

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 85 (2010)

Artikel: Dynamics on the unit disk : short geodesics and simple cycles
Autor: McMullen, C.T.
DOI: <https://doi.org/10.5169/seals-130677>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 16.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Dynamics on the unit disk: Short geodesics and simple cycles

Curtis T. McMullen*

Mathematics Subject Classification (2010). 30D05, 37E45.

Keywords. Blaschke products, complex dynamics, rotation numbers.

Contents

| | | |
|---|----------------------------------|-----|
| 1 | Introduction | 723 |
| 2 | Simple cycles | 727 |
| 3 | Blaschke products | 732 |
| 4 | The thin part of $f(z)$ | 736 |
| 5 | Bounds on repelling cycles | 739 |
| 6 | Short cycles and short geodesics | 744 |
| 7 | Binding and renormalization | 747 |
| | References | 748 |

1. Introduction

In this paper we show that rotation cycles on S^1 for a proper holomorphic map $f: \Delta \rightarrow \Delta$ share several of the analytic, geometric and topological features of simple closed geodesics on a compact hyperbolic surface.

Dynamics on the unit disk. Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For $d > 1$ let $\mathcal{B}_d \cong \Delta^{(d-1)}$ denote the space of all proper holomorphic maps $f: \Delta \rightarrow \Delta$ of the form

$$f(z) = z \prod_1^{d-1} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right),$$

$|a_i| < 1$. Every degree d holomorphic map $g: \Delta \rightarrow \Delta$ with a fixed point in the disk can be put into the form above, by normalizing so its fixed point is $z = 0$.

The maps $f \in \mathcal{B}_d$ have the property that $f|S^1$ is measure-preserving and $|f'| > 1$ on the circle. Moreover, there is a unique *marking* homeomorphism $\phi_f: S^1 \rightarrow S^1$

*Research supported in part by the NSF.

that varies continuously with f , conjugates f to $p_d(z) = z^d$, and satisfies $\phi_f(z) = z$ when $f = p_d$. We define the *length* on f of a periodic cycle C for p_d by

$$L(C, f) = \log |(f^q)'(z)|, \quad (1.1)$$

where $q = |C|$ and $\phi_f(z) \in C$.

The *degree* of a cycle C is the least $e > 0$ such that $p_d|C$ extends to a covering map of the circle of degree e . We say C is *simple* if $\deg(p_d|C) = 1$; equivalently, if $p_d|C$ preserves its cyclic ordering. A finite collection of cycles C_i is *binding* if $\deg(\bigcup C_i) = d$ and if $\bigcup C_i$ is not renormalizable (§7).

In this paper we establish four main results.

Theorem 1.1. *Any cycle with $L(C, f) < \log 2$ is simple. All such cycles C_i have the same rotation number, and $p_d|\bigcup C_i$ preserves the cyclic ordering of $\bigcup C_i$.*

Theorem 1.2. *Every $f \in \mathcal{B}_d$ has a simple cycle C with $L(C, f) = O(d)$.*

Theorem 1.3. *Let $(C_i)_1^n$ be a binding collection of cycles. Then for any $M > 0$, the set of $f \in \mathcal{B}_d$ with $\sum_1^n L(C_i, f) \leq M$ has compact closure in the moduli space of all rational maps of degree d .*

Theorem 1.4. *The closure $E \subset S^1$ of the simple cycles for a given $f \in \mathcal{B}_d$ has Hausdorff dimension zero.*

See Theorems 4.1, 5.8, 7.1 and 2.2 below.

Hyperbolic surfaces. The results above echo the following fundamental facts about compact hyperbolic surfaces X of genus $g > 1$:

- (1) The closed geodesics on X of length less than $\log(3 + 2\sqrt{2})$ are simple and disjoint.
- (2) There exists a simple closed geodesic on X with length $O(\log g)$.
- (3) If $(\gamma_i)_1^n$ is a binding collection of closed curves, then the locus in Teichmüller space \mathcal{T}_g where $\sum L(\gamma_i, X) \leq M$ is compact for any $M > 0$.¹
- (4) The union of the simple geodesics on $X = \Delta/\Gamma$ is a closed set of Hausdorff dimension one.

See [Bus, §4, §5], [Ker, Lemma 3.1] and [BS] for proofs. Thus simple cycles behave in many ways like simple closed geodesics.

¹ A collection of closed curves is *binding* if their geodesic representatives cut X into disks.

Rotation numbers and slopes. Next we formulate a more direct connection between short cycles and short geodesics. Suppose $f \in \mathcal{B}_d$ satisfies $\alpha = f'(0) = \exp(2\pi i \tau) \neq 0$. The action of $\langle f \rangle$ on Δ (with the orbit of $z = 0$ removed) determines a natural *quotient torus*, isomorphic to

$$X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) \cong \mathbb{C}^*/\alpha^\mathbb{Z}.$$

Let $L(p/q, X_\tau)$ denote the length of a closed geodesic on X_τ in the homotopy class $(-p, q)$, for the flat metric of area one. The slope $p/q \bmod 1$ which minimizes $L(p/q, X_\tau)$ depends only on $f'(0) \in \Delta^*$. The regions $T(p/q) \subset \Delta^*$ where a given slope is shortest rest on the corresponding roots of unity, and form a tiling of Δ^* (see Figure 1).

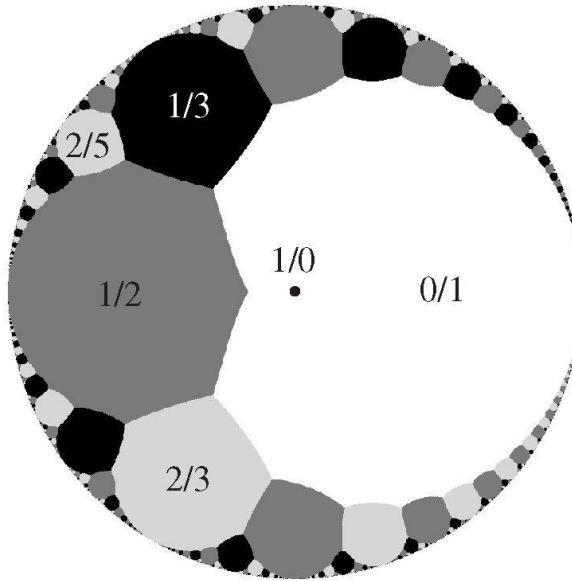


Figure 1. Tiling of Δ^* according to the slope of the shortest loop on the torus $\mathbb{C}^*/\alpha^\mathbb{Z}$.

In §6 we will show:

Theorem 1.5. *For any $f \in \mathcal{B}_d$ with $f'(0) \in T(p/q)$, there is a nonempty collection of compatible simple cycles C_i with rotation number p/q such that*

$$\frac{1}{L(p/q, X_\tau)^2} \leq \sum \frac{\pi}{L(C_i, f)} \leq \frac{1}{L(p/q, X_\tau)^2} + O(d),$$

and all other cycles satisfy $L(C, f) > \epsilon_d > 0$.

(Compatibility is defined in §2.) This result implies Theorem 1.2 and gives an alternate proof of Theorem 1.1 (with $\log 2$ replaced by ϵ_d); it also yields:

Corollary 1.6. *If a sequence $f_n \in \mathcal{B}_d$ satisfies $L(C, f_n) \rightarrow 0$, then $f'_n(0) \rightarrow \exp(2\pi i p/q)$ where p/q is the rotation number of C .*

On the other hand, we will see in §3:

Proposition 1.7. *If $f_n \in \mathcal{B}_d$ and $f'_n(0) \rightarrow \exp(2\pi i \theta)$ where θ is irrational, then $L(C, f_n) \rightarrow \infty$ for every cycle C .*

Thus the cycles of moderate length guaranteed by Theorem 1.2 may be forced to have very large periods.

Petals. The proof of Theorem 1.5 is illustrated in Figure 2. Consider a map $f \in \mathcal{B}_2$ with $f'(0) = \exp(2\pi i \tau) \in T(1/3)$, $\tau = 1/3 + i/10$. The dark petals shown in the figure form the preimage $\tilde{A} \subset \Delta$ of an annulus A in the homotopy class $[3\tau - 1]$ on the quotient torus for the attracting fixed point at $z = 0$. Any two adjacent rectangles within a petal give a fundamental domain for the action of f . The three largest petals join $z = 0$ to the repelling cycle on S^1 labeled by $C = (1/7, 2/7, 4/7)$. Thus a copy of A embeds in the quotient torus for the repelling cycle as well; by the method of extremal length (§5), this gives an upper bound for $L(C, f)$ in terms of $L(1/3, X_\tau)$. (The lower bound comes from the holomorphic Lefschetz fixed-point theorem.)

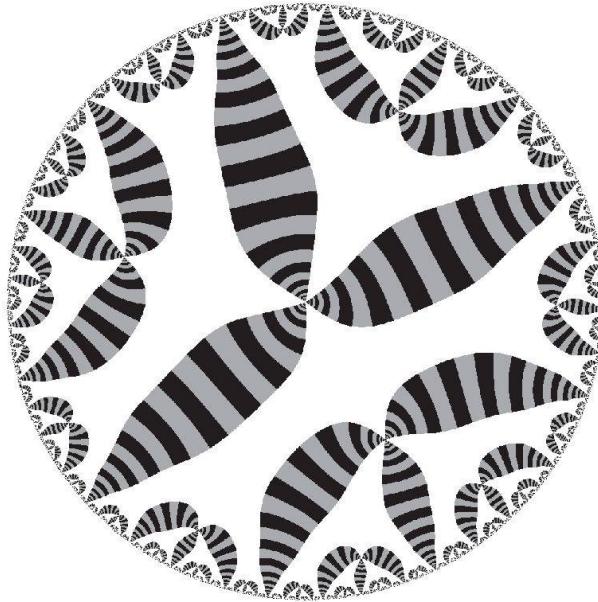


Figure 2. Petals joining $z = 0$ to the $(1, 2, 4)/7$ cycle on S^1 .

