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Autor: González, Josep / Lario, Joan-C.
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Modular elliptic directions with complex multiplication (with an application to Gross's elliptic curves)

Josep González and Joan-C. Lario*

Abstract. Let A_f be the abelian variety attached by Shimura to a normalized newform $f \in S_2(\Gamma_1(N))$ and assume that A_f has elliptic quotients. The paper deals with the determination of the one dimensional subspaces (elliptic directions) in $S_2(\Gamma_1(N))$ corresponding to the pullbacks of the regular differentials of all elliptic quotients of A_f . For modular elliptic curves over number fields without complex multiplication (CM), the directions were studied by the authors in [8]. The main goal of the present paper is to characterize the directions corresponding to elliptic curves with CM. Then we apply the results obtained to the case $N = p^2$, for primes $p > 3$ and $p \equiv 3 \pmod{4}$. For this case we prove that if f has CM, then all optimal elliptic quotients of A_f are also optimal in the sense that its endomorphism ring is the maximal order of $\mathbb{Q}(\sqrt{-p})$. Moreover, if f has trivial Nebentypus then all optimal quotients are Gross's elliptic curve $A(p)$ and its Galois conjugates. Among all modular parametrizations $J_0(p^2) \rightarrow A(p)$, we describe a canonical one and discuss some of its properties.

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Keywords. Modular abelian varieties, complex multiplication, Größencharacter, optimal elliptic quotient.

1. Introduction

Let \mathbb{Q}^{alg} be a fixed algebraic closure of \mathbb{Q} . An elliptic curve C defined over \mathbb{Q}^{alg} is said to be modular if there is a non-constant homomorphism $\pi: J_1(N) \rightarrow C$, where $J_1(N)$ denotes the jacobian of the modular curve $X_1(N)$. Every modular elliptic curve over \mathbb{Q}^{alg} is a quotient of some modular abelian variety A_f attached by Shimura to a normalized newform f . From now on, we shall always consider parametrizations $\pi: J_1(N) \rightarrow C$ which factorize through such abelian varieties A_f , called in this paper modular abelian varieties of *elliptic type*.

A modular parametrization $\pi: J_1(N) \rightarrow C$ defined over a number field $L \subseteq \mathbb{Q}^{\text{alg}}$ induces an injection $\pi^*: \Omega^1(C/L) \hookrightarrow \Omega^1(J_1(N)/L)$. In what follows, we shall

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identify $\Omega^1(J_1(N)/L)$ with the subspace of cusp forms in $S_2(\Gamma_1(N))$ whose q -expansion lies in $L[[q]]$, via $h dq/q \mapsto h$ where $q = \exp(2\pi iz)$.

The determination of the normalized cusp forms in $S_2(\Gamma_1(N))$ associated with the pullbacks $\pi^*(\Omega^1(C))$ was discussed by the authors in [8] for elliptic curves without complex multiplication. In this paper, we shall deal with the complex multiplication case that needs techniques *ad hoc*. The present case is substantially richer since it requires the intervention of class field theory as well as the main theorem of complex multiplication.

Shimura shows in [16] that all elliptic curves with complex multiplication (CM) are modular. Due to Ribet [12], we know that A_f has an elliptic quotient with CM by an imaginary quadratic field $K \subset \mathbb{Q}^{\text{alg}}$ if and only if $f = f \otimes \chi$, where χ is the quadratic Dirichlet character attached to K . In this case, there is a primitive Hecke character $\psi: I(\mathfrak{m}) \rightarrow \mathbb{Q}^{\text{alg}}$ of conductor an ideal \mathfrak{m} of K such that the q -expansion of the CM normalized newform f is given by

$$f = \sum_{(\alpha, \mathfrak{m})=1} \psi(\alpha) q^{N(\alpha)} = \sum_{n=1}^{\infty} a_n q^n.$$

Here, $I(\mathfrak{m})$ denotes the multiplicative group of fractional ideals of K relatively prime to \mathfrak{m} , and the first summation is over integral ideals. The level of f is $N = N(\mathfrak{m}) |\Delta_K|$, the norm of \mathfrak{m} times the absolute value of the discriminant of K . We consider the number fields $E_f = \mathbb{Q}(\{a_n\})$ and $E = \mathbb{Q}(\{\psi(\alpha)\})$, generated by the images of ψ . One has $E = E_f \cdot K$, and we shall denote by Φ the set of its K -embeddings $E \hookrightarrow \mathbb{Q}^{\text{alg}}$. The number field E is a CM field. Through the paper, for all CM fields we shall denote by $\bar{}$ the canonical complex conjugation.

For future use, we recall that an abelian variety Y is called an optimal quotient of an abelian variety X over a field k if there is a surjective morphism $\pi: X \rightarrow Y$ defined over k whose kernel is an abelian variety. In this case, every endomorphism of X which leaves stable $\ker \pi$ induces an endomorphism of Y . The property of being an optimal quotient is transitive. Hereafter, every A_f is taken to be an optimal quotient of $J_1(N)$.

The plan of the paper is as follows. In Section 2, we study the decomposition of A_f over the quadratic field K for f with CM as before. This is an intermediate step necessary to determine the elliptic directions we are interested in. We shall prove

Theorem 1.1. *Let $f \in S_2(\Gamma_1(N))$ be a newform with CM and keep the above notations. There is an abelian variety (A, ι) of CM type Φ defined over K , with $\iota: E \hookrightarrow \text{End}_K^0(A)$, satisfying the following properties:*

- (i) *A is an optimal quotient of A_f over K and the pullback of $\Omega^1(A)$ corresponds with the subspace generated by $\{\sigma f : \sigma \in \Phi\}$;*
- (ii) *$\iota(\psi(\alpha))^*(\sigma f) = \sigma \psi(\alpha)^\sigma f$, for all $\alpha \in I(\mathfrak{m})$ and $\sigma \in \Phi$;*

- (iii) ι is an isomorphism;
- (iv) if \mathfrak{p} is a prime ideal of K with $\mathfrak{p} \nmid N$, then the lifting of the Frobenius endomorphism acting on the reduction of $A \bmod \mathfrak{p}$ is $\iota(\psi(\mathfrak{p}))$ or $\iota(\psi(\bar{\mathfrak{p}}))$ depending on $K \not\subseteq E_f$ or $K \subseteq E_f$, respectively.

We remark that the above abelian variety A is simple over K , and that A is A_f over K when $K \not\subseteq E_f$, while A_f is isogenous over K to $A \times \bar{A}$ when $K \subseteq E_f$. To encode both cases of part (iv) in Theorem 1.1, we shall denote by ψ' the primitive Hecke character mod $\bar{\mathfrak{m}}$ defined as

$$\psi'(\alpha) = \begin{cases} \psi(\alpha) & \text{if } K \not\subseteq E_f; \\ \overline{\psi(\bar{\alpha})} & \text{if } K \subseteq E_f. \end{cases}$$

As it will be shown, one has $\mathfrak{m} = \bar{\mathfrak{m}}$ in the first case.

Then we study the splitting field of A ; that is, the smallest number field where all endomorphisms of A are defined. We make use of class field theory to build a certain abelian extension L/K attached to the Hecke character ψ' ; the field L is a cyclic extension of the Hilbert class field of K and it is contained in the ray class field mod $\bar{\mathfrak{m}}$. To simplify notation, the Artin automorphism $(\frac{L/K}{\alpha})$ in $\text{Gal}(L/K)$ will be often denoted by the same symbol representing the ideal α . In particular, one has

$$\mathfrak{p} \beta \equiv \beta^{N(\mathfrak{p})} \pmod{\mathfrak{F}}$$

for all $\beta \in \mathcal{O}_L$, where \mathfrak{F} is an unramified prime ideal of L over a prime ideal \mathfrak{p} of K . The extension L/K is characterized by the property that α viewed in $\text{Gal}(L/K)$ is trivial if and only if $\psi'(\alpha) \in K^*$. The main result of Section 3 is the following

Theorem 1.2. *Let A be as above. Then the following holds:*

- (i) *There is an elliptic curve C defined over L with complex multiplication by the ring of integers \mathcal{O}_K and such that A is isogenous over L to $C^{\dim A}$.*
- (ii) *The field L is the smallest number field satisfying $\text{End}_{\mathbb{Q}^{\text{alg}}}^0(A) = \text{End}_L^0(A)$.*
- (iii) *There is a one-cocycle $\lambda: I(\bar{\mathfrak{m}}) \rightarrow L^*$ satisfying $\lambda(\alpha) = \psi'(\alpha)$ for all $\alpha \in I(\bar{\mathfrak{m}})$ with $(\frac{L/K}{\alpha}) = \text{id}$ in $\text{Gal}(L/K)$. The class of λ in $H^1(I(\bar{\mathfrak{m}}), L^*)$ is uniquely determined by this condition.*

In view of (iii), the cohomology class of λ depends intrinsically on A , and we shall denote it by $[A] \in H^1(I(\bar{\mathfrak{m}}), L^*)$. Section 4 is devoted to determining the elliptic directions in $\Omega^1(A)$ in terms of $[A]$. To this end, for each one-cocycle $\lambda \in [A]$ and $\sigma \in \Phi$, we introduce the sums

$$g_\sigma(\lambda) := \sum_{\alpha \in \text{Gal}(L/K)} \frac{\alpha^{-1} \lambda(\alpha)}{\sigma \psi'(\alpha)} \in {}^\sigma E \cdot L,$$

and also its Φ -trace

$$\text{tr}_\Phi(\lambda) := \sum_{\sigma \in \Phi} g_\sigma(\lambda) \in L.$$

Theorem 1.3. *With the above notations, the following holds.*

- (1) *If $\sum_{n \geq 1} \gamma_n q^n \in S_2(\Gamma_1(N))$ corresponds to an elliptic direction attached to a modular parametrization $\pi \in \text{Hom}_L(A, C)$, then $\gamma_1 \neq 0$.*
- (2) *The following statements are equivalent:*

- (i) *the normalized cusp form*

$$h = q + \sum_{n \geq 2} \gamma_n q^n \in S_2(\Gamma_1(N))$$

gives an elliptic direction attached to some $\pi \in \text{Hom}_L(A, C)$;

- (ii) *there is a one-cocycle $\lambda \in [A]$ with $\text{tr}_\Phi(\lambda) = [L : K]$ and such that*

$$h = \frac{1}{[L : K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda) \cdot {}^\sigma f.$$

The q -expansion of this elliptic direction is then given by

$$h = \begin{cases} \sum_{(\alpha, \mathfrak{m})=1} \alpha^{-1} \lambda(\alpha) q^{N(\alpha)} & \text{if } K \not\subseteq E_f; \\ \sum_{(\alpha, \mathfrak{m})=1} \frac{N(\alpha)}{\lambda(\bar{\alpha})} q^{N(\alpha)} & \text{if } K \subseteq E_f. \end{cases}$$

Moreover, all other elliptic directions are $\iota(a)^(h)$, for $a \in E^*$, and the equality $\iota(\psi'(\alpha))^*h = \alpha^{-1} \lambda(\alpha) \alpha^{-1} h$ holds for every $\alpha \in I(\bar{\mathfrak{m}})$.*

We shall say that a one-cocycle $\lambda \in [A]$ is *modular* if one has $\text{tr}_\Phi(\lambda) = [L : K]$. According to Theorem 1.3, these are precisely the one-cocycles that provide the elliptic directions. In Section 3, we also describe how to obtain all modular one-cocycles in $[A]$ explicitly way by means of a K -linear projector, and close the section by raising some open questions.

In the last three sections, we deal with the particular case concerning the level $N = p^2$ where $p > 3$ is a prime with $p \equiv 3 \pmod{4}$. The relevance of this case is in connection with the elliptic curves $A(p)$ studied by Gross in [9] and [10]. For convenience of the reader, we recall here its definition. Let $K = \mathbb{Q}(\sqrt{-p})$ and let \mathcal{O}_K be its ring of integers. Let H denote the Hilbert class field of K , and let

$H_0 = \mathbb{Q}(j(\mathcal{O}_K))$ be its maximal real subfield. The elliptic curve $A(p)$ is defined over H_0 and given by the Weierstrass equation

$$y^2 = x^3 + \frac{mp}{2^4 \cdot 3} x - \frac{np^2}{2^5 \cdot 3^3},$$

where m and n are the real numbers satisfying

$$m^3 = j(\mathcal{O}_K), \quad n^2 = \frac{j(\mathcal{O}_K) - 1728}{-p}, \quad \text{sgn } n = \left(\frac{2}{p}\right).$$

The elliptic curve $A(p)$ admits a global minimal model over H_0 with discriminant $-p^3$ and whose invariants are $c_4 = -mp$ and $c_6 = np^2$.

Given any intermediate modular subgroup Γ between $\Gamma_1(p^2)$ and $\Gamma_0(p^2)$ and a normalized newform $f \in S_2(\Gamma)$, we denote by $A_f^{(\Gamma)}$ its associated optimal quotient of $\text{Jac}(X_\Gamma)$, where X_Γ denotes the modular curve over \mathbb{Q} attached to Γ . According to this terminology, we have $A_f^{(\Gamma_1(p^2))} = A_f$. In Section 5, we prove:

Theorem 1.4. *With the above notations, the following holds.*

- (i) *For every positive divisor d of $(p - 1)/2$ there is a unique abelian variety A_f of CM elliptic type in $J_1(p^2)$ such that the Nebentypus of f has order d ; one has $K \not\subseteq E_f$, $\dim A_f = [H : K]\varphi(d)$, where φ is the Euler function, and the splitting field of A_f is the intermediate field between H and $H \cdot \mathbb{Q}(e^{2\pi i/p})$ of degree d .*
- (ii) *Let f be a CM normalized newform in $S_2(\Gamma_1(p^2))$ and let Γ satisfy*

$$\Gamma_1(p^2) \subseteq \Gamma \subseteq \Gamma_\varepsilon := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p^2) : \varepsilon(d) = 1 \right\},$$

where ε is the Nebentypus of f . Then all optimal elliptic quotients of $A_f^{(\Gamma)}$ have complex multiplication by \mathcal{O}_K . Moreover, if f belongs to $S_2(\Gamma_0(p^2))$, then all optimal quotients of $A_f^{(\Gamma)}$ are defined over H and are precisely the elliptic curve $A(p)$ and its Galois conjugates.

Among all modular parametrizations $J_0(p^2) \rightarrow A(p)$ one stands out. In Section 6, we discuss this canonical parametrization and give some of its arithmetical properties.

Theorem 1.5. *Set $\mathfrak{p} = \sqrt{-p} \mathcal{O}_K$. Let $\delta: I(\mathfrak{p}) \rightarrow H$ be the unique map defined by the conditions $\delta(\alpha)^{12} = \Delta(\mathcal{O}_K)/\Delta(\alpha)$ and $\left(\frac{N_{H/K}(\delta(\alpha))}{\mathfrak{p}}\right) = 1$. Let ω denote a Néron differential of $A(p)$, and let ψ be any Hecke character attached to $A(p)$. Then:*

- (i) *There is an optimal quotient $\pi : J_0(p^2) \rightarrow A(p)$ such that $\pi^*(\omega) = c g(q) dq/q$ where the elliptic direction is given by*

$$g(q) = \sum_{(\alpha, p)=1} \delta(\alpha)q^{N(\alpha)} \in S_2(\Gamma_0(p^2)),$$

and $c \in \mathbb{Z}$ is a unit in $\mathbb{Z}[\frac{1}{2p}]$.

