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Autor: Ambrosio, Luigi / Katz, Mikhail G.
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Flat currents modulo p in metric spaces and filling radius inequalities

Luigi Ambrosio and Mikhail G. Katz*

Abstract. We adapt the theory of currents in metric spaces, as developed by the first-mentioned author in collaboration with B. Kirchheim, to currents with coefficients in $\mathbb{Z}_p$. We obtain isoperimetric inequalities $\operatorname{mod}(p)$ in Banach spaces and we apply these inequalities to provide a proof of Gromov's filling radius inequality which applies also to nonorientable manifolds. With this goal in mind, we use the Ekeland principle to provide quasi-minimizers of the mass $\operatorname{mod}(p)$ in the homology class, and use the isoperimetric inequality to give lower bounds on the growth of their mass in balls.

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Our aim is the extension of the theory of rectifiable currents in metric and infinite-dimensional Banach spaces to the case of coefficients in $\mathbb{Z}_p$. Such an extension can be applied to give transparent proofs of Gromov's filling radius and filling volume inequalities which apply to nonorientable manifolds, as well.

1. Current history

Following the classical paper by H. Federer and W. Fleming [21], as well as Federer's treatise [20] on the theory of currents, in the last few years the theory has undergone two important developments:

- B. White's theory [46], inspired by Fleming's paper [23], of rectifiable flat chains with coefficients in a general group, in Euclidean spaces;
- the theory developed by the first author and B. Kirchheim in [3], and inspired by E. De Giorgi [13], of real and integer rectifiable currents in general metric spaces.

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A unified picture (general coefficients in general spaces) seemed to be still missing, but after the completion of this paper we learned of the paper by T. De Pauw and R. Hardt [16] and the earlier paper by T. Adams [1], developed in the same spirit of the Fleming–White theory (but with no discussion of isoperimetric inequalities). Another valuable contribution to the literature came even more recently with S. Wenger's papers [44], [45] on the isoperimetric inequalities. The classical approach [21] to proving these inequalities in arbitrary dimension and codimension goes back to the deformation theorem. A different technique was introduced by M. Gromov [24] and fully exploited in [3]. It is based on the fact that, in finite-dimensional spaces, one can prove isoperimetric inequalities independent not only of the codimension, but also of the norm in the space. Such a technique allows one to prove the inequality in suitable metric spaces and in infinite-dimensional spaces, provided a finite-dimensional approximation scheme exists.

Wenger [44] introduced a new "global" technique, based on covering arguments and independent of deformation theorems and finite-dimensional schemes. His technique allows one to treat also the case of Banach spaces to which the results in [3] do not apply. White's isoperimetric inequality [48] applies to chains in finite-dimensional Banach spaces with coefficients in general groups. However, White's inequality is based on the deformation theorem in the corresponding Euclidean space, and therefore does not provide universal constants depending only on the dimension of the chain.

In the present text, we follow the approach of [20] (see also W. Ziemer [49] for the case $p = 2$, still in Euclidean space) to achieve an extension of the metric theory of [3] to currents with coefficients in $\mathbb{Z}_p$: the initial idea is simply to identify currents which differ by pT, with T integer rectifiable. But then, since we want this equivalence to be stable under the action of the boundary operator, it turns out that larger equivalence classes and a suitable topology (induced by the so-called flat distances) are needed. In any case, our currents arise as quotient classes $[T]$ of currents T akin to those considered in [3], which extend to general spaces those of the Federer–Fleming theory.

In the simplest case $p = 2$, it is well-known that one can use currents modulo 2 to describe possibly nonorientable manifolds. In particular, we will prove in Theorem 13.1 that to any compact n-dimensional Riemannian manifold without boundary M one can associate a canonical equivalence class

$$[[\![M]\!]]$$

(notice that the current $[\![M]\!]$ itself is by no means canonical) whose boundary is zero, still mod(2). In particular, after embedding M in a linear space, we can consider chains whose boundary mod(2) coincides with the image of $[\![M]\!]$.

2. Gromov's inequalities

A quarter century ago, M. Gromov [24] initiated the modern period in systolic geometry by proving a curvature-free 1-systolic lower bound for the total volume of an essential Riemannian manifold M of dimension n. Recall that the 1-systole, denoted "Sys", of a space is the least length of a loop that cannot be contracted to a point in the space. Here the term "curvature-free" refers to a bound independent of curvature invariants, with a constant depending on the dimension of M (and possibly on the topology of M), but not on its geometry. Such a bound is given by the inequality between the leftmost and the rightmost terms in (2.2) below, and can be thought of as a far-reaching generalisation of Loewner's classical torus inequality

$$\mathrm{Sys}^2 \le \frac{2}{\sqrt{3}}\mathrm{Area}, \tag{2.1}$$

satisfied by every metric on the 2-torus, cf. [42]. It is conjectured that the bound (2.1) is satisfied by every surface of negative Euler characteristic, see [30] for a detailed discussion. Recent publications in systolic geometry include [5], [8], [9], [10], [11], [19], [30], [35], [36], [43], [27], [31], [26].

The main ingredient in the proof of the inequality is Gromov's filling inequality. There is a certain amount of confusion in the literature as to what constitutes Gromov's "filling inequality". Gromov actually proved several inequalities:

- an inequality relating the filling radius and the volume. It is this inequality that's immediately relevant to Gromov's systolic inequality;
- the inequality between the filling volume (an $(n+1)$-dimensional invariant) and the volume (n-dimensional invariant) of M. Such an inequality can be more appropriately referred to as an isoperimetric inequality.

Marcel Berger performed a great deal of propaganda for systolic geometry (see most recently [7], [8]). The success of the field is certainly due to his efforts. In one of his popularisation talks, he presented the following string of three inequalities:

$$\mathrm{Sys} \le 6\,\mathrm{Fillrad} \le \mathrm{Const}\cdot \mathrm{FillVol}^{1/(n+1)} \le \mathrm{Const}\cdot \mathrm{Vol}^{1/n}. \tag{2.2}$$

(Here the last inequality corresponds to the isoperimetric inequality, while the first one is sharp [33].) Berger's presentation was intended for pedagogic purposes, but eventually led to a slight confusion. Namely, this string of inequalities gave the impression that the proof breaks up into three stages, each requiring separate treatment. In reality, the last two inequalities are proved simultaneously. The technique is essentially a more precise version of Federer–Fleming's deformation theorem.

As a matter of fact, proving the isoperimetric inequality alone does not *directly* lead to any simplification of the proof. Consider, for example, the familiar picture of the pseudosphere in $\mathbb{R}^3$, with a cusp along an asymptote given by the z-axis. We think of it as a "filling" of the unit circle in the (x, y)-plane. Alternatively, truncate the pseudosphere at large height $z = H$, to obtain a filling which is topologically a disk. One immediately realizes that the *filling volume* stays uniformly bounded, but the *filling radius* (with respect to this particular filling) tends to infinity.

Gromov's original proof starts by imbedding the manifold M into the space $L^\infty(M)$ of bounded Borel functions on M. Here a point $x \in M$ is sent to the function f_x defined by

$$f_x(y) = \mathrm{dist}(x, y), \tag{2.3}$$

where "dist" is the Riemannian distance function in M. The fact that the space $L^\infty(M)$ is infinite-dimensional has given some readers the impression that infinite-dimensionality of the imbedding is an essential aspect of Gromov's proof of the systolic inequality. In fact, this is not the case. Indeed, we can choose a maximal ϵ-net $N \subset M$ with $|N| < \infty$ points. We choose ϵ satisfying $\epsilon < \frac{1}{10}\mathrm{Sys}(M)$. This results in an imbedding

$$M \to \ell^\infty(N) \tag{2.4}$$

where the systole goes down by a factor at most 5, see [34], p. 97. Thus the systolic problem can easily be reduced to finite-dimensional imbeddings. Similarly, by choosing a sufficiently fine ϵ-net, one can force the map (2.4) to be $(1+\epsilon)$–bi-Lipschitz, for all $\epsilon > 0$ (see [31] and Proposition 5.1 below). Hence finite-dimensional approximations work well for our filling radius, as well, provided the estimates one proves are independent of N.

Gromov's original proof is difficult (a recent generalisation is provided by L. Guth in [25]; see also [26] and [32]). Only the experts possess a complete understanding of the proof. It would thus be desirable to write down a detailed proof of Gromov's influential theorem, and to sort out some of the confusion in the literature.

3. Summary of main results

In Section 6, we introduce flat currents and flat currents modulo p, following the traditional procedure in [49], [20]. The only difference is that the initial objects we complete with respect to the flat topology are the currents of [3], whose main properties are recalled in the appendix. Then, we see that in this class a slice operator

$$[T] \mapsto \langle [T], u, r\rangle$$

and a boundary operator $[T] \mapsto \partial[T]$ are well defined. This allows us to state a list of properties that a suitable class of currents, together with a suitable notion of mass, should satisfy, as in [45], in order to obtain the isoperimetric inequality. The idea is to start from the 1-dimensional isoperimetric inequality, which needs to be directly checked, and then make a bootstrap argument based on a clever covering argument. Actually, as in [44], we use the covering argument even to establish the 1-dimensional isoperimetric inequality (trivial in the case of Lipschitz images of 1-dimensional simplexes considered in [45], but not trivial in our case). Then, we show in Section 5 and Section 6 that our class of currents, together with a suitable notion of p-mass, denoted by $\mathbf{M}_p$, do satisfy the list of properties, so that an isoperimetric inequality holds in this class.

Definition 3.1. The filling radius

$$r([L], M)$$

of a n-dimensional cycle mod(2) in a space M is the infimum of the numbers $r > 0$ such that, for all Banach spaces F and all isometric embedding i of M into F there exists an $(n+1)$ current $[T]$ mod(2) in F such that $\partial[T] = i_\sharp[L]$ and the support of $[T]$ is contained in the r-neighbourhood of the support of $i_\sharp[L]$.

Of course this definition makes sense only specifying the cycles we are dealing with: they are equivalence classes mod(2) of currents $L \in \mathcal{I}_n(E)$ whose boundary is zero, still mod(2). Analogously, the admissible fillings T are equivalence classes mod(2) of currents in $\mathcal{I}_{n+1}(E)$ whose boundary is equivalent mod(2) to L (see Section 10 for a precise definition of the additive group $\mathcal{I}_n(E)$ of integer rectifiable n-currents in E).

One of the main result of our paper, achieved as a particular case of our Theorem 11.1 below, is the universal upper bound

$$r([L], M) \le c(n)\big[\mathbf{M}_2([L])\big]^{1/n}.$$

When M is a compact Riemannian manifold without boundary, applying this result to the canonical n-cycle $[L] = \big[[\![M]\!]\big]$ in M and setting

$$r(M) = r\big(M, \big[[\![M]\!]\big]\big) \tag{3.1}$$

we obtain the following result.

Theorem 3.2. *For any compact n-dimensional Riemannian manifold without boundary the universal upper bound $r(M) \le c(n)[\mathrm{Vol}(M)]^{1/n}$ holds.*

Remark 3.3. Up to the proof of the isoperimetric inequalities no completeness of our spaces of currents is really needed (closure under the action of the slicing operator suffices). However, the proof of the universal upper bound seems really to require some form of completeness, and justifies the whole mathematical apparatus developed in this paper (however, we left out many mathematical questions concerning currents with coefficients in $\mathbb{Z}_p$ that we plan to investigate in the forthcoming paper [4]). In order to prove our result we use as in [3] the Ekeland principle (valid in complete metric spaces, see Section 12 for a precise statement) to find "quasi-minimizers" of the $\mathbf{M}_p$-mass in the homology class

$$\{[T] : \partial[T] = i_\sharp[L]\}$$

and prove, using the isoperimetric inequality, that any such minimizer has support close to the support of $i_\sharp[L]$. Notice also that the same argument, based on the isoperimetric inequalities, applies to orientable manifolds: in this case the filling radius invariant (possibly a larger one) could also be defined using the currents in [3] and no quotient $\mathrm{mod}(p)$ is needed.

4. Filling radius and systole

The invariant defined in (3.1) is related to the systole by means of the following inequality of Gromov's [24], which turns out to be sharp [33]. Recall that a closed manifold M is called *essential* if it admits a continuous map an Eilenberg–MacLane space $K(\pi, 1)$ such that the induced homomorphism in top-dimensional homology sends the fundamental homology class of M to a nonzero class.

Theorem 4.1 (M. Gromov). *Every essential M satisfies $r(M) \geq \frac{1}{6}\mathrm{Sys}(M)$.*

Proof. The idea of Gromov's proof is to build a retraction skeleton-by-skeleton. We will outline the essential idea of the argument first, so as not to overburden the presentation with technical details, which will be explained later.