Rational maps. Here is a related result from §5 for general rational maps $f: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. Let $L(f) = \inf \log |\beta|$, where β ranges over the multipliers of all repelling and indifferent periodic cycles for f .

Theorem 1.8. *If $f_n \in \text{Rat}_d$ and $L(f_n) \rightarrow \infty$, then the maps f_n have fixed points z_n with $f'(z_n) \rightarrow 0$.*

Questions. We conclude with some natural questions suggested by the analogy with hyperbolic surfaces.

- (1) Let C be a simple cycle. Is the function $L(C, f)$ free of critical points in \mathcal{B}_d ?
- (2) Let (C_i) be a binding collection of cycles. Does $\sum L(C_i, f)$ achieve its minimum at a unique point $f \in \mathcal{B}_d$?
- (3) Let \mathcal{QB}_d denote the space rational maps of the form

$$f(z) = z \prod_1^{d-1} \left(\frac{z - a_i}{1 - b_i z} \right)$$

such that $\prod |a_i| < 1$, $\prod |b_i| < 1$, and $J(f)$ is a Jordan curve. Each $f \in \mathcal{QB}_d$ can be regarded as a *marked quasiblaschke product*, obtained by gluing together a pair of maps $f_1, f_2 \in \mathcal{B}_d$ using their markings on S^1 .

Does there exist an $\epsilon_d > 0$ such for all $f \in \mathcal{QB}_d$, all cycles of length shorter than ϵ_d are simple?

- (4) Suppose the cycles (C_1, C_2) are binding. Does the set of $f \in \mathcal{QB}_d$ with $L(C_1, f_1) + L(C_2, f_2) \leq M$ have compact closure in the moduli space of all rational maps of degree d ?

The analogous questions for hyperbolic surfaces and quasifuchsian groups are known to have positive answers [Ker, §3], [Ot], [Th, Theorem 4.4].

Notes and references. This paper is a sequel to [Mc4] and [Mc5] in which we construct a Weil–Petersson metric on \mathcal{B}_d and an embedding of \mathcal{B}_d into the space of invariant measures for $p_d(z) = z^d$.

Simple cycles in degree two play a central role in the combinatorics of the Mandelbrot set [DH], [Ke], and are studied for higher degree in [Gol] and [GM]. Extremal length arguments similar to those we use in §5 are well-known both in the theory of Kleinian groups [Bers, Theorem 3], [Th, Proposition 1.3], [Mc1, §6.3], [Pet1], [Mil2] and rational maps [Pom], [Lev], [Hub], [Pet2]. The quotient Riemann surface of a general rational map is discussed in [McS]; other aspects of the dictionary between rational maps and Kleinian groups are presented in [Mc2]. See [PL] for a related discussion of spinning degenerations of the quotient torus.

2. Simple cycles

In this section we discuss the combinatorics of periodic cycles for the map $p_d(t) = d \cdot t \bmod 1$, and prove the closure of the simple cycles has Hausdorff dimension zero.

Degree and rotation number. Let $S^1 = \mathbb{R}/\mathbb{Z}$. Given $a \neq b \in S^1$, let $[a, b] \subset S^1$ denote the unique subinterval that is positively oriented from a to b . We write $a < c < b$ if $c \in [a, b]$. The length of an interval is denoted $|I|$.

Let $f: S^1 \rightarrow S^1$ be a topological covering map of degree $d > 0$, and suppose $f(X) = X$. The *degree* of $f|X$, denoted $\deg(f|X)$, is the least $e > 0$ such that $f|X$ extends to a topological covering $g: S^1 \rightarrow S^1$ of degree e .

Note that $\deg(f|X) = 1$ iff f preserves the cyclic ordering of X , in which case $f|X$ also has a well-defined *rotation number* $\rho(f|X) \in S^1$. If X is finite then $\rho(f|X) = p/q$ is rational and the orbits of $f|X$ have size q .

Example. Suppose $X = \{x_0, x_1, \dots, x_n = x_0\}$ in increasing cyclic order, and $f|X$ is a permutation; then we have

$$\deg(f|X) = \sum_0^{n-1} |[f(x_i), f(x_{i+1})]|.$$

Indeed, an extension of $f|X$ of minimal degree is obtain by mapping $[x_i, x_{i+1}]$ homeomorphically to $[f(x_i), f(x_{i+1})]$. The degree is thus a variant of the number of *descents* of a permutation (see e.g. [St, §1.3]).

The model map and its modular group. Now fix $d > 1$, and let $p_d(t) = d \cdot t \bmod 1$. Any expanding map $f: S^1 \rightarrow S^1$ of degree d is topologically conjugate to p_d [Sh].

The *modular group* $\text{Mod}_d \subset \text{Aut}(S^1)$ is the cyclic group of rotations generated by $t \mapsto 1/(d-1) + t \bmod 1$; it coincides with the group of (degree one) topological automorphisms of p_d . Note that Mod_d acts transitively on the fixed points of p_d .

Simple cycles. A finite set $C \subset S^1$ is a *cycle* of degree d if $p_d|C$ is a transitive permutation. As in §1, we say a cycle is *simple* if $\deg(p_d|C) = 1$. Simple cycles (C_1, \dots, C_m) are *compatible* if $\deg(p_d| \bigcup C_i) = 1$.

It is elementary to see:

Proposition 2.1. *The simple cycles (C_1, \dots, C_m) are compatible iff they are pairwise compatible.*

We let \mathcal{C}_d denote the set of all cycles of degree d , and $\mathcal{C}_d(p/q) \subset \mathcal{C}_d$ the simple cycles with rotation number p/q .

Portraits of fixed points. The *fixed-point portrait* [Gol] of a simple cycle $C \in \mathcal{C}_d(p/q)$ is the monotone increasing function

$$\sigma: \{1, \dots, d-2\} \rightarrow \{0, 1, \dots, q\}$$

given by

$$\sigma(j) = |C \cap [0, j/(d-1)]|.$$

This invariant specifies how C is interleaved between the fixed points of p_d , which are all of the form $j/(d-1) \bmod 1$.

Basic properties. The following results are immediate from [Gol] (see especially Lemma 2 and Theorem 7).

- (1) A simple cycle $C \in \mathcal{C}_d(p/q)$ is uniquely determined by its fixed-point portrait $\sigma(j)$, and all possible monotone increasing functions $\sigma(j)$ arise.
- (2) The number of simple cycles of degree d and rotation number p/q is $\binom{d+q-2}{q}$.
- (3) The number of cycles of period q grows like d^q , while the number of simple cycles is $O(q^{d-1})$; so most cycles are not simple.
- (4) Cycles $C_1, C_2 \in \mathcal{C}_d(p/q)$ are compatible iff their fixed-point portraits satisfy

$$\sigma_1(j) \leq \sigma_2(j) \leq \sigma_1(j) + 1$$

for $0 \leq j \leq d-2$, or the same with σ_1 and σ_2 reversed.

- (5) Every maximal collection of compatible cycles has cardinality $d-1$.

From portraits to cycles. A simple cycle $C \in \mathcal{C}_d$ can be reconstructed explicitly from its rotation number p/q and its fixed-point portrait σ as follows. Let τ be the ‘transpose’ of σ , namely the monotone function $\tau: \{0, 1, \dots, q-1\} \rightarrow \{0, 1, \dots, d-1\}$ given by

$$\tau(i) = |\{j : \sigma(j) \leq i\}|, \quad (2.1)$$

and let

$$\tau'(i) = \tau(i) + \begin{cases} 0 & \text{if } 0 \leq i < q-p, \text{ and} \\ 1 & \text{otherwise,} \end{cases}$$

where i is taken $\bmod q$. Then the periodic point given by $t = 0.\tau'(0)\tau'(p)\tau'(2p)\dots$ in base d generates C ; indeed, t is the ‘first point’ in the cycle C .

Examples. To simplify notation, let $(p_1/q, \dots, p_m/q) = (p_1, \dots, p_m)/q$, and let $\sigma = n_1 \dots n_{d-1}$ denote the function with values $\sigma(j) = n_j$.

Degree $d = 2$. In the quadratic case, σ is trivial and hence there is a unique simple cycle $C(p/q)$ for each possible rotation number; e.g.

$$\begin{aligned} C(1/2) &= (1, 2)/3, \\ C(1/3) &= (1, 2, 4)/7, \\ C(2/5) &= (5, 10, 20, 9, 18)/31. \end{aligned}$$

The only cycle of period ≤ 4 which is not simple is $C = (1, 2, 4, 3)/5$. For period 5 there are two such, namely C and $-C$ where $C = (3, 6, 12, 24, 17)/31$. Any two distinct quadratic simple cycles are incompatible.

