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# Asymptotic isoperimetry on groups and uniform embeddings into Banach spaces

Romain Tessera

**Abstract.** We characterize the possible asymptotic behaviors of the compression associated to a uniform embedding into some  $L^p$ -space, with  $1 < p < \infty$ , for a large class of groups including connected Lie groups with exponential growth and word-hyperbolic finitely generated groups. In particular, the Hilbert compression exponent of these groups is equal to 1. This also provides new and optimal estimates for the compression of a uniform embedding of the infinite 3-regular tree into some  $L^p$ -space. The main part of the paper is devoted to the explicit construction of affine isometric actions of amenable connected Lie groups on  $L^p$ -spaces whose compressions are asymptotically optimal. These constructions are based on an asymptotic lower bound of the  $L^p$ -isoperimetric profile inside balls. We compute the asymptotic behavior of this profile for all amenable connected Lie groups and for all  $1 \leq p < \infty$ , providing new geometric invariants of these groups. We also relate the Hilbert compression exponent with other asymptotic quantities such as volume growth and probability of return of random walks.

**Mathematics Subject Classification (2010).** 20F65, 43A85.

**Keywords.** Isoperimetric profile on groups, Hilbert compression.

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## 1. Introduction

The study of uniform embeddings of locally compact groups into Banach spaces and especially of those associated to proper affine isometric actions plays a crucial role in various fields of mathematics ranging from K-theory to geometric group theory.

Recall that a locally compact group is called a-T-menable if it admits a proper affine action by isometries on a Hilbert space (for short: a proper isometric Hilbert action). An amenable  $\sigma$ -compact locally compact group is always a-T-menable [CCJJV]; but the converse is false since for instance non-amenable free groups are a-T-menable. However, if a locally compact, compactly generated group  $G$  admits a proper isometric Hilbert action whose compression  $\rho$  satisfies

$$\rho(t) \succ t^{1/2},$$

then  $G$  is amenable<sup>1</sup>. On the other hand, in [CTV], we prove that non-virtually abelian polycyclic groups cannot have proper isometric Hilbert actions with linear compression. These results motivate a systematic study of the possible asymptotic behaviors of compression functions, especially for amenable groups.

In this paper, we “characterize” the asymptotic behavior of the  $L^p$ -compression, with  $1 < p < \infty$ , for a large class of groups including all connected Lie groups with exponential growth. Some partial results in this direction for  $p = 2$  had been obtained in [GK] and [BrSo] by completely different methods.

**1.1.  $L^p$ -compression: optimal estimates.** Let us recall some basic definitions. Let  $G$  be some locally compact compactly generated group. Equip  $G$  with the word length function  $|\cdot|_S$  associated to a compact symmetric generating subset  $S$  and consider a uniform embedding  $F$  of  $G$  into some Banach space. The compression  $\rho$  of  $F$  is the nondecreasing function defined by

$$\rho(t) = \inf_{|g^{-1}h|_S \geq t} \|F(g) - F(h)\|.$$

Let  $f, g: \mathbb{R}_+ \rightarrow \bar{\mathbb{R}}_+$  be nondecreasing, nonzero functions. We write respectively  $f \preceq g$ ,  $f \prec g$  if there exists  $C > 0$  such that  $f(t) = O(g(Ct))$ , resp. for all  $c > 0$ ,  $f(t) = o(g(ct))$  when  $t \rightarrow \infty$ . We write  $f \approx g$  if both  $f \preceq g$  and  $g \preceq f$ . The asymptotic behavior of  $f$  is its class modulo the equivalence relation  $\approx$ .

Note that the asymptotic behavior of the compression of a uniform embedding does not depend on the choice of  $S$ .

In the sequel, an  $L^p$ -space denotes a Banach space of the form  $L^p(X, m)$  where  $(X, m)$  is a measure space. An  $L^p$ -representation of  $G$  is a continuous linear  $G$ -action on some  $L^p$ -space. Let  $\pi$  be an isometric  $L^p$ -representation of  $G$  and consider a 1-cocycle  $b \in Z^1(G, \pi)$ , or equivalently an affine isometric action of  $G$  with linear part  $\pi$ : see the preliminaries for more details. The compression of  $b$  is defined by

$$\rho(t) = \inf_{|g|_S \geq t} \|b(g)\|_p.$$

---

<sup>1</sup>This was proved for finitely generated groups in [GK]. In [CTV], we give a shorter argument that applies to all locally compact compactly generated groups.

In this paper, we mainly focus our attention on groups in the two following classes. Denote  $(\mathcal{L})$  the class of groups including

- (1) polycyclic groups and connected amenable Lie groups;
- (2) semidirect products  $\mathbb{Z}[\frac{1}{mn}] \rtimes_{\frac{m}{n}} \mathbb{Z}$ , with  $m, n$  co-prime integers with<sup>2</sup>  $|mn| \geq 2$  (if  $n = 1$  this is the Baumslag–Solitar group  $BS(1, m)$ ); semidirect products  $(\mathbb{R} \oplus \bigoplus_{p \in P} \mathbb{Q}_p) \rtimes_{\frac{m}{n}} \mathbb{Z}$  with  $m, n$  coprime integers and  $P$  a finite family of primes dividing  $mn$ ;
- (3) wreath products  $F \wr \mathbb{Z}$  for  $F$  a finite group.

Denote  $(\mathcal{L}')$  the class of groups including groups in the class  $(\mathcal{L})$  and

- (1) connected Lie groups and their cocompact lattices;
- (2) irreducible lattices in semisimple groups of rank  $\geq 2$ ;
- (3) hyperbolic finitely generated groups.

Let  $\mu$  be a left Haar measure on the locally compact group  $G$  and write  $L^p(G) = L^p(G, \mu)$ . The group  $G$  acts by isometry on  $L^p(G)$  via the left regular representation  $\lambda_{G,p}$  defined by

$$\lambda_{G,p}(g)\varphi = \varphi(g^{-1}\cdot).$$

**Theorem 1.** *Fix some  $1 \leq p < \infty$ . Let  $G$  be a group of the class  $(\mathcal{L})$  and let  $f$  be an increasing function  $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying*

$$\int_1^\infty \left( \frac{f(t)}{t} \right)^p \frac{dt}{t} < \infty. \quad (C_p)$$

*Then there exists a 1-cocycle  $b \in Z^1(G, \lambda_{G,p})$  whose compression  $\rho$  satisfies*

$$\rho \succeq f.$$

**Corollary 2.** *Fix some  $1 \leq p < \infty$ . Let  $G$  be a group of the class  $(\mathcal{L}')$  and let  $f$  be an increasing function  $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying Property  $(C_q)$ , with  $q = \max\{p, 2\}$ . Then there exists a uniform embedding of  $G$  into some  $L^p$ -space whose compression  $\rho$  satisfies*

$$\rho \succeq f.$$

Let us sketch the proof of the corollary. First, recall [W], III.A.6, that for  $1 \leq p \leq 2$ ,  $L^2([0, 1])$  is isomorphic to a subspace of  $L^p([0, 1])$ . It is thus enough to prove the theorem for  $2 \leq p < \infty$ . This is an easy consequence of Theorem 1 since every group of class  $(\mathcal{L}')$  quasi-isometrically embeds into a group of  $(\mathcal{L})$ . Indeed, any connected Lie group admits a closed cocompact connected solvable subgroup.

<sup>2</sup>This condition guaranties that the group is compactly generated.



On the other hand, irreducible lattices in semisimple groups of rank  $\geq 2$  are quasi-isometrically embedded [LMR]. Finally, any hyperbolic finitely generated group quasi-isometrically embeds into the real hyperbolic space  $\mathbf{H}^n$  for  $n$  large enough [BoS] which is itself quasi-isometric to  $\mathrm{SO}(n, 1)$ .

The particular case of nonabelian free groups, which are quasi-isometric to 3-regular trees, can also be treated by a more direct method. More generally that method applies to any simplicial<sup>3</sup> tree with possibly infinite degree.

**Theorem 3** (see Theorem 7.3). *Let  $T$  be a simplicial tree. For every increasing function  $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying*

$$\int_1^\infty \left( \frac{f(t)}{t} \right)^p \frac{dt}{t} < \infty, \quad (C_p)$$

*there exists a uniform embedding  $F$  of  $T$  into  $\ell^p(T)$  with compression  $\rho \succeq f$ .*

**Remark 1.1.** In [BuSc1], [BuSc2], it is shown that real hyperbolic spaces and word hyperbolic groups quasi-isometrically embed into finite products of (simplicial) trees. Thus the restriction of Corollary 2 to word hyperbolic groups and to simple Lie groups of rank 1 can be deduced from Proposition 7.3. Nevertheless, not every connected Lie group quasi-isometrically embeds into a finite product of trees. Namely, a finite product of trees is a CAT(0) space, and in [Pau] it is proved that a non-abelian simply connected nilpotent Lie group cannot quasi-isometrically embed into any CAT(0) space.

**Theorem 4.** *Let  $T_N$  be the binary rooted tree of depth  $N$ . Let  $\rho$  be the compression of some 1-Lipschitz map from  $T_N$  to some  $L^p$ -space for  $1 < p < \infty$ . Then there exists  $C < \infty$ , depending only on  $p$ , such that*

$$\int_1^{2N} \left( \frac{\rho(t)}{t} \right)^q \frac{dt}{t} \leq C,$$

*where  $q = \max\{p, 2\}$ .*

Although this result is a strengthening (see Corollary 6.3) of Theorem 1 in [Bou], its proof is based on the same arguments. As a consequence, we have

**Corollary 5.** *Assume that the 3-regular tree quasi-isometrically embeds into some metric space  $X$ . Then, the compression  $\rho$  of any uniform embedding of  $X$  into any  $L^p$ -space for  $1 < p < \infty$  satisfies  $(C_q)$  for  $q = \max\{p, 2\}$ .*

---

<sup>3</sup>By simplicial, we mean that every edge has length 1.

In [BeSc], Theorem 1.5, it is proved that the 3-regular tree quasi-isometrically embeds into any graph with bounded degree and positive Cheeger constant (e.g. any non-amenable finitely generated group). On the other hand, in a work in preparation with Cornulier [CT], we prove that finitely generated linear groups with exponential growth, and finitely generated solvable groups with exponential growth admit quasi-isometrically embedded free non-abelian sub-semigroups. Together with the above corollary, they lead to the optimality of Theorem 1 (resp. Corollary 2) when the group has exponential growth and when  $2 \leq p < \infty$  (resp.  $1 < p < \infty$ ).

**Corollary 6.** *Let  $G$  be a finitely generated group with exponential growth which is either virtually solvable or non-amenable. Let  $\varphi$  be a uniform embedding of  $G$  into some  $L^p$ -space for  $1 < p < \infty$ . Then its compression  $\rho$  satisfies Condition  $(C_q)$  for  $q = \max\{p, 2\}$ .*

**Corollary 7.** *Let  $G$  be a group of class  $(\mathcal{L}')$  with exponential growth. Consider an increasing map  $f$  and some  $1 < p < \infty$ ; then  $f$  satisfies Condition  $(C_q)$  with  $q = \max\{p, 2\}$  if and only if there exists a uniform embedding of  $G$  into some  $L^p$ -space whose compression  $\rho$  satisfies  $\rho \succeq f$ .*

Note that the 3-regular tree cannot uniformly embed into a group with subexponential growth. So the question of the optimality of Theorem 1 for non-abelian nilpotent connected Lie groups remains open.

