \{*\} \to \{1, \ldots, 2d+1\}$ be a bijection such that neither $i(\mathcal{A}_l) \subset \mathcal{A}_r$ nor $i(\mathcal{A}_r) \subset \mathcal{A}_l$, where $\mathcal{A}_l = \{\alpha \in \mathcal{A},\ \pi(\alpha) < \pi(*)\}$ and $\mathcal{A}_r = \{\alpha \in \mathcal{A}_r,\ \pi(\alpha) > \pi(*)\}$. The combinatorial data can be viewed as a row where the elements of $\mathcal{A} \cup \{*\}$ are displayed in the order $(\pi^{-1}(1), \ldots, \pi^{-1}(2d+1))$.

Length data: Let $\lambda \in \mathbb{R}_+^{\underline{\mathcal{A}}}$ be a vector satisfying

$$\sum_{\pi(\alpha)<\pi(*)} \lambda_{\underline{\alpha}} = \sum_{\pi(\alpha)>\pi(*)} \lambda_{\underline{\alpha}} \tag{5}$$

(it is easy to find such a vector λ).

Let $\mathfrak{S} = \mathfrak{S}(\mathcal{A}, i, *)$ be the set of all bijections π as above.
The transformation is then defined as follows:

(1) Let $I \subset \mathbb{R}$ be the interval (all intervals will be assumed to be closed at the left and open at the right) centered on 0 and of length $|I| \equiv \sum_{\alpha\in\mathcal{A}} \lambda_{\underline{\alpha}}$ (notice that $|I| = 2\sum_{\underline{\alpha}\in\underline{\mathcal{A}}} \lambda_{\underline{\alpha}}$).

(2) Let $\overline{\pi}\colon \mathcal{A} \cup \{*\} \to \{1, \ldots, 2d+1\}$ be defined by $\overline{\pi}(*) = 2d + 2 - \pi(*)$ and $\overline{\pi}(\alpha) = 2d + 2 - \pi(i(\alpha))$.

(3) Break I into $2d$ subintervals I_α of length $\lambda_{\underline{\alpha}}$, ordered according to π.

(4) Rearrange the subintervals inside I in the order given by $\overline{\pi}$.

3.1.2. Rauzy classes with involution. We define two operations, the *left* and the *right* on $\mathfrak{S}$ as follows. Let α and β be the leftmost and the rightmost letters of the row representing π, respectively. If $\beta \neq i(\alpha)$ and taking β and putting it into the position immediately after $i(\alpha)$ results in a row representing an element π' of $\mathfrak{S}$, we say that the left operation is defined at π, and it takes π to π'. In this case, we say that α wins and β loses. Similarly, if $\alpha \neq i(\beta)$ and taking α and putting it into the position immediately before $i(\beta)$ results in a row representing an element π' of $\mathfrak{S}$, we say that the right operation is defined at π, and it takes π to π'. In this case, we say that β wins and α loses.

Remark 3.1. Notice that $\beta \neq i(\alpha)$ and $\alpha \neq i(\beta)$ are equivalent conditions since the involution i is a bijection. But to define left (respectively right) operation we also ask that the row obtained after moving β (respectively α) represents an element of $\mathfrak{S}$. So we can have none of the operations defined at some permutation, or just one, or both.

Consider an oriented diagram with vertices which are the elements of $\mathfrak{S}$ and oriented arrows representing the operations left and right starting and ending at two vertices of $\mathfrak{S}$. We will say that such an arrow has type left or right, respectively. We will call this diagram by *Rauzy diagram with involution*. A *path* γ of length $m \geq 0$ is a sequence of m arrows, $a_1, \ldots, a_m$, joining $m+1$ vertices, $v_0, \ldots, v_m$, respectively. In this case we say that γ starts at v_0, it ends at v_m and pass through $v_1, \ldots, v_{m-1}$. Let γ_1 and γ_2 be two paths such that the end of γ_1 is the start of γ_2. We define their concatenation denoted by $\gamma_1\gamma_2$, which also is a path. A path of length zero is identified with a vertex and if it has length one we identify it with an arrow.

A *Rauzy class with involution* $\mathfrak{R}$ is a minimal non-empty subset of $\mathfrak{S}$, which is invariant under the left and the right operations, and such that any involution class admits a representative which is the winner of some arrow starting (and ending) in $\mathfrak{R}$. Elements of Rauzy classes with involution are said to be *irreducible*. We denote by $\mathfrak{S}^0 = \mathfrak{S}^0(\mathcal{A}, i, *) \subset \mathfrak{S}$ the set of irreducible permutations and let $\Pi(\mathfrak{R})$ be the set of all paths.

Lemma 3.2. *If π is irreducible then the left operation (respectively the right operation) is defined at π if and only if there exists $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ satisfying (5) such that $\lambda_{\underline{\alpha}} > \lambda_{\underline{\beta}}$ (respectively $\lambda_{\underline{\beta}} > \lambda_{\underline{\alpha}}$) where α and β are the leftmost and the rightmost elements of π.*

Proof. Assume that the left operation is defined at π and let π' be the image of π. Let $\lambda' \in \mathbb{R}_+^{\mathcal{A}}$ be a vector satisfying $\sum_{\pi'(\xi)<\pi'(*)} \lambda'_{\underline{\xi}} = \sum_{\pi'(\xi)>\pi'(*)} \lambda'_{\underline{\xi}}$. Let $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ be given by $\lambda_{\underline{\alpha}} = \lambda'_{\underline{\alpha}} + \lambda'_{\underline{\beta}}$ and $\lambda_{\underline{\xi}} = \lambda'_{\underline{\xi}}$, $\underline{\xi} \neq \underline{\alpha}$. Then λ satisfies (5) and we have $\lambda_{\underline{\alpha}} > \lambda_{\underline{\beta}}$.

Assume that $\lambda_{\underline{\alpha}} > \lambda_{\underline{\beta}}$. Let $\lambda'_{\underline{\alpha}} = \lambda_{\underline{\alpha}} - \lambda_{\underline{\beta}}$. Let $\pi'(x) = \pi(x)$ for $\pi(x) \leq \pi(i(\alpha))$, $\pi'(\beta) = \pi(i(\alpha)) + 1$ and $\pi'(x) = \pi(x) + 1$ for $\pi(i(\alpha)) < \pi(x) < 2d + 1$. We need to show that $\pi' \in \mathfrak{S}$.

Let $\mathcal{A}_l = \{\xi : \pi(\xi) < \pi(*)\}$, $\mathcal{A}_r = \{\xi : \pi(\xi) > \pi(*)\}$, $\mathcal{A}'_l = \{\xi : \pi'(\xi) < \pi'(*)\}$, $\mathcal{A}'_r = \{\xi : \pi'(\xi) > \pi'(*)\}$. Notice that $\sum_{\pi'(\xi)<\pi'(*)} \lambda'_{\underline{\xi}} = \sum_{\pi'(\xi)>\pi'(*)} \lambda'_{\underline{\xi}}$, so $i(\mathcal{A}'_l)$ can not be properly contained or properly contain $\mathcal{A}'_r$. If $i(\mathcal{A}'_l) = \mathcal{A}'_r$, then $\pi'(i(\alpha)) > \pi'(*)$, so $\pi(i(\alpha)) > \pi(*)$ as well. This implies that $\mathcal{A}'_l = \mathcal{A}_l$ and $\mathcal{A}'_r = \mathcal{A}_r$, and since $\pi \in \mathfrak{S}$ we have $\pi' \in \mathfrak{S}$. □

3.1.3. Linear action. Given a Rauzy class $\mathfrak{R}$, we associate to each path $\gamma \in \Pi(\mathfrak{R})$ a linear map $B_\gamma \in \mathrm{SL}(\underline{\mathcal{A}}, \mathbb{Z})$. If γ is a vertex we take $B_\gamma = \mathrm{id}$. If γ is an arrow with

winner α and loser β then we define $B_\gamma \cdot e_{\underline{\xi}} = e_{\underline{\xi}}$ for $\underline{\xi} \in \underline{\mathcal{A}} \backslash \{\underline{\alpha}\}$, $B_\gamma \cdot e_{\underline{\alpha}} = e_{\underline{\alpha}} + e_{\underline{\beta}}$, where $\{e_{\underline{\xi}}\}_{\underline{\xi} \in \underline{\mathcal{A}}}$ is the canonical basis of $\mathbb{R}^{\underline{\mathcal{A}}}$. If γ is a path, of the form $\gamma = \gamma_1 \dots \gamma_n$, where γ_i are arrows for all $i = 1, \dots, n$, we take $B_\gamma = B_{\gamma_1 \dots \gamma_n} = B_{\gamma_n} \dots B_{\gamma_1}$.

3.2. Rauzy algorithm with involution. Given a Rauzy class $\mathfrak{R} \subset \mathfrak{S}$, consider the set

$$\mathcal{S}_\pi = \left\{ \lambda \in \mathbb{R}^{\underline{\mathcal{A}}} : \sum_{\pi(\alpha) < \pi(*)} \lambda_{\underline{\alpha}} = \sum_{\pi(\alpha) > \pi(*)} \lambda_{\underline{\alpha}} \right\}.$$

We define

$$\mathcal{S}_\pi^+ = \mathcal{S}_\pi \cap \mathbb{R}_+^{\underline{\mathcal{A}}}, \quad \Delta_\pi = \mathcal{S}_\pi^+ \times \{\pi\}, \quad \Delta_{\mathfrak{R}}^0 = \bigcup_{\pi \in \mathfrak{R}} \Delta_\pi.$$

Let (λ, π) be an element of $\Delta_{\mathfrak{R}}^0$. We say that we can apply Rauzy algorithm with involution to (λ, π) if $\lambda_{\underline{\alpha}} \neq \lambda_{\underline{\beta}}$, where $\alpha, \beta \in \mathcal{A}$ are the leftmost and the rightmost elements of π, respectively. Then we define (λ', π') as follows:

(1) Let $\gamma = \gamma(\lambda, \pi)$ be an arrow representing the left or the right operation at π, according to whether $\lambda_{\underline{\alpha}} > \lambda_{\underline{\beta}}$ or $\lambda_{\underline{\beta}} > \lambda_{\underline{\alpha}}$.

(2) Let $\lambda'_{\underline{\xi}} = \lambda_{\underline{\xi}}$ if $\underline{\xi}$ is not the class of the winner of γ, and $\lambda'_{\underline{\xi}} = |\lambda_{\underline{\alpha}} - \lambda_{\underline{\beta}}|$ if $\underline{\xi}$ is the class of the winner of γ, i.e., $\lambda = B_\gamma^* \cdot \lambda'$ (here and in the following we will use the notation A^* to the transpose of a matrix A).

(3) Let π' be the end of γ.

We say that (λ', π') is obtained from (λ, π) by applying Rauzy algorithm with involution, of type left or right depending on whether the operation is left or right. We have $(\lambda', \pi') \in \Delta_{\mathfrak{R}}^0$. In this way we define a map $Q\colon (\lambda, \pi) \mapsto (\lambda', \pi')$ which is called *Rauzy induction map with involution*. Its domain of definition is the set of all $(\lambda, \pi) \in \Delta_{\mathfrak{R}}^0$ such that $\lambda_{\underline{\alpha}} \neq \lambda_{\underline{\beta}}$ (where α and β are the leftmost and the rightmost letters of π) and we denote it by $\Delta_{\mathfrak{R}}^1$. The connected components $\Delta_\pi \subset \Delta_{\mathfrak{R}}^0$ are naturally labeled by elements of $\mathfrak{R}$ and the connected components Δ_γ of $\Delta_{\mathfrak{R}}^1$ are naturally labeled by arrows, i.e., paths in $\Pi(\mathfrak{R})$ of length 1.

We associate to (λ, π) and to (λ', π') two interval exchange transformations with involution $f\colon I \to I$ and $f'\colon I' \to I'$, respectively. The relation between (λ, π) and (λ', π') implies a relation between the interval exchange transformations with involution, namely, the map f' is the first return map of f to a subinterval of I, obtained by cutting two subintervals from the beginning and from the end of I with the same length $\lambda_{\underline{\xi}}$, where ξ is the loser of γ.

Let $\Delta_{\mathfrak{R}}^n$ be the domain of Q^n, $n \geq 2$. The connected components of $\Delta_{\mathfrak{R}}^n$ are naturally labeled by paths in $\Pi(\mathfrak{R})$ of length n: if γ is obtained by following a sequence of arrows $\gamma_1, \dots, \gamma_n$, then $\Delta_\gamma = \{x \in \Delta_{\mathfrak{R}}^0 : Q^{k-1}(x) \in \Delta_{\gamma_k},\ 1 \leq k \leq n\}$. Notice that if γ starts at π and ends at π' then $\Delta_\gamma = (B_\gamma^* \cdot \mathcal{S}_{\pi'}^+) \times \{\pi\}$.

If γ is a path in $\Pi(\mathfrak{R})$ of length n ending at $\pi \in \mathfrak{R}$, let

$$Q^\gamma = Q^n \colon \Delta_\gamma \to \Delta_\pi.$$

Let $\Delta_{\mathfrak{R}}^\infty = \bigcap_{n\geq 0} \Delta_{\mathfrak{R}}^n$.

Definition 3.3. A path γ is said to be *complete* if all involution classes $\underline{\alpha} \in \underline{\mathcal{A}}$ are winners of some arrow composing γ.

The concatenation of k complete paths is said to be *k-complete*.

A path $\gamma \in \Pi(\mathfrak{R})$ is *positive* if B_γ is given, in the canonical basis of $\mathbb{R}^{\mathcal{A}}_+$, by a matrix with all entries positive.

Lemma 3.4. *A $(2d-3)$-complete path $\gamma \in \Pi(\mathfrak{R})$ is positive.*

Proof. Let $\gamma = \gamma^1\gamma^2 \dots \gamma^N$ where γ^i is an arrow starting at π^{i-1} and ending at π^i. Since γ is l-complete we also can represent it as $\gamma = \gamma_{(1)}\gamma_{(2)} \dots \gamma_{(l)}$ where each $\gamma_{(i)}$ is a complete path passing through $\pi_1^{(i-1)}, \dots, \pi_{n(i)}^{(i-1)}, \pi_1^{(i)}$.

Let $B^*_{\gamma_{(i)}}$ be the matrix such that $\lambda_1^{(i)} = B^*_{\gamma_{(i)}} \cdot \lambda_1^{(i+1)}$. And let $B^*(\underline{\alpha}, \underline{\beta}, i)$ be the coefficient on row $\underline{\alpha}$ and on column $\underline{\beta}$ of the matrix $B^*_{\gamma_{(i)}}$. Fix $k < l$. We denote $C^*(k) = B^*_{\gamma_{(1)}} \dots B^*_{\gamma_{(k)}}$. Let $C^*(\underline{\alpha}, \underline{\beta}, k)$ be the coefficient on row $\underline{\alpha}$ and on column $\underline{\beta}$ of the matrix $C^*(k)$. We want to prove that for all $\underline{\alpha}, \underline{\beta} \in \underline{\mathcal{A}}$ we have $C^*(\underline{\alpha}, \underline{\beta}, l) > 0$. For $r \geq 0$ denote $\widehat{C}(r) = B^*_{\gamma^1} \dots B^*_{\gamma^r}$.

Since the diagonal elements of the matrices $B^*_{\tilde{\gamma}}$, where $\tilde{\gamma}$ is an arrow, are one and all other terms are non-negative integers, we obtain that the sequence $C^*(\underline{\alpha}, \underline{\beta}, k)$ is non-decreasing in k, thus:

$$C^*(\underline{\alpha}, \underline{\beta}, k) > 0 \Longrightarrow C^*(\underline{\alpha}, \underline{\beta}, k+1) > 0. \tag{6}$$

Fix any $\underline{\alpha}, \underline{\beta} \in \underline{\mathcal{A}}$. We will reorder the involution classes of $\underline{\mathcal{A}}$ as $\underline{\alpha} = \underline{\alpha_1}$, $\underline{\alpha_2}, \dots, \underline{\alpha_d} = \underline{\beta}$ with associate numbers $0 = r_1 < r_2 < \dots < r_d$ such that

$$C^*(\underline{\alpha_1}, \underline{\alpha_j}, r) > 0 \quad \text{for all } r \geq r_j. \tag{7}$$

If $\underline{\alpha} = \underline{\beta}$ we take $d = 1$ and $r_1 = 0$ and therefore we have (7). Otherwise we choose the smallest positive integer r_2 such that the winner of γ^{r_2} is $\underline{\alpha_1} = \underline{\alpha}$ and let $\underline{\alpha_2}$ be the loser of the same arrow. Notice that $\underline{\alpha_1} \neq \underline{\alpha_2}$ by irreducibility, and $B^*(\underline{\alpha_1}, \underline{\alpha_2}, r_2) = 1$, hence $C^*(\underline{\alpha_1}, \underline{\alpha_2}, r) > 0$ for every $r \geq r_2$. This gives the result for $d = 2$.

Now we will see the general case. Assume that $\underline{\alpha_1}, \dots, \underline{\alpha_j}$ and $r_1, \dots, r_j$ have been constructed with $\underline{\beta} \neq \underline{\alpha_m}$ for $1 \leq m \leq j$. Let r'_j be the smallest integer greater than r_j such that the winner of $\gamma^{r'_j}$ does not belong to $\{\underline{\alpha_1}, \dots, \underline{\alpha_j}\}$ and let r_{j+1} be the smallest integer greater than r'_j such that the winner of $\gamma^{r_{j+1}}$ is in $\{\underline{\alpha_1}, \dots, \underline{\alpha_j}\}$.

Let $\underline{\alpha_{j+1}}$ be the loser of $\gamma^{r_{j+1}}$. Then $\underline{\alpha_{j+1}}$ is the winner of $\gamma^{r_{j+1}-1}$ and therefore $\underline{\alpha_{j+1}} \notin \{\underline{\alpha_1}, \ldots, \underline{\alpha_j}\}$. Thus, for some $1 \le m \le j$ we have $B^*(\underline{\alpha_m}, \underline{\alpha_{j+1}}, r_{j+1}) = 1$ and $C^*(\underline{\alpha_1}, \underline{\alpha_m}, r_{j+1} - 1) > 0$, since $r_{j+1} > r_m$. Thus

$$C^*(\underline{\alpha_1}, \underline{\alpha_{j+1}}, r) > 0 \quad \text{for all } r \ge r_{j+1}.$$

Following this process, we will obtain $\underline{\alpha_d} = \underline{\beta}$.

Now, we will see how many complete paths we need until define r_d. We need a complete path to define each r_j and another one to define each r'_j, for $2 \le j \le d-1$. And we need another complete path to define r_d. Thus we need at most $2(d-2)+1 = 2d-3$ complete paths composing γ to conclude it is positive. □

3.3. Zippered rectangles. Let π be a permutation in a Rauzy class $\mathfrak{R} \subset \mathfrak{S}^0$. Let $\Theta_\pi \subset \mathcal{S}_\pi$ be the set of all τ such that

$$\begin{aligned} \sum_{\pi(*)<\pi(\xi)\le k_r} \tau_{\underline{\xi}} > 0 \quad &\text{for all } \pi(*) < k_r < 2d+1, \\ \sum_{k_l \le \pi(\xi)<\pi(*)} \tau_{\underline{\xi}} < 0 \quad &\text{for all } 1 < k_l < \pi(*). \end{aligned} \tag{8}$$

Observe that Θ_π is an open convex polyhedral cone and we will see later it is non-empty.

Given a letter $\alpha \in \mathcal{A}$, we define $M_\pi(\alpha) = \max\{\pi(\alpha), \pi(i(\alpha))\}$ and $m_\pi(\alpha) = \min\{\pi(\alpha), \pi(i(\alpha))\}$.

Define the linear operator $\Omega(\pi)$ on $\mathbb{R}^{\underline{\mathcal{A}}}$ as follows:

$$\Omega(\pi)_{\underline{x},\underline{y}} = \begin{cases} 2 & \text{if } M_\pi(x) < m_\pi(y), \\ -2 & \text{if } M_\pi(y) < m_\pi(x), \\ 1 & \text{if } m_\pi(x) < m_\pi(y) < M_\pi(x) < M_\pi(y), \\ -1 & \text{if } m_\pi(y) < m_\pi(x) < M_\pi(y) < M_\pi(x), \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Observe that $\Omega(\pi)$ is well-defined, since $(\Omega(\pi)\cdot\lambda)_\alpha = (\Omega(\pi)\cdot\lambda)_{i(\alpha)}$ for all $\alpha \in \mathcal{A}$.

We define the vector $w \in \mathbb{R}^{\underline{\mathcal{A}}}$ by $w = \Omega(\pi)\cdot\lambda$ and the vector $h \in \mathbb{R}^{\underline{\mathcal{A}}}$ by $h = -\Omega(\pi)\cdot\tau$. For each $\underline{\alpha} \in \underline{\mathcal{A}}$ define $\zeta_{\underline{\alpha}} = \lambda_{\underline{\alpha}} + i\,\tau_{\underline{\alpha}}$.

Lemma 3.5. *If γ is an arrow between (λ, π) and (λ', π'), then $w' = B_\gamma \cdot w$.*

Proof. We will consider the case when γ is a left arrow. The other case is entirely analogous. Let $\underline{\alpha(l)}$ and $\underline{\alpha(r)}$ be the leftmost and the rightmost letters in π, respectively. Thus $\underline{\alpha(l)}$ is the winner and $\underline{\alpha(r)}$ is the loser.