By a *strongly isometric imbedding* we mean an imbedding of metric spaces $M \to V$ such that the instrinsic distance in M coincides with the ambient distance in V among points of M.

We can assume without loss of generality that a piecewise linear strongly isometric (up to epsilon) imbedding $M \to \ell^\infty$ satisfies $\dim(\ell^\infty) < \infty$ (see Remark 4.3 and Proposition 5.1). If $6r(M) < \mathrm{Sys}$, we set

$$\epsilon = \frac{1}{10}(\mathrm{Sys} - 6r(M)). \tag{4.1}$$

Consider a triangulation, extending that of (the image of) M, of ℓ^∞ so each simplex has diameter at most ϵ. If C is a current with support in the neighborhood $U_r M$, let C_{fat} be the union of all simplices meeting the support of C. Then C_{fat} lies in the $(r + \epsilon)$-neighborhood of M. Let

$$C_{\mathrm{fat}}^{(k)} \subset C_{\mathrm{fat}}$$

be its k-skeleton. A map

$$f^{(0)}: C_{\text{fat}}^{(0)} \to M$$

on the 0-skeleton is constructed by sending each vertex to a nearest point of M. Next, we extend $f^{(0)}$ to a map

$$f^{(1)}: C_{\text{fat}}^{(1)} \to M$$

by sending each edge to a shortest path joining the images of its endpoints under $f^{(0)}$, in such a way that $f^{(1)}$ is the identity on each edge contained in M itself (here we are assuming that the edges of the triangulation of M are minimizing paths). Since the distances in M coincide with the ambient distances in ℓ^∞, each edge of $C_{\text{fat}}^{(1)}$ is mapped to a path of length at most $(r+\epsilon)+\epsilon+(r+\epsilon)=2r+3\epsilon$. Next, given a 2-simplex abc in $C_{\text{fat}}^{(2)}$, note that its boundary is mapped to a loop L_{abc} of length at most

$$3(2r+3\epsilon)=6r+9\epsilon<\text{Sys},$$

by (4.1), and hence L_{abc} is contractible by definition of the systole. We can therefore extend $f^{(1)}$ to a map

$$f^{(2)}: C_{\text{fat}}^{(2)} \to M$$

whose restriction to the intersection $M^{(2)} \cap C_{\text{fat}}$ is the identity. Every essential manifold M (see [24]) by definition admits a classifying map

$$g: M \to B\pi$$

to the classifying space $B\pi = K(\pi, 1)$, such that

- $\pi = \pi_1(M)$;
- $\pi_i(B\pi) = 0$ for $i \geq 2$,
- $g_*([M]) \neq 0$, where $[M]$ is the fundamental class.

Therefore the composed map

$$g \circ f^{(2)}: C_{\text{fat}}^{(2)} \to B\pi$$

extends to a map

$$h: C_{\text{fat}} \to B\pi$$

in such a way that h coincides with g on $M \subset C_{\text{fat}}^{(2)}$ (see Lemma 4.2 for a more detailed statement in the simplicial category). Since

$$h_*([M]) = g_*([M]) \neq 0,$$

we conclude that the neighborhood C_{fat} cannot contain a current filling M, proving the inequality. $\square$

The proof above is formulated in the category of continuous maps, which is the most convenient one in the context of classifying spaces. On the other hand, a simplicial approximation can easily be constructed if one works with finite skeleta of the classifying space. The following essential lemma is standard.

Lemma 4.2. *Consider finite dimensional simplicial complexes M, Y, Z, where $M \subset Y$ is a subcomplex,* $\dim(Y) = N$, *and $g: M \to Z$ is continuous and simplicial, where $\pi_i(Z) = 0$ for $i = 2, \dots, N-1$. Then given a simplicial map $f^{(2)}: Y^{(2)} \to M$ which is the identity on $M^{(2)}$, the composition $g \circ f^{(2)}$ extends to a simplicial map $h: Y \to Z$ whose restriction to $M \subset Y$ satisfies $h|_M = g$.*

Remark 4.3. Let N be a maximal ϵ-net in M, and consider the finite dimensional imbedding $\iota: M \to \ell^\infty(N)$ whose coordinate functions are the distance functions f_p from points $p \in N$. The imbedding is not quite strongly isometric, since $d(p,q) = \|f_p - f_q\|$ but the functions f_p and f_q only occur as coordinates in ℓ^∞ if p, q belong to the net. However, choosing nearby points p_0, q_0 of the maximal net, we obtain by the triangle inequality

$$d(p,q) \leq d(p_0,q_0) + 2\epsilon = \|f_{p_0} - f_{q_0}\| + 2\epsilon \leq \|\iota(p) - \iota(q)\| + 4\epsilon.$$

Thus upper bounds on distances in ℓ^∞ entail upper bounds on intrinsic distances in M, up to arbitrarily small error. A more detailed discussion may be found in Proposition 5.1.

Remark 4.4 (Gromov's scheme). Gromov's scheme, outlined in Berger [6], p. 298, is to fill a manifold $M = M^d$ in ℓ^∞ by a minimal $(d+1)$-submanifold N. Next, N contains a point x at distance at least r from each point of M. Since N is minimal, the volume of the distance spheres from x grows sufficiently fast. Finally, the total volume of N is at least that of a ball of radius r in N, hence at least a constant times r^{d+1}. But $\mathrm{Vol}(M) \geq \mathrm{Const} \cdot \mathrm{Vol}^{d/d+1}(N)$ by the isoperimetric inequality for minimal submanifolds (with boundary) in ℓ^∞. Combined with the inequality of Theorem 4.1, this would complete the proof of Gromov's systolic inequality.

Of course, lacking a completeness result, no notion of minimal submanifold in Banach space was available at the time, which accounts in part for the complications in Gromov's original proof [24]. In some sense, the scheme outlined by Berger is made rigorous in the present text, where we do have completeness, cf. Remark 3.3.

5. Approximation by finite-dimensional imbeddings

Proposition 5.1. *Let M be a compact Riemannian manifold without boundary. For every $\varepsilon > 0$, there exists a $(1+\varepsilon)$–bi-Lipschitz finite-dimensional imbedding of M, approximating its isometric imbedding in $L^\infty(M)$.*

Proof. For each $n \in \mathbb{N}$, choose a maximal $\frac{1}{n}$-separated net

$$\mathcal{M}_n \subset M,$$

and imbed M in ℓ^∞ by the distance functions from the points in the net by the 1-Lipschitz map

$$\iota_n \colon M \to \ell^\infty(\mathcal{M}_n). \tag{5.1}$$

If there exists a real $\varepsilon > 0$ such that the inverse of ι_n is not $(1-\varepsilon)^{-1}$–Lipschitz, then there is a pair of points $x_n, y_n \in M$ such that the distance $d(x_n, y_n)$ satisfies

$$|\iota_n(x_n) - \iota_n(y_n)| \le (1-\varepsilon) d(x_n, y_n), \tag{5.2}$$

meaning

$$|d(x_n, z) - d(y_n, z)| \le (1-\varepsilon) d(x_n, y_n) \quad \text{for all } z \in \mathcal{M}_n. \tag{5.3}$$

Since M is compact, we can assume with no loss of generality that $x_n \to x$ and $y_n \to y$, and if $x \neq y$ we can contradict (5.3) by choosing $z_n \in M_n$ at distance less than $1/n$ from x and n large enough. So, $x = y$ and we denote $s_n = d(x_n, y_n) \to 0$.

Since M is compact and locally bi-Lipschitz to an Euclidean space (with Lipschitz constant close to 1 provided we choose sufficiently small neighbourhoods), for any $\delta > 0$ we can find $\bar{b} > 0$ such that all (geodesic) triangles in M with side lengths less than $\bar{b}$ have sum of the internal angles less $2\pi + \delta$; we choose δ in such a way that $1 - \varepsilon/2 < \cos\delta$ and we assume with no loss of generality that $\bar{b} \le \mathrm{InjRad}(M)$.

Let $v_n \in T_{x_n}M$ be the unit vector such that $y_n = \exp_{x_n}(s_n v_n)$, set $q_n := \exp_{x_n}(\frac{1}{2}\bar{b} v_n)$ and denote by $a_n \in \mathcal{M}_n$ a point of the maximal net nearest to q_n. Denoting by α_n be the angle at x_n of the geodesic triangle having a_n, y_n, x_n as vertices,

$$\alpha_n := \angle a_n x_n y_n,$$

we have the Taylor expansion

$$d(a_n, \exp_{x_n}(s v_n)) = d(a_n, x_n) - s\cos\alpha_n + s\omega_n(s) \tag{5.4}$$

where, thanks to the smoothness of d in both variables, $\sup_n |\omega_n(s)|$ is infinitesimal as $s \downarrow 0$. We claim that $\alpha_n < \delta$ for n large enough; indeed, the angle at y_n in the geodesic triangle having a_n, y_n, q_n as vertices tends to 0 because the length of the side from q_n to a_n tends to 0, while the length of the other two sides does not. As a consequence the angle at y_n in the geodesic triangle having a_n, y_n, x_n as vertices tends to π. Since all sides of the latter triangle are shorter than $\bar{b}$ for n large enough, our choice of $\bar{b}$ ensures that the angle α_n is less than δ for n large enough. Putting $s = s_n$ in (5.4) we get

$$\begin{aligned} |d(a_n, y_n) - d(a_n, x_n)| &= s_n\cos\alpha_n + o(s_n) > (1-\varepsilon/2)s_n + s_n\omega_n(s_n) \\ &= (1-\varepsilon/2)s_n + o(s_n) \end{aligned}$$

contradicting (5.3) for n large. □

6. Preliminary definitions

Let (E, d_E) be a metric space and $k \geq 0$ integer. We assume, since this suffices for our purposes, that (E, d_E) is separable; this assumption is needed to avoid subtle measurability problems (assuming that the cardinality of E is an Ulam number this assumption could be avoided, see [20], 2.1.6, and Lemma 2.9 in [3]). We use the standard notation $B_r(x)$ for the open balls in E, $\mathrm{Lip}(E)$ for the space of Lipschitz real-valued functions, relative to d_E, and $\mathrm{Lip}_b(E)$ for bounded Lipschitz functions.

We consider, as in [3], the space $MF_k(E)$ of k-dimensional currents in E. We denote by $\mathbf{M}(T)$ the mass of $T \in MF_k(E)$, possibly infinite. We recall the basic definitions of mass, support, push-forward, restriction, boundary in the appendix.

Spaces of currents in E are defined as in [3], with the same notation, we will only use $\mathcal{I}_k(E)$ (integer rectifiable currents with finite mass) and $\mathbf{I}_k(E)$ (currents in $\mathcal{I}_k(E)$ whose boundary belongs to $\mathcal{I}_{k-1}(E)$), see Section 10. In the sequel $p \geq 2$ is a given integer.

6.1. Flat integer currents. We shall denote by $\mathscr{F}_k(E)$ the currents in $MF_k(E)$ that can by written as $R + \partial S$ with $R \in \mathcal{I}_k(E)$ and $S \in \mathcal{I}_{k+1}(E)$. It is obviously an additive (Abelian) group and

$$T \in \mathscr{F}_k(E) \implies \partial T \in \mathscr{F}_{k-1}(E). \tag{6.1}$$

$\mathscr{F}_k(E)$ is a metric space when endowed with the distance $d(T_1, T_2) = \mathscr{F}(T_1 - T_2)$, where

$$\mathscr{F}(T) := \inf \{\mathbf{M}(R) + \mathbf{M}(S) : R \in \mathcal{I}_k(E),\ S \in \mathcal{I}_{k+1}(E),\ T = R + \partial S\} .$$

The subadditivity of $\mathscr{F}$, namely $\mathscr{F}(nT) \leq n\mathscr{F}(T)$, ensures that d is a distance, and the completeness of the groups $\mathcal{I}_k(E)$, when endowed with the mass norm, ensures that $\mathscr{F}_k(E)$ is complete. Also, whenever $\mathbf{I}_k(E)$ is dense in $\mathcal{I}_k(E)$ (see Proposition 14.7 for sufficient conditions), the subset

$$\{R + \partial S : R \in \mathbf{I}_k(E),\ S \in \mathbf{I}_{k+1}(E)\} \subset \mathbf{I}_k(E)$$

is dense in $\mathscr{F}_k(E)$. For the special class of currents T in $\mathscr{F}_k(E)$ with finite mass the density result can be strengthened: indeed, if $T = T_i + R_i + \partial S_i$ with $T_i \in \mathbf{I}_k(E)$, $R_i \in \mathcal{I}_k(E)$, $S_i \in \mathcal{I}_{k+1}(E)$ and $\mathbf{M}(R_i) + \mathbf{M}(S_i) \to 0$, then Theorem 10.2 gives $S_i \in \mathbf{I}_{k+1}(E)$ (because ∂S_i has finite mass) hence $T_i + \partial S_i \in \mathbf{I}_k(E)$. So, T can be approximated in the stronger mass norm by the currents $T_i + \partial S_i$ and this yields

$$\{T \in \mathscr{F}_k(E) : \mathbf{M}(T) < \infty\} = \mathcal{I}_k(E). \tag{6.2}$$

Notice also that

$$\mathscr{F}(\partial T) \leq \mathscr{F}(T) \quad \text{for all } T \in \mathscr{F}_k(E). \tag{6.3}$$

In addition, since $\partial(\varphi_\sharp S) = \varphi_\sharp(\partial S)$ we get

$$\mathscr{F}(\varphi_\sharp T) \le [\mathrm{Lip}(\varphi)]^k \mathscr{F}(T) \tag{6.4}$$

for all $T \in \mathscr{F}_k(E)$, $\varphi \in \mathrm{Lip}(E, \mathbb{R}^k)$.