Degree $d = 3$. In the cubic case p_d has two fixed points, 0 and $1/2$, and three cycles of period two, given by

$$\begin{aligned} C(1/2, 0) &= (5, 7)/8, \\ C(1/2, 1) &= (1, 3)/4 \end{aligned}$$

and

$$C(1/2, 2) = (1, 3)/8.$$

The first and last are incompatible, while the other pairs are compatible. In general there are $q + 1$ cubic simple cycles with rotation number p/q , whose fixed-point portraits are given by $\sigma(1) = 0, 1, \dots, q$. Only the pairs with adjacent values of $\sigma(1)$ are compatible.

Degree $d = 4$. In the quartic case there are six cycles in $\mathcal{C}_4(1/2)$, generated by $t = p/15$ with $p = 1, 2, 3, 6, 7$ and 11. The compatibility relation between these cycles is shown in Figure 3. The four visible triangles give the four distinct triples of compatible simple cycles with rotation number $1/2$. Note that the modular group $\text{Mod}_4 \cong \mathbb{Z}/3$ acts by rotations on this diagram.

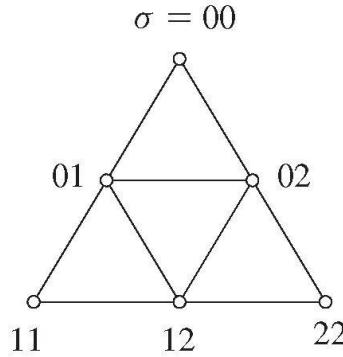


Figure 3. Compatibility of degree 4 cycles of the form $C(1/2, \sigma)$.

In general $\mathcal{C}_d(p/q)$ can be identified with the vertices of the q -fold barycentric subdivision of a $(d-2)$ -simplex, with the top-dimensional cells corresponding to maximal collections of compatible cycles.

Sample computation in degree $d = 5$. To compute $C(3/7, 013)$, we first use equation (2.1) to compute the ‘transpose’ $\tau = 1223333$ of $\sigma = 013$. Note that the graphs of σ and τ , shown in white and black in Figure 4, fit together to form a rectangle. Evaluating $\tau' = 1223444$ along the sequence $ip \bmod q$, $i = 0, 1, 2, \dots$ we obtain the base 5 expansion $t = 0.\overline{1342424}_5 = 6966/19531$ for a generator of C .

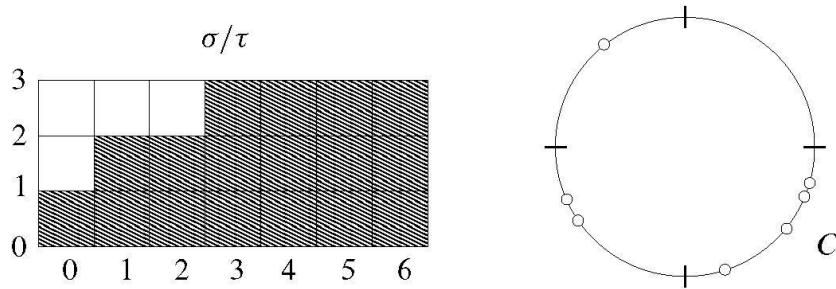


Figure 4. The degree 5 simple cycle with rotation number $3/7$ and $\sigma = 013$.

The cycle C , along with the four fixed points of p_5 , is drawn at the right in Figure 4. Note that $\sigma = 013$ gives the running total of the number of points of C in the first three quadrants.

Comparison with simple geodesics. The simple cycles for $p_d|S^1$ behave in many ways like simple closed geodesics on a compact hyperbolic surface $X = \Delta/\Gamma$ of genus g , with compatible cycles corresponding to disjoint geodesics. For example, every maximal collection of disjoint simple closed curves on X has $3g - 3$ elements, just as every maximal collection of compatible cycles for p_d has $d - 1$ elements.

It is also known that the endpoints of lifts of simple geodesics lie in a closed set $E \subset S^1$ of Hausdorff dimension zero [BS]. The analogous statement for simple cycles is:

Theorem 2.2. *The closure E of the union of all simple cycles $C \subset S^1$ of degree d has Hausdorff dimension zero.*

Proof. Let us say a finite set $P \subset S^1$ is a *precycle* if it is the forward orbit of preperiodic point $x \in S^1$ under p_d . We say P is *simple*, with rotation number p/q , if $p_d|P$ extends to a continuous, monotone increasing map $f: S^1 \rightarrow S^1$ with rotation number p/q . Then $q \leq n$ and the periodic part C of P is a simple cycle.

Let $\mathcal{P}_d(n, p/q)$ denote the set of all simple precycles of length n and rotation number p/q . The argument that shows $|\mathcal{C}_d(p/q)| = O(q^{d-2})$ can be adapted to show that $|\mathcal{P}_d(n, p/q)| = O(n^{d-2})$ as well.

Now fix $N > 0$. We claim that every $x \in E$ lies within distance $O(d^{-N})$ of a simple precycle P with $|P| \leq N$. To find this precycle, simply increase x continuously until two of the points among $x, f(x), \dots, f^N(x)$ coincide. This requires moving x only slightly, since $|(f^N)'(x)| = d^N$.

Thus E is contained in a neighborhood of diameter $O(d^{-N})$ of the union E_N of all simple precycles with $|P| \leq N$. Since $|E_N| = O(N^{d+2})$ grows only like a polynomial in N , this implies $\dim(E) = 0$. \square

Proof of Theorem 1.4. The Hölder continuous conjugacy ϕ_f between f and p_d preserves sets of Hausdorff dimension zero. \square

Remark: Invariant measures. The basic properties of simple cycles can also be developed using the correspondence between invariant measures and covering relations established in [Mc5]. For example, any union $D = \bigcup C_i$ of compatible cycles in $\mathcal{C}_d(p/q)$ arises as the support of an invariant measure ν for $p_d|S^1$. Invariant measures, in turn, correspond bijectively to covering relations (F, S) of degree d . In the case at hand, $F(t) = t + p/q \bmod 1$ and S is a divisor on S^1 of degree $d-1$. By perturbing S so its points have multiplicity one, we obtain a nearby invariant measure ν' whose support $D' \supset D$ is a maximal union of exactly $(d-1)$ compatible cycles (property (5) above).

The compactification of the space of Blaschke products by covering relations (F, S) is discussed in the following section.

Question. Is there a useful notion of intersection number for a pair of cycles?

3. Blaschke products

This section presents basic facts about marked Blaschke products, their derivatives and their images in the moduli space of all rational maps. See [Mc5] for related background material.

Blaschke products. Identify $S^1 = \mathbb{R}/\mathbb{Z}$ with the unit circle in the complex plane, using the coordinate $z = \exp(2\pi i t)$. Let $\Delta = \{z : |z| < 1\}$ be the unit disk, and $\Delta^{(n)}$ its n -fold symmetric product.

Given $d > 1$, let $\mathcal{B}_d \cong \Delta^{(d-1)}$ denote the space of Blaschke products $f : \Delta \rightarrow \Delta$ of the form

$$f(z) = z \prod_1^{d-1} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)$$

with $a_i \in \Delta$. Note that f extends to a rational map on the whole Riemann sphere, and $f|S^1$ is a covering map of degree d .

A proper holomorphic map $g : \Delta \rightarrow \Delta$ of degree $d > 1$ is conjugate to some $f \in \mathcal{B}_d$ iff g has a fixed point.

Derivatives and measure. By logarithmic differentiation, any $f \in \mathcal{B}_d$ satisfies

$$|f'(z)| = 1 + \sum_1^{d-1} \frac{1 - |a_i|^2}{|z - a_i|^2} \quad (3.1)$$

for $z \in S^1$. In particular, $f|S^1$ is expanding.

More importantly, $f|S^1$ preserves normalized Lebesgue measure λ on the circle; equivalently, $f_*(dz/z) = dz/z$, as can be verified by residue considerations. This means

$$\sum_{f(w)=z} |f'(w)|^{-1} = 1 \quad (3.2)$$

for any $z \in S^1$.

Markings. All $f \in \mathcal{B}_d$ are topologically conjugate to the model mapping $p_d(z) = z^d$. A *marking* for f the choice of one such conjugacy, i.e. the choice of a degree one homeomorphisms $\phi: S^1 \rightarrow S^1$ such that

$$f(z) = \phi^{-1} \circ p_d \circ \phi(z).$$

There is a unique marking ϕ_f which varies continuously in f and satisfies $\phi_f(z) = z$ when $f = p_d$. Thus \mathcal{B}_d can be regarded as the space of *marked Blaschke products*.

The modular group $\text{Mod}_d \cong \mathbb{Z}/(d-1)$ acts on \mathcal{B}_d by $(a_i) \mapsto (\zeta a_i)$ where $\zeta^{d-1} = 1$. Its orbits correspond to different markings of the same map. Thus $f_1, f_2 \in \mathcal{B}_d$ are conformally conjugate on Δ iff they are in the same orbit of the modular group.

Lengths. The canonical marking allows one to label the cycles of f by the cycles of p_d . We define the *length on f* of a cycle $C \in \mathcal{C}_d$ of period q by

$$L(C, f) = \log |(f^q)'(z)|$$

for any $z \in S^1$ with $\phi_f(z) \in C$.