- (ii) *The complex lattice $\{2\pi i \int_{\gamma} g(z)dz : \gamma \in H_1(X_0(p^2), \mathbb{Z})\}$ is*

$$\frac{1}{c} \cdot i^{(p+1)/4} \cdot \sqrt[h]{\rho \cdot (2\pi)^{(2h+1-p)/4} \cdot \sqrt{p}^{(1-3h)/2} \cdot \prod_{\substack{1 \leq m < p \\ \chi(m)=1}} \Gamma\left(\frac{m}{p}\right)} \cdot \mathcal{O}_K$$

where h is the class number of K , the h -th root is taken to be real, Γ is the Gamma function, and $\rho = \prod_{\alpha \in \text{Gal}(H/K)} \frac{\delta(\alpha)}{\psi(\alpha)}$ is a positive unit of H_0 .

Finally, in Section 7 we discuss how to compute the modular elliptic directions for A_f when $f \in S_2(\Gamma_1(p^2))$ has CM and its Nebentypus is nontrivial.

2. The abelian variety A

We shall adhere to the notations in the Introduction and prove Theorem 1.1. Let $\psi : I(\mathfrak{m}) \rightarrow \mathbb{Q}^{\text{alg}}$ be the fixed primitive Hecke character, and let

$$f = \sum_{(\alpha, \mathfrak{m})=1} \psi(\alpha)q^{N(\alpha)} = \sum_{n=1}^{\infty} a_n q^n$$

be its associated CM newform in $S_2(\Gamma_1(N))$. The optimal quotient A_f of $J_1(N)$ is defined over \mathbb{Q} by $A_f = J_1(N)/I_f(J_1(N))$, where $I_f(J_1(N))$ is the annihilator of f in the Hecke algebra acting on $J_1(N)$. In particular, the pullback of $\Omega^1(A_f/\mathbb{Q}^{\text{alg}})$ is $\{\{\sigma f\}\}$ where σ runs over $\text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$. Recall that $E_f = \mathbb{Q}(\{a_n\})$ and $E = \mathbb{Q}(\{\psi(\alpha)\})$. We fix an isomorphism

$$\iota : E_f \hookrightarrow \text{End}_{\mathbb{Q}}^0(A_f),$$

in such a way that $\iota(a_n)$ corresponds to the Hecke operator T_n acting on A_f . The Nebentypus of f is the mod N Dirichlet character $\varepsilon(d) = \chi(d)\psi((d))/d$, where χ is the quadratic character attached to K . We recall that $\iota(\varepsilon(d))$ is the diamond operator $\langle d \rangle$ acting on A_f . One has

$$\dim A_f = [E_f : \mathbb{Q}] = \begin{cases} [E : K] & \text{if } K \not\subseteq E_f; \\ 2[E : K] & \text{if } K \subseteq E_f. \end{cases}$$

Notice that $E = K \cdot E_f$. Now, we proceed to construct the abelian variety A over K of dimension $[E : K]$ with the properties required in Theorem 1.1. According to Shimura’s Proposition 8 in [17], there exists $u \in \text{End}_K^0(A_f)$ such that

$$u^*(\sigma f) = \sqrt{\Delta_K} \cdot \sigma f$$

for all σ in $\text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$. Here, the choice of the square root $\sqrt{\Delta_K}$ fixes u up to a sign. For the case $K \not\subseteq E_f$, we let $A = A_f$ and extend ι to E ,

$$\iota: E \hookrightarrow \text{End}_K^0(A_f),$$

via $\iota(\sqrt{\Delta_K}) = u$. For the second case, we proceed as follows. Since now $K \subseteq E_f$, there is $\alpha \in E_f$ such that $\iota(\alpha) \in \text{End}_{\mathbb{Q}}^0(A_f)$ acts as

$$\iota(\alpha)^*(\sigma f) = \sigma \sqrt{\Delta_K} \cdot \sigma f$$

for all σ in $\text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$. Then consider the involution $w := \iota(\alpha)u^{-1} \in \text{End}_K^0(A_f)$. Let A be the optimal quotient of $J_1(N)$ defined by A_f/B , where $B = (1 - w)A_f$. Clearly, the abelian variety A is defined over K , and $\Omega^1(A/K)$ is identified with $\langle \sigma f \rangle_{\sigma \in \Phi}$. Since B is stable by $\iota(E)$, the isomorphism $\iota: E \hookrightarrow \text{End}_{\mathbb{Q}}^0(A_f)$ induces in a natural way an embedding still denoted by the same letter

$$\iota: E \hookrightarrow \text{End}_K^0(A)$$

such that $\iota(\gamma)^*(\sigma f) = \sigma \gamma \cdot \sigma f$ for all γ in E and all K -embeddings σ in Φ . From the equality $\bar{w} = -w$, it follows that $\bar{B} = (1 + w)A_f$. Note that \bar{B} is K -isogenous to A .

A case-by-case argument, employing that $\text{End}_K^0(X) \hookrightarrow \text{End}_{\mathbb{Q}}^0(\text{Res}_{K/\mathbb{Q}}(X))$ for any abelian variety X/K , shows that the abelian variety A is K -simple in both cases. Therefore, it follows that ι is an isomorphism. In both cases, A is an abelian variety of CM type Φ and satisfies (i), (ii), and (iii) of Theorem 1.1.

To conclude the proof, it remains to check the property (iv) relative to the Frobenius liftings. To this end, let p be a prime such that $p \nmid N$ and denote by Frob_p and Ver_p the Frobenius and the Verschiebung acting on the reduction of A_f modulo p , which satisfy $\text{Frob}_p \cdot \text{Ver}_p = p$. By the Eichler–Shimura congruence, we know that

$$\tilde{T}_p = \text{Frob}_p + \text{Ver}_p \cdot \langle \tilde{p} \rangle,$$

where \tilde{T}_p and $\langle \tilde{p} \rangle$ denote the reductions of the Hecke operator T_p and the diamond operator $\langle p \rangle$ acting on $A_f \bmod p$. Let us consider the two cases separately.

Case $K \not\subseteq E_f$: first, assume that $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K . Since

$$\iota(a_p) = \iota(\psi(\mathfrak{p})) + \iota(\psi(\bar{\mathfrak{p}})), \quad \iota(\psi(\mathfrak{p})) \cdot \iota(\psi(\bar{\mathfrak{p}})) = p \langle p \rangle,$$

and $\widetilde{T}_p = \iota(\widetilde{a_p})$, it follows that the lifting of Frob_p is either $\iota(\psi(p))$ or $\iota(\psi(\bar{p}))$. Since a certain power of $\psi(p)$ belongs to \mathfrak{p} , one concludes that the lifting of $\text{Frob}_p = \text{Frob}_p$ is $\iota(\psi(p))$. A similar argument works when $p\mathcal{O}_K = \mathfrak{p}$ is inert in K , taking into account that $\text{Frob}_p = \text{Frob}_p^2 = -p \langle \widetilde{p} \rangle = \iota(\psi(\langle p \rangle))$.

Case $K \subseteq E_f$: since $\iota(E)$ leaves the abelian subvariety B stable, applying the same arguments as before, it follows that $\iota(\psi(p))$ is the lifting of Frob_p acting on the reduction of $B \bmod \mathfrak{p}$. Since A is K -isogenous to \bar{B} , the statement (iv) holds in this case as well. This completes the proof of Theorem 1.1.

The following lemma will be used in the next sections.

Lemma 2.1. *If $K \not\subseteq E_f$, then $\mathfrak{m} = \bar{\mathfrak{m}}$.*

Proof. Since $K \not\subseteq E_f$, there is σ in $\text{Gal}(\mathbb{Q}^{\text{alg}}/K)$ such that $\sigma f = \bar{f}$. First, we prove that the Hecke characters ${}^\sigma\psi$ and ψ_c given by ${}^\sigma\psi(\alpha) = \sigma(\psi(\alpha))$ and $\psi_c(\alpha) = \overline{\psi(\bar{\alpha})}$ coincide on $I(\mathfrak{m} \bar{\mathfrak{m}})$. Indeed, since $\sigma \varepsilon = \varepsilon^{-1}$ the assertion is immediate for prime ideals $\mathfrak{p} \mid p$ when p is inert. For the case that p splits completely in K , from the equalities ${}^\sigma a_p = \bar{a}_p$ and ${}^\sigma \varepsilon(p) = \varepsilon^{-1}(p)$, that is,

$${}^\sigma\psi(\mathfrak{p}) + {}^\sigma\psi(\bar{\mathfrak{p}}) = \psi_c(\mathfrak{p}) + \psi_c(\bar{\mathfrak{p}}) \quad \text{and} \quad {}^\sigma\psi(\mathfrak{p}) \cdot {}^\sigma\psi(\bar{\mathfrak{p}}) = \psi_c(\mathfrak{p}) \cdot \psi_c(\bar{\mathfrak{p}}),$$

it follows that ${}^\sigma\psi(\mathfrak{p})$ is either $\psi_c(\mathfrak{p})$ or $\psi_c(\bar{\mathfrak{p}})$. Again, we obtain that ${}^\sigma\psi(\mathfrak{p})$ and $\psi_c(\mathfrak{p})$ are equal because a certain power of them lie in \mathfrak{p} . Both Hecke characters being primitive of conductor \mathfrak{m} and $\bar{\mathfrak{m}}$ respectively, we must have $\mathfrak{m} = \bar{\mathfrak{m}}$. \square

3. Splitting field of A

We first introduce an abelian extension L/K that will play a key role in the splitting of the abelian variety A over \mathbb{Q}^{alg} . Let ψ' be the primitive Hecke character mod $\bar{\mathfrak{m}}$,

$$\psi' : I(\bar{\mathfrak{m}}) \rightarrow \mathbb{Q}^{\text{alg}},$$

given by $\psi'(\alpha) = \psi(\alpha)$ if $K \not\subseteq E_f$ or $\psi'(\alpha) = \overline{\psi(\bar{\alpha})}$ otherwise. We consider the character $\eta : (\mathcal{O}_K/\bar{\mathfrak{m}})^* \rightarrow \mathbb{Q}^{\text{alg}}$ defined by

$$\eta(a) = \frac{\psi'(\langle a \rangle)}{a}, \quad \text{for all } a \in \mathcal{O}_K \text{ with } (a, \bar{\mathfrak{m}}) = 1.$$

One easily checks that η is well defined. Recall that the existence of a Hecke character mod $\bar{\mathfrak{m}}$ is equivalent to the condition that the composition $\mathcal{O}_K^* \hookrightarrow \mathcal{O}_K \rightarrow \mathcal{O}_K/\bar{\mathfrak{m}}$ is a group monomorphism (see [16]) and thus $\ker \eta \cap \mathcal{O}_K^* = \{1\}$. By class field theory, to the congruence subgroup

$$P_\eta(\bar{\mathfrak{m}}) = \{(a) \in I(\bar{\mathfrak{m}}) : a \bmod \bar{\mathfrak{m}} \in \ker(\eta)\}$$

there corresponds an abelian extension L/K . It is easy to check that, for $\alpha \in I(\bar{m})$, one has $\alpha \in P_\eta(\bar{m})$ if and only if $\psi'(\alpha) \in K$. Let $K_{\bar{m}}$ denote the ray class field of $K \pmod{\bar{m}}$. Since the map $a \mapsto a\mathcal{O}_K$ provides an isomorphism between $\ker \eta$ and $P_\eta(\bar{m})/P_1(\bar{m})$, by using the exact sequence

$$1 \rightarrow \mathcal{O}_K^* \rightarrow (\mathcal{O}_K/\bar{m})^* \rightarrow I(\bar{m})/P_1(\bar{m}) \rightarrow I(\mathcal{O}_K)/P(\mathcal{O}_K) \rightarrow 1,$$

one readily shows that $L = K_{\bar{m}}^{\ker \eta}$ and $\text{Gal}(L/H)$ is isomorphic to the cyclic group $\text{im}(\eta)/\mathcal{O}_K^*$. Recall that here H denotes the Hilbert class field of K and, as usual, for any integral ideal \mathfrak{n} we denote by $P(\mathfrak{n})$ the subgroup of $I(\mathfrak{n})$ formed by principal ideals and the subscript 1 is for the subgroup of principal ideals with a generator congruent to one mod \mathfrak{n} . An alternate route to define the extension L/K is as follows. For every $\sigma \in \Phi$, the character

$$\chi_\sigma : \text{Gal}(K_{\bar{m}}/K) \rightarrow \mathbb{Q}^{\text{alg}*}, \quad \chi_\sigma(\alpha) = \frac{\sigma \psi'(\alpha)}{\psi'(\alpha)}$$

is well defined via the Artin isomorphism $\text{Gal}(K_{\bar{m}}/K) \simeq I(\bar{m})/P_1(\bar{m})$. Due to the fact that $\bigcap_{\sigma \in \Phi} \ker \chi_\sigma = P_\eta(\bar{m})/P_1(\bar{m})$, it follows that

$$L = K_{\bar{m}}^{\bigcap_{\sigma \in \Phi} \ker \chi_\sigma}.$$

Notice that L/\mathbb{Q} is not necessarily a normal extension; in fact, this is so if and only if $L = \bar{L}$.

Proposition 3.1. *There is an elliptic curve C defined over L such that:*

- (i) $\text{End}_L(C) \simeq \mathcal{O}_K$;
- (ii) *its Grössencharacter ψ_C coincides with $\psi' \circ N_{L/K}$;*
- (iii) *C is isogenous over L to all its $\text{Gal}(L/K)$ -conjugates;*
- (iv) *the abelian variety A is isogenous over L to the power $C^{[E:K]}$.*

Proof. The extreme cases $L = H$ and $L = K_{\bar{m}}$ are proved by Gross in [9] and by de Shalit in [6], respectively. For the general case, one can follow the same arguments. Let C_1 be any elliptic curve over L such that $\text{End}_L(C) \simeq \mathcal{O}_K$. Let \mathfrak{n} be its conductor. Once we fix an isomorphism $\theta : K \rightarrow \text{End}_L^0(C_1)$, we can consider the Grössencharacter $\psi_{C_1} : I_L(\mathfrak{n}) \rightarrow K^*$ attached to the pair (C_1, θ) . For a prime ideal \mathfrak{P} of L relatively prime to \mathfrak{n} , we know that $\theta(\psi_{C_1}(\mathfrak{P}))$ is the lifting of the \mathfrak{P} -Frobenius acting on the reduction of $C_1 \pmod{\mathfrak{P}}$. Recall also that if $\mathfrak{P} \in P_{1,L}(\mathfrak{n})$ then $\psi_{C_1}(\mathfrak{P}) = N_{L/K}(\beta)$, where $\mathfrak{P} = (\beta)$ with $\beta \equiv 1 \pmod{\mathfrak{P}}$.

By class field theory, the composition $\psi' \circ N_{L/K}$ takes values in K^* and the equality $\psi' \circ N_{L/K}(\mathfrak{P}) = N_{L/K}(\beta)$ holds for every $\mathfrak{P} = (\beta)$ with $\beta \equiv 1 \pmod{\bar{m}\mathcal{O}_L}$. Hence

the quotient $(\psi' \circ N_{L/K})/\psi_{C_1}$ defines a character $\delta: I_L(\mathfrak{m}\mathcal{O}_L)/P_{1,L}(\mathfrak{m}\mathcal{O}_L) \rightarrow \mathcal{O}_K^*$ of finite order. The twist $C := C_1 \otimes \delta$ satisfies (i) and (ii). Now, (iii) follows from the fact that $\psi_C = \psi_{\alpha C}$ for all $\alpha \in \text{Gal}(L/K)$ due to (ii).