It should also be emphasized that the concepts introduced in this section are sensitive to the ambient space, namely if E embeds isometrically in F then, for $T \in \mathscr{F}_k(E)$, $\mathscr{F}(i_\sharp T)$ can well be strictly smaller than $\mathscr{F}(T)$; the same remark applies to the $\mathbf{M}_p$ mass, built in Section 9. This is not the case for the concepts of mass, a genuine isometric invariant, see [3].

6.2. Flat distance modulo p. For $T \in \mathscr{F}_k(E)$ we define

$$\mathscr{F}_p(T) := \inf\left\{\mathscr{F}(T - pQ) : Q \in \mathscr{F}_k(E)\right\}.$$

The definition of $\mathscr{F}$ gives

$$\begin{aligned}\mathscr{F}_p(T) = \inf\big\{\mathbf{M}(R) + \mathbf{M}(S) : T = R + \partial S + pQ,\ R \in \mathscr{I}_k(E),\\ S \in \mathscr{I}_{k+1}(E),\ Q \in \mathscr{F}_k(E)\big\}.\end{aligned}$$

Furthermore, whenever $\mathbf{I}_k(E)$ is dense $\mathscr{F}_k(E)$, both infima are unchanged if Q runs in $\mathbf{I}_k(E)$.

Obviously $\mathscr{F}_p(T) \le \mathscr{F}(T)$, and (6.3) together with (6.1) give

$$\mathscr{F}_p(\partial T) \le \mathscr{F}_p(T), \quad T \in \mathscr{F}_k(E), \tag{6.5}$$

while (6.4) gives

$$\mathscr{F}_p(\varphi_\sharp T) \le [\mathrm{Lip}(\varphi)]^k \mathscr{F}_p(T) \tag{6.6}$$

for all $T \in \mathscr{F}_k(E)$, $\varphi \in \mathrm{Lip}(E, \mathbb{R}^k)$.

We now introduce an equivalence relation $\mathrm{mod}(p)$ in $\mathscr{F}_k(E)$, compatible with the group structure, by saying that $T = \tilde{T}\ \mathrm{mod}(p)$ if $\mathscr{F}_p(T - \tilde{T}) = 0$, and denote by $\mathscr{F}_{p,k}(E)$ the quotient group. Clearly $T = 0\ \mathrm{mod}(p)$ if $T = pQ$ for some $Q \in \mathscr{F}_k(E)$, but the converse implication is not known, not even in Euclidean spaces.

The equivalence classes are closed in $\mathscr{F}_k(E)$ and by (6.5) the boundary operator can be defined also in the quotient spaces $\mathscr{F}_{p,k}(E)$ in such a way that

$$\partial[T] = [\partial T] \in \mathscr{F}_{p,k-1}(E) \quad \text{for all } T \in \mathscr{F}_k(E).$$

The same holds, thanks to (6.6), for the push-forward operator, defined in such a way to commute with the equivalence relation $\mathrm{mod}(p)$. We emphasize that $\mathscr{F}_{p,k}(E)$, when endowed with the distance induced by $\mathscr{F}_p$, is a complete metric space: to see this, let $([T_h]) \subset \mathscr{F}_{p,k}(E)$ be a Cauchy sequence and assume with no loss of generality that

$$\sum_h \mathscr{F}_p(T_{h+1} - T_h) < \infty;$$

we can find $R_h \in \mathcal{I}_k(E)$, $S_h \in \mathcal{I}_{k+1}(E)$ and $Q_h \in \mathcal{F}_k(E)$ such that

$$T_{h+1} = T_h + R_h + \partial S_h + pQ_h \quad \text{and} \quad \sum_{h=1}^{\infty} \mathbf{M}(R_h) + \mathbf{M}(S_h) < \infty.$$

Setting $\widetilde{T}_h := T_h - p\sum_0^{h-1} Q_h$ it follows that $\widetilde{T}_h = T_h \bmod(p)$ and since $\widetilde{T}_{h+1} - \widetilde{T}_h = R_h + \partial S_h$ it follows that $(\widetilde{T}_h)$ is a Cauchy sequence in $\mathcal{F}_k(E)$. Denoting by T its limit, by the inequality $\mathcal{F}_p \leq \mathcal{F}$ we infer $[T_h] = [\widetilde{T}_h] \to [T]$ in $\mathcal{F}_{p,k}(E)$.

7. Restriction, slicing

The restriction and slicing operators can be easily extended to the set $\mathcal{F}_k^*(E)$, defined as the closure in $\mathcal{F}_k(E)$ of currents in $\mathbf{I}_k(E)$, using a completion argument. In the cases considered in Proposition 14.7, this closure coincides with the whole of $\mathcal{F}_k(E)$ and, in any case, it is easily seen that ∂ maps $\mathcal{F}_k^*(E)$ into $\mathcal{F}_{k-1}^*(E)$.

Recall from [3] that, for $u \in \mathrm{Lip}(E)$ and T having finite mass and boundary of finite mass, the slice operator $\langle T, u, r\rangle \in MF_{k-1}(E)$ is defined by

$$\langle T, u, r\rangle := \partial(T \llcorner \{u < r\}) - (\partial T) \llcorner \{u < r\}.$$

Notice that $\partial\langle T, u, r\rangle = -\langle \partial T, u, r\rangle$. It turns out that for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$ $\langle T, u, r\rangle$ has finite mass, and

$$\mathbf{M}(\langle T, u, r\rangle) \leq \mathrm{Lip}(u)\frac{d}{dr}\|T\|(\{u < r\}). \tag{7.1}$$

Now, let T be with finite mass; since $T = R + \partial S$ with $R \in \mathcal{I}_k(E)$ and $S \in \mathcal{I}_{k+1}(E)$ imply that ∂S has finite mass we can apply the slicing operator to S to obtain

$$\begin{aligned} T \llcorner \{u < r\} &= R \llcorner \{u < r\} + (\partial S) \llcorner \{u < r\} \\ &= R \llcorner \{u < r\} + \partial(S \llcorner \{u < r\}) - \langle S, u, r\rangle. \end{aligned}$$

Since $\langle S, u, r\rangle$ belongs to $\mathcal{I}_k(E)$ for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$, thanks to Proposition 10.3, by integration between m and ℓ we obtain

$$\begin{aligned} \int_m^{*\ell} \mathcal{F}(T \llcorner \{u < r\})\,dr \leq \int_m^{\ell} &\mathbf{M}(R \llcorner \{u < r\}) \\ &+ \mathbf{M}(S \llcorner \{u < r\})\,dr + \mathrm{Lip}(u)\|S\|(\{u < \ell\}) \\ \leq\ &(\ell - m)\mathbf{M}(R) + (\ell - m + \mathrm{Lip}(u))\mathbf{M}(S) \end{aligned}$$

where $\int^*$ denoted the upper integral (we use it to avoid the discussion of the measurability of the map $r \mapsto \mathcal{F}(T \llcorner \{u < r\})$). Since R and S are arbitrary we get

$$\int_m^{*\ell} \mathcal{F}(T \llcorner \{u < r\})\,dr \leq (\ell - m + \mathrm{Lip}(u))\mathcal{F}(T). \tag{7.2}$$

Now, let $T \in \mathscr{F}_k(E)$, assume that there exist $T_n \in \mathscr{F}_k(E)$ with finite mass convergent to T in $\mathscr{F}_k(E)$ (this surely holds if $T \in \mathscr{F}_k^*(E)$), with $\sum_n \mathscr{F}(T_n - T) < \infty$, and let $u \in \mathrm{Lip}(E)$. By adding the inequalities (7.2) relative to $T_{n+1} - T_n$, and taking into account the subadditivity of the outer integral and the fact that ℓ and m are arbitrary, we obtain that $(T_{n+1} \llcorner \{u < r\})$ is a Cauchy sequence in $\mathscr{F}_k(E)$ for $\mathscr{L}^1$-a.e. $r \in \mathbb{R}$.

It follows that for any such T we can define

$$T \llcorner \{u < r\} := \lim_{n\to\infty} T_n \llcorner \{u < r\} \in \mathscr{F}_k(E) \tag{7.3}$$

whenever the limit exists. By construction the operator $T \mapsto T \llcorner \{u < r\}$ is additive and (7.2) still holds when $T \in \mathscr{F}_k(E)$. A similar argument shows that this definition is independent, up to Lebesgue negligible sets, on the chosen approximating sequence (T_n), provided the "fast convergence" condition $\sum_n \mathscr{F}(T_n - T) < \infty$ holds.

Having defined the restriction, the slice operator, mapping currents in $\mathscr{F}_k^*(E)$ into currents in $\mathscr{F}_{k-1}^*(E)$, can be again defined by

$$\langle T, u, r\rangle := \partial(T \llcorner \{u < r\}) - (\partial T) \llcorner \{u < r\})$$

whenever the right hand side is defined. We still have the property $\partial\langle T, u, r\rangle = -\langle \partial T, u, r\rangle$.

From (7.2) we immediately get

$$\int_m^{*\ell} \mathscr{F}_p(T \llcorner \{u < r\})\, dr \le (\ell - m + \mathrm{Lip}(u))\mathscr{F}_p(T). \tag{7.4}$$

In particular $\mathscr{F}_p(T) = 0$ implies $\mathscr{F}_p(T \llcorner \{u < r\}) = 0$ for $\mathscr{L}^1$-a.e. $r \in \mathbb{R}$, so that the restriction operator can also be viewed as an operator in the quotient spaces

$$\mathscr{F}_{p,k}^*(E) := \{[T] : T \in \mathscr{F}_k^*(E)\},$$

with the property

$$[T] \llcorner \{u < r\} = [T \llcorner \{u < r\}] \quad \text{for } \mathscr{L}^1\text{-a.e. } r \in \mathbb{R}.$$

Hence, the same holds for the slice operator, satisfying $\partial\langle [T], u, r\rangle = -\langle \partial[T], u, r\rangle$ and

$$\langle [T], u, r\rangle = [\langle T, u, r\rangle] \quad \text{for } \mathscr{L}^1\text{-a.e. } r \in \mathbb{R}.$$

8. Isoperimetric inequalities

In this section we discuss the validity of isoperimetric inequalities mod(p) in suitable subspaces $\mathbf{C}_{p,k}(E) \subset \mathscr{F}_{p,k}^*(E)$ analogous to those valid in the case of currents with integer coefficients. We follow, as in [45], an axiomatic approach: we assume the existence, given these subspaces $\mathbf{C}_{p,k}(E)$, of a notion of p-mass $\mathbf{M}_p\colon \mathbf{C}_{p,k}(E) \to \mathbb{R}$ satisfying the following property:

Definition 8.1 (Additivity). For all $[T] \in \mathbf{C}_{p,k}(E)$ there exists a σ-additive Borel measure $\|T\|_p$ satisfying

$$\mathbf{M}_p([T] \llcorner \{u < r\}) = \|T\|_p(\{u < r\}) \quad \text{for } \mathscr{L}^1\text{-a.e. } r \in \mathbb{R}$$

for all $u \in \mathrm{Lip}(E)$.

Strictly speaking, we should use the notation $\|[T]\|$ to emphasize that the measure depends only on the equivalence class of T, but we opted for a simpler notation.

