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## Complete constant mean curvature surfaces in homogeneous spaces

José M. Espinar\* and Harold Rosenberg

**Abstract.** In this paper we classify complete surfaces of constant mean curvature whose Gaussian curvature does not change sign in a simply connected homogeneous manifold with a 4-dimensional isometry group.

**Mathematics Subject Classification (2010).** 53A10, 53C21.

**Keywords.** Constant mean curvature, homogeneous spaces.

### 1. Introduction

In 1966, T. Klotz and R. Ossermann showed the following:

**Theorem ([KO]).** *A complete  $H$ -surface in  $\mathbb{R}^3$  whose Gaussian curvature  $K$  does not change sign is either a sphere, a minimal surface, or a right circular cylinder.*

The above result was extended to  $\mathbb{S}^3$  by D. Hoffman [H], and to  $\mathbb{H}^3$  by R. Tribuzy [T] with an extra hypothesis if  $K$  is non-positive. The additional hypothesis says that, when  $K \leq 0$ , one has  $H^2 - K - 1 > 0$ .

In recent years, the study of  $H$ -surfaces in product spaces and, more generally, in a homogeneous three-manifold with a 4-dimensional isometry group is quite active (see [AR], [AR2], [CoR], [ER], [FM], [FM2], [DH] and references therein).

The aim of this paper is to extend the above theorem to homogeneous spaces with a 4-dimensional isometry group. These homogeneous spaces are denoted by  $\mathbb{E}(\kappa, \tau)$ , where  $\kappa$  and  $\tau$  are constant and  $\kappa - 4\tau^2 \neq 0$ . They can be classified as  $\mathbb{M}^2(\kappa) \times \mathbb{R}$  if  $\tau = 0$ , with  $\mathbb{M}^2(\kappa) = \mathbb{S}^2(\kappa)$  if  $\kappa > 0$  ( $\mathbb{S}^2(\kappa)$  the sphere of curvature  $\kappa$ ), and  $\mathbb{M}^2(\kappa) = \mathbb{H}^2(\kappa)$  if  $\kappa < 0$  ( $\mathbb{H}^2(\kappa)$  the hyperbolic plane of curvature  $\kappa$ ). If  $\tau$  is not equal to zero,  $\mathbb{E}(\kappa, \tau)$  is a Berger sphere if  $\kappa > 0$ , a Heisenberg space if  $\kappa = 0$  (of

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bundle curvature  $\tau$ ), and the universal cover of  $\mathrm{PSL}(2, \mathbb{R})$  if  $\kappa < 0$ . Henceforth we will suppose  $\kappa$  is plus or minus one or zero.

The paper is organized as follows. In Section 2, we establish the definitions and necessary equations for an  $H$ -surface. We also state here two classification results for  $H$ -surfaces. We prove them in Section 5 and Section 6 for the sake of completeness.

Section 3 is devoted to the classification of  $H$ -surfaces with non-negative Gaussian curvature,

**Theorem 3.1.** *Let  $\Sigma \subset \mathbb{E}(\kappa, \tau)$  be a complete  $H$ -surface with  $K \geq 0$ . Then,  $\Sigma$  is either a rotational sphere (in particular,  $4H^2 + \kappa > 0$ ), or a complete vertical cylinder over a complete curve of geodesic curvature  $2H$  on  $\mathbb{M}^2(\kappa)$ .*

In Section 4 we continue with the classification of  $H$ -surfaces with non-positive Gaussian curvature.

**Theorem 4.1.** *Let  $\Sigma \subset \mathbb{E}(\kappa, \tau)$  be a complete  $H$ -surface with  $K \leq 0$  and  $H^2 + \tau^2 - |\kappa - 4\tau^2| > 0$ . Then,  $\Sigma$  is a complete vertical cylinder over a complete curve of geodesic curvature  $2H$  on  $\mathbb{M}^2(\kappa)$ .*

The above theorem is not true without the inequality; for example, any complete minimal surface in  $\mathbb{H}^2 \times \mathbb{R}$  that is not a vertical cylinder.

In the Appendix, we give a result, which we think is of independent interest, concerning differential operators on a Riemannian surface  $\Sigma$  of the form  $\Delta + g$ , acting on  $C^2(\Sigma)$ -functions, where  $\Delta$  is the Laplacian with respect to the Riemannian metric on  $\Sigma$  and  $g \in C^0(\Sigma)$ .

## 2. The geometry of surfaces in homogeneous spaces

Henceforth  $\mathbb{E}(\kappa, \tau)$  denotes a complete simply connected homogeneous three-manifold with 4-dimensional isometry group. Such a three-manifold can be classified in terms of a pair of real numbers  $(\kappa, \tau)$  satisfying  $\kappa - 4\tau^2 \neq 0$ . In fact, these manifolds are Riemannian submersions over a complete simply-connected surface  $\mathbb{M}^2(\kappa)$  of constant curvature  $\kappa$ ,  $\pi: \mathbb{E}(\kappa, \tau) \rightarrow \mathbb{M}^2(\kappa)$ , and translations along the fibers are isometries, therefore they generate a Killing field  $\xi$ , called the *vertical field*. Moreover,  $\tau$  is the real number such that  $\bar{\nabla}_X \xi = \tau X \wedge \xi$  for all vector fields  $X$  on the manifold. Here,  $\bar{\nabla}$  is the Levi-Civita connection of the manifold and  $\wedge$  is the cross product.