Now we check (iv). By Faltings’s criterion (for instance, see §2, Corollary 2, of [3]), it suffices to prove that for every prime \mathfrak{P} of L not dividing N nor the conductor of C , the reductions of the abelian varieties A and $C^{\dim A}$ modulo \mathfrak{P} are isogenous over the residue field $\mathcal{O}_L/\mathfrak{P}$. We write $\mathfrak{p}^f = N_{L/K} \mathfrak{P}$, where with no risk of confusion now f is the residue degree of \mathfrak{P} over K . On the one hand, the characteristic polynomial of the endomorphism $\text{Frob}_{\mathfrak{P}}$ acting on the l -adic Tate module of the reduction of A/L modulo \mathfrak{P} , for a prime $l \neq p$, is the characteristic polynomial of the complex representation of $\iota(\psi'(\mathfrak{p}^f))$:

$$P_{A,\mathfrak{P}}(x) = \prod_{\sigma \in \Phi} (x - \sigma \psi'(\mathfrak{p}^f))(x - \overline{\sigma \psi'(\mathfrak{p}^f)}).$$

On the other hand, the corresponding Frobenius characteristic polynomial for C at \mathfrak{P} is

$$P_{C,\mathfrak{P}}(x) = (x - \psi_C(\mathfrak{P}))(x - \overline{\psi_C(\mathfrak{P})}) = (x - \psi'(\mathfrak{p}^f))(x - \overline{\psi'(\mathfrak{p}^f)}).$$

Since $\psi'(\mathfrak{p}^f)$ belongs to K , we obtain $P_{A,\mathfrak{P}}(x) = P_{C,\mathfrak{P}}(x)^{\dim A}$. Thus, A is isogenous over L to $C^{\dim A}$. □

Proposition 3.2. *The field L is the smallest number field satisfying $\text{End}_{\mathbb{Q}^{\text{alg}}}^0(A) = \text{End}_L^0(A)$.*

Proof. Since A is isogenous over L to the $[E : K]$ -th power of the elliptic curve C , we have $\text{End}_{\mathbb{Q}^{\text{alg}}}^0(A) = \text{End}_L^0(A)$. That L is the smallest number field with this property can be deduced from the following fact. For every $\varphi \in \text{End}_L^0(A)$, one has the explicit version of the Skolem–Noether theorem:

$${}^{\mathfrak{p}}\varphi = \iota(\psi'(\mathfrak{p})) \cdot \varphi \cdot \iota(\psi'(\mathfrak{p}))^{-1},$$

for all $\mathfrak{p} \in I(\mathfrak{m})$ not dividing N . To check this equality, it is enough to verify that it holds reduced modulo a prime ideal \mathfrak{P} of L over \mathfrak{p} . The smallest field of definition for all endomorphisms of A is the fixed field L^G , where

$$G = \{v \in \text{Gal}(L/K) : {}^v\phi = \phi \text{ for all } \phi \in \text{End}_L^0(A)\}.$$

By the Čebotarev density theorem, every v in $\text{Gal}(L/K)$ can be written as $v = (\frac{L/K}{\mathfrak{p}})$ for some prime ideal \mathfrak{p} relatively prime to N . We have that $v \in G$ if and only if $\iota(\psi'(\mathfrak{p}))$ is in the center of $\text{End}_L^0(A)$; that is, when $\psi'(\mathfrak{p}) \in K$ and this fact implies that \mathfrak{p} splits completely in L , so that $v = \text{id}$. □

Let C be an elliptic curve defined over L as in Proposition 3.1. The main theorem of complex multiplication (Theorem 5.4 in [15]) implies the existence of a system of isogenies $\{\mu_\alpha: C \rightarrow {}^\alpha C\}$ over L , $(\alpha, \bar{m}) = 1$, satisfying the following properties:

- (i) $\mu_{\alpha\beta} = {}^\alpha \mu_\beta \mu_\alpha$;
- (ii) if C has good reduction at a prime ideal $\mathfrak{P} \mid \mathfrak{p}$, then $\mu_\mathfrak{p}$ is the lifting of the Frobenius map between the reductions of C and ${}^\mathfrak{p}C \bmod \mathfrak{P}$.

Attached to the system of isogenies $\{\mu_\alpha\}$, a one-cocycle can be defined as follows (see also [7]). For a non-zero regular differential ω in $\Omega^1(C/L)$, let $\lambda_\omega: I(\bar{m}) \rightarrow L^*$ be the map given by

$$\mu_\alpha^*({}^\alpha \omega) = \lambda_\omega(\alpha)\omega,$$

where ${}^\alpha \omega$ denotes the differential in ${}^\alpha C$ corresponding to ω by conjugation. It follows that λ_ω is a one-cocycle, and for all $u \in L^*$ one has

$$\lambda_{u\omega}(\alpha) = \lambda_\omega(\alpha) {}^\alpha u/u.$$

Clearly, the class of λ_ω in $H^1(I(\bar{m}), L^*)$ does not depend on the particular choice of ω . Note that if $\alpha \in P_\eta(\bar{m})$, then we have $\lambda_\omega(\alpha) = \psi'(\alpha)$. The class λ_ω in $H^1(I(\bar{m}), L^*)$ can be characterized from ψ' as follows:

Proposition 3.3. *Let $\lambda: I(\bar{m}) \rightarrow L^*$ be any one-cocycle satisfying $\lambda(\alpha) = \psi'(\alpha)$ for all $\alpha \in I(\bar{m})$ with $(\frac{L/K}{\alpha}) = \text{id}$ in $\text{Gal}(L/K)$. Then $[\lambda] = [\lambda_\omega]$.*

Proof. Assume that $\lambda \in H^1(I(\bar{m}), L^*)$ satisfies $\lambda(\alpha) = \psi'(\alpha)$ for all $\alpha \in P_\eta(\bar{m})$. The quotient λ/λ_ω defines a one-cocycle in $H^1(\text{Gal}(L/K), L^*)$. By Hilbert’s 90 theorem, we know that there is $u \in L^*$ such that $\lambda(\alpha)/\lambda_\omega(\alpha) = {}^\alpha u/u$ for all $\alpha \in I(\bar{m})$. Thus, we have $[\lambda] = [\lambda_\omega]$. □

This completes the proof of Theorem 1.2 in the Introduction. From now on, we shall denote by $[A]$ in $H^1(I(\bar{m}), L^*)$ the cohomology class of λ_ω .

4. Modular one-cocycles and elliptic directions

In this section we keep the notations as above and tackle the problem of determining the elliptic directions in $\Omega^1(A)$. The goal is to prove Theorem 1.3 that will be deduced from the next three Propositions after the following

Lemma 4.1. *Let $\pi \in \text{Hom}_L(A, C)$ be a non-constant modular parametrization, and let $\omega \in \Omega^1(C/L)$ be any non-zero regular differential. Denote by*

$$h = \sum_{n \geq 1} \gamma_n q^n \in S_2(\Gamma_1(N))$$

the cusp form associated with the pullback $\pi^*(\omega)$. Then:

- (i) $\gamma_1 \in L^*$;
- (ii) for all $\alpha \in I(\bar{m})$ relatively prime to N , one has $\iota(\psi'(\alpha))^*h = \alpha^{-1}\lambda_\omega(\alpha)\alpha^{-1}h$;
- (iii) we have the identity $h = \frac{1}{[L:K]} \sum_{\alpha \in \text{Gal}(L/K)} \sum_{\sigma \in \Phi} \frac{\alpha^{-1}\lambda_\omega(\alpha)\alpha^{-1}}{\sigma\psi'(\alpha)} h$;
- (iv) $\{\psi'(\alpha_i)\}$ is a K -basis of E if and only if $\{\alpha_i^{-1}h\}$ is an L -basis of $\Omega^1(A/L)$.

Proof. (i) Since π and ω are defined over L , the cusp form h associated with $\pi^*(\omega)$ has q -expansion $\sum_{n \geq 1} \gamma_n q^n$ with coefficients in L . Since the abelian variety A is simple over K , we have that A is a K -factor of the Weil restriction $\text{Res}_{L/K}(C)$. Thus, the set $\{\alpha h : \alpha \in \text{Gal}(L/K)\}$ generates $\Omega^1(A/L)$. This implies $\gamma_1 \neq 0$.

(ii) It is enough to consider the case when $\alpha = \mathfrak{p}$ is a prime ideal not dividing N . Then the claim follows from the commutativity of the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\iota(\psi'(\mathfrak{p}))} & A \\
 \alpha^{-1}\pi \downarrow & & \downarrow \pi \\
 \alpha^{-1}C & \xrightarrow{\alpha^{-1}\mu_\alpha} & C
 \end{array}$$

due to the fact that $\iota(\psi'(\mathfrak{p}))$ and $\alpha^{-1}\mu_\alpha$ are liftings of the corresponding \mathfrak{p} -Frobenius morphisms at a prime ideal $\mathfrak{P} | \mathfrak{p}$ of L .

(iii) Write $h = \sum_{v \in \Phi} c_v \nu f$, with $c_v \in \mathbb{Q}^{\text{alg}}$. By applying (ii), for all $\sigma \in \Phi$ and $\alpha \in \text{Gal}(L/K)$, one has

$$\frac{\alpha^{-1}\lambda_\omega(\alpha)}{\sigma\psi'(\alpha)} \alpha^{-1}h = \frac{1}{\sigma\psi'(\alpha)} \left(\sum_{v \in \Phi} c_v \nu \psi'(\alpha) \nu f \right) = \sum_{v \in \Phi} c_v (\chi_v \cdot \chi_\sigma^{-1})(\alpha) \nu f.$$

Thus, it holds

$$\begin{aligned}
 \sum_{\alpha} \sum_{\sigma} \frac{\alpha^{-1}\lambda_\omega(\alpha)}{\sigma\psi'(\alpha)} \alpha^{-1}h &= \sum_{\sigma, v} \sum_{\alpha} c_v (\chi_v \cdot \chi_\sigma^{-1})(\alpha) \nu f \\
 &= [L : K] \sum_v c_v \nu f = [L : K] h.
 \end{aligned}$$

(iv) If $\{\psi'(\alpha_1), \dots, \psi'(\alpha_r)\}$ is a K -basis of E , then for every $\alpha \in I(\bar{m})$ we can write $\psi'(\alpha) = \sum_{i=1}^r \alpha_i \psi'(\alpha_i)$ with $\alpha_i \in K$. Thus, we obtain

$$\alpha^{-1}\lambda_\omega(\alpha)\alpha^{-1}h = \iota(\psi'(\alpha))^*(h) = \sum_{i=1}^r \alpha_i \iota(\psi'(\alpha_i))^*h = \sum_{i=1}^r \alpha_i \alpha_i^{-1} \lambda_\omega(\alpha_i) \alpha_i^{-1} h.$$

Since $\{\alpha h : \alpha \in \text{Gal}(L/K)\}$ generates $\Omega^1(A/L)$ and $\dim(A) = [E : K]$, it follows that $\{\alpha_1^{-1} h, \dots, \alpha_r^{-1} h\}$ is a L -basis of $\Omega^1(A/L)$.

Conversely, assume that $\{\alpha_1^{-1} h, \dots, \alpha_r^{-1} h\}$ is a L -basis of $\Omega^1(A/L)$. By using part (ii), if $\sum_{i=1}^r \alpha_i \psi'(\alpha_i) = 0$ for some $\alpha_i \in K$, then $\sum_{i=1}^r \alpha_i \alpha_i^{-1} \lambda_\omega(\alpha_i) \alpha_i^{-1} h = 0$. This implies that all $\alpha_i = 0$. Since $\dim(A) = [E : K] = r$, the proof is done. \square

Due to part (i) in the above Lemma 4.1, there is a unique $\omega \in \Omega^1(C/L)$ such that the pullback $\pi^*(\omega)$ gives a normalized cusp form, say

$$h = q + \sum_{n \geq 2} \gamma_n q^n.$$

This particular λ_ω will be called *modular with respect to π* or, simply, *π -modular*. For every 1-cocycle $\lambda \in [A]$, we consider the following sums. Let $\sigma \in \Phi$, and set

$$g_\sigma(\lambda) := \sum_{\alpha \in \text{Gal}(L/K)} \frac{\alpha^{-1} \lambda(\alpha)}{\sigma \psi'(\alpha)}.$$

Notice that $g_\sigma(\lambda)$ is well defined and $g_\sigma(\lambda) \in {}^\sigma E \cdot L$.

Remark 4.1. The sum $g_\sigma(\lambda)$ can be interpreted as a sort of Gauss sum, in the sense that we have

$$g_\sigma(\lambda) = \sum_{\alpha \in \text{Gal}(L/K)} \chi_\sigma^{-1}(\alpha) u_\alpha$$

where $u_\alpha = \alpha^{-1} \lambda(\alpha) / \psi'(\alpha)$. If C admits a global minimal Weierstrass equation over L , then the one-cocycle λ attached to a Néron differential satisfies the capitulation property $\lambda(\alpha) \mathcal{O}_L = \alpha \mathcal{O}_L$ (see Remark 10.3 in [7]). Then u_α^e is an unit in \mathcal{O}_L^* where e is the order of α in $\text{Gal}(L/K)$.

We shall denote the Φ -trace of $g_\sigma(\lambda)$ by

$$\text{tr}_\Phi(\lambda) = \sum_{\sigma \in \Phi} g_\sigma(\lambda) \in L.$$

Remark 4.2. Recall that we have defined $\lambda \in [A]$ to be modular if $\text{tr}_\Phi(\lambda) = [L : K]$ in the Introduction. As it will be shown, both terms (modular and π -modular) turn out to be equivalent.

For every $\gamma \in L^*$ and $\lambda \in [A]$, let λ_γ denote the twisted one-cocycle in $[A]$ given by $\lambda_\gamma(\alpha) = \lambda(\alpha) \gamma / \alpha \gamma$. Writing $\lambda = \lambda_\omega$ with some $\omega \in \Omega^1(C/L)$, then $\lambda_\gamma = \lambda_{\frac{1}{\gamma} \omega}$. We shall need the following lemma.

Lemma 4.2. *For all $\alpha \in I(\bar{m})$ and $\sigma \in \Phi$, one has*

- (i) $g_\sigma(\lambda_{\alpha^{-1}\lambda(\alpha)}) \cdot \alpha^{-1}\lambda(\alpha) = g_\sigma(\lambda) \cdot \sigma\psi'(\alpha)$;
- (ii) $\text{tr}_\Phi(\lambda_{\alpha^{-1}\lambda(\alpha)}) = \alpha^{-1}\text{tr}_\Phi(\lambda)$.

