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**Autor:** McMullen, C.T.  
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# Dynamics on the unit disk: Short geodesics and simple cycles

Curtis T. McMullen\*

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## 1. Introduction

In this paper we show that rotation cycles on  $S^1$  for a proper holomorphic map  $f: \Delta \rightarrow \Delta$  share several of the analytic, geometric and topological features of simple closed geodesics on a compact hyperbolic surface.

**Dynamics on the unit disk.** Let  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ . For  $d > 1$  let  $\mathcal{B}_d \cong \Delta^{(d-1)}$  denote the space of all proper holomorphic maps  $f: \Delta \rightarrow \Delta$  of the form

$$f(z) = z \prod_1^{d-1} \left( \frac{z - a_i}{1 - \bar{a}_i z} \right),$$

$|a_i| < 1$ . Every degree  $d$  holomorphic map  $g: \Delta \rightarrow \Delta$  with a fixed point in the disk can be put into the form above, by normalizing so its fixed point is  $z = 0$ .

The maps  $f \in \mathcal{B}_d$  have the property that  $f|_{S^1}$  is measure-preserving and  $|f'| > 1$  on the circle. Moreover, there is a unique *marking* homeomorphism  $\phi_f: S^1 \rightarrow S^1$

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that varies continuously with  $f$ , conjugates  $f$  to  $p_d(z) = z^d$ , and satisfies  $\phi_f(z) = z$  when  $f = p_d$ . We define the *length* on  $f$  of a periodic cycle  $C$  for  $p_d$  by

$$L(C, f) = \log |(f^q)'(z)|, \quad (1.1)$$

where  $q = |C|$  and  $\phi_f(z) \in C$ .

The *degree* of a cycle  $C$  is the least  $e > 0$  such that  $p_d|_C$  extends to a covering map of the circle of degree  $e$ . We say  $C$  is *simple* if  $\deg(p_d|_C) = 1$ ; equivalently, if  $p_d|_C$  preserves its cyclic ordering. A finite collection of cycles  $C_i$  is *binding* if  $\deg(\bigcup C_i) = d$  and if  $\bigcup C_i$  is not renormalizable (§7).

In this paper we establish four main results.

**Theorem 1.1.** *Any cycle with  $L(C, f) < \log 2$  is simple. All such cycles  $C_i$  have the same rotation number, and  $p_d|_{\bigcup C_i}$  preserves the cyclic ordering of  $\bigcup C_i$ .*

**Theorem 1.2.** *Every  $f \in \mathcal{B}_d$  has a simple cycle  $C$  with  $L(C, f) = O(d)$ .*

**Theorem 1.3.** *Let  $(C_i)_1^n$  be a binding collection of cycles. Then for any  $M > 0$ , the set of  $f \in \mathcal{B}_d$  with  $\sum_1^n L(C_i, f) \leq M$  has compact closure in the moduli space of all rational maps of degree  $d$ .*

**Theorem 1.4.** *The closure  $E \subset S^1$  of the simple cycles for a given  $f \in \mathcal{B}_d$  has Hausdorff dimension zero.*

See Theorems 4.1, 5.8, 7.1 and 2.2 below.

**Hyperbolic surfaces.** The results above echo the following fundamental facts about compact hyperbolic surfaces  $X$  of genus  $g > 1$ :

- (1) The closed geodesics on  $X$  of length less than  $\log(3 + 2\sqrt{2})$  are simple and disjoint.
- (2) There exists a simple closed geodesic on  $X$  with length  $O(\log g)$ .
- (3) If  $(\gamma_i)_1^n$  is a binding collection of closed curves, then the locus in Teichmüller space  $\mathcal{T}_g$  where  $\sum L(\gamma_i, X) \leq M$  is compact for any  $M > 0$ .<sup>1</sup>
- (4) The union of the simple geodesics on  $X = \Delta/\Gamma$  is a closed set of Hausdorff dimension one.

See [Bus, §4, §5], [Ker, Lemma 3.1] and [BS] for proofs. Thus simple cycles behave in many ways like simple closed geodesics.

<sup>1</sup> A collection of closed curves is *binding* if their geodesic representatives cut  $X$  into disks.