By definition,

$$w'_{\underline{\alpha}} = \sum_{\pi'(\xi)>\pi'(i(\alpha))} \lambda'_\xi - \sum_{\pi'(\xi)<\pi'(\alpha)} \lambda'_\xi.$$

Since $\lambda'_{\underline{\alpha}} = \lambda_{\underline{\alpha}}$ for all $\underline{\alpha} \neq \underline{\alpha(l)}$ and $\lambda'_{\underline{\alpha(l)}} = \lambda_{\underline{\alpha(l)}} - \lambda_{\underline{\alpha(r)}}$, it is easy to verify that

$$w'_{\underline{\alpha}} = \sum_{\pi(\xi)>\pi(i(\alpha))} \lambda_\xi - \sum_{\pi(\xi)<\pi(\alpha)} \lambda_\xi = w_{\underline{\alpha}} \quad \text{if } \underline{\alpha} \neq \underline{\alpha(r)}.$$

And if $\underline{\alpha} = \underline{\alpha(r)}$, we have

$$\begin{aligned} w'_{\underline{\alpha(r)}} &= \sum_{\pi'(\xi)>\pi'(i(\alpha(r)))} \lambda'_\xi - \sum_{\pi'(\xi)<\pi'(\alpha(r))} \lambda'_\xi \\ &= \sum_{\pi'(\xi)>\pi'(\alpha(r))} \lambda'_\xi - \sum_{\pi'(\xi)<\pi'(i(\alpha(r)))} \lambda'_\xi = \sum_{\pi(\xi)>\pi(i(\alpha(l)))} \lambda_\xi - \sum_{\pi(\xi)<\pi(i(\alpha(r)))} \lambda_\xi \\ &= \sum_{\pi(\xi)>\pi(i(\alpha(l)))} \lambda_\xi - \sum_{\pi(\xi)<\pi(\alpha(l))} \lambda_\xi + \sum_{\pi(\xi)>\pi(\alpha(r))} \lambda_\xi - \sum_{\pi(\xi)<\pi(i(\alpha(r)))} \lambda_\xi \\ &= w_{\underline{\alpha(l)}} + w_{\underline{\alpha(r)}}. \end{aligned}$$

Therefore, $w' = B_\gamma \cdot w$. □

Let $H(\pi) = \Omega(\pi) \cdot \mathcal{S}_\pi$. According to the previous lemma, given a path $\gamma \in \Pi(\mathfrak{R})$ starting at π and ending at π', we have $B_\gamma \cdot H(\pi) = H(\pi')$.

Lemma 3.6. *If $\pi \in \mathfrak{R}$ and $\tau \in \Theta_\pi$ then $h \in \mathbb{R}^{\mathcal{A}}_+$.*

Proof. Let $\alpha \in \mathcal{A}$. We have

$$h_{\underline{\alpha}} = \sum_{\pi(\xi)<\pi(\alpha)} \tau_\xi - \sum_{\pi(\xi)>\pi(i(\alpha))} \tau_\xi.$$

Suppose $\pi(\alpha), \pi(i(\alpha)) < \pi(*)$:

$$\begin{aligned} h_{\underline{\alpha}} &= \sum_{\pi(\xi)<\pi(\alpha)} \tau_\xi - \sum_{\pi(\xi)>\pi(i(\alpha))} \tau_\xi \\ &= \sum_{\pi(\xi)<\pi(\alpha)} \tau_\xi - \sum_{\pi(i(\alpha))<\pi(\xi)<\pi(*)} \tau_\xi - \sum_{\pi(*)<\pi(\xi)\leq 2d+1} \tau_\xi \\ &= -\sum_{\pi(\alpha)\leq\pi(\xi)<\pi(*)} \tau_\xi - \sum_{\pi(i(\alpha))<\pi(\xi)<\pi(*)} \tau_\xi > 0. \end{aligned}$$

Analogously, if $\pi(\alpha), \pi(i(\alpha)) > \pi(*)$ we have $h_{\underline{\alpha}} > 0$.

Now we will suppose that $\pi(\alpha) < \pi(*) < \pi(i(\alpha))$. In this case, we have:

$$\begin{aligned} h_{\underline{\alpha}} &= \sum_{\pi(\xi)<\pi(\alpha)} \tau_{\underline{\xi}} - \sum_{\pi(\xi)>\pi(i(\alpha))} \tau_{\underline{\xi}} \\ &= \sum_{1\le\pi(\xi)<\pi(*)} \tau_{\underline{\xi}} - \sum_{\pi(\alpha)\le\pi(\xi)<\pi(*)} \tau_{\underline{\xi}} - \sum_{\pi(*)<\pi(\xi)\le 2d+1} \tau_{\underline{\xi}} + \sum_{\pi(*)<\pi(\xi)\le\pi(i(\alpha))} \tau_{\underline{\xi}} \\ &= -\sum_{\pi(\alpha)<\pi(\xi)<\pi(*)} \tau_{\underline{\xi}} + \sum_{\pi(*)<\pi(\xi)<\pi(i(\alpha))} \tau_{\underline{\xi}} > 0. \end{aligned}$$

So, $h \in \mathbb{R}^{\underline{\mathcal{A}}}_{+}$. □

Lemma 3.7. *If $\gamma \in \Pi(\mathfrak{R})$ is an arrow starting at π and ending at π' then $(B^*_\gamma)^{-1} \cdot \Theta_\pi \subset \Theta_{\pi'}$.*

Proof. We will suppose that γ is a right arrow and the other case is entirely analogous. Let $\tau \in \Theta_\pi$ and let $\underline{\alpha} \in \underline{\mathcal{A}}$ be the winner and $\underline{\beta} \in \underline{\mathcal{A}}$ be the loser of γ.

Notice we have $\tau'_{\underline{\xi}} = \tau_{\underline{\xi}}$ for all $\underline{\xi} \in \underline{\mathcal{A}}\backslash\{\underline{\alpha}\}$. Let $m = \pi(i(\alpha))$. Notice that

$$h_{\underline{\alpha}} = h_\alpha = h_{i(\alpha)} = \sum_{\pi(\xi)<\pi(i(\alpha))} \tau_{\underline{\xi}} - \sum_{\pi(\xi)>\pi(\alpha)} \tau_{\underline{\xi}} = \sum_{\pi(\xi)<m} \tau_{\underline{\xi}}. \tag{10}$$

Suppose $m < \pi(*)$. Since $\pi'(\xi) = \pi(\xi)$ for all $\xi \in \mathcal{A}$ such that $\pi(\xi) \ge m$ we have that the first inequalities of (8) are satisfied and

$$\sum_{k_l\le\pi'(\xi)<\pi'(*)} \tau'_{\underline{\xi}} = \sum_{k_l\le\pi(\xi)<\pi(*)} \tau_{\underline{\xi}} < 0 \quad \text{for all } m < k_l < \pi(*).$$

Thus, it remains to prove the last inequalities to $1 < k_l \le m$. Let $1 < k_l < m$. Since $\tau'_{\underline{\alpha}} = \tau_{\underline{\alpha}} - \tau_{\underline{\beta}}$,

$$\sum_{k_l\le\pi'(\xi)<\pi'(*)} \tau'_{\underline{\xi}} = \sum_{k_l\le\pi(\xi)<\pi(*)} \tau_{\underline{\xi}} < 0 \quad \text{for all } 1 < k_l < m.$$

If $k_l = m$, by (10)

$$\sum_{m\le\pi'(\xi)<\pi'(*)} \tau'_{\underline{\xi}} = \sum_{m\le\pi(\xi)<\pi(*)} \tau_{\underline{\xi}} - \tau_{\underline{\beta}} = \sum_{2\le\pi(\xi)<\pi(*)} \tau_{\underline{\xi}} - h_{\underline{\alpha}} < 0. \tag{11}$$

Now suppose $m > \pi(*)$ This case is analogous to the first one. We will just do the part corresponding to (11).

$$\sum_{\pi'(*)<\pi'(\xi)\le m-1} \tau'_{\underline{\xi}} = \sum_{\pi(*)<\pi(\xi)\le m-1} \tau_{\underline{\xi}} + \tau_{\underline{\beta}} = h_{\underline{\alpha}} - \sum_{2\le\pi(\xi)<\pi(*)} \tau_{\underline{\xi}} > 0.$$

Thus $\tau' \in \Theta_{\pi'}$, as we wanted to prove. □

Definition 3.8. Let $\Theta'_\pi \subset \mathcal{S}_\pi$ be the set of all $\tau \neq 0$ such that

$$\begin{aligned} \sum_{\pi(*)<\pi(\xi)\leq k_r} \tau_{\underline{\xi}} \geq 0 \quad \text{for all } \pi(*) < k_r < 2d+1, \\ \sum_{k_l \leq \pi(\xi)<\pi(*)} \tau_{\underline{\xi}} \leq 0 \quad \text{for all } 1 < k_l < \pi(*). \end{aligned} \tag{12}$$

Let $\gamma \in \Pi(\mathfrak{R})$ be a path starting at π_s and ending at π_e. In the same way we showed that $(B^*_\gamma)^{-1} \cdot \Theta_{\pi_s} \subset \Theta_{\pi_e}$ in the previous lemma, one sees that $(B^*_\gamma)^{-1} \cdot \Theta'_{\pi_s} \subset \Theta'_{\pi_e}$.

Definition 3.9. Let $\pi \in \Pi(\mathfrak{R})$ and $\alpha \in \mathcal{A}$. We say that α is a *simple letter* if $\pi(\alpha) < \pi(*) < \pi(i(\alpha))$ or if $\pi(i(\alpha)) < \pi(*) < \pi(\alpha)$. We say that α is a *double letter* if $\pi(\alpha)$ and $\pi(i(\alpha))$ are either both smaller or either both greater than $\pi(*)$. If $\pi(\alpha), \pi(i(\alpha)) < \pi(*)$ we say that α is a *left double letter* or has *left type*, otherwise we say that α is a *right double letter* or has *right type*.

Lemma 3.10. *If π is irreducible then Θ'_π is non-empty.*

Proof. By invariance and irreducibility, it is enough to find some $\pi \in \mathfrak{R}$ such that Θ'_π is non-empty.

Given $\alpha, \beta \in \mathcal{A}$ suppose we have $\pi \in \mathfrak{R}$ with one of the two following forms:

$$\cdots \alpha \cdots i(\alpha) \cdot \beta \cdot i(\beta) \cdot * \cdots \tag{13}$$

or

$$\cdots \alpha \cdots \beta \cdot i(\alpha) \cdot i(\beta) \cdot * \cdots. \tag{14}$$

We can define $\tau \in \Theta'_\pi$ by choosing $\tau_{\underline{\alpha}} = -\tau_{\underline{\beta}} = 1$ and $\tau_{\underline{\xi}} = 0$ for all $\underline{\xi} \in \underline{\mathcal{A}} \backslash \{\underline{\alpha}, \underline{\beta}\}$.

Let us show that there exists some $\pi \in \mathfrak{R}$ satisfying this property.

By definition of permutation in $\mathfrak{S}$, there exist at least one double letter of each one of the types, i.e., there exist $\alpha, \beta \in \mathcal{A}$ such that α is left double letter and β is right double letter.

If there exists more than one double letter of both types, we can obtain another irreducible permutation π' which has at most one double letter of each one of the types, as follows. First we apply left or right operations until we obtain one double letter in the leftmost or the rightmost position, which is possible by irreducibility. We will assume, without loss of generality, that such a letter is at rightmost position. If there is at most one left double letter, we take the permutation obtained to be π'. But, if there are more than one left double letter, we apply right operations, until we find a permutation with just one left double letter. Those right operations are well-defined since we have more than one double letter of both types.

Suppose that $\alpha \in \mathcal{A}$ is the unique left double letter. Then, if it is necessary, we apply right operations until obtain $\pi(\alpha) = 1$.

Let $\beta \in \mathcal{A}$ such that $\pi(\beta) = 2d + 1$, i.e.,

$$\alpha \cdot \cdot \cdot i(\alpha) \cdot * \cdot \cdot \beta.$$

If β is simple applying the left operation we obtain a permutation of type (13) or (14) depending on $\pi(i(\beta)) > \pi(i(\alpha))$ or $\pi(i(\beta)) < \pi(i(\alpha))$, respectively. If β is double we apply the left operation until we obtain a simple letter in the rightmost position of π and we are in the same conditions as in the previous case. □

Definition 3.11. Let us say that a path $\gamma \in \Pi(\mathfrak{R})$, starting at π_s and ending at π_e, is *strongly positive* if it is positive and $(B_\gamma^*)^{-1} \cdot \Theta'_{\pi_s} \subset \Theta_{\pi_e}$.

Lemma 3.12. *Let γ be a $(4d-6)$-complete path. Then γ is strongly positive.*

Proof. Let $d = \#\underline{\mathcal{A}}$. Fix $\tau \in \Theta'_{\pi_s} \backslash \{0\}$. Write γ as a concatenation of arrows $\gamma = \gamma_1 \dots \gamma_n$, and let π^{i-1} and π^i denote the start and the end of γ_i. Let $\tau^0 = \tau$, $\tau^i = (B_{\gamma_i}^*)^{-1} \cdot \tau^{i-1}$. We must show that $\tau^n \in \Theta_{\pi^n}$.

Let $h^i = -\Omega(\pi^i) \cdot \tau^i$. Notice that $\tau \in \Theta'_{\pi^0} \backslash \{0\}$ implies that $h^0 \in \overline{\mathbb{R}_+^{\mathcal{A}}} \backslash \{0\}$. Indeed, since $\tau \in \Theta'_{\pi^0}$, for every $\underline{\xi} \in \underline{\mathcal{A}}$, we have

$$\sum_{\pi^0(*) < \pi^0(\alpha) < \pi^0(\xi)} \tau_{\underline{\alpha}} \geq 0 \quad \text{and} \quad \sum_{\pi^0(\xi) < \pi^0(\alpha) < \pi^0(*)} \tau_{\underline{\alpha}} \leq 0.$$

Moreover, since $\tau \neq 0$, there exist $1 \leq k^l \leq \pi^0(*)$ maximal and $\pi^0(*) \leq k^r \leq 2d + 1$ minimal such that $\tau_{(\pi^0)^{-1}(k^l)} \neq 0$ and $\tau_{(\pi^0)^{-1}(k^r)} \neq 0$. Since π^0 is irreducible, $k^r - k^l < 2d - 1$. Remember that $h^0_{\underline{\xi}} \geq 0$ for all $\underline{\xi}$ and the inequality is strict if $\pi^0(\xi) = k^r + 1$ and $k^r < 2d + 1$ or if $\pi^0(\xi) = k^l - 1$ and $1 < k^l < \pi^0(*)$. Since $h^i = B_{\gamma_i} \cdot h^{i-1}$ we can consider a positive path $\gamma_1 \dots \gamma_i$ and then $h^i \in \mathbb{R}_+^{\mathcal{A}}$.

Let $\pi^i(*) \leq k_i^r \leq 2d$ be maximal and $2 \leq k_i^l \leq \pi^i(*)$ be minimal such that

$$\sum_{\pi^i(*) < \pi^i(\xi) \leq k} \tau^i_{\underline{\xi}} > 0 \quad \text{for all } \pi^i(*) < k \leq k_i^r,$$

$$\sum_{k \leq \pi^i(\xi) < \pi^i(*)} \tau^i_{\underline{\xi}} < 0 \quad \text{for all } k_i^l \leq k < \pi^i(*).$$

We claim that

(1) if $h^{i-1} \in \mathbb{R}_+^{\mathcal{A}}$ then $k_i^r - \pi^i(*) \geq k_{i-1}^r - \pi^{i-1}(*)$ and $\pi^i(*) - k_i^l \geq \pi^{i-1}(*) - k_{i-1}^l$, in particular $k_i^r - k_i^l \geq k_{i-1}^r - k_{i-1}^l$;

(2) if $h^{i-1} \in \mathbb{R}_+^{\mathcal{A}}$ and the winner of γ_i is one of the first $k_{i-1}^r + 1 - \pi^{i-1}(*)$ letters after $*$ in π^{i-1} then $k_i^r - k_i^l \geq \min\{k_{i-1}^r - k_{i-1}^l + 1, 2d - k_{i-1}^l\}$;

(3) if $h^{i-1} \in \mathbb{R}_+^{\mathcal{A}}$ and the winner of γ_i is one of the last $\pi^{i-1}(*) - k_{i-1}^l + 1$ letters before $*$ in π^{i-1} then $k_i^r - k_i^l \geq \min\{k_{i-1}^r - k_{i-1}^l + 1, k_{i-1}^r - 2\}$.

Notice that $2 < \pi^i(*) < 2d$ for all i.

Let us see that (1), (2) and (3) imply the result, which is equivalent to the statement that $k_n^r - k_n^l \geq 2d - 2$. Let us write $\gamma = \gamma_{(1)} \dots \gamma_{(4d-7)}$ where $\gamma_{(j)}$ is complete and each $\gamma_{(j)} = \gamma_{s_j} \dots \gamma_{e_j}$. By Lemma 3.4, $h^k \in \mathbb{R}_+^{\mathcal{A}}$ for $k \geq e_{2d-3}$. From the definition of a complete path, for each $j > 2d-3$, there exists $e_j < i_1 \leq e_{j+1}$ such that the winner of γ_{i_1} is one of the letters in position m_1 at π^{i_1-1} such that $\pi^{i_1-1}(*) < m_1 < k_{e_j}^r + 1$. It follows that $k_{e_{j+1}}^r - k_{e_{j+1}}^l \geq \min\{k_{i_1-1}^r - k_{i_1-1}^l + 1, 2d - k_{i_1-1}^l\}$, so

$$k_{e_{j+1}}^r - k_{e_{j+1}}^l \geq \min\{k_{e_j}^r - k_{e_j}^l + 1, 2d - k_{e_j}^l\}. \tag{15}$$

In the same way there exists $e_{j-1} < i_2 \leq e_j$ such that the winner of γ_{i_2} is one of the letters in position m_2 at π^{i_2-1} such that $k_{e_j}^r + 1 < m_2 < \pi^{i_2-1}(*)$. It follows that $k_{e_j}^r - k_{e_j}^l \geq \min\{k_{i_2-1}^r - k_{i_2-1}^l + 1, k_{i_2-1}^r - 2\}$, thus

$$k_{e_j}^r - k_{e_j}^l \geq \min\{k_{e_{j-1}}^r - k_{e_{j-1}}^l + 1, k_{e_{j-1}}^r - 2\}. \tag{16}$$

By (15) and (16), we see that:

$$k_{e_{j+1}}^r - k_{e_{j+1}}^l \geq \min\{k_{e_{j-1}}^r - k_{e_{j-1}}^l + 2, 2d - (k_{e_{j-1}}^l - 1), (k_{e_{j-1}}^r + 1) - 2, 2d - 2\}.$$

Therefore, we obtain $k_n^r - k_n^l = k_{e_{2d-3+2d-4}}^r - k_{e_{2d-3+2d-4}}^l \geq \min\{k_{e_{2d-3}}^r - k_{e_{2d-3}}^l + 2d - 2, 2d - (k_{e_{2d-3}}^l - 2d - 4), (k_{e_{2d-3}}^r + 2d - 4) - 2, 2d - 2\} = 2d - 2$.

We now check (1), (2) and (3). Assume that $h^{i-1} \in \mathbb{R}_+^{\mathcal{A}}$, and that γ_i is a right arrow, the other case being analogous. Let α be the rightmost letter of π^{i-1} which is the winner of γ_i, and let β be the leftmost letter of π^{i-1} which is the loser of γ_i.

Case 1: Suppose $\pi^{i-1}(i(\alpha)) < \pi^{i-1}(*)$.

If the winner of γ_i is not one of the $\pi^{i-1}(*) - k_{i-1}^l + 1$ last letters on the left side of $*$ in π^{i-1}, then for every $\underline{\xi} \in \underline{\mathcal{A}}$ such that $k_{i-1}^l \leq \pi^{i-1}(\xi) \leq 2d + 1$, we have $\pi^{i-1}(\xi) = \pi^i(\xi)$ and $\tau_{\underline{\xi}}^{i-1} = \tau_{\underline{\xi}}^i$ for all $k_{i-1}^l \leq \pi^{i-1}(\xi) \leq 2d$. Hence $k_i^r - \pi^i(*) \geq k_{i-1}^r - \pi^{i-1}(*)$ and $\pi^i(*) - k_i^l \geq \pi^{i-1}(*) - k_{i-1}^l$.

If the winner $\underline{\alpha}$ of γ_i appears in the k-th position counting from $*$ to the left in π^{i-1} with $k_{i-1}^l - 1 \leq k < \pi^{i-1}(*)$, then

$$\sum_{j \leq \pi^i(\xi) < \pi^i(*)} \tau_{\underline{\xi}}^i = \sum_{j \leq \pi^{i-1}(\xi) < \pi^{i-1}(*)} \tau_{\underline{\xi}}^{i-1} < 0 \quad \text{for all } k+1 \leq j < \pi(*),$$

$$\sum_{j\le\pi^i(\xi)<\pi^i(*)} \tau^i_{\underline{\xi}} = \sum_{j+1\le\pi^{i-1}(\xi)<\pi^{i-1}(*)} \tau^{i-1}_{\underline{\xi}} < 0 \quad \text{for all } k^l_{i-1}-1 \le j \le k-1,$$

$$\sum_{k\le\pi^i(\xi)<\pi^i(*)} \tau^i_{\underline{\xi}} = \sum_{2\le\pi^{i-1}(\xi)<\pi^{i-1}(*)} \tau^{i-1}_{\underline{\xi}} - h^{i-1}_{\underline{\alpha}} \le -h^{i-1}_{\underline{\alpha}} < 0,$$

which implies that $\pi^i(*) - k^l_i \ge \min\{\pi^{i-1}(*) - 2, \pi^{i-1}(*) + 1 - k^l_{i-1}\}$, hence $k^r_i - k^l_i \ge \min\{k^r_{i-1} - k^l_{i-1} + 1, k^r_{i-1} - 2\}$.

This shows that (1) holds and (3) holds. Moreover, (2) also holds since its hypothesis can only be satisfied if $k^r_{i-1} = 2d$.

Case 2: Suppose $\pi^{i-1}(i(\alpha)) > \pi^{i-1}(*)$.

If the winner of γ_i is not one of the $k^r_{i-1} - \pi^{i-1}(*) + 1$ first letters on the right side of $*$ in π^{i-1}, then for every $\underline{\xi} \in \underline{\mathcal{A}}$ such that $1 < \pi^{i-1}(\xi) \le k^r_{i-1}$, we have $\pi^i(\xi) = \pi^{i-1}(\xi) - 1$ and $\tau^{i-1}_{\underline{\xi}} = \tau^i_{\underline{\xi}}$, so $k^r_i - \pi^i(*) \ge k^r_{i-1} - \pi^{i-1}(*)$ and $\pi^i(*) - k^l_i \ge \pi^{i-1}(*) - k^l_{i-1}$.