**About Condition  $(C_p)$ .** First, note that if  $p \leq q$ , then  $(C_p)$  implies  $(C_q)$ : this immediately follows from the fact that a nondecreasing function  $f$  satisfying  $(C^p)$  also satisfies  $f(t)/t = O(1)$ .

Let us give examples of functions  $f$  satisfying Condition  $(C_p)$ . Clearly, if  $f$  and  $h$  are two increasing functions such that  $f \leq h$  and  $h$  satisfies  $(C_p)$ , then  $f$  satisfies  $(C_p)$ . The function  $f(t) = t^a$  satisfies  $(C_p)$  for every  $a < 1$  but not for  $a = 1$ . More precisely, the function

$$f(t) = \frac{t}{(\log t)^{1/p}}$$

does not satisfy  $(C_p)$  but

$$f(t) = \frac{t}{((\log t)(\log \log t)^a)^{1/p}}$$

satisfies  $(C_p)$  for every  $a > 1$ . In comparison, in [BrSo], the authors construct a uniform embedding of the free group of rank 2 into a Hilbert space with compression larger than

$$\frac{t}{((\log t)(\log \log t)^2)^{1/2}}.$$

As  $t/(\log t)^{1/p}$  does not satisfy  $(C_p)$ , one may wonder if  $(C_p)$  implies

$$\rho(t) \leq \frac{t}{(\log t)^{1/p}}.$$

The following proposition answers negatively to this question. We say that a function  $f$  is sublinear if  $f(t)/t \rightarrow 0$  when  $t \rightarrow \infty$ .

**Proposition 8** (see Proposition 7.5). *For any increasing sublinear function  $h: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and every  $1 \leq p < \infty$ , there exists a nondecreasing function  $f$  satisfying  $(C_p)$ , a constant  $c > 0$  and an increasing sequence of integers  $(n_i)$  such that*

$$f(n_i) \geq ch(n_i) \quad \forall i \in \mathbb{N}.$$

In particular, it follows from Theorem 1 that the compression  $\rho$  of a uniform embedding of a 3-regular tree in a Hilbert space does not satisfy any *a priori* majoration by any sublinear function.

**1.2. Isoperimetry and compression.** To prove Theorem 1, we observe a general relation between the  $L^p$ -isoperimetry inside balls and the  $L^p$ -compression. Let  $G$  be a locally compact compactly generated group and consider some compact symmetric generating subset  $S$ . For every  $g \in G$ , write<sup>4</sup>

$$|\tilde{\nabla}\varphi|(g) = \sup_{s \in S} |\varphi(sg) - \varphi(g)|.$$

Let  $2 \leq p < \infty$  and let us call the  $L^p$ -isoperimetric profile inside balls the nondecreasing function  $J_{G,p}^b$  defined by

$$J_{G,p}^b(t) = \sup_{\varphi} \frac{\|\varphi\|_p}{\|\tilde{\nabla}\varphi\|_p},$$

where the supremum is taken over all measurable functions in  $L^p(G)$  with support in the ball  $B(1, t)$ . Note that the group  $G$  is amenable if and only if  $\lim_{t \rightarrow \infty} J_{G,p}^b(t) = \infty$ . Theorem 1 results from the two following theorems.

**Theorem 9** (see Theorem 5.1). *Let  $G$  be a group of class  $(\mathcal{L})$ . Then  $J_{G,p}^b(t) \approx t$ .*

**Theorem 10** (see Corollary 4.6). *Let  $G$  be a locally compact compactly generated group and let  $f$  be a nondecreasing function satisfying*

$$\int_1^\infty \left( \frac{f(t)}{J_{G,p}^b(t)} \right)^p \frac{dt}{t} < \infty \quad (CJ_p)$$

---

<sup>4</sup>We write  $\tilde{\nabla}$  instead of  $\nabla$  because this is not a “metric” gradient. The gradient associated to the metric structure would be the right gradient:  $|\nabla\varphi|(g) = \sup_{s \in S} |\varphi(gs) - \varphi(g)|$ . This distinction is only important when the group is non-unimodular.

for some  $1 < p < \infty$ . Then there exists a 1-cocycle  $b \in Z^1(G, \lambda_{G,p})$  whose compression  $\rho$  satisfies  $\rho \succeq f$ .

Theorem 9 may sound as a “functional” property of groups of class  $(\mathcal{L})$ . Nevertheless, our proof of this result is based on a purely geometric construction. Namely, we prove that these groups admit controlled Følner pairs (see Definition 4.8). In particular, when  $p = 1$  we obtain the following corollary of Theorem 9, which has its own interest.

**Theorem 11** (see Remark 4.10 and Theorem 5.1). *Let  $G$  be a group of class  $(\mathcal{L})$  and let  $S$  be some compact generating subset of  $G$ . Then  $G$  admits a sequence of compact subsets  $(F_n)_{n \in \mathbb{N}}$  satisfying the two following conditions:*

(i) *there is a constant  $c > 0$  such that*

$$\mu(sF_n \triangle F_n) \leq c\mu(F_n)/n \quad \forall s \in S, \forall n \in \mathbb{N};$$

(ii) *for every  $n \in \mathbb{N}$ ,  $F_n$  is contained<sup>5</sup> in  $S^n$ .*

*In particular,  $G$  admits a controlled Følner sequence in the sense of [CTV].*

This theorem is a strengthening of the well-known construction by Pittet [Pit]. It is stronger first because it does not require the group to be unimodular, second because the control (ii) of the diameter is really a new property that was not satisfied in general by the sequences constructed in [Pit].

**1.3. Compression, subexponential growth, and random walks.** Let  $\pi$  be an isometric  $L^p$ -representation of  $G$ . Denote by  $B_\pi(G)$  the supremum of all  $\alpha$  such that there exists a 1-cocycle  $b \in Z^1(G, \pi)$  whose compression  $\rho$  satisfies  $\rho(t) \succeq t^\alpha$ . Denote by  $B_p(G)$  the supremum of  $B_\pi(G)$  over all isometric  $L^p$ -representations  $\pi$ . For  $p = 2$ ,  $B_2(G) = B(G)$  has been introduced in [GK] where it was called the equivariant Hilbert compression rate (we suggest that the term *exponent* would be more appropriate here than the term *rate*). On the other hand, define

$$\alpha_{G,p} = \liminf_{t \rightarrow \infty} \frac{\log J_{G,p}^b(t)}{\log t}.$$

As a corollary of Theorem 1, we have

**Corollary 12.** *For every  $1 \leq p < \infty$ , and every group  $G$  of the class  $(\mathcal{L})$ , we have  $B_p(G) = 1$ .*

The following result is a corollary of Theorem 10.

---

<sup>5</sup>Actually, they also satisfy  $S^{[cn]} \subset F_n$  for a constant  $c > 0$ .

**Corollary 13** (see Corollary 4.6). *Let  $G$  be a locally compact compactly generated group. For every  $0 < p < \infty$ , we have*

$$B_{\lambda_{G,p}}(G) \geq \alpha_{G,p}.$$

The interest of this corollary is illustrated by the two following propositions. Recall the volume growth of  $G$  is the  $\approx$  equivalence class  $V_G$  of the function  $r \mapsto \mu(B(1, r))$ .

**Proposition 14** (see Proposition 7.1). *Assume that there exists  $\beta < 1$  such that  $V_G(r) \leq e^{r^\beta}$ . Then*

$$\alpha_{G,p} \geq 1 - \beta.$$

As an example we obtain that  $B(G) \geq 0,19$  for the first Grigorchuk's group (see [Ba] for the best known upper bound of the growth function of this group).

Let  $G$  be a finitely generated group and let  $\nu$  be a symmetric finitely supported probability measure on  $G$ . Write  $\nu^{(n)} = \nu * \dots * \nu$  ( $n$  times). Recall that  $\nu^{(n)}(1)$  is the probability of return of the random walk starting at 1 whose probability transition is given by  $\nu$ .

**Proposition 15** (see Proposition 7.2). *Assume that there exists  $\gamma < 1$  such that  $\nu^{(n)}(1) \geq e^{-n^\gamma}$ . Then*

$$\alpha_{G,2} \geq (1 - \gamma)/2.$$

In [PS], it is proved that if  $G$  is a finitely generated extension

$$1 \rightarrow K \rightarrow G \rightarrow N \rightarrow 1$$

where  $K$  is abelian and  $N$  is abelian with  $\mathbb{Q}$ -rank  $d$ . Then

$$\limsup_n \log(-\log(\nu^{(n)}(1))) \leq 1 - 2/(d + 2)$$

for any symmetric finitely supported probability on  $G$ .

**Corollary 16.** *Assume that  $G$  is a finitely generated extension  $1 \rightarrow K \rightarrow G \rightarrow N \rightarrow 1$  where  $K$  is abelian and  $N$  is abelian with  $\mathbb{Q}$ -rank  $d$ . Then*

$$B(G) \geq 1/(d + 2).$$

*In particular,  $B(G) > 0$  for any finitely generated metabelian group  $G$ .*

**1.4. The case of  $\mathbb{Z} \wr \mathbb{Z}$ .** Combining the construction of Theorem 1 for  $C_2 \wr \mathbb{Z}$  with the cocycle induced by the morphism of  $\mathbb{Z}^{(\mathbb{Z})} \rightarrow \ell^p(\mathbb{Z})$ , we obtain (see Proposition 7.6 for the details).

**Theorem 17.** *Fix some  $1 \leq p < \infty$ . Let  $G = \mathbb{Z} \wr \mathbb{Z}$  and let  $f$  be an increasing function  $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying*

$$\int_1^\infty \left( \frac{f(t)}{t^{p/(2p-1)}} \right)^p \frac{dt}{t} < \infty. \quad (C_p)$$

*Then there exists a 1-cocycle  $b \in Z^1(G, \lambda_{G,p})$  whose compression  $\rho$  satisfies*

$$\rho \succeq f.$$

*In particular,*

$$B_p(\mathbb{Z} \wr \mathbb{Z}) \geq \frac{p}{2p-1}.$$

In a previous version of this paper, we stated the lower bound  $B(\mathbb{Z} \wr \mathbb{Z}) \geq 2/3$ , but the proof that we gave relied on a wrong version of Proposition 15 (we stated  $\alpha_{G,2} \geq 1 - \gamma$ , which is wrong as shown by a counter-example in [NP]). The mistake, together with a proof of the full statement  $B_p(\mathbb{Z} \wr \mathbb{Z}) \geq \frac{p}{2p-1}$  (see [NP], Lemma 7.8) was communicated to us by Naor and Peres. The proof that we propose here is essentially the same as the one of [NP], but it was actually also known by the author.

## 1.5. Questions

**Question 1.2** (Condition  $(C_p)$  for nilpotent connected Lie groups.). Let  $N$  be a simply connected non-abelian nilpotent Lie group and let  $\rho$  be the compression of a 1-cocycle with values in some  $L^p$ -space (resp. of a uniform embedding into some  $L^p$ -space) for  $2 \leq p < \infty$ . Does  $\rho$  always satisfies Condition  $(C_p)$ ?