Then, we assume that $\mathbf{C}_{p,k}(E)$ and $\mathbf{M}_p$ are well-behaved with respect to the slice operator, and satisfy the isoperimetric inequality for 1-dimensional currents and the homogeneous version of the isoperimetric inequality (typically achieved by a simple cone construction):

(i) For $k \geq 1$ the slice operator $\langle [T], u, r\rangle$ maps $\mathbf{C}_{p,k}(E)$ into $\mathbf{C}_{p,k-1}(E)$ and

$$\mathrm{Lip}(u)\frac{d}{dr}\mathbf{M}_p([T] \llcorner \{u < r\}) \geq \mathbf{M}_p(\langle [T], u, r\rangle) \quad \text{for } \mathscr{L}^1\text{-a.e. } r \in \mathbb{R}. \tag{8.1}$$

(ii) For some constant c^* the following holds: for all $[L] \in \mathbf{C}_{p,1}(E)$ with $\partial[L] = 0$ and bounded support there exists $[T] \in \mathbf{C}_{p,2}(E)$ with $\partial[T] = [L]$ and

$$\mathbf{M}_p([T]) \leq c^*\big[\mathbf{M}_p([L])\big]^2.$$

In addition, if $[L]$ is supported in a ball B, we may choose $[T]$ supported in the same ball.

(iii) For some constant c_k the following holds: for all $[L] \in \mathbf{C}_{p,k}(E)$ with $\partial[L] = 0$ and support contained in a ball with radius R there exists $[T] \in \mathbf{C}_{p,k+1}(E)$ supported in the same ball with $\partial[T] = [L]$ and

$$\mathbf{M}_p([T]) \leq c_k R\mathbf{M}_p([L]).$$

(iv) For some constant $A_k > 0$, the following holds: for all $[T] \in \mathbf{C}_{p,k}(E)$ we have

$$\liminf_{r\downarrow 0} \frac{\|T\|_p(B_r(x))}{r^k} \geq A_k, \quad \|T\|_p\text{-a.e.}$$

Given these properties, the nice and constructive decomposition argument in [44], [45] (that we reproduce in part in Theorem 10.6 to prove the initial isoperimetric inequality (ii)) provides the following result:

Theorem 8.2 (Isoperimetric inequality mod(p)). *Assume that E, $\mathbf{C}_{p,k}(E)$ and $\mathbf{M}_p$ fulfil the additivity property and conditions* (i), (ii), (iii), (iv). *Then, for $k \geq 1$ there* *exist constants γ_k such that, if $[L] \in \mathbf{C}_{p,k}(E)$ has bounded support and satisfies $\partial[L] = 0$, there exists $[T] \in \mathbf{C}_{p,k+1}(E)$ with $\partial[T] = [L]$ and*

$$\mathbf{M}_p([T]) \leq \gamma_k \big[\mathbf{M}_p([L])\big]^{(k+1)/k}.$$

For $k \geq 2$ the constant γ_k depends on γ_{k-1}, c_k, A_k.

Proof. The proof is by induction on $k \geq 1$; in order to apply the construction of [45] one needs to assume inductively that $[T]$ can be chosen with support in a ball B whenever $[L]$ is supported in the ball. The case $k = 1$ being covered by assumption (i) and the induction step goes exactly as in [45]. □

9. Definition of $\mathbf{M}_p$

For $T \in \mathscr{F}_k(E)$, its (relaxed) mass modulo p is defined by

$$\mathbf{M}_p(T) := \inf\Big\{\liminf_{h\to\infty} \mathbf{M}(T_h) : T_h \in \mathscr{I}_k(E),\ \mathscr{F}_p(T_h - T) \to 0\Big\} \tag{9.1}$$

with the convention $\mathbf{M}_p(T) = +\infty$ if no approximating sequence (T_h) with finite mass exists. If $\mathbf{I}_k(E)$ is dense in $\mathscr{I}_k(E)$ in mass norm then, as we already observed, $\mathscr{F}_k^*(E) = \mathscr{F}_k(E)$ and flat chains with finite mass can be approximated in mass by currents in $\mathbf{I}_k(E)$. Therefore, under this assumption, the infimum is unchanged is we require the approximating currents T_h to be in $\mathbf{I}_k(E)$.

Obviously $\mathbf{M}_p \leq \mathbf{M}$ and $\mathbf{M}_p(\tilde{T}) = \mathbf{M}_p(T)$ if $\mathscr{F}_p(\tilde{T} - T) = 0$; finally, $T \mapsto \mathbf{M}_p(T)$ is lower semicontinuous with respect to $\mathscr{F}_p$-convergence. Actually, it is easy to check that $\mathbf{M}_p$ is the largest functional, among those bounded above by $\mathbf{M}$, with all these properties: it follows in particular that $\mathbf{M}_p(T) \geq \mathscr{F}_p(T)$. We can think of $\mathbf{M}_p$ also as a map defined in the quotient groups $\mathscr{F}_{p,k}(E)$ and we shall not use a distinguished notation for it.

Theorem 9.1. *Assume that E is compact. For all $[T] \in \mathscr{F}_{p,k}(E)$ with $\mathbf{M}_p([T]) < \infty$ there exists a finite, nonnegative and σ-additive Borel measure $\|T\|_p$ such that*

$$\mathbf{M}_p([T] \llcorner \{u < r\}) = \|T\|_p(\{u < r\}) \quad \textit{for } \mathscr{L}^1\textit{-a.e. } r \in \mathbb{R} \tag{9.2}$$

for all $u \in \mathrm{Lip}(E)$.

Proof. Let $(T_i) \subset \mathscr{I}_k(E)$ be such that $\mathbf{M}(T_i) \to \mathbf{M}_p(T)$ and $\mathscr{F}_p(T_i - T) \to 0$. Possibly extracting a subsequence we can assume without loss of generality that

$$\sum_i \mathscr{F}_p(T_i - T) < \infty$$

and that $\|T_i\|$ weakly converge, in the duality with $C(E)$, to some finite, nonnegative and σ-additive Borel measure ν. Obviously $\nu(E) = \mathbf{M}_p(T)$ and we claim that ν fulfills (9.2). Indeed, let $u \in \mathrm{Lip}(E)$ be fixed and let us adopt the notation

$$R \llcorner \{u > r\}$$

for $R \llcorner \{-u < -r\}$; by (7.4) we infer that for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$, one has that $(T_i \llcorner \{u < r\})$ and $(T_i \llcorner \{u > r\})$ are Cauchy sequences with respect to $\mathcal{F}_p$ and the sum of their limits is T (indeed, since T_h have finite mass,

$$T_h = T_h \llcorner \{u < r\} + T_h \llcorner \{u > r\}$$

with at most countably many exceptions). Then, denoting by $T \llcorner \{u < r\}$ and $T \llcorner \{u > r\}$ the respective limits, the lower semicontinuity of $\mathbf{M}_p$ gives

$$\mathbf{M}_p(T \llcorner \{u < r\}) \leq \liminf_{i\to\infty} \|T_i\|(\{u < r\}),$$

and

$$\mathbf{M}_p(T \llcorner \{u > r\}) \leq \liminf_{i\to\infty} \|T_i\|(\{u > r\}).$$

The subadditivity of $\mathbf{M}_p$ yields

$$\begin{aligned}
\mathbf{M}_p(T) &\leq \mathbf{M}_p(T \llcorner \{u < r\}) + \mathbf{M}_p(T \llcorner \{u > r\}) \\
&\leq \liminf_{i\to\infty} \|T_i\|(\{u < r\}) + \liminf_{i\to\infty} \|T_i\|(\{u > r\}) \\
&\leq \limsup_{i\to\infty} \|T_i\|(\{u < r\}) + \liminf_{i\to\infty} \|T_i\|(\{u > r\}) \\
&\leq \limsup_{i\to\infty} \|T_i\|(E) = \mathbf{M}_p(T).
\end{aligned}$$

It follows that all inequalities are equalities. Hence,

$$\|T_i\|(\{u > r\}) \to \mathbf{M}_p(T \llcorner \{u > r\})$$

for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$. But, thanks to the weak convergence of $\|T_i\|$ to ν, we have also

$$\|T_i\|(\{u > r\}) \to \nu(\{u > r\})$$

with at most countably many exceptions (corresponding to the numbers r such that $\nu(\{u = r\}) > 0$, see for instance [2]), Proposition 1.62 (b). This proves (9.2). $\square$

Using the measure $\|T\|_p$ we can define the support of $T \in \mathcal{F}^*_{p,k}(E)$, when T has finite $\mathbf{M}_p$ mass.

Definition 9.2 (Support). Assume that E is compact and that $[T] \in \mathcal{F}^*_{p,k}(E)$ has finite $\mathbf{M}_p$ mass. We denote by $\operatorname{supp}[T]$ the support of the measure $\|T\|_p$, namely $x \in \operatorname{supp}[T]$ if and only if $\|T\|_p(B_r(x)) > 0$ for all $r > 0$.

10. Definitions of $\mathbf{I}_{p,k}(E)$

In this section we define classes $\mathbf{I}_{p,k}(E)$ in such a way that the properties listed in Section 8 hold with $\mathbf{C}_{p,k}(E) = \mathbf{I}_{p,k}(E)$, so that the isoperimetric inequality holds in $\mathbf{C}_{p,k}(E)$.

10.1. Currents $[\![\theta]\!]$. Recall that, for $\theta \in L^1(\mathbb{R}^k)$, $[\![\theta]\!] \in MF_k(\mathbb{R}^k)$ is the k-current in $\mathbb{R}^k$ defined by

$$[\![\theta]\!](f_0\, d\pi_1 \wedge \cdots \wedge d\pi_k) = \int_{\mathbb{R}^k} \theta f_0 \mathrm{det} \nabla\pi\, dx.$$

The change of variables formula for Lipschitz maps immediately gives

$$f_\sharp [\![\theta]\!] = [\![(\sigma\theta) \circ f^{-1}]\!] \tag{10.1}$$

whenever f is a Lipschitz and injective map from $\{f \neq 0\} \subset \mathbb{R}^k$ to $\mathbb{R}^k$. Here $\sigma(x) \in \{-1, 1\}$ is the sign of the jacobian determinant of $\nabla f(x)$ (recall that points x where $\sigma(x)$ is not defined, i.e. $\nabla f(x)$ is singular, are mapped to a Lebesgue negligible set, and so they are irrelevant).

10.2. Countably $\mathcal{H}^k$-rectifiable sets and integer rectifiable currents. Denoting by $\mathcal{H}^k$ the Hausdorff k-dimensional measure in E, we recall also that a set $S \subset E$ is said to be countably $\mathcal{H}^k$-rectifiable if we can find countably many Borel sets $B_i \subset \mathbb{R}^k$ and Lipschitz maps $f_i : B_i \to E$ such that

$$\mathcal{H}^k\big(S \setminus \textstyle\bigcup_i f_i(B_i)\big) = 0.$$

More precisely, we can also find by an exhaustion argument compact sets $K_i \subset \mathbb{R}^k$ and Lipschitz maps $f_i : K_i \to E$ such that $f_i(K_i)$ are pairwise disjoint and $\mathcal{H}^k(S \setminus \bigcup_i f_i(K_i)) = 0$. Furthermore, possibly refining once more the partition, one can assume that $f_i : K_i \to f_i(K_i)$ are invertible with a Lipschitz inverse (in short, bi-Lipschitz), see [3], Lemma 4.1. In the case $k = 0$ we identify countably $\mathcal{H}^k$-rectifiable sets with finite or countable sets.

Definition 10.1 (Rectifiable and integer rectifiable currents). We say that T in $MF_k(E)$ with finite mass is rectifiable if $\|T\|$ vanishes on $\mathcal{H}^k$-negligible sets and it is concentrated on a countably $\mathcal{H}^k$-rectifiable set. We say that T is integer rectifiable if, in addition, for all $\varphi \in \mathrm{Lip}(E, \mathbb{R}^k)$ and all Borel sets A it holds $\varphi_\sharp(T \llcorner A) = [\![\theta]\!]$ for some integer valued $\theta \in L^1(\mathbb{R}^k)$.

In the case $k = 0$ rectifiable currents are finite or countable series of Dirac masses, with integer coefficients in the integer case, see Theorem 4.3 in [3]. In this latter case, finiteness of mass implies that the sum is finite.