If the winner $\underline{\alpha}$ of γ_i appears in the k-th in π^{i-1} with $\pi^{i-1}(*) < k \le k^r_{i-1} + 1$, then

$$\sum_{\pi^i(*)<\pi^i(\xi)\le j} \tau^i_{\underline{\xi}} = \sum_{\pi^{i-1}(*)<\pi^{i-1}(\xi)\le j-1} \tau^{i-1}_{\underline{\xi}} > 0 \quad \text{for all } \pi(*) \le j < k-1,$$

$$\sum_{\pi^i(*)<\pi^i(\xi)\le j} \tau^i_{\underline{\xi}} = \sum_{\pi^{i-1}(*)<\pi^{i-1}(\xi)\le j} \tau^{i-1}_{\underline{\xi}} > 0 \quad \text{for all } k \le j \le k^r_{i-1} + 1,$$

$$\sum_{\pi^i(*)<\pi^i(\xi)\le k-1} \tau^i_{\underline{\xi}} = -\sum_{2\le\pi^{i-1}(\xi)<\pi^{i-1}(*)} \tau^{i-1}_{\underline{\xi}} + h^{i-1}_{\underline{\alpha}} \ge h^{i-1}_{\underline{\alpha}} > 0,$$

which implies that $k^r_i - \pi^i(*) \ge \min\{k^r_{i-1} + 1 - \pi^{i-1}(*), 2d - \pi^{i-1}(*)\}$, hence $k^r_i - k^l_i \ge \min\{k^r_{i-1} - k^l_{i-1} + 1, 2d - k^l_{i-1}\}$.

This shows that both (1) and (2) holds. Moreover, (3) also holds since its hypothesis can only be satisfied if $k^l_{i-1} = 2$. □

Corollary 3.13. *If π is irreducible then Θ_π is non-empty.*

Proof. Let $\gamma \in \Pi(\mathfrak{R})$ be a strongly positive path starting and ending at π, which exists by Lemma 3.12. Then $(B^*_\gamma)^{-1} \cdot \Theta'_\pi \subset \Theta_\pi$ and by Lemma 3.10 the set Θ'_π is non-empty. Therefore Θ_π is non-empty. □

Given that Θ_π is non-empty, it is easy to see that $\Theta'_\pi \cup \{0\}$ is in fact just the closure of Θ_π.

3.3.1. Extension of induction to the space of zippered rectangles. Let $\gamma \in \Pi(\mathfrak{R})$ be a path starting at π and define Θ_γ satisfying:

$$B_\gamma^* \cdot \Theta_\gamma = \Theta_\pi.$$

If γ is a right arrow ending at π', then $\Theta_\gamma = \{\tau \in \Theta_{\pi'} \mid \sum_{x\in\mathcal{A}} \tau_x < 0\}$, and if γ is a left arrow ending at π', then $\Theta_\gamma = \{\tau \in \Theta_{\pi'} \mid \sum_{x\in\mathcal{A}} \tau_x > 0\}$. Indeed, if γ is a right arrow and $B_\gamma^* \cdot \tau' = \tau$, then

$$\begin{aligned}\sum_{\pi'(*)<\pi'(\xi)\le 2d+1} \tau'_{\underline{\xi}} &= \tau_{\underline{\alpha}} - \tau_{\underline{\beta}} + \sum_{\pi(*)<\pi(\xi)\le 2d} \tau'_{\underline{\xi}} \\ &= \sum_{1\le\pi(\xi)<\pi(*)} \tau'_{\underline{\xi}} - \tau_{\underline{\beta}} \\ &= \sum_{1<\pi(\xi)<\pi(*)} \tau'_{\underline{\xi}} < 0,\end{aligned}$$

where α is the winner and β is the loser of γ. The case of a left arrow is analogous.

Thus, the map

$$\widehat{Q}^\gamma \colon \Delta_\gamma \times \Theta_\pi \to \Delta_{\pi'} \times \Theta_\gamma, \quad \widehat{Q}^\gamma(\lambda,\pi,\tau) = (Q(\lambda,\pi), (B_\gamma^*)^{-1} \cdot \tau)$$

is invertible. With this we define an invertible skew-product $\widehat{Q}$ over Q considering all $\widehat{Q}^\gamma$ for every arrow γ. So, we obtain a map from $\bigcup(\Delta_\gamma \times \Theta_\pi)$ (where the union is taken over all $\pi \in \mathfrak{R}$ and all arrows γ starting at π) to $\bigcup(\Delta_{\pi'} \times \Theta_\gamma)$ (where the union is taken over all $\pi' \in \mathfrak{R}$ and all arrows ending at π'). Denote $\widehat{\Delta}_\mathfrak{R} = \bigcup_{\pi\in\mathfrak{R}}(\Delta_\pi \times \Theta_\pi)$.

Let $(e_{\underline{\alpha}})_{\underline{\alpha}\in\mathcal{A}}$ be the canonical basis of $\mathbb{R}^{\mathcal{A}}$. We will consider a measure in $\widehat{\Delta}_\mathfrak{R}$ defined as follows. Let $\{v_1,\dots,v_d\}$ be a basis of $\mathbb{R}^d$. We have a volume form given by $\omega(v_1,\dots,v_d) = \det(v_1,\dots,v_d)$. We want to define a volume form in $\widehat{\Delta}_\mathfrak{R}$ coherent with ω. Given the subspace $\mathcal{S}_\pi$ define the orthogonal vector $v_\pi = \sum_{\pi(x)<\pi(*)} e_{\underline{x}} - \sum_{\pi(x)>\pi(*)} e_{\underline{x}}$. We can view this vector like a linear functional $\psi_\pi \colon \mathbb{R}^d \to \mathbb{R}$ defined by $\psi_\pi(x) = \langle v_\pi, x\rangle$. Now we define a form ω^{v_π} on $\mathcal{S}_\pi$ such that $\omega = \omega^{v_\pi} \wedge \psi_\pi$ (where $\wedge$ denotes the exterior product). We have that ω^{v_π} is a $(d-1)$-form and it is well defined in the orthogonal complement of ψ_π (i.e., $\psi_\pi(\mathbb{R}^d)^\perp = \ker(\psi_\pi) = \mathcal{S}_\pi$). Notice that, given $\pi,\pi' \in \mathfrak{R}$ and a path $\gamma \in \Pi(\mathfrak{R})$ joining π to π', $B_\gamma \cdot v_\pi = v_{\pi'}$ and

$$(B_\gamma^*)^{-1} \cdot \left(\frac{v_\pi}{\langle v_\pi, v_\pi\rangle}\right) - \frac{v_{\pi'}}{\langle v_{\pi'}, v_{\pi'}\rangle} \in \mathcal{S}_{\pi'}.$$

So, the pull-back of $\omega^{v_{\pi'}}$ is equal to ω^{v_π}, i.e.,

$$[(B_\gamma^*)^{-1}]^* \omega^{v_{\pi'}} = \omega^{v_\pi}.$$

Consider the volume form ω^{ν_π} and the corresponding Lebesgue measure Leb_π on $\mathcal{S}_\pi$. So we have a natural volume measure $\hat{m}_\mathfrak{R}$ on $\hat{\Delta}_\mathfrak{R}$ which is a product of a counting measure on $\mathfrak{R}$ and the restrictions of Leb_π on $\mathcal{S}_\pi^+$ and Θ_π.

Let $\phi(\lambda, \pi, \tau) = \|\lambda\| = \sum_{\underline{\alpha} \in \underline{\mathcal{A}}} \lambda_{\underline{\alpha}}$. The subset $\mathcal{V}_\mathfrak{R} \subset \hat{\Delta}_\mathfrak{R}$ of all x such that either

- $\hat{Q}(x)$ is defined and $\phi(\hat{Q}(x)) < 1 \leq \phi(x)$,
- $\hat{Q}(x)$ is not defined and $\phi(x) \geq 1$,
- $\hat{Q}^{-1}(x)$ is not defined and $\phi(x) < 1$.

is a fundamental domain for the action of $\hat{Q}$: each orbit of $\hat{Q}$ intersects $\mathcal{V}_\mathfrak{R}$ exactly once.

Let $\mathcal{V}_\mathfrak{R}^{(1)}$ be the subset of $\mathcal{V}_\mathfrak{R}$ such that $A(\lambda, \pi, \tau) = 1$. Let $m_\mathfrak{R}^{(1)}$ be the restriction of the measure $\hat{m}_\mathfrak{R}$ to the subset $\mathcal{V}_\mathfrak{R}^{(1)}$.

3.3.2. The Veech flow with involution. Using the coordinates introduced above, we define a flow $\mathcal{TV} = (\mathcal{TV}_t)_{t \in \mathbb{R}}$ on $\hat{\Delta}_\mathfrak{R}$ given by $\mathcal{TV}_t(\lambda, \pi, \tau) = (e^t \lambda, \pi, e^{-t} \tau)$. It is clear that $\mathcal{TV}$ commutes with the map $\hat{Q}$. The *Veech flow with involution* is defined by $\mathcal{VF}_t \colon \mathcal{V}_\mathfrak{R} \to \mathcal{V}_\mathfrak{R}$, $\mathcal{VF}_t(x) = \hat{Q}^n(\mathcal{TV}_t(x))$, where n is the unique value such that $\hat{Q}^n(\mathcal{TV}_t(x)) \in \mathcal{V}_\mathfrak{R}$.

Notice that the Veech flow with involution leaves invariant the space of zippered rectangles of area one. So, the restriction $\mathcal{VF}_t \colon \mathcal{V}_\mathfrak{R}^{(1)} \to \mathcal{V}_\mathfrak{R}^{(1)}$ leaves invariant the volume form which, as we will see later, is finite.

4. The distortion estimate

We will introduce a class of measures involving the Lebesgue measure and its forward iterates under the Rauzy induction map with involution.

For $q \in \mathbb{R}_+^{\mathcal{A}}$, let $\Lambda_{\pi,q} = \{\lambda \in \mathcal{S}_\pi^+ : \langle \lambda, q \rangle < 1\}$. Let $\nu_{\pi,q}$ be the measure on the σ-algebra of subsets of $\mathcal{S}_\pi$ which are invariant under multiplication by positive scalars, given by $\nu_{\pi,q}(A) = \mathrm{Leb}_\pi(A \cap \Lambda_{\pi,q})$. If γ is a path starting at π and ending at π' then

$$\nu_{\pi,q}(B_\gamma^* \cdot A) = \mathrm{Leb}_\pi((B_\gamma^* \cdot A) \cap \Lambda_{\pi,q}) = \mathrm{Leb}_{\pi'}(A \cap \Lambda_{\pi', B_\gamma \cdot q}) = \nu_{\pi', B_\gamma \cdot q}(A).$$

We will obtain estimates for $\nu_{\pi,q}(\Delta_\gamma)$.

Let $\mathfrak{R} \subset \mathfrak{S}^0(\mathcal{A})$ be a Rauzy class, $\gamma \in \Pi(\mathfrak{R})$, let π and π' be the start and the end of γ, respectively. We denote $\Lambda_{\gamma,q} \times \{\pi\} = (\Lambda_{\pi,q} \times \{\pi\}) \cap \Delta_\gamma$, so that $B_\gamma^* \cdot \Lambda_{\pi', B_\gamma \cdot q} = \Lambda_{\gamma,q}$.

For $\mathcal{A}' \subset \mathcal{A}$ non-empty and invariant by involution, let $\mathrm{M}_{\mathcal{A}'}(q) = \max_{\alpha \in \mathcal{A}'} q_\alpha$ and $\mathrm{M}(q) = \mathrm{M}_\mathcal{A}(q)$. Consider also $\mathrm{m}_{\mathcal{A}'}(q) = \min_{\alpha \in \mathcal{A}'} q_\alpha$ and $\mathrm{m}(q) = \mathrm{m}_\mathcal{A}(q)$.

If $\Gamma \subset \Pi(\mathfrak{R})$ is a set of paths starting at the same $\pi \in \mathfrak{R}$, we denote $\Lambda_{\Gamma,q} = \bigcup_{\gamma\in\Gamma} \Lambda_{\gamma,q}$. Given $\Gamma \subset \Pi(\mathfrak{R})$ and $\gamma_s \in \Pi(\mathfrak{R})$ we define $\Gamma_{\gamma_s} = \{\gamma \in \Gamma : \gamma_s \text{ is the start of } \gamma\}$.

We say that a vertex is *simple* or *double* depending whether it is labelled by a simple or a double letter, respectively. Notice that $\Lambda_{\pi,q}$ is a convex open polyhedron which vertices are

- the trivial vertex 0;
- the simple vertices $q_{\underline{\alpha}}^{-1} e_{\underline{\alpha}}$, where α is simple;
- the double vertices $(q_{\underline{\alpha}} + q_{\underline{\beta}})^{-1}(e_{\underline{\alpha}} + e_{\underline{\beta}})$, where α, β are double and $\pi(\alpha) < \pi(*) < \pi(\beta)$ or $\pi(\beta) < \pi(*) < \pi(\alpha)$.

A simple vertex $v = q_{\underline{\alpha}}^{-1} e_{\underline{\alpha}}$ is called of type $\underline{\alpha}$ and weight $w(v) = q_{\underline{\alpha}}$, and a double vertex $v = (q_{\underline{\alpha}} + q_{\underline{\beta}})^{-1}(e_{\underline{\alpha}} + e_{\underline{\beta}})$ is called of type $\{\underline{\alpha}, \underline{\beta}\}$ and of weight $w(v) = q_{\underline{\alpha}} + q_{\underline{\beta}}$.

An *elementary subsimplex* of $\Lambda_{\pi,q}$ is an open simplex whose vertices are also vertices of $\Lambda_{\pi,q}$, and one of them is 0. Notice that $\Lambda_{\pi,q}$ can be always written as a union of at most $C_1(d)$ elementary simplices, up to a set of codimension one.

A set of non-trivial vertices of $\Lambda_{\pi,q}$ is contained in the set of vertices of some elementary subsimplex if and only if the vertices are linearly independent. If α is simple then any elementary subsimplex must have a vertex of type $\underline{\alpha}$ and if α is double then any elementary subsimplex must have a vertex of type $\{\underline{\alpha}, \underline{x}\}$ for some $\underline{x} \neq \underline{\alpha}$ with x double. If Λ is an elementary subsimplex with simple vertices of type $\underline{\alpha_i}$ and double vertices of type $\{\underline{\beta_{j_1}}, \underline{\xi_{j_2}}\}$ then $\nu_{\pi,q}(\Lambda) = k(\pi, \Lambda) \prod q_{\underline{\alpha_i}}^{-1} \prod (q_{\underline{\beta_{j_1}}} + q_{\underline{\xi_{j_2}}})^{-1}$ where $k(\pi, \Lambda)$ is a positive integer only depending on v_π and on the types of the double vertices of Λ. In particular there is an integer $C_2(d)$ such that $k(\pi, \Lambda) \leq C_2(d)$.

Let $\gamma \in \Pi(\mathfrak{R})$ be an arrow starting at π and ending at π'. If $\Gamma \subset \Pi(\mathfrak{R})$ then we define

$$P_q(\Gamma \mid \gamma) = \frac{\nu_{\pi', B_\gamma \cdot q}(\Lambda_{\Gamma_\gamma, q})}{\nu_{\pi,q}(\Lambda_{\gamma,q})}$$

and

$$P_q(\gamma \mid \pi) = \frac{\nu_{\pi', B_\gamma \cdot q}(\Lambda_{\gamma,q})}{\nu_{\pi,q}(\Lambda_{\pi,q})}.$$

We have that $P_q(\Gamma \mid \gamma) = P_{B_\gamma \cdot q}(\Gamma_\gamma \mid \pi')$.

We define a partial order in the set of paths as follows. Let $\gamma, \gamma_s \in \Pi(\mathfrak{R})$ be two paths. We say that $\gamma_s \leq \gamma$ if and only if γ_s is the start of γ. A family $\Gamma_s \subset \Pi(\mathfrak{R})$ is called *disjoint* if no two elements are comparable by the partial order defined before. If Γ_s is disjoint and $\Gamma \subset \Pi(\mathfrak{R})$ is a family such that any $\gamma \in \Gamma$ starts by some element $\gamma_s \subset \Gamma_s$, then for every $\pi \in \mathfrak{R}$,

$$P_q(\Gamma \mid \pi) = \sum_{\gamma_s \in \Gamma_s} P_q(\Gamma \mid \gamma_s) P_q(\gamma_s \mid \pi) \leq P_q(\Gamma_s \mid \pi) \sup_{\gamma_s \in \Gamma_s} P_q(\Gamma \mid \gamma_s).$$

Lemma 4.1. *There exists $C_3(d) < 1$ with the following property. Let $q \in \mathbb{R}_+^{\mathcal{A}}$, $\gamma \in \Pi(\mathfrak{R})$ be an arrow starting at π with loser β. If $C \geq 1$ is such that $q_{\underline{\beta}} > C^{-1}M(q)$ then*

$$P_q(\gamma \mid \pi) > C_3(d)C^{-(d-1)}.$$

Proof. Let α be the winner of γ and let π' be the end of γ. Let Λ be an elementary subsimplex of $\Lambda_{\pi,q}$. We are going to show that there exists an elementary subsimplex $\Lambda' \subset B_\gamma(\Lambda)$ of $\Lambda_{\pi',B_\gamma \cdot q}$ such that $\mathrm{Leb}_{\pi'}(\Lambda') \geq C'^{-1}\mathrm{Leb}_\pi(\Lambda)$, which implies the result by decomposition.

We will separate the proof in four cases depending on whether the winner and the loser are simple or double.

Suppose that α and β are simple. Let Λ be an elementary subsimplex of $\Lambda_{\pi,q}$. Then $\Lambda' = B_\gamma \cdot (\Lambda_{\pi,q} \cap \Lambda)$ is an elementary subsimplex of $\Lambda_{\pi',B_\gamma \cdot q}$. The set of vertices of Λ' differs from the set of vertices of Λ just by replacing the vertex $q_{\underline{\beta}}^{-1}e_{\underline{\beta}}$ by $(q_{\underline{\alpha}} + q_{\underline{\beta}})^{-1}e_{\underline{\beta}}$. It follows that $\mathrm{Leb}_{\pi'}(\Lambda')/\mathrm{Leb}_\pi(\Lambda) = q_{\underline{\beta}}/(q_{\underline{\alpha}} + q_{\underline{\beta}}) > 1/(C + 1)$. By considering a decomposition into elementary subsimplices we conclude that $P_q(\gamma|\pi) = q_{\underline{\beta}}/(q_{\underline{\alpha}} + q_{\underline{\beta}}) > 1/(C + 1)$.

Suppose that the winner is simple and the loser is double. Let Λ be an elementary subsimplex of $\Lambda_{\pi,q}$. Then $\Lambda' = B_\gamma \cdot (\Lambda_{\pi,q} \cap \Lambda)$ is an elementary subsimplex of $\Lambda_{B_\gamma \cdot q}$. The set of vertices of Λ' differs from the set of vertices of Λ just by replacing the vertices $(q_{\underline{x}} + q_{\underline{\beta}})^{-1}(e_{\underline{x}} + e_{\underline{\beta}})$ by $(q_{\underline{\alpha}} + q_{\underline{x}} + q_{\underline{\beta}})^{-1}(e_{\underline{x}} + e_{\underline{\beta}})$. It follows that $\mathrm{Leb}_{\pi'}(\Lambda')/\mathrm{Leb}_\pi(\Lambda) = \prod(q_{\underline{x}} + q_{\underline{\beta}})/(q_{\underline{\alpha}} + q_{\underline{x}} + q_{\underline{\beta}}) > (1/(1 + C))^{d-1}$, where the product is over all $\underline{x}$ such that $\bar{\Lambda}$ has a vertex of type $\{\underline{x}, \underline{\beta}\}$. Thus $P_q(\gamma|\pi) > (1/(1 + C))^{d-1}$.

If the winner is double and the loser is simple, let γ' be the other arrow starting at π. Analogous to the previous case,

$$\begin{aligned} P_q(\gamma' \mid \pi) &= \prod(q_{\underline{x}} + q_{\underline{\alpha}})/(q_{\underline{\alpha}} + q_{\underline{x}} + q_{\underline{\beta}}) \\ &= \prod \left(1 - q_{\underline{\beta}}/(q_{\underline{\alpha}} + q_{\underline{x}} + q_{\underline{\beta}})\right) \\ &< (2C/(1 + 2C))^{d-1}, \end{aligned}$$

so $P_q(\gamma|\pi) > (2C/(1 + 2C))^{d-1}$.

Finally, suppose that the winner and the loser are both double. Let Λ be an elementary subsimplex with $\mathrm{Leb}_\pi(\Lambda) \geq \mathrm{Leb}_\pi(\Lambda_{\pi,q})/C_1(d)$. Let Z be the set of vertices of Λ and let $\tilde{Z} \subset Z$ be the set of double vertices of type $\{q_{\underline{x}}, q_{\underline{\beta}}\}$ with $\underline{x} \neq \underline{\alpha}$. Notice that $B_\gamma \cdot (Z \backslash \tilde{Z})$ is a subset of the set of vertices of $\Lambda_{\pi',B_\gamma \cdot q}$. Since $Z \backslash \tilde{Z}$ is linearly independent, $B_\gamma \cdot (Z \backslash \tilde{Z})$ is also. Thus there exists an elementary subsimplex Λ' of $\Lambda_{\pi',B_\gamma \cdot q}$ whose set Z' of vertices contains $B_\gamma \cdot (Z \backslash \tilde{Z})$. Let $\tilde{Z}' = Z' \backslash B_\gamma \cdot (Z \backslash \tilde{Z})$. The weight of a vertice $v \in Z \backslash \tilde{Z}$ is the same weight as the weight

of $B_\gamma \cdot v$. Notice that each vertex of $\widetilde{Z}$ has weight at least $C^{-1}\mathrm{M}(q)$ and each vertex of $\widetilde{Z}'$ has weight at most $2\mathrm{M}(B_\gamma \cdot q) \leq 4\mathrm{M}(q)$. Thus

$$\frac{\mathrm{Leb}_{\pi'}(\Lambda')}{\mathrm{Leb}_\pi(\Lambda)} = \frac{k(\pi',\Lambda')}{k(\pi,\Lambda)} \frac{\prod_{v\in\widetilde{Z}} w(v)}{\prod_{v\in\widetilde{Z}'} w(v)} > C_2(d)^{-1}(4C)^{1-d}.$$

Thus $P_q(\gamma|\pi) > C_1(d)^{-1}C_2(d)^{-1}(4C)^{1-d}$. □

The proof of the recurrence estimates is based on the analysis of the Rauzy renormalization map. The key step involves a control on the measure of sets which present big distortion after some long (Teichmüller) time.