Let  $\Sigma$  be a complete  $H$ -surface immersed in  $\mathbb{E}(\kappa, \tau)$ . By passing to a 2-sheeted covering space of  $\Sigma$ , we can assume  $\Sigma$  is orientable. Let  $N$  be a unit normal to  $\Sigma$ . In terms of a conformal parameter  $z$  of  $\Sigma$ , the first,  $\langle \cdot, \cdot \rangle$ , and second,  $II$ , fundamental

forms are given by

$$\begin{aligned} \langle \cdot, \cdot \rangle &= \lambda |dz|^2 \\ H &= p dz^2 + \lambda H |dz|^2 + \bar{p} d\bar{z}^2, \end{aligned} \tag{2.1}$$

where  $p dz^2 = \langle -\nabla_{\partial_z} N, \partial_z \rangle dz^2$  is the Hopf differential of  $\Sigma$ .

Set  $\nu = \langle N, \xi \rangle$  and  $T = \xi - \nu N$ , i.e.,  $\nu$  is the normal component of the vertical field  $\xi$ , called the *angle function*, and  $T$  is the tangent component of the vertical field.

First we state the following necessary equations on  $\Sigma$  which were obtained in [FM].

**Lemma 2.1.** *Given an immersed surface  $\Sigma \subset \mathbb{E}(\kappa, \tau)$ , the following equations are satisfied:*

$$K = K_e + \tau^2 + (\kappa - 4\tau^2) \nu^2, \tag{2.2}$$

$$p_{\bar{z}} = \frac{\lambda}{2} (H_z + (\kappa - 4\tau^2) \nu A), \tag{2.3}$$

$$A_{\bar{z}} = \frac{\lambda}{2} (H + i\tau) \nu, \tag{2.4}$$

$$\nu_z = -(H - i\tau) A - \frac{2}{\lambda} p \bar{A}, \tag{2.5}$$

$$|A|^2 = \frac{1}{4} \lambda (1 - \nu^2), \tag{2.6}$$

$$A_z = \frac{\lambda_z}{\lambda} A + p \nu, \tag{2.7}$$

where  $A = \langle \xi, \partial_z \rangle$ ,  $K_e$  the extrinsic curvature and  $K$  the Gauss curvature of  $\Sigma$ .

For an immersed  $H$ -surface  $\Sigma \subset \mathbb{E}(\kappa, \tau)$  there is a globally defined quadratic differential, called the *Abresch–Rosenberg differential*, which in these coordinates is given by (see [AR2]):

$$Q dz^2 = (2(H + i\tau) p - (\kappa - 4\tau^2) A^2) dz^2,$$

following the notation above.

It is not hard to verify this quadratic differential is holomorphic on an  $H$ -surface using (2.3) and (2.4),

**Theorem 2.1** ([AR], [AR2]).  *$Q dz^2$  is a holomorphic quadratic differential on any  $H$ -surface in  $\mathbb{E}(\kappa, \tau)$ .*

Associated to the Abresch–Rosenberg differential we define the smooth function  $q: \Sigma \rightarrow [0, +\infty)$  given by

$$q = \frac{4|Q|^2}{\lambda^2}.$$

By means of Theorem 2.1,  $q$  either has isolated zeroes or vanishes identically. Note that  $q$  does not depend on the conformal parameter  $z$ , hence  $q$  is globally defined on  $\Sigma$ .

We continue this section establishing some formulae relating the angle function,  $q$  and the Gaussian curvature.

**Lemma 2.2.** *Let  $\Sigma$  be an  $H$ -surface immersed in  $\mathbb{E}(\kappa, \tau)$ . Then the following equations are satisfied:*

$$\begin{aligned} \|\nabla v\|^2 &= \frac{4H^2 + \kappa - (\kappa - 4\tau^2)v^2}{4(\kappa - 4\tau^2)}(4(H^2 - K_e) \\ &\quad + (\kappa - 4\tau^2)(1 - v^2)) - \frac{q}{\kappa - 4\tau^2}, \end{aligned} \quad (2.8)$$

$$\Delta v = -(4H^2 + 2\tau^2 + (\kappa - 4\tau^2)(1 - v^2) - 2K_e)v. \quad (2.9)$$

Moreover, away from the isolated zeroes of  $q$ , we have

$$\Delta \ln q = 4K. \quad (2.10)$$

*Proof.* From (2.5)

$$|v_z|^2 = \frac{4|p|^2|A|^2}{\lambda^2} + (H^2 + \tau^2)|A|^2 + \frac{2(H + i\tau)}{\lambda}p\bar{A}^2 + \frac{2(H - i\tau)}{\lambda}\bar{p}A^2,$$

and taking into account that

$$\begin{aligned} |Q|^2 &= 4(H^2 + \tau^2)|p|^2 + (\kappa - 4\tau^2)^2|A|^4 - (\kappa - 4\tau^2)(2(H + i\tau)p\bar{A}^2 \\ &\quad + 2(H - i\tau)\bar{p}A^2), \end{aligned}$$

we obtain, using also (2.6), that

$$\begin{aligned} |v_z|^2 &= (H^2 + \tau^2)|A|^2 + (H^2 - K_e)|A|^2 + (\kappa - 4\tau^2)\frac{|A|^4}{\lambda} \\ &\quad + 4\left(\frac{H^2 + \tau^2}{\kappa - 4\tau^2}\right)\frac{|p|^2}{\lambda} - \frac{|Q|^2}{(\kappa - 4\tau^2)\lambda} \end{aligned}$$

where we have used that  $4|p|^2 = \lambda^2(H^2 - K_e)$  and  $\kappa - 4\tau^2 \neq 0$ . Thus

$$\begin{aligned} \|\nabla v\|^2 &= \frac{4}{\lambda}|v_z|^2 = (2H^2 - K_e + \tau^2)(1 - v^2) + \frac{\kappa - 4\tau^2}{4}(1 - v^2)^2 \\ &\quad + 4\left(\frac{H^2 + \tau^2}{\kappa - 4\tau^2}\right)(H^2 - K_e) - \frac{q}{\kappa - 4\tau^2}, \end{aligned}$$

and finally, re-ordering in terms of  $H^2 - K_e$ , we obtain the first expression.