Limits of lower degree. The space of Blaschke products has a natural compactification $\bar{\mathcal{B}}_d \cong \bar{\Delta}^{(d-1)}$, whose boundary points (a_i) can be interpreted as pairs (F, S) consisting of a Blaschke product

$$F(z) = z \prod_{|a_i| < 1} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right) \cdot \prod_{|a_i| = 1} (-a_i)$$

and a divisor of *sources*

$$S = \sum_{|a_i|=1} 1 \cdot a_i \in \text{Div}(S^1),$$

satisfying $\deg F + \deg S = d$. It is easy to see:

Proposition 3.1. *A sequence $f_n \in \mathcal{B}_d$ converges to $(F, S) \in \partial \mathcal{B}_d$ iff*

- (i) $f_n(z) \rightarrow F(z)$ uniformly on compact subsets of $\widehat{\mathbb{C}} - \text{supp } S$, and
- (ii) the zeros $Z(f_n)$ converge to $Z(F) + S$ as divisors on $\widehat{\mathbb{C}}$.

More generally, the space Rat_d of degree d rational maps $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ has a compactification $\overline{\text{Rat}}_d \cong \mathbb{P}^{2d+1}$, whose boundary points (F, S) are pairs consisting of a rational map F and an effective divisor $S \in \text{Div}(\widehat{\mathbb{C}})$ with $\deg(F) + \deg(S) = d$. We have $f_n \rightarrow (F, S)$ in $\overline{\text{Rat}}_d$ iff their graphs satisfy

$$\text{gr}(f_n) \rightarrow \text{gr}(F) + S \times \widehat{\mathbb{C}}$$

as divisors of degree $(1, d)$ on $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ (cf. [D, §1]).

Radial bounds on $f'(z)$. The following elementary observation is useful for studying limits as above.

Proposition 3.2. *For any proper holomorphic map $f: \Delta \rightarrow \Delta$ and $\zeta \in S^1$, we have*

$$\sup_{r \in [0, 1]} |f'(r\zeta)| \leq 4|f'(\zeta)|.$$

Note that we do not require that $f(0) = 0$. This bound is sharp, as can be seen by considering $f(z) = (z + a)/(1 + az)$ as $a \rightarrow 1-$.

Proof. We can write

$$f(z) = e^{i\theta} \prod_1^d M_i(z), \quad (3.3)$$

where $M_i(z) = (z - a_i)/(1 - \bar{a}_i z)$ and $a_i \in \Delta$. Composing with a rotation, we can also assume that $\zeta = 1$. For $r \in [0, 1]$ we have

$$\left| \frac{M'_i(r)}{M'_i(1)} \right| = \frac{|1 - a_i|^2}{|1 - r a_i|^2},$$

and therefore

$$|M'_i(r)| \leq 4|M'_i(1)|,$$

since the distance from 1 to a_i is never more than twice the distance from 1 to $r a_i$. Differentiating the product (3.3) and using the fact that $|\prod_{j \neq i} M_j(r)| \leq 1$, we obtain:

$$|f'(r)| \leq \sum |M'_i(r)| \leq 4 \sum |M'_i(1)| = 4|f'(1)|.$$

□

Corollary 3.3. *If $f_n \rightarrow (F, S) \in \bar{\mathcal{B}}_d$, $z_n \in S^1$, $z_n \rightarrow z$ and $|f'_n(z_n)| = O(1)$, then $\lim f_n(z_n) = F(z)$.*

Proof. Suppose $\sup |f'_n(z_n)| = M$; then for any $r < 1$ we have

$$\begin{aligned} \limsup |f_n(z_n) - F(z)| &\leq \limsup |f_n(rz_n) - F(z)| + 4M(1-r) \\ &= |F(rz) - F(z)| + 4M(1-r); \end{aligned}$$

now let $r \rightarrow 1$. □

Irrational rotations. As a sample application, we prove the following result stated in the Introduction:

Corollary 3.4. *If $f_n \in \mathcal{B}_d$ satisfies $f'_n(0) \rightarrow \exp(2\pi i \theta)$ where θ is irrational, then $L(C, f_n) \rightarrow \infty$ for every cycle C .*

Proof. Suppose to the contrary that $L(C, f_n)$ is bounded for some cycle C . Let $C_n \subset S^1$ be the corresponding periodic cycle for f_n . Pass to a subsequence such that $f_n \rightarrow (F, S) \in \partial \mathcal{B}_d$ and $C_n \rightarrow D \subset S^1$ in the Hausdorff topology. Then $F(z) = \exp(2\pi i \theta)z$ and by Corollary 3.3 we have $F(D) = D$, contradicting the irrationality of θ . □

Variants. Here are two useful variants of the results above:

Proposition 3.5. *For any proper holomorphic map $f : \mathbb{H} \rightarrow \mathbb{H}$ and $x \in \mathbb{R}$, we have*

$$\sup_y |f'(x + iy)| \leq f'(x).$$

Proposition 3.6. *Assume $f_n \in \text{Rat}_d$ converges to $(F, S) \in \overline{\text{Rat}}_d$, $z_n \rightarrow z$, and $\|Df_n(z_n)\| = O(1)$ in the spherical metric on $\hat{\mathbb{C}}$. Then we have*

$$f_n(z_n) \rightarrow F(z)$$

provided z_n belongs to a circle T_n with $f_n^{-1}(T_n) = T_n$, and $\inf_n \text{diam}(T_n) > 0$.

Proofs. The first result follows directly from the representation $f(z) = a_0z + b_0 + \sum_1^{d-1} a_i/(b_i - z)$ with $a_i > 0$ and $b_i \in \mathbb{R}$, and the second follows by the same argument as Corollary 3.3. □

The maps $f_n(z) = 1/(1 + nz^2)$ satisfy $f'_n(0) = 0$ and $\lim f_n(0) = 1 \neq F(0) = 0$; thus some extra hypothesis is needed to interchange limits as in Proposition 3.6.

Moduli space of rational maps. Let $\text{MRat}_d = \text{Rat}_d / \text{Aut}(\hat{\mathbb{C}})$ denote the moduli space of holomorphic conjugacy classes of rational maps of degree $d > 1$. A pair of Blaschke products are conjugate iff they are related by the modular group or by $z \mapsto 1/z$; thus we have an inclusion

$$\mathcal{B}_d / (\text{Mod}_d \times \mathbb{Z}/2) \hookrightarrow \text{MRat}_d.$$

The next result shows this inclusion is almost proper.

Theorem 3.7. *If $f_n \rightarrow (F, S) \in \partial \mathcal{B}_d$ but $[f_n]$ remains bounded in MRat_d , then $F(z) = z$ and $\text{supp } S$ is a single point. In particular, we have $f'_n(0) \rightarrow 1$.*

Proof. Pass to a subsequence such $[f_n] \rightarrow [g] \in \text{MRat}_d$ and $f_n \rightarrow (F, S) \in \partial \mathcal{B}_d$. Then there are conjugates $h_n = A_n f_n A_n^{-1} \rightarrow g$. Since f_n diverges in \mathcal{B}_d , $A_n \rightarrow \infty$ in $\text{Aut}(\hat{\mathbb{C}})$. On the other hand, the measures of maximal entropy satisfy $\mu(h_n) \rightarrow \mu(g)$ and $\mu(f_n) \rightarrow \mu(F, S)$, by [D, Theorem 0.1] (see also [Mc5]). Since $\mu(g)$ is nonatomic, this implies $\mu(F, S) = \lim A_n^*(\mu(h_n))$ is supported at a single point. But $\text{supp } \mu(F, S)$ is F -invariant and includes $\text{supp } S$; thus $F(z) = z$ and $\text{supp } S = \{s\}$ is itself a single point. \square

Example. The sequence $f_n(z) = z(z + a_n)/(1 + a_n z)$, with $a_n = 1 - 1/n$, is divergent in \mathcal{B}_2 but convergent in MRat_2 . To see this, normalize so the origin is a critical point instead of a fixed point; then $f_n(z)$ is conjugate to $h_n(z) = (z^2 + b_n)/(1 + b_n z^2)$, and $b_n = a_n/(2 + a_n) \rightarrow 1/3$ as $a_n \rightarrow 1$.

4. The thin part of $f(z)$

Let us define the *thin part* of $f \in \mathcal{B}_d$ by

$$S_{\text{thin}}^1(f) = \{z \in S^1 : |f'(z)| < 2\}.$$

In this section we will show:

Theorem 4.1. *For any $f \in \mathcal{B}_d$, the map $f|_{S_{\text{thin}}^1(f)}$ extends to a degree one homeomorphism of the circle.*

Corollary 4.2. *All cycles of f with $L(C, f) < \log 2$ are simple and compatible.*

Visual angles. The derivative of

$$f(z) = z \prod_1^{d-1} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)$$

can be conveniently analyzed using the *hyperbolic visual angle*, defined for $a, z \in \bar{\Delta}$ by

$$\alpha(z, a) = 2 \arg(z - a) - \arg(z).$$

This is the angle at a of the hyperbolic geodesic \overline{az} . For $z \in S^1$ we have $\arg(1 - \bar{a}z) = \arg(z) - \arg(z - a)$, and thus

$$\arg(f(z)) = \arg(z) + \sum_1^{d-1} \alpha(z, a_i). \quad (4.1)$$

(Note this simplifies to $\arg(f(z)) = 2 \arg(z - a_1)$ when $d = 2$.) Letting $\theta = \arg(z)$ and $\dot{\alpha} = d\alpha/d\theta$, we then obtain

$$|f'(z)| = 1 + \sum_1^{d-1} \dot{\alpha}(z, a_i) \quad (4.2)$$

for $z \in S^1$.