A positive answer would lead to the optimality of Theorem 1. On the contrary, one should wonder if it is possible, for any increasing sublinear function  $f$ , to find a 1-cocycle (resp. a uniform embedding) in  $L^p$  with compression  $\rho \succeq f$ . This would also be optimal since we know [Pau] that  $N$  cannot quasi-isometrically embed into any uniformly convex Banach space. Namely, the main theorem in [Pau] states that such a group cannot quasi-isometrically embed into any CAT(0)-space. So this only directly applies to Hilbert spaces, but the key argument, consisting in a comparison between the large scale behavior of geodesics (not exactly in the original spaces but in tangent cones of ultra-products of them) is still valid if the target space is a Banach space with unique geodesics, a property satisfied by uniformly convex Banach spaces.

**Question 1.3** (Quasi-isometric embeddings into  $L^1$ -spaces.). Which connected Lie groups quasi-isometrically embed into some  $L^1$ -space?

It is easy to quasi-isometrically embed a simplicial tree  $T$  into  $\ell^1$  (see for instance [GK]). In [BuSc1], [BuSc2], it is proved that every semisimple Lie group of rank 1 quasi-isometrically embeds into a finite product of simplicial trees, hence into a  $\ell^1$ -space. The above question is of particular interest for simply-connected non-abelian nilpotent Lie groups since they do not quasi-isometrically embed into any finite product of trees. Kleiner and Cheeger recently announced a proof that the Heisenberg group cannot quasi-isometrically embed into any  $L^1$ -space.

**Question 1.4.** If  $G$  is an amenable group, is it true that

$$B_p(G) = \alpha_{G,p}?$$

We conjecture that this is true for  $\mathbb{Z} \wr \mathbb{Z}$ , i.e. that  $B(\mathbb{Z} \wr \mathbb{Z}) = 2/3$ . A first step to prove this is done by Proposition 3.9 which, applied to  $G = \mathbb{Z} \wr \mathbb{Z}$  says that

$$B(\mathbb{Z} \wr \mathbb{Z}) = B_{\lambda_{G,2}}(\mathbb{Z} \wr \mathbb{Z}).$$

As a variant of the above question, we may wonder if the weaker equality  $B_{\lambda_{G,p}}(G) = \alpha_{G,p}$  holds, in other words if Corollary 13 is optimal for all amenable groups. Possible counterexamples would be wreath products of the form  $G = \mathbb{Z} \wr H$  where  $H$  has non-linear growth (e.g.  $H = \mathbb{Z}^2$ ).

**Question 1.5.** Does there exist an amenable group  $G$  with  $B(G) = 0$ ?

A candidate would be the wreath product  $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$  since the probability of return of any non-degenerate random walk in this group satisfies

$$\nu^{(n)}(1) \prec e^{-n^\gamma}$$

for every  $\gamma < 1$  ([Er], Theorem 2). It is proved in [AGS] that  $B(\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})) \leq 1/2$ .

**Question 1.6.** Let  $G$  be a compactly generated locally compact group. If  $G$  admits an isometric action on some  $L^p$ -space,  $p \geq 2$ , with compression  $\rho(t) \asymp t^{1/p}$ , does it imply that  $G$  is amenable?

Recall that this was proved in [GK], [CTV] for  $p = 2$ . The generalization to every  $p \geq 2$  would be of great interest. For instance, this would prove the optimality of a recent result of Yu [Yu] saying that every finitely generated hyperbolic group admits a proper isometric action on some  $\ell^p$ -space for large  $p$  enough, with<sup>6</sup> compression  $\rho(t) \approx t^{1/p}$ .

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<sup>6</sup>This is clear in the proof.



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## 2. Preliminaries

**2.1. Compression.** Let us recall some definitions. Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. A map  $F: X \rightarrow Y$  is called a uniform embedding of  $X$  into  $Y$  if

$$d_X(x, y) \rightarrow \infty \iff d_Y(F(x), F(y)) \rightarrow \infty.$$

Note that this property only concerns the large-scale geometry. A metric space  $(X, d)$  is called *quasi-geodesic* if there exist  $\delta > 0$  and  $\gamma \geq 1$  such that for all  $x, y \in X$ , there exists a chain  $x = x_0, x_1, \dots, x_n = y$  satisfying:

$$\sum_{k=1}^n d(x_{k-1}, x_k) \leq \gamma d(x, y),$$

$$\forall k = 1, \dots, n, \quad d(x_{k-1}, x_k) \leq \delta.$$

If  $X$  is quasi-geodesic and if  $F: X \rightarrow Y$  is a uniform embedding, then it is easy to see that  $F$  is large-scale Lipschitz, i.e. there exists  $C \geq 1$  such that

$$\forall x, y \in X, \quad d_Y(F(x), F(y)) \leq C d_X(x, y) + C.$$

Nevertheless, such a map is not necessarily large scale bi-Lipschitz (in other words, quasi-isometric).

**Definition 2.1.** We define the *compression*  $\rho: \mathbb{R}_+ \rightarrow [0, \infty]$  of a map  $F: X \rightarrow Y$  by

$$\forall t > 0, \quad \rho(t) = \inf_{d_X(x, y) \geq t} d_Y(F(x), F(y)).$$

Clearly, if  $F$  is large-scale Lipschitz, then  $\rho(t) \leq t$ .



**2.2. Length functions on a group.** Now, let  $G$  be a group. A length function on  $G$  is a function  $L: G \rightarrow \mathbb{R}_+$  satisfying  $L(1) = 0$ ,  $L(gh) \leq L(g) + L(h)$ , and  $L(g) = L(g^{-1})$ . If  $L$  is a length function, then  $d(g, h) = L(g^{-1}h)$  defines a left-invariant pseudo-metric on  $G$ . Conversely, if  $d$  is a left-invariant pseudo-metric on  $G$ , then  $L(g) = d(1, g)$  defines a length function on  $G$ .

Let  $G$  be a locally compact compactly generated group and let  $S$  be some compact symmetric generating subset of  $G$ . Equip  $G$  with a proper, quasi-geodesic length function by

$$|g|_S = \inf \{n \in \mathbb{N} : g \in S^n\}.$$

Denote  $d_S$  the associated left-invariant distance. Note that any proper, quasi-geodesic left-invariant metric is quasi-isometric to  $d_S$ , and so belongs to the same “asymptotic class”.

**2.3. Affine isometric actions and first cohomology.** Let  $G$  be a locally compact group, and  $\pi$  an isometric representation (always assumed continuous) on a Banach space  $E = E_\pi$ . The space  $Z^1(G, \pi)$  is defined as the set of continuous functions  $b: G \rightarrow E$  satisfying, for all  $g, h$  in  $G$ , the 1-cocycle condition  $b(gh) = \pi(g)b(h) + b(g)$ . Observe that, given a continuous function  $b: G \rightarrow \mathcal{H}$ , the condition  $b \in Z^1(G, \pi)$  is equivalent to saying that  $G$  acts by affine isometries on  $\mathcal{H}$  by  $\alpha(g)v = \pi(g)v + b(g)$ . The space  $Z^1(G, \pi)$  is endowed with the topology of uniform convergence on compact subsets.

The subspace of coboundaries  $B^1(G, \pi)$  is the subspace (not necessarily closed) of  $Z^1(G, \pi)$  consisting of functions of the form  $g \mapsto v - \pi(g)v$  for some  $v \in E$ . In terms of affine actions,  $B^1(G, \pi)$  is the subspace of affine actions fixing a point.

The first cohomology space of  $\pi$  is defined as the quotient space

$$H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi).$$

Note that if  $b \in Z^1(G, \pi)$ , the map  $(g, h) \rightarrow \|b(g) - b(h)\|$  defines a left-invariant pseudo-distance on  $G$ . Therefore the compression of a 1-cocycle  $b: (G, d_S) \rightarrow E$  is simply given by

$$\rho(t) = \inf_{|g|_S \geq t} \|b(g)\|.$$

The compression of an affine isometric action is defined as the compression of the corresponding 1-cocycle.

**Remark 2.2.** When the space  $E$  is a Hilbert space<sup>7</sup>, it is well known [HV], §4.a, that  $b \in B^1(G, \pi)$  if and only if  $b$  is bounded on  $G$ .

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<sup>7</sup>The same proof holds for uniformly convex Banach spaces.

### 3. The maximal $L^p$ -compression functions $M\rho_{G,p}$ and $M\rho_{\lambda_{G,p}}$

**3.1. Definitions and general results.** Let  $(G, d_S, \mu)$  be a locally compact compactly generated group, generated by some compact symmetric subset  $S$  and equipped with a left Haar measure  $\mu$ . Denote by  $Z^1(G, p)$  the collection of all 1-cocycles with values in any  $L^p$ -representation of  $G$ . Denote by  $\rho_b$  the compression function of a 1-cocycle  $b \in Z^1(G, p)$ .

**Definition 3.1.** We call *maximal  $L^p$ -compression function* of  $G$  the nondecreasing function  $M\rho_{G,p}$  defined by

$$M\rho_{G,p}(t) = \sup \left\{ \rho_b(t) : b \in Z^1(G, p), \sup_{s \in S} \|b(s)\| \leq 1 \right\}.$$

We call maximal *regular  $L^p$ -compression function* of  $G$  the nondecreasing function  $M\rho_{\lambda_{G,p}}$  defined by

$$M\rho_{\lambda_{G,p}} = \sup \left\{ \rho_b(t) : b \in Z^1(G, \lambda_{G,p}), \sup_{s \in S} \|b(s)\| \leq 1 \right\}.$$

Note that the asymptotic behaviors of both  $M\rho_{G,p}$  and  $M\rho_{\lambda_{G,p}}$  do not depend on the choice of the compact generating set  $S$ . Moreover, we have

$$M\rho_{\lambda_{G,p}}(t) \leq M\rho_{G,p}(t) \leq t.$$

Let  $\varphi$  be a measurable function on  $G$  such that  $\varphi - \lambda(s)\varphi \in L^p(G)$  for every  $s \in S$ . For every  $t > 0$ , define

$$\text{Var}_p(\varphi, t) = \inf_{|g|_S \geq t} \|\varphi - \lambda(g)\varphi\|_p.$$

The function  $\varphi$  and  $p$  being fixed, the map  $t \mapsto \text{Var}_p(\varphi, t)$  is nondecreasing.

**Proposition 3.2.** *We have*

$$M\rho_{\lambda_{G,p}}(t) = \sup_{\|\tilde{\nabla}\varphi\|_p \leq 1} \text{Var}_p(\varphi, t).$$

*Proof.* We trivially have

$$M\rho_{\lambda_{G,p}}(t) \geq \sup_{\|\tilde{\nabla}\varphi\|_p \leq 1} \text{Var}_p(\varphi, t).$$

Let  $b$  be an element of  $Z^1(G, \lambda_{G,p})$ . By convoluting  $b(g)$ , for every  $g$ , on the right by a Dirac approximation, one can approximate  $b$  by a cocycle  $b'$  such that  $x \rightarrow b'(g)(x)$  is continuous for every  $g$  in  $G$ . Hence, we can assume that  $b(g)$  is

continuous for every  $g$  in  $G$ . Now, setting  $\varphi(g) = b(g)(g)$ , we define a measurable function satisfying

$$b(g) = \varphi - \lambda(g)\varphi.$$

So we have

$$\rho(t) = \text{Var}_p(\varphi, t) \leq M\rho_{\lambda_{G,p}}(t)$$

where  $\rho$  is the compression of  $b$ . □

**Remark 3.3.** It is not difficult to prove that the asymptotic behavior of  $M\rho_{\lambda_{G,p}}$  is invariant under quasi-isometry between finitely generated groups.