Next, by differentiating (2.5) with respect to  $\bar{z}$  and using (2.7), (2.4) and (2.3), one gets

$$v_{z\bar{z}} = -(\kappa - 4\tau^2)v|A|^2 - \frac{2}{\lambda}|p|^2v - \frac{H^2 + \tau^2}{2}\lambda v.$$

Then, from (2.6),

$$v_{z\bar{z}} = -\frac{\lambda v}{4} \left( (\kappa - 4\tau^2)(1 - v^2) + \frac{8|p|^2}{\lambda^2} + 2(H^2 + \tau^2) \right),$$

thus

$$\Delta v = \frac{4}{\lambda}v_{z\bar{z}} = -\left( (\kappa - 4\tau^2)(1 - v^2) + 2(H^2 - K_e) + 2(H^2 + \tau^2) \right)v.$$

Finally,

$$\Delta \ln q = \Delta \ln \frac{4|Q|^2}{\lambda^2} = -2\Delta \ln \lambda = 4K,$$

where we have used that  $Q dz^2$  is holomorphic and the expression of the Gaussian curvature in terms of a conformal parameter. □

**Remark 2.1.** Note that (2.9) is nothing but the Jacobi equation for the Jacobi field  $v$ .

Next, we recall a definition in these homogeneous spaces.

**Definition 2.1.** We say that  $\Sigma \subset \mathbb{E}(\kappa, \tau)$  is a vertical cylinder over  $\alpha$  if  $\Sigma = \pi^{-1}(\alpha)$ , where  $\alpha$  is a curve on  $\mathbb{M}^2(\kappa)$ .

It is not hard to verify that if  $\alpha$  is a complete curve of geodesic curvature  $2H$  on  $\mathbb{M}^2(\kappa)$ , then  $\Sigma = \pi^{-1}(\alpha)$  is complete and has constant mean curvature  $H$ . Moreover, these cylinders are characterized by  $v \equiv 0$ .

We now state two results about the classification of  $H$ -surfaces. They will be used in Sections 3 and 4, but we prove them in Section 5 and Section 6 for the sake of clarity. The first one concerns  $H$ -surfaces for which the angle function is constant. However, we need to introduce a family of surfaces that appear in the classification.

**Definition 2.2.** Denote by  $\mathcal{S}_{\kappa,\tau}$  a family of complete  $H$ -surfaces in  $\mathbb{E}(\kappa, \tau)$ ,  $\kappa < 0$ , satisfying for any  $\Sigma \in \mathcal{S}_{\kappa,\tau}$ :

- $4H^2 + \kappa < 0$ .
- $q$  vanishes identically on  $\Sigma \in \mathcal{S}_{\kappa,\tau}$ , i.e.,  $\Sigma$  is invariant by a one parameter family of isometries.
- $0 < v^2 < 1$  is constant along  $\Sigma$ .
- $K_e = -\tau^2$  and  $K = (\kappa - 4\tau^2)v^2 < 0$  are constants along  $\Sigma$ .

An anonymous referee indicated to us the preprint “Hypersurfaces with a parallel higher fundamental form” by S. Verpoort who observed that we mistakenly omitted the surfaces  $S_{\kappa, \tau}$  in a first draft of this paper.

**Theorem 2.2.** *Let  $\Sigma \subset \mathbb{E}(\kappa, \tau)$  be a complete  $H$ -surface with constant angle function. Then  $\Sigma$  is either a vertical cylinder over a complete curve of curvature  $2H$  on  $\mathbb{M}^2(\kappa)$ , a slice in  $\mathbb{H}^2 \times \mathbb{R}$  or  $\mathbb{S}^2 \times \mathbb{R}$ , or  $\Sigma \in S_{\kappa, \tau}$  with  $\kappa < 0$ .*

**Remark 2.2.** Theorem 2.2 improves Lemma 2.3 in [ER] for surfaces in  $\mathbb{H}^2 \times \mathbb{R}$ .

Of special interest for us are those  $H$ -surfaces for which the Abresch–Rosenberg differential is constant.

**Theorem 2.3.** *Let  $\Sigma \subset \mathbb{E}(\kappa, \tau)$  be a complete  $H$ -surface with  $q$  constant.*

- *If  $q = 0$ , then  $\Sigma$  is invariant by a one-parameter group of isometries of  $\mathbb{E}(\kappa, \tau)$ , and if  $H = 0 = \tau$ , then  $\Sigma$  is a slice in  $\mathbb{H}^2 \times \mathbb{R}$  or  $\mathbb{S}^2 \times \mathbb{R}$ .*

*Moreover, the Gauss curvature of these examples is as follows.*

- *If  $4H^2 + \kappa > 0$ , then  $K = 0$ , and they are rotationally invariant spheres.*
- *If  $4H^2 + \kappa = 0$  and  $v \equiv 0$ , then  $K \equiv 0$  and  $\Sigma$  is either a vertical plane in  $\text{Nil}_3$ , or a vertical cylinder over a horocycle in  $\mathbb{H}^2 \times \mathbb{R}$  or  $\overline{\text{PSL}(2, \mathbb{C})}$ .*
- *There exists a point with negative Gauss curvature in the remaining cases.*
- *If  $q \neq 0$  on  $\Sigma$ , then  $\Sigma$  is a vertical cylinder over a complete curve of curvature  $2H$  on  $\mathbb{M}^2(\kappa)$ .*

### 3. Complete $H$ -surfaces $\Sigma$ with $K \geq 0$

Here we prove

**Theorem 3.1.** *Let  $\Sigma \subset \mathbb{E}(\kappa, \tau)$  be a complete  $H$ -surface with  $K \geq 0$ . Then,  $\Sigma$  is either a rotational sphere (in particular,  $4H^2 + \kappa > 0$ ), or a complete vertical cylinder over a complete curve of geodesic curvature  $2H$  on  $\mathbb{M}^2(\kappa)$ .*

*Proof.* The proof goes as follows: First, we prove that  $\Sigma$  is a topological sphere or a complete non-compact parabolic surface. We show that when the surface is a topological sphere then it is a rotational sphere. If  $\Sigma$  is a complete non-compact parabolic surface, we prove that it is a vertical cylinder by means of Theorem 2.3.