Proof. It follows straightforward from the definitions and by using the cocycle relations for λ . □

Proposition 4.3. *Assume that $\lambda \in [A]$ is modular with respect to $\pi \in \text{Hom}_L(A, C)$. Then $\text{tr}_\Phi(\lambda) = [L : K]$ and*

$$h = \frac{1}{[L : K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda) \cdot \sigma f$$

is the normalized elliptic direction in $\pi^(\Omega^1(C/L))$.*

Proof. Since λ is π -modular, there is a non-zero regular differential $\omega \in \Omega^1(C/L)$ such that $\pi^*(\omega)$ is a normalized cusp form $h = q + \sum_{n \geq 2} \gamma_n q^n$ and $\lambda = \lambda_\omega$. By comparing the first Fourier coefficient in the equality at Lemma 4.1 (iii), we have that $\text{tr}_\Phi(\lambda) = [L : K]$. For every $\sigma \in \Phi$, set

$$F_\sigma = \sum_{\mathfrak{b} \in \text{Gal}(L/K)} \frac{\mathfrak{b}^{-1}\lambda(\mathfrak{b})}{\sigma\psi'(\mathfrak{b})} \mathfrak{b}^{-1}h.$$

Also by Lemma 4.1 (iii), we know that $\sum_{\sigma \in \Phi} F_\sigma = [L : K]h$. From the equality $\iota(\psi'(\alpha))^*(\mathfrak{b}^{-1}h) = (\mathfrak{b}\cdot\alpha)^{-1}\lambda(\alpha)(\mathfrak{b}\cdot\alpha)^{-1}h$, one obtains

$$\begin{aligned} \iota(\psi'(\alpha))^*(F_\sigma) &= \sum_{\mathfrak{b} \in \text{Gal}(L/K)} \frac{\mathfrak{b}^{-1}\lambda(\mathfrak{b})}{\sigma\psi'(\mathfrak{b})} (\mathfrak{b}\cdot\alpha)^{-1}\lambda(\alpha)(\mathfrak{b}\cdot\alpha)^{-1}h \\ &= \sum_{\mathfrak{b} \in \text{Gal}(L/K)} \frac{(\mathfrak{b}\cdot\alpha)^{-1}\lambda(\mathfrak{b}\cdot\alpha)}{\sigma\psi'(\mathfrak{b})} (\mathfrak{b}\cdot\alpha)^{-1}h = \sigma\psi'(\alpha)F_\sigma. \end{aligned}$$

Hence F_σ and σf differ by a scalar multiple. Since the q -expansion of F_σ begins as $g_\sigma(\lambda)q + \dots$, it follows that $F_\sigma = g_\sigma(\lambda) \cdot \sigma f$, and then $h = \frac{1}{[L:K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda) \cdot \sigma f$. □

Now, we shall prove that the modular one-cocycles λ in $[A]$ with respect to some modular parametrization π are precisely those that satisfy the trace condition $\text{tr}_\Phi(\lambda) = [L : K]$. To this end, for a given one-cocycle $\lambda \in [A]$ (not necessarily modular), let us consider the K -linear map $\text{pr}: L \rightarrow L$,

$$\text{pr}(u) := \sum_{\alpha \in \text{Gal}(L/K)} \left(\sum_{\sigma \in \Phi} \frac{1}{\sigma\psi'(\alpha)} \right)^{\alpha^{-1}} \lambda(\alpha)^{\alpha^{-1}} u = \begin{cases} u \cdot \text{tr}_\Phi(\lambda_u) & \text{if } u \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Consider the eigenspace $\mathcal{M} = \{u \in L : \text{pr}(u) = [L : K] \cdot u\}$. Notice that λ_u is modular if and only if $u \in \mathcal{M} \setminus \{0\}$. In particular, we know that $\dim_K(\mathcal{M}) > 0$ and it does not depend on the particular choice of $\lambda \in [A]$ used to define the K -linear map pr .

Proposition 4.4. *One has*

- (i) $\text{pr}^2 = [L : K] \text{pr}$;
- (ii) $\dim_K(\mathcal{M}) = [E : K]$;
- (iii) if λ is modular, then $\mathcal{M} = \{\{\alpha^{-1} \lambda(\alpha)\}\}_K$ where α runs over $\text{Gal}(L/K)$.

Proof. The first claim comes from the computation:

$$\begin{aligned} \text{pr}^2(u) &= \sum_{\alpha} \left(\sum_{\sigma} \frac{1}{\sigma \psi'(\alpha)} \right)^{\alpha^{-1}} \lambda(\alpha)^{\alpha^{-1}} \left[\sum_{\mathfrak{b}} \left(\sum_{\tau} \frac{1}{\tau \psi'(\mathfrak{b})} \right)^{\mathfrak{b}^{-1}} \lambda(\mathfrak{b})^{\mathfrak{b}^{-1}} u \right] \\ &= \sum_{\alpha} \sum_{\mathfrak{b}} \left(\sum_{\sigma} \frac{1}{\sigma \psi'(\alpha)} \right) \left(\sum_{\tau} \frac{1}{\tau \psi'(\mathfrak{b})} \right)^{(\alpha \mathfrak{b})^{-1}} \lambda(\alpha \mathfrak{b})^{(\alpha \mathfrak{b})^{-1}} u \\ &= \sum_{\alpha} \sum_{\mathfrak{b}} \left(\sum_{\sigma} \frac{1}{\sigma \psi'(\alpha)} \right) \left(\sum_{\tau} \frac{1}{\tau \psi'(\alpha^{-1} \mathfrak{b})} \right)^{\mathfrak{b}^{-1}} \lambda(\mathfrak{b})^{\mathfrak{b}^{-1}} u \\ &= \sum_{\mathfrak{b}} \sum_{\alpha} \left(\sum_{\sigma, \tau} (\chi_{\sigma} \chi_{\tau}^{-1})(\alpha) \right) \frac{\lambda(\mathfrak{b})^{\mathfrak{b}^{-1}}}{\tau \psi'(\mathfrak{b})} u \\ &= [L : K] \text{pr}(u). \end{aligned}$$

Let us prove (ii) and (iii) simultaneously. Since $\dim_K(\mathcal{M})$ is independent of the one-cocycle λ chosen in $[A]$, we can (and do) assume that λ is modular. Set

$$h = \frac{1}{[L : K]} \sum_{\sigma \in \Phi} g_{\sigma}(\lambda) \cdot \sigma f = 1 + \sum_{n>1} \gamma_n q^n.$$

Let $W = \{\{\alpha^{-1} \lambda(\alpha)\}\}_K$ where α runs over $\text{Gal}(L/K)$. We need to show that $W = \mathcal{M}$ and $\dim_K(W) = [E : K]$. Choose $\alpha_1, \dots, \alpha_r \in I(\bar{\mathfrak{m}})$ such that $\{\psi'(\alpha_1), \dots, \psi'(\alpha_r)\}$ is a K -basis of E . We claim that $\{\alpha_1^{-1} \lambda(\alpha_1), \dots, \alpha_r^{-1} \lambda(\alpha_r)\}$ is a K -basis of W . Indeed, if $\sum_{i=1}^r \alpha_i \alpha_i^{-1} \lambda(\alpha_i) = 0$ for some α_i in K , then consider $\alpha := \sum_{i=1}^r \alpha_i \psi'(\alpha_i) \in E$. It is easy to check that $\iota(\alpha)^*(h) = \sum_{n \geq 1} \gamma'_n q^n$ with $\gamma'_1 = 0$. This forces $\alpha = 0$, since otherwise we get a contradiction from Lemma 4.1 (i) applied

to $\iota(\alpha)^*(h)$. Therefore, all $\alpha_i = 0$ which implies that $\alpha_1^{-1}\lambda(\alpha_1), \dots, \alpha_r^{-1}\lambda(\alpha_r)$ are linearly independent. Now, for every ideal $\alpha \in I(\bar{m})$, one has $\psi'(\alpha) = \sum_{i=1}^r \alpha_i \psi'(\alpha_i)$ for some $\alpha_i \in K$. By taking q -expansions in the equality

$$\alpha^{-1}\lambda(\alpha)\alpha^{-1}h = \sum_{i=1}^r \alpha_i \alpha_i^{-1}\lambda(\alpha_i)\alpha_i^{-1}h,$$

we obtain $\alpha^{-1}\lambda(\alpha) = \sum_{i=1}^r \alpha_i \alpha_i^{-1}\lambda(\alpha_i)$. So far, we have $\dim_K(W) = [E : K]$ and the inclusion $W \subseteq \mathcal{M}$ follows from Lemma 4.2 (ii).

To easy notation, set $u_i = \alpha_i^{-1}\lambda(\alpha_i)$ for $1 \leq i \leq r$ and let us show that they generate \mathcal{M} . For any nonzero $u \in \mathcal{M}$, consider the normalized cusp form

$$h_u = \frac{1}{[L : K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda_u) \cdot^\sigma f.$$

Since $\{h_{u_1}, \dots, h_{u_n}\}$ is a L -basis of $\Omega^1(A/L)$ by Lemma 4.1 (iv), there are $\gamma_i \in L$ such that $h_u = \sum_{i=1}^r \gamma_i h_{u_i}$. Notice that $\sum_{i=1}^r \gamma_i = 1$. By applying $\iota(\psi'(\alpha))^*$ to h_u , and then conjugate by α , we obtain

$$\lambda_u(\alpha)h_u = \sum_{i=1}^r \alpha \gamma_i \lambda_{u_i}(\alpha) h_{u_i}.$$

Therefore, we have

$$\gamma_i = \alpha \gamma_i \frac{\lambda_{u_i}(\alpha)}{\lambda_u(\alpha)} = \alpha \gamma_i \frac{\alpha u}{\alpha u_i u}$$

for all α and $1 \leq i \leq r$. That is, $\beta_i := \gamma_i u/u_i \in K$. Then $u = \sum_{i=1}^r \beta_i u_i$ since $\sum_{i=1}^r \gamma_i = 1$. The statement (iii) follows. \square

Proposition 4.5. *Let $\lambda' \in [A]$ such that $\text{tr}_\Phi(\lambda') = [L : K]$. Then λ' is modular with respect to some $\pi' \in \text{Hom}_L(A, C)$.*

Proof. We shall prove that there is $\pi' \in \text{Hom}_L(A, C)$ and $\omega' \in \Omega^1(C/L)$ such that $\pi'^*(\omega')$ corresponds to the normalized cusp form

$$h' = \frac{1}{[L : K]} \sum_{\sigma \in \Phi} \sum_{\alpha \in \text{Gal}(L/K)} \frac{\alpha^{-1}\lambda'(\alpha)}{\sigma \psi'(\alpha)} \cdot^\sigma f.$$

Consider any non-constant $\pi \in \text{Hom}_L(A, C)$ and take $\omega \in \Omega^1(C/L)$ such that $\pi^*(\omega)$ corresponds to the normalized cusp form

$$h = \frac{1}{[L : K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda) \cdot^\sigma f,$$

where $\lambda = \lambda_\omega$. Let $L = \ker(\text{pr}) \oplus \mathcal{M}$ be the decomposition corresponding to the projector pr attached to λ . Now, there is $\gamma \in \mathcal{M}$ such that $\lambda' = \lambda_\gamma$ and

$$h' = \frac{1}{[L : K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda_\gamma) \cdot^\sigma f$$

with $\gamma = \sum_{\alpha \in \text{Gal}(L/K)} r_\alpha \alpha^{-1} \lambda(\alpha)$ for some $r_\alpha \in K$ due to Proposition 4.4 (iii). We claim that

$$\left(\sum_{\alpha \in \text{Gal}(L/K)} r_\alpha \alpha^{-1} \lambda(\alpha) \right) h' = \iota \left(\sum_{\alpha \in \text{Gal}(L/K)} r_\alpha \psi'(\alpha) \right)^* h. \tag{1}$$

Letting $\Psi = \iota \left(\sum_{\alpha \in \text{Gal}(L/K)} r_\alpha \psi'(\alpha) \right) \in \text{End}_K^0(A)$, then it follows

$$h' = \Psi^* \left(\pi^* \left(\frac{1}{\gamma} \omega \right) \right) = (\pi \circ \Psi)^* \left(\frac{1}{\gamma} \omega \right),$$

which implies that λ' is modular. To check (1), we use Lemma 4.2 (i):

$$\begin{aligned} \gamma h' &= \frac{1}{[L : K]} \sum_{\sigma} \sum_{\mathfrak{b}} \frac{\mathfrak{b}^{-1} \lambda(\mathfrak{b}) \mathfrak{b}^{-1} \gamma}{\sigma \psi'(\mathfrak{b})} \sigma f \\ &= \frac{1}{[L : K]} \sum_{\sigma} \sum_{\mathfrak{b}} \sum_{\alpha} \frac{\mathfrak{b}^{-1} \lambda(\mathfrak{b}) r_\alpha (\alpha \mathfrak{b})^{-1} \lambda(\alpha)}{\sigma \psi'(\mathfrak{b})} \sigma f \\ &= \frac{1}{[L : K]} \sum_{\sigma} \sum_{\alpha} r_\alpha g_\sigma(\lambda_{\alpha^{-1} \lambda(\alpha)})^{\alpha^{-1}} \lambda(\alpha) \sigma f \\ &= \frac{1}{[L : K]} \sum_{\sigma} \sum_{\alpha} r_\alpha \sigma \psi'(\alpha) g_\sigma(\lambda) \sigma f \\ &= \frac{1}{[L : K]} \Psi^* \left(\sum_{\sigma} g_\sigma(\lambda) \sigma f \right) = \Psi^*(h). \quad \square \end{aligned}$$

The transitivity of the action of $\iota(E^*)$ on the set of elliptic directions follows from the equality (1). To finish the proof of Theorem 1.3, it remains to determine the q -expansions of the normalized elliptic directions. For it, first we need a technical lemma.