We shall denote by $\mathcal{I}_k(E)$ the space of integer rectifiable currents. We shall also denote by $\mathbf{I}_k(E)$ the subspace

$$\mathbf{I}_k(E) := \{T \in \mathcal{I}_k(E) : \partial T \in \mathcal{I}_{k-1}(E)\} .$$

In connection with integer rectifiable currents, let us recall the following important result (see [3], Theorem 8.6):

Theorem 10.2 (Boundary rectifiability). *If T is integer rectifiable and has boundary with finite mass, then ∂T is integer rectifiable.*

If E is a closed convex subset of a Banach space the slicing operator makes sense in $\mathcal{I}_k(E)$, thanks to Proposition 14.7, and it enjoys the following properties (see [3], Theorem 5.7):

Proposition 10.3 (Slices of integer rectifiable currents). *Let E be a closed convex subset of a Banach space, $T \in \mathcal{I}_k(E)$ and $u \in \mathrm{Lip}(E)$. Then $\langle T, u, r\rangle \in \mathcal{I}_{k-1}(E)$ for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$ and*

$$T \llcorner du = \int_{\mathbf{R}} \langle T, u, r\rangle \, dr, \quad \|T \llcorner du\| = \int_{\mathbf{R}} \|\langle T, u, r\rangle\| \, dr.$$

It turns out that the minimal (in $\mathcal{H}^k$-measure) set S on which T is concentrated is

$$S_T := \big\{x \in E : \liminf_{r\downarrow 0} r^{-k}\|T\|(B_r(x)) > 0\big\}. \tag{10.2}$$

10.3. Multiplicity of integer rectifiable currents and reductions mod(p). The *multiplicity* θ of a rectifiable current $T \in MF_k(E)$ can be defined as follows: when $E = \mathbb{R}^k$ the multiplicity of $[\![\theta]\!]$ is θ; in general, let us represent a Borel set S on which $\|T\|$ is concentrated (i.e. $\|T\|(E \setminus S) = 0$) as $\bigcup_i f_i(K_i)$ with $K_i \subset \mathbb{R}^k$ compact, $f_i \colon K_i \to f_i(K_i)$ bi-Lipschitz and $f_i(K_i)$ pairwise disjoint. Then, denoting by $g_i \colon E \to \mathbb{R}^k$ Lipschitz maps such that $g_i \circ f_i(x) = x$ on K_i, we define $\theta(y)$ at $y \in f_i(K_i)$ as the multiplicity of $(g_i)_\sharp(T \llcorner f_i(K_i))$ at $g_i(y) \in K_i$. Using (10.1) it is not difficult to check that this definition is well posed on S up to the sign and up to $\mathcal{H}^k$-negligible sets, i.e. that $|\theta|$ does not depend on the chosen partition and on the Lipschitz maps f_i up to $\mathcal{H}^k$-negligible sets (when E is a linear space see also §9 of [3] for a definition of multiplicity closer to the one of the Federer–Fleming theory; since this definition uses the quite technical concept of approximate tangent space here we avoid it). Notice also that we allow, for simplicity, the multiplicity to vanish: but the multiplicity is nonzero $\mathcal{H}^k$-a.e. on the set S_T.

If $m \in \mathbb{Z}$ we call reduction of m mod(p) an integer $\widetilde{m}$ which minimizes $|q|$ among all $q \in [-p/2, p/2]$ with $m - q \in p\mathbb{Z}$. The integer $\widetilde{m}$ is possibly not unique if p

is even (for instance $\widetilde{-1} = -1$ or $\widetilde{-1} = 1$ if $p = 2$), nevertheless $|\tilde{m}|$ is uniquely determined, and $|\widetilde{-m}| = |\tilde{m}|$.

We define *reduction of* T mod(p) a current obtained from T by taking the reduction of its multiplicity mod(p), namely

$$T^p := \sum_{i=1}^{\infty} (f_i)_\sharp [\![(\tilde{\theta} \circ f_i) \chi_{K_i}]\!]$$

whenever $T = \sum_i (f_i)_\sharp [\![(\theta \circ f_i) \chi_{K_i}]\!]$. Obviously any reduction T^p has integer multiplicity in $[-p/2, p/2]$ and it is equivalent to T mod(p). The reduction is not unique, because of the ambiguity on the sign of the multiplicity and on the choice of the reduction from $\mathbb{Z}$ to $[-p/2, p/2]$, but since $|\widetilde{-m}| = |\tilde{m}|$ it turns out that $|\tilde{\theta}|$ is nonzero and uniquely determined by T on S_T, up to $\mathcal{H}^k$-negligible sets.

The following proposition shows that elements of $\mathcal{F}_{p,0}(E)$ are equivalence classes of currents in $\mathcal{I}_0(E)$ and provides a basic lower semicontinuity property.

Proposition 10.4 (Characterization of $\mathcal{F}_{p,0}(E)$). *Let E be a compact length space, let $[R] \in \mathcal{F}_{p,0}(E)$ and let $T_h \in \mathcal{I}_0(E)$ be such that $[T_h] \to [R]$ in $\mathcal{F}_{p,0}(E)$ and* $\sup_h \|T_h\|(E)$ *is finite. Then there exists $T \in \mathcal{I}_0(E)$ such that $[T] = [R]$ and* $\liminf \|T_h^p\|(E) \geq \|T\|(E)$.

Proof. We assume without loss of generality that the lim inf is a finite limit and write

$$T_h = \sum_{i=1}^{N_h} \theta_{h,i} \delta_{x(h,i)}$$

with $\theta_{h,i} \in \mathbb{Z} \setminus \{0\}$. We can also assume, possibly replacing T_h by their reductions, that $\theta_{h,i} \in [-p/2, p/2]$, so that $T_h = T_h^p$. We have $N_h \leq \sup_h \|T_h\|(E)$ and we can assume (possibly extracting once more a subsequence) that $N_h = N$ is independent of h. Furthermore, we can also assume that $x(h,i) \to x(i)$ as $h \to \infty$ and

$$\theta_{h,i} = \theta_i \in [-p/2, p/2] \setminus \{0\} \quad \text{for } h \text{ large enough}$$

for all $i = 1, \dots, N$. Since E is a length space we can find currents $G_{h,i} \in \mathbf{I}_1(E)$ (induced by geodesics joining $x(h,i)$ to x_i) with $\partial G_{h,i} = \delta_{x(h,i)} - \delta_{x(i)}$ and $\mathbf{M}(G_{h,i}) \to 0$, for $i = 1, \dots, N$. Since $T_h - \sum_i \theta_i \delta_{x_i} = \sum \partial G_{h,i}$, it turns out that

$$\mathcal{F}(T_h - \sum_{i=1}^{N} \theta_i \delta_{x(i)}) \to 0,$$

whence $[R] = [\sum_1^N \theta_i \delta_{x_i}]$ mod(p). Also, it follows that

$$\| \sum_{i=1}^{N} \theta_i \delta_{x_i} \|(E) \leq \sum_{i=1}^{N} |\theta_i| \leq \liminf_{h\to\infty} \sum_{i=1}^{N} |\theta_{h,i}| = \liminf_{h\to\infty} \|T_h\|(E). \qquad \square$$

In the next theorem we characterize $\mathbf{M}_p$ on $\mathcal{I}_k(E)$.

Theorem 10.5. *Let $T \in \mathcal{I}_k(E)$, with E compact length space. Then $\mathbf{M}_p(T) = \|T^p\|(E)$, where T^p is any reduction of T modulo p. In particular, the additivity property holds with $\|T\|_p = \|T^p\|$.*

Proof. The inequality $\mathbf{M}_p(T) \le \|T^p\|(E)$ is obvious, because $T^p = T \operatorname{mod}(p)$. We shall prove the converse inequality by induction on k. Without loss of generality we can assume that E is a compact convex subset of a Banach space (indeed, an isometric embedding does not increase the $\mathbf{M}_p$ mass, while leaving $\|T^p\|(E)$ unchanged). The inequality is equivalent to the lower semicontinuity of $T \mapsto \|T^p\|(E)$ under $\mathcal{F}_p$-convergence. More generally, we shall prove by induction on k that

$$\|T^p\|(A) \le \liminf_{h\to\infty} \|T_h^p\|(A)$$

for all open sets $A \subset E$ whenever $\mathcal{F}_p(T_h - T) \to 0$.

$k = 0$. Let $T \in \mathcal{I}_0(E)$ and let $T_h \in \mathcal{I}_0(E)$ be satisfying $\mathcal{F}_p(T_h - T) \to 0$; we fix an open set $A \subset E$ and we assume with no loss of generality that the lim inf above is a limit and that $T_h = T_h^p$. Then, we are allowed to extract further subsequences and we can assume that the fast convergence condition $\sum_h \mathcal{F}_p(T_h - T) < \infty$ holds. Let u be the distance function from $E \setminus A$ and apply for $\mathcal{L}^1$-a.e. $r > 0$ Proposition 10.4 to $T_h \llcorner \{u > r\}$ and $[T \llcorner \{u > r\}]$ to obtain the existence of $S_r \in \mathcal{I}_0(E)$ with $S_r = T \llcorner \{u > r\} \operatorname{mod}(p)$ and

$$\|S_r\|(E) \le \liminf_{h\to\infty} \|T_h\|(\{u > r\}).$$

Since $S_r = T^p \llcorner \{u > r\} \operatorname{mod}(p)$ as well, it follows that

$$\|T^p\|(\{u > r\}) \le \|S_r\|(E) \le \liminf_{h\to\infty} \|T_h\|(\{u > r\}) \le \liminf_{h\to\infty} \|T_h\|(A).$$

Letting $r \downarrow 0$ the lower semicontinuity property on A follows.

Induction step. Let us prove that the induction assumption gives

$$\liminf_h \|T_h^p \llcorner du\|(A) \ge \|T^p \llcorner du\|(A)$$

whenever $T_h \to T$ in $\mathcal{F}_{p,k}(E)$. Indeed, assuming with no loss of generality that

$$\sum_h \mathcal{F}_p(T_h - T) < \infty,$$

we know from the definition of the slice operator and (7.4) that

$$\lim_{h\to\infty} \langle T_h, u, r\rangle = \langle T, u, r\rangle \quad \text{in } \mathcal{F}_{p,k}(E)$$

for $\mathscr{L}^1$-a.e. $r \in \mathbb{R}$, hence Proposition 10.3 gives

$$\begin{aligned}\liminf_{h\to\infty} \|T_h^p \llcorner du\|(A) &= \liminf_{h\to\infty} \int_{\mathbf{R}} \|\langle T_h^p, u, r\rangle\|(A)\,dr \\ &\geq \int_{\mathbf{R}} \liminf_{h\to\infty} \|\langle T_h^p, u, r\rangle\|(A)\,dr \\ &\geq \int_{\mathbf{R}} \|\langle T^p, u, r\rangle\|(A)\,dr \\ &= \|T^p \llcorner du\|(A).\end{aligned}$$

By applying Proposition 14.8 to $T^p \llcorner A$ we have

$$\|T^p\|(A) = \sup\Big\{\sum_{i=1}^{N} \|T^p \llcorner d\pi^i\|(A_i)\Big\},$$

where the supremum runs among all finite disjoint families of open sets $A_1, \dots, A_N \subset A$ and all N-ples of 1-Lipschitz maps π^i. By the previous step all the finite sums are lower semicontinuous with respect to $\mathscr{F}_p$ convergence, whence the lower semicontinuity of $T \mapsto \|T^p\|(A)$ follows.

This concludes the proof of the equality $\mathbf{M}_p(T) = \|T^p\|(E)$. Since for $T \in \mathcal{I}_k(E)$ and $u \in \mathrm{Lip}(E)$ it holds

$$(T \llcorner\{u < r\})^p = T^p \llcorner\{u < r\} \quad \text{for } \mathscr{L}^1\text{-a.e. } r \in \mathbb{R},$$

it follows that the additivity property is fulfilled with $\|T\|_p := \|T^p\|$. □

10.4. Isoperimetric inequalities mod(p). Having defined $\mathcal{I}_k(E)$, we define

$$\mathcal{I}_{p,k}(E) := \{[T] : T \in \mathcal{I}_k(E)\}\,.$$

An open problem, in connection with the $\mathbf{M}_p$, mass is the validity of the analogous of (6.2), namely

$$\big\{[T] \in \mathscr{F}_{p,k}(E) : \mathbf{M}_p([T]) < \infty\big\} = \mathcal{I}_{p,k}(E).$$

We plan to investigate this in [4].