Since  $K \geq 0$  and  $\Sigma$  is complete, Lemma 5 in [KO] implies that  $\Sigma$  is either a sphere or non-compact and parabolic.

If  $\Sigma$  is a sphere, then it is a rotational example (see [AR2] or [AR]). Thus, we can assume that  $\Sigma$  is non-compact and parabolic.

We can assume that  $q$  does not vanish identically in  $\Sigma$ . If  $q$  does vanish, then  $\Sigma$  is either a vertical cylinder over a straight line in  $\text{Nil}_3$  or a vertical cylinder over a horocycle in  $\mathbb{H}^2 \times \mathbb{R}$  or  $\widetilde{\text{PLS}}(2, \mathbb{C})$ . Note that we have used here that  $K \geq 0$  and Theorem 2.3.

On the one hand, from the Gauss equation (2.2)

$$0 \leq K = K_e + \tau^2 + (\kappa - 4\tau^2)v^2 \leq K_e + \tau^2 + |\kappa - 4\tau^2|,$$

hence

$$H^2 - K_e \leq H^2 + \tau^2 + |\kappa - 4\tau^2|. \tag{3.1}$$

On the other hand, using the very definition of  $Q dz^2$ , (3.1) and the inequality  $|\xi_1 + \xi_2|^2 \leq 2(|\xi_1|^2 + |\xi_2|^2)$  for  $\xi_1, \xi_2 \in \mathbb{C}$ , we obtain

$$\begin{aligned} \frac{q}{2} &= \frac{2|Q|^2}{\lambda^2} \leq 4(H^2 + \tau^2)\frac{4|p|^2}{\lambda^2} + (\kappa - 4\tau^2)^2\frac{4|A|^4}{\lambda^2} \\ &= 4(H^2 + \tau^2)(H^2 - K_e) + \frac{(\kappa - 4\tau^2)^2}{4}(1 - v^2)^2 \\ &\leq 4(H^2 + \tau^2)(H^2 - K_e) + \frac{(\kappa - 4\tau^2)^2}{4} \\ &\leq 4(H^2 + \tau^2)(H^2 + \tau^2 + |\kappa - 4\tau^2|) + \frac{(\kappa - 4\tau^2)^2}{4}. \end{aligned}$$

So, from (2.10),  $\Delta \ln q = 4K \geq 0$  and  $\ln q$  is a bounded subharmonic function on a non-compact parabolic surface  $\Sigma$  and since the value  $-\infty$  is allowed at isolated points (see [AS]),  $q$  is a positive constant (recall that we are assuming that  $q$  does not vanish identically). Therefore, Theorem 2.3 gives the result.  $\square$

#### 4. Complete $H$ -surfaces $\Sigma$ with $K \leq 0$

**Theorem 4.1.** *Let  $\Sigma \subset \mathbb{E}(\kappa, \tau)$  be a complete  $H$ -surface with  $K \leq 0$  and  $H^2 + \tau^2 - |\kappa - 4\tau^2| > 0$ . Then,  $\Sigma$  is a complete vertical cylinder over a complete curve of geodesic curvature  $2H$  on  $\mathbb{M}^2(\kappa)$ .*

*Proof.* We divide the proof into two cases,  $\kappa - 4\tau^2 < 0$  and  $\kappa - 4\tau^2 > 0$ .

*Case  $\kappa - 4\tau^2 < 0$ :* On the one hand, since  $K \leq 0$ , we have

$$H^2 - K_e \geq H^2 + \tau^2 + (\kappa - 4\tau^2)v^2 \geq H^2 + \kappa - 3\tau^2,$$

from the Gauss equation (2.2). Therefore, from (2.8) and  $\kappa - 4\tau^2 < 0$ , we obtain:

$$\begin{aligned} q &\geq 4(H^2 + \tau^2)(H^2 - K_e) + (\kappa - 4\tau^2)(1 - \nu^2) \\ &\quad \cdot \left( H^2 + \tau^2 + H^2 - K_e + \frac{\kappa - 4\tau^2}{4}(1 - \nu^2) \right) \\ &= (H^2 - K_e)(4H^2 + 4\tau^2 + (\kappa - 4\tau^2)(1 - \nu^2)) \\ &\quad + (H^2 + \tau^2)(\kappa - 4\tau^2)(1 - \nu^2) + \frac{(\kappa - 4\tau^2)^2}{4}(1 - \nu^2)^2 \\ &\geq (H^2 + \tau^2 + (\kappa - 4\tau^2)\nu^2)(4H^2 + 4\tau^2 + (\kappa - 4\tau^2)(1 - \nu^2)) \\ &\quad + (H^2 + \tau^2)(\kappa - 4\tau^2)(1 - \nu^2) + \frac{(\kappa - 4\tau^2)^2}{4}(1 - \nu^2)^2; \end{aligned}$$

note that the last inequality holds since  $4H^2 + 4\tau^2 + (\kappa - 4\tau^2)(1 - \nu^2) \geq 4H^2 + \kappa > 0$ .  $4H^2 + \kappa > 0$  follows from

$$0 < 4(H^2 + \tau^2) - |\kappa - 4\tau^2| = 4H^2 + \kappa.$$

Set  $a := H^2 + \tau^2$  and  $b := \kappa - 4\tau^2$ . Define the real smooth function  $f: [-1, 1] \rightarrow \mathbb{R}$  as

$$f(x) = (a + bx^2)(4a + b(1 - x^2)) + ab(1 - x^2) + \frac{b^2}{4}(1 - x^2)^2. \quad (4.1)$$