**Rotation numbers and slopes.** Next we formulate a more direct connection between short cycles and short geodesics. Suppose  $f \in \mathcal{B}_d$  satisfies  $\alpha = f'(0) = \exp(2\pi i \tau) \neq 0$ . The action of  $\langle f \rangle$  on  $\Delta$  (with the orbit of  $z = 0$  removed) determines a natural *quotient torus*, isomorphic to

$$X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) \cong \mathbb{C}^*/\alpha^\mathbb{Z}.$$

Let  $L(p/q, X_\tau)$  denote the length of a closed geodesic on  $X_\tau$  in the homotopy class  $(-p, q)$ , for the flat metric of area one. The slope  $p/q \bmod 1$  which minimizes  $L(p/q, X_\tau)$  depends only on  $f'(0) \in \Delta^*$ . The regions  $T(p/q) \subset \Delta^*$  where a given slope is shortest rest on the corresponding roots of unity, and form a tiling of  $\Delta^*$  (see Figure 1).

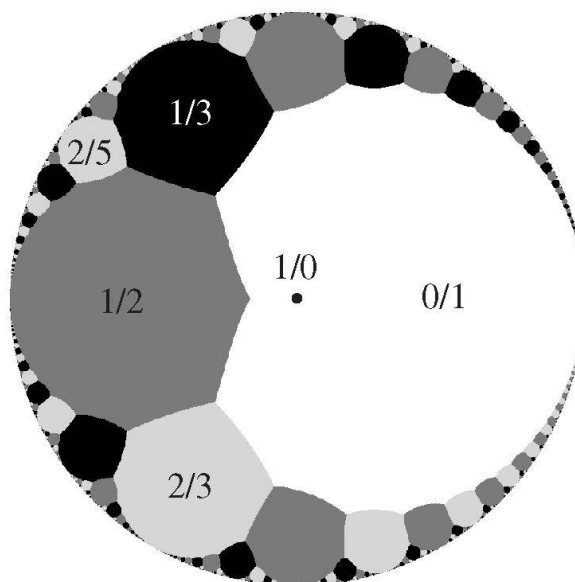


Figure 1. Tiling of  $\Delta^*$  according to the slope of the shortest loop on the torus  $\mathbb{C}^*/\alpha^\mathbb{Z}$ .

In §6 we will show:

**Theorem 1.5.** *For any  $f \in \mathcal{B}_d$  with  $f'(0) \in T(p/q)$ , there is a nonempty collection of compatible simple cycles  $C_i$  with rotation number  $p/q$  such that*

$$\frac{1}{L(p/q, X_\tau)^2} \leq \sum \frac{\pi}{L(C_i, f)} \leq \frac{1}{L(p/q, X_\tau)^2} + O(d),$$

*and all other cycles satisfy  $L(C, f) > \epsilon_d > 0$ .*

(Compatibility is defined in §2.) This result implies Theorem 1.2 and gives an alternate proof of Theorem 1.1 (with  $\log 2$  replaced by  $\epsilon_d$ ); it also yields:



**Corollary 1.6.** *If a sequence  $f_n \in \mathcal{B}_d$  satisfies  $L(C, f_n) \rightarrow 0$ , then  $f'_n(0) \rightarrow \exp(2\pi i p/q)$  where  $p/q$  is the rotation number of  $C$ .*

On the other hand, we will see in §3:

**Proposition 1.7.** *If  $f_n \in \mathcal{B}_d$  and  $f'_n(0) \rightarrow \exp(2\pi i \theta)$  where  $\theta$  is irrational, then  $L(C, f_n) \rightarrow \infty$  for every cycle  $C$ .*

Thus the cycles of moderate length guaranteed by Theorem 1.2 may be forced to have very large periods.