Lemma 4.6. *Let $\ell : I(\mathfrak{m}) \rightarrow L^*$ be a map such that $\ell(\alpha) = \psi(\alpha)$ for all $\alpha = \text{id}$ in $\text{Gal}(\bar{L}/K)$. Let $\tau : \text{Gal}(\bar{L}/K) \rightarrow \text{Gal}(L/K)$ be a map such that $\ell(\alpha \mathfrak{b}) =$*

$\ell(\alpha)^{\tau(\alpha)}\ell(\mathfrak{b})$ for all $\alpha \in I(\mathfrak{m})$. Then the identity

$$\frac{1}{[L : K]} \sum_{\sigma \in \Phi} \beta_{\sigma}^{\sigma} \psi(c) = \ell(c) \quad (2)$$

holds for all $c \in I(\mathfrak{m})$ if and only if

$$\beta_{\sigma} = \sum_{\alpha \in \text{Gal}(\bar{L}/K)} \frac{\ell(\alpha)}{\sigma \psi(\alpha)} \quad \text{and} \quad \sum_{\sigma \in \Phi} \beta_{\sigma} = [L : K]. \quad (3)$$

Proof. Assume (3). For every $c \in I(\mathfrak{m})$, we have

$$\begin{aligned} \sum_{\sigma \in \Phi} \left(\sum_{\alpha \in \text{Gal}(\bar{L}/K)} \frac{\ell(\alpha)}{\sigma \psi(\alpha)} \right)^{\sigma} \psi(c) &= \sum_{\sigma \in \Phi} \left(\sum_{\alpha \in \text{Gal}(\bar{L}/K)} \frac{\ell(\alpha c)}{\sigma \psi(\alpha c)} \right)^{\sigma} \psi(c) \\ &= \ell(c) \sum_{\sigma \in \Phi} \left(\sum_{\alpha \in \text{Gal}(\bar{L}/K)} \frac{\tau(c)\ell(\alpha)}{\sigma \psi(\alpha)} \right) \\ &= \ell(c)^{\tau(c)} \left(\sum_{\alpha \in \text{Gal}(\bar{L}/K)} \ell(\alpha) \left(\sum_{\sigma \in \Phi} \frac{1}{\sigma \psi(\alpha)} \right) \right) \\ &= \ell(c)^{\tau(c)} \left(\sum_{\sigma \in \Phi} \beta_{\sigma} \right) = \ell(c) [L : K]. \end{aligned}$$

Now, suppose (2). Fix $\nu \in \Phi$. Note that for $\sigma \in \Phi$, the characters χ_{σ} and χ_{ν} are equal if and only if $\sigma = \nu$. For every $\alpha \in \text{Gal}(\bar{L}/K)$, one has

$$\begin{aligned} \frac{\ell(\alpha)}{\nu \psi(\alpha)} &= \frac{1}{[L : K]} \left(\beta_{\nu} + \sum_{\sigma \in \Phi \setminus \{\nu\}} \beta_{\sigma} \frac{\sigma \psi(\alpha)}{\nu \psi(\alpha)} \right) \\ &= \frac{1}{[L : K]} \left(\beta_{\nu} + \sum_{\sigma \in \Phi \setminus \{\nu\}} \beta_{\sigma} (\chi_{\sigma} \chi_{\nu}^{-1})(\alpha) \right). \end{aligned}$$

Summing over all α , then

$$\sum_{\alpha \in \text{Gal}(\bar{L}/K)} \frac{\ell(\alpha)}{\nu \psi(\alpha)} = \beta_{\nu} + \frac{1}{[L : K]} \left(\sum_{\sigma \in \Phi \setminus \{\nu\}} \beta_{\sigma} \sum_{\alpha \in \text{Gal}(\bar{L}/K)} (\chi_{\sigma} \chi_{\nu}^{-1})(\alpha) \right) = \beta_{\nu}.$$

The condition $\sum_{\sigma \in \Phi} \beta_{\sigma} = [L : K]$ is obtained by replacing α with \emptyset in (2). \square

Proposition 4.7. *Assume that $\lambda \in [A]$ satisfies $\text{tr}_\Phi(\lambda) = [L : K]$. Consider the normalized cusp form*

$$h = \frac{1}{[L : K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda) \cdot \sigma f.$$

Then:

(i) *one has*

$$h = \begin{cases} \sum_{(\alpha, \mathfrak{m})=1} \alpha^{-1} \lambda(\alpha) q^{N(\alpha)} & \text{if } K \not\subseteq E_f; \\ \sum_{(\alpha, \mathfrak{m})=1} \frac{N(\alpha)}{\lambda(\bar{\alpha})} q^{N(\alpha)} & \text{if } K \subseteq E_f; \end{cases}$$

(ii) *for all $c \in I(\bar{\mathfrak{m}})$, we have $\iota(\psi'(c))^*(h) = c^{-1} \lambda(c)^{c-1} h$.*

Proof. For all $\alpha \in I(\mathfrak{m})$, set

$$\ell(\alpha) = \begin{cases} \alpha^{-1} \lambda(\alpha) & \text{if } K \not\subseteq E_f; \\ \frac{N(\alpha)}{\lambda(\bar{\alpha})} & \text{if } K \subseteq E_f. \end{cases}$$

It is clear that $\ell(\alpha\beta)$ is $\ell(\alpha)\alpha^{-1}\ell(\beta)$ or $\ell(\alpha)\bar{\alpha}\ell(\beta)$ depending on whether $K \not\subseteq E_f$ or not, respectively. Since for the case $K \subseteq E_f$ one has

$$\frac{\ell(\alpha^{-1})}{\sigma \psi(\alpha^{-1})} = \frac{\bar{\alpha}^{-1} (1/\ell(\alpha))}{\sigma \psi(\alpha^{-1})} = \frac{\bar{\alpha}^{-1} (N(\alpha)/\ell(\alpha))}{N(\alpha)/\sigma \psi(\alpha)} = \frac{\bar{\alpha}^{-1} \lambda(\bar{\alpha})}{\sigma \psi'(\bar{\alpha})},$$

for all $\sigma \in \Phi$, then in both cases it follows that $g_\sigma(\lambda) = \sum_{\alpha \in \text{Gal}(\bar{L}/K)} \ell(\alpha)/\sigma \psi(\alpha)$. By using Lemma 4.6, a case-by-case computation shows that for all $\alpha \in I(\mathfrak{m})$ and $c \in I(\bar{\mathfrak{m}})$ one has

$$\frac{1}{[L : K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda) \sigma \psi(\alpha) \sigma \psi'(c) = c^{-1} \lambda(c)^{c-1} \ell(\alpha). \tag{4}$$

Plugging $c = 1$ in (4) it follows part (i). Part (ii) follows from part (i) and (4). □

Now, Theorem 1.3 in the Introduction follows from Propositions 4.3, 4.5 and 4.7. Note that due to Proposition 4.4, all one-cocycles in $[A]$ are modular if and only if $[E : K] = [L : K]$; i.e., when A is K -isogenous to $\text{Res}_{L/K}(C)$. In general, in order to determine a modular one-cocycle in $[A]$ a strategy emerges from the previous results. Indeed, first one can build a one-cocycle $\lambda \in [A]$ by solving and combining norm equations. If $\text{tr}_\Phi(\lambda) \neq 0$, then $\lambda_{\text{tr}_\Phi(\lambda)}$ is modular since its Φ -trace equals $[L : K]$.

Alternatively, if $\text{tr}_\Phi(\lambda) = 0$ or in any circumstance, the nullspace of the K -linear map $\text{pr} - [L : K] \text{Id}$ provides all $u \in L$ such that λ_u is modular.

We also remark that for the case $K \subseteq E_f$, there are elliptic quotients of A_f that do not factor through neither A nor \bar{A} . These quotients can be obtained using the above results plus the Weil involution acting on A_f .

We conclude this section with three open questions: one concerning about the isomorphism $\iota: E \rightarrow \text{End}_K^0(A)$ and the others about the elliptic optimal quotients of A . All the results of the paper hold when we replace $J_1(N)$ with $\text{Jac}(X_\Gamma)$, where Γ is an intermediate congruence subgroup between $\Gamma_1(N)$ and $\Gamma_0(N)$ such that f in $S_2(\Gamma)$ and X_Γ is the modular curve attached to this subgroup. Although the optimal quotient A of $A_f^{(\Gamma)}$ does depend on Γ , it is known that $\iota(T_p) \in \text{End}_{\mathbb{Q}}(A_f^{(\Gamma)})$ and, thus, $\iota(T_p)$ belongs to $\text{End}_K(A)$ for all Γ .

Question 4.8. Is $\iota(\psi(\alpha)) \in \text{End}_K(A)$ for all integral ideals α and all Γ ?

We ask ourselves whether the j -invariants of optimal modular parametrizations of CM elliptic curves are not far from being also *optimal* in the sense of having CM by the maximal order of K . Of course, if $\iota(\mathcal{O}_K) \subset \text{End}_K(A)$ all optimal elliptic quotients have multiplication by \mathcal{O}_K . If $\iota(\eta(\alpha)) \in \text{End}_K(A)$ for all integral ideals $\alpha \in I(\bar{\mathfrak{m}})$, then the j -invariants of all optimal elliptic quotients are in the Hilbert class field H . From Cremona's tables ($N < 130000$), we have checked that all optimal elliptic quotients over \mathbb{Q} with CM of $J_0(N)$ have complex multiplication by \mathcal{O}_K . Also, the same experimental result has been obtained in all examples over \mathbb{Q}^{alg} collected by the authors.

Question 4.9. Assume that $\pi \in \text{Hom}_L(A, C)$ is optimal. Does C have complex multiplication by \mathcal{O}_K ?

And the last question is related to the above Remark 4.1.

Question 4.10. Is it true that the existence of an optimal elliptic quotient of A having global minimal model over L is equivalent to the existence of a modular one-cocycle $\lambda \in [A]$ with values $\lambda(\alpha)$ in the ring of integers \mathcal{O}_L for all integral ideals $\alpha \in I(\bar{\mathfrak{m}})$?

In the next sections, we apply the above results and focus our attention on Gross's elliptic curves $A(p)$. We also give a positive answer to the second question mentioned above for the particular case of level $N = p^2$.

5. CM elliptic optimal quotients of $J_1(p^2)$

In the sequel p is a prime > 3 and such that $p \equiv 3 \pmod{4}$. The discriminant of $K = \mathbb{Q}(\sqrt{-p})$ is $-p$. Set $\mathfrak{p} = \sqrt{-p} \mathcal{O}_K$. Let \mathcal{X} denote the set of Hecke

characters mod \mathfrak{p} and let \mathcal{Y} be the set of Dirichlet characters $\eta: (\mathcal{O}_K/\mathfrak{p})^* \rightarrow \mathbb{C}^*$ such that $\eta(-1) = -1$.

To every Hecke character $\psi \in \mathcal{X}$, we attach its eta-character η in \mathcal{Y} defined as in Section 3 by $\eta(a) = \psi((a))/a$, and it can be easily seen that this map $\mathcal{X} \rightarrow \mathcal{Y}$ is surjective. The Nebentypus $\varepsilon: (\mathbb{Z}/p\mathbb{Z})^* \rightarrow \mathbb{C}^*$ of the newform $f \in S_2(\Gamma_1(p^2))$ associated with ψ is given by $\varepsilon(n) = \chi(n)\eta(n)$, where χ is the quadratic Dirichlet character associated with K . In this case, we have that $\text{ord } \varepsilon = (\text{ord } \eta)/2$.

By the results in Section 3, we know that the elliptic optimal quotients of the abelian variety A_f are defined over a number field L , which is a cyclic extension of H of degree $\text{ord } \varepsilon$ contained in $K_{\mathfrak{p}}$.

Proposition 5.1. *The ray class field $K_{\mathfrak{p}}$ satisfies $[K_{\mathfrak{p}} : H] = (p - 1)/2$ and we have $K_{\mathfrak{p}} = H \cdot \mathbb{Q}(\zeta_p)$, where $\zeta_p = e^{2\pi i/p}$.*

Proof. From the exact sequence

$$1 \longrightarrow (\mathcal{O}_K/\mathfrak{p})^*/\mathcal{O}_K^* \longrightarrow I(\mathfrak{p})/P_1(\mathfrak{p}) \longrightarrow I(\mathcal{O}_K)/P(\mathcal{O}_K) \longrightarrow 1,$$

we know that the Galois group $\text{Gal}(K_{\mathfrak{p}}/H)$ is isomorphic to $(\mathcal{O}_K/\mathfrak{p})^*/\mathcal{O}_K^*$ and, thus, one has $[K_{\mathfrak{p}} : H] = (p - 1)/2$. Consider the morphism $\Phi_{\mathfrak{p}}: I(\mathfrak{p}) \rightarrow \text{Gal}(H \cdot \mathbb{Q}(\zeta_p)/K)$ given by the Artin symbol. We claim that $\Phi_{\mathfrak{p}}$ has kernel $P_1(\mathfrak{p})$, which implies that $K_{\mathfrak{p}} \subseteq H \cdot \mathbb{Q}(\zeta_p)$. Indeed, for any ideal $\alpha \in I(\mathfrak{p})$, we have that $\Phi_{\mathfrak{p}}(\alpha)$ acts trivially on H if and only if $\alpha \in P(\mathfrak{p})$, that is $\alpha = a\mathcal{O}$. Moreover, $\Phi_{\mathfrak{p}}(a\mathcal{O})$ acts trivially on $\mathbb{Q}(\zeta_p)$ if and only if the Artin symbol $(\frac{\mathbb{Q}(\zeta_p)/\mathbb{Q}}{N(a)})$ is the identity; i.e., $N(a) \equiv 1 \pmod{\mathfrak{p}}$ which is equivalent to $\alpha \in P_1(\mathfrak{p})$ since $N(a) \equiv a^2 \pmod{\mathfrak{p}}$. Finally, for any subfield F of $\mathbb{Q}(\zeta_p)$ which contains K we have that $H \cap F = K$ since either $F = K$ or F/K is ramified at \mathfrak{p} . Hence, one has the equality $[H \cdot \mathbb{Q}(\zeta_p) : H] = (p - 1)/2 = [K_{\mathfrak{p}} : H]$ and the statement follows. \square

We shall need the following lemma.

Lemma 5.2. *Let $\psi \in \mathcal{X}$ and denote by η and f its eta-character and newform, respectively. Then the following holds:*

(i) *For every ideal $\alpha \in I(\mathfrak{p})$, one has*

$$\text{Tr}_{E/K}(\psi(\alpha)) = \begin{cases} a \sum_{\sigma \in \Phi} \sigma \eta(a) & \text{if } \alpha = a\mathcal{O}_K, \\ 0 & \text{if } \alpha \notin P(\mathfrak{p}). \end{cases}$$

(ii) *Let η' and f' denote the eta-character and newform associated with $\psi' \in \mathcal{X}$. Then $f' = {}^{\sigma} f$ for some $\sigma \in \text{Gal}(\mathbb{Q}^{\text{alg}}/K)$ if and only if $\ker \eta' = \ker \eta$.*

Proof. First, let us prove (i). When $\alpha = a\mathcal{O}_K$, the claim on the trace is clear since ${}^\sigma\psi((a)) = a^\sigma\eta(a)$. Suppose that $\alpha \notin P(\mathfrak{p})$, and let n be the order of α in $I(\mathfrak{p})/P(\eta)$. Notice that $n > 1$ and $\psi(\alpha) \notin K$. For every $\sigma \in \Phi$, we have ${}^\sigma\psi(\alpha) = \psi(\alpha)\zeta_\sigma$ for some $\zeta_\sigma \in \mu_n$, where μ_n denotes the group of n -th roots of unity. Thus, we have

$$\sum_{\sigma \in \Phi} {}^\sigma\psi(\alpha) = \psi(\alpha) \sum_{\sigma \in \Phi} \zeta_\sigma \in K.$$

Therefore, either $\text{Tr}_{E/K}(\psi(\alpha)) = 0$ or $\psi(\alpha) \in K(\mu_n)$. Let us see that the last possibility does not occur. For it, assume that $\psi(\alpha) \in K(\mu_n)$ which implies that the extension $K(\psi(\alpha))/K$ is normal. Since n is the minimum positive integer such that $\psi(\alpha)^n \in K$, it follows that either $\mu_n \subset K$ or $\psi(\alpha)^{2n} \in K^n$ (see Proposition 2 in [14]). Since $\psi(\alpha) \notin K$, we must have that $\psi(\alpha^{2n}) = b^n = \psi((b\mathcal{O}_K)^n)$ for some $b \in K$ and, hence, $\alpha^2 = b\mathcal{O}_K$. The class number of K being odd, we get a contradiction.