Note that  $q \geq f(\nu)$  on  $\Sigma$ ,  $f(\nu)$  is just the last part in the above inequality involving  $q$ . It is easy to verify that the only critical point of  $f$  in  $(-1, 1)$  is  $x = 0$ . Moreover,

$$f(0) = (4a + b)^2/4 > 0 \quad \text{and} \quad f(\pm 1) = 4a(a + b) > 0.$$

Actually,  $f: \mathbb{R} \rightarrow \mathbb{R}$  has two others critical points,  $x = \pm \sqrt{\frac{4a+b}{3|b|}}$ , but here we have used that

$$\frac{4a + b}{3|b|} > 1,$$

since  $0 < 4(H^2 + \kappa - 3\tau^2) = (4H^2 + \kappa) - 3|\kappa - 4\tau^2| = (4a + b) - 3|b|$ .

So, set  $c = \min \{f(0), f(\pm 1)\} > 0$ , then

$$q \geq f(\nu) \geq c > 0.$$

Now, from (2.10) and  $q \geq c > 0$  on  $\Sigma$ , it follows that  $ds^2 = \sqrt{q}I$  is a complete flat metric on  $\Sigma$  and

$$\Delta^{ds^2} \ln q = \frac{1}{\sqrt{q}} \Delta \ln q = \frac{4K}{\sqrt{q}} \leq 0.$$

Since  $q$  is bounded below by a positive constant and  $(\Sigma, ds^2)$  is parabolic, then  $\ln q$  is constant which implies that  $q$  is a positive constant. Thus, the result follows from Theorem 2.3. The case  $\kappa - 4\tau^2 < 0$  is proved.

Case  $\kappa - 4\tau^2 > 0$ : Set  $w_1 := 2(H + i\tau)\frac{\rho}{\lambda}$  and  $w_2 := (\kappa - 4\tau^2)\frac{A^2}{\lambda}$ , i.e.,  $q = 4|w_1 - w_2|^2$ . Then

$$|w_1|^2 = (H^2 + \tau^2)(H^2 - K_e) \geq (H^2 + \tau^2)^2,$$

$$|w_2|^2 = \frac{(\kappa - 4\tau^2)^2}{16}(1 - \nu^2)^2 \leq \left(\frac{\kappa - 4\tau^2}{4}\right)^2,$$

where we have used that  $H^2 - K_e \geq H^2 + \tau^2 + (\kappa - 4\tau^2)\nu^2 \geq H^2 + \tau^2$ , since  $K \leq 0$  and  $\kappa - 4\tau^2 > 0$ .

We recall a well-known inequality for complex numbers. Let  $\xi_1, \xi_2 \in \mathbb{C}$ , then  $|\xi_1 + \xi_2|^2 \geq ||\xi_1| - |\xi_2||^2$ . Thus,

$$\begin{aligned} \frac{1}{4}q &\geq ||w_1| - |w_2||^2 \geq \left| (H^2 + \tau^2) - \frac{|\kappa - 4\tau^2|}{4} \right|^2 \\ &= \frac{1}{16} |4(H^2 + \tau^2) - |\kappa - 4\tau^2||^2 > 0. \end{aligned}$$

So, as  $q$  is bounded below by a positive constant, then, arguing as in the previous case,  $q$  is a constant. Thus, the result follows from Theorem 2.3. The case  $\kappa - 4\tau^2 > 0$  is proved.  $\square$

**Remark 4.1.** Note that in the above theorem, in the case  $\kappa - 4\tau^2 > 0$ , we only need to assume that  $4(H^2 + \tau^2) - |\kappa - 4\tau^2| > 0$ .

### 5. Complete $H$ -surfaces with constant angle function

We classify here the complete  $H$ -surfaces in  $\mathbb{E}(\kappa, \tau)$  with constant angle function. The purpose is to take advantage of this classification result in the next section.

**Theorem 2.2.** *Let  $\Sigma \subset \mathbb{E}(\kappa, \tau)$  be a complete  $H$ -surface with constant angle function. Then  $\Sigma$  is either a vertical cylinder over a complete curve of curvature  $2H$  on  $\mathbb{M}^2(\kappa)$ , a slice in  $\mathbb{H}^2 \times \mathbb{R}$  or  $S^2 \times \mathbb{R}$ , or  $\Sigma \in \mathcal{S}_{\kappa, \tau}$  with  $\kappa < 0$  (see Definition 2.2).*

*Proof.* We can assume that  $\nu \leq 0$ . We will divide the proof into three cases:

- $\nu = 0$ : In this case,  $\Sigma$  must be a vertical cylinder over a complete curve of geodesic curvature  $2H$  on  $\mathbb{M}^2(\kappa)$ .

- $\nu = -1$ : From (2.4),  $\tau = 0$  and  $H = 0$ , then  $\Sigma$  is a slice in  $\mathbb{H}^2 \times \mathbb{R}$  or  $\mathbb{S}^2 \times \mathbb{R}$ .
- $-1 < \nu < 0$ : We prove here that  $\Sigma \in \mathcal{S}_{\kappa, \tau}$  with  $\kappa < 0$ . From (2.5) we have

$$(H - i\tau)A = -\frac{2p}{\lambda}\bar{A}, \quad (5.1)$$

then

$$H^2 + \tau^2 = \frac{4|p|^2}{\lambda^2} = H^2 - K_e$$

since  $|A|^2 \neq 0$  from (2.6), so  $K_e = -\tau^2$  on  $\Sigma$ .