**Petals.** The proof of Theorem 1.5 is illustrated in Figure 2. Consider a map  $f \in \mathcal{B}_2$  with  $f'(0) = \exp(2\pi i \tau) \in T(1/3)$ ,  $\tau = 1/3 + i/10$ . The dark petals shown in the figure form the preimage  $\tilde{A} \subset \Delta$  of an annulus  $A$  in the homotopy class  $[3\tau - 1]$  on the quotient torus for the attracting fixed point at  $z = 0$ . Any two adjacent rectangles within a petal give a fundamental domain for the action of  $f$ . The three largest petals join  $z = 0$  to the repelling cycle on  $S^1$  labeled by  $C = (1/7, 2/7, 4/7)$ . Thus a copy of  $A$  embeds in the quotient torus for the repelling cycle as well; by the method of extremal length (§5), this gives an upper bound for  $L(C, f)$  in terms of  $L(1/3, X_\tau)$ . (The lower bound comes from the holomorphic Lefschetz fixed-point theorem.)



Figure 2. Petals joining  $z = 0$  to the  $(1, 2, 4)/7$  cycle on  $S^1$ .

**Rational maps.** Here is a related result from §5 for general rational maps  $f: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ . Let  $L(f) = \inf \log |\beta|$ , where  $\beta$  ranges over the multipliers of all repelling and indifferent periodic cycles for  $f$ .

**Theorem 1.8.** *If  $f_n \in \text{Rat}_d$  and  $L(f_n) \rightarrow \infty$ , then the maps  $f_n$  have fixed points  $z_n$  with  $f'(z_n) \rightarrow 0$ .*

**Questions.** We conclude with some natural questions suggested by the analogy with hyperbolic surfaces.

- (1) Let  $C$  be a simple cycle. Is the function  $L(C, f)$  free of critical points in  $\mathcal{B}_d$ ?
- (2) Let  $(C_i)$  be a binding collection of cycles. Does  $\sum L(C_i, f)$  achieve its minimum at a unique point  $f \in \mathcal{B}_d$ ?
- (3) Let  $\mathcal{QB}_d$  denote the space rational maps of the form

$$f(z) = z \prod_1^{d-1} \left( \frac{z - a_i}{1 - \bar{b}_i z} \right)$$

such that  $\prod |a_i| < 1$ ,  $\prod |b_i| < 1$ , and  $J(f)$  is a Jordan curve. Each  $f \in \mathcal{QB}_d$  can be regarded as a *marked quasiblaschke product*, obtained by gluing together a pair of maps  $f_1, f_2 \in \mathcal{B}_d$  using their markings on  $S^1$ .

Does there exist an  $\epsilon_d > 0$  such for all  $f \in \mathcal{QB}_d$ , all cycles of length shorter than  $\epsilon_d$  are simple?

- (4) Suppose the cycles  $(C_1, C_2)$  are binding. Does the set of  $f \in \mathcal{QB}_d$  with  $L(C_1, f_1) + L(C_2, f_2) \leq M$  have compact closure in the moduli space of all rational maps of degree  $d$ ?

The analogous questions for hyperbolic surfaces and quasifuchsian groups are known to have positive answers [Ker, §3], [Ot], [Th, Theorem 4.4].

**Notes and references.** This paper is a sequel to [Mc4] and [Mc5] in which we construct a Weil–Petersson metric on  $\mathcal{B}_d$  and an embedding of  $\mathcal{B}_d$  into the space of invariant measures for  $p_d(z) = z^d$ .

Simple cycles in degree two play a central role in the combinatorics of the Mandelbrot set [DH], [Ke], and are studied for higher degree in [Gol] and [GM]. Extremal length arguments similar to those we use in §5 are well-known both in the theory of Kleinian groups [Bers, Theorem 3], [Th, Proposition 1.3], [Mc1, §6.3], [Pet1], [Mil2] and rational maps [Pom], [Lev], [Hub], [Pet2]. The quotient Riemann surface of a general rational map is discussed in [McS]; other aspects of the dictionary between rational maps and Kleinian groups are presented in [Mc2]. See [PL] for a related discussion of spinning degenerations of the quotient torus.

## 2. Simple cycles

In this section we discuss the combinatorics of periodic cycles for the map  $p_d(t) = d \cdot t \bmod 1$ , and prove the closure of the simple cycles has Hausdorff dimension zero.

**Degree and rotation number.** Let  $S^1 = \mathbb{R}/\mathbb{Z}$ . Given  $a \neq b \in S^1$ , let  $[a, b] \subset S^1$  denote the unique subinterval that is positively oriented from  $a$  to  $b$ . We write  $a < c < b$  if  $c \in [a, b]$ . The length of an interval is denoted  $|I|$ .