The visual density. The *visual density* $\dot{\alpha}(z, a)$ is essentially the Poisson kernel; for $a = r \geq 0$ it is given by

$$\dot{\alpha}(z, r) = \frac{1 - r^2}{1 + r^2 - 2r \cos \theta}, \quad (4.3)$$

where $\theta = \arg z$. Geometrically, $(\dot{\alpha}(z, a)/2\pi) d\theta$ is the hitting measure on the circle for a random hyperbolic geodesic starting at a .

For fixed $z \in S^1$, the level sets of $\dot{\alpha}(z, a)$ are horocycles resting on z . Thus

$$J(a) = \{z \in S^1 : \dot{\alpha}(z, a) < 1\}$$

is the large arc cut off by the chord perpendicular to $\overline{0a}$. This follows from the fact that the horocycle resting on one of the endpoints of $J(a)$ and passing through 0 also passes through a (see Figure 5).

Proposition 4.3. *The visual density $\dot{\alpha}(z, a)|_{J(a)}$ is strictly convex, and decreases as a moves radially towards the circle. In other words, we have*

$$\ddot{\alpha}(z, a) > 0 \quad \text{and} \quad \frac{d}{ds} \dot{\alpha}(z, sa) \Big|_{s=1} < 0$$

for all $z \in J(a)$.

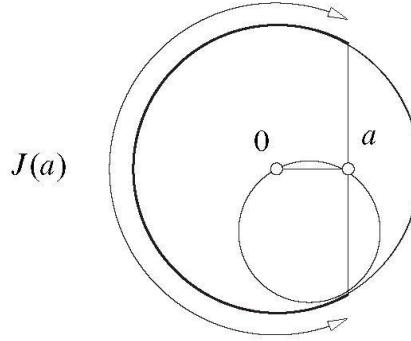


Figure 5. The arc $J(a)$ where $\dot{\alpha}(z, a) < 1$.

Proof. To verify convexity, consider the case where $a = r \in [0, 1)$. By (4.3), in this case we have $\dot{\alpha} = (1 - r^2)/u$ where $u = 1 + r^2 - 2r \cos \theta$. We may assume $\theta \in (0, \pi)$. Cross-multiplying and differentiating, we obtain

$$\begin{aligned}\dot{\alpha}u &= 1 - r^2, \\ \ddot{\alpha}u + \dot{\alpha}(2r \sin \theta) &= 0, \quad \text{and} \\ \ddot{\alpha}u + \ddot{\alpha}(4r \sin \theta) + \dot{\alpha}(2r \cos \theta) &= 0.\end{aligned}$$

Since r, u and $\sin \theta$ are all positive, we have $\dot{\alpha} > 0$ and $\ddot{\alpha} < 0$. Comparing the last two equations, we find the sign of $\ddot{\alpha}$ is the same as the sign of the determinant

$$D = \det \begin{pmatrix} 2r \sin \theta & u \\ 2r \cos \theta & 4r \sin \theta \end{pmatrix} = 8r^2 \sin^2 \theta - 2ru \cos \theta.$$

We claim $D > 0$ when $z \in J(r)$, i.e. when $u = |z - r|^2 > 1 - r^2$. The claim is evident if $\cos \theta$ is negative, so assume $\theta \in (0, \pi/2)$; then

$$u = |z - r|^2 \leq |z - 1|^2 \leq 2(\operatorname{Im} z)^2 = 2 \sin^2 \theta.$$

We also have $\cos \theta = \operatorname{Re}(z) < r$ for $z \in J(r)$, and thus:

$$D \geq 4r^2 u - 2r^2 u > 0.$$

The proof of the density decreasing property is straightforward. \square

Properties of the thin part of f . We can now show that $f|S_{\text{thin}}^1(f)$ acts like a rotation. We first observe:

Proposition 4.4. *For any $f \in \mathcal{B}_d$,*

- (i) *the map $f|S_{\text{thin}}^1(f)$ is injective,*
- (ii) *we have $S_{\text{thin}}^1(f) \subset \bigcap J(a_i)$,*

- (iii) $S_{\text{thin}}^1(f)$ consists of at most $(d - 1)$ disjoint open intervals, and
- (iv) $S_{\text{thin}}^1(f)$ increases as the zeros a_i of f move radially towards the circle.

Proof. If $f(x_1) = f(x_2)$ for two distinct points in $S_{\text{thin}}^1(f)$, then $|f'(x_1)| + |f'(x_2)| > 1/2 + 1/2 = 1$, which violates the measure-preserving property (3.2) of f ; thus $f|S_{\text{thin}}^1(f)$ is injective. Equation (4.2) implies (ii). Since $\bigcup(S^1 - J(a_i))$ has at most $(d - 1)$ components, so does $I = \bigcap J(a_i)$. By Proposition 4.3, $|f'(z)|$ is locally convex on I ; thus the intersection of $S_{\text{thin}}^1(f)$ with any component of I is connected, and (iii) follows. The density decreasing property stated in Proposition 4.3 implies (iv). \square

Proof of Theorem 4.1. By moving the points (a_i) radially to the circle, we obtain a smooth 1-parameter family of maps $f_t \in \bar{\mathcal{B}}_d$, $t \in [0, 1]$, with $f_0 = f$ and $f_1 = (F, S)$. Since $\deg(S) = d - 1$, we have $\deg(F) = 1$. Proposition 4.4 implies that $f_t|T_t = S_{\text{thin}}^1(f_t)$ is injective, $T_s \subset T_t$ when $s < t$, and $\text{supp } S \cap T_t = \emptyset$. Thus for any three distinct points $x_i \in S_{\text{thin}}^1(f)$, the triple $(f_t(x_1), f_t(x_2), f_t(x_3))$ moves by isotopy as t increases from 0 to 1, and converges to $(F(x_1), F(x_2), F(x_3))$ as $t \rightarrow 1$. Since F is a rotation, it preserves the cyclic ordering of the points (x_i) , so the same is true of f . Consequently f extends from $S_{\text{thin}}^1(f)$ to an orientation-preserving homeomorphism of the circle. \square

5. Bounds on repelling cycles

In this section we show that every $f \in \mathcal{B}_d$ has a simple cycle with $L(C, f) = O(d)$, and obtain related results for general rational maps.

Moduli and tori. We begin by summarizing some well-known facts about extremal length on tori.

Any point $\tau \in \mathbb{H}$ determines a complex torus

$$X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$$

with a flat metric inherited from the plane, and a distinguished basis $\langle 1, \tau \rangle$ for its fundamental group. Factoring the covering map $\mathbb{C} \rightarrow X_\tau$ through the map $\xi: \mathbb{C} \rightarrow \mathbb{C}^* \cong \mathbb{C}/\mathbb{Z}$ given by $\xi(z) = \exp(2\pi i z)$, we have

$$X_\tau = \mathbb{C}^*/\alpha^\mathbb{Z}$$

where $\alpha = \xi(\tau)$ satisfies $0 < |\alpha| < 1$. The same construction can be made when $-\tau \in \mathbb{H}$; then $|\alpha| > 1$.

Given a slope $p/q \in \mathbb{Q} \cup \{\infty\}$, let $\gamma_{p/q} \subset X_\tau$ denote the simple closed geodesic obtained as the projection of the line $\mathbb{R} \cdot (\tau - p/q)$ from \mathbb{C} to X_τ . Its preimage $\tilde{\gamma}_{p/q}$

in the intermediate cover \mathbb{C}^* consists of q arcs joining 0 to ∞ , cyclically permuted with rotation number p/q by $z \mapsto \alpha z$.

Any annulus A is conformally equivalent to a right cylinder, which is unique up to scale. The ratio $\text{mod}(A) = h/c$ between the height and circumference of this cylinder is the *modulus* of A .

The maximum modulus of an annulus $A \subset X_\tau$ homotopic to $\gamma_{p/q}$ is given by

$$\text{mod}(p/q, X_\tau) = \frac{\text{area}(X_\tau)}{L(\gamma_{p/q}, X_\tau)^2} = \frac{|\text{Im } \tau|}{|q\tau - p|^2} \quad (5.1)$$

(assuming $\gcd(p, q) = 1$). This maximum is realized by taking $A = X_\tau \setminus \gamma_{p/q}$. The set of $\tau \in \mathbb{H}$ with $\text{mod}(p/q, X_\tau) \geq m$ is a horoball of diameter $1/(mq^2)$ resting on the real axis at p/q . For $p/q = 1/0$ we have

$$\text{mod}(\infty, X_\tau) = |\text{Im } \tau|.$$

The *intersection inequality*

$$\text{mod}(p/q, X_\tau) \text{mod}(r/s, X_\tau) \leq \left(\det \begin{pmatrix} p & q \\ r & s \end{pmatrix} \right)^{-2} \quad (5.2)$$

is easily verified by considering the determinant of the lattice $\mathbb{Z}(q\tau - p) \oplus \mathbb{Z}(s\tau - r)$. This inequality implies:

There is at most one slope with $\text{mod}(p/q, X_\tau) > 1$.