Thus, from (2.9), we have

$$4H^2 + 4\tau^2 + (\kappa - 4\tau^2)(1 - \nu^2) = 0. \quad (5.2)$$

Now, using the definition of  $q$ , (5.1), (5.2) and  $K_e = -\tau^2$ , we have

$$\begin{aligned} q &= \frac{4|Q|^2}{\lambda^2} = 4(H^2 + \tau^2)\frac{4|p|^2}{\lambda^2} + (\kappa - 4\tau^2)^2\frac{4|A|^4}{\lambda^2} \\ &\quad - 4\frac{\kappa - 4\tau^2}{\lambda^2} (2(H + i\tau)p\bar{A}^2 + 2(H - i\tau)\bar{p}A^2) \\ &= 4(H^2 + \tau^2)(H^2 - K_e) + (\kappa - 4\tau^2)^2\frac{(1 - \nu^2)^2}{4} \\ &\quad + 2(\kappa - 4\tau^2)(1 - \nu^2)(H^2 + \tau^2) \\ &= \frac{1}{4} (4H^2 + (\kappa - 4\tau^2)(1 - \nu^2) + 4\tau^2)^2 = 0, \end{aligned}$$

that is,  $q$  vanishes identically on  $\Sigma$ . Moreover, from (5.2), we can see that  $4H^2 + \kappa < 0$ , that is,  $\kappa < 0$ . Therefore,  $\Sigma \in \mathcal{S}_{\kappa, \tau}$ ,  $\kappa < 0$ .  $\square$

## 6. Complete $H$ -surfaces with $q$ constant

Here, we prove the classification result for complete  $H$ -surfaces in  $\mathbb{E}(\kappa, \tau)$  employed in the proof of Theorem 3.1 and Theorem 4.1.

**Theorem 2.3.** *Let  $\Sigma \subset \mathbb{E}(\kappa, \tau)$  be a complete  $H$ -surface with  $q$  constant.*

- *If  $q = 0$  on  $\Sigma$ , then  $\Sigma$  is either a slice in  $\mathbb{H}^2 \times \mathbb{R}$  or  $\mathbb{S}^2 \times \mathbb{R}$  if  $H = 0 = \tau$ , or  $\Sigma$  is invariant by a one-parameter group of isometries of  $\mathbb{E}(\kappa, \tau)$ .*

*Moreover, the Gauss curvature of these examples is as follows.*

- *If  $4H^2 + \kappa > 0$ , then  $K > 0$  they are the rotationally invariant spheres.*

- If  $4H^2 + \kappa = 0$  and  $v \equiv 0$ , then  $K \equiv 0$  and  $\Sigma$  is either a vertical plane in  $\text{Nil}_3$ , or a vertical cylinder over a horocycle in  $\mathbb{H}^2 \times \mathbb{R}$  or  $\overline{\text{PSL}(2, \mathbb{C})}$ .
- There exists a point with negative Gauss curvature in the remaining cases.
- If  $q \neq 0$  on  $\Sigma$ , then  $\Sigma$  is a vertical cylinder over a complete curve of curvature  $2H$  on  $\mathbb{M}^2(\kappa)$ .

The case  $q = 0$  has been treated extensively when the target manifold is a product space, but it has not been established explicitly when  $\tau \neq 0$ . So, we assemble the results in [AR], [AR2] for the reader's convenience.

**Lemma 6.1.** *Let  $\Sigma \subset \mathbb{E}(\kappa, \tau)$  be a complete  $H$ -surface whose Abresch–Rosenberg differential vanishes. Then  $\Sigma$  is either a slice in  $\mathbb{H}^2 \times \mathbb{R}$  or  $\mathbb{S}^2 \times \mathbb{R}$  if  $H = 0 = \tau$ , or  $\Sigma$  is invariant by a one-parameter group of isometries of  $\mathbb{E}(\kappa, \tau)$ .*

*Moreover, the Gauss curvature of these examples is as follows.*

- If  $4H^2 + \kappa > 0$ , then  $K > 0$  they are the rotationally invariant spheres.
- If  $4H^2 + \kappa = 0$  and  $v \equiv 0$ , then  $K \equiv 0$  and  $\Sigma$  is either a vertical plane in  $\text{Nil}_3$ , or a vertical cylinder over a horocycle in  $\mathbb{H}^2 \times \mathbb{R}$  or  $\overline{\text{PSL}(2, \mathbb{C})}$ .
- There exists a point with negative Gauss curvature in the remaining cases.

*Proof.* The idea of the proof for product spaces that we use below can be found in [dCF] and [FM].

If  $H = 0 = \tau$ , from the definition of the Abresch–Rosenberg differential, we have

$$0 = -(\kappa - 4\tau)A^2,$$

that is,  $v^2 = \pm 1$  using (2.6). Thus,  $\Sigma$  is a slice in  $\mathbb{H}^2 \times \mathbb{R}$  or  $\mathbb{S}^2 \times \mathbb{R}$ .