**Proposition 3.4.** *The group  $G$  admits a proper<sup>8</sup> 1-cocycle with values in some  $L^p$ -representation if and only if  $M\rho_{G,p}(t)$  goes to infinity as  $t \rightarrow \infty$ .*

*Proof.* The “only if” part is trivial. Assume that  $M\rho_{G,p}(t)$  goes to infinity. Let  $(t_k)$  be an increasing sequence growing fast enough so that

$$\sum_{k \in \mathbb{N}} \frac{1}{t_k^p} < \infty.$$

For every  $k \in \mathbb{N}$ , choose some  $b_k \in Z^1(G, p)$  whose compression  $\rho_k$  satisfies

$$\rho_k(t_k) \geq \frac{M\rho_{G,p}(t_k)}{2}$$

and such that

$$\sup_{s \in S} \|b_k(s)\| \leq 1.$$

Clearly, we can define a 1-cocycle  $b \in Z^1(G, p)$  by

$$b = \bigoplus_k^{\ell^p} \frac{1}{t_k} b_k.$$

That is, if for every  $k$ ,  $b_k$  takes values in the representation  $\pi_k$ , then  $b$  takes values in the direct sum  $\bigoplus_k^{\ell^p} \pi_k$ . Now, observe that for  $|g| \geq t_k$  and  $j \leq k$ , we have  $\|b_j(g)\| \geq 1/2$ , so that

$$\|b(g)\|^p \geq k/2^p.$$

Thus the cocycle  $b$  is proper. □

The following proposition, which is a quantitative version of the previous one, plays a crucial role in the sequel.

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<sup>8</sup>For  $p = 2$ , this means that  $G$  is a-T-menable if and only if  $M\rho_{G,2}$  goes to infinity. It should be compared to the role played by the H-metric (see § 2.6 in [C], and § 7.4) for Property (T).

**Proposition 3.5.** *Let  $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a nondecreasing map satisfying*

$$\int_1^\infty \left( \frac{f(t)}{M\rho_{G,p}(t)} \right)^p \frac{dt}{t} < \infty, \quad (CM_p)$$

Then,

(1) *there exists a 1-cocycle  $b \in Z^1(G, p)$  such that*

$$\rho \succeq f;$$

(2) *if one replaces  $M\rho_{G,p}$  by  $M\rho_{\lambda_{G,p}}$  in Condition  $(CM_p)$ , then  $b$  can be chosen in  $Z^1(G, \lambda_{G,p})$ .*

*Proof.* (1): For every  $k \in \mathbb{N}$ , choose some  $b_k \in Z^1(G, p)$  (for (2), we take  $b_k \in Z^1(G, \lambda_{G,p})$ ) whose compression  $\rho_k$  satisfies

$$\rho_k(2^{k+1}) \geq \frac{M\rho_{G,p}(2^{k+1})}{2}$$

and such that

$$\sup_{s \in S} \|b_k(s)\| \leq 1.$$

Then define another sequence of cocycles  $\tilde{b}_k \in Z^1(G, p)$  by

$$\tilde{b}_k = \frac{f(2^k)}{M\rho_{G,p}(2^{k+1})} b_k.$$

Since  $M\rho_{G,p}$  and  $f$  are nondecreasing, for any  $2^k \leq t \leq 2^{k+1}$ , we have

$$\frac{f(2^k)}{M\rho_{G,p}(2^{k+1})} \leq \frac{f(t)}{M\rho_{G,p}(t)}.$$

Hence, for  $s \in S$ ,

$$\begin{aligned} \sum_k \|\tilde{b}_k(s)\|_p^p &\leq \sum_k \left( \frac{f(2^k)}{M\rho_{G,p}(2^{k+1})} \right)^p \\ &\leq 2 \int_1^\infty \left( \frac{f(t)}{M\rho_{G,p}(t)} \right)^p \frac{dt}{t} < \infty \end{aligned}$$

So we can define a 1-cocycle on  $b \in Z^1(G, p)$  by

$$b = \bigoplus_k \tilde{b}_k. \quad (3.1)$$

On the other hand, if  $|g|_S \geq 2^{k+1}$ , then

$$\begin{aligned} \|b(g)\|_p &\geq \|\tilde{b}_k(g)\|_p \\ &\geq \frac{f(2^k)}{M\rho_{\lambda_{G,p}}(2^{k+1})} \rho_k(2^{k+1}) \\ &\geq f(2^k). \end{aligned}$$

So if  $\rho$  is the compression of the 1-cocycle  $b$ , we have  $\rho \succeq f$ .

(2): We keep the previous notation. Assume that  $f$  satisfies

$$\int_1^\infty \left( \frac{f(t)}{M\rho_{\lambda_{G,p}}(t)} \right)^p \frac{dt}{t} < \infty.$$

The cocycle  $b$  provided by the proof of (1) has the expected compression but it takes values in an infinite direct sum of regular representation  $\lambda_{G,p}$ . Now, we would like to replace the direct sum  $b = \oplus_k b_k$  by a mere sum, in order to obtain a cocycle in  $Z^1(G, \lambda_{G,p})$ . Since  $G$  is not assumed unimodular, the measure  $\mu$  is not necessarily right-invariant. However, one can define an isometric representation  $r_{G,p}$  on  $L^p(G)$ , called the right regular representation by

$$r_{G,p}(g)\varphi = \Delta(g)^{-1}\varphi(\cdot g) \quad \forall \varphi \in L^p(G),$$

where  $\Delta$  is the modular function of  $G$ . We will use the following well-known property of the representation  $r_{G,p}$ , for  $p > 1$ . To simplify, let us write  $r(g)$  instead of  $r_{G,p}(g)$ . For every  $(\varphi, \psi) \in L^p(G) \times L^p(G)$ , we have

$$\lim_{|g| \rightarrow \infty} \|r(g)\varphi + \psi\|_p^p = \|\varphi\|_p^p + \|\psi\|_p^p. \quad (3.2)$$

Moreover, this limit is uniform on compact subsets of  $(L^p(G))^2$ . As  $r_{G,p}$  and  $\lambda_{G,p}$  commute,  $r_{G,p}$  acts by isometries on  $Z^1(G, \lambda_{G,p})$ .

**Lemma 3.6.** *There exists a sequence  $(g_k)$  of elements of  $G$  such that  $b' = \sum r(g_k)b_k$  defines a cocycle in  $Z^1(G, \lambda_{G,p})$  and such that*

$$\left| \|b'(g)\|_p^p - \left\| \sum_{j=0}^{k-1} r(g_j)b_j(g) \right\|_p^p - \sum_{j \geq k} \|b_j(g)\|_p^p \right| \leq 1 \quad (3.3)$$

for any  $k$  large enough and every  $g \in B(1, 2^{k+2})$ .

*Proof of Lemma 3.6.* By an immediate induction, using (3.2), we construct a sequence  $(g_k) \in G^{\mathbb{N}}$  satisfying, for every  $K \geq 0, s \in S$ ,

$$\left\| \sum_{k=0}^K r(g_k)b_k(s) \right\|_p^p \leq \sum_{k=0}^K \|b_k(s)\|_p^p + \sum_{k=0}^K 2^{-k-1} \leq 1,$$

which implies that  $b'$  is a well-defined 1-cocycle in  $Z^1(G, \lambda_{G,p})$ . Similarly, one can choose  $(g_k)$  satisfying the additional property that, for every  $k \in \mathbb{N}$ ,  $|g| \leq 2^{k+2}$ ,

$$\left| \left\| \sum_{j=0}^k r(g_j) b_j(g) \right\|_p^p - \left\| \sum_{j=0}^{k-1} r(g_j) b_j(g) \right\|_p^p - \|b_k(g)\|_p^p \right| \leq 2^{-k-1}.$$

Fixing  $k \in \mathbb{N}$ , an immediate induction over  $K$  shows that for every  $|g| \leq 2^{k+2}$  and every  $K \geq k$ ,

$$\left| \left\| \sum_{j=0}^K r(g_j) b_j(g) \right\|_p^p - \left\| \sum_{j=0}^{k-1} r(g_j) b_j(g) \right\|_p^p - \sum_{j=k}^K \|b_j(g)\|_p^p \right| \leq \sum_{j=k}^K 2^{-j-1}.$$

This proves (3.3).  $\square$

By the lemma, for  $|g| \leq 2^{k+2}$ ,

$$\|b'(g)\|_p^p \geq \|b_k(g)\|_p^p - 1.$$

Then, for  $2^{k+1} \leq |g| \leq 2^{k+2}$ , we have

$$\|b'(g)\|_p^p \geq f(2^k) - 1$$

Therefore, the compression  $\rho'$  of  $b'$  satisfies

$$\rho' \succeq f$$

and we are done.  $\square$

We have the following immediate consequence.

**Corollary 3.7.** *For every  $1 \leq p < \infty$ ,*

$$B(G, p) = \liminf_{t \rightarrow \infty} \frac{\log M_{\rho_{G,p}}(t)}{\log t}.$$

**Example 3.8.** Let  $F_r$  be the free group of rank  $r \geq 2$  and let  $A(F_r)$  be the set of edges of the Cayley graph of  $F_r$  associated to the standard set of generators. The standard isometric affine action of  $F_r$  on  $\ell^p(A(F_r))$ , whose linear part is isomorphic to a direct sum  $\lambda_{G,p} \oplus_{\ell^p} \cdots \oplus_{\ell^p} \lambda_{G,p}$  of  $r$  copies of  $\lambda_{G,p}$  has compression  $\approx t$ . This shows that  $M_{\rho_{\lambda_{F_r,p}}}(t) \succeq t^{1/p}$ .

**3.2. Reduction to the regular representation for  $p = 2$ .** In the Hilbert case, we prove that if a group admits a 1-cocycle with large enough compression, then  $M\rho_{G,2} = M\rho_{\lambda_{G,2}}$ . This result is mainly motivated by Question 1.4 since it implies that

$$B(\mathbb{Z} \wr \mathbb{Z}) = B_{\lambda_{G,2}}(\mathbb{Z} \wr \mathbb{Z}).$$

**Proposition 3.9.** *Let  $\pi$  be a unitary representation of the group  $G$  on a Hilbert space  $\mathcal{H}$  and let  $b \in Z^1(G, \pi)$  be a cocycle whose compression  $\rho$  satisfies*

$$\rho(t) \succ t^{1/2}.$$

*Then*<sup>9</sup>

$$\rho \leq M\rho_{\lambda_{G,2}}.$$

*In particular,*

$$M\rho_2 = M\rho_{\lambda_{G,2}}.$$

combining with Proposition 3.5, we obtain

**Corollary 3.10.** *With the same hypotheses, we have*

$$B(G) = B(G, \lambda_{G,2}) = \liminf_{t \rightarrow \infty} \frac{\log M\rho_{\lambda_{G,2}}(t)}{\log t}.$$

*Proof of Proposition 3.9.* For every  $t > 0$ , define

$$\varphi_t(g) = e^{-\|b(g)\|^2/t^2}.$$

By Schoenberg's Theorem (Appendix C in [BHV]),  $\varphi_t$  is positive definite. It is easy to prove that  $\varphi_t$  is square-summable (see [CTV], Theorem 4.1). By [Dix], Théorème 13.8.6, it follows that there exists a positive definite, square-summable function  $\psi_t$  on  $G$  such that  $\varphi_t = \psi_t * \psi_t$ , where  $*$  denotes the convolution product. In other words,  $\varphi_t = \langle \lambda(g)\psi_t, \psi_t \rangle$ . In particular,

$$\varphi_t(1) = 1 = \|\psi_t\|_2^2$$

and for every  $s \in S$ ,

$$\begin{aligned} \|\psi_t - \lambda(s)\psi_t\|_2^2 &= 2(\|\psi_t\|_2^2 - \langle \lambda(s)\psi_t, \psi_t \rangle) \\ &= 2(1 - \varphi_t(s)) \\ &= 2(1 - e^{-\|b(s)\|^2/t^2}) \\ &\leq 1/t^2 \end{aligned}$$

---

<sup>9</sup>Note that the hypotheses of the proposition also imply that  $G$  is amenable [CTV] (Theorem 4.1), [GK].