On the other hand we have:

Proposition 5.1. *For any $\tau \in \mathbb{H}$, there exists a slope $p/q \in \mathbb{Q} \cup \{\infty\}$ such that*

$$\text{mod}(p/q, X_\tau) \geq \sqrt{3}/2.$$

Proof. Since the statement is invariant under the action of $\text{SL}_2(\mathbb{Z})$ on \mathbb{H} , it suffices to verify it when τ lies in the fundamental domain $|\tau| \geq 1, |\text{Re } \tau| \leq 1/2$; and in this case, we have $\text{mod}(\infty, X_\tau) = \text{Im } \tau \geq \sqrt{3}/2$. \square

Rational maps. Now let $f: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational map of degree $d > 1$. If $z \in \hat{\mathbb{C}}$ is a point of period q , its *multiplier* is given by $\beta = (f^q)'(z)$. The *grand orbit* of z is the set $\bigcup_{i,j>0} f^{-i} \circ f^j(z)$.

Suppose f has a fixed point at $z = 0$ and a periodic point $w \neq 0$ with period q . We say w has *rotation number* p/q relative to $z = 0$ if there are arcs $(\delta_i)_0^{q-1} \subset \hat{\mathbb{C}}$ joining $z = 0$ to $f^i(w)$, meeting only at $z = 0$, which are cyclically permuted by f with rotation number p/q .

Theorem 5.2. *Let f be a rational map with an attracting fixed-point at $z = 0$, with multiplier*

$$\alpha = f'(0) = \exp(2\pi i \tau) \neq 0.$$

Let e be the number of grand orbits of critical points in the immediate basin Ω of $z = 0$. Then for each $p/q \in \mathbb{Q}$, there exists a repelling or parabolic periodic point $w \in \partial\Omega$ such that

- (1) *the rotation number of w relative to $z = 0$ is p/q ; and*
- (2) *its multiplier has the form $\beta = (f^q)'(w) = \exp(-2\pi i \sigma)$, where $\sigma = 0$ or*

$$\frac{\operatorname{Im} \sigma}{|\sigma|^2} \geq \frac{\operatorname{mod}(p/q, X_\tau)}{e}. \quad (5.3)$$

In particular, we have

$$|\beta| \leq \left(\exp \left(\frac{2\pi}{\operatorname{mod}(p/q, X_\tau)} \right) \right)^e. \quad (5.4)$$

Proof. Let Ω^* denote the immediate basin of $z = 0$ with the grand orbits of all critical points in Ω and of $z = 0$ deleted. Then $f: \Omega^* \rightarrow \Omega^*$ is a covering map. Moreover, the holomorphic linearizing map

$$\phi(z) = \lim \alpha^{-n} f^n(z)$$

is defined for all $z \in \Omega^*$, and satisfies $\phi(f(z)) = \alpha \phi(z)$. Consequently ϕ descends to an inclusion of the space of grand orbits $Y = \Omega^*/\langle f \rangle$ into the torus $X_\tau = \mathbb{C}^*/\alpha^\mathbb{Z}$, making the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{\phi} & \mathbb{C}^* \\ \downarrow & & \downarrow \\ \Omega/\langle f \rangle = Y & \hookrightarrow & X_\tau = \mathbb{C}^*/\alpha^\mathbb{Z} \end{array}$$

commute. By assumption we have $|Y - X_\tau| = e$.

For a given $p/q \in \mathbb{Q}$, the geodesics parallel to $\gamma_{p/q}$ passing through the punctures of Y cut it into $\leq e$ parallel annuli, one of which satisfies

$$\operatorname{mod}(A) \geq \operatorname{mod}(p/q, X_\tau)/e. \quad (5.5)$$

Let $\delta \subset A$ be the core curve of A , and $\delta_0 \subset \Omega^*$ one of its lifts which is incident to $z = 0$. Let $\delta_i = f^i(\delta_0)$. By construction, the arc δ_0 is invariant under f^q , and $f^q|\delta_0$ is a bounded translation in the hyperbolic metric on Ω^* . Consequently δ_0 must join

$z = 0$ to another fixed point w of f^q in $\partial\Omega$. By the Snail Lemma [Mil1, Lem. 16.2], w is repelling or parabolic.

We have seen that the preimage of $\gamma_{p/q}$ on \mathbb{C}^* consists of q arcs, cyclically permuted with rotation number p/q by $z \mapsto \alpha z$. Since ϕ is a homeomorphism near $z = 0$, the arcs $\delta_0, \dots, \delta_{q-1}$ are also cyclically permuted with rotation number p/q by f . In particular w has rotation number p/q relative to $z = 0$.

Now suppose w is repelling, with multiplier β . Choose an injective branch of f^{-q} defined on a punctured neighborhood U^* of w such that $f^{-q}: U^* \rightarrow U^*$ and

$$Z = U^*/\langle f^{-q} \rangle \cong \mathbb{C}^*/\beta^{\mathbb{Z}} = X_{\sigma},$$

where $\sigma = \log(\beta/2\pi i)$. There is a unique choice of the logarithm such that the invariant arc $\delta_0 \cap U^*$ descends to a loop isotopic to γ_0 on X_{σ} .

By construction, $A \subset Y$ is covered by a strip $A_0 \subset \Omega^*$ which retracts to δ_0 , and hence we have an inclusion

$$A \cong A_0/\langle f^q \rangle \hookrightarrow Z \cong X_{\sigma}$$

in the same homotopy class as γ_0 . This implies

$$\text{mod}(0, X_{\sigma}) \geq \text{mod}(A),$$

and the bound (5.3) follows from equations (5.1) and (5.5). \square

Corollary 5.3. *If $f \in \text{Rat}_d$ has an attracting fixed point with multiplier satisfying*

$$|\alpha| > \exp(-\pi\sqrt{3}) = 0.0043\dots,$$

then it also has a repelling or parabolic cycle with multiplier satisfying

$$|\beta| \leq \exp(4\pi/\sqrt{3})^{2d-2} \leq 1416^{2d-2}.$$

Proof. The lower bound on $|\alpha|$ implies $\text{Im}(\tau) = \text{mod}(\infty, X_{\tau}) < \sqrt{3}/2$, where $\tau = (\log \alpha)/2\pi i$. Hence $\text{mod}(p/q, X_{\tau}) \geq \sqrt{3}/2$ for some $p/q \in \mathbb{Q}$, by Proposition 5.1. Now apply equation (5.4) and note that $e \leq 2d - 2$. \square

Corollary 5.4. *If a map $f \in \text{Rat}_d$ has an attracting fixed point with multiplier α , then it also has a repelling or parabolic cycle with multiplier satisfying*

$$|\beta| \leq (\exp(4\pi/\sqrt{3})/|\alpha|)^{2d-2}.$$

Proof. Choose $\tau = (\log \alpha)/2\pi i = x + iy$ with $x \in [-1/2, 1/2]$. The previous corollary shows the desired bound holds when $y < \sqrt{3}/2$. For $y \geq \sqrt{3}/2$ we have

$$m = \text{mod}(0, X_{\tau})^{-1} \leq \frac{x^2 + y^2}{y} \leq \frac{1}{2\sqrt{3}} + y < \frac{2}{\sqrt{3}} + y,$$

which implies $\exp(2\pi/m) \leq \exp(4\pi/\sqrt{3})/|\alpha|$; thus by (5.4) the desired bound holds in this case as well. \square

The bottom of the spectrum. Here is a qualitative consequence of the preceding corollary.

Let the *spectrum* $S(f) \subset \mathbb{C}$ be the set of all multipliers β that arise from periodic points of $f \in \text{Rat}_d$, and let

$$L(f) = \inf \{\log |\beta| : \beta \in S(f) \text{ and } |\beta| \geq 1\}.$$

By the fixed-point formula for rational maps [Mil1, Theorem 12.4], the multipliers of f at its fixed points satisfy

$$\sum \frac{1}{\mu_j - 1} = 1, \quad (5.6)$$

provided no $\mu_j = 1$; in particular, $|\mu_j| \leq d + 1$ for some j . Thus if f has no attracting fixed points, it satisfies

$$L(f) \leq \log(d + 1).$$

Combining this observation with Corollary 5.4, we obtain:

Corollary 5.5. *Let $f_n \in \text{Rat}_d$ be a sequence of rational maps with $L(f_n) \rightarrow \infty$. Then the maps f_n have fixed points with multipliers $\alpha_n \rightarrow 0$.*

Examples. It is easy to see that $f_n(z) = z^2 + n^2$ satisfies $L(f_n) \rightarrow \infty$ as $n \rightarrow \infty$, since its Julia set lies close to $\pm n$. Of course f_n has a fixed point at infinity with multiplier $\alpha_n = 0$.

Parabolics must be included in the definition of $L(f)$ to obtain Corollary 5.5. In fact, if we let $L^*(f) = \inf \{\log |\beta| : \beta \in S(f), |\beta| > 1\}$, then $f_n(z) = z - 1/z + n$ satisfies $L^*(f_n) \rightarrow \infty$ even though f_n has no attracting fixed point. (The map $f_n(z)$ behaves like the Hecke group $\langle z \mapsto -1/z, z \mapsto z + n \rangle$; cf. [Mc3, Theorem 6.2].)

Question. Does Corollary 5.5 remain true if only parabolic and repelling multipliers are included in the definition of $L(f)$?

Blaschke products. We now return to the setting of a proper map $f: \Delta \rightarrow \Delta$ fixing $z = 0$. In this case formula (5.6) implies:

Proposition 5.6. *The multipliers $(\lambda_i)_{1}^{d-1}$ of $f \in \mathcal{B}_d$ at its fixed points on the circle satisfy*

$$\sum_{i=1}^{d-1} \frac{1}{\lambda_i - 1} = \frac{1 - |\alpha|^2}{|1 - \alpha|^2},$$

where $\alpha = f'(0)$.