Let us prove (ii). If $f' = {}^\sigma f$ for some $\sigma \in \text{Gal}(\mathbb{Q}^{\text{alg}}/K)$ then the statement is clear since $\eta' = {}^\sigma\eta$. Now, suppose that $\ker \eta' = \ker \eta$. We claim that

$$\{{}^\sigma f : \sigma \in \Phi\} \cap \{{}^\sigma f' : \sigma \in \Phi'\} \neq \emptyset,$$

where Φ' is the corresponding set of K -embeddings $\mathbb{Q}(\psi') \hookrightarrow \mathbb{C}$. Let us consider the normalized cusp forms

$$h = \frac{1}{|\Phi|} \sum_{\sigma \in \Phi} {}^\sigma f = q + \dots,$$

$$h' = \frac{1}{|\Phi'|} \sum_{\sigma \in \Phi'} {}^\sigma f' = q + \dots$$

in $S_2(\Gamma_1(p^2))^{\text{new}}$. Since $K \not\subseteq \mathbb{Q}(\text{im } \eta)$ and $\ker \eta' = \ker \eta$, there is $\tau \in \Phi$ such that ${}^\tau\eta(a) = \eta'(a)$ for all $a \in \mathcal{O}_K$ coprime with \mathfrak{p} . By applying (i), we obtain the equality

$$h = \sum_{\alpha \in P(\mathfrak{p})} \frac{\text{Tr}_{E/K}(\psi(\alpha))}{|\Phi|} q^{N(\alpha)} = \sum_{\alpha \in P(\mathfrak{p})} \frac{\text{Tr}_{E/K}(\psi'(\alpha))}{|\Phi'|} q^{N(\alpha)} = h'.$$

Therefore, the \mathbb{Q}^{alg} -vector spaces generated by $\{{}^\sigma f : \sigma \in \Phi\}$ and $\{{}^\sigma f' : \sigma \in \Phi'\}$ have a common non-zero cusp form, which implies that $f' = {}^\sigma f$ for some $\sigma \in \text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$ (cf. Proposition 3.2 in [1]). Since $h \in \langle {}^\tau f : \tau \in \text{Gal}(\mathbb{Q}^{\text{alg}}/K) \rangle \cap \langle {}^\tau f' : \tau \in \text{Gal}(\mathbb{Q}^{\text{alg}}/K) \rangle$, it follows that $\sigma \in \text{Gal}(\mathbb{Q}^{\text{alg}}/K)$. \square

Proposition 5.3. *For every positive divisor d of $(p - 1)/2$ there is a unique abelian variety A_f of CM elliptic type of level p^2 such that the Nebentypus of f has order d ; this abelian variety satisfies that $K \not\subseteq E_f$ and $\dim A_f = [H : K]\varphi(d)$, where φ is the Euler function.*

Proof. Let d be a divisor of $(p - 1)/2$ and take $\psi \in \mathcal{X}$ such that its eta-character has order $2d$. Let us denote by f the newform attached to ψ , whose Nebentypus ε has order d . First, let us show that $K \not\subseteq E_f$. Indeed, let $\psi_c \in \mathcal{X}$ defined by $\psi_c(\alpha) = \overline{\psi(\bar{\alpha})}$. The eta-character and the normalized newform attached to ψ_c are clearly $\bar{\eta}$ and \bar{f} , respectively. Since $\ker \bar{\eta} = \ker \eta$, Lemma 5.2(ii) ensures that $\bar{f} \in \{\sigma f : \sigma \in \Phi\}$, which implies $K \not\subseteq E_f$. The same argument can be applied to another newform f' obtained from $\psi' \in \mathcal{X}$ whose associated character η' has order $2d$ to show that f' belongs to $\{\sigma f : \sigma \in \Phi\}$, which proves that A_f is unique when the order of ε has been fixed.

Since $K \not\subseteq E_f$, the equality $\dim A_f = [E_f : \mathbb{Q}] = [E : K]$ holds. Now, we have that $[E : K] = |\{\sigma f : \sigma \in \Phi\}| = |\{\sigma \psi : \sigma \in \Phi\}|$. Again using part (ii) of Lemma 5.2, we obtain

$$[E : K] = |\{\sigma \in \Phi : \eta = \sigma \eta\}| \cdot |\{\sigma \eta : \sigma \in \Phi\}| = |\{\sigma \in \Phi : \eta = \sigma \eta\}| \cdot \varphi(d).$$

Since the condition $\sigma \eta = \eta$ is equivalent to $\psi/\sigma \psi$ being a character of $\text{Gal}(H/K)$, it follows $\dim A_f = [H : K]\varphi(d)$. □

Remark 5.1. Note that the number of abelian varieties A_f of CM elliptic type of level p^2 is the number of divisors of $(p - 1)/2$. Also for every number field L intermediate between H and $H \cdot \mathbb{Q}(\zeta_p)$ there is a unique abelian variety A_f of CM elliptic type and level p^2 for which L is its splitting field as defined in Section 3.

Next, in order to show that the CM elliptic optimal quotients of A_f in $J_1(p^2)$ have endomorphism ring isomorphic to \mathcal{O}_K , we shall need to use some auxiliary congruence subgroups of $\text{SL}_2(\mathbb{Z})$ of level p^2 . To this end, fix a newform f in $S_2(\Gamma_1(p^2))$ attached to a Hecke character $\psi \in \mathcal{X}$. Let ε denote the Nebentypus of f . Let us consider the following congruence subgroups of level p^2 :

$$\Gamma_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p^2) : a \equiv d \equiv 1 \pmod{p} \right\},$$

and Γ_ε as in the introduction; i.e.,

$$\Gamma_\varepsilon = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p^2) : \varepsilon(d) = 1 \right\}.$$

It is clear that $\Gamma_1(p^2) \subseteq \Gamma_p \subseteq \Gamma_\varepsilon$ and $f \in S_2(\Gamma_\varepsilon)$. For any intermediate congruence subgroup Γ of level p^2 satisfying $\Gamma_1(p^2) \subseteq \Gamma \subseteq \Gamma_\varepsilon$, let X_Γ be the modular curve over \mathbb{Q} attached to Γ . We shall denote by $A_f^{(\Gamma)}$ the optimal quotient of the jacobian of X_Γ attached to f by Shimura. More precisely, let I_f be the annihilator of f in the Hecke algebra acting on $\text{Jac}(X_\Gamma)$. Then

$$A_f^{(\Gamma)} = \text{Jac}(X_\Gamma) / I_f(\text{Jac}(X_\Gamma)).$$

Proposition 5.4. *Let f and Γ be as above. Then all elliptic optimal quotients of $A_f^{(\Gamma)}$ have complex multiplication by \mathcal{O}_K .*

Proof. Fix an elliptic direction in $\Omega^1(A_f)$ and let C_Γ be an elliptic optimal quotient attached to this direction. By Proposition 5.3 and Theorem 1.2, we know that $K \not\subseteq E_f$ and thus all endomorphisms of $A_f^{(\Gamma)}$ are defined over its splitting field, say L , that satisfies $L \subseteq K_p$. Let c_Γ denote the conductor of the order $\mathcal{O}_\Gamma \simeq \text{End}_L(C_\Gamma)$ in \mathcal{O}_K . We want to show that $c_\Gamma = 1$, and split the proof in three steps.

Step 1: $c_\Gamma \mid 2$ for all Γ . Since $\text{End}(C_\Gamma) = \text{End}_L(C_\Gamma)$, one has that L contains the ring class field of \mathcal{O}_Γ , say K_Γ . Notice that $K_\Gamma \subseteq L \subseteq K_p$. But $p \nmid c_\Gamma$, since otherwise p must divide $[L : H]$ (cf. Proposition 7.24 in [4]) and this degree is a divisor of $(p - 1)/2$. Hence, K_Γ is an unramified extension of the Hilbert class field and, therefore, it must coincide with H . Again by Proposition 7.24 in [4], we obtain that $c_\Gamma \mid 2$.

Step 2: c_Γ does not depend on Γ . We consider the natural projection $\pi : X_\Gamma \rightarrow X_{\Gamma_\varepsilon}$. The degree of π is odd since it divides $[\Gamma_1(p^2) : \Gamma_0(p^2)/\{\pm 1\}] = p(p - 1)/2$ and $p \equiv 3 \pmod{4}$.

Let $\pi_{\Gamma, \Gamma_\varepsilon} : \text{Jac}(X_\Gamma) \rightarrow A_{\Gamma, \Gamma_\varepsilon}$ be the optimal quotient over \mathbb{Q} for which there is an isogeny $\nu : A_{\Gamma, \Gamma_\varepsilon} \rightarrow \text{Jac}(X_{\Gamma_\varepsilon})$ defined over \mathbb{Q} rendering the following diagram

$$\begin{array}{ccc}
 \text{Jac}(X_\Gamma) & \xrightarrow{\pi_*} & \text{Jac}(X_{\Gamma_\varepsilon}) \\
 \searrow \pi_{\Gamma, \Gamma_\varepsilon} & & \nearrow \nu \\
 & A_{\Gamma, \Gamma_\varepsilon} &
 \end{array}$$

commutative. Since every element of the group $H_1(X_{\Gamma_\varepsilon}, \mathbb{Z})/\pi_*(H_1(X_\Gamma, \mathbb{Z}))$ has order dividing $\deg \pi$, the cardinality of this group is odd. From the group isomorphism $\ker \nu \simeq H_1(X_{\Gamma_\varepsilon}, \mathbb{Z})/\pi_*(H_1(X_\Gamma, \mathbb{Z}))$, it follows that $\deg \nu$ is odd. Since $A_f^{(\Gamma)}$ is an optimal quotient of $A_{\Gamma, \Gamma_\varepsilon}$, there is an isogeny $\nu_f : A_f^{(\Gamma)} \rightarrow A_f^{(\Gamma_\varepsilon)}$ whose degree divides $\deg \nu$. Hence, for every optimal elliptic quotient $\pi_\Gamma : A_f^{(\Gamma)} \rightarrow C_\Gamma$ there is an optimal elliptic quotient $\pi_\varepsilon : A_f^{(\Gamma_\varepsilon)} \rightarrow C_{\Gamma_\varepsilon}$ and an isogeny $\mu : C_\Gamma \rightarrow C_{\Gamma_\varepsilon}$ rendering the diagram

$$\begin{array}{ccc}
 A_f^{(\Gamma)} & \xrightarrow{\nu_f} & A_f^{(\Gamma_\varepsilon)} \\
 \pi_\Gamma \downarrow & & \downarrow \pi_\varepsilon \\
 C_\Gamma & \xrightarrow{\mu} & C_{\Gamma_\varepsilon}
 \end{array}$$

commutative. It is clear that $\deg \mu$ is odd since it divides $\deg \nu_f$. So c_{Γ_ε} and c_Γ can only differ by an odd factor, which implies that c_Γ is independent of the group Γ .

Step 3: $c_\Gamma = 1$ for all Γ . Now, it suffices to prove $c_\Gamma = 1$ for a particular subgroup Γ . We consider $\Gamma = \Gamma_p$. Following Shimura in [17], we know that the matrix

$$\begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}$$

lies in the normalizer of Γ_p in $SL_2(\mathbb{R})$ and provides an automorphism u of X_{Γ_p} of order p . Set

$$G = \sum_{\substack{1 \leq i < p \\ \chi(i)=1}} (u^*)^i \in \text{End Jac}(X_{\Gamma_p}).$$

We claim that G leaves stable the subvariety $I_f(\text{Jac}(X_{\Gamma_p}))$, which is equivalent to saying that G leaves stable the vector space generated by the set of eigenforms in $S_2(\Gamma_p)$ which are not Galois conjugates of f . In fact, the action of G on all eigenforms of $S_2(\Gamma_p)$ can be described as follows. It is well-known that if we denote by New_Γ the set of normalized newforms in $S_2(\Gamma)$, then the set of normalized eigenforms in $S_2(\Gamma_p)$ is the disjoint union of New_{Γ_p} , \mathcal{S}_1 , and \mathcal{S}_2 , where $\mathcal{S}_1 = \text{New}_{\Gamma_1(p)} \cap S_2(\Gamma_p)$, $\mathcal{S}_2 = B_p(\text{New}_{\Gamma_1(p)}) \cap S_2(\Gamma_p)$, and B_p is the operator acting as $B_p(h(q)) = h(q^p)$. With $\zeta_p = e^{2\pi i/p}$ and from the equality

$$\sum_{\substack{1 \leq i < p \\ \chi(i)=1}} \zeta_p^i = \frac{-1 + \sqrt{-p}}{2},$$

it can be easily checked that every eigenform $h(q) = \sum_{n \geq 1} b_n q^n \in S_2(\Gamma_p)$ satisfies:

$$G^*(h) = \begin{cases} \frac{-1 + \sqrt{-p}}{2} h + \frac{p - \sqrt{-p}}{2} b_p B_p(h) & \text{if } h \in \text{New}_{\Gamma_p} \cup \mathcal{S}_1, \\ \frac{p - 1}{2} h & \text{if } h \in \mathcal{S}_2. \end{cases}$$

The claim follows from the fact that all $h \in \text{New}_{\Gamma_p}$ have level p^2 and Nebentypus whose conductor divides p and, thus, $b_p = 0$ (see Subsection 1.8 in [5]).

Since G leaves stable the subvariety $I_f(\text{Jac}(X_{\Gamma_p}))$, then G induces an endomorphism of $A_f^{(\Gamma_p)}$, which we still denote by G . Due to the fact that G acts on $\Omega^1(A_f^{(\Gamma_p)})$ as the multiplication by $(-1 + \sqrt{-p})/2$, it follows that G leaves stable all subvarieties of $A^{(\Gamma_p)}$. Thus, $(-1 + \sqrt{-p})/2 \in \mathcal{O}_{\Gamma_p}$ and the statement follows. \square

As for Gross’s elliptic curves, we obtain the following result, which concludes the proof of Theorem 1.4.

Corollary 5.5. *Let f be a CM normalized newform with trivial Nebentypus. The elliptic curve $A(p)$ and its Galois conjugates are the optimal quotients of $A_f^{(\Gamma)}$ over the Hilbert class field H , for all subgroups Γ with $\Gamma_1(p^2) \subseteq \Gamma \subseteq \Gamma_0(p^2)$.*

Proof. By Theorem 20.1 in [9], we know that $A(p)$ is a quotient of $J_0(p^2)$ defined over H , attached to a newform f with trivial Nebentypus. Notice that the corresponding field L coincides with the Hilbert class field H . Since we have $K \not\subseteq E_f$, by Theorems 1.1 and 1.2, every elliptic optimal quotient C_Γ of $A_f^{(\Gamma)}$ is defined over H and the abelian variety $A_f^{(\Gamma)}$ is simple over K . Since $\dim A_f^{(\Gamma)} = [H : K]$, it follows that $A_f^{(\Gamma)}$ is K -isogenous to the Weil restriction $\text{Res}_{H/K} C_\Gamma$. In [9], Gross shows that $A_f^{(\Gamma)}$ is K -isogenous to $\text{Res}_{H/K} A(p)$. Therefore, on the one hand, there is $\sigma \in \text{Gal}(H/K)$ such that $A(p)$ and ${}^\sigma C_\Gamma$ are \mathbb{Q}^{alg} -isomorphic. On the other hand, by Theorem 5.4, $A(p)$ and ${}^\sigma C_\Gamma$ are H -isogenous. Hence, $A(p)$ is H -isomorphic to ${}^\sigma C_\Gamma$ and the claim follows. \square

6. Canonical CM elliptic direction for $A(p)$

When the class number of K is greater than one, there are infinitely many elliptic directions in $S_2(\Gamma_0(p^2))$ attached to different parametrizations $J_0(p^2) \rightarrow A(p)$. Here, we shall emphasize one of them (we call it canonical) in terms of a particular one-cocycle that can be constructed by means of the Dedekind eta-function.