Let  $f: S^1 \rightarrow S^1$  be a topological covering map of degree  $d > 0$ , and suppose  $f(X) = X$ . The *degree* of  $f|X$ , denoted  $\deg(f|X)$ , is the least  $e > 0$  such that  $f|X$  extends to a topological covering  $g: S^1 \rightarrow S^1$  of degree  $e$ .

Note that  $\deg(f|X) = 1$  iff  $f$  preserves the cyclic ordering of  $X$ , in which case  $f|X$  also has a well-defined *rotation number*  $\rho(f|X) \in S^1$ . If  $X$  is finite then  $\rho(f|X) = p/q$  is rational and the orbits of  $f|X$  have size  $q$ .

**Example.** Suppose  $X = \{x_0, x_1, \dots, x_n = x_0\}$  in increasing cyclic order, and  $f|X$  is a permutation; then we have

$$\deg(f|X) = \sum_0^{n-1} |[f(x_i), f(x_{i+1})]|.$$

Indeed, an extension of  $f|X$  of minimal degree is obtain by mapping  $[x_i, x_{i+1}]$  homeomorphically to  $[f(x_i), f(x_{i+1})]$ . The degree is thus a variant of the number of *descents* of a permutation (see e.g. [St, §1.3]).

**The model map and its modular group.** Now fix  $d > 1$ , and let  $p_d(t) = d \cdot t \bmod 1$ . Any expanding map  $f: S^1 \rightarrow S^1$  of degree  $d$  is topologically conjugate to  $p_d$  [Sh].

The *modular group*  $\text{Mod}_d \subset \text{Aut}(S^1)$  is the cyclic group of rotations generated by  $t \mapsto 1/(d-1) + t \bmod 1$ ; it coincides with the group of (degree one) topological automorphisms of  $p_d$ . Note that  $\text{Mod}_d$  acts transitively on the fixed points of  $p_d$ .

**Simple cycles.** A finite set  $C \subset S^1$  is a *cycle* of degree  $d$  if  $p_d|C$  is a transitive permutation. As in §1, we say a cycle is *simple* if  $\deg(p_d|C) = 1$ . Simple cycles  $(C_1, \dots, C_m)$  are *compatible* if  $\deg(p_d|\bigcup C_i) = 1$ .

It is elementary to see:

**Proposition 2.1.** *The simple cycles  $(C_1, \dots, C_m)$  are compatible iff they are pairwise compatible.*

We let  $\mathcal{C}_d$  denote the set of all cycles of degree  $d$ , and  $\mathcal{C}_d(p/q) \subset \mathcal{C}_d$  the simple cycles with rotation number  $p/q$ .

**Portraits of fixed points.** The *fixed-point portrait* [Gol] of a simple cycle  $C \in \mathcal{C}_d(p/q)$  is the monotone increasing function

$$\sigma: \{1, \dots, d-2\} \rightarrow \{0, 1, \dots, q\}$$

given by

$$\sigma(j) = |C \cap [0, j/(d-1))|.$$

This invariant specifies how  $C$  is interleaved between the fixed points of  $p_d$ , which are all of the form  $j/(d-1) \bmod 1$ .

**Basic properties.** The following results are immediate from [Gol] (see especially Lemma 2 and Theorem 7).

- (1) A simple cycle  $C \in \mathcal{C}_d(p/q)$  is uniquely determined by its fixed-point portrait  $\sigma(j)$ , and all possible monotone increasing functions  $\sigma(j)$  arise.
- (2) The number of simple cycles of degree  $d$  and rotation number  $p/q$  is  $\binom{d+q-2}{q}$ .
- (3) The number of cycles of period  $q$  grows like  $d^q$ , while the number of simple cycles is  $O(q^{d-1})$ ; so most cycles are not simple.
- (4) Cycles  $C_1, C_2 \in \mathcal{C}_d(p/q)$  are compatible iff their fixed-point portraits satisfy

$$\sigma_1(j) \leq \sigma_2(j) \leq \sigma_1(j) + 1$$

for  $0 \leq j \leq d-2$ , or the same with  $\sigma_1$  and  $\sigma_2$  reversed.