Corollary 5.7. *If $|\alpha| < 1/2$, then f has a repelling fixed point with multiplier satisfying $1 < \beta \leq 1 + (d - 1)/3$.*

Theorem 5.8. *Every $f \in \mathcal{B}_d$ has a simple cycle with $L(C, f) = O(d)$.*

Proof. Combine Corollaries 5.4 and 5.7. □

6. Short cycles and short geodesics

In this section we use the fixed-point formula for rational maps to obtain the following more detailed connection between the short cycles for f and the short geodesics on its quotient torus.

Theorem 6.1. *Given $f \in \mathcal{B}_d$ with $f'(0) = \exp(2\pi i \tau)$, choose $p/q \in \mathbb{Q}$ to maximize $\text{mod}(p/q, X_\tau)$. Then there exist compatible simple cycles C_i with rotation number p/q , such that*

(1) *their lengths satisfy*

$$\text{mod}(p/q, X_\tau) \leq \pi \sum L(C_i, f)^{-1} \leq \text{mod}(p/q, X_\tau) + O(d); \quad (6.1)$$

(2) *all other cycles satisfy $L(C, f) > \epsilon_d > 0$; and*

(3) *for any $r > 0$, the multipliers of f^r at its repelling fixed points satisfy*

$$\frac{1}{r} \sum' \frac{1}{\lambda_j - 1} = O(d), \quad (6.2)$$

where the prime indicates that fixed points in $\bigcup C_i$ are excluded.

In qualitative terms, the construction shows:

Corollary 6.2. *All cycles with $L(C, f) < \epsilon_d$ arise from short geodesics on the quotient torus for f .*

Tiling of Δ^* . The slope $p/q \bmod 1$ appearing in the theorem above depends only on $\alpha = f'(0) \in \Delta^*$. Figure 1 of the Introduction shows the regions $T(p/q) \subset \Delta^*$ where a given slope maximizes the value of $\text{mod}(p/q, X_\tau) = \text{mod}(p/q, \mathbb{C}/\alpha^\mathbb{Z})$.

This picture is nothing more than the image, under the covering map $\xi: \mathbb{H} \rightarrow \Delta^*$ given by $\xi(\tau) = \exp(2\pi i \tau)$, of the tiling of \mathbb{H} by $\text{SL}_2(\mathbb{Z})$ translates of the Dirichlet region

$$F = \{\tau \in \mathbb{H} : |\tau - n| \geq 1 \ \forall n \in \mathbb{Z}\}$$

for the cusp $\tau = \infty$. The tile $T(\infty) = \xi(F)$ lies in a ball of radius $\exp(-\pi\sqrt{3}) \approx 1/230$ about the origin. In this tile the short curve is $\gamma_\infty \subset X_\tau$, which lifts to a loop around $z = 0$ rather than a path connecting $z = 0$ to a periodic point. Thus the length of γ_∞ can go to zero without any cycle getting short.

Each remaining tile $T(p/q)$ contains a horocycle H resting on the root of unity $\exp(2\pi i p/q) \in S^1$. Within a still smaller horocycle $H' \subset H$, $\gamma_{p/q}$ becomes very short, and hence f has a very short cycle with rotation number p/q .

Moduli and multipliers. We begin the proof of Theorem 6.1 by connecting Diophantine properties of $\alpha \in \Delta^*$ to lengths of geodesics on $\mathbb{C}^*/\alpha^\mathbb{Z}$.

Lemma 6.3. *For any $\alpha = \exp(2\pi i \tau) \in \Delta^*$ and $q > 0$, we have*

$$\sup_p \frac{\text{mod}(p/q, X_\tau)}{\gcd(p, q)^2} = \frac{\pi}{q} \frac{1 - |\alpha^q|^2}{|1 - \alpha^q|^2} + O(1).$$

Proof. First consider the case $q = 1$, and assume τ is chosen so $|\text{Re } \tau| \leq 1/2$. Then we have $2\pi i \tau \approx 1 - \alpha$ when either side is small, and hence

$$\sup_p \text{mod}(p, X_\tau) = \frac{\text{Im } \tau}{|\tau|^2} = \pi \frac{1 - |\alpha|^2}{|1 - \alpha|^2} + O(1).$$

The general case follows using the fact that

$$\frac{\text{mod}(p/q, X_\tau)}{\gcd(p, q)^2} = \frac{\text{mod}(p, X_{q\tau})}{q}. \quad \square$$

Proof of Theorem 6.1. Choose p so that $\text{mod}(p/q, X_\tau)$ is maximized. As in Theorem 5.2, by cutting the torus X_τ open along $e \leq d - 1$ geodesics parallel to $\gamma_{p/q}$ we obtain annuli $A_1, \dots, A_e \subset Y$ with

$$\text{mod}(p/q, X_\tau) = \sum \text{mod}(A_i).$$

Each annulus A_i , when lifted to the unit disk, connects $z = 0$ to a simple cycle C_i for f with rotation number p/q and multiplier $\beta_i > 1$.

The lifts of the annuli A_i are disjoint, so the cycles C_i are compatible. Assume for the moment they are also distinct. Since two copies of A_i embed in the quotient torus $\mathbb{C}^*/\beta_i^\mathbb{Z}$ (one for the inside of the disk and one for the outside), we have

$$2\text{mod}(A_i) \leq \frac{2\pi}{\log \beta_i} = \frac{2\pi}{L(C_i, f)}.$$

The combination of these inequalities yields:

$$\text{mod}(p/q, X_\tau) \leq \pi \sum L(C_i, f)^{-1}.$$

This lower bound also holds when the cycles are not distinct; then we simply have more annuli A_i embedded in a given torus $\mathbb{C}^*/\beta_j^\mathbb{Z}$.

For the upper bound, let (λ_j) denote the multipliers of the repelling fixed points of f^q . Note that each cycle C_i contributes q fixed points, each with multiplier β_i . Combining Proposition 5.6 and Lemma 6.3, we obtain:

$$\begin{aligned} \frac{1}{q} \sum \frac{1}{\lambda_j - 1} &= \frac{1}{q} \sum' \frac{1}{\lambda_j - 1} + \sum \frac{1}{\beta_i - 1} = \frac{1 - |\alpha^q|^2}{q|1 - \alpha^q|^2} \\ &= \pi^{-1} \text{mod}(p/q, X_\tau) + O(1). \end{aligned}$$

(Again, the prime indicates fixed points in $\bigcup C_i$ are excluded.) Since the cycles C_i are compatible, there are no more than $d - 1$ of them, and hence

$$\sum \frac{1}{\beta_i - 1} = \sum \left(\frac{1}{\log \beta_i} + O(1) \right) = \left(\sum L(C_i)^{-1} \right) + O(d).$$

This yields the upper bound in (6.1); and it also implies

$$\frac{1}{q} \sum' \frac{1}{\lambda_j - 1} = O(d).$$

That is, equation (6.2) holds for $r = q$.

To obtain (6.2) for other values of r , recall that by (5.2) we have $\text{mod}(s/r, X_\tau) < 1$ whenever $s/r \neq p/q$. Thus if q does not divide r , Lemma 6.3 implies

$$\frac{1}{r} \sum' (\lambda_j - 1)^{-1} \leq \frac{1 - |\alpha^r|^2}{r|1 - \alpha^r|^2} \leq \sup_s \text{mod}(s/r, X_\tau) + O(1) = O(1);$$

while for $r = nq$ we obtain

$$\frac{1}{r} \sum' \frac{1}{\lambda_j - 1} + \frac{q}{r} \sum \frac{1}{\beta_i^n - 1} = \frac{1 - |\alpha^r|^2}{r|1 - \alpha^r|^2} = \frac{\text{mod}(p/q, X_\tau)}{\pi n^2} + O(1),$$

which again implies (6.2), since (6.1) gives

$$\frac{q}{r} \sum \frac{1}{\beta_i^n - 1} = \frac{1}{n} \left(\sum \frac{1}{nL(C_i, f)} + O(1) \right) = \frac{\text{mod}(p/q, X_\tau)}{\pi n^2} + O(d).$$

Finally note that equation (6.2) implies $L(C, f) > \epsilon_d \asymp 1/d > 0$, since any cycle C of period r and multiplier β , not among the C_i , contributes $1/(\beta - 1)$ to the sum $(1/r) \sum' (\lambda_j - 1)^{-1}$. \square

7. Binding and renormalization

We conclude by proving the following compactness result.

Theorem 7.1. *Let $(C_i)_1^n$ be a binding set of cycles of degree d . Then for any $M > 0$, the set of $f \in \mathcal{B}_d$ such that $\sum_i L(C_i, f) \leq M$ has compact closure in MRat_d .*

Corollary 7.2. *The set of $f \in \mathcal{B}_d$ such that $\sum_i L(C_i, f) \leq M$ and $|f'(0) - 1| \geq 1/M$ is compact.*

Proof. By Theorem 3.7, the only way a sequence f_n can diverge in \mathcal{B}_d but remain bounded in MRat_d is if $f'_n(0) \rightarrow 1$. \square

Definitions. Sets $A, B \subset S^1$ are *unlinked* if they lie in disjoint connected sets; equivalently, if their convex hulls in the unit disk are disjoint. A map $f: X \rightarrow X$ with $X \subset S^1$ is *renormalizable* if there is a nontrivial partition of X into disjoint, unlinked subsets X_1, \dots, X_n , such that every $f(X_i)$ lies in some X_j .