Let \mathcal{O}_H be the ring of integers of the Hilbert class field H . For all $a \in K$ coprime with \mathfrak{p} , we denote by $\left(\frac{a}{\mathfrak{p}}\right)$ the Jacobi symbol $\left(\frac{m}{p}\right)$, where m is an integer such that $a \equiv m \pmod{\mathfrak{p}}$. One has $\eta(a) = \left(\frac{a}{\mathfrak{p}}\right)$. By [10], we know that there is a unique map $\delta: I(\mathfrak{p}) \rightarrow H$ with the following two requirements:

- (i) $\delta(\alpha)^{12} = \Delta(\mathcal{O})/\Delta(\alpha)$,
- (ii) $\left(\frac{N_{H/K}(\delta(\alpha))}{\mathfrak{p}}\right) = 1$,

for all $\alpha \in I(\mathfrak{p})$. Moreover, this map also satisfies the following conditions:

- (iii) $\delta(\alpha)\mathcal{O}_H = \alpha\mathcal{O}_H$,
- (iv) $\delta(\alpha \cdot \mathfrak{b}) = \delta(\alpha) \cdot \alpha^{-1} \delta(\mathfrak{b})$ for all $\alpha, \mathfrak{b} \in I(\mathfrak{p})$,
- (v) $\delta(\bar{\alpha}) = \overline{\delta(\alpha)}$ for all $\alpha \in I(\mathfrak{p})$.

By taking into account conditions (ii) and (iv), and since $[H : K]$ is odd, we also obtain:

- (vi) for all $\alpha \in P(\mathfrak{p})$, one has $\delta(\bar{\alpha}) \in K$ and $\left(\frac{\delta(\alpha)}{\mathfrak{p}}\right) = 1$.

For every $\alpha \in I(\mathfrak{p})$, we set

$$\lambda(\alpha) := {}^\alpha \delta(\alpha) = \frac{N(\alpha)}{\delta(\bar{\alpha})}. \tag{5}$$

The map $\lambda: I(\mathfrak{p}) \rightarrow H$ also satisfies conditions (ii), (iii), (v), and (vi). But now conditions (i) and (iv) are replaced with

$$(i') \lambda(\alpha)^{12} = N(\alpha)^{12} \frac{\Delta(\bar{\alpha})}{\Delta(\mathcal{O}_K)},$$

and the one-cocycle condition:

$$(iv') \lambda(\alpha \cdot \mathfrak{b}) = \lambda(\alpha) \cdot {}^\alpha\lambda(\mathfrak{b}), \text{ for all } \alpha, \mathfrak{b} \in I(\mathfrak{p}).$$

Conditions (vi) and (iv') imply that the one-cocycle λ belongs to $[A_f]$ for all A_f of CM elliptic type and level p^2 .

Remark 6.1. Notice that the above one-cycle λ can be effectively computed by using the Dedekind eta-function on ideals (as Rodríguez-Villegas does in [13]), and it coincides with what Hajir denotes ϕ in Definition 2.3 in [11].

Let f denote the normalized newform in $S_2(\Gamma_0(p^2))$ attached to a Hecke character ψ whose eta-character has order 2. By Section 3, the splitting field L of A_f is H . Let $S_2(A_f)$ be the \mathbb{C} -vector space generated by the Galois conjugates of the newform f attached to ψ and let ω denote a Néron differential of Gross's elliptic curve $A(p)$.

Proposition 6.1. *Let f be as above. There is an optimal quotient $\pi : J_0(p^2) \rightarrow A(p)$ such that $\pi^*(\omega) = c g(q) dq/q$, where*

$$g(q) = \sum_{(\alpha, \mathfrak{p})=1} \delta(\alpha) q^{N(\alpha)} \in S_2(A_f),$$

and $c \in \mathbb{Z}$ is a unit in $\mathbb{Z}[\frac{1}{2p}]$.

Proof. By Lemma 5.3, we have $[E : K] = [L : K]$ and, thus, all one-cocycles in $[A_f]$ are modular. Therefore, by Theorem 1.3 we have that

$$g(q) = \sum_{(\alpha, \mathfrak{p})=1} \alpha^{-1} \lambda(\alpha) q^{N(\alpha)} = \sum_{(\alpha, \mathfrak{p})=1} \delta(\alpha) q^{N(\alpha)}$$

is a normalized cusp form in $S_2(A_f)$ for which there is an optimal elliptic quotient C_λ given by the lattice

$$\Lambda_g = \{2\pi i \int_\gamma g(z) dz : \gamma \in H_1(X_0(p^2), \mathbb{Z})\}.$$

Since $\delta(\bar{\alpha}) = \overline{\delta(\alpha)}$ for all $\alpha \in I(\mathfrak{p})$, it follows that $g(q) \in H_0[[q]]$. Thus, $g(q) dq/q \in \Omega^1(X_0(p^2))/H_0$. Hence, the natural morphism $\pi : X_0(p^2) \rightarrow C_\lambda$ is defined over H_0 . Notice that necessarily one has $\Lambda_g = \Omega \cdot \mathcal{O}_K$, for some $\Omega \in \mathbb{C}^*$. Indeed, \mathcal{O}_K is the only ideal α such that $j(\alpha) = \overline{j(\alpha)}$ since $[H : K]$ is odd. Thus, we have $j(\Lambda_g) = j(\mathcal{O}_K)$. Since A_f is \mathbb{Q} -isogenous to $\text{Res}_{H_0/\mathbb{Q}}(C_\lambda)$ and to $\text{Res}_{H_0/\mathbb{Q}}(A(p))$, it follows that C_λ and $A(p)$ are H_0 -isogenous and, therefore, H_0 -isomorphic. Therefore, there exists $c \in H_0^*$ such that $\pi^*(\omega) = c g(q) dq/q$. It is clear that $\Delta(\Lambda_g) = -p^3 c^{12}$.

The Manin ideal attached to π is $c \mathcal{O}_{H_0}$ (we refer to Section 4 in [8] for more details on the Manin ideal). By Propositions 4.1 and 4.2 in [8], we know that $c \mathcal{O}_{H_0}$ is an integral ideal and it can only be divided by primes lying over 2 or p . Now, we want to prove that $c \in \mathbb{Z}$. Since $\pi^*(\omega/c) = g dq/q$, the one-cocycle attached to ω/c is λ . This means that for every $\alpha \in I(\mathfrak{p})$ there is an isogeny of degree $N(\alpha)$,

$$\mu: \alpha^{-1} C_\lambda \rightarrow C_\lambda,$$

such that $\mu^*(\omega/c) = \alpha^{-1} \lambda(\alpha) \cdot \alpha^{-1}(\omega/c)$. Taking into account that $j(\alpha) = \alpha^{-1} j(\mathcal{O}_K)$, we obtain that the lattice corresponding to $\alpha^{-1} C_\lambda$ is $\frac{1}{\delta(\alpha)} \cdot \Omega\alpha$. Finally, we have that

$$\alpha^{-1} \Delta(\Omega \mathcal{O}_K) = \Delta\left(\frac{1}{\delta(\alpha)} \Omega\alpha\right) = \delta(\alpha)^{12} \Delta(\Omega\alpha) = \frac{\Delta(\mathcal{O}_K)}{\Delta(\alpha)} \Delta(\Omega\alpha) = \Delta(\Omega \mathcal{O}_K).$$

Therefore, $\Delta(\Lambda) \in K \cap H_0 = \mathbb{Q}$ and $c^{12} \in \mathbb{Q}$. Since $\mathbb{Q}(c) \subseteq H$ is unramified outside p and there is not a real quadratic field of discriminant p , it follows that $c^3 \in \mathbb{Q}$. Finally, since H does not contain the 3rd roots of unity (recall $p > 3$), one obtains $c \in \mathbb{Q}$. □

Remark 6.2. Since $c \in K^*$, the one-cocycle attached to ω is also λ . In this sense, we say that the normalized cusp form g is the canonical cusp form attached to $A(p)$.

For when the class number of K is 1 (that is, $p = 7, 11, 19, 43, 67, 163$), one has that π is defined over \mathbb{Q} and c coincides with the (classical) Manin constant. Then $c = \pm 1$ in these cases since Manin’s conjecture has been checked for all elliptic curves over \mathbb{Q} with conductor ≤ 130000 in Cremona’s tables. We have computed c for the remaining primes $p \leq 100$ (that is, $p = 23, 31, 47, 59, 71, 79, 83$) and we have also obtained that $c = \pm 1$. It seems reasonable to expect $c = \pm 1$ for all $A(p)$.

Remark 6.3. In general, as already mentioned, there are infinitely many normalized cusp forms $g' \in S_2(A_f)$ whose directions are pullbacks of $\Omega^1(A(p))$ under modular parametrizations $\pi': A_f \rightarrow A(p)$. For each one of them, there is a one-cocycle λ' (cohomologous to λ) such that

$$g' = \sum_{\alpha} \alpha^{-1} \lambda'(\alpha) q^{N(\alpha)} \in S_2(A_f),$$

and a constant $c' \in H_0$ with $\pi'^*(\omega) = c' g'$. The concern on whether the constant c is ± 1 is already in [9], see Question 23.2.2 on p. 81, but without fixing π' . However, $c' \neq \pm 1$ unless $\pi' = \pi$ (canonical) as in Theorem 6.1, although the Manin ideal attached to any $\pi' \neq \pi$ might still be \mathcal{O}_K as well.

We end this section giving an expression for the transcendental $\Omega \in \mathbb{C}^*$ attached to the lattice Λ of $A(p)$, which generalizes the one given by Gross in [9] for when K has class number one. Keeping the above notations, we set

$$\rho := \prod_{\substack{\mathfrak{b} \in \text{Gal}(H/K) \\ (\mathfrak{b}, p) = 1}} \frac{\delta(\mathfrak{b})}{\psi(\mathfrak{b})}.$$

It is clear that ρ is well defined, independent of the Galois conjugate of ψ , and $\rho \in \mathcal{O}_H^*$. Let h denote the class number of K , and consider

$$\{\mathcal{O}_K, \mathfrak{b}_1, \dots, \mathfrak{b}_{(h-1)/2}, \dots, \bar{\mathfrak{b}}_1, \dots, \bar{\mathfrak{b}}_{(h-1)/2}\}$$

a set of representatives of $\text{Gal}(H/K)$ with $(\mathfrak{b}_i, p) = 1$. Then we can rewrite

$$\rho = \prod_{i=1}^{(h-1)/2} \frac{\delta(\mathfrak{b}_i) \delta(\bar{\mathfrak{b}}_i)}{N(\mathfrak{b}_i)}. \tag{6}$$

Indeed, since $\delta(\mathfrak{b})/\psi(\mathfrak{b})$ is independent of the class of \mathfrak{b} in $\text{Gal}(H/K)$, it suffices to prove that $\psi(\mathfrak{b}) \cdot \psi(\bar{\mathfrak{b}}) = N(\mathfrak{b})$. But this is a consequence of

$$\left(\frac{N(\mathfrak{b})}{p}\right) = \left(\frac{N(\mathfrak{b})}{p}\right)^h = \left(\frac{\beta}{\mathfrak{p}}\right) \left(\frac{\bar{\beta}}{\mathfrak{p}}\right) = \left(\frac{\beta}{\mathfrak{p}}\right)^2 = 1,$$

where $\beta \in K$ is a generator of \mathfrak{b}^h . Observe that ρ is a positive unit in $\mathcal{O}_{H_0}^*$.

Proposition 6.2. *Let $\Lambda = \Omega \cdot \mathcal{O}_K$ be the lattice attached to $A(p)$. Then*

$$\Omega = \pm i^{(p+1)/4} \sqrt[h]{\rho \cdot (2\pi)^{(2h+1-p)/4} \cdot \sqrt{p}^{(1-3h)/2} \cdot \prod_{\substack{1 \leq m < p \\ \chi(m)=1}} \Gamma\left(\frac{m}{p}\right)},$$

where the h -th root is taken to be real.

Proof. By the Chowla–Selberg formula [2], we know that

$$\prod_{\alpha \in \text{Gal}(H/K)} N(\alpha)^{-6} \Delta(\tau_\alpha) = \left(\frac{2\pi}{p}\right)^{6h} \left(\prod_{m=1}^{p-1} \Gamma\left(\frac{m}{p}\right)^{\chi(m)}\right)^6,$$

where $\langle 1, \tau_\alpha \rangle = \frac{1}{N(\alpha)}\alpha$. Since λ is the one-cocycle attached to ω , we have that

$$\Delta(\tau_\alpha) = N(\alpha)^{12} \Delta(\alpha) = N(\alpha)^{12} \Delta\left(\frac{\Omega}{\delta(\alpha)} \alpha\right) \frac{\Omega^{12}}{\delta(\alpha)^{12}} = -p^3 \frac{N(\alpha)^{12}}{\delta(\alpha)^{12}} \Omega^{12}. \tag{7}$$

Combining (6), (7), and Gauss’s identity

$$\prod_{i=1}^{n-1} \Gamma\left(\frac{i}{n}\right) = (2\pi)^{(n-1)/2} n^{-1/2},$$

the statement follows by taking into account that Ω lies in \mathbb{R} or $i\mathbb{R}$ according to $p \equiv -1 \pmod{8}$ or not (cf. [9]). □

As a result, we obtain the following fact, which concludes the proof of Theorem 1.5.

Corollary 6.3. *With the above notations, one has*

$$\{2\pi i \int_{\gamma} g(z) dz : \gamma \in H_1(X_0(p^2), \mathbb{Z})\} = \frac{1}{c} \cdot \Omega \cdot \mathcal{O}_K.$$

7. CM elliptic directions for non-trivial Nebentypus

In this section, we shall consider arbitrary Hecke characters mod \mathfrak{p} . Let ψ in \mathcal{X} and let η be its eta-character. Let f denote the normalized newform attached to ψ . In order to find the elliptic directions in $S_2(A_f)$, one needs to determine the modular one-cocycles λ_u in $[A_f]$. Then the normalized cusp forms

$$g_u = \sum_{(\alpha, \mathfrak{p})=1} \alpha^{-1} \lambda_u(\alpha) q^{N(\alpha)}$$

are the elliptic directions in $S_2(A_f)$. Recall that in the particular case $\eta^2 = 1$, all one-cocycles are modular. In general, as explained above, to find the modular one-cocycles amounts to an eigenvector problem. In our particular setting, the following lemma will be useful since it will allow to handle certain linear systems by means of a quotient polynomial ring.

Lemma 7.1. *Let M/F be a cyclic field extension of degree k . Fix a generator τ of $\text{Gal}(M/F)$, and let μ_k be the group of k -th roots of unity. Let $\mathcal{E} = \text{End}_{F[\text{Gal}(M/F)]}(M)$ be the F -algebra of $\text{Gal}(M/F)$ -equivariant F -linear endomorphisms of M . One has*

(i) *the map $\Theta: F[X]/(X^k - 1) \rightarrow \mathcal{E}$ given by*

$$\Theta\left(\sum_{i=1}^k a_i X^i\right)(u) = \sum_{i=1}^k a_i \tau^i u, \quad \text{for all } u \in M,$$

is well defined and an isomorphism of F -algebras.

(ii) For every $p(X) \in F[X]/(X^k - 1)$, let $\mathcal{Z} = \{\zeta \in \mu_k : p(\zeta) = 0\}$. Then the endomorphism $G = \Theta(p(X))$ diagonalizes and its characteristic polynomial is

$$(-1)^k \prod_{i=1}^k (X - p(\zeta_k^i)),$$

where $\zeta_k = e^{2\pi i/k}$. We have $\dim_F \ker G = |\mathcal{Z}|$, and

$$\ker G = \Theta \left(\frac{X^k - 1}{\prod_{\zeta \in \mathcal{Z}} (X - \zeta)} \right) (M). \tag{8}$$

Proof. It is obvious that Θ is well defined and a morphism of F -algebras. Choose $\alpha \in M$ such that $\{\tau^i \alpha\}_{1 \leq i \leq k}$ is a F -basis of M . The morphism Θ is injective because $\Theta(q(X)) = 0$ implies that $\Theta(q(X))(\alpha) = 0$ and, then, $q(X) = 0$. For a given $G \in \mathcal{E}$, we have that $G(\alpha) = \sum_{i=1}^k a_i \tau^i \alpha$ for some $a_i \in F$ and, thus, $G(u) = \sum_{i=1}^k a_i \tau^i u$ for all $u \in M$. Therefore, Θ is surjective and part (i) is proved.