On the other hand, for  $g$  such that  $\rho(|g|_S) \geq t$ , we have

$$\begin{aligned}\|\psi_t - \lambda(g)\psi_t\|_2^2 &= 2(1 - e^{-\|b(g)\|^2/t^2}) \\ &\geq 2(1 - e^{-\rho(|g|_S)^2/t^2}) \\ &\geq 2(1 - 1/e).\end{aligned}$$

So, we have

$$\frac{\|\psi_t - \lambda(g)\psi_t\|_2}{\|\tilde{\nabla}\psi_t\|_2} \geq ct$$

where  $c$  is a constant. In other words,

$$\text{Var}_2(\psi_t, \rho^{-1}(t)) \geq ct.$$

It follows from the definitions that  $M\rho_{\lambda_{G,2}} \geq \rho$ . □

#### 4. $L^p$ -isoperimetry inside balls

**4.1. Comparing  $J_{G,p}^b$  and  $M\rho_{\lambda_{G,p}}$ .** Let  $G$  be a locally compact compactly generated group and let  $S$  be a compact symmetric generating subset of  $G$ . Let  $A$  be a subset of the group  $G$ . One defines the  $L^p$ -isoperimetric profile inside  $A$  by

$$J_p(A) = \sup_{\varphi} \frac{\|\varphi\|_p}{\|\tilde{\nabla}\varphi\|_p}$$

where the supremum is taken over nonzero functions in  $L^p(G)$  with support included in  $A$ .

**Definition 4.1.** The  $L^p$ -isoperimetric profile inside balls is the nondecreasing function  $J_{G,p}^b$  defined by

$$J_{G,p}^b(t) = J_p(B(1, t)).$$

**Remark 4.2.** The usual  $L^p$ -isoperimetric profile of  $G$  (see for example [Cou]) is defined by

$$j_{G,p}(t) = \sup_{\mu(A)=t} J_p(A).$$

Note that our notion of isoperimetric profile depends on the diameter of the subsets instead of their measure.

**Remark 4.3.** The asymptotic behavior of  $J_{p,G}^b$  is invariant under quasi-isometry between compactly generated groups [T]. In particular, it is also invariant under passing to a cocompact lattice [CS].



**Remark 4.4.** Using basic  $L^p$ -calculus, one can easily prove [Cou] that if  $p \leq q$ , then

$$(J_{G,p}^b)^{p/q} \leq J_{G,q}^b \leq J_{G,p}^b.$$

Now let us compare  $J_{p,G}^b$  and  $M\rho_{\lambda_{G,p}}$  introduced in § 3.

**Proposition 4.5.** *For every  $2 \leq p < \infty$ , we have*

$$M\rho_{\lambda_{G,p}} \succeq J_{G,p}^b.$$

*Proof.* Fix some  $t > 0$  and choose some  $\varphi \in L^p(X)$  whose support lies in  $B(1, t)$  such that

$$\frac{\|\varphi\|_p}{\|\tilde{\nabla}\varphi\|_p} \geq J_{G,p}^b(t)/2.$$

Take  $g \in G$  satisfying  $|g|_S \geq 3t$ . Note that  $B(1, t) \cap \lambda(g)B(1, t) = \emptyset$ . So  $\varphi$  and  $\lambda(g)\varphi$  have disjoint supports. In particular,

$$\|\varphi - \lambda(g)\varphi\|_p \geq \|\varphi\|_p$$

and

$$\|\tilde{\nabla}(\varphi - \lambda(g)\varphi)\|_p = 2^{1/p} \|\tilde{\nabla}\varphi\|_p.$$

This clearly implies the proposition. □

Combining with Proposition 3.5, we obtain

**Corollary 4.6.** *Let  $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  a nondecreasing map be satisfying*

$$\int_1^\infty \left( \frac{f(t)}{J_{G,p}^b(t)} \right)^p \frac{dt}{t} < \infty \tag{CJ_p}$$

*for some  $1 \leq p < \infty$ . Then there exists a 1-cocycle  $b$  in  $Z^1(G, \lambda_{G,p})$  such that*

$$\rho \succeq f.$$

**Question 4.7.** For which groups  $G$  do we have  $M\rho_{\lambda_{G,p}} \approx J_{G,p}^b$ ?

We show that the question has positive answer for groups of class  $(\mathcal{L})$ . On the contrary, note that the group  $G$  is nonamenable if and only if  $J_{G,p}^b$  is bounded. But we have seen in the previous section that for a free group of rank  $\geq 2$ ,  $M\rho_{\lambda_{G,p}}(t) \succeq t^{1/p}$ . More generally, the answer to Question 4.7 is no for every nonamenable group admitting a proper 1-cocycle with values in the regular representation. This question is therefore only interesting for amenable groups.

**4.2. Sequences of controlled Følner pairs.** In this section, we give a method, adapted<sup>10</sup> from [CGP] to estimate  $J_p^b$ .

**Definition 4.8.** Let  $G$  be a compactly generated, locally compact group equipped with a left invariant Haar measure  $\mu$ . Let  $\alpha = (\alpha_n)$  be a nondecreasing sequence of integers. A sequence of  $\alpha$ -controlled Følner pairs of  $G$  is a family  $(H_n, H'_n)$  where  $H_n$  and  $H'_n$  are nonempty compact subsets of  $G$  satisfying for some constant  $C > 0$  the following conditions:

- (1)  $S^{\alpha_n} H_n \subset H'_n$
- (2)  $\mu(H'_n) \leq C\mu(H_n)$ ;
- (3)  $H'_n \in B(1, Cn)$

If  $\alpha_n \approx n$ , we call  $(H_n, H'_n)$  a controlled sequence of Følner pairs.

**Proposition 4.9.** Assume that  $G$  admits a sequence of  $\alpha$ -controlled Følner pairs. Then

$$J_{G,p}^b \succeq \alpha.$$

*Proof.* For every  $n \in \mathbb{N}$ , consider the function  $\varphi_n: G \rightarrow \mathbb{R}_+$  defined by

$$\varphi_n(g) = \min\{k \in \mathbb{N} : g \in S^k(H'_n)^c\}$$

where  $A^c = G \setminus A$ . Clearly,  $\varphi_n$  is supported in  $H'_n$ . It is easy to check that

$$\|\tilde{\nabla} \varphi_n\|_p \leq (\mu(H'_n))^{1/p}$$

and that

$$\|\varphi_n\|_p \geq \alpha_n(\mu(H_n))^{1/p}.$$

Hence by (2),

$$\|\varphi_n\|_p \geq C^{-1/p} \alpha_n \|\tilde{\nabla} \varphi_n\|_p,$$

so we are done.  $\square$

**Remark 4.10.** Note that if  $H$  and  $H'$  are subsets of  $G$  such that  $S^k H \subset H'$  and  $\mu(H') \leq C\mu(H)$ , then there exists by pigeonhole principle an integer  $0 \leq j \leq k-1$  such that

$$\mu(\partial S^j H) = \mu(S^{j+1} H \setminus S^j H) \leq \frac{C}{k} \mu(S^j H).$$

So in particular if  $(H_n, H'_n)$  is a  $\alpha$ -controlled sequence of Følner pairs, then there exists a Følner sequence  $(K_n)$  such that  $H_n \subset K_n \subset H'_n$  and

$$\frac{\mu(\partial K_n)}{\mu(K_n)} \leq C/\alpha_n.$$

Moreover, if  $\alpha_n \approx n$ , then one obtains a controlled Følner sequence in the sense of [CTV], Definition 4.8.

<sup>10</sup>In [CGP], the authors are interested in estimating the  $L^2$ -isoperimetric profile of a group.

## 5. Isoperimetry in balls for groups of class $(\mathcal{L})$

The purpose of this section is to prove the following theorem.

**Theorem 5.1.** *Let  $G$  be a group belonging to the class  $(\mathcal{L})$ . Then,  $G$  admits a controlled sequence of Følner pairs. In particular,  $J_{G,p}^b(t) \approx t$ .*

Note that Theorem 1 follows from Theorem 5.1 and Corollary 4.6.

**5.1. Wreath products  $F \wr \mathbb{Z}$ .** Let  $F$  be a finite group. Consider the wreath product  $G = F \wr \mathbb{Z} = \mathbb{Z} \ltimes F^{(\mathbb{Z})}$ , the group law being defined as  $(n, f)(m, g) = (n + m, \tau_m f + g)$  where  $\tau_m f(x) = f(m + x)$ . As a set,  $G$  is a Cartesian product  $\mathbb{Z} \times U$  where  $U$  is the direct sum  $F^{(\mathbb{Z})} = \bigoplus_{n \in \mathbb{Z}} F_n$  of copies  $F_n$  of  $F$ . The set  $S = S_F \cup S_{\mathbb{Z}}$ , where  $S_F = F_0$  and  $S_{\mathbb{Z}} = \{-1, 0, 1\}$  is clearly a symmetric generating set for  $G$ . Define

$$H_n = I_n \times U_n$$

and

$$H'_n = I_{2n} \times U_n$$

where  $U_n = F^{[-2n, 2n]}$  and  $I_n = [-n, n]$ .

Let us prove that  $(H_n, H'_n)_n$  is a sequence of controlled Følner pairs. We therefore have to show that

- (1)  $S^n H_n \subset H'_n$
- (2)  $|H'_n| \leq 2|H_n|$ ;
- (3) there exists  $C > 0$  such that  $H'_n \subset B(1, Cn)$

Property (2) is trivial. To prove (1) and (3), recall that the length of an element of  $g = (k, u)$  of  $G$  equals  $L(\gamma) + \sum_{h \in \mathbb{Z}} |u(h)|_F$  where  $L(\gamma)$  is the length of a shortest path  $\gamma$  from 0 to  $k$  in  $\mathbb{Z}$  passing through every element of the support of  $u$  (see [Par], Theorem 1.2). In particular,

$$|(u, k)|_S \leq 2L(\gamma).$$

Thus, if  $g \in H_n$ , then  $L(\gamma) \leq 30n$ . So (3) follows. On the other hand, if  $g = (k, u) \in S^n$ , then

$$|k|_{\mathbb{Z}} \leq L(\gamma) \leq n$$

and

$$\text{Supp}(u) \subset I_n.$$

So  $H_n g \subset H'_n$ . □

**Remark 5.2.** Note that the proof still works replacing  $\mathbb{Z}$  by any group with linear growth. On the other hand, replacing it by a group of polynomial growth of degree  $d$  yields a sequence of  $n^{1/d}$ -controlled Følner pairs. For instance, as a corollary, we obtain that  $B(F \wr \mathbb{Z}^d) \geq 1/d$ .