Theorem 4.2. *There exists $C_4(d) > 1$ with the following property. Let $q \in \mathbb{R}^{\mathcal{A}}_+$. Then for every $\pi \in \mathfrak{R}$,*

$$P_q\big(\gamma \in \Pi(\mathfrak{R}),\ \mathrm{M}(B_\gamma \cdot q) > C_4(d)\mathrm{M}(q) \text{ and } \mathrm{m}(B_\gamma \cdot q) < \mathrm{M}(q) \mid \pi\big) < 1 - C_4(d)^{-(d-1)}.$$

Proof. For $1 \leq k \leq d$, let $\mathrm{m}_k(q) = \max \mathrm{m}_{\mathcal{A}'}(q)$ where the maximum is taken over all involution invariant sets $\mathcal{A}' \subset \mathcal{A}$ such that $\#\mathcal{A}' = 2k$. In particular $\mathrm{m}(q) = \mathrm{m}_d(q)$. We will show that for $1 \leq k \leq d$ there exists $D > 1$ such that

$$P_q(\gamma \in \Pi(\mathfrak{R}),\ \mathrm{M}(B_\gamma \cdot q) > D\mathrm{M}(q) \text{ and } \mathrm{m}_k(B_\gamma \cdot q) < \mathrm{M}(q) \mid \pi) < 1 - D^{-(d-1)}. \tag{17}$$

(the case $k = d$ implies the desired statement). The proof is by induction on k. For $k = 1$ it is obvious. Assume that it is proved for some $1 \leq k < d$ with $D = D_0$.

Let Γ be the set of minimal paths γ starting at π with $\mathrm{M}(B_\gamma \cdot q) > D_0\mathrm{M}(q)$. Consider the set Γ_0 of minimal paths γ starting at π satisfying $\mathrm{M}(B_\gamma \cdot q) > D_0\mathrm{M}(q)$ and $\mathrm{m}_k(B_\gamma \cdot q) < \mathrm{M}(q)$. Since $\Gamma_0 \subset \Gamma$, by (17), $P_q(\Gamma \cap \Gamma_0) = P_q(\Gamma_0) < 1 - D_0^{-(d-1)}$. By definition of Γ we have $P_q(\Gamma) = 1$, then $P_q(\Gamma \setminus \Gamma_0) = 1 - P_q(\Gamma \cap \Gamma_0) > D_0^{-(d-1)}$. Moreover, if $\gamma \in \Gamma \setminus \Gamma_0$, we have $m_k(B_\gamma \cdot q) \geq M(q)$. Then there exists $\Gamma_1 \subset \Gamma$ with $P_q(\Gamma_1 \mid \pi) > D_0^{-(d-1)}$ and an involution invariant set $\mathcal{A}' \subset \mathcal{A}$ with $\#\mathcal{A}' = 2k$ such that if $\gamma \in \Gamma_1$ then $\mathrm{m}_{\mathcal{A}'}(B_\gamma \cdot q) \geq \mathrm{M}(q)$.

For $\gamma_s \in \Gamma_1$, choose a path $\gamma = \gamma_s\gamma_e$ with minimal length such that γ ends at a permutation π_e such that either the first or the last element of π_e is an element of $\mathcal{A}\setminus\mathcal{A}'$. Let Γ_2 be the collection of the $\gamma = \gamma_s\gamma_e$ thus obtained. Then $P_q(\Gamma_2 \mid \pi) > D_1^{-(d-1)}$ and $\mathrm{M}(B_\gamma \cdot q) < D_1\mathrm{M}(q)$ for $\gamma \in \Gamma_2$.

Let Γ_3 be the space of minimal paths $\gamma = \gamma_s\gamma_e$ with $\gamma_s \in \Gamma_2$ and $\mathrm{M}(B_\gamma \cdot q) > 2dD_1\mathrm{M}(q)$. Let $\Gamma_4 \subset \Gamma_3$ be the set of all $\gamma = \gamma_s\gamma_e$ where all the arrows of γ_e have as loser an element of $\mathcal{A}'$. For each $\gamma_s \in \Gamma_2$, there exists at most one $\gamma = \gamma_s\gamma_e \in \Gamma_4$, and if $P_q(\Gamma_4 \mid \gamma_s) < \frac{1}{2d}$, it follows that $P_q(\Gamma_3\setminus\Gamma_4 \mid \pi) > \left(1 - \frac{1}{2d}\right) D_1^{-(d-1)}$. It remains to prove that $P_q(\Gamma_4 \mid \gamma_s) < \frac{1}{2d}$.

Let $\gamma = \gamma_s\gamma_e \in \Gamma_4$ such that $\gamma_s \in \Gamma_2$. Let $\pi_e \in \Pi(\mathfrak{K})$ be the end of γ_s and let α and β be the winner and the loser of π_e, respectively. By definition, we have that $\alpha \in \mathcal{A}\backslash\mathcal{A}'$ and $\beta \in \mathcal{A}'$. Besides, all losers of γ_e are in $\mathcal{A}'$.

We claim that α is simple. Suppose this is not the case. Assume, without lost of generality, $\pi_e(\alpha) = 1$ and $\pi_e(\beta) = 2d + 1$. Applying Rauzy algorithm with involution one time we would obtain $\pi'_e(\beta) < \pi'_e(*)$ and to keep the same winner α we just can apply Rauzy algorithm with involution at most $2d - 4$ times. But even if we could apply Rauzy algorithm with involution those number of times, we will have

$$\mathrm{M}(B_\gamma \cdot q) < (2d - 3)D_1\mathrm{M}(q) < 2dD_1\mathrm{M}(q)$$

what contradicts that $\gamma \in \Gamma_3$. Then α is simple as we claim.

We are considering a path $\gamma = \gamma_s\gamma_e \in \Gamma_4$ and a permutation $\pi_e \in \Pi(\mathfrak{K})$ which is the end of γ_s. Suppose $\gamma_s = \gamma_1 \dots \gamma_m$ and $\gamma_e = \gamma_{m+1} \dots \gamma_n$, where $\gamma_i \in \Pi(\mathfrak{K})$ for all $i = 1, \dots, n$ and each γ_i is an arrow joining permutations $\pi_e^{(i-1)}$ and $\pi_e^{(i)}$ for $m + 1 \le i \le n$, where $\pi_e^m = \pi_e$. We have $\gamma_s \in \Gamma_2$, so, to obtain $\mathrm{M}(B_\gamma \cdot q) > 2dD_1\mathrm{M}(q)$ we need $n \ge 2d + 1$. We also have $\mathcal{S}_{\pi_e} = \mathcal{S}_{\pi_e^{(i)}}$ for all $i \in \{m, \dots, n\}$.

Let $\Lambda = \Lambda_{\pi_e, B_{\gamma_s} \cdot q}$ which is a finite union of elementary simplices Λ_j. Let $\beta_0 = \beta$ and β_i be the loser of $\pi_e^{(i)}$ for $i = m, \dots, n$. For each $i \in \{m, \dots, n\}$ if β_i is a simple vertex then all Λ_j has a vertex of type $\underline{\beta_i}$ and if β_i is double then all Λ_j has a vertex of type $\{\underline{\beta_i}, \underline{x}\}$ for some $\underline{x} \notin \{\underline{\alpha}, \underline{\beta_i}\}$. Let $\Lambda_j^{(n)} = B_{\gamma_e}(\Lambda_j)$. Notice that the type of vertices of $\Lambda_j^{(n)}$ coincides with type of vertices of Λ_j. Let Z be the set of vertices of Λ_j and $Z^{(n)}$ be the set of vertices of $\Lambda_j^{(n)}$. Then

$$\frac{\mathrm{Leb}_{\pi_e^{(n)}}(\Lambda_j^{(n)})}{\mathrm{Leb}_{\pi_e}(\Lambda_j)} = \frac{\prod_{v \in Z} w(v)}{\prod_{v \in Z^{(n)}} w(v)}. \tag{18}$$

But $\mathrm{M}(B_{\gamma_s} \cdot q) < D_1\mathrm{M}(q)$ and $\mathrm{M}(B_\gamma \cdot q) > 2dD_1\mathrm{M}(q)$, so there is one term in (18) which is less then $\frac{1}{2d}$. Thus

$$P_q(\Gamma_4 \mid \gamma_s) < \frac{1}{2d}.$$

Let $\gamma = \gamma_s\gamma_e \in \Gamma_3\backslash\Gamma_4$. Let us show that $\mathrm{m}_{k+1}(B_\gamma \cdot q) > \mathrm{M}(q)$, which implies (17) with $k + 1$ in place of k and $D = 2dD_1$. Assume that this is not the case. In this case, the last arrow composing γ_e must have as loser an element of $\mathcal{A}'$. Moreover, no arrow composing γ_e has as winner an element of $\mathcal{A}'$ (otherwise, the loser β of the first such arrow does not belong to $\mathcal{A}'$ and is such that $m_{\mathcal{A}' \cup \{\beta\}}(B_\gamma \cdot q) > \mathrm{M}(q)$). Let $\gamma_e = \gamma_{e,s}\gamma_{e,e}$ where $\gamma_{e,s}$ is maximal such that all losers of $\gamma_{e,s}$ are in $\mathcal{A}'$. Then all losers in $\gamma_{e,s}$ are distinct and $\mathrm{M}(B_{\gamma_s\gamma_{e,s}} \cdot q) < 2D_1\mathrm{M}(q)$. Let $\gamma_{e,e} = \gamma_{e,1} \dots \gamma_{e,l}$ where $\gamma_{e,j} = \gamma_{e,j,s}\gamma_{e,j,e}$ with $\gamma_{e,j,s}$ and $\gamma_{e,j,e}$ non-trivial such that all the losers of

$\gamma_{e,j,s}$ are in $\mathcal{A}\backslash\mathcal{A}'$ and all the losers of $\gamma_{e,j,e}$ are in $\mathcal{A}'$. Let $\gamma_j = \gamma_s\gamma_{e,s}\gamma_{e,1}\dots\gamma_{e,j}$, $0 \le j \le l$. Notice that for each j, $\gamma_{e,j,e}$ has distinct losers, and the same winner $\alpha \in \mathcal{A}\backslash\mathcal{A}'$ which is also the last winner of $\gamma_{e,j,s}$. Let $\beta \in \mathcal{A}\backslash\mathcal{A}'$ be the last loser of $\gamma_{e,j,s}$. Then

$$\mathrm{M}(B_{\gamma_{j+1}}\cdot q) - \mathrm{M}(B_{\gamma_j}\cdot q) \le \mathrm{M}_{\underline{\beta}}(B_{\gamma_{j+1}}\cdot q) - \mathrm{M}_{\underline{\beta}}(B_{\gamma_j}\cdot q)$$

which implies that

$$\begin{aligned}(2d-1)D_1\mathrm{M}(q) &\le \mathrm{M}(B_\gamma\cdot q) - \mathrm{M}(B_{\gamma_s}\cdot q)\\ &\le \sum_{\beta\in\mathcal{A}\backslash\mathcal{A}'} \mathrm{M}_\beta(B_\gamma\cdot q) - \mathrm{M}_\beta(B_{\gamma_s}\cdot q) \le dD_1\mathrm{M}(q)\end{aligned}$$

which is a contradiction. □

5. Recurrence estimates

The goal of this section is to prove proposition 5.2. It provides quantitative decay of the measure of the set of points which take long to enter a given simplex $\Lambda_{\tilde{\gamma}}$ under iteration of the Rauzy induction map with involution.

Lemma 5.1. *For every $\hat{\gamma} \in \Pi(\mathfrak{R})$, there exist $M \ge 0$, $\rho < 1$ such that for every $\pi \in \mathfrak{R}$, $q \in \mathbb{R}_+^{\mathcal{A}}$,*

$$P_q\big(\gamma \text{ can not be written as } \gamma_s\hat{\gamma}\gamma_e \text{ and } \mathrm{M}(B_\gamma\cdot q) > 2^M\mathrm{M}(q) \mid \pi\big) \le \rho.$$

Proof. Fix $M_0 \ge 0$ large and let $M = 2M_0$. Let Γ be the set of all minimal paths γ starting at π which can not be written as $\gamma_s\hat{\gamma}\gamma_e$ and such that $\mathrm{M}(B_\gamma\cdot q) > 2^M\mathrm{M}(q)$. Any path $\gamma \in \Gamma$ can be written as $\gamma = \gamma_1\gamma_2$ where γ_1 is minimal with $\mathrm{M}(B_{\gamma_1}\cdot q) > 2^{M_0}\mathrm{M}(q)$. Let Γ_1 collect the possible γ_1. Then Γ_1 is disjoint, by minimality. Let $\widetilde{\Gamma}_1 \subset \Gamma_1$ be the set of all γ_1 such that $\mathrm{m}(B_{\gamma_1}\cdot q) \ge \mathrm{M}(q)$. By Theorem 4.2, if M_0 is sufficiently large we have

$$P_q(\Gamma_1\backslash\widetilde{\Gamma}_1 \mid \pi) < 1 - C_4(d)^{-(d-1)}.$$

For $\pi_e \in \mathfrak{R}$, let γ_{π_e} be a shortest possible path starting at π_e with $\gamma_{\pi_e} = \gamma_s\hat{\gamma}$. Let π_f be the end of γ_{π_e}. If M_0 is sufficiently large then $\|B_{\gamma_{\pi_e}}\| < \frac{1}{d-1}2^{M_0-1}$. It follows that if $\gamma_1 \in \Gamma_1$ ends at π_e then

$$P_q(\Gamma \mid \gamma_1) \le 1 - P_{B_{\gamma_1}\cdot q}(\gamma_{\pi_e} \mid \pi_e).$$

Let $\Lambda \subset \Lambda_{\pi_e, B_{\gamma_1}\cdot q}$ be an elementary subsimplex with

$$\mathrm{Leb}_{\pi_e}(\Lambda) \ge \frac{\mathrm{Leb}_{\pi_e}(\Lambda_{\pi_e, B_{\gamma_1}\cdot q})}{C_1(d)}.$$

Choose an elementary subsimplex Λ' of $\Lambda_{\pi_f, B_{\gamma_1\gamma\pi_e}\cdot q}$, such that for all $\underline{\alpha} \in \underline{\mathcal{A}}$ there exists a vertice v' of Λ' of type $\underline{\alpha}$ or of type $\{\underline{\alpha}, \underline{\xi}\}$ for some $\underline{\xi} \in \underline{\mathcal{A}}$ and $\mathrm{Leb}_{\pi_f}(\Lambda') \geq \mathrm{Leb}_{\pi_f}(B_{\gamma_{\pi_e}} \cdot \Lambda_{\pi_e, B_{\gamma_1}\cdot q})/C_1(d)$. Let Z and Z' be the set of vertices of Λ and Λ', respectively. If furthermore $\gamma_1 \in \widetilde{\Gamma}_1$ then

$$\frac{\mathrm{Leb}_{\pi_f}(\Lambda')}{\mathrm{Leb}_{\pi_e}(\Lambda)} = \frac{k(\pi_f, \Lambda')\prod_{v\in Z} w(v)}{k(\pi_e, \Lambda)\prod_{v\in Z'} w(v)} \geq \frac{\mathrm{M}(q)^{d-1}}{(2^{2M_0}\mathrm{M}(q))^{d-1}} = 2^{-2(d-1)M_0}.$$

So, $P_{B_{\gamma_1}\cdot q}(\gamma_{\pi_e} \mid \pi_e) \geq 2^{-2(d-1)M_0}$ and $P_q(\Gamma \mid \pi) \leq 1 - C_4(d)^{-(d-1)}2^{-2(d-1)M_0}$. □

Proposition 5.2. *For every $\hat{\gamma} \in \Pi(\mathfrak{R})$, there exist $\delta > 0$, $C > 0$ such that for every $\pi \in \mathfrak{R}$, $q \in \mathbb{R}_+^{\mathcal{A}}$ and for every $T > 1$*

$$P_q\big(\gamma \text{ can not be written as } \gamma_s\hat{\gamma}\gamma_e \text{ and } \mathrm{M}(B_\gamma \cdot q) > T\mathrm{M}(q) \mid \pi\big) \leq CT^{-\delta}.$$

Proof. Let M and ρ be as in the previous lemma. Let k be maximal with $T \geq 2^{k(M+1)}$. Let Γ be the set of minimal paths γ such that γ is not of the form $\gamma_s\hat{\gamma}\gamma_e$ and $\mathrm{M}(B_\gamma \cdot q) > 2^{k(M+1)}\mathrm{M}(q)$. Any path $\gamma \in \Gamma$ can be written as $\gamma_1 \dots \gamma_k$ where $\gamma_{(i)} = \gamma_1 \dots \gamma_i$ is minimal with $\mathrm{M}(B_{\gamma_{(i)}} \cdot q) > 2^{i(M+1)}\mathrm{M}(q)$. Let $\Gamma_{(i)}$ collect the $\gamma_{(i)}$. Then the $\Gamma_{(i)}$ are disjoint. Moreover, by Lemma 5.1, for all $\gamma_{(i)} \in \Gamma_{(i)}$,

$$P_q(\Gamma_{(i+1)} \mid \gamma_{(i)}) \leq \rho.$$

This implies that $P_q(\Gamma \mid \pi) \leq \rho^k$. The result follows. □

Remark 5.3. Notice that in the case of [AGY], they obtain a better recurrence estimate. In fact, they obtain $T^{-(\delta-1)}$ instead $T^{-\delta}$. But our estimate will be enough.

6. Construction of an excellent hyperbolic semi-flow

6.1. The Veech flow with involution as a suspension over the Rauzy renormalization. Let $\widehat{\Upsilon}_{\mathfrak{R}}$ be the subset of $\mho_{\mathfrak{R}}$ of all (λ, π, τ) with $\phi(\lambda, \pi, \tau) = \|\lambda\| = \sum_{\underline{\alpha}\in\underline{\mathcal{A}}} \lambda_{\underline{\alpha}} = 1$. We denote by $\widehat{\Upsilon}_\pi$ the connected components of $\widehat{\Upsilon}_{\mathfrak{R}}$. Consider $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)} = \mho_{\mathfrak{R}}^{(1)} \cap \widehat{\Upsilon}_{\mathfrak{R}}$ and $\widehat{\Upsilon}_\pi^{(1)} = \mho_{\mathfrak{R}}^{(1)} \cap \widehat{\Upsilon}_\pi$. Let $\hat{m}_{\mathfrak{R}}^{(1)}$ be the induced Lebesgue measure to $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)}$.

We have that $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)}$ is transverse to the Veech flow with involution on $\mho_{\mathfrak{R}}^{(1)}$ which is given by a certain iterate of the Rauzy induction with involution, after applying the flow $\mathcal{T}\mathcal{V}_t$. We are interested in the first return map $\widehat{R}$ to this section. The domain of $\widehat{R}$ is the intersection of $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)}$ with the domain of definition of $\widehat{Q}$, and we have

$$\widehat{R}(\lambda, \pi, \tau) = (e^r\lambda', \pi', e^{-r}\tau'),$$

where $(\lambda', \pi', \tau') = \widehat{Q}(\lambda, \pi, \tau)$ and

$$r = r(\lambda, \pi) = -\log \|\lambda'\| = -\log \phi(\widehat{Q}(\lambda, \pi, \tau))$$

is the first return time. Notice that the map $\widehat{R}$ is a skew-product $\widehat{R}(\lambda, \pi, \tau) = (R(\lambda, \pi), e^{-r}\tau')$ over the non-invertible map R defined by $R(\lambda, \pi) = (e^r\lambda', \pi)$. The map $\widehat{R}$ is called the *Rauzy renormalization map with involution* and it preserves the measure $\widehat{m}^{(1)}_{\mathfrak{R}}$. The renormalization map $\widehat{R}$ is an "invertible extension" of the map R.

We can see the Veech flow with involution as a suspension over the renormalization map $\widehat{R}$. In this suspension model, we lose the control of the orbits which do not return to $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$. However, this does not cause any problem to our considerations because the set of such orbits has zero Lebesgue measure.

6.2. Precompact section. The suspension model for the Veech flow with involution presented above is obtained over a discrete transformation $\widehat{R}$ which is not sufficiently hyperbolic. In general, $\widehat{R}$ can not be expected to be uniformly hyperbolic, in fact, it does not even have appropriate distortion properties. This is related to the fact that the domain is not compact. The approach taken in [AGY] and other recent works as [AF] and [AV] is to introduce a class of suitably small (precompact in $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$) sections, and to prove that the corresponding return maps have good distortion properties.

So, we will choose a specific precompact section which is the intersection of $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$ with (finite unions of) sets of the form $\Delta_\gamma \times \Theta_{\gamma'}$. Let γ be a path starting at π_s and ending at π_e. Precompactness in the λ direction is equivalent to having $B^*_\gamma \cdot (\overline{\mathcal{S}}^+_{\pi_e} \setminus \{0\}) \subset \mathcal{S}^+_{\pi_s}$, which is a necessary condition if γ is a positive path. To take care of both the λ and the τ direction, we have already introduced the notion of strongly positive.