We say a collection of degree d cycles C_1, \dots, C_m is *binding* if $\deg(p_d| \bigcup C_i) = d$ and $p_d| \bigcup C_i$ is not renormalizable.

Proof of Theorem 7.1. Suppose to the contrary that we have a sequence $f_n \in \mathcal{B}_d$ with $\sum_i L(C_i, f_n) \leq M$ that is divergent in moduli space. Let

$$D_n = \phi_{f_n}^{-1} \left(\bigcup C_i \right) \subset S^1$$

be the finite f_n -invariant set corresponding to the binding cycles. Since $f_n|S^1$ is expanding, we have $|f'_n| \leq e^M$ on D_n .

Next we conjugate the entire picture by an affine transformation depending on n , so that $0 \in D_n$ and $\text{diam}(D_n) = 1$. Then S^1 goes over to a circle $T_n \supset D_n$ invariant by f_n , and we still have $|f'_n|D_n| \leq e^M$.

Pass to a subsequence such that $f_n \rightarrow (F, S) \in \overline{\text{Rat}}_d$. Since f_n diverges in MRat_d , we have $\deg(F) < d$. Passing to a further subsequence, we can find a finite set D containing zero and a circle $T \subset \hat{\mathbb{C}}$ such that $D_n \rightarrow D$ and $T_n \rightarrow T$ in the Hausdorff topology. Note that $|D| > 1$ since $\text{diam } D = 1$.

By Proposition 3.6, the map $f_n|D_n$ converges to $F|D$. But if $|D| = |\bigcup C_i|$, the map $F|(D \subset T)$ is combinatorially the same as $p_d|(\bigcup C_i \subset S^1)$, contradicting our assumption that $\deg(p_d| \bigcup C_i) = d$. Similarly, if $|D| < |\bigcup C_i|$, then the collapse of D_n to D provides an invariant partition for $\bigcup C_i$, contradicting our assumption that $p_d| \bigcup C_i$ is not renormalizable. \square

Examples. The single cycle $C = (3, 6, 12, 24, 17)/31$ in degree 2 is already binding, as is any cycle of prime order with $\deg(p_d|C) = d$.

The first renormalizable cycle in degree 2 is $C = (1, 2, 4, 3)/5$. Although $\deg(p_2|C) = 2$, $L(C, f_n)$ remains bounded as $f_n \in \mathcal{B}_2$ diverges along the sequence specified by $f'_n(0) = -1 + 1/n$. Indeed, f_n^2 can be renormalized so that C converges to the cycle of period 2 for $G(z) = z - 1/z$ [Ep]; and thus $L(C, f_n) \rightarrow \log 9$. For more details, see [Mc6, §14].

References

- [Bers] L. Bers, On boundaries of Teichmüller spaces and on Kleinian groups. I. *Ann. of Math.* **91** (1970), 570–600. [Zbl 0197.06001](#) [MR 0297992](#)
- [BS] J. S. Birman and C. Series, Geodesics with bounded intersection number on surfaces are sparsely distributed. *Topology* **24** (1985), 217–225. [Zbl 0568.57006](#) [MR 0793185](#)
- [Bus] P. Buser, *Geometry and Spectra of Compact Riemann Surfaces*. Progr. Math. 106, Birkhäuser, Boston, Mass., 1992. [Zbl 0770.53001](#) [MR 1183224](#)
- [D] L. DeMarco, Iteration at the boundary of the space of rational maps. *Duke Math. J.* **130** (2005), 169–197. [Zbl 05004325](#) [MR 2176550](#) [MR 0762431](#) [MR 0812271](#)
- [DH] A. Douady and J. Hubbard, *Étude dynamique des polynômes complexes*. Pub. Math. d’Orsay 84–2, 85–4, Université de Paris-Sud, Département de Mathématiques, Orsay, 1984, 1985. [Zbl 0552.30018](#) [Zbl 0571.30026](#) [MR 0762431](#) [MR 0812271](#)
- [Ep] A. L. Epstein, Bounded hyperbolic components of quadratic rational maps. *Ergodic Theory Dynam. Systems* **20** (2000), 727–748. [Zbl 0963.37041](#) [MR 1764925](#)
- [Gol] L. Goldberg, Fixed points of polynomial maps I. Rotation subsets of the circles. *Ann. Sci. École Norm. Sup.* **25** (1992), 679–685. [Zbl 0771.30027](#) [MR 1198093](#)
- [GM] L. R. Goldberg and J. Milnor, Fixed points of polynomial maps II: Fixed point portraits. *Ann. Sci. École Norm. Sup.* **26** (1993), 51–98. [Zbl 0771.30028](#) [MR 1209913](#)
- [Hub] J. H. Hubbard, Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. In *Topological Methods in Modern Mathematics*, ed. by L. R. Goldberg and A. V. Phillips, Publish or Perish, Inc., 1993, 467–511. [Zbl 0797.58049](#) [MR 1215974](#)
- [Ke] K. Keller, *Invariant Factors, Julia Equivalences and the (Abstract) Mandelbrot Set*, Lecture Notes in Math. 1732, Springer-Verlag, Berlin 2000. [Zbl 0970.37032](#) [MR 1761576](#)
- [Ker] S. Kerckhoff, The Nielsen realization problem. *Ann. of Math.* **177** (1983), 235–265. [Zbl 0528.57008](#) [MR 0690845](#)
- [Lev] G. M. Levin, On Pommerenke’s inequality for the eigenvalues of fixed points. *Colloq. Math.* **62** (1991), 167–177. [Zbl 0742.30026](#) [MR 1114630](#)
- [Mc1] C. McMullen, Iteration on Teichmüller space. *Invent. Math.* **99** (1990), 425–454. [Zbl 0695.57012](#) [MR 1031909](#)
- [Mc2] C. McMullen, The classification of conformal dynamical systems. In *Current Developments in Mathematics, 1995*, International Press, Cambridge, Mass., 1995, 323–360. [Zbl 0908.30028](#) [MR 1474980](#)

- [Mc3] C. McMullen, Hausdorff dimension and conformal dynamics III: Computation of dimension. *Amer. J. Math.* **120** (1998), 691–721. [Zbl 0953.30026](#) [MR 1637951](#)
- [Mc4] C. McMullen, Thermodynamics, dimension and the Weil-Petersson metric. *Invent. Math.* **173** (2008), 365–425. [Zbl 1156.30035](#) [MR 2415311](#)
- [Mc5] C. McMullen, A compactification of the space of expanding maps on the circle. *Geom. Funct. Anal.* **18** (2008), 2101–2119. [Zbl 05603850](#) [MR 2491699](#)
- [Mc6] C. McMullen, Ribbon \mathbb{R} -trees and holomorphic dynamics on the unit disk. *J. Topol.* **2** (2009), 23–76. [Zbl 05549240](#) [MR 2499437](#)
- [McS] C. McMullen and D. Sullivan, Quasiconformal homeomorphisms and dynamics III: The Teichmüller space of a holomorphic dynamical system. *Adv. Math.* **135** (1998), 351–395. [Zbl 0926.30028](#) [MR 1620850](#)
- [Mil1] J. Milnor, *Dynamics in One Complex Variable: Introductory Lectures*. Vieweg, Wiesbaden 1999. [Zbl 0946.30013](#) [MR 1721240](#)
- [Mil2] J. Milnor, Local connectivity of Julia sets: expository lectures. In *The Mandelbrot Set, Theme and Variations*, ed. by Tan Lei, Cambridge University Press, Cambridge 2000, 67–116. [Zbl 1107.37305](#) [MR 1765085](#)
- [Ot] J.-P. Otal, Sur le nouage des géodésiques dans les variétés hyperboliques. *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995), 847–852. [Zbl 0840.57008](#) [MR 1326694](#)
- [Pet1] C. L. Petersen, A PLY inequality for Kleinian groups. *Ann. Acad. Sci. Fenn.* **18** (1993), 23–29. [Zbl 0783.30033](#) [MR 1207892](#)
- [Pet2] C. L. Petersen, No elliptic limits for quadratic maps. *Ergodic Theory Dynam. Systems* **19** (1999), 127–141. [Zbl 0921.30019](#) [MR 1676926](#)
- [PL] K. Pilgrim and T. Lei, Spinning deformations of rational maps. *Conform. Geom. Dyn.* **8** (2004), 52–86. [Zbl 1084.37039](#) [MR 2060378](#)
- [Pom] Ch. Pommerenke, On conformal mapping and iteration of rational functions. *Complex Variables Theory Appl.* **5** (1986), 117–126. [MR 0846481](#)
- [Sh] M. Shub, Endomorphisms of compact differentiable manifolds. *Amer. J. Math.* **91** (1969), 175–199. [Zbl 0201.56305](#) [MR 0240824](#)
- [St] R. P. Stanley, *Enumerative Combinatorics*. Vol. I, Wadsworth Brooks/Cole Math. Ser., Wadsworth & Brooks, Monterey, CA, 1986. [Zbl 0608.05001](#) [MR 0847717](#)
- [Th] W. P. Thurston, Hyperbolic structures on 3-manifolds II: Surface groups and 3-manifolds which fiber over the circle. Preprint, 1986.

Received August 12, 2007

Curtis T. McMullen, Mathematics Department, Harvard University, 1 Oxford St, Cambridge, MA 02138-2901, U.S.A.