**5.2. Semidirect products  $(\mathbb{R} \oplus \bigoplus_{p \in P} \mathbb{Q}_p) \rtimes_{\frac{m}{n}} \mathbb{Z}$ .** Note that discrete groups of type (2) of the class  $(\mathcal{L})$  are cocompact lattices of a group of the form

$$G = \mathbb{Z} \rtimes_{\frac{m}{n}} \left( \mathbb{R} \oplus \bigoplus_{p \in P} \mathbb{Q}_p \right)$$

with  $m, n$  coprime integers and  $P$  a finite set of primes (possibly infinite) dividing  $mn$ . To simplify notation, we will only consider the case when  $P = \{p\}$  is reduced to one single prime, the generalization presenting no difficulty. The case where  $p = \infty$  will result from the case of connected Lie groups (see next section) since  $\mathbb{Z} \rtimes_{\frac{m}{n}} \mathbb{R}$  embeds as a closed cocompact subgroup of the group of positive affine transformations  $\mathbb{R} \ltimes \mathbb{R}$ .

So consider the group  $G = \mathbb{Z} \rtimes_{1/p} \mathbb{Q}_p$ . Define a compact symmetric generating set by  $S = S_{\mathbb{Q}_p} \cup S_{\mathbb{Z}}$  where  $S_{\mathbb{Q}_p} = \mathbb{Z}_p$  and  $S_{\mathbb{Z}} = \{-1, 0, 1\}$ . Define  $(H_k, H'_k)$  by

$$H_k = I_k \times p^{-2k} \mathbb{Z}_p$$

and

$$H'_k = I_{2k} \times p^{-2k} \mathbb{Z}_p,$$

where  $I_k = [-k, k]$ . Using the same kind of arguments as previously for  $F \wr \mathbb{Z}$ , one can prove easily that  $(H_k, H'_k)$  is a controlled sequence of Følner pairs.  $\square$

**5.3. Amenable connected Lie groups.** Let  $G$  be a solvable simply connected Lie group. Let  $S$  be a compact symmetric generating subset. In [Gu] (see also [O]), it is proved that  $G$  admits a maximal normal connected subgroup such that the quotient of  $G$  by this subgroup has polynomial growth. This subgroup is called the exponential radical and is denoted  $\text{Exp}(G)$ . We have  $\text{Exp}(G) \subset N$ , where  $N$  is the maximal nilpotent normal subgroup of  $G$ . Let  $T$  be a compact symmetric generating subset of  $\text{Exp}(G)$ . An element  $g \in G$  is called strictly exponentially distorted if the  $S$ -length of  $g^n$  grows as  $\log |n|$ . The subset of strictly exponentially distorted elements of  $G$  coincides with  $\text{Exp}(G)$ . That is,

$$\text{Exp}(G) = \{g \in G : |g^n|_S \approx \log |n|\} \cup \{1\}.$$

Moreover,  $\text{Exp}(G)$  is strictly exponentially distorted in  $G$  in the sense that there exists  $\beta \geq 1$  such that for every  $h \in \text{Exp}(G) \setminus \{1\}$ ,

$$\beta^{-1} \log(|h|_T + 1) - \beta \leq |h|_S \leq \beta \log(|h|_T + 1) + \beta \quad (5.1)$$

where  $T$  is a compact symmetric generating subset of  $\text{Exp}(G)$ .

We will need the following two lemmas.

**Lemma 5.3.** *Let  $G$  be a locally compact group. Let  $H$  be a closed normal subgroup. Let  $\lambda$  and  $\nu$  be respectively left Haar measures of  $H$  and  $G/H$ . Let  $i$  be a measurable*

left-section of the projection  $\pi: G \rightarrow G/H$ , i.e.  $G = \sqcup_{x \in G/H} i(x)H$ . Identify  $G$  with the cartesian product  $G/H \times H$  via the map  $(x, h) \mapsto i(x)h$ . Then the product measure  $\nu \otimes \lambda$  is a left Haar measure on  $G$ .

*Proof.* We have to prove that  $\nu \otimes \lambda$  is left-invariant on  $G$ . Fix  $g$  in  $G$ . Define a measurable map  $\sigma_g$  from  $G/H$  to  $H$  by

$$\sigma_g(x) = (i(\pi(g)x)^{-1}gi(x)).$$

In other words,  $\sigma_g(x)$  is the unique element of  $H$  such that

$$gi(x) = i(\pi(g)x)\sigma_g(x).$$

Let  $\varphi: G \rightarrow \mathbb{R}$  be a continuous, compactly supported function. We have

$$\int_{G/H \times H} \varphi[gi(x)h]d\nu(x)d\lambda(h) = \int_{G/H \times H} \varphi[i(\pi(g)x)\sigma_g(x)h]d\nu(x)d\lambda(h).$$

As  $\nu$  and  $\lambda$  are respectively left Haar measures on  $G/H$  and  $H$ , the Jacobian of the transformation  $(x, h) \mapsto (\pi(g)x, \sigma_g(x)h)$  is equal to 1. Hence,

$$\int_{G/H \times H} \varphi[i(\pi(g)x)\sigma_g(x)h]d\nu(x)d\lambda(h) = \int_{G/H \times H} \varphi[i(x)h]d\nu(x)d\lambda(h).$$

Thus  $\nu \otimes \lambda$  is left-invariant. □

**Lemma 5.4.** *Let  $G$  be a connected Lie group and  $H$  be a normal subgroup. Consider the projection  $\pi: G \rightarrow G/H$ . There exists a compact generating set  $S$  of  $G$  and a  $\sigma$ -compact cross-section  $\sigma$  of  $G/H$  inside  $G$  such that  $\sigma(\pi(S)^n) \subset S^{n+1}$ .*

*Proof.* Since  $\pi$  is a submersion, there exists a compact neighborhood  $S$  of 1 in  $G$  such that  $\pi(S)$  admits a continuous cross-section  $\sigma_1$  in  $S$ . Now, let  $X$  be a minimal (discrete) subset of  $G/H$  satisfying  $G/H = \bigcup_{x \in X} x\pi(S)$ . Since this covering is locally finite and  $\pi(S)$  is compact, one can construct by induction a partition  $(A_x)_{x \in X}$  of  $G/H$  such that every  $A_x$  is a constructible, and therefore  $\sigma$ -compact subset of  $x\pi(S)$ . Let  $\sigma_2: X \rightarrow G$  be a cross-section of  $X$  satisfying  $\sigma_2(X \cap \pi(S)^n) \subset S^n$ . Now, for every  $z \in A_x$ , define

$$\sigma(z) = \sigma_2(x)\sigma_1(x^{-1}z).$$

Clearly,  $\sigma$  satisfies to the hypotheses of the lemma. □

Equip the group  $P = G/\text{Exp}(G)$  with a Haar measure  $\nu$  and with the symmetric generating subset  $\pi(S)$ , where  $\pi$  is the projection on  $P$ . Assume that  $S$  satisfies to the hypotheses of Lemma 5.4 and let  $\sigma$  be a  $\sigma$ -compact cross-section of  $P$  inside  $G$

such that  $\sigma(\pi(S)^n) \subset S^{n+1}$ . For every  $n \in \mathbb{N}$ , write  $F_n = \sigma(\pi(S)^n)$ . Let  $\alpha$  be some large enough positive number that we will determine later. Denote by  $\lfloor x \rfloor$  the integer part of a real number  $x$ . Define, for every  $n \in \mathbb{N}$ ,

$$H_n = S^n T^{\lfloor \exp(\alpha n) \rfloor}$$

and

$$H'_n = S^{2n} T^{\lfloor \exp(\alpha n) \rfloor}.$$

Note that  $H'_n = S^n H_n$ . On the other hand, since  $\text{Exp}(G)$  is strictly exponentially distorted, there exists  $a \geq 1$  only depending on  $\alpha$  and  $\beta$  such that, for every  $n \in \mathbb{N}$ ,

$$S^n T^{\lfloor \exp(\alpha n) \rfloor} \subset S^{an}.$$

Hence, to prove that  $(H_n, H'_n)$  is a sequence of controlled Følner pairs, it suffices to show that  $\mu(H'_n) \leq C\mu(H_n)$ . Consider another sequence  $(A_n, A'_n)$  defined by, for every  $n \in \mathbb{N}^*$ ,

$$A_n = F_{n-1} T^{\lfloor \exp(\alpha n) \rfloor}$$

and

$$A'_n = F_{2n} T^{2\lfloor \exp(\alpha n) \rfloor}.$$

As  $F_n$  is  $\sigma$ -compact,  $A_n$  and  $A'_n$  are measurable. To compute the measures of  $A_n$  and  $A'_n$ , we choose a normalization of the Haar measure  $\lambda$  on  $\text{Exp}(G)$  such that the measure  $\mu$  disintegrates over  $\lambda$  and the pull-back measure of  $\nu$  on  $\sigma(P)$  as in Lemma 5.3. We therefore obtain

$$\mu(A_n) = \nu(\pi(S)^{n-1})\lambda(T^{\lfloor \exp(\alpha n) \rfloor})$$

and

$$\mu(A'_n) = \nu(\pi(S)^{2n})\lambda(T^{2\lfloor \exp(\alpha n) \rfloor}).$$

Since  $P$  and  $\text{Exp}(G)$  have both polynomial growth, there is a constant  $C$  such that, for every  $n \in \mathbb{N}^*$ ,

$$\mu(A'_n) \leq C\mu(A_n).$$

So now, it suffices to prove that

$$A_n \subset H_n \subset H'_n \subset A'_n,$$

where the only nontrivial inclusion is  $H'_n \subset A'_n$ . Let  $g \in S^{2n}$ ; let  $f \in F_{2n}$  be such that  $\pi(g) = \pi(f)$ . Since  $F_{2n} \subset S^{2n+2} \subset S^{3n}$ ,

$$gf^{-1} \in S^{6n} \cap \text{Exp}(G).$$

On the other hand, by (5.1),

$$S^{6n} \cap \text{Exp}(G) \subset T^{2\lfloor \exp(6\beta n) \rfloor}.$$

Therefore, for every  $n \in \mathbb{N}^*$ ,

$$H'_n \subset F_{2n} T^{2\lfloor \exp(6\beta n) \rfloor} T^{\lfloor \exp(\alpha n) \rfloor} = F_{2n} T^{2\lfloor \exp(6\beta n) \rfloor + \lfloor \exp(\alpha n) \rfloor}.$$

Hence, choosing  $\alpha \geq 6\beta + \log 2$ , we have

$$H'_n \subset F_{2n} T^{2\lfloor \exp(\alpha n) \rfloor} = A'_n,$$

and we are done.  $\square$

## 6. On embedding of finite trees into uniformly convex Banach spaces

**Definition 6.1.** A Banach space  $X$  is called  $q$ -uniformly convex ( $q > 0$ ) if there is a constant  $a > 0$  such that for any two points  $x, y$  in the unit sphere satisfying  $\|x - y\| \geq \varepsilon$ , we have

$$\left\| \frac{x + y}{2} \right\| \leq 1 - a\varepsilon^q.$$

Note that by a theorem of Pisier [Pis], every uniformly convex Banach space is isomorphic to some  $q$ -uniformly convex Banach space.

In this section, we prove that the compression of a Lipschitz embedding of a finite binary rooted tree into a  $q$ -uniformly convex space  $X$  always satisfies condition  $(C_q)$ . Theorem 4 follows from the fact that an  $L^p$ -space is  $\max\{p, 2\}$ -uniformly convex.