- (5) Every maximal collection of compatible cycles has cardinality  $d-1$ .

**From portraits to cycles.** A simple cycle  $C \in \mathcal{C}_d$  can be reconstructed explicitly from its rotation number  $p/q$  and its fixed-point portrait  $\sigma$  as follows. Let  $\tau$  be the ‘transpose’ of  $\sigma$ , namely the monotone function  $\tau: \{0, 1, \dots, q-1\} \rightarrow \{0, 1, \dots, d-1\}$  given by

$$\tau(i) = |\{j : \sigma(j) \leq i\}|, \quad (2.1)$$

and let

$$\tau'(i) = \tau(i) + \begin{cases} 0 & \text{if } 0 \leq i < q-p, \text{ and} \\ 1 & \text{otherwise,} \end{cases}$$

where  $i$  is taken  $\bmod q$ . Then the periodic point given by  $t = 0.\tau'(0)\tau'(p)\tau'(2p)\dots$  in base  $d$  generates  $C$ ; indeed,  $t$  is the ‘first point’ in the cycle  $C$ .

**Examples.** To simplify notation, let  $(p_1/q, \dots, p_m/q) = (p_1, \dots, p_m)/q$ , and let  $\sigma = n_1 \dots n_{d-1}$  denote the function with values  $\sigma(j) = n_j$ .

*Degree  $d = 2$ .* In the quadratic case,  $\sigma$  is trivial and hence there is a unique simple cycle  $C(p/q)$  for each possible rotation number; e.g.

$$C(1/2) = (1, 2)/3,$$

$$C(1/3) = (1, 2, 4)/7,$$

$$C(2/5) = (5, 10, 20, 9, 18)/31.$$

The only cycle of period  $\leq 4$  which is not simple is  $C = (1, 2, 4, 3)/5$ . For period 5 there are two such, namely  $C$  and  $-C$  where  $C = (3, 6, 12, 24, 17)/31$ . Any two distinct quadratic simple cycles are incompatible.

*Degree  $d = 3$ .* In the cubic case  $p_d$  has two fixed points, 0 and  $1/2$ , and three cycles of period two, given by

$$C(1/2, 0) = (5, 7)/8,$$

$$C(1/2, 1) = (1, 3)/4$$

and

$$C(1/2, 2) = (1, 3)/8.$$

The first and last are incompatible, while the other pairs are compatible. In general there are  $q + 1$  cubic simple cycles with rotation number  $p/q$ , whose fixed-point portraits are given by  $\sigma(1) = 0, 1, \dots, q$ . Only the pairs with adjacent values of  $\sigma(1)$  are compatible.

*Degree  $d = 4$ .* In the quartic case there are six cycles in  $\mathcal{C}_4(1/2)$ , generated by  $t = p/15$  with  $p = 1, 2, 3, 6, 7$  and 11. The compatibility relation between these cycles is shown in Figure 3. The four visible triangles give the four distinct triples of compatible simple cycles with rotation number  $1/2$ . Note that the modular group  $\text{Mod}_4 \cong \mathbb{Z}/3$  acts by rotations on this diagram.

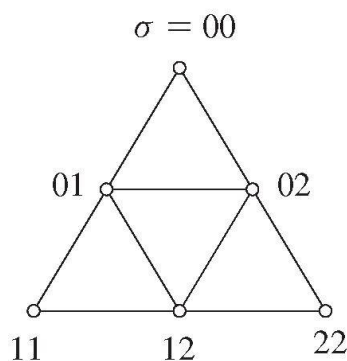


Figure 3. Compatibility of degree 4 cycles of the form  $C(1/2, \sigma)$ .

In general  $\mathcal{C}_d(p/q)$  can be identified with the vertices of the  $q$ -fold barycentric subdivision of a  $(d - 2)$ -simplex, with the top-dimensional cells corresponding to maximal collections of compatible cycles.