We also define

$$\mathbf{I}_{p,k}(E) := \big\{[T] : [T] \in \mathcal{I}_{p,k}(E),\ [\partial T] \in \mathcal{I}_{p,k}(E)\big\}\,. \tag{10.3}$$

Theorem 10.6. *Let E be a compact convex subset of a separable Banach space. Then $\mathbf{M}_p$ and $\mathbf{I}_{p,k}(E)$, as defined in* (9.1) *and* (10.3) *respectively, satisfy conditions* (i), (ii), (iii), (iv) *of Section* 8 *with constants depending on k only.*

Proof. (i) The fact that the slice operator maps $\mathcal{I}_{p,k}(E)$ into $\mathcal{I}_{p,k-1}(E)$ follows by the fact the slice preserves integer rectifiability, see Proposition 10.3. Since the boundary operator and the slice commute (up to a change of sign) the slice operator maps also $\mathbf{I}_{p,k}(E)$ into $\mathbf{I}_{p,k-1}(E)$. In order to prove (8.1) we consider the inequality in an integral form, namely

$$\int_a^{*b} \mathbf{M}_p\big(\langle [T], u, r\rangle\big)\, dr \le \mathrm{Lip}(u)\big(\|T\|_p(\{u < b\}) - \|T\|_p(\{u < a\})\big), \qquad -\infty < a \le b < +\infty. \tag{10.4}$$

For $S \in \mathbf{I}_k(E)$ we can apply Theorem 5.6 of [3] to obtain

$$\int_a^{b} \mathbf{M}\big(\langle S, u, r\rangle\big)\, dr \le \mathrm{Lip}(u)\big(\|S\|(\{u < b\}) - \|S\|(\{u < a\})\big).$$

Now, let $(S_i) \subset \mathbf{I}_k(E)$ be such that $\sum_i \mathcal{F}_p(S_i - T) < \infty$ and $\mathbf{M}(S_i) \to \mathbf{M}_p([T])$; we have seen in the proof of Theorem 9.1 that there exists an at most countable set N such that $\mathbf{M}(S_i \llcorner \{u < r\}) \to \|T\|_p(\{u < r\})$ for all $r \in \mathbb{R} \setminus N$; in addition, the fast convergence assumption ensures that $\mathcal{F}_p(\langle S_i, u, r\rangle - \langle T, u, r\rangle) \to 0$ for $\mathcal{L}^1$-a.e. $r > 0$. So, passing to the limit in the previous inequality with $S = S_i$, Fatou's lemma and the lower semicontinuity of $\mathbf{M}_p$ provide (10.4) when $a,\, b \notin N$. In the general case the inequality can be recovered by monotone approximation.

(ii) In the proof of this property we shall use properties (i), (iii) and (iv) which are established independently of (ii). In the case $k = 1$, property (iv) holds with the explicit constant $A_k = 2$; furthermore (iii) holds with $c^* = 2$. For all $[L] \in \mathbf{I}_{p,1}(E)$ with $\partial [L] = 0$ we shall be able to construct a family of currents $[L_i]$ with the same properties satisfying

$$\mathbf{M}_p\Big([L] - \sum_{i=1}^{\infty} [L_i]\Big) = 0, \quad \mathbf{M}_p([L]) = \sum_{i=0}^{\infty} \mathbf{M}_p([L_i]) \tag{10.5}$$

and $\mathrm{diam}(\mathrm{supp}([L_i])) \le 8\mathbf{M}_p([L_i])$. Given this decomposition, an application of property (iii) to all $[L_i]$ provides currents $[T_i]$ with $\partial [T_i] = [L_i]$ and $\mathbf{M}_p([T_i]) \le 16[\mathbf{M}_p([L_i])]^2$ and we can apply the property (iii) to find $[S_N]$ with $\partial S_N = [L] - \sum_1^N [L_i]$ and $\mathbf{M}_p([S_N]) \to 0$; it turns that for N large enough the current

$$[T] := \sum_{i=1}^{N} [T_i] + S_N$$

has the required property.

In order to achieve the decomposition (10.5) it suffices to find finitely many, say N, currents $[L_i]$ with $\operatorname{diam}(\operatorname{supp}([L_i])) \leq 8\mathbf{M}_p([L_i])$,

$$\mathbf{M}_p([L]-\sum_{i=1}^{N}[L_i]) \leq \frac{4}{5}\mathbf{M}_p([L]), \quad \mathbf{M}_p([L]) = \mathbf{M}_p([L]-\sum_{i=1}^{N}[L_i]) + \sum_{i=1}^{N}\mathbf{M}_p([L_i]) \tag{10.6}$$

and then iterate this decomposition (first to $[L]-\sum_1^N [L_i]$ and so on) countably many times. In order to obtain the decomposition (10.6) we apply Lemma 3.2 of [44] with $F = 1/2$ and $\mu = \|T^p\|$ (since $A_1 = 2 > F$ this choice ensures that for μ-a.e. x there exists $r > 0$ such that $\mu(B_r(x)) \geq Fr$) to obtain finitely many points $y_1, \dots, y_N$ and corresponding radii $r_i > 0$ satisfying:

(a) $\mu(B_{r_i}(y_i)) \geq Fr_i$ and $\mu(B_s(y_i)) < Fs$ for all $s > r_i$;

(b) the balls $B_{2r_i}(y_i)$ are disjoint;

(c) $5\sum_1^N \mu(B_{r_i}(y_i)) \geq \mu(E)$.

Since (a) gives

$$\int_{r_i}^{*2r_i} \mathbf{M}_p(\langle [L], d(\cdot, y_i), r\rangle)\, dr \leq \mathbf{M}_p([L] \llcorner B_{2r_i}(y_i)) < 2Fr_i = r_i$$

we know that $\mathbf{M}_p(\langle [L], d(\cdot, y_i), r\rangle) < 1$ in a set of positive $\mathscr{L}^1$-measure in $(r_i, 2r_i)$. But since the slices belong to $\mathbf{I}_{p,0}(E)$ it follows that $\mathbf{M}_p(\langle [L], d(\cdot, y_i), r\rangle) = 0$ in a set of positive $\mathscr{L}^1$-measure in $(r_i, 2r_i)$. Choosing $\eta_i \in (r_i, 2r_i)$ in such a way that $\langle [L], d(\cdot, y_i), \eta_i\rangle) = 0$ we can define

$$[L_i] := [L] \llcorner \{d(\cdot, y_i) < \eta_i\}, \quad 1 \leq i \leq N.$$

Our choice of η_i ensures that $\partial[L_i] = 0$ and property (b) ensures that the supports of these chains are pairwise disjoint. Also,

$$\operatorname{diam}(\operatorname{supp}([L_i])) \leq 2\eta_i \leq 4r_i \leq 8\mu(B_{r_i}(y_i)) \leq 8\mathbf{M}_p([L_i]).$$

Property (c) ensures that $5\sum_1^N \mathbf{M}_p([L_i]) \geq \mathbf{M}_p([L])$, so that (10.6) holds.

(iii) This can be easily achieved by a cone construction as, for instance, in [3], Proposition 10.2. This construction provides the constant $c^* = 2$.

(iv) If $T \in \mathcal{I}_k(E)$ and T^p is a reduction $\operatorname{mod}(p)$, since its multiplicity is at least 1 we know by [3], Theorem 9.5, that

$$\|T^p\| \geq \lambda \mathscr{H}^k \llcorner S,$$

where $S = S(T^p)$ is defined in (10.2) with T^p in place of T and λ is an "area factor" depending only on S. In addition, Lemma 9.2 in [3] provides the universal lower bound $\lambda \geq k^{-k/2}$. Finally, taking into account (see [37]) that any countably $\mathcal{H}^k$-rectifiable set with finite $\mathcal{H}^k$-measure S satisfies

$$\liminf_{r\downarrow 0} \frac{\mathcal{H}^k(S \cap B_r(x))}{\omega_k r^k} = 1 \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in S,$$

with ω_k equal to the Lebesgue measure of the unit ball in $\mathbb{R}^k$, we obtain that (iv) holds with $A_k = k^{-k/2}\omega_k$. □

As a consequence, we can obtain isoperimetric inequalities in the case when the cycle belongs to $\mathbf{I}_{p,k}(E)$ (resp. $\mathcal{F}_{p,k}(E)$) and the filling belongs to $\mathbf{I}_{k+1}(E)$ (resp. $\mathcal{F}_{p,k+1}(E)$). In this connection, notice that in the class of integer multiplicity currents we have that $L \in \mathcal{F}_k(E)$ with finite mass and $\partial L = 0$ implies $L \in \mathbf{I}_k(E)$: indeed, writing $L = A + \partial B$ with $A \in \mathcal{I}_k(E)$ and $B \in \mathcal{I}_{k+1}(E)$, we have $\partial A = 0$ and so $A = \partial R$ for some $R \in \mathbf{I}_{k+1}(E)$. Since $L = \partial(R + B)$ the boundary rectifiability theorem gives that $L \in \mathbf{I}_k(E)$. We plan to investigate the boundary rectifiability theorem and further properties of currents mod(p) in [4].

Corollary 10.7 (Isoperimetric inequality mod(p) in $\mathbf{I}_{p,k}(E)$ and $\mathcal{F}_{p,k}(E)$). *Let E be a compact convex subset of a separable Banach space. For $k \geq 1$ there exist constants δ_k such that, if $[L] \in \mathbf{I}_{p,k}(E)$ is a non zero current with bounded support and $\partial[L] = 0$ then*

$$\inf\left\{\frac{\mathbf{M}_p([T])}{\left[\mathbf{M}_p([L])\right]^{(k+1)/k}} : [T] \in \mathbf{I}_{p,k+1}(E),\ \partial[T] = [L]\right\} \leq \delta_k.$$

The same property holds when $[L] \in \mathcal{F}_{p,k}(E)$, taking the infimum among all $[T] \in \mathcal{F}_{p,k+1}(E)$ with $\partial[T] = [L]$.

Proof. If $[L] \in \mathbf{I}_{p,k}(E)$, we want to apply Theorem 8.2. To this aim, it suffices to combine Theorem 10.6 and Theorem 10.5. In the general case $[L] \in \mathcal{F}_{p,k}(E)$, let $P_i \in \mathbf{I}_k(E)$ be satisfying $\mathcal{F}_p(P_i - L) \to 0$ and $\mathbf{M}(P_i) \to \mathbf{M}_p(L)$. Let us write

$$P_i = L + A_i + \partial B_i + pQ_i$$

with $A_i \in \mathcal{I}_k(E)$, $B_i \in \mathcal{I}_{k+1}(E)$, $Q_i \in \mathcal{F}_k(E)$ and $\mathbf{M}(A_i)+\mathbf{M}(B_i) \to 0$. We have $[\partial P_i] = [\partial A_i]$, and since $[P_i - A_i] \in \mathbf{I}_{p,k}(E)$ we can find currents $[P_i'] \in \mathbf{I}_{p,k+1}(E)$ with $\partial[P_i'] = [P_i - A_i]$ and

$$\mathbf{M}_p([P_i']) \leq \delta_k\left[\mathbf{M}_p([P_i - A_i])\right]^{(k+1)/k} \leq \delta_k\left[\mathbf{M}_p([L])\right]^{(k+1)/k} + \omega_i$$

with ω_i infinitesimal. It is now immediate to check that $\partial[P_i' - B_i] = [L]$, so that $[P_i' - B_i] \in \mathcal{F}_{p,k+1}(E)$, and that

$$\limsup_{i\to\infty} \mathbf{M}_p([P_i' - B_i]) \leq \delta_k\left[\mathbf{M}_p([L])\right]^{(k+1)/k}.$$

□

11. Filling radius inequality

In this section we investigate the validity of a filling radius inequality, which complements the isoperimetric inequality of Corollary 10.7. To this aim, for $[L] \in \mathbf{I}_{p,k}(E)$ with $\partial[L] = 0$ we consider the subspace $\mathcal{M}$ defined by

$$\mathcal{M} := \left\{[T] \in \mathcal{F}_{p,k+1}(E) : \partial[T] = [L],\ \mathbf{M}_p([T]) < \infty\right\}. \tag{11.1}$$

By Corollary 10.7 $\mathcal{M}$ contains $[\bar{T}] \in \mathbf{I}_{p,k+1}(E)$ with $\mathbf{M}_p([\bar{T}]) \le \delta_k [\mathbf{M}_p([L])]^{(k+1)/k}$.

Theorem 11.1. *Assume that E is a compact convex subset of a separable Banach space. Let $[L] \in \mathbf{I}_{p,k}(E)$ with $\mathbf{M}_p([L]) < \infty$ and $\partial[L] = 0$. Then, the infimum of the numbers r such that there exists $[T] \in \mathbf{I}_{p,k+1}(E)$ satisfying $\partial[T] = [L]$ whose support is contained in the r-neighbourhood of* supp $[L]$ *is not greater than $C_k[\mathbf{M}_p([L])]^{1/k}$.*

The constant C_k depends only on k and on the constant δ_k in Corollary 10.7.