If  $H \neq 0$  or  $\tau \neq 0$ , we have

$$2(H + i\tau)p = (\kappa - 4\tau^2)A^2, \tag{6.1}$$

from where we obtain, taking modulus,

$$H^2 - K_e = \frac{(\kappa - 4\tau^2)^2(1 - v^2)^2}{16(H^2 + \tau^2)}. \tag{6.2}$$

Inserting (6.1) in (2.5),

$$(H + i\tau)v_z = -\frac{1}{4}(4H^2 + \kappa - (\kappa - 4\tau^2)v^2)A,$$

and taking modulus,

$$|v_z|^2 = g(v)^2|A|^2, \quad g(v) = \frac{4H^2 + \kappa - (\kappa - 4\tau^2)v^2}{4\sqrt{H^2 + \tau^2}}. \tag{6.3}$$

Assume that  $\nu$  is not constant. Let  $p \in \Sigma$  be a point where  $\nu_z(p) \neq 0$  and let  $\mathcal{U}$  be a neighborhood of that point  $p$  where  $\nu_z \neq 0$  (we can assume  $\nu^2 \neq 1$  at  $p$ ). In particular,  $g(\nu) \neq 0$  in  $\mathcal{U}$  from (6.3). Now, inserting (6.3) in (2.6), we obtain

$$\lambda = \frac{4|\nu_z|^2}{(1-\nu^2)g(\nu)^2}. \quad (6.4)$$

Thus, putting (6.2) and (6.4) in the Jacobi equation (2.9)

$$\nu_{z\bar{z}} = -2 \frac{\nu|\nu_z|^2}{1-\nu^2}. \quad (6.5)$$

So, define the real function  $s := \operatorname{arctgh}(\nu)$  on  $\mathcal{U}$ . Such a function is harmonic by means of (6.5), thus we can consider a new conformal parameter  $w$  for the first fundamental form so that  $s = \operatorname{Re}(w)$ ,  $w = s + it$ .

Since  $\nu = \operatorname{tgh}(s)$  by the definition of  $s$ , we have that  $\nu \equiv \nu(s)$ , i.e., it only depends on one parameter. Thus, we have  $\lambda \equiv \lambda(s)$  and  $T \equiv T(s)$  from (6.4) and (6.3) respectively, and  $p \equiv p(s)$  by the definition of the Abresch–Rosenberg differential. That is, all the fundamental data of  $\Sigma$  depend only on  $s$ .

Now, let  $\mathcal{U}$  be a simply connected domain on  $\Sigma$  and  $\mathcal{V} \subset \mathbb{R}^2$  a simply connected domain of a surface  $S$  so that  $\psi_0: \mathcal{V} \rightarrow \mathcal{U} \subset \mathbb{E}(\kappa, \tau)$ . We parametrize  $\mathcal{V}$  by the parameters  $(s, t)$  obtained above. Then, the fundamental data (see [FM], Theorem 2.3)  $\{\lambda_0, p_0, T_0, \nu_0\}$  of  $\psi_0$  are given by

$$\begin{cases} \lambda_0(s, t) = \lambda(s), \\ p_0(s, t) = p(s), \\ T_0(s, t) = a(s)\partial_s, \\ \nu_0(s, t) = \nu(s), \end{cases}$$

where  $a(s)$  is a smooth function.

Let  $\bar{t} \in \mathbb{R}$  and let  $\mathbf{i}_{\bar{t}}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the diffeomorphism given by

$$\mathbf{i}_{\bar{t}}(s, t) := (s, t + \bar{t}),$$

and define  $\psi_{\bar{t}} := \psi_0 \circ \mathbf{i}_{\bar{t}}$ . Then, the fundamental data  $\{\lambda_{\bar{t}}, p_{\bar{t}}, T_{\bar{t}}, \nu_{\bar{t}}\}$  of  $\psi_{\bar{t}}$  are given by

$$\begin{cases} \lambda_{\bar{t}}(s, t) = \lambda(s), \\ p_{\bar{t}}(s, t) = p(s), \\ T_{\bar{t}}(s, t) = a(s)\partial_s, \\ \nu_{\bar{t}}(s, t) = \nu(s), \end{cases}$$

that is, both fundamental data match at any point  $(s, t) \in \mathcal{V}$ . Therefore, using [D], Theorem 4.3, there exists an ambient isometry  $\mathcal{I}_{\bar{t}}: \mathbb{E}(\kappa, \tau) \rightarrow \mathbb{E}(\kappa, \tau)$  so that

$$\mathcal{I}_{\bar{t}} \circ \psi_0 = \psi_0 \circ \mathbf{i}_{\bar{t}} \quad \text{for all } \bar{t} \in \mathbb{R},$$

thus the surface is invariant by a one parameter group of isometries.

Let us prove the claim about the Gauss curvature. Using the Gauss equation (2.2) in (6.2), one gets

$$H^2 + \tau^2 + (\kappa - 4\tau^2)v^2 - K = \frac{(\kappa - 4\tau^2)^2(1 - v^2)^2}{16(H^2 + \tau^2)}.$$

Set  $a := 4(H^2 + \tau^2)$  and  $b := \kappa - 4\tau^2$ , then one can check easily that the above equality can be expressed as

$$4aK = a^2 - b^2 + (2a + b)^2 - (2a + b(1 - v^2))^2. \tag{6.6}$$

So, if  $4H^2 + \kappa > 0$  then  $a > |b|$  and  $K > 0$ , that is,  $\Sigma$  is a topological sphere since it is complete. If  $4H^2 + \kappa = 0$ ,  $a = -b$  and the equation reads as

$$4aK = a^2(1 - (1 + v^2)^2),$$

that is,  $\Sigma$  has a point with negative Gauss curvature unless  $v \equiv 0$ .

If  $4H^2 + \kappa < 0$ , one can check that  $a^2 - b^2 = (a - b)(a + b) < 0$  since  $a + b > 0$  and  $a - b < 0$ . So, if  $\inf_{\Sigma}\{v^2\} = 0$  then, from (6.6),  $\Sigma$  has a point with negative curvature. Therefore, to finish this lemma, we shall prove the following

*Claim.* There are no complete constant mean curvature surfaces in  $\mathbb{E}(\kappa, \tau)$  with  $4H^2 + \kappa < 0$ ,  $q \equiv 0$ ,  $K \geq 0$ , and  $\inf\{v^2\} = c > 0$ .

*Proof of the Claim.* Assume such a surface  $\Sigma$  exists. Since we are assuming that  $K \geq 0$  and  $\Sigma$  is complete, then  $\Sigma$  is parabolic and noncompact. If  $\Sigma$  were compact we would have a contradiction with the fact that  $\inf_{\Sigma}\{v^2\} = c > 0$  and  $4H^2 + \kappa < 0$ .