*Sample computation in degree  $d = 5$ .* To compute  $C(3/7, 013)$ , we first use equation (2.1) to compute the ‘transpose’  $\tau = 1223333$  of  $\sigma = 013$ . Note that the graphs of  $\sigma$  and  $\tau$ , shown in white and black in Figure 4, fit together to form a rectangle. Evaluating  $\tau' = 1223444$  along the sequence  $ip \bmod q, i = 0, 1, 2, \dots$  we obtain the base 5 expansion  $t = 0.\overline{1342424}_5 = 6966/19531$  for a generator of  $C$ .



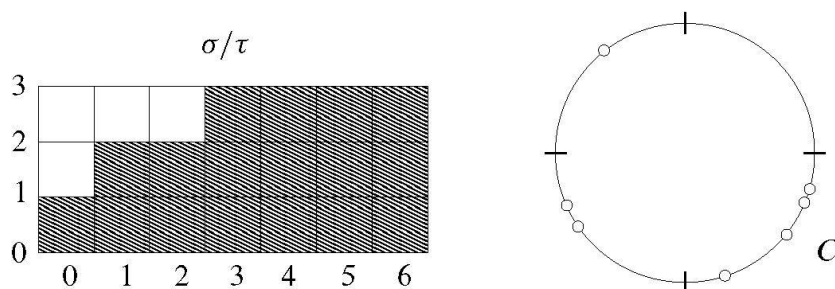


Figure 4. The degree 5 simple cycle with rotation number  $3/7$  and  $\sigma = 013$ .

The cycle  $C$ , along with the four fixed points of  $p_5$ , is drawn at the right in Figure 4. Note that  $\sigma = 013$  gives the running total of the number of points of  $C$  in the first three quadrants.

**Comparison with simple geodesics.** The simple cycles for  $p_d|S^1$  behave in many ways like simple closed geodesics on a compact hyperbolic surface  $X = \Delta/\Gamma$  of genus  $g$ , with compatible cycles corresponding to disjoint geodesics. For example, every maximal collection of disjoint simple closed curves on  $X$  has  $3g - 3$  elements, just as every maximal collection of compatible cycles for  $p_d$  has  $d - 1$  elements.

It is also known that the endpoints of lifts of simple geodesics lie in a closed set  $E \subset S^1$  of Hausdorff dimension zero [BS]. The analogous statement for simple cycles is:

**Theorem 2.2.** *The closure  $E$  of the union of all simple cycles  $C \subset S^1$  of degree  $d$  has Hausdorff dimension zero.*

*Proof.* Let us say a finite set  $P \subset S^1$  is a *precycle* if it is the forward orbit of preperiodic point  $x \in S^1$  under  $p_d$ . We say  $P$  is *simple*, with rotation number  $p/q$ , if  $p_d|P$  extends to a continuous, monotone increasing map  $f: S^1 \rightarrow S^1$  with rotation number  $p/q$ . Then  $q \leq n$  and the periodic part  $C$  of  $P$  is a simple cycle.

Let  $\mathcal{P}_d(n, p/q)$  denote the set of all simple precycles of length  $n$  and rotation number  $p/q$ . The argument that shows  $|\mathcal{C}_d(p/q)| = O(q^{d-2})$  can be adapted to show that  $|\mathcal{P}_d(n, p/q)| = O(n^{d-2})$  as well.

Now fix  $N > 0$ . We claim that every  $x \in E$  lies within distance  $O(d^{-N})$  of a simple precycle  $P$  with  $|P| \leq N$ . To find this precycle, simply increase  $x$  continuously until two of the points among  $x, f(x), \dots, f^N(x)$  coincide. This requires moving  $x$  only slightly, since  $|(f^N)'(x)| = d^N$ .

Thus  $E$  is contained in a neighborhood of diameter  $O(d^{-N})$  of the union  $E_N$  of all simple precycles with  $|P| \leq N$ . Since  $|E_N| = O(N^{d+2})$  grows only like a polynomial in  $N$ , this implies  $\dim(E) = 0$ .  $\square$

*Proof of Theorem 1.4.* The Hölder continuous conjugacy  $\phi_f$  between  $f$  and  $p_d$  preserves sets of Hausdorff dimension zero.  $\square$

**Remark: Invariant measures.** The basic properties of simple cycles can also be developed using the correspondence between invariant measures and covering relations established in [Mc5]. For example, any union  $D = \bigcup C_i$  of compatible cycles in  $\mathcal{C}_d(p/q)$  arises as the support of an invariant measure  $\nu$  for  $p_d|S^1$ . Invariant measures, in turn, correspond bijectively to covering relations  $(F, S)$  of degree  $d$ . In the case at hand,  $F(t) = t + p/q \bmod 1$  and  $S$  is a divisor on  $S^1$  of degree  $d - 1$ . By perturbing  $S$  so its points have multiplicity one, we obtain a nearby invariant measure  $\nu'$  whose support  $D' \supset D$  is a maximal union of exactly  $(d - 1)$  compatible cycles (property (5) above).