Proof. We claim that the infimum is unchanged if we look for fillings in the more general class $\mathcal{F}_{p,k+1}(E)$. Indeed, let $[S] \in \mathcal{F}_{p,k+1}(E)$ with $\partial[S] = [L]$ whose support is contained in the r-neighbourhood of K, and let u be the distance function from K, the support of $[L]$. We now consider a sequence $(S_i) \subset \mathbf{I}_{k+1}(E)$ with $\sum_i \mathcal{F}_p(S_i - S) < \infty$ and $r' > r$. We know that for $\mathcal{L}^1$-a.e. $\rho \in (r, r')$ we still have $[S_i \llcorner \{u < \rho\}] \to [S \llcorner \{u < \rho\}]$ in $\mathcal{F}_{p,k+1}(E)$, and since $[S \llcorner \{u < \rho\}] = [S] \llcorner \{u < \rho\} = [S]$ we see that, possibly replacing S_i by $S_i \llcorner \{u < \rho\}$, there is no loss of generality in assuming that the supports of S_i are contained in the ρ-neighbourhood of K, for some $\rho < r'$. Now, let us fix i and write

$$S - S_i = A + \partial B + pQ$$

with $A \in \mathcal{I}_{k+1}(E)$, $B \in \mathcal{I}_{k+2}(E)$, $Q \in \mathcal{F}_{k+1}(E)$. For $\mathcal{L}^1$-a.e. $t \in (\rho, r')$ we can restrict both sides to $\{u < t\}$ to obtain

$$S - S_i = A \llcorner \{u < t\} - \langle B, u, t\rangle + \partial(B \llcorner \{u < t\}) + pQ \llcorner \{u < t\}.$$

It follows that the current $[S_i + A \llcorner \{u < t\} - \langle B, u, t\rangle] \in \mathbf{I}_{p,k+1}(E)$ has boundary $[L]$ and support contained in the r'-neighbourhood of K. Since $r' > r$ is arbitrary, this proves the claim. $\square$

So, from now on we look for $[S] \in \mathcal{F}_{p,k+1}(E)$ with $\partial[S] = [L]$ and we set

$$c := \delta_k [\mathbf{M}_p([L])]^{(k+1)/k}.$$

12. Ekeland principle

Let us recall the Ekeland variational principle [17] (see also the proof in [18], using only the countable axiom of choice): If (X, d) is a complete metric space and $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and bounded from below, then for all $\varepsilon > 0$ there exists $y \in X$ such that $x \mapsto f(x) + \varepsilon d(x, y)$ attains its minimum value at $x = y$. Since $\mathbf{M}_p \geq \mathscr{F}_p$ and is $\mathscr{F}_p$ lower semicontinuous, we know that $\mathcal{M}$ is a complete metric space, when endowed with the distance induced by $\mathbf{M}_p$. Let $\varepsilon > 0$ be fixed; the lower semicontinuity of $[T] \mapsto \mathbf{M}_p([T])$ ensures that we can apply the Ekeland variational principle to find $[S] \in \mathcal{M}$ such that

$$[T] \mapsto \mathbf{M}_p([T]) + \varepsilon \mathbf{M}_p([T] - [S]), \quad [T] \in \mathcal{M},$$

is minimal at $[T] = [S]$. If $\varepsilon \leq 1/2$, the minimality of $[S]$ gives

$$\mathbf{M}_p([S]) \leq \frac{1+\varepsilon}{1-\varepsilon} \mathbf{M}_p([\bar{T}]) \leq 3c. \tag{12.1}$$

Let us now prove the density lower bound

$$\|S\|_p(B_\varrho(x)) \geq \frac{(3\delta_k)^{-k}}{(k+1)^{k+1}} \varrho^{k+1} \quad \text{for all } \varrho \in (0, \tau(x)) \tag{12.2}$$

for any $x \in \operatorname{supp}[S] \setminus K$; here $\tau(x) = \operatorname{dist}(x, K) > 0$. In order to prove (12.2) we use a standard comparison argument based on the isoperimetric inequalities: let $x \in \operatorname{supp}[S] \setminus K$: for $\mathscr{L}^1$-a.e. $\varrho > 0$ the slice

$$[S_\varrho] := \langle [S], d(\cdot, x), \varrho \rangle = \partial([S] \llcorner \{d(\cdot, x) < \varrho\}) - (\partial[S]) \llcorner \{d(\cdot, x) < \varrho\}$$

belongs to $\mathscr{F}_{p,k}(E)$ and has no boundary, because the conditions $\rho < \tau(x)$ and $\partial[S] = [L]$ imply

$$(\partial[S]) \llcorner \{d(\cdot, x) < \varrho\} = 0.$$

By Corollary 10.7 we can find $[R] \in \mathscr{F}_{p,k+1}(E)$ with $\partial[R] = [S_\varrho]$ and

$$\mathbf{M}_p([R]) \leq \delta_k \big[\mathbf{M}_p([S_\varrho])\big]^{(k+1)/k}. \tag{12.3}$$

Comparing $[S]$ with

$$[S'] := [S] \llcorner \big(E \setminus B_\varrho(x)\big) + [R]$$

we find

$$\begin{aligned}
\mathbf{M}_p([S]) &\leq \mathbf{M}_p([S']) + \varepsilon \mathscr{F}_p([S] \llcorner B_\varrho(x) - R) \\
&\leq \mathbf{M}_p([R]) + \mathbf{M}_p([S] \llcorner (E \setminus B_\varrho(x))) + \varepsilon \mathbf{M}_p([S] \llcorner B_\varrho(x)) + \varepsilon \mathbf{M}_p([R]),
\end{aligned}$$

so that

$$\mathbf{M}_p([S] \llcorner B_\varrho(x)) \le \frac{1+\varepsilon}{1-\varepsilon}\mathbf{M}_p([R]) \le 3\mathbf{M}_p([R]). \tag{12.4}$$

By (12.3) and (12.4) it follows that

$$\|S\|_p(B_\varrho(x)) \le 3\delta_k \left[\frac{d}{d\varrho}\|S\|_p(B_\varrho(x))\right]^{(k+1)/k}$$

for $\mathscr{L}^1$-a.e. $\varrho > 0$. Since $\|S\|_p(B_\varrho(x)) > 0$ for any $\varrho > 0$ (because $x \in \operatorname{supp}[S]$), this proves that

$$\varrho \mapsto \left(\|S\|_p(B_\varrho(x))\right)^{1/(k+1)} - (3\delta_k)^{-k/(k+1)}\varrho/(k+1)$$

nondecreasing, and hence nonnegative, in $(0, \tau(x))$.

To obtain the estimate on the support of $[S]$ it suffices to take a sequence $\varrho_i \uparrow \tau(x)$ and to use the inequalities

$$\|T\|_p(B_\varrho(x)) \le \mathbf{M}_p([S]) \le 3c \le 3\delta_k \mathbf{M}_p([L])]^{(k+1)/k}$$

to obtain that $\tau(x)$ can be bounded by a multiplicative constant times $[\mathbf{M}_p([L])]^{1/k}$. Since x is arbitrary this proves that the support of $[S]$ is contained in the r-neighbourhood of K, with $r \le C_k[\mathbf{M}_p([L])]^{1/k}$. □

Remark 12.1 (Extension to $\mathscr{F}_{p,k}(E)$). The same property holds, with the same proof, in the classes $\mathscr{F}_{p,k}(E)$, namely: for all $[L] \in \mathscr{F}_{p,k}(E)$ with $\mathbf{M}_p([L]) < \infty$ and $\partial[L] = 0$ the infimum of the numbers r such that there exists $[T] \in \mathscr{F}_{p,k+1}(E)$ satisfying $\partial[T] = [L]$ whose support is contained in the r-neighbourhood of $\operatorname{supp}[L]$ is not greater than $C_k[\mathbf{M}_p([L])]^{1/k}$.

13. Nonorientable manifolds and currents mod(2)

Let (M, g) be a compact n-dimensional Riemannian manifold without boundary and let τ be a Borel orientation of M, i.e. a Borel choice of unit vectors $\tau_1, \dots, \tau_n$ spanning the tangent space and mutually orthogonal (the construction can be easily achieved in local coordinates and gluing, by the minimal Borel regularity imposed on τ, is not a problem), possibly up to $\mathscr{H}^n$-negligible sets. Here $\mathscr{H}^n$ is the Hausdorff n-dimensional measure induced by the Riemannian distance. Of course, when M is not orientable any orientation τ is necessarily discontinuous and it is by no means canonical. In any case, given this orientation, we can define a current $[\![M]\!] \in \mathcal{I}_n(M)$ as follows:

$$[\![M]\!](f\,d\pi_1 \wedge \cdots \wedge d\pi_k) := \int_M f \det\left(\frac{\partial \pi_i}{\partial \tau_j}\right) d\mathscr{H}^n.$$

While $[\![M]\!]$ is not canonical, its equivalence class mod(2) obviously is, because different orientations induce currents $[\![M]\!]$ equivalent mod(2). In connection with mass measures, it is not difficult to check that

$$\|[\![M]\!]\|(B) = \mathcal{H}^n(B) \quad \text{for all } B \subset M \text{ Borel}$$

(or, it suffices to apply Lemma 9.2 and Theorem 9.5 of [3], valid in a much more general context). In turn, $\mathcal{H}^n$ coincides with the Riemannian volume measure, see for instance [20], 3.2.46. Passing to the equivalence class the same is true, because $[\![M]\!]$ is already reduced mod(2), hence $\|[\![M]\!]\|_2 = \|[\![M]\!]\|$ and their total mass is $\mathrm{Vol}(M)$.

We are now going to show that $\partial[\![M]\!] = 0 \bmod(2)$, and we prove this fact building a "nice" current on M as the image of the exponential map Exp_O at some base point $O \in M$. As the referee pointed out, for the purpose of proving $\partial[\![M]\!] = 0 \bmod(2)$ simpler proofs are possible, which apply to Lipschitz manifolds as well; on the other hand, we believe that this global construction (which uses some properties of the cut locus established only recently) might have an independent interest.

Theorem 13.1. *Let (M, g) be a compact n-dimensional Riemannian manifold with no boundary. Then $\partial\big[[\![M]\!]\big] = 0$ and, in particular, $\big[[\![M]\!]\big] \in \mathbf{I}_{2,n}(M)$.*

Proof. We fix a base point $O \in M$ and consider the distance function u from O. We consider the tangent cut locus TC at O, namely $v \in T_O M$ belongs to TC if and only if $\exp_O(tv)$ is the unique minimizing geodesic in $[0, \tau]$ for all $\tau < 1$, and it is nonminimizing in $[0, \tau]$ for all $\tau > 1$. It turns out that TC is locally a Lipschitz graph [28], [39], and that the boundary of the star-shaped region

$$\Omega := \{tv : v \in TC,\ t \in [0, 1]\}$$

is contained in TC. Of course the exponential map Exp_O maps TC into the cut locus, that we shall denote by C.

Next, we consider some additional regularity properties of u, besides 1-Lipschitz continuity: this function is locally semiconcave out of O, namely in local coordinates its second derivatives are locally bounded from above in $M \setminus \{O\}$. This implies, by standard results about semiconcave functions and viscosity solutions to the Hamilton–Jacobi equation $g_x(\nabla u, \nabla u) = 1$ the following facts (for (i), (ii), (iii) see for instance [40]; for the more delicate property (iv) see [41], Theorem 4.12, or the appendix of [22]):

(i) for all $x \neq O$ the set of supergradients

$$\partial^+ u(x) := \{v \in T_x M : u(\exp_x(w)) \le u(x) + g_x(v, w) + o(|w|)\}$$

is convex and not empty, and u is differentiable at x if and only if $\partial^+ u(x)$ is a singleton;

(ii) for all $x \neq O$ the closed convex hull of $\partial^+ u(x) \cap \{v \in T_x M : g_x(v,v) = 1\}$ coincides with $\partial^+ u(x)$ and the former set is in 1-1 correspondence with final speeds of minimizing unit speed geodesics joining O to x;

(iii) for j integer the set $\{x \in M : \dim(\partial^+ u(x)) \geq j\}$ has σ-finite $\mathcal{H}^{n-j}$-measure;

(iv) the set of points $x \in C$ where u is differentiable is $\mathcal{H}^{n-1}$-negligible.

Now, we fix an orientation of $T_O M$ and we consider the canonical (Euclidean) n-current $[[\Omega]] \in \mathcal{I}_n(T_O M)$, with multiplicity 1 on Ω and 0 on $T_O M \setminus \Omega$ induced by this orientation; since

$$\mathcal{H}^{n-1}(\partial\Omega) \leq \mathcal{H}^{n-1}(TC) < \infty$$

we know that $[[\Omega]] \in \mathbf{I}_n(T_O M)$ and its boundary is supported on TC. Then, we consider its image $T = (\exp_O)_\sharp[[\Omega]] \in \mathbf{I}_n(M)$ via the exponential map. We are going to prove that:

(a) $T = [[M]]$ for some orientation of M;

(b) $\partial T = 2R$ for some $R \in \mathbf{I}_{n-1}(M)$.