Let $\Pi(\pi) \subset \Pi(\mathfrak{R})$ be the set of paths starting and ending at the same $\pi \in \mathfrak{R}$. Let $\pi \in \mathfrak{R}$ and let $\gamma_* \in \Pi(\pi)$ be a strongly positive path. Assume further that if $\gamma_* = \gamma_s\gamma = \gamma\gamma_e$ then either $\gamma = \gamma_*$ or γ is trivial. We will say that γ_* is *neat*. If for example γ_* ends by a left arrow and starts by a sufficiently long (at least half the length of γ_*) sequence of right arrows then the last condition of being neat is automatically satisfied.

Let $\widehat{\Xi} = \widehat{\Upsilon}^{(1)}_{\mathfrak{R}} \cap (\Delta_{\gamma_*} \times \Theta_{\gamma_*})$ and let $\Xi = \Upsilon^0_{\mathfrak{R}} \cap \Delta_{\gamma_*}$. We will study the first return map $T_{\widehat{\Xi}}$ to the section $\widehat{\Xi}$ under the Veech flow with involution. Notice that the connected components of its domain are given by $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}} \cap (\Delta_{\gamma\gamma^*} \times \Theta_{\gamma_*})$, where γ is either γ_*, or a minimal path of the form $\gamma_*\gamma_0\gamma_*$ not beginning by $\gamma_*\gamma_*$. The restriction of $T_{\widehat{\Xi}}$ to each connected component of its domain has the expression

$$T_{\widehat{\Xi}}(\lambda, \pi, \tau) = \left(\frac{(B^*_\gamma)^{-1} \cdot \lambda}{\|(B^*_\gamma)^{-1} \cdot \lambda\|}, \pi, \|(B^*_\gamma)^{-1} \cdot \lambda\| (B^*_\gamma)^{-1} \cdot \tau\right),$$

and the return time function is given by

$$r_{\widehat{\Xi}}(\lambda, \pi, \tau) = r_{\Xi}(\lambda, \pi) = -\log \|(B_\gamma^*)^{-1} \cdot \lambda\|.$$

The map $T_{\widehat{\Xi}}(\lambda, \pi, \tau) = (\lambda', \pi, \tau')$ is a skew-product over a non-invertible transformation $T_{\Xi}(\lambda, \pi) = (\lambda', \pi)$.

Analogously to the case of the renormalization map, the Veech flow with involution can be seen as a suspension over $T_{\widehat{\Xi}}$, with roof function $r_{\widehat{\Xi}}$. But considering this suspension model, we lose the control of many more orbits which do not come back to $\widehat{\Xi}$. Still, due to ergodicity of the Veech flow with involution, almost every orbit is captured by the suspension model.

6.3. Hyperbolic properties. The reason to choose the section $\widehat{\Xi}$ is because the transformation $T_{\widehat{\Xi}}$ has better hyperbolic properties than transformations considering larger sections and we can also describe easily the connected components of its domain.

Lemma 6.1. *$T_{\widehat{\Xi}}$ is a hyperbolic skew-product over T_{Ξ}.*

Recalling the Definition 2.5, we observe that associated to a hyperbolic skew-product we have: a probability measure $\hat{\nu}$ (which we chose as the normalized restriction of $\widehat{m}_{\mathfrak{R}}^{(1)}$ to $\widehat{\Xi}$) and a Finsler metric $\|\cdot\|_{\widehat{\Xi}}$ (which we will choose in order to obtain the hyperbolic properties we want from $T_{\widehat{\Xi}}$). At first we will introduce a complete Finsler metric on $\widehat{\Upsilon}_{\pi}^{(1)}$, and then we will consider its restriction denoting it by $\|\cdot\|_{\widehat{\Xi}}$. Since γ_* is strongly positive, the section $\widehat{\Xi}$ is a precompact open subset of $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)}$, therefore $\widehat{\Xi}$ will have bounded diameter with respect to such metric.

6.4. Hilbert metric. Now, we will introduce the Hilbert projective metric and state some of its properties which we will use after. This notion can be defined for a general convex cone C in any vector space, but in our case we only need $C = \mathbb{R}_+^{\mathcal{A}}$.

We call the Hilbert pseudo-metric on $\mathbb{R}_+^2$ the function $\mathrm{dist}_{\mathbb{R}_+^2}$ defined by

$$\mathrm{dist}_{\mathbb{R}_+^2}(x, y) = \log \max_{1 \le i,j \le 2} \frac{x_i y_j}{x_j y_i}, \text{ for each } x, y \in \mathbb{R}_+^2.$$

Suppose we are given a linear operator $B \in \mathrm{GL}(2, \mathbb{R})$ such that $B \cdot \mathbb{R}_+^2 \subset \mathbb{R}_+^2$, or equivalently, such that all coefficients of the matrix B are non-negative. Then we have $\mathrm{dist}_{\mathbb{R}_+^2}(B \cdot x, B \cdot y) \le \mathrm{dist}_{\mathbb{R}_+^2}(x, y)$ for all $x, y \in \mathbb{R}_+^2$, which means that B contracts weakly the Hilbert pseudo-metric. In particular, the Hilbert pseudo-metric is invariant under linear isomorphisms of $\mathbb{R}_+^2$.

In general, for an open convex cone $C \subset \mathbb{R}^{\mathcal{A}} \setminus \{0\}$ whose closure does not contain any one-dimensional subspace of $\mathbb{R}^{\mathcal{A}}$, we define the Hilbert pseudo-metric on C by

$\mathrm{dist}_C(x,y) = 0$ if x and y are collinear and $\mathrm{dist}_C(x,y) = \mathrm{dist}_{\mathbb{R}^2_+}(\psi(x),\psi(y))$ otherwise, where ψ is any isomorphism between $\mathbb{R}^2_+$ and the intersection of C with the subspace generated by x and y (this isomorphism exist since x and y are not collinear). If $C = \mathbb{R}^{\mathcal{A}}_+$ then we define $\mathrm{dist}_C(x,y) = \max_{\underline{\alpha},\underline{\beta}\in\underline{\mathcal{A}}} \log \frac{x_{\underline{\alpha}} y_{\underline{\beta}}}{x_{\underline{\beta}} y_{\underline{\alpha}}}$.

Given two convex cones C and C' such that $C' \subset C$ then $\mathrm{dist}_C(x,y) \leq \mathrm{dist}_{C'}(x,y)$, i.e., the inclusion map $C' \to C$ is a weak contraction of the respective Hilbert pseudo-metrics. But if the diameter of C' with respect to dist_C is bounded by some M then we have an uniform contraction by some constant $\delta = \delta(M) < 1$, i.e., $\mathrm{dist}_C(x,y) \leq \delta\, \mathrm{dist}_{C'}(x,y)$.

It is clear that $\mathrm{dist}_C(x,y) = 0$ if and only if there exists $t > 0$ such that $y = tx$. If we restrict the Hilbert pseudo-metric on a convex cone C to the space of rays $\{tx : t \in \mathbb{R}_+\} \subset C$ we have the Hilbert metric, which is a complete Finsler metric.

6.5. Uniform expansion and contraction. Recall that $\widehat{\Upsilon}^{(1)}_\pi$ is contained in $\Delta_\pi \times \Theta_\pi$, which is a product of two convex cones. In $\Delta_\pi \times \Theta_\pi$, we have the product Hilbert pseudo-metric

$$\mathrm{dist}((\lambda,\pi,\tau),(\lambda',\pi,\tau')) = \mathrm{dist}_{\Delta_\pi}((\lambda,\pi),(\lambda',\pi)) + \mathrm{dist}_{\Theta_\pi}(\tau,\tau').$$

Each product of rays $\{(a\lambda,\pi,b\tau) : a,b \in \mathbb{R}_+\} \subset \Delta_\pi \times \Theta_\pi$ intersects transversely $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$ in a unique point. It follows that the product Hilbert pseudo-metric induces a metric dist on $\widehat{\Upsilon}^{(1)}_\pi$. It is a complete Finsler metric.

Lemma 6.2. *Given $\pi \in \mathfrak{R}$, let $g^\pi : \mathcal{S}^+_\pi \to \mathbb{R}$ be a functional defined by $g^\pi(\lambda) = \sum_{\underline{\beta}} g^\pi_{\underline{\beta}} \lambda_{\underline{\beta}}$, where $g^\pi_{\underline{\beta}} \geq 0$ for all $\underline{\beta} \in \underline{\mathcal{A}}$. Then $\log g^\pi(\lambda)$ is 1-Lipschitz relative to the Hilbert metric.*

Proof. Given $(\lambda,\pi),(\lambda',\pi) \in \Delta^1_\pi$, we have

$$\frac{g^\pi(\lambda)}{g^\pi(\lambda')} = \frac{\sum_{\underline{\beta}} g^\pi_{\underline{\beta}} \lambda_{\underline{\beta}}}{\sum_{\underline{\beta}} g^\pi_{\underline{\beta}} \lambda'_{\underline{\beta}}} \leq \sup_{\underline{\beta}} \frac{\lambda_{\underline{\beta}}}{\lambda'_{\underline{\beta}}} \leq e^{\mathrm{dist}_{\Delta_\pi}((\lambda,\pi),(\lambda',\pi))}.$$

Thus $\log g^\pi(\lambda)$ is 1-Lipschitz with respect to $\mathrm{dist}_{\Delta_\pi}$. □

Proof of Lemma 6.1. Let us first show that T_Ξ is a uniformly expanding Markov map (the underlying Finsler metric being the restriction of $\mathrm{dist}_{\Delta_\pi}$, and the underlying measure Leb being the induced Lebesgue measure). It is clear that Ξ is a John domain.

Condition (1) of Definition 2.2 is easily verified, except for the definite contraction of inverse branches. To check this property, we notice that an inverse branch can be written as $h(\lambda,\pi) = \left(\frac{B^*_\gamma \cdot \lambda}{\|B^*_\gamma \cdot \lambda\|}, \pi\right)$. Since γ_* is neat, we can write $B^*_\gamma = B^*_{\gamma_*} B^*_{\gamma_0}$ for

some γ_0. Thus h can be written as (the restriction of) the composition of two maps $\Delta^1_\pi \to \Delta^1_\pi$, $h = h_* \circ h_0$, where h_0 is weakly contracting and h_* is uniformly contracting by precompactness of Ξ in Δ_π (which is a consequence of strong positivity of γ_*).

To check condition (2) of Definition 2.2, let $h(\lambda, \pi)$ be an inverse branch of T_Ξ.

Let $V = \{v \in \mathcal{S}_\pi : \sum v_{\underline{\alpha}} = 0\}$ be the hyperplane tangent to Δ^1_π at a point $(\lambda, \pi) \in \Delta^1_\pi$. Since the coordinate π is fixed by h we can dismiss it. Thus the simplified expression of h is $h(\lambda) = \frac{B^*_\gamma \cdot \lambda}{\|B^*_\gamma \cdot \lambda\|}$. We will denote $\phi(B^*_\gamma \cdot \lambda) = \|B^*_\gamma \cdot \lambda\| = \sum_{\underline{\alpha},\underline{\beta}} (B^*_\gamma)_{\underline{\alpha},\underline{\beta}} \lambda_{\underline{\beta}}$, where $(B^*_\gamma)_{\underline{\alpha},\underline{\beta}}$ is the coefficient of B^*_γ in the line $\underline{\alpha}$ and the column $\underline{\beta}$. So,

$$Dh(\lambda) \cdot v = \frac{B^*_\gamma \cdot v}{\phi(B^*_\gamma \cdot \lambda)} - \frac{B^*_\gamma \cdot \lambda}{\phi(B^*_\gamma \cdot \lambda)} \frac{\sum_{\underline{\alpha}} (B^*_\gamma \cdot v)_{\underline{\alpha}}}{\phi(B^*_\gamma \cdot \lambda)}.$$

So $Dh(\lambda) = P_\lambda \circ \phi(B^*_\gamma \cdot \lambda)^{-1} \circ B^*_\gamma$, where $B^*_\gamma \colon V \to B^*_\gamma \cdot V$, $\phi(B^*_\gamma \cdot \lambda)^{-1}$ is the division by the scalar $\phi(B^*_\gamma \cdot \lambda)$ on $B^*_\gamma \cdot V$ and $P_\lambda \colon B^*_\gamma \cdot V \to V$ is the projection on V along the direction $B^*_\gamma \cdot \lambda$. The Jacobian of h at (λ, π) is $J \circ h(\lambda) = \det Dh(\lambda)$, so,

$$\log J \circ h = \log \det P_\lambda - (d-2) \log \det \phi(B^*_\gamma \cdot \lambda) + \log \det B^*_\gamma$$

We want to prove that $\log J \circ h$ is Lipschitz relative to the Hilbert metric. We have that $\log \det B^*_\gamma$ is constant and, by Lemma 6.2, $\log \det \phi(B^*_\gamma \cdot \lambda)$ is 1-Lipschitz. Now we have to verify what happens with $\log \det P_\lambda$. We have that

$$\det P_\lambda = \frac{\langle n_1, B^*_\gamma \cdot \lambda \rangle}{\langle n_0, B^*_\gamma \cdot \lambda \rangle}$$

where n_0 and n_1 are unit vectors in $\mathcal{S}_\pi$ orthogonal to the hyperplanes V and $B^*_\gamma \cdot V$. Indeed, the vector n_0 and the vector $B_\gamma \cdot n_1$ are collinear with the orthogonal projection of $(1, \ldots, 1)$ on $\mathcal{S}_\pi$. Note that n_0 has no negative coefficients, so neither $B_\gamma \cdot n_0$ and $B_\gamma \cdot n_1$ have. Once again by Lemma 6.2, we have that each $\log(\langle n_i, B^*_\gamma \cdot \lambda \rangle)$ is 1-Lipschitz. Therefore, $\log J \circ h$ is d-Lipschitz with respect to $\mathrm{dist}_{\Delta_\pi}$.

To see that $T_{\widehat{\Xi}}$ is a hyperbolic skew-product over T_Ξ, one checks the conditions (1-4) of Definition 2.5. Condition (1) is obvious, and condition (4) follows from precompactness of $\widehat{\Xi}$ in $\Delta_\pi \times \Theta_\pi$ as before. Since $T_{\widehat{\Xi}}$ is a first return map, the restriction of $\widehat{m}^{(1)}_{\mathfrak{R}}$ to $\widehat{\Xi}$ is $T_{\widehat{\Xi}}$-invariant. Its normalization is the probability measure $\hat{\nu}$ of condition (2). In order to check condition (3), it is convenient to trivialize $\widehat{\Xi}$ to a product (via the natural diffeomorphism $\widehat{\Xi} \to \Xi \times \mathbb{P}\Theta_{\gamma_*}$, where $\mathbb{P}V$ denotes the projective space of V). Since $\hat{\nu}$ has a smooth density with respect to the product of the Lebesgue measure on the factors, condition (3) follows by the Leibniz rule. □

Our results give the finiteness of the measure and the integrability of the cocycle.

Proposition 6.3. *The space $\mathcal{V}^{(1)}_{\mathfrak{R}}$ has finite volume.*

Proof. Consider the section $\widehat{\Xi}$. Notice that this section has positive measure and almost every orbit return to $\widehat{\Xi}$. We have already mention the probability measure $\hat{\nu}$ on $\widehat{\Xi}$, which is the normalized restriction of $\hat{m}^{(1)}_{\mathfrak{R}}$ to $\widehat{\Xi}$.

We want to compute $\int_{\widehat{\Xi}} \log \|(B^*_\gamma)^{-1} \cdot x\| \, d\hat{\nu}$.

A connected component of the domain of $T_{\widehat{\Xi}}$ which intersects the set $\{x \in \widehat{\Xi} : r_{\widehat{\Xi}}(x) > \log T\}$ is of the form $(\Delta_\gamma \times \Theta_{\gamma_*}) \cap \widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$ where γ can not be written as $\gamma_s \hat{\gamma} \gamma_e$ with $\hat{\gamma} = \gamma_* \gamma_* \gamma_* \gamma_*$, $\mathrm{M}(B_\gamma \cdot q_0) \geq D^{-1} T$, where $q_0 = (1, \ldots, 1)$, and $D = D(\gamma_*)$ is some constant.

The projection of $\hat{\nu}|_{(\Delta_\gamma \times \Theta_{\gamma_*}) \cap \widehat{\Upsilon}^{(1)}_{\mathfrak{R}}}$ on $\Upsilon^{(1)}_{\mathfrak{R}}$ is absolutely continuous with a bounded density, so

$$\begin{aligned} &\hat{\nu}\{x \in (\Delta_\gamma \times \Theta_{\gamma_*}) \cap \widehat{\Upsilon}^{(1)}_{\mathfrak{R}} : r_{\widehat{\Xi}}(x) > \log T\} \\ &\quad \leq P_{q_0}\big(\gamma \text{ can not be written as } \gamma_s \hat{\gamma} \gamma_e \text{ and } \mathrm{M}(B_\gamma \cdot q_0) \geq D^{-1} T \mid \pi\big) \end{aligned}$$

and the result follows by Proposition 5.2. □

6.6. Properties of the roof function. Recall $H(\pi) = \Omega(\pi) \cdot \mathcal{S}_\pi$. As we have observed, given a path $\gamma \in \Pi(\pi)$, $H(\pi)$ is invariant under the map B_γ. By Lemma 3.6, if $\tau \in \Theta_\pi$ then $-\Omega(\pi) \cdot \tau \in \mathbb{R}^{\underline{\mathcal{A}}}_+$ and by Corollary 3.13, Θ_π is a non-empty set, so $H(\pi) \cap \mathbb{R}^{\underline{\mathcal{A}}}_+ \neq \emptyset$.

Lemma 6.4. *Let π be an irreducible permutation and $\gamma \in \Pi(\mathfrak{R})$. The subspace $H(\pi)$ has dimension greater than one.*

Proof. Let A be a minimal double letter in the sense that A is a left double letter and there is no double letter Z such that $\pi(Z) < \pi(A)$ or $\pi(i(Z)) < \pi(A)$. Let B be a maximal double letter in the sense that B is a right double letter and there is no double letter Z such that $\pi(B) < \pi(Z)$ or $\pi(B) < \pi(i(Z))$.

We have

$$\mathcal{S}_\pi = \left\{\lambda \in \mathbb{R}^{\underline{\mathcal{A}}} : \lambda_{\underline{A}} = \lambda_{\underline{B}} + \sum \epsilon_{\underline{\alpha}} \lambda_{\underline{\alpha}}\right\}$$

where $\epsilon_{\underline{\alpha}} = 0$ if α is simple, $\epsilon_{\underline{\alpha}} = -1$ if α is left double letter and $\epsilon_{\underline{\alpha}} = 1$ if α is right double letter.

Given $\{e_{\underline{\alpha}}\}_{\underline{\alpha} \in \underline{\mathcal{A}}}$ the canonical basis of $\mathbb{R}^{\underline{\mathcal{A}}}$, we consider the basis $\{\tilde{e}_{\underline{\alpha}}\}_{\underline{\alpha} \in \underline{\mathcal{A}} \setminus \{A\}}$ of $\mathcal{S}_\pi$ obtained as follows: $\tilde{e}_{\underline{\alpha}} = e_{\underline{\alpha}}$ if α is a simple letter, $\tilde{e}_{\underline{\alpha}} = e_{\underline{\alpha}} + e_{\underline{A}}$ if α is a right double letter and $\tilde{e}_{\underline{\alpha}} = e_{\underline{\alpha}} - e_{\underline{A}}$ if α is a left double letter. We define the induced matrix $\Omega(\pi)|_{\mathcal{S}_\pi}$ as the matrix with $d - 1$ columns equal to $\Omega(\pi) \cdot \tilde{e}_{\underline{\alpha}}$, with $\underline{\alpha} \in \underline{\mathcal{A}} \setminus \{A\}$.

Case I: Let C be a simple letter. Denote $x_C = \Omega(\pi)_{AC} \in \{0,1,2\}$ and $y_C = \Omega(\pi)_{CB} \in \{0,1,2\}$. Thus the matrix $\Omega(\pi)$ has a submatrix of the form

$$\begin{pmatrix} 0 & 2 & x_C \\ -2 & 0 & -y_C \\ -x_C & y_C & 0 \end{pmatrix}.$$

So, the induced matrix $\Omega(\pi)|_{\mathcal{S}_\pi}$ has a submatrix of the form

$$\begin{pmatrix} 2 & x_C \\ -2 & -y_C \\ -x_C + y_C & 0 \end{pmatrix}.$$

In this case the induced matrix $\Omega(\pi)|_{\mathcal{S}_\pi}$ has rank 2, except if $x_C = y_C$.

Case II: Let D be a left double letter (the other case is analogous). Denote $z_D = \Omega(\pi)_{AD} \in \{-2,-1,0,1,2\}$. Notice that $\Omega(\pi)_{DB} = 2$. In this case, the matrix $\Omega(\pi)$ has a submatrix of the form

$$\begin{pmatrix} 0 & 2 & z_D \\ -2 & 0 & -2 \\ -z_D & 2 & 0 \end{pmatrix},$$

therefore, the induced matrix $\Omega(\pi)|_{\mathcal{S}_\pi}$ has a submatrix of the form

$$\begin{pmatrix} 2 & 2+z_D \\ -2 & -2 \\ 2-z_D & 2 \end{pmatrix}.$$

In this case the induced matrix $\Omega(\pi)|_{\mathcal{S}_\pi}$ has rank 2, except if $z_D = 0$.

Suppose that if C is a simple letter of the permutation π then $x_C = y_C$ and if D is a double letter of the permutation π then $z_D = 0$. Since π is irreducible, $x_C = y_C = 1$. Thus π has the form

$$A \cdots i(A) * i(B) \cdots B.$$

If we apply the right or the left operation we obtain a reducible permutation.

So, there exists a simple letter C such that $x_C \neq y_C$ or there exists a double letter D such that $z_D \neq 0$. □

Recall that $v_\pi = \sum_{\pi(x)<\pi(*)} e_{\underline{x}} - \sum_{\pi(x)>\pi(*)} e_{\underline{x}}$ is the orthogonal vector to $\mathcal{S}_\pi$.