The compactification of the space of Blaschke products by covering relations  $(F, S)$  is discussed in the following section.

**Question.** Is there a useful notion of intersection number for a pair of cycles?

### 3. Blaschke products

This section presents basic facts about marked Blaschke products, their derivatives and their images in the moduli space of all rational maps. See [Mc5] for related background material.

**Blaschke products.** Identify  $S^1 = \mathbb{R}/\mathbb{Z}$  with the unit circle in the complex plane, using the coordinate  $z = \exp(2\pi i t)$ . Let  $\Delta = \{z : |z| < 1\}$  be the unit disk, and  $\Delta^{(n)}$  its  $n$ -fold symmetric product.

Given  $d > 1$ , let  $\mathcal{B}_d \cong \Delta^{(d-1)}$  denote the space of Blaschke products  $f : \Delta \rightarrow \Delta$  of the form

$$f(z) = z \prod_1^{d-1} \left( \frac{z - a_i}{1 - \bar{a}_i z} \right)$$

with  $a_i \in \Delta$ . Note that  $f$  extends to a rational map on the whole Riemann sphere, and  $f|S^1$  is a covering map of degree  $d$ .

A proper holomorphic map  $g : \Delta \rightarrow \Delta$  of degree  $d > 1$  is conjugate to some  $f \in \mathcal{B}_d$  iff  $g$  has a fixed point.

**Derivatives and measure.** By logarithmic differentiation, any  $f \in \mathcal{B}_d$  satisfies

$$|f'(z)| = 1 + \sum_1^{d-1} \frac{1 - |a_i|^2}{|z - a_i|^2} \quad (3.1)$$

for  $z \in S^1$ . In particular,  $f|S^1$  is expanding.

More importantly,  $f|S^1$  preserves normalized Lebesgue measure  $\lambda$  on the circle; equivalently,  $f_*(dz/z) = dz/z$ , as can be verified by residue considerations. This means

$$\sum_{f(w)=z} |f'(w)|^{-1} = 1 \quad (3.2)$$

for any  $z \in S^1$ .

**Markings.** All  $f \in \mathcal{B}_d$  are topologically conjugate to the model mapping  $p_d(z) = z^d$ . A *marking* for  $f$  the choice of one such conjugacy, i.e. the choice of a degree one homeomorphism  $\phi: S^1 \rightarrow S^1$  such that

$$f(z) = \phi^{-1} \circ p_d \circ \phi(z).$$

There is a unique marking  $\phi_f$  which varies continuously in  $f$  and satisfies  $\phi_f(z) = z$  when  $f = p_d$ . Thus  $\mathcal{B}_d$  can be regarded as the space of *marked Blaschke products*.

The modular group  $\text{Mod}_d \cong \mathbb{Z}/(d-1)$  acts on  $\mathcal{B}_d$  by  $(a_i) \mapsto (\zeta a_i)$  where  $\zeta^{d-1} = 1$ . Its orbits correspond to different markings of the same map. Thus  $f_1, f_2 \in \mathcal{B}_d$  are conformally conjugate on  $\Delta$  iff they are in the same orbit of the modular group.

**Lengths.** The canonical marking allows one to label the cycles of  $f$  by the cycles of  $p_d$ . We define the *length on  $f$*  of a cycle  $C \in \mathcal{C}_d$  of period  $q$  by

$$L(C, f) = \log |(f^q)'(z)|$$

for any  $z \in S^1$  with  $\phi_f(z) \in C$ .