**Theorem 6.2.** Let  $T_J$  be the binary rooted tree of depth  $J$  and let  $1 < q < \infty$ . Let  $F$  be a 1-Lipschitz map from  $T_J$  to some  $q$ -uniformly convex Banach space  $X$  and let  $\rho$  be the compression of  $F$ . Then there exists  $C = C(q) < \infty$  such that

$$\int_1^{2^J} \left( \frac{\rho(t)}{t} \right)^q \frac{dt}{t} \leq C. \quad (6.1)$$

**Corollary 6.3.** Let  $F$  be any uniform embedding of the 3-regular tree  $T$  into some  $q$ -uniformly convex Banach space. Then the compression  $\rho$  of  $F$  satisfies Condition  $(C_q)$ .  $\square$

As a corollary, we also reobtain the theorem of Bourgain.

**Corollary 6.4** ([Bou], Theorem 1). With the notation of Theorem 6.2, there exists at least two vertices  $x$  and  $y$  in  $T_J$  such that

$$\frac{\|F(x) - F(y)\|}{d(x, y)} \leq \left( \frac{C}{\log J} \right)^{1/q}.$$

*Proof.* For every  $1 \leq t \leq 2J$ , there exist  $z, z' \in T_J$ ,  $d(z, z') \geq t$  such that:

$$\frac{\rho(t)}{t} = \frac{\|F(z) - F(z')\|}{t} \geq \frac{\|F(z) - F(z')\|}{d(z, z')}.$$

Therefore

$$\min_{z \neq z' \in T_J} \frac{\|F(z) - F(z')\|}{d(z, z')} \leq \min_{1 \leq u \leq 2J} \frac{\rho(u)}{u}.$$

But by (6.1)

$$\left( \min_{1 \leq u \leq 2J} \frac{\rho(u)}{u} \right)^q \int_1^{2J} \frac{1}{t} dt \leq \int_1^{2J} \left( \frac{\rho(t)}{t} \right)^q \frac{dt}{t} \leq C.$$

We then have

$$\min_{z \neq z' \in T_J} \frac{\|F(z) - F(z')\|}{d(z, z')} \leq \left( \frac{C}{\log J} \right)^{1/q}. \quad \square$$

**Proof of Theorem 6.2.** Since the proof follows closely the proof of Theorem 1 in [Bou], we keep the same notation to allow the reader to compare them. For  $j = 1, 2, \dots$ , denote  $\Omega_j = \{-1, 1\}^j$  and  $T_j = \bigcup_{j' \leq j} \Omega_{j'}$ . Thus  $T_j$  is the finite tree with depth  $j$ . Denote  $d$  the tree-distance on  $T_j$ .

**Lemma 6.5** ([Pis], Proposition 2.4). *There exists  $C = C(q) < \infty$  such that if  $(\xi_s)_{s \in \mathbb{N}}$  is an  $X$ -valued martingale on some probability space  $\Omega$ , then*

$$\sum_s \|\xi_{s+1} - \xi_s\|_q^q \leq C \sup_s \|\xi_s\|_q^q \quad (6.2)$$

where  $\|\cdot\|_q$  stands for the norm in  $L_X^q(\Omega)$ .

Lemma 6.5 is used to prove

**Lemma 6.6.** *If  $x_1, \dots, x_J$ , with  $J = 2^r$ , is a finite system of vectors in  $X$ , then*

$$\sum_{s=1}^r 2^{-qs} \min_{2^s < j \leq J-2^s} \|2x_j - x_{j-2^s} - x_{j+2^s}\|^q \leq C \sup_{1 \leq j \leq J-1} \|x_{j+1} - x_j\|^q. \quad (6.3)$$

Denote  $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \dots \subset \mathcal{D}_r$  the algebras of intervals on  $[0, 1]$  obtained by successive dyadic refinements. Define the  $X$ -valued function

$$\xi = \sum_{1 \leq j \leq J-1} 1_{[\frac{j}{J}, \frac{j+1}{J}[} (x_{j+1} - x_j)$$



and consider expectations  $\xi_s = \mathbf{E}[\xi | \mathcal{D}_s]$  for  $s = 1, \dots, r$ . Since  $\xi_s$  form a martingale ranging in  $X$ , it satisfies inequality (6.2). On the other hand

$$\begin{aligned} \|\xi_{s+1} - \xi_s\|_q^q &= 2^{-r+s} 2^{qs} \sum_{1 < t \leq 2^{r-s}} 2^{-qs} \|2x_{t2^s} - x_{(t-1)2^s} - x_{(t+1)2^s}\|^q \\ &\leq 2^{-qs} \min_{2^s < j \leq J-2^s} \|2x_j - x_{j-2^s} - x_{j+2^s}\|^q. \end{aligned}$$

So (6.3) follows from the fact that

$$\|\xi_s\|_q^q \leq \|\xi_{s+1} - \xi_s\|_\infty^q = \sup_j \|x_{j+1} - x_j\|^q. \quad \square$$

**Lemma 6.7.** *If  $f_1, \dots, f_J$ , with  $J = 2^r$ , is a finite system of functions in  $L_X^\infty(\Omega)$ . Then*

$$\sum_{s=1}^r 2^{-qs} \min_{2^s < j \leq J-2^s} \|2f_j - f_{j-2^s} - f_{j+2^s}\|^q \leq C \sup_{1 \leq j \leq J-1} \|f_{j+1} - f_j\|_\infty^q. \quad (6.4)$$

*Proof.* Replace  $X$  by  $L_X^q(\Omega)$ , for which (6.2) remains valid, and use (6.3).  $\square$

**Lemma 6.8.** *Let  $f_1, \dots, f_J$ , with  $J = 2^r$ , be a sequence of functions on  $\{1, -1\}^J$  where  $f_j$  only depends on  $\varepsilon_1, \dots, \varepsilon_j$ . Then*

$$\begin{aligned} \sum_{s=1}^r 2^{-qs} \min_{2^s < j \leq J-2^s} &\left( \int_{\Omega_j \times \Omega_{2^s} \times \Omega_{2^s}} \|f_{j+2^s}(\varepsilon, \delta) - f_{j+2^s}(\varepsilon, \delta')\|^q d\varepsilon d\delta d\delta' \right) \\ &\leq 2^q C \sup_{1 \leq j \leq J-1} \|f_{j+1} - f_j\|_\infty^q. \end{aligned}$$

*Proof.* For every  $d < j \leq J-d$ , using the triangle inequality, we obtain

$$\begin{aligned} \|2f_j - f_{j-d} - f_{j+d}\|_q^q &= \int_{\Omega_j \times \Omega_d} \|2f_j - f_{j-d} - f_{j+d}\|^q d\varepsilon d\delta \\ &\geq 2^{-q} \int_{\Omega_j \times \Omega_d \times \Omega_d} \|f_{j+2^s}(\varepsilon, \delta) - f_{j+2^s}(\varepsilon, \delta')\|^q d\varepsilon d\delta d\delta'. \end{aligned}$$

The lemma then follows from (6.4).  $\square$

Now, let us prove Theorem 6.2. Fix  $J$  and consider a 1-Lipschitz map  $F: T_J \rightarrow X$ . Apply Lemma 6.8 to the functions  $f_1, \dots, f_J$  defined by

$$\forall \alpha \in \Omega_j, \quad f_j(\alpha) = F(\alpha).$$

By definition of the compression, we have

$$\rho(d((\varepsilon, \delta), (\varepsilon, \delta'))) \leq \|f_{j+2^s}(\varepsilon, \delta) - f_{j+2^s}(\varepsilon, \delta')\| \quad (6.5)$$

where  $\varepsilon \in \Omega_j$  and  $\delta, \delta' \in \Omega_{2^s}$ .

But, on the other hand, with probability  $1/2$ , we have

$$d((\varepsilon, \delta), (\varepsilon, \delta')) = 2 \cdot 2^s.$$

So combining this with Lemma 6.8, (6.5) and with the fact that  $F$  is 1-Lipschitz, we obtain

$$\sum_{s=1}^r 2^{-qs} \rho(2^s)^q \leq 2^{q+1} C$$

But since  $\rho$  is decreasing, we have

$$2^{-qs} \rho(2^s)^q \geq 2^{-q-1} \int_{2^{s-1}+1}^{2^s} \frac{1}{t} \left( \frac{\rho(t)}{t} \right)^q dt.$$

So (6.1) follows.  $\square$

## 7. Applications and further results

**7.1. Hilbert compression, volume growth and random walks.** Let  $G$  be a locally compact group generated by a symmetric compact subset  $S$  containing 1. Let us denote  $V(n) = \mu(S^n)$  and  $S(n) = V(n+1) - V(n) = \mu(S^{n+1} \setminus S^n)$ . Extend  $V$  as a piecewise linear function on  $\mathbb{R}_+$  such that  $V'(t) = S(n)$  for  $t \in ]n, n+1[$ .

**Proposition 7.1.** *Let  $G$  be a compactly generated locally compact group. For any  $2 \leq p < \infty$ ,*

$$J_{G,p}(t) \leq \frac{t}{\log V(t)}.$$

*Proof.* For every  $n \in \mathbb{N}$ , define

$$k(n) = \sup\{k, V(n-k) \geq V(n)/2\}$$

and

$$j(n) = \sup_{1 \leq j \leq n} k(j).$$

For every positive integer  $l \leq n/j(n)$ ,

$$V(n) \geq 2^l V(n - lj(n)).$$

Hence, as  $V(0) = 1$ ,

$$V(n) \geq 2^{n/(j(n)+1)}.$$

Thus, there is a constant  $c > 0$  such that

$$j(n) \geq \frac{cn}{\log V(n)}.$$

Let  $q_n \leq n$  be such that  $j(n) = k(q_n)$ . Now define

$$\varphi_n = \sum_{k=1}^{q_n-1} 1_{B(1,k)}.$$

Note that the subsets  $SB(1,k) \triangle B(1,k) = B(1,k+1) \setminus B(1,k)$ , for  $k \in \mathbb{N}$ , are piecewise disjoint. Thus, an easy computation shows that

$$\|\tilde{\nabla} \varphi_n\|_p \leq V(q_n)^{1/p}.$$

On the other hand

$$\|\varphi_n\|_p \geq j(n)V(q_n - j(n))^{1/p} \geq \frac{cn}{\log V(n)}(V(q_n)/2)^{1/p}.$$

Since  $J_{G,p}^b(n) \geq \|\varphi_n\|_p / \|\tilde{\nabla} \varphi_n\|_p$ , we conclude that  $J_{G,p}^b(n) \geq n / \log V(n)$ .  $\square$

Now, consider a symmetric probability measure  $\nu$  on a finitely generated group  $G$ , supported by a finite generating subset  $S$ . Given an element  $\varphi$  of  $\ell^2(G)$ , a simple calculation shows that

$$\frac{1}{2} \iint |\varphi(sx) - \varphi(x)|^2 d\nu^{(2)}(s) d\mu(x) = \int (\varphi - \nu^{(2)} * \varphi) \varphi d\mu = \|\varphi\|_2^2 - \|\nu * \varphi\|_2^2$$

where  $\mu$  denotes the counting measure on  $G$ . Let us introduce a (left) gradient on  $G$  associated to  $\nu$ . Let  $\varphi$  be a function on  $G$ ; define

$$|\tilde{\nabla} \varphi|_2^2(g) = \int |\varphi(sg) - \varphi(g)|^2 d\nu^{(2)}(s).$$