Since  $q$  vanishes identically on  $\Sigma$ ,  $\operatorname{arctanh}(v)$  is a bounded harmonic function on  $\Sigma$  and so  $v$  is constant. So, the projection  $\pi : \Sigma \rightarrow \mathbb{M}^2(\kappa)$  is a global diffeomorphism and a quasi-isometry. This is impossible since  $\Sigma$  is parabolic and  $\mathbb{M}^2(\kappa)$ ,  $\kappa < 0$ , is hyperbolic. Therefore, the Claim is proved and so the lemma is proved.  $\square$

*Proof of Theorem 2.3.* We focus on the case  $q \neq 0$  because Lemma 6.1 gives the classification when  $q = 0$ .

Suppose  $v$  is not constant in  $\Sigma$ . Since  $q = c^2 > 0$ , we can consider a conformal parameter  $z$  so that  $\langle \cdot, \cdot \rangle = |dz|^2$  and  $Q dz^2 = c dz^2$  on  $\Sigma$ . Thus,

$$Q = c = 2(H + i\tau)p - (\kappa - 4\tau^2)A^2.$$

First, note that we can assume that  $H \neq 0$  or  $\tau \neq 0$ , otherwise  $v$  would be constant. So, from (2.5), we have

$$(H + i\tau)v_z = -\left(H^2 + \tau^2 + \frac{\kappa - 4\tau^2}{4}(1 - v^2)\right)A - c\bar{A},$$

where we have used  $2(H + i\tau)p = c + (\kappa - 4\tau^2)A^2$ . That is,

$$16(H^2 + \tau^2) \|\nabla v\|^2 = (g(v) + 4c)^2 (1 - v^2), \quad (6.7)$$

where

$$g(v) := 4H^2 + \kappa - (\kappa - 4\tau^2)v^2. \quad (6.8)$$

From (2.10),  $\Sigma$  is flat and  $H^2 - K_e = H^2 + \tau^2 + (\kappa - 4\tau^2)v^2$  by (2.2), joining this last equation to (2.8) we obtain using the definition of  $g(v)$  given in (6.8)

$$\|\nabla v\|^2 = \frac{g(v)^2}{4(\kappa - 4\tau^2)} + v^2 g(v) - \frac{c^2}{\kappa - 4\tau^2}. \quad (6.9)$$

Putting together (6.7) and (6.9) we obtain a polynomial expression in  $v^2$  with coefficients depending on  $a := 4(H^2 + \tau^2)$ ,  $b := \kappa - 4\tau^2$  and  $c$ :

$$P(v^2) := C(a, b, c)v^6 + \text{lower terms} = 0,$$

but one can easily check that the coefficient of  $v^6$  is  $C(a, b, c) = -a^{-1}b^2 \neq 0$ , a contradiction. Thus  $v$  is constant, and so, by means of Theorem 2.2,  $\Sigma$  is a vertical cylinder over a complete curve of curvature  $2H$ .  $\square$

## 7. Appendix

Let  $\Sigma$  be a connected Riemannian surface. We establish in this Appendix a result which we think is of independent interest, concerning differential operators of the form  $\Delta + g$ , acting on  $C^2(\Sigma)$ -functions, where  $\Delta$  is the Laplacian with respect to the Riemannian metric on  $\Sigma$  and  $g \in C^0(\Sigma)$ .

**Lemma 7.1.** *Let  $g \in C^0(\Sigma)$ ,  $v \in C^2(\Sigma)$  such that  $\|\nabla v\|^2 \leq h v^2$  on  $\Sigma$ ,  $h$  is a non-negative continuous function on  $\Sigma$ , and  $\Delta v + g v = 0$  in  $\Sigma$ . Then either  $v$  never vanishes or  $v$  vanishes identically on  $\Sigma$ .*

*Proof.* Set  $\Omega = \{p \in \Sigma : v(p) = 0\}$ . We will show that either  $\Omega = \emptyset$  or  $\Omega = \Sigma$ .

So, let us assume that  $\Omega \neq \emptyset$ . If we prove that  $\Omega$  is an open set then, since  $\Omega$  is closed and  $\Sigma$  is connected,  $\Omega = \Sigma$ . Let  $p \in \Omega$  and  $\mathcal{B}(R) \subset \Sigma$  be the geodesic ball centered at  $p$  of radius  $R$ . Such a geodesic ball is relatively compact in  $\Sigma$ .

Set  $\phi = v^2/2 \geq 0$ . Then

$$\Delta \phi = v \Delta v + \|\nabla v\|^2 = -g v^2 + \|\nabla v\|^2 \leq -2(g - h)\phi,$$

that is,

$$-\Delta \phi - 2(g - h)\phi \geq 0. \quad (7.1)$$

Define  $\beta := \min \{\inf_{\Omega} \{2(g - h)\}, 0\} \leq 0$ . Then,  $\psi = -\phi$  satisfies

$$\Delta\psi + \beta\psi = -\Delta\phi - \beta\phi \geq -\Delta\phi - 2(g - h)\phi \geq 0,$$

where we have used (7.1).

Since we are assuming that  $v$  has a zero at an interior point of  $\mathcal{B}(R)$ ,  $\beta \leq 0$  and  $\psi$  has a non-negative maximum at  $p$ , the Maximum Principle [GT], Theorem 3.5, implies that  $\psi$  is constant and so  $v$  is constant as well, i.e.,  $v \equiv 0$  in  $\mathcal{B}(R)$ . Then  $\mathcal{B}(R) \subset \Omega$ , and  $\Omega$  is an open set. Thus  $\Omega = \Sigma$ .  $\square$

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