We consider the F -algebra monomorphism $\Psi : \mathcal{E} \rightarrow \text{End}_F F[X]/(X^k - 1)$ defined by $\Psi(G) = \widehat{G}$, where

$$\widehat{G}(q(X)) = \Theta^{-1}(G) \cdot q(X), \quad \text{for all } q(X) \in F[X]/(X^k - 1). \tag{9}$$

Now it suffices to prove part (ii) for \widehat{G} . Note that for any field extension F_0/F , the relation (9) allows us to consider \widehat{G} as a F_0 -linear endomorphism of $F_0[X]/(X^k - 1)$.

Let $G = \Theta(p(X))$. The set of eigenvalues of \widehat{G} is $\{p(\zeta_k^i) : 1 \leq i \leq k\}$. Indeed, if $\beta \in F_0$ is an eigenvalue of eigenvector $q(X) \in F_0[X]/(X^k - 1)$, then there exists $\zeta \in \mu_k$ such that $q(\zeta) \neq 0$ and, thus, $\beta = p(\zeta)$. Conversely, if $\beta = p(\zeta)$ for some $\zeta \in \mu_k$ then $q(X) = \prod_{\zeta' \in \mu_k \setminus \{\zeta\}} (X - \zeta')$ is an eigenvector with eigenvalue β . Notice that all eigenvalues of \widehat{G} are in $F_0 = F(\mu_k)$.

Now, let $\beta = p(\zeta)$ for some $\zeta \in \mu_k$ and we will prove that

$$\dim_{F_0} \ker(\widehat{G} - \beta \text{id}) = |\{\zeta \in \mu_k : p(\zeta) = \beta\}|,$$

which implies part (ii) except for the equality (8). Note that by a translation of \widehat{G} , we can (and do) assume $\beta = 0$. Then one has

$$\begin{aligned} \ker \widehat{G} &= \{q(X) \in F_0[X]/(X^k - 1) : q(\zeta) = 0 \text{ for all } \zeta \in \mu_k \setminus \mathcal{Z}\} \\ &= \{q(X) \in F_0[X]/(X^k - 1) : q(X) = \prod_{\zeta \in \mu_k \setminus \mathcal{Z}} (X - \zeta) r(X), \deg r < |\mathcal{Z}|\}. \end{aligned}$$

It follows that $\dim_{F_0} \ker \widehat{G} = |\mathcal{Z}|$ and $\ker \widehat{G} = \ker(\Psi \circ \Theta)(\prod_{\zeta \in \mathcal{Z}} (X - \zeta))$. Finally, the equality (8) is a consequence of the fact that $q(X) = \prod_{\zeta \in \mathcal{Z}} (X - \zeta) \in F[X]$ is coprime with $r(X) = (X^k - 1)/p(X)$ and $q(X) \cdot r(X)$ is zero in $F[X]/(X^k - 1)$. \square

Now, we focus our attention on the Hecke character $\psi \in \mathcal{X}$. For the sake of simplicity, let us assume that its eta-character satisfies $\text{ord}(\eta) = p - 1$. Since $\ker \eta$ is trivial, the corresponding field L is the ray class field of $K \bmod \mathfrak{p}$; that is, $L = H \cdot \mathbb{Q}(\zeta_p)$ (cf. Proposition 5.1). The cyclic group $\text{Gal}(L/H)$ has order $k := (p - 1)/2$. Also, let $\mathcal{E} = \text{End}_{H[\text{Gal}(L/H)]}(L)$ be the H -algebra of $\text{Gal}(L/H)$ -equivariant endomorphisms. After fixing a generator τ of $\text{Gal}(L/H)$, consider Θ as in Lemma 7.1. Finally, let $\lambda: I(\mathfrak{p}) \rightarrow L^*$ be the one-cocycle in Section 6. To find the elliptic directions in $S_2(A_f)$ turns out to be equivalent to find the twisted one-cocycles $\lambda_u(\alpha) = \lambda(\alpha) u / \alpha u$ which are modular. Note that now λ is not modular in $[A_f]$.

Proposition 7.2. *For all $u \in L^*$, the following conditions are equivalent:*

- (i) *the one-cocycle $\lambda_u(\alpha) = \lambda(\alpha) \frac{u}{\alpha u}$ is modular;*
- (ii) *$u = \Theta\left(\frac{X^k - 1}{\Phi_k(X)}\right)(v)$, for some $v \notin \ker \Theta\left(\frac{X^k - 1}{\Phi_k(X)}\right)$.*

In particular, for $u = \Theta\left(\frac{X^k - 1}{\Phi_k(X)}\right)(\zeta_p)$ the one-cocycle λ_u is modular. Here, $\Phi_k(X)$ denotes the k -th cyclotomic polynomial.

Proof. The values $u \in L^*$ for which λ_u is modular are the eigenvectors of the K -linear map

$$\text{pr}(u) = \sum_{\alpha \in \text{Gal}(L/K)} \alpha^{-1} \lambda(\alpha) \left(\sum_{\sigma \in \Phi} \frac{1}{\sigma \psi(\alpha)} \right)^{\alpha^{-1}} u \tag{10}$$

with eigenvalue equal to $[L : K]$. Also, by Proposition 4.4, we know that $\text{pr}/[L : K]$ is a projector, pr diagonalizes, and its characteristic polynomial is

$$([L : K] - X)^{[E:K]} X^{[L:K]-[E:K]} = (([L : K] - X)^{\varphi(k)} X^{k-\varphi(k)})^{[H:K]}.$$

By part (i) of Lemma 5.2, we can rewrite

$$\text{pr}(u) = \sum_{\alpha \in \text{Gal}(L/H)} \alpha^{-1} \lambda(\alpha) \left(\sum_{\sigma \in \Phi} \frac{1}{\sigma \psi(\alpha)} \right)^{\alpha^{-1}} u.$$

Let $g \in \mathbb{Z}$ be a primitive root of $(\mathbb{Z}/p\mathbb{Z})^*$ such that $\eta(g) = \zeta$, where $\zeta = e^{\frac{\pi i}{k}}$. Since the set of principal ideals $\{\alpha_j = g^{2j} \mathcal{O}_K : 1 \leq j \leq k\}$ is a set of representatives of $\text{Gal}(L/H)$ and $\lambda(g^{2j} \mathcal{O}_K) = g^{2j}$, we have

$$G(u) := \frac{\text{pr}(u)}{[H : K]} = \frac{1}{[H : K]} \sum_{j=1}^k \left(\sum_{\sigma \in \Phi} \sigma \zeta^{-2j} \right)^{\alpha_j^{-1}} u = \sum_{j=1}^k \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta^{-2j})^{\alpha_j^{-1}} u.$$

Hence, G belongs to \mathcal{E} and its characteristic polynomial has roots 0 and k with multiplicities $k - \varphi(k)$ and $\varphi(k)$, respectively.

Now, we fix the generator $\tau = g^{-2}\theta_K$ of $\text{Gal}(L/H)$ and apply Lemma 7.1 to the endomorphism $G - k \text{Id} \in \mathcal{E}$. It follows that the set

$$\mathcal{Z} = \{\zeta' \in \mu_k : \sum_{j=1}^k \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta^{-2j})(\zeta')^{2j} - k = 0\}$$

has cardinality $|\mathcal{Z}| = \varphi(k)$. Letting $\zeta_k = \zeta^2$, we claim that

$$\mathcal{Z} = \{\zeta_k^j : 1 \leq j < k, \text{gcd}(j, k) = 1\}.$$

Since $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ acts transitively on \mathcal{Z} and $|\mathcal{Z}| = \varphi(k)$, it suffices to prove that $\zeta_k \in \mathcal{Z}$. Indeed, one checks:

$$\sum_{j=1}^k \left(\sum_{i \in (\mathbb{Z}/k\mathbb{Z})^*} \zeta_k^{-ji} \right) \zeta_k^j = \sum_{j=1}^k \left(\sum_{i \in (\mathbb{Z}/k\mathbb{Z})^*} \zeta_k^{(1-i)j} \right) = \sum_{j=1}^k \left(\sum_{i \in (\mathbb{Z}/k\mathbb{Z})^*} \zeta_k^{ij} \right) = k.$$

Then, from Lemma 7.1, we obtain

$$\{u \in L : \text{pr}(u) = [L : K]u\} = \left\{ u = \Theta\left(\frac{X^k-1}{\Phi_k(X)}\right)(v) : v \in L \right\}.$$

Note that the image of $\Theta\left(\frac{X^k-1}{\Phi_k(X)}\right)$ is independent of the choice of the generator τ in $\text{Gal}(L/H)$. It can be easily checked that $\Theta((X^k - 1)/\Phi_k(X))$ vanishes on H , which implies that $\Theta((X^k - 1)/\Phi_k(X))(\zeta_p)$ is non-zero since the class of the polynomial $(X^k - 1)/\Phi_k(X)$ in $L[X]/(X^k - 1)$ is non-zero. \square

Example. Take $p = 7$, so that $K = \mathbb{Q}(\sqrt{-7})$ has class number one. Let ψ in \mathcal{X} with eta-character satisfying $\eta(3) = e^{2\pi i/6}$. Its corresponding newform $f = \sum \psi((a))q^{N(a)} \in S_2(\Gamma_1(49))$ has Nebentypus ε of order 3; note that $\psi((a)) = a\eta(a)$ for all $a \in \mathcal{O}_K$. The one-cocycle λ satisfies $\lambda((a)) = a$ with the unique choice of sign for a such that the symbol $(a/\sqrt{-7}) = 1$. This one-cocycle is not modular for ψ (in fact, it is modular for the Hecke character in \mathcal{X} with eta-character of order 2 in which case the (unique) elliptic direction coincides with the rational newform in $S_2(\Gamma_0(49))$ giving rise to the elliptic curve 49A1 in Cremona’s notation.) Thus, we need to twist λ by a coboundary in order to get a modular one-cocycle. According to Proposition 7.2, we can take, for instance, $u = \Theta(X - 1)(\zeta_7) = \zeta_7^2 - \zeta_7$ and the cuspidal form $g_u = \sum \alpha^{-1} \lambda_u(\alpha)q^{N(\alpha)} = \sum \lambda((a))^{(a^2)}u/u q^{N(a)} \in S_2(\Gamma_1(49))$ is an elliptic direction of A_f . A computer calculation shows the lattice Λ for the corresponding elliptic optimal quotient from $\text{Jac}(X_{\Gamma_\varepsilon})$ satisfies: $c_4(\Lambda) = c_4(A(7))u^4$, and $c_6(\Lambda) = c_6(A(7))u^6$.

References

[1] M. H. Baker, E. González-Jiménez, J. González, and B. Poonen, Finiteness results for modular curves of genus at least 2. *Amer. J. Math.* **127** (6) (2005), 1325–1387. [Zbl 1127.11041](#) [MR 2183527](#)

- [2] S. Chowla and A. Selberg, On Epstein's zeta-function. *J. Reine Angew. Math.* **227** (1967), 86–110. [Zbl 0166.05204](#) [MR 0215797](#)
- [3] G. Cornell and J. H. Silverman (eds.), *Arithmetic geometry*. Papers from the conference held at the University of Connecticut, Storrs, Connecticut, July 30–August 10, 1984 Springer-Verlag, New York 1986. [Zbl 0596.00007](#) [MR 0861969](#)
- [4] D. A. Cox, *Primes of the form $x^2 + ny^2$* . John Wiley & Sons Inc., New York 1989. [Zbl 0701.11001](#) [MR 1028322](#)
- [5] P. Deligne and J.-P. Serre, Formes modulaires de poids 1. *Ann. Sci. École Norm. Sup.* (4) **7** (1974), 507–530. [Zbl 0321.10026](#) [MR 0379379](#)
- [6] E. de Shalit, *Iwasawa theory of elliptic curves with complex multiplication. p -adic L functions*. *Perspect. Math.* 3, Academic Press Inc., Boston, Mass., 1987. [Zbl 0674.12004](#) [MR 0917944](#)
- [7] C. Goldstein and N. Schappacher, Séries d'Eisenstein et fonctions L de courbes elliptiques à multiplication complexe. *J. Reine Angew. Math.* **327** (1981), 184–218. [Zbl 0456.12007](#) [MR 0631315](#)
- [8] J. González and J.-C. Lario, \mathbf{Q} -curves and their Manin ideals. *Amer. J. Math.* **123** (3) (2001), 475–503. [Zbl 1035.14009](#) [MR 1833149](#)
- [9] B. H. Gross, *Arithmetic on elliptic curves with complex multiplication*. With an appendix by B. Mazur, *Lecture Notes in Math.* 776, Springer-Verlag, Berlin 1980. [Zbl 0433.14032](#) [MR 0563921](#)
- [10] B. H. Gross, Minimal models for elliptic curves with complex multiplication. *Compositio Math.* **45** (2) (1982), 155–164. [Zbl 0541.14010](#) [MR 0651979](#)
- [11] F. Hajir, On units related to the arithmetic of elliptic curves with complex multiplication. *Arch. Math. (Basel)* **66** (4) (1996), 280–291. [Zbl 0855.11027](#) [MR 1380570](#)
- [12] K. A. Ribet, Galois representations attached to eigenforms with Nebentypus. In *Modular functions of one variable, V* (Proc. Second Internat. Conf., Universität Bonn, 1976), *Lecture Notes in Math.* 601, Springer-Verlag, Berlin 1977, 17–51. [Zbl 0363.10015](#) [MR 0453647](#)
- [13] F. Rodríguez Villegas, On the square root of special values of certain L -series. *Invent. Math.* **106** (3) (1991), 549–573. [Zbl 0773.11034](#) [MR 1134483](#)
- [14] D. E. Rohrlich, Galois conjugacy of unramified twists of Hecke characters. *Duke Math. J.* **47** (3) (1980), 695–703. [Zbl 0446.12011](#) [MR 0587174](#)
- [15] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*. Kanô Memorial Lectures 1, Publications of the Mathematical Society of Japan 11, Iwanami Shoten, Publishers, Tokyo 1971. [Zbl 0221.10029](#) [MR 0314766](#)
- [16] G. Shimura, On elliptic curves with complex multiplication as factors of the Jacobians of modular function fields. *Nagoya Math. J.* **43** (1971), 199–208. [Zbl 0225.14015](#) [MR 0296050](#)
- [17] G. Shimura, On the factors of the Jacobian variety of a modular function field. *J. Math. Soc. Japan* **25** (1973), 523–544. [Zbl 0266.14017](#) [MR 0318162](#)

Josep González Rovira, Departament Matemàtica Aplicada 4, UPC, Vilanova i la Geltrú,
Av. Víctor Balaguer, s/n., 08800 Vilanova i la Geltrú, Spain

E-mail: josepg@ma4.upc.edu

Joan-Carles Lario, Departament Matemàtica Aplicada 2, UPC, Barcelona, Jordi Girona, 1-3,
08034 Barcelona, Spain

E-mail: joan.carles.lario@upc.edu