Lemma 6.5. *Let π be an irreducible permutation and $\gamma \in \Pi(\Re)$. If $v_\pi \in H(\pi)$, then the subspace $H(\pi)$ has dimension greater than two.*

Proof. Let A and B be the leftmost and the rightmost letters of π.

Since $H(\pi)$ and v_π are invariant under B_γ, we can suppose that A is simple.

If B is simple, then π has the form

$$A \cdot \cdot i(B) \cdot \cdot * \cdot i(A) \cdot \cdot B.$$

We have that $\Omega(\pi)\cdot e_A$, $\Omega(\pi)\cdot e_B$ and v_π are linearly independent, since $(v_\pi)_A = (v_\pi)_B = 0$.

If $\pi(i(B)) < \pi(i(A))$ and B is double, we take a left double letter C, i.e., π has the form

$$A \cdot C \cdot \cdot i(C) \cdot * \cdot i(B) \cdot i(A) \cdot \cdot B.$$

And in the case that $\pi(i(B)) > \pi(i(A))$, π has the form

$$A \cdot C \cdot \cdot i(C) \cdot * \cdot \cdot i(A) \cdot i(B) \cdot B.$$

In these last two cases, we have $\Omega(\pi)\cdot e_A$, $\Omega(\pi)\cdot(e_B + e_C)$ and v_π are linearly independent, since $(v_\pi)_A = 0$ and $(v_\pi)_B = -(v_\pi)_C = 1$. □

Lemma 6.6. *The roof function r_Ξ is good (in the sense of Definition 2.3).*

Proof. Let $\Gamma \subset \Pi(\mathfrak{R})$ be the set of all γ such that γ is either γ_*, or a minimal path of the form $\gamma_*\gamma_0\gamma_*$ not beginning by $\gamma_*\gamma_*$. Notice that Γ consists of positive paths.

The set $\mathcal{H}$ of inverse branches h of T_Ξ is in bijection with Γ, since each inverse branch is of the form $h(\lambda,\pi) = \left(\frac{B^*_{\gamma_h}\cdot\lambda}{\|B^*_{\gamma_h}\cdot\lambda\|}, \pi\right)$ for some $\gamma_h \in \Gamma$.

Let $h \in \mathcal{H}$. Then $r_\Xi(h(\lambda,\pi)) = \log\|B^*_{\gamma_h}\cdot\lambda\|$. Since γ_h is positive, $r_\Xi \geq \log 2$, which implies condition (1). By Lemma 6.2, $r_\Xi(h(\lambda,\pi))$ is 1-Lipschitz with respect to $\mathrm{dist}_{\Delta_\pi}$, so (2) follows.

Let us check condition (3). We identify the tangent space to Ξ at a point $(\lambda,\pi)\in\Xi$ with $V = \{\lambda \in \mathcal{S}_\pi : \sum \lambda_\alpha = 0\}$. Assume that we can write $r_\Xi = \psi + \phi\circ T_\Xi - \phi$ with ϕ C^1 and ψ locally constant. Write $r^{(n)}(\lambda,\pi) = \sum_{j=0}^{n-1} r_\Xi(T^j_\Xi(\lambda,\pi))$. Then $D(r^{(n)}\circ h^n) = D\phi - D(\phi\circ h^n)$, which can be rewritten as

$$\frac{\|(B^*_{\gamma_h})^n\cdot v\|}{\|(B^*_{\gamma_h})^n\cdot\lambda\|} = D\phi(\lambda,\pi)\cdot v - D(\phi\circ h^n)(\lambda,\pi)\cdot v, \quad (\lambda,\pi)\in\Xi,\ v\in V. \tag{19}$$

If we define

$$w_{n,h} = \frac{B^n_{\gamma_h}\cdot(1,\dots,1)}{\langle\lambda, B^n_{\gamma_h}\cdot(1,\dots,1)\rangle},$$

we replace (19) by

$$\langle v, w_{n,h}\rangle = D\phi(\lambda,\pi)\cdot v - D(\phi\circ h^n)(\lambda,\pi)\cdot v, \quad (\lambda,\pi)\in\Xi,\ v\in V.$$

We have that $\langle \lambda, w_{n,h} \rangle = 1$ for all $\lambda \in \mathcal{S}_\pi$ and $w_{n,h}$ are vectors with all coordinates positive. By the Perron–Frobenius Theorem $w_{n,h}$ converges to some w_h collinear with the unique positive eigenvector of B_{γ_h} (which also corresponds to the largest eigenvalue). And $w_h = w_{0,h} + t_h v_\pi$, where $w_{0,h} \in \mathcal{S}_\pi \backslash \{0\}$ is the orthogonal projection of w_h in $\mathcal{S}_\pi$ and v_π is the orthogonal vector of $\mathcal{S}_\pi$, which is invariant under B_{γ_h} for all γ_h.

Since $Dh^n \to 0$, we conclude that

$$\langle v, w_h \rangle = D\phi(\lambda, \pi) \cdot v, \quad (\lambda, \pi) \in \Xi,\ v \in V.$$

Since $\langle \lambda, w_{0,h} \rangle = 1$, we have $w_{0,h} = w_0$. Thus $w_h = w_0 + t_h v_\pi$ where $w_0 \in \mathcal{S}_\pi \backslash \{0\}$.

Recalling that $H(\pi)$ is invariant under B_{γ_h}, and intersects $\mathbb{R}_+^{\mathcal{A}}$, it follows that $w_h \in H(\pi)$ and $\mathbb{R}_+ w_{n,h}$ is converging to $\mathbb{R}_- (\Omega(\pi) \cdot \tau)$.

Let $W = \mathbb{R} w_0 \oplus \mathbb{R} v_\pi$. We have that $W \cap H(\pi) \neq \emptyset$ is closed and invariant by B_{γ_h}. By the previous two lemmas, there exists $\tau \in \Theta_\pi$ such that $\Omega(\pi) \cdot \tau \notin W$. But, given such τ, we can construct paths γ_n such that $B_{\gamma_n} \cdot \Omega(\pi) \cdot \Theta_\pi$ converges to $\mathbb{R}_- (\Omega(\pi) \cdot \tau)$ as follows. We have already observed that $\widehat{Q}^{-1}$ is recurrent, so given $(\lambda, \pi, \tau) \in \widehat{\Xi}$, we apply $\widehat{Q}^{-1}$ until obtain $(\lambda^{(-1)}, \pi, \tau^{(-1)}) \in \widehat{\Xi}$. We denote by γ_1 the path obtained previously, starting at $(\lambda^{(-1)}, \pi, \tau^{(-1)})$ and ending at (λ, π, τ). We follow the same procedure to obtain γ_n starting at $(\lambda^{(-n)}, \pi, \tau^{(-n)})$ and ending at (λ, π, τ).

By definition of $\widehat{\Xi}$, we have that such paths γ_n are strongly positive, so the image of $B_{\gamma_n}^* \cdot \Theta_{\gamma_n}$ is contracted, relatively to the Hilbert metric. Thus we have that $B_{\gamma_n} \cdot \Omega(\pi) \cdot \Theta_\pi$ is converging to $\mathbb{R}_- (\Omega(\pi) \cdot \tau)$. Thus we have a contradiction. □

Theorem 6.7. *The roof function r_Ξ has exponential tails.*

Proof. Let π be the start of γ_*. The push-forward under radial projection of the Lebesgue measure on Λ_{π,q_0} onto $\Delta_\pi \cap \Upsilon_{\mathfrak{R}}^{(1)}$ yields a smooth measure $\tilde{\nu}$. It is enough to show that $\tilde{\nu}\{x \in \Xi : r_\Xi(x) \geq \log T\} \leq C T^{-\delta}$, for some $C > 0, \delta > 0$. A connected component of the domain of T_Ξ that intersects the set $\{x \in \Xi : r_\Xi(x) \geq \log T\}$ is of the form $\Delta_\gamma \cap \Upsilon_{\mathfrak{R}}^{(1)}$ where γ can not be written as $\gamma_s \hat{\gamma} \gamma_e$ with $\hat{\gamma} = \gamma_* \gamma_* \gamma_* \gamma_*$ and $\mathrm{M}(B_\gamma \cdot q_0) \geq C^{-1} T$, where $q_0 = (1, \ldots, 1)$ and C is a constant depending on γ_*. Thus

$$\begin{aligned} &\tilde{\nu}\{x \in \Xi : r_\Xi(x) \geq \log T\} \\ &\quad \leq P_{q_0}\big(\gamma \text{ can not be written as } \gamma_s \hat{\gamma} \gamma_e \text{ and } \mathrm{M}(B_\gamma \cdot q_0) \geq C^{-1} T \mid \pi\big). \end{aligned}$$

The result follows from Proposition 5.2. □

Using both the map $T_{\widehat{\Xi}}$ and the roof function r_Ξ we will define a flow $\widehat{T}_t$ on the space $\widehat{\Delta}_r = \{(x, y, s) : (x, y) \in \widehat{\Xi},\ T_{\widehat{\Xi}}(x, y) \text{ is defined and } 0 \leq s < r_\Xi(x)\}$.

Since $T_{\widehat{\Xi}}$ is a hyperbolic skew-product (Lemma 6.1), and r_Ξ is a good roof function (Lemma 6.6) with exponential tails (Theorem 6.7), $\widehat{T}_t$ is an excellent hyperbolic semi-flow. By Theorem 2.7, we get exponential decay of correlations

$$C_t(\tilde{f},\tilde{g}) = \int \tilde{f}\cdot\tilde{g}\circ\widehat{T}_t\,\mathrm{d}\nu - \int \tilde{f}\,\mathrm{d}\nu\int\tilde{g}\,\mathrm{d}\nu,$$

for C^1 functions $\tilde{f},\tilde{g}$, that is

$$|C_t(\tilde{f},\tilde{g})| \le Ce^{-3\delta t}\|\tilde{f}\|_{C^1}\|\tilde{g}\|_{C^1}, \tag{20}$$

for some $C>0, \delta>0$.

7. The Teichmüller flow

7.1. Half-translation surfaces. Let S be a compact oriented surface of genus $g\ge 0$, let Σ be a finite non-empty subset of S, which we call the *singular set*. Let $l=\{l_x\}_{x\in\Sigma}$ (the *multiplicity vector*) be such that $l_x\in\{-1\}\cup\mathbb{N}$ and $\sum l_x = 4g-4$. We say that l_x is the multiplicity of the singular point x. Consider a maximal atlas $\mathfrak{A}=\{(\mathcal{U}_\lambda,\phi_\lambda\colon \mathcal{U}_\lambda\to\mathcal{V}_\lambda\subset\mathbb{C})\}$ of orientation preserving charts on $S\setminus\Sigma$ such that for all λ_1,λ_2 with $\mathcal{U}_{\lambda_1}\cap\mathcal{U}_{\lambda_2}\neq\emptyset$ we have $\phi_{\lambda_1}\phi_{\lambda_2}^{-1}(z)=\pm z+constant$, i.e., coordinate changes are compositions of rotations by 180° and translations. We call these coordinates the *regular charts*. We also assume that each singular point x has an open neighborhood $\mathcal{U}$ which is isomorphic to the $\frac{l_x+2}{2}$-folded cover of an open neighborhood $\mathcal{V}\subset\mathbb{C}$ of 0, that is, there exists a homeomorphism, called a *singular chart*, $\phi\colon\mathcal{U}\to\mathcal{V}$ such that any branch of $z\mapsto\phi(z)^{(l_x+2)/2}$ is a regular chart. Under these conditions, we say that the atlas $\mathfrak{A}$ defines a *half-translation structure* on (S,Σ) with multiplicity vector l, and we call S a *half-translation surface*.

Since the change of coordinates preserves families of parallel lines in the plane, we have a well-defined singular foliation $\mathcal{F}_\theta$ of S, for each direction $\theta\in\mathbb{PR}^2$ (the projective space of $\mathbb{R}^2$). In particular, we have well-defined vertical and horizontal directions. Notice that we can pullback the Euclidean metric in $\mathbb{R}^2$ by the regular charts to define a flat metric on $S\setminus\Sigma$. This flat metric does not extend smoothly to Σ except at points with $l_x=0$. The other points of Σ are genuine *conical singularities* with total angle $\pi(l_x+2)$. The corresponding volume form on $S\setminus\Sigma$ has finite total mass.

Notice that from each $x\in\Sigma$, there are l_x+2 *horizontal separatrices* alternating with l_x+2 *vertical separatrices* emanating from x. A half-translation surface together with the choice of some $x_0\in\Sigma$ and of one of the horizontal separatrices X emanating from x_0 is called a *marked half-translation surface*.

7.2. Translation surfaces. If there exists a compatible atlas such that the coordinate changes are just translations, then any maximal such atlas is said to define a *translation structure* on (S, Σ) compatible with the half-translation structure, and we call S a *translation surface*. A half-translation surface has thus either 0 or 2 compatible translation structures. Locally, each half-translation structure is compatible with a translation structure, but in general it is not true globally. Given a half-translation surface S, we can associate a number ε where $\varepsilon = 1$ or $\varepsilon = -1$ according to whether the half-translation structure is, or is not, compatible with a translation structure (on the other hand, obviously each translation structure is compatible with a unique half-translation structure). Notice that if $\varepsilon = 1$ then $l_x \in 2\mathbb{N}$ for every $x \in \Sigma$ (and thus necessarily $g \geq 1$), but the converse is not generally true.

Given a translation surface S, each oriented direction $\theta \in \mathbb{S}^1$ determines a singular oriented foliation $\mathcal{F}_\theta$ on S. From every singularity thus emanate $(l_x + 2)/2$ eastbound (respectively, northbound, westbound, southbound) oriented separatrices. A translation surface together with the choice of some $x_0 \in \Sigma$ and of a eastbound separatrix X emanating from x_0 is called a *marked translation surface*.

7.3. Translation surfaces with involution. Let $\widetilde{S}$ be a compact oriented surface of genus $g \geq 1$, let $\widetilde{\Sigma}$ be a finite non-empty subset of $\widetilde{S}$, and let $I\colon \widetilde{S} \to \widetilde{S}$ be an involution preserving $\widetilde{\Sigma}$ and whose fixed points are contained in $\widetilde{\Sigma}$. A *translation structure with involution* on $(\widetilde{S}, \widetilde{\Sigma}, I)$ is a translation structure such that for every regular chart ϕ of the translation structure, $-\phi \circ I$ is also a regular chart.

Notice that given $(\widetilde{S}, \widetilde{\Sigma}, I)$ we can consider the canonical projection $p\colon \widetilde{S} \to S = \widetilde{S}/I$. Denote $\Sigma = \widetilde{\Sigma}/I$. We see that any translation structure with involution on (S, Σ, I) induces by p a half-translation structure on (S, Σ), with $\varepsilon = -1$ if $\widetilde{S}$ is connected.

Conversely, given (S, Σ) and a multiplicity vector l such that there exists a half-translation structure on (S, Σ) with such multiplicity vector and $\varepsilon = -1$, there exists a ramified double covering $p\colon (\widetilde{S}, \widetilde{\Sigma}) \to (S, \Sigma)$ which is unramified in $\widetilde{S}\backslash\widetilde{\Sigma}$. Indeed, given such a half-translation structure, we can define $\widetilde{S}\backslash\widetilde{\Sigma}$ to be the set of pairs (z, α) where $z \in S\backslash\Sigma$ and α is an orientation of the horizontal direction through z (the assumption that $\varepsilon = -1$ guarantees that $\widetilde{S}$ is connected). It is then easy to define the missing set $\widetilde{\Sigma}$ necessary to compactify: each $x \in \Sigma$ with odd l_x giving rise to a single point of $\widetilde{\Sigma}$ with multiplicity $2l_x + 2$ and each $x \in \Sigma$ with even l_x giving rise to a pair points of $\widetilde{\Sigma}$ with multiplicity l_x each one. To each half-translation surface we can associate a combinatorial data $\tilde{l}$, which is the multiplicity vector considered up to labelling. The construction above gives rise to a translation surface $\widetilde{S}$ with singularity set $\widetilde{\Sigma}$ and there is a natural involution defined, interchanging points (z, α) with fixed $z \in \widetilde{S}$.

A translation surface with involution together with the choice of some $\tilde{x}_0 \in \widetilde{\Sigma}$ and of one of the horizontal separatrices $\widetilde{X}$ emanating from $\tilde{x}_0$ to east is called a *marked translation surface with involution*. We say that $\tilde{x}_0$ is the start point of $\widetilde{X}$. It

is obvious that fixing $\tilde{x}_0$ and $\widetilde{X}$ we also fix $I(\tilde{x}_0)$ and $I(\widetilde{X})$.

Notice that when we do the double covering construction above we can do it in such a way that $p(\widetilde{X}) = X$ and $p(\tilde{x}_0) = x_0$, i.e., the marked separatrix and its start point are preserved.

As we will see in Section 8.2, we can obtain combinatorial and length data (λ, π, τ) (as in the Section 3.1) associated to a marked translation surface with involution $(\widetilde{S}, \widetilde{\Sigma}, I)$.

7.4. Moduli spaces. Let S be a surface with singular set Σ and genus g. To consider the space of surfaces with fixed genus, singularity set, multiplicity vector and the marked separatrix, we can define equivalence relations on those surfaces, obtaining moduli spaces. Although moduli spaces are not manifolds, we can see them as quotients of less restricted spaces, which have a complex affine manifold structure, by the modular group of (S, Σ), i.e., the group of orientation preserving diffeomorphisms of S fixing Σ modulo those isotopic to the identity. Thus, moduli spaces are complex affine orbifolds.

7.4.1. Moduli space of marked translation surfaces. Given $g \geq 1$, a function $\kappa\colon \mathbb{N} \to 2\mathbb{N}$ with finite support and $\sum_{i\geq 0} i\kappa(i) = 4g - 4$, and an integer $j \geq 0$ with $\kappa(j) \geq 0$, we let $\mathcal{MH}(g, \kappa, j)$ to be the *moduli space of marked translation surfaces* (S, Σ, x_0, X) with genus g, $\#\{x \in \Sigma : l_x = i\} = \kappa(i)$ and $l_{x_0} = j$. Thus two surfaces (S, Σ, x_0, X) and (S', Σ', x_0', X') are equivalent if there exists a homeomorphism $\phi\colon (S, \Sigma, x_0, X) \to (S', \Sigma', x_0', X')$ preserving the translation structure, the marked point and the given preferred separatrix.

An alternative way to view $\mathcal{MH}(g, \kappa, j)$ is as follows. Given a fixed surface S, with finite singular set Σ, a multiplicity vector l satisfying $\sum l_i = 2g - 2$, a fixed point $x_0 \in \Sigma$ and some horizontal separatrix X starting from x_0 going east, consider the space $\mathcal{TH}(S, \Sigma, x_0, X)$ of all marked translation surfaces modulo the following equivalence relation: two surfaces (S, Σ, x_0, X) and (S', Σ', x_0', X') are equivalent if there exists a homeomorphism $\phi\colon (S, \Sigma, x_0, X) \to (S', \Sigma', x_0', X')$ isotopic to the identity relatively to Σ, which preserves the translation structure. The space $\mathcal{MH}(g, \kappa, j)$ is recovered in this way by taking the quotient by an appropriate modular group, i.e., the group of orientation preserving diffeomorphisms of S, fixing Σ modulo those isotopic to the identity. The advantage of seeing the moduli space as a quotient like this, is that it inherits a structure of complex affine orbifold, since charts in $\mathcal{TH}(S, \Sigma, x_0, X)$ are complex affine. Indeed, given a path $\gamma \in C^\circ([0, T], S)$, we can lift it in $\mathbb{C}$. Since we have the translation structure, we can do this lifting everywhere. Thus, we can obtain a linear map $H_1(S, \Sigma; \mathbb{Z}) \to \mathbb{C}$, which we can see as an element of the relative cohomology group $H^1(S, \Sigma; \mathbb{C})$. This map is a local homeomorphism, thus it is a local coordinate chart. So $\mathcal{TH}(S, \Sigma, x_0, X)$ has a complex affine manifold structure.

The Lebesgue measure on space $H^1(S, \Sigma; \mathbb{C})$ (normalized so that the integer lattice $H^1(S, \Sigma; \mathbb{Z}) \oplus iH^1(S, \Sigma; \mathbb{Z})$ has covolume one) can be pulled back via these local coordinates, and we obtain a smooth measure on the space $\mathcal{TH}(S, \Sigma, x_0, X)$. In charts, the modular group acts (discretely and properly discontinuously) by complex affine maps preserving the integer lattice (and hence the Lebesgue measure). This exhibits $\mathcal{MH}(g, \kappa, j)$ as a complex affine orbifold, with a canonical absolutely continuous measure $\nu_{\mathcal{MH}}$. We denote by $\mathcal{MH}^{(1)}$ the moduli space of marked translation surfaces with area one and by $\nu^{(1)}_{\mathcal{MH}}$ the measure induced by $\nu_{\mathcal{MH}}$ on $\mathcal{MH}^{(1)}$.

The moduli spaces $\mathcal{MH}$ are also called strata and they can be disconnected. Kontsevich and Zorich ([KZ]) classified these connected components and they proved that there are at most three for each strata.

7.4.2. Moduli space of marked translation surfaces with involution. Given $g \geq 1$, functions $\tilde{\kappa}, \eta\colon \mathbb{N} \to \mathbb{N}$ with finite support and $\sum_{i \geq 0} i\tilde{\kappa}(i) = 4\tilde{g} - 4$, and an integer $\tilde{j} \geq 0$ with $\tilde{\kappa}(\tilde{j}) \geq 0$, we let $\mathcal{MHI}(\tilde{g}, \tilde{\kappa}, \eta, \tilde{j})$ to be the *moduli space of marked translation surfaces with involution* $(\widetilde{S}, \widetilde{\Sigma}, I, \tilde{x}_0, \widetilde{X})$ with genus $\tilde{g}$, an involution $I\colon \widetilde{S} \to \widetilde{S}$ preserving $\widetilde{\Sigma}$ and whose fixed points are contained in $\widetilde{\Sigma}$, $\#\{x \in \widetilde{\Sigma} : l_x = i\} = \tilde{\kappa}(i)$, $l_{x_0} = \tilde{j}$, and $\#\{x \in \widetilde{\Sigma} : l_x = 2i \text{ and } I(x) = x\} = \eta(2i)$. Thus two surfaces $(\widetilde{S}, \widetilde{\Sigma}, I, \tilde{x}_0, \widetilde{X})$ and $(\widetilde{S}', \widetilde{\Sigma}', I', \tilde{x}'_0, \widetilde{X}')$ are equivalent if there exists a homeomorphism $\phi\colon (\widetilde{S}, \widetilde{\Sigma}, I, \tilde{x}_0, \widetilde{X}) \to (\widetilde{S}', \widetilde{\Sigma}', I', \tilde{x}'_0, \widetilde{X}')$ preserving the translation structure and preserving the involution, in the sense that $\phi \circ I = I' \circ \phi$. The marked point and the chosen separatrix are also preserved.