These two facts imply the stated properties of $[[M]]$. In connection with (a), notice first that $\exp_O(\Omega) = M$, because for each point $x \in M$ there is at least one minimizing geodesic to O, and it is unique before reaching x. Moreover, Rademacher's theorem implies that $\mathcal{H}^n$-a.e. point $x \in M$ is a differentiability point of u, so that $\partial u^+(x) = \{\nabla u(x)\}$ is a singleton and there is a unique minimizing constant speed geodesic between O and x (since its final speed is uniquely determined, ODE uniqueness applies); if v is the initial speed of this geodesic, it turns out that $x = \exp_O(d(O,x)v)$ and $td(O,x)v \in \Omega$ for all $t < 1$, hence $d(O,x)v \in \Omega$. This proves that $\exp_O$ has a unique inverse $\mathcal{H}^n$-a.e.; these facts imply that $T = [[M]]$ provided we choose as orientation of M the one induced by $T_O M$ via the exponential map $\exp_O$.

In connection with (b), we know that $\partial T = (\exp_O)_\sharp(\partial[[\Omega]])$ and that $\partial[[\Omega]]$ is a current with unit multiplicity $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, because TC is locally a Lipschitz graph. We claim that for $\mathcal{H}^{n-1}$-a.e. $x \in C$ the pre-image $\exp_O^{-1}(x)$ contains exactly two points. Since the multiplicity of ∂T at x can be obtained adding the properly multiplicities of $\partial[[\Omega]]$ at $\exp_O^{-1}(x)$, this proves that ∂T has an even multiplicity. To prove the claim, we know by (iv) that for $\mathcal{H}^{n-1}$-a.e. $x \in C$ the number of minimizing geodesics is strictly greater than 1; on the other hand, (iii) with $j = 2$ gives that for $\mathcal{H}^{n-1}$-a.e. $x \in C$ the dimension of $\partial^+ u(x)$ is at most 1, hence the extreme points are at most two: therefore there exist precisely two minimizing geodesics from O to x at $\mathcal{H}^{n-1}$-a.e. $x \in C$. □

Proof of Theorem 3.2. It suffices to apply Theorem 11.1 with $k = n$. To this aim, we consider the canonical current $\left[[[M]]\right]$ associated to M. By Theorem 13.1 this current belongs to $\mathbf{I}_{2,n}(M)$ and it is a cycle mod(2). Then, given an isometric embedding i of

M into a (separable) Banach space F, we consider the closed convex hull E of $i(M)$ (which is a compact set, by the compactness of $i(M)$), and apply Theorem 11.1 to the cycle $[L] = i_\sharp\big[[\![M]\!]\big] \in \mathbf{I}_{2,n}(E)$, whose $\mathbf{M}_2$ mass is (by the isometric invariance of the $\mathbf{M}_2$-mass of rectifiable currents) equal to $\mathbf{M}_2([\![M]\!]) = \mathrm{Vol}(M)$.

14. Appendix

In this appendix we recall the basic definitions of the metric theory developed in [3].

Definition 14.1. Let $k \geq 1$ be an integer. We denote by $\mathcal{D}^k(E)$ the set of all $(k+1)$-ples $\omega = (f, \pi_1, \ldots, \pi_k)$ of Lipschitz real valued functions in E with the first function f in $\mathrm{Lip}_b(E)$. In the case $k = 0$ we set $\mathcal{D}^0(E) = \mathrm{Lip}_b(E)$.

Definition 14.2 (Metric functionals). We call k-dimensional metric current any function $T\colon \mathcal{D}^k(E) \to \mathbb{R}$ satisfying the following three axioms:

(a) T is multilinear;

(b) $T(f, \pi_1^n, \ldots, \pi_k^n) \to T(f, \pi_1, \ldots, \pi_k)$ whenever $\pi_i^n \to \pi_i$ pointwise and $\sup_n \mathrm{Lip}(\pi_i^n)$ is finite, for $1 \leq i \leq k$;

(c) $T(f, \pi_1, \ldots, \pi_k) = 0$ if, for some $i \in \{1, \ldots, k\}$, π_i is constant in a neighbourhood of the support of f.

We denote by $MF_k(E)$ the vector space of k-dimensional metric currents.

A consequence of these axioms is that T is alternating in $(\pi_1, \ldots, \pi_k)$, so the differential forms notation $f d\pi_1 \wedge \cdots \wedge d\pi_k$ can be used. We can now define an "exterior differential"

$$d\omega = d(f d\pi_1 \wedge \cdots \wedge d\pi_k) := df \wedge d\pi_1 \wedge \cdots \wedge \pi_k$$

mapping $\mathcal{D}^k(E)$ into $\mathcal{D}^{k+1}(E)$ and, for $\varphi \in \mathrm{Lip}(E, F)$, a pull back operator

$$\varphi^\sharp\omega = \varphi^\sharp(f d\pi_1 \wedge \cdots \wedge d\pi_k = f \circ \varphi d\pi_1 \circ \varphi \wedge \cdots \wedge d\pi_k \circ \varphi$$

mapping $\mathcal{D}^k(F)$ on $\mathcal{D}^k(E)$. These operations induce in a natural way a boundary operator and a push forward map for metric functionals.

Definition 14.3 (Boundary). Let $k \geq 1$ be an integer and let $T \in MF_k(E)$. The boundary of T, denoted by ∂T, is the $(k-1)$-dimensional metric current in E defined by $\partial T(\omega) = T(d\omega)$ for any $\omega \in \mathcal{D}^{k-1}(E)$.

Definition 14.4 (Push-forward). Let $\varphi\colon E \to F$ be a Lipschitz map and let $T \in MF_k(E)$. Then, we can define a k-dimensional metric current in F, denoted by $\varphi_\sharp T$, setting $\varphi_\sharp T(\omega) = T(\varphi^\sharp\omega)$ for any $\omega \in \mathcal{D}^k(F)$.

We notice that, by construction, $\varphi_\sharp$ commutes with the boundary operator, i.e.

$$\varphi_\sharp(\partial T) = \partial(\varphi_\sharp T). \tag{14.1}$$

Definition 14.5 (Restriction). Let $T \in MF_k(E)$ and let $\omega = g\,d\tau_1 \wedge \cdots \wedge d\tau_m \in \mathcal{D}^m(E)$, with $m \le k$ ($\omega = g$ if $m = 0$). We define a $(k-m)$-dimensional metric current in E, denoted by $T \llcorner \omega$, setting

$$T \llcorner \omega(f\,d\pi_1 \wedge \cdots \wedge d\pi_{k-m}) := T(fg\,d\tau_1 \wedge \cdots \wedge d\tau_m \wedge d\pi_1 \wedge \cdots \wedge d\pi_{k-m}).$$

Definition 14.6 (Currents with finite mass). Let $T \in MF_k(E)$; we say that T has finite mass if there exists a finite Borel measure μ in E satisfying

$$|T(f\,d\pi_1 \wedge \cdots \wedge d\pi_k)| \le \prod_{i=1}^{k} \mathrm{Lip}(\pi_i) \int_E |f|\,d\mu \tag{14.2}$$

for any $f\,d\pi_1 \wedge \cdots \wedge d\pi_k \in \mathcal{D}^k(E)$, with the convention $\prod_i \mathrm{Lip}(\pi_i) = 1$ if $k = 0$.

It can be shown that there is a minimal measure μ satisfying (14.2), which will be denoted by $\|T\|$ (indeed one checks, using the subadditivity of T with respect to the first variable, that if $\{\mu_i\}_{i\in I} \subset \mathcal{M}(E)$ satisfy (14.3) also their infimum satisfies the same condition). We call mass of T the total mass of $\|T\|$, namely $\mathbf{M}(T) = \|T\|(E)$.

By the density of $\mathrm{Lip}_b(E)$ in $L^1(E, \|T\|)$, which contains the class of bounded Borel functions, any $T \in MF_k(E)$ with finite mass can be uniquely extended to forms $f\,d\pi$ with f bounded Borel, in such a way that

$$|T(f\,d\pi_1 \wedge \cdots \wedge d\pi_k)| \le \prod_{i=1}^{k} \mathrm{Lip}(\pi_i) \int_E |f|\,d\|T\| \tag{14.3}$$

for any f bounded Borel, $\pi_1, \dots, \pi_k \in \mathrm{Lip}(E)$. Since this extension is unique we do not introduce a distinguished notation for it.

Functionals with finite mass are well behaved under the push-forward map: in fact, if $T \in MF_k(E)$ the functional $\varphi_\sharp T$ has finite mass, satisfying

$$\|\varphi_\sharp T\| \le [\mathrm{Lip}(\varphi)]^k \varphi_\sharp \|T\| \ . \tag{14.4}$$

If either φ is an isometry or $k = 0$ it is easy to check, using (14.6) below, that equality holds in (14.4). It is also easy to check that the identity

$$\varphi_\sharp T(f\,d\pi_1 \wedge \cdots \wedge d\pi_k) = T(f \circ \varphi\,d\pi_1 \circ \varphi \wedge \cdots \wedge d\pi_k \circ \varphi)$$

remains true if f is bounded Borel and $\pi_i \in \mathrm{Lip}(E)$.

Functionals with finite mass are also well behaved with respect to the restriction operator: in fact, the definition of mass easily implies

$$\|T \llcorner \omega\| \leq \sup|g| \prod_{i=1}^{m} \mathrm{Lip}(\tau_i)\|T\| \quad \text{with } \omega = g\,d\tau_1 \wedge \cdots \wedge d\tau_m. \tag{14.5}$$

For metric functionals with finite mass, the restriction operator $T \llcorner \omega$ can be defined even though $\omega = (g, \tau_1, \ldots, \tau_m)$ with g bounded Borel, and still (14.5) holds; the restriction will be denoted by $T \llcorner A$ in the special case $m = 0$ and $g = \chi_A$.

Finally, we will use the following approximation results.

Proposition 14.7. *Let E be a closed convex set of a Banach space. Then $\mathbf{I}_k(E)$ is dense in $\mathcal{I}_k(E)$ in mass norm. As a consequence $\mathbf{I}_k(E)$ is dense in $\mathcal{F}_k(E)$ in flat norm. The same holds in metric spaces F that are Lipschitz retracts of E.*

Proof. We argue as in Theorem 4.5 of [3], reducing ourselves to the approximation of currents $T \in \mathcal{I}_k(E)$ of the form $f_\sharp[\![\theta]\!]$ with $\theta \in L^1(\mathbb{R}^k, \mathbb{Z})$, $B \subset \mathbb{R}^k$ Borel, $f: B \to E$ Lipschitz and $\theta = 0$ $\mathcal{L}^k$-a.e. out of B. Since E is closed and convex, the construction of [29] provides a Lipschitz extension of f to the whole of $\mathbb{R}^k$, still with values in E. For $\varepsilon > 0$ given, we can choose $\theta' \in BV(\mathbb{R}^k; \mathbb{Z})$ such that $\int_{\mathbb{R}^k} |\theta - \theta'|\,dx < \varepsilon$ to obtain that the current $\widetilde{T} := f_\sharp[\![\theta']\!] \in \mathbf{I}_k(E)$ satisfies $\mathbf{M}(T - \widetilde{T}) < \varepsilon[\mathrm{Lip}(f)]^k$.

If $T \in \mathcal{I}_k(F)$ and $i: E \to F$ is a Lipschitz retraction, then we can find a sequence $(T_n) \subset \mathbf{I}_k(E)$ converging in mass to T. Then, the sequence $(i_\sharp T_n) \subset \mathbf{I}_k(F)$ provides the desired approximation. □

Proposition 14.8 (Characterization of mass). *Let $T \in MF_k(E)$ with finite mass. Then $\|T\|(E)$ is representable by*

$$\sup\Big\{\sum_{i=1}^{N} \|T \llcorner d\pi^i\|(A_i)\Big\}, \tag{14.6}$$

where the supremum runs among all finite disjoint families of open sets $A_1, \ldots, A_N$ and all N-ples of 1-Lipschitz maps π^i.

Proof. In [3], Proposition 2.7, it is proved that

$$\|T\|(E) = \sup\Big\{\sum_{i=1}^{N} \|T \llcorner d\pi_1^i \wedge \cdots \wedge d\pi_k^i\|(A_i)\Big\},$$

where the supremum runs among all finite disjoint families (A_i) of Borel sets and 1-Lipschitz maps π_j^i, $1 \leq i \leq N$ and $1 \leq j \leq k$. Approximating Borel sets from inside with compact sets, and then compacts sets from outside with open sets, one can see that the supremum is the same if (A_i) runs among all finite disjoint families of open sets. By the inequalities

$$\|T \llcorner dq_1 \wedge \cdots \wedge dq_k\| \leq \|T \llcorner dq_1\| \leq \|T\|$$

with q_j 1-Lipschitz we obtain (14.6). □

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Luigi Ambrosio, Scuola Normale Superiore, Piazza Cavalieri 7, 56100 Pisa, Italy
E-mail: l.ambrosio@sns.it

Mikhail G. Katz, Department of Mathematics, Bar Ilan University, Ramat Gan 52900, Israel
E-mail: katzmik@macs.biu.ac.il