This gradient satisfies

$$\| |\tilde{\nabla} \varphi|_2 \|_2^2 = 2(\|\varphi\|_2^2 - \|\nu * \varphi\|_2^2).$$

We have

$$\mu(S)^{-1/2} |\tilde{\nabla} \varphi|_2 \leq |\tilde{\nabla} \varphi| \leq |\tilde{\nabla} \varphi|_2.$$

**Proposition 7.2.** *Assume that  $\nu^{(n)}(1) \geq e^{-Cn^b}$  for some  $b < 1$ . Then*

$$J_{G,2}^b(t) \geq Ct^{1-b}.$$

*Proof.* Let us prove that there exists a constant  $C' < \infty$  such that for every  $n \in \mathbb{N}$ , there exists  $n \leq k \leq 2n$  such that

$$\frac{\| |\tilde{\nabla} v^{(2k)}|_2 \|_2^2}{\| v^{(2k)} \|_2^2} \leq C' n^{b-1}.$$

Since  $v^{(2k)}$  is supported in  $S^{2k} \subset S^{4n}$ , this will prove the proposition. Let  $C_n$  be such that for every  $n \leq q \leq 2n$ ,

$$\frac{\| |\tilde{\nabla} v^{(2q)}|_2 \|_2^2}{\| v^{(2q)} \|_2^2} \geq C_n n^{b-1}.$$

Since the function defined by  $\psi(q) = \| v^{(2q)} \|_2^2$  satisfies

$$\psi(q+1) - \psi(q) = -\frac{1}{2} \| |\tilde{\nabla} v^{(2q)}|_2 \|_2^2,$$

we can extend  $\psi$  as a piecewise linear function on  $\mathbb{R}_+$  such that

$$\psi'(t) = \frac{1}{2} \| |\tilde{\nabla} v^{(2q)}|_2 \|_2^2$$

for every  $t \in [q, q+1[$ . Then, for every  $n \leq t \leq 2n$  we have

$$-\frac{\psi'(t)}{\psi(t)} \geq C_n n^{b-1}$$

which integrates in

$$-\log \left( \frac{\psi(2n)}{\psi(n)} \right) \geq C_n n^b.$$

Since  $\psi(n) < 1$ , this implies

$$\psi(2n) \leq e^{-C_n n^b}.$$

But on the other hand,

$$\psi(2n) \geq \| v^{(4n)} \|_2^2 \geq v^{(8n)}(1) \geq e^{-8C n^b}.$$

So  $C_n \leq 8C$ . □

**7.2. A direct construction to embed trees.** Here, we propose to show that the method used in [Bou], [GK], [BrSo] to embed trees in  $L^p$ -spaces can also be exploited to obtain optimal estimates (i.e. a converse to Theorem 6.2). Moreover, no hypothesis of local finitude is required for this construction.

**Theorem 7.3.** *Let  $T$  be a simplicial tree. For every increasing function  $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying, for  $1 \leq p < \infty$*

$$\int_1^\infty \left( \frac{f(t)}{t} \right)^p \frac{dt}{t} < \infty, \quad (C_p)$$

*there exists a uniform embedding  $F$  of  $T$  into  $\ell^p(T)$  with compression  $\rho \succeq f$ .*

*Proof.* Let us start with a lemma.

**Lemma 7.4.** *For every nonnegative sequence  $(\xi_n)$  such that*

$$\sum_n |\xi_{n+1} - \xi_n|^p < \infty,$$

*there exists a Lipschitz map  $F: T \rightarrow \ell^p(T)$  whose compression  $\rho$  satisfies*

$$\forall n \in \mathbb{N}, \quad \rho(n) \geq \left( \sum_{j=0}^n \xi_j^p \right)^{1/p}.$$

*Proof.* The following construction is a generalization of those carried out in [GK] and [BrSo]. Fix a vertex  $o$ . For every  $y \in T$ , denote  $\delta_y$  the element of  $\ell^p(T)$  that takes value 1 on  $y$  and 0 elsewhere. Let  $x$  be a vertex of  $T$  and let  $x_0 = x, x_1, \dots, x_l = o$  be the minimal path joining  $x$  to  $o$ . Define

$$F(x) = \sum_{i=1}^l \xi_i \delta_{x_i}.$$

To prove that  $F$  is Lipschitz, it suffices to prove that  $\|F(x) - F(y)\|_p$  is bounded for neighbor vertices in  $T$ . So let  $x$  and  $y$  be neighbor vertices in  $T$  such that  $d(o, y) = d(x, o) + 1 = l + 1$ . We have

$$\|F(y) - F(x)\|_p^p \leq \xi_0^p + \sum_{j=0}^l |\xi_{n+1} - \xi_n|^p.$$

On the other hand, let  $x$  and  $y$  be two vertices in  $T$ . Let  $z$  be the last common vertex of the two geodesic paths joining  $o$  to  $x$  and  $y$ . We have

$$d(x, y) = d(x, z) + d(z, y)$$

and

$$\begin{aligned} \|F(x) - F(y)\|_p^p &= \|F(x) - F(z)\|_p^p + \|F(z) - F(y)\|_p^p \\ &\geq \max\{\|F(x) - F(z)\|_p^p, \|F(z) - F(y)\|_p^p\}. \end{aligned}$$

Let  $k = d(z, x)$ ; we have

$$\|F(x) - F(z)\|_p^p \geq \sum_{j=0}^k \xi_j^p,$$

which proves the lemma.  $\square$

Now, let us prove the theorem. Define  $(\xi_j)$  by

$$\xi_0 = \xi_1 = 0;$$

$$\forall j \geq 1, \quad \xi_{j+1} - \xi_j = \frac{1}{j^p} \frac{f(j)}{j}$$

and consider the associated Lipschitz map  $F$  from  $T$  to  $\ell^p(T)$ . Clearly, we have

$$\sum |\xi_{n+1} - \xi_n|^p < \infty$$

and

$$\sum_{j=0}^n \xi_j^p \geq \sum_{j=[n/2]}^n \left( \sum_{k=0}^{j-1} |\xi_{k+1} - \xi_k| \right)^p \geq n/2 \left( \sum_{k=0}^{[n/2]-1} |\xi_{k+1} - \xi_k| \right)^p \geq cf([n/2])$$

using the fact that  $f$  is nondecreasing. So the theorem now follows from the lemma.  $\square$

### 7.3. Cocycles with lacunar compression

**Proposition 7.5.** *For any increasing sublinear function  $h: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and every  $2 \leq p < \infty$ , there exists a function  $f$  satisfying  $(C_p)$ , a constant  $c > 0$  and an increasing sequence of integers  $(n_i)$  such that*

$$\forall i \in \mathbb{N}, \quad f(n_i) \geq ch(n_i).$$

*Proof.* Choose a sequence  $(n_i)$  such that

$$\sum_{i \in \mathbb{N}} \left( \frac{h(n_i)}{n_i} \right)^p < \infty$$

Define

$$\forall i \in \mathbb{N}, \quad n_i \leq t < n_{i+1}, \quad f(t) = h(n_i)$$

We have

$$\int_1^\infty \frac{1}{t} \left( \frac{f(t)}{t} \right)^p dt \leq \sum_i (h(n_i))^p \int_{n_i}^{n_{i+1}} \frac{dt}{t^{p+1}} \leq (p+1) \sum_i \left( \frac{h(n_i)}{n_i} \right)^p < \infty$$

So we are done.  $\square$

**7.4. The case of  $\mathbb{Z} \wr \mathbb{Z}$ .** The proof of Theorem 17 follows from Proposition 3.5 and from the following observation.

**Proposition 7.6.** *For all  $1 \leq p < \infty$ , the maximal  $\ell^p$ -compression function of the group  $G = \mathbb{Z} \wr \mathbb{Z}$  satisfies*

$$M\rho_{G,p}(t) \succeq t^{p/(2p-1)}.$$

*Proof.* Denote by  $\theta$  the projection  $\mathbb{Z} \wr \mathbb{Z} \rightarrow C_2 \wr \mathbb{Z}$ . Fix two word lengths on  $\mathbb{Z} \wr \mathbb{Z}$  and  $C_2 \wr \mathbb{Z}$ , which for simplicity, we will both denote by  $|g|$ .

Consider the unique cocycle  $b: \mathbb{Z} \wr \mathbb{Z} \rightarrow \ell^p(\mathbb{Z})$  which extends the natural injective morphism  $\mathbb{Z}^{(\mathbb{Z})} \rightarrow \ell^p(\mathbb{Z})$ . For any  $g = (k, u) \in \mathbb{Z} \wr \mathbb{Z} = \mathbb{Z} \ltimes \mathbb{Z}^{(\mathbb{Z})}$ , we therefore have  $\|b(g)\| = \|u\|_p$ . Taking the  $\ell^p$ -direct sum of this cocycle with every cocycle of  $\mathbb{Z} \wr \mathbb{Z}$  factorizing through  $\theta$ , and since  $M\rho_{C_2 \wr \mathbb{Z},p}(t) \approx t$ , we obtain

$$M\rho_{\mathbb{Z} \wr \mathbb{Z},p}(t) \succeq \inf_{g \in \mathbb{Z} \wr \mathbb{Z}, |g| \geq t} \max\{|p(g)|, \|b(g)\|\}. \quad (7.1)$$

Up to multiplicative constants, (see [Par], Theorem 1.2), the word length of an element  $g = (k, u) \in \mathbb{Z} \wr \mathbb{Z}$  is given by

$$L(\gamma) + \sum_{h \in \mathbb{Z}} |u(h)| = L(\gamma) + \|u\|_1,$$

where  $L(\gamma)$  is the length of a shortest path  $\gamma$  from 0 to  $k$  passing through every element of the support of  $u$ . Similarly,  $|p(g)| \approx L(\gamma) + |\text{Supp}(u)|$ . Hence by (7.1), we can assume that  $L(\gamma) \leq |g|/2$ , so that  $\|u\|_1 \geq |g|/2$ . By Hölder's inequality, we have  $\|u\|_1 \leq \|u\|_p |\text{Supp}(u)|^{1-1/p}$ , which is less than a constant times  $\|b(g)\| |p(g)|^{1-1/p}$ . Therefore

$$M\rho_{\mathbb{Z} \wr \mathbb{Z},p}(t) \succeq \inf_{g \in \mathbb{Z} \wr \mathbb{Z}, |g| \geq t} \max\{|p(g)|, |g|/|p(g)|^{1-1/p}\},$$

which immediately implies the proposition.  $\square$

**7.5. H-metric.** Let  $G$  be a locally compact, compactly generated group and let  $S$  be a compact symmetric generating set. A Hilbert length function is a length function associated to some Hilbert 1-cocycle  $b$ , i.e.  $L(g) = \|b(g)\|$ . Consider the supremum of all Hilbert length functions on  $G$ , bounded by 1 on  $S$ : it defines a length function on  $G$  which in general is no longer a Hilbert length function. This length function has been introduced by Cornuier [C], § 2.6, who called the corresponding metric “H-metric”. Observe that if the group  $G$  satisfies  $M\rho_{G,2}(t) \approx t$ , then the H-metric of  $G$  is quasi-isometric to the word length. As a consequence of Theorem 5.1 and Proposition 4.5, we get

**Proposition 7.7.** *For every group in the class  $(\mathcal{L})$ , the H-metric is quasi-isometric to the word length.*



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