Analogous to the previous case, we will consider the moduli space of marked translation surfaces with involution $\mathcal{MHI}(\tilde{g}, \tilde{\kappa}, \eta, \tilde{j})$ as a larger space, which has an affine complex manifold structure, quotiented by a modular group. Consider a fixed translation surface $\widetilde{S}$, an associated involution $I\colon \widetilde{S} \to \widetilde{S}$, a finite singular set $\widetilde{\Sigma}$ invariant by I, with a multiplicity vector $\tilde{l}$ satisfying $\sum l_i = 4\tilde{g} - 4$, together with some fixed $\tilde{x}_0 \in \widetilde{\Sigma}$ and one fixed horizontal separatrix $\widetilde{X}$ emanating from $\tilde{x}_0$. Let $\mathcal{THI}(\widetilde{S}, \widetilde{\Sigma}, I, \tilde{x}_0, \widetilde{X})$ be the set of $(\widetilde{S}, \widetilde{\Sigma}, I, \tilde{x}_0, \widetilde{X})$ modulo homeomorphism ϕ isotopic to the identity relatively to Σ, which preserves the translation structure with involution, in particular $\phi \circ I = I \circ \phi$.

Let $I\colon \widetilde{S} \to \widetilde{S}$ be the involution as defined before. Consider the induced involution $I^*\colon H^1(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C}) \to H^1(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C})$ on the relative cohomology group. We can decompose the cohomology group into a direct sum $H^1(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C}) = H^1_+(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C}) \oplus H^1_-(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C})$, where $H^1_+(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C})$ and $H^1_-(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C})$ are, respectively, the invariant and the anti-invariant subspaces of I^*. Observe that, since the involution changes the orientation of regular charts, the element of $H^1(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C})$ which represents $\widetilde{S}$ is in $H^1_-(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C})$ and a small neighborhood of it gives a local coordinate chart of a neighborhood of $\widetilde{S}$ in $\mathcal{THI}(\widetilde{S}, \widetilde{\Sigma}, I, \tilde{x}_0, \widetilde{X})$. Notice that we are considering that the translation surface with involution $\widetilde{S}$ can have some regular points in $\widetilde{\Sigma}$. But if we consider the set $\hat{\Sigma} \subset \widetilde{\Sigma}$ such that $\hat{\Sigma}$ has no regular points, we have that the canonical

homomorphism $H^1_-(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C}) \to H^1_-(\widetilde{S}, \hat{\Sigma}; \mathbb{C})$ induced by the inclusion $\hat{\Sigma} \hookrightarrow \widetilde{\Sigma}$ is an isomorphism. So we can choose $\hat{\Sigma}$ or $\widetilde{\Sigma}$ to define the coordinate charts (see [MZ]). Since the modular group acts discretely and properly discontinuously, we obtain a complex affine structure of orbifold to $\mathcal{MHI}(\tilde{g}, \tilde{\kappa}, \eta, \tilde{j})$. The space $H^1_-(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C})$ has a smooth standard measure which we can transport to $\mathcal{THI}(\widetilde{S}, \widetilde{\Sigma}, I, \tilde{x}_0, \widetilde{X})$ obtaining a smooth measure in this space. Hence, the space $\mathcal{MHI}$ inherits a smooth measure $\mu_{\mathcal{MHI}}$ and the moduli space of surfaces with area one $\mathcal{MHI}^{(1)}$ inherits the induced measure $\mu^{(1)}_{\mathcal{MHI}}$.

7.4.3. Moduli space of marked half-translation surfaces. Given $g \geq 0$, a function $\kappa\colon \mathbb{N} \cup \{-1\} \to \mathbb{N}$ with finite support and $\sum_{i \geq -1} i\kappa(i) = 4g - 4$, $\varepsilon \in \{-1, 1\}$, and an integer $j \geq -1$ with $\kappa(j) > 0$, we let $\mathcal{MHQ}(g, \kappa, \varepsilon, j)$ to be the *moduli space of marked half-translation surfaces* (S, Σ, x_0, X) with genus g, $\#\{x \in \Sigma : l_x = i\} = \kappa(i)$ and $l_{x_0} = j$. Two surfaces (S, Σ, x_0, X) and (S', Σ', x'_0, X') are equivalent if there exists a homeomorphism $\phi\colon (S, \Sigma, x_0, X) \to (S', \Sigma', x'_0, X')$ preserving the half-translation structure, the marked point and the fixed separatrix.

If $\varepsilon = 1$, the half-translation structure is compatible with two translation structures (corresponding to both possible orientations) and there exists a natural map $\mathcal{MH}(g, \kappa, j) \to \mathcal{MHQ}(g, \kappa, 1, j)$ which forgets the polarization. This map is a ramified double cover of orbifolds.

Given a half-translation structure which is not compatible with a translation structure, we will associate a translation structure using the (ramified) double covering construction. Define $\tilde{\kappa}\colon \mathbb{N} \cup \{-1\} \to \mathbb{N}$ by $\tilde{\kappa}(2i-1) = 0$, $\tilde{\kappa}(4i) = 2\kappa(4i) + \kappa(2i-1)$, $\tilde{\kappa}(4i+2) = 2\kappa(4i+2)$. Let $\tilde{g} = 4 + \sum_{i \geq -1} i\tilde{\kappa}(i) = 2g - 1 + \frac{1}{2}\sum_{i \geq 0} \kappa(2i-1)$, $\tilde{j} = j$ if j is even or $\tilde{j} = 2j + 2$ if j is odd. Thus, we obtain a canonical injective map $\mathcal{MHQ}(g, \kappa, -1, j) \to \mathcal{MH}(\tilde{g}, \tilde{\kappa}, \tilde{j})$. In fact, by construction, the image of this map is $\mathcal{MHI}(\tilde{g}, \tilde{\kappa}, \eta, \tilde{j})$, where the map $\eta\colon \mathbb{N} \cup \{-1\} \to \mathbb{N}$ is such that $\eta(2i) = \#\{x \in \widetilde{\Sigma} : l_x = 2i \text{ and } I(x) = x\}$ and $\eta(2i+1) = 0$.

We also can define the quotient map $\mathcal{MHI}(\tilde{g}, \tilde{\kappa}, \eta, \tilde{j}) \to \mathcal{MHQ}(g, \kappa, -1, j)$, such that, to each structure $(\widetilde{S}, \widetilde{\Sigma}, I, \tilde{x}_0, \widetilde{X})$ associates the quotient structure $(S, \Sigma, x_0, X) = (\widetilde{S}/I, \widetilde{\Sigma}/I, \tilde{x}_0/I, \widetilde{X}/I)$. Notice that this map is well-defined and it is injective, since $\widetilde{S}$ is connected. Thus, we have a bijection between marked half-translation surfaces which are not translation surfaces and connected marked translation surfaces with involution.

As in the case of translation surfaces, the moduli spaces of marked half-translation surfaces are called strata. Lanneau classified the connected components of each strata, which are at most two ([L1], [L3]).

7.5. Teichmüller flow. The group $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathcal{MHI}$ (or more generally, on the space of marked translation surfaces with involution) by postcomposition in the charts. This action preserves the hypersurface $\mathcal{MHI}^{(1)}$ and measures $\mu_{\mathcal{MHI}}$ on

$\mathcal{MHI}$ and $\mu^{(1)}_{\mathcal{MHI}}$ on $\mathcal{MHI}^{(1)}$.

The *Teichmüller flow* is the particular action of the diagonal subgroup $\mathcal{TF}_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ and it is measure-preserving.

Notice that, the Veech flow $\mathcal{VF}$ introduced in Section 3.3.2 lifts the Teichmüller flow. This is readily seen by its expression as a quotient of the flow $\mathcal{TV}$ which is expressly given by the diagonal flow.

Recall that the Veech flow $\mathcal{VF}$ preserves the standard Lebesgue measure on $\mathcal{V}_{\mathfrak{R}}$.

Theorem 7.1 (Masur, Veech). *The Teichmüller flow is mixing on each connected component of each stratum of the moduli space $\mathcal{MHQ}^{(1)}$, with respect to the finite equivalent Lebesgue measure, $\mu^{(1)}$.*

Theorem 1.2, in the setting of translation surfaces, was proved by Avila, Gouëzel and Yoccoz [AGY]. So, we will restrict the proof just to the case of half-translation surfaces which are not translation surfaces. Thus, we can prove it, just considering marked translation surfaces with involution. In this setting, the Theorem 1.2 is equivalent to:

Theorem 7.2. *The Teichmüller flow is exponential mixing on each connected component of the moduli space $\mathcal{MHI}^{(1)}$ with respect to the measure $\mu^{(1)}_{\mathcal{MHI}}$ for observables in the Ratner class.*

8. From the model to the Teichmüller flow

8.1. Zippered rectangles construction. Consider an irreducible permutation $\pi \in \mathfrak{R}$ and length data $\lambda \in \mathcal{S}_\pi$, $\tau \in \Theta_\pi$ and $h \in \mathbb{R}^{\underline{\mathcal{A}}}_+$ defined by $h = -\Omega(\pi) \cdot \tau$. Notice that $h = (h_{\underline{\alpha}})_{\underline{\alpha} \in \underline{\mathcal{A}}}$ is such that $h_{\underline{\alpha}} > 0$ for all $\underline{\alpha} \in \underline{\mathcal{A}}$. Let $\alpha(l)$ and $\alpha(r)$ be the leftmost and the rightmost letters of π, i.e., $\pi(\alpha(l)) = 1 = \overline{\pi}(i(\alpha(r)))$ and $\pi(\alpha(r)) = 2d + 1 = \overline{\pi}(i(\alpha(l)))$.

Define the following sets:

$$\mathcal{B}_\pi(\alpha) = \begin{cases} \{\beta \in \mathcal{A} : \pi(\alpha) < \pi(\beta) < \pi(*)\} & \text{if } 1 \leq \pi(\alpha) < \pi(*), \\ \{\beta \in \mathcal{A} : \pi(*) < \pi(\beta) < \pi(\alpha)\} & \text{if } \pi(*) < \pi(\alpha) \leq 2d + 1; \end{cases}$$

$$\mathcal{B}'_\pi(\alpha) = \begin{cases} \{\beta \in \mathcal{A} : \pi(\alpha) \leq \pi(\beta) < \pi(*)\} & \text{if } 1 \leq \pi(\alpha) < \pi(*), \\ \{\beta \in \mathcal{A} : \pi(*) < \pi(\beta) \leq \pi(\alpha)\} & \text{if } \pi(*) < \pi(\alpha) \leq 2d + 1; \end{cases}$$

$$\mathcal{B}_{\overline{\pi}}(\alpha) = \begin{cases} \{\beta \in \mathcal{A} : \overline{\pi}(\alpha) < \overline{\pi}(\beta) < \overline{\pi}(*)\} & \text{if } 1 \leq \overline{\pi}(\alpha) < \overline{\pi}(*), \\ \{\beta \in \mathcal{A} : \overline{\pi}(*) < \overline{\pi}(\beta) < \overline{\pi}(\alpha)\} & \text{if } \overline{\pi}(*) < \overline{\pi}(\alpha) \leq 2d + 1; \end{cases}$$

$$\mathcal{B}'_{\overline{\pi}}(\alpha) = \begin{cases} \{\beta \in \mathcal{A} : \overline{\pi}(\alpha) \le \overline{\pi}(\beta) < \overline{\pi}(*)\} & \text{if } 1 \le \overline{\pi}(\alpha) < \overline{\pi}(*), \\ \{\beta \in \mathcal{A} : \overline{\pi}(*) < \pi(\beta) \le \overline{\pi}(\alpha)\} & \text{if } \overline{\pi}(*) < \overline{\pi}(\alpha) \le 2d+1. \end{cases}$$

For each $\alpha \in \mathcal{A}$ consider the rectangles with horizontal sides λ_α and vertical sides h_α defined by

$$\begin{aligned} R_\alpha^{t,r} &= \Big(\sum_{\beta \in \mathcal{B}_\pi(\alpha)} \lambda_{\underline{\beta}}, \sum_{\beta \in \mathcal{B}'_\pi(\alpha)} \lambda_{\underline{\beta}} \Big) \times [0, h_\alpha], \\ R_\alpha^{t,l} &= \Big(- \sum_{\beta \in \mathcal{B}'_\pi(\alpha)} \lambda_{\underline{\beta}}, - \sum_{\beta \in \mathcal{B}_\pi(\alpha)} \lambda_{\underline{\beta}} \Big) \times [0, h_\alpha], \\ R_\alpha^{b,r} &= \Big(\sum_{\beta \in \mathcal{B}_{\overline{\pi}}(\alpha)} \lambda_{\underline{\beta}}, \sum_{\beta \in \mathcal{B}'_{\overline{\pi}}(\alpha)} \lambda_{\underline{\beta}} \Big) \times [-h_\alpha, 0], \\ R_\alpha^{b,l} &= \Big(- \sum_{\beta \in \mathcal{B}'_{\overline{\pi}}(\alpha)} \lambda_{\underline{\beta}}, - \sum_{\beta \in \mathcal{B}_{\overline{\pi}}(\alpha)} \lambda_{\underline{\beta}} \Big) \times [-h_\alpha, 0]. \end{aligned}$$

If $\alpha \notin \{\alpha(l), \alpha(r)\}$, also consider the vertical segments:

$$\begin{aligned} S_\alpha^{t,r} &= \Big\{ \sum_{\beta \in \mathcal{B}'_\pi(\alpha)} \lambda_{\underline{\beta}} \Big\} \times \Big[0, \sum_{\beta \in \mathcal{B}'_\pi(\alpha)} \tau_{\underline{\beta}} \Big], \\ S_\alpha^{t,l} &= \Big\{ - \sum_{\beta \in \mathcal{B}'_\pi(\alpha)} \lambda_{\underline{\beta}} \Big\} \times \Big[0, - \sum_{\beta \in \mathcal{B}'_\pi(\alpha)} \tau_{\underline{\beta}} \Big], \\ S_\alpha^{b,r} &= \Big\{ \sum_{\beta \in \mathcal{B}'_{\overline{\pi}}(\alpha)} \lambda_{\underline{\beta}} \Big\} \times \Big[\sum_{\beta \in \mathcal{B}'_{\overline{\pi}}(\alpha)} \tau_{\underline{\beta}}, 0 \Big], \\ S_\alpha^{b,l} &= \Big\{ - \sum_{\beta \in \mathcal{B}'_{\overline{\pi}}(\alpha)} \lambda_{\underline{\beta}} \Big\} \times \Big[- \sum_{\beta \in \mathcal{B}'_{\overline{\pi}}(\alpha)} \tau_{\underline{\beta}}, 0 \Big]. \end{aligned}$$

If $L_\pi = \sum_{\pi(*) < \pi(\beta) \le \pi(\alpha(r))} \tau_{\underline{\beta}} > 0$ we define

$$\begin{aligned} S_{\alpha(r)}^{t,r} &= -S_{i(\alpha(r))}^{b,l} = \Big\{ \sum_{\pi(*) < \pi(\beta) \le \pi(\alpha(r))} \lambda_{\underline{\beta}} \Big\} \times [0, L_\pi], \\ S_{\alpha(l)}^{t,l} &= S_{i(\alpha(l))}^{b,r} = \emptyset. \end{aligned}$$

If $L_\pi < 0$ we define

$$\begin{aligned} S_{\alpha(l)}^{t,l} &= -S_{i(\alpha(l))}^{b,r} = \Big\{ \sum_{\pi(\alpha(l)) \le \pi(\beta) < \pi(*)} \lambda_{\underline{\beta}} \Big\} \times [0, L_\pi], \\ S_{i(\alpha(r))}^{b} &= S_{\alpha(r)}^{t} = \emptyset. \end{aligned}$$

Otherwise, if $L_\pi = 0$ we take

$$S^{t,l}_{\alpha(l)} = S^{b,l}_{i(\alpha(r))} = \Big\{ - \sum_{\pi(\alpha(l))\le\pi(\beta)<\pi(*)} \lambda_\beta \Big\} \times \{0\},$$

$$S^{t,r}_{\alpha(r)} = S^{b,r}_{i(\alpha(l))} = \Big\{ \sum_{\pi(*)<\pi(\beta)\le\pi(\alpha(r))} \lambda_\beta \Big\} \times \{0\}.$$

Notice that, for each $\alpha \in \mathcal{A}$, the labels l and r in $X^{\epsilon,l}_\alpha$ and $X^{\epsilon,r}_\alpha$, where $\epsilon \in \{t, b\}$ and $X \in \{R, S\}$, are just to make clear when $\pi(\alpha) < \pi(*)$ or $\pi(*) < \pi(\alpha)$. When it does not lead to confusion, we will omit l and r.

Example 8.1. Figure 1 represents a zippered rectangle associated to

$$\pi = D \quad i(B) \quad i(D) \quad C \quad i(C) \quad * \quad A \quad i(A) \quad B.$$

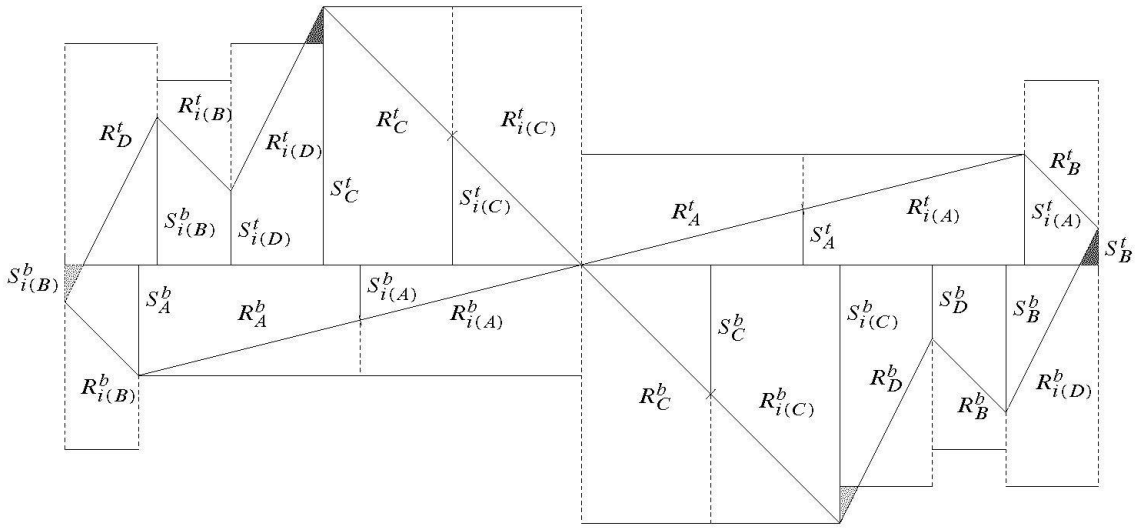

Figure 1. Zippered rectangle.

Define the set

$$R_{(\lambda,\pi,\tau)} = \bigcup_{\alpha\in\mathcal{A}} \bigcup_{\epsilon\in\{l,r\}} \left(R^\epsilon_\alpha \cup S^\epsilon_\alpha\right).$$

We will identify, by translation, the rectangle R^t_α with R^b_α for all $\alpha \in \mathcal{A}$.

If $L_\pi > 0$ we identify $S^t_{\alpha(r)}$ with the vertical segment S_1 of length L_π at the bottom of the right side of the rectangle $R^b_{\alpha(r)}$ if $\alpha(r)$ is the winner of π or at the top of the right side of the rectangle $R^t_{i(\alpha(l))}$ if $\alpha(r)$ is the loser of π. Symmetrically, we identify $S^b_{i(\alpha(r))}$ with $-S_1$.

If $L_\pi < 0$ we identify $S^b_{i(\alpha(l))}$ with the vertical segment S_2 of length $-L_\pi$ at the bottom of the right side of the rectangle $R^b_{\alpha(r)}$ if $\alpha(r)$ is the winner of π or at the top of the right side of the rectangle $R^t_{i(\alpha(l))}$ if $\alpha(r)$ is the loser of π. Symmetrically, we identify $S^t_{\alpha(l)}$ with $-S_2$.

Let $\widetilde{S}^*(\lambda, \pi, \tau)$ be the topological space obtained from $R_{(\lambda,\pi,\tau)}$ by these identifications. Thus, $\widetilde{S}^*(\lambda, \pi, \tau)$ inherits from $\mathbb{R}^2 = \mathbb{C}$ the structure of a Riemann surface and also a holomorphic 1-form ω (given by dz).

For each $\alpha \in \mathcal{A}$ recall $\zeta_\alpha = \lambda_\alpha + i\,\tau_\alpha$. We call *vertices* the extreme points at the top of segments S^t_α and the extremes at the bottom of segments S^b_α, for all $\alpha \in \mathcal{A}$. So, the vertices are points with following coordinates:

$$\xi^t_\alpha = \begin{cases} \displaystyle\sum_{\pi(\alpha)\le\pi(\beta)<\pi(*)} -\zeta_\beta & \text{if } \pi(\alpha) < \pi(*), \\ \displaystyle\sum_{\pi(*)<\pi(\beta)\le\pi(\alpha)} \zeta_\beta & \text{if } \pi(*) < \pi(\alpha); \end{cases}$$

$$\xi^b_\alpha = \begin{cases} \displaystyle\sum_{\overline{\pi}(\alpha)\le\overline{\pi}(\beta)<\overline{\pi}(*)} -\zeta_\beta & \text{if } \overline{\pi}(\alpha) < \overline{\pi}(*), \\ \displaystyle\sum_{\overline{\pi}(*)<\overline{\pi}(\beta)\le\overline{\pi}(\alpha)} \zeta_\beta & \text{if } \overline{\pi}(*) < \overline{\pi}(\alpha). \end{cases}$$

Now we will define a relation to identify vertices between them. Define the set of all pairs (α, Y) with $\alpha \in \mathcal{A} \cup \{*\}$ and $Y \in \{L, R\}$. Consider the following identification:

$$(\pi(\pi^{-1}(*) + 1), L) \sim (*, R) \sim (\overline{\pi}(\overline{\pi}^{-1}(*) + 1), L),$$
$$(\pi(\pi^{-1}(*) - 1), R) \sim (*, L) \sim (\overline{\pi}(\overline{\pi}^{-1}(*) - 1), R),$$

$$(\alpha(r), R) \sim (i(\alpha(l)), R),$$
$$(\alpha(l), L) \sim (i(\alpha(r)), L).$$

We say that these pairs are *irregular* and all other pairs we call *regular*. We also identify

$$(\alpha(r), R) \sim (\beta, L) \quad \text{if } \pi(\alpha) + 1 = \pi(\beta),$$
$$(\alpha(r), R) \sim (\beta, L) \quad \text{if } \overline{\pi}(\alpha) + 1 = \overline{\pi}(\beta).$$

We can extend $\sim$ to an equivalence relation in the set of pairs (α, Y). This equivalence relation describes how half-planes are identified when one winds around an end of $\widetilde{S}^*(\lambda, \pi, \tau)$. Let $\widetilde{\Sigma}$ be the set of equivalence classes relative to the relation $\sim$. Thus to each $c \in \widetilde{\Sigma}$ we have one, and only one, end v_c of $\widetilde{S} = \widetilde{S}^*(\lambda, \pi, \tau)$. When it does not lead to confusion we will use $\widetilde{S}$ to mean $\widetilde{S}(\lambda, \pi, \tau)$. From the local

structure around v_c, the compactification

$$\widetilde{S}(\lambda,\pi,\tau) = \widetilde{S}^*(\lambda,\pi,\tau) \cup \Big(\bigcup_{\widetilde{\Sigma}} \{v_c\}\Big)$$

is a compact Riemann surface with marked points $\{v_c\}$. The 1-form ω extends to a holomorphic 1-form on $\widetilde{S}(\lambda,\pi,\tau)$ such that at the points v_c we have marked zeroes of angle $2k_c\pi$ where $2k_c$ is the cardinality of the equivalence class of c.

Given $(x,0)$ on the bottom side of the rectangle R^t_α, we can transport this point vertically and when we reach the top side, which is the point (x,h_α) we identify it with the point $(x+\omega_\alpha,0)$, where $\omega = \Omega(\pi)\cdot\lambda$, in the top side of R^b_α. So, we have the vertical flow well-defined almost everywhere (except in the points which reach singularities in finite time). It is clear that the return time of points in the rectangle R^t_α is equal to h_α and the area of the surface $\widetilde{S}(\lambda,\pi,\tau)$ is $A(\lambda,\pi,\tau) = -2\langle\lambda,\Omega(\pi)\cdot\tau\rangle$.

When we constructed the surface $\widetilde{S}$, we have an implicit relation between the horizontal coordinates λ_α and $\lambda_{i(\alpha)}$ and the vertical coordinates τ_α and $\tau_{i(\alpha)}$. Indeed, we have an involution $I\colon \widetilde{S} \to \widetilde{S}$, with a fixed point at the origin, defined as follows. Given any point $x \in \widetilde{S}$ there exists $\alpha \in \mathcal{A}$ such that $x \in R^t_\alpha$ or $x \in S^t_\alpha \cup S^b_\alpha$. Thus $-x \in R^b_{i(\alpha)}$ or $-x \in S^b_{i(\alpha)} \cup S^t_{i(\alpha)}$, respectively. So $I(x)$ is identified with $-x$.

Let $S(\lambda,\pi,\tau)$ be the surface $\widetilde{S}(\lambda,\pi,\tau)$ quotiented by the involution I and let Σ to be the set $\widetilde{\Sigma}$ quotiented by the involution I. We can see that this identification by involution implies that, for each $\alpha \in \mathcal{A}$, the rectangle R^t_α is identified with the rectangle $R^t_{i(\alpha)}$ by a translation composed with a rotation of 180 degrees. So, the top side (resp. the bottom side) of the rectangle R^t_α is identified with the bottom side (resp. the top side) of the rectangle $R^t_{i(\alpha)}$.

8.2. Coordinates. Let $(\widetilde{S},\widetilde{\Sigma},I,\tilde{x}_0,\widetilde{X})$ be a marked translation surface with an involution I. The marked separatrix $\widetilde{X}$ starts at $\tilde{x}_0$ and it goes to east. A segment σ adjacent to $\tilde{x}_0$ contained in $\widetilde{X}$ is called *admissible* if the vertical geodesic Y passing through the right endpoint $\tilde{z}$ of σ meets a singularity in the positive or in the negative direction before returning to $\sigma \cup I(\sigma)$. Symmetrically, if σ is an admissible segment then $I(\sigma)$ starting at $I(\tilde{x}_0)$ going west and ending at $I(\tilde{z})$ (which has a vertical geodesic meeting a singularity in the negative or in the positive direction before returning to $\sigma \cup I(\sigma)$), also is an admissible segment if we consider the marked separatrix $I(\widetilde{X})$ instead of considering $\widetilde{X}$.

We call a separatrix *incoming* if its natural orientation points towards the associated singularity and we call it *outgoing* otherwise. Let σ^+ be the set of points of first intersection of incoming vertical separatrices with $\sigma \cup I(\sigma)$. Analogously, let σ^- be the set of points of first intersection of outgoing vertical separatrices with $\sigma \cup I(\sigma)$. Notice that $\tilde{x}_0$ and $I(\tilde{x}_0)$ are in both sets σ^+ and $\sigma^-$and if Y is incoming (resp. outgoing), then $\tilde{z} \in \sigma^+$ (resp. $\tilde{z} \in \sigma^-$). We extend the definition of σ^+ and σ^- in order to both $\tilde{z}$ and $I(\tilde{z})$ be in the sets σ^+ and σ^-.

Notice that $p \in \sigma^-$ if and only if $I(p) \in \sigma^+$, for all $p \in \sigma^-$. Thus we will consider just the set σ^+ which determines the set σ^- by involution.

Let $|\lambda|$ be the length of σ, which also coincides with length of $I(\sigma)$. Let $\phi_r\colon [0, |\lambda|] \to \widetilde{S}$ and $\phi_l\colon [-|\lambda|, 0] \to \widetilde{S}$ be the arc-length parametrizations of σ and $I(\sigma)$, respectively, such that $\phi_r(0) = \tilde{x}_0$ and $\phi_l(0) = I(\tilde{x}_0)$.

We can write

$$\sigma^+ = \{I(\tilde{z}) = p^+_{-l} > \cdots > p^+_{-1} > p^+_{0-} = I(\tilde{x}_0)\}$$
$$\cup \{\tilde{x}_0 = p^+_{0+} < p^+_1 < \cdots < p^+_r = \tilde{z}\}$$

where $<$ and $>$ refer to the natural orientation on σ and $I(\sigma)$, respectively.

Therefore, we have numbers

$$-|\lambda| = a^+_{-l} < \cdots < a^+_{-1} < a^+_{0-} = 0 = a^+_{0+} < a^+_1 < \cdots < a^+_r = |\lambda|$$

such that $\phi_r(p^+_j) = a^+_j$, for all $j \in \{-l \ldots, -1, 1 \ldots r\}$, $\phi_r(p^+_{0-}) = a^+_{0-} = 0$ and $\phi_r(p^+_{0+}) = a^+_{0+} = 0$.

Let $a^+_0 = a^+_{0+} = a^+_{0-}$ and define $\lambda_j = |I_j| = a^+_{j+1} - a^+_j$ for $-l \le j \le r-1$. Let τ_j be the length of the vertical segment from the horizontal section to the singularity corresponding to the point p^+_j. Notice that $\tau_0 = 0$.

It is possible to verify that the first return map to the cross-section $\sigma \cup I(\sigma)$ is well-defined except at the points p^+_j. Moreover the first return time is constant on each open interval (a^+_j, a^+_{j+1}). So, we can consider the interval exchange transformation with involution π associated to the cross-section $\sigma \cup I(\sigma)$ where the points defined above are the points of discontinuity.

Let h_j be the first return time of the points in the interval (a_{j-1}, a_j). We can define the zippered rectangle which represents $(\widetilde{S}, \widetilde{\Sigma}, I, \tilde{x}_0, \widetilde{X})$ by

$$ZR(\lambda, \pi, \tau, h) = \bigcup_j (a_{j-1}, a_j) \times [0, h_j].$$

Lemma 8.2. *If two admissible segments σ and $\tilde{\sigma}$ with the same left extreme point $\tilde{x}_0$ are such that $\tilde{\sigma} \subset \sigma$, then the corresponding zippered rectangles (λ, π, τ, h) and $(\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}, \tilde{h})$ satisfy*

$$(\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}) = \widehat{Q}^n(\lambda, \pi, \tau) \quad \textit{for some } n \in \mathbb{N}.$$

Proof. Let σ and $\tilde{\sigma}$ be admissible segments and let the respective zippered rectangles representations $ZR(\lambda, \pi, \tau, h)$ and $ZR(\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}, \tilde{h})$.

Consider a sequence of maximal admissible segments σ^{i+1} strictly contained in σ^i such that $\sigma^1 = \sigma$. Let z_i be the right endpoint of σ^i. The right endpoint z_2 of σ^2 corresponds to a discontinuity point of the first return map of the vertical flow to the section σ^1 and there is no other discontinuity point between z_2 and z_1. By

maximality, we conclude that, up to relabeling, $\widehat{Q}(\lambda, \pi, \tau)$ is the representation of such first return map. We follow this process until obtain $\sigma^n = \tilde{\sigma}$ for some $n \in \mathbb{N}$. For such n we have $\widehat{Q}^n(\lambda, \pi, \tau) = (\tilde{\lambda}, \tilde{\pi}, \tilde{\tau})$. □

Corollary 8.3. *Let $(\widetilde{S}, \widetilde{\Sigma}, I, \tilde{x}_0, \widetilde{X})$ be a marked translation surface with involution and $ZR(\lambda, \pi, \tau, h)$ and $ZR(\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}, \tilde{h})$ be two zippered rectangle representations of the surface. Then there exists $n \in \mathbb{Z}$ such that $(\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}) = \widehat{Q}^n(\lambda, \pi, \tau)$.*

Proof. Let σ and $\tilde{\sigma}$ be admissible segments of $ZR(\lambda, \pi, \tau, h)$ and $ZR(\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}, \tilde{h})$, respectively. By definition, the initial points of σ and $\tilde{\sigma}$ are the same $\tilde{x}_0$. Suppose, without loss of generality, that $\tilde{\sigma} \subset \sigma$.

By the previous lemma there exists $n \in \mathbb{N}$ such that $(\tilde{\lambda}, \tilde{\pi}, \tilde{\tau}) = \widehat{Q}^n(\lambda, \pi, \tau)$. Thus, the result follows. □

Given a marked translation surface with involution $(\widetilde{S}, \widetilde{\Sigma}, I, \tilde{x}, \widetilde{X})$ with zippered rectangle representation $ZR(\lambda, \pi, \tau, h)$, we can cut and paste it appropriately until we obtain a surface $(\widetilde{S}', \widetilde{\Sigma}', I', \tilde{x}', \widetilde{X}')$ which representation in zippered rectangles is an iterated by Rauzy induction with involution of the first marked translation surface with involution. Since these operations preserve the relation between parallel sides, then $\widetilde{S}$ and $\widetilde{S}'$ are isomorphic and the marked separatrix is mapped to one another. Moreover if we have a marked translation surface with involution $(\widetilde{S}_1, \widetilde{\Sigma}_1, I_1, \tilde{x}_1, \widetilde{X}_1)$ which is near $(\widetilde{S}, \widetilde{\Sigma}, I, \tilde{x}, \widetilde{X})$, by the continuity of the marked separatrix and of the singularities, we will obtain, up to relabel, a zippered rectangle construction $ZR(\lambda_1, \pi_1, \tau_1, h_1)$ near $ZR(\lambda, \pi, \tau, h)$. So, the zippered rectangle construction, gives a system of local coordinates in each stratum of the moduli space.

Using the zippered rectangles construction, we obtain a finite covering ZR, of a stratum of the moduli space of marked translation surfaces with involution, $\mathcal{MHI}(\tilde{g}, \tilde{\kappa}, \eta, \tilde{j})$. Under the condition $\langle \lambda, h \rangle = 1$, we get the space of zippered rectangles of area one covering the space $\mathcal{MHI}^{(1)}(\tilde{g}, \tilde{\kappa}, \eta, \tilde{j})$. We have a bijection between Rauzy classes with involution and a connected component of a stratum of the moduli space of translation surfaces with involution (see [BL]). Thus we have a well-defined map proj: $\mathcal{V}_{\mathfrak{R}} \to \mathcal{C}$, where $\mathcal{C} = \mathcal{C}(\mathfrak{R})$ is a connected component of $\mathcal{MHI}(\tilde{g}, \tilde{\kappa}, \eta, \tilde{j})$ and $\text{proj} \circ \widehat{Q} = \text{proj}$. The fibers of this map are almost everywhere finite (with constant cardinality). The projection of the standard Lebesgue measure on $\mathcal{V}_{\mathfrak{R}}$ is (up to scaling) the standard volume form on $\mathcal{C}$.

The subset $\mathcal{V}_{\mathfrak{R}}^{(1)} = \text{proj}^{-1}(\mathcal{C}^{(1)})$ of surfaces with area one is invariant by the Veech flow. So, the restriction $\mathcal{TV}_t(x)$: $\mathcal{V}_{\mathfrak{R}}^{(1)} \to \mathcal{V}_{\mathfrak{R}}^{(1)}$ leaves invariant the volume form that projects, up to scaling, to the invariant volume form on $\mathcal{C}^{(1)}$. It was proved by Veech that this volume form is finite using the lift measure on $\mathcal{V}_{\mathfrak{R}}^{(1)}$.

8.2.1. Homology and cohomology. For each $\alpha \in \mathcal{A}$ consider the curve c_α which is a path in $R_{(\lambda,\pi,\tau)}$ joining $\xi_\alpha^t - \zeta_\alpha$ to ξ_α^t if $\pi(\alpha) > \pi(*)$ or joining ξ_α^t to $\xi_\alpha^t + \zeta_\alpha$ if

$\pi(\alpha) < \pi(*)$. Note that $I(c_\alpha) = -c_{i(\alpha)}$.

Consider the relative homology group $H_1(\widetilde{S}, \widetilde{\Sigma}; \mathbb{Z})$ of the surface $\widetilde{S}$ relative to the finite set of singularities $\widetilde{\Sigma}$. We have a decomposition of the relative homology group into an invariant subgroup $H_1^+(\widetilde{S}, \widetilde{\Sigma}; \mathbb{Z})$ and an anti-invariant subgroup $H_1^-(\widetilde{S}, \widetilde{\Sigma}; \mathbb{Z})$, with respect to the involution I. Following Masur and Zorich ([MZ]), we can choose a basis in $H_1^-(\widetilde{S}, \widetilde{\Sigma}; \mathbb{Z})$ which has dimension $d - 1$, where d is the number of classes of $\underline{\mathcal{A}}^1$. The elements of the basis will be lifts of a collection of saddle connections on S, where S is the surface $\widetilde{S}$ quotiented by involution.

Analogously, the first (de Rham) cohomology group $H^1(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C})$, is decomposed into an invariant subspace $H^1_+(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C})$ and an anti-invariant subspace $H^1_-(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C})$, under the induced involution $I^*\colon H^1(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C}) \to H^1(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C})$ Notice that $[\omega]$ is anti-invariant under the induced involution, so $[\omega] \in H^1_-(\widetilde{S}, \widetilde{\Sigma}, \mathbb{C})$ and we also have:

$$\int_{c_\alpha} \omega = \zeta_\alpha.$$

In Section 7.4.2 we have observed that $H^1_-(\widetilde{S}, \widetilde{\Sigma}; \mathbb{C})$ yields local coordinates of an element of a stratum of the moduli space of translation surfaces with involution. So if we consider the set of $\zeta_{\underline{\alpha}} = \lambda_{\underline{\alpha}} + i\tau_{\underline{\alpha}}$ such that $\lambda, \tau \in \mathcal{S}_\pi$ we obtain coordinates which describe $\widetilde{S}(\lambda, \pi, \tau)$. And as we have seen in Section 8.2, for any other pair $(\widetilde{S}', \omega')$ in a neighborhood of $(\widetilde{S}, \omega)$ we can find coordinates (λ', π', τ') of $(\widetilde{S}', \omega')$, and so we can define the vectors $\zeta'_{\underline{\alpha}}$ as in Section 8.1. For more details in the construction of coordinates see [Ve3].

8.3. Teichmüller flow is exponential mixing. In Section 7.5 we have seen the relation between the Teichmüller flow and the Veech flow, which is naturally identified with the first return map of the renormalization operator to the section $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$.

We will identify $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}} \times \mathbb{R}$ with a connected component $\mathcal{C}^{(1)}$ by the map $P\colon \widehat{\Upsilon}^{(1)}_{\mathfrak{R}} \times \mathbb{R} \to \mathcal{C}^{(1)}$ defined by $P(z, s) = \mathcal{T}\mathcal{F}_s(\mathrm{proj}(z))$, where $z = (\lambda, \pi, \tau)$ and proj: $\widehat{\Delta}_{\mathfrak{R}} \to \mathcal{C}$ is the natural projection.

Lemma 8.4. *Let $f\colon \mathcal{C}^{(1)} \to \mathbb{R}$ be a C^1 compactly supported function and let $\delta > 0$ be as in* (20). *There exists $\epsilon_0 > 0$ and $C > 0$ such that for every $t > 0$, there exists a C^1 function $f^{(t)}\colon \widehat{\Delta}_r \to \mathbb{R}$, such that $\|f \circ P - f^{(t)}\|_{L^2(\nu)} \le Ce^{-\epsilon_0 t}$ and $\|f^{(t)}\|_{C^1(\widehat{\Delta}_r)} \le Ce^{\delta t}$.*

Proof. Let $\delta_0 > 0$ be small and let $Y_t \subset \widehat{\Delta}_r$ be the union of connected components of $\widehat{\Delta}_r$ which contain points (λ, π, τ, s) with $s > \delta_0 t$. Let $f^{(t)} = 0$ in Y_t and $f^{(t)} = f \circ P$ in the complement. The estimate $\|f \circ P - f^{(t)}\|_{L^2(\nu)} \le Ce^{-\epsilon_0 t}$ is

[1]The complex dimension of the moduli space of half-translation surfaces of genus g with σ singularities is $2g + \sigma - 2$ and $d = 2g + \sigma - 1$.

then clear since $\|f \circ P - f^{(t)}\|_{C^0} \leq \|f\|_{C^0}$, while the support of $f \circ P - f^{(t)}$ has exponentially small ν measure (since the roof function has exponential tails).

For the other estimate, it is enough to show that if $(z,s) \in \widehat{\Delta}_r$ and $P(z,s)$ belongs to any fixed compact set $K \subset \mathcal{C}^{(1)}$ then P is locally Lipschitz near (z,s), with constant bounded by $C(K)e^{C(K)s}$. Here we fix some arbitrary Finsler metric in $\mathcal{C}^{(1)}$ (the precise choice is irrelevant since K is compact). This result is obvious if we impose some bound on s, say $0 \leq s \leq 1$, since P is smooth. If $s_0 > 0$ is such that $s_0 < s < s_0 + 1$, notice that for (z',s') in a neighborhood of (z,s), $P(z',s')$ is obtained from $P(z,s'-s_0)$ by applying the Teichmüller flow for time s_0. Thus, it is enough to show that if x and $\mathcal{TF}_{s_0}(x)$ belong to some fixed compact set of $\mathcal{C}^{(1)}$ then $\mathcal{TF}_{s_0}$ is locally Ce^{Cs_0} Lipschitz in a neighborhood of x. This is a well known estimate, for instance, we can define a Finsler metric on $\mathcal{C}^{(1)}$ such that $\mathcal{TF}_{s_0}$ is globally Lipschitz with Lipschitz constant e^{2s_0} (see [AGY], §2.2.2, for the construction of a metric in the whole strata of squares, the Finsler metric we need here being just the restriction to the substrata). □

Lemma 8.5. *If $f : \mathcal{C}^{(1)} \to \mathbb{R}$ is C^1 and compactly supported with $\int f dv_{\mathcal{C}^{(1)}} = 0$ then there exist $C > 0$, $\epsilon > 0$ such that for $t > 0$,*

$$\int f \cdot (f \circ \mathcal{TF}_t)\, dv_{\mathcal{C}^{(1)}} \leq Ce^{-\epsilon t}. \tag{21}$$

Proof. We can estimate (21) with exponentially small error by comparison with the correlations $\int f^{(t)} \cdot f^{(t)} \circ \widehat{T}_t dv - (\int f^{(t)} dv)^2$, where $f^{(t)}$ is provided by the previous lemma. Those decays exponentially by (20). □

Finally, we are in a position to prove the main theorem:

Proof of Theorem 1.2. Let H be the Hilbert space of $\mathrm{SO}(2,\mathbb{R})$ invariant $L^2(\nu_{\mathcal{C}^{(1)}})$ functions with zero mean. As shown in Appendix B of [AGY], exponential decay of correlations for the Ratner class follows from the existence of a dense set of f in H such that (21) decays exponentially fast. Since compactly supported smooth functions are dense in H, the result follows. □

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