**Limits of lower degree.** The space of Blaschke products has a natural compactification  $\bar{\mathcal{B}}_d \cong \bar{\Delta}^{(d-1)}$ , whose boundary points  $(a_i)$  can be interpreted as pairs  $(F, S)$  consisting of a Blaschke product

$$F(z) = z \prod_{|a_i| < 1} \left( \frac{z - a_i}{1 - \bar{a}_i z} \right) \cdot \prod_{|a_i| = 1} (-a_i)$$

and a divisor of *sources*

$$S = \sum_{|a_i|=1} 1 \cdot a_i \in \text{Div}(S^1),$$

satisfying  $\deg F + \deg S = d$ . It is easy to see:



**Proposition 3.1.** *A sequence  $f_n \in \mathcal{B}_d$  converges to  $(F, S) \in \partial \mathcal{B}_d$  iff*

- (i)  $f_n(z) \rightarrow F(z)$  uniformly on compact subsets of  $\widehat{\mathbb{C}} - \text{supp } S$ , and
- (ii) the zeros  $Z(f_n)$  converge to  $Z(F) + S$  as divisors on  $\widehat{\mathbb{C}}$ .

More generally, the space  $\text{Rat}_d$  of degree  $d$  rational maps  $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  has a compactification  $\overline{\text{Rat}}_d \cong \mathbb{P}^{2d+1}$ , whose boundary points  $(F, S)$  are pairs consisting of a rational map  $F$  and an effective divisor  $S \in \text{Div}(\widehat{\mathbb{C}})$  with  $\deg(F) + \deg(S) = d$ . We have  $f_n \rightarrow (F, S)$  in  $\overline{\text{Rat}}_d$  iff their graphs satisfy

$$\text{gr}(f_n) \rightarrow \text{gr}(F) + S \times \widehat{\mathbb{C}}$$

as divisors of degree  $(1, d)$  on  $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$  (cf. [D, §1]).

**Radial bounds on  $f'(z)$ .** The following elementary observation is useful for studying limits as above.

**Proposition 3.2.** *For any proper holomorphic map  $f: \Delta \rightarrow \Delta$  and  $\zeta \in S^1$ , we have*

$$\sup_{r \in [0,1]} |f'(r\zeta)| \leq 4|f'(\zeta)|.$$

Note that we do not require that  $f(0) = 0$ . This bound is sharp, as can be seen by considering  $f(z) = (z + a)/(1 + az)$  as  $a \rightarrow 1-$ .

*Proof.* We can write

$$f(z) = e^{i\theta} \prod_1^d M_i(z), \tag{3.3}$$

where  $M_i(z) = (z - a_i)/(1 - \bar{a}_i z)$  and  $a_i \in \Delta$ . Composing with a rotation, we can also assume that  $\zeta = 1$ . For  $r \in [0, 1]$  we have

$$\left| \frac{M'_i(r)}{M'_i(1)} \right| = \frac{|1 - a_i|^2}{|1 - ra_i|^2},$$

and therefore

$$|M'_i(r)| \leq 4|M'_i(1)|,$$

since the distance from 1 to  $a_i$  is never more than twice the distance from 1 to  $ra_i$ . Differentiating the product (3.3) and using the fact that  $|\prod_{j \neq i} M_j(r)| \leq 1$ , we obtain:

$$|f'(r)| \leq \sum |M'_i(r)| \leq 4 \sum |M'_i(1)| = 4|f'(1)|.$$

□

**Corollary 3.3.** *If  $f_n \rightarrow (F, S) \in \overline{\mathcal{B}}_d$ ,  $z_n \in S^1$ ,  $z_n \rightarrow z$  and  $|f'_n(z_n)| = O(1)$ , then  $\lim f_n(z_n) = F(z)$ .*

*Proof.* Suppose  $\sup |f'_n(z_n)| = M$ ; then for any  $r < 1$  we have

$$\begin{aligned} \limsup |f_n(z_n) - F(z)| &\leq \limsup |f_n(rz_n) - F(z)| + 4M(1-r) \\ &= |F(rz) - F(z)| + 4M(1-r); \end{aligned}$$

now let  $r \rightarrow 1$ . □

**Irrational rotations.** As a sample application, we prove the following result stated in the Introduction:

**Corollary 3.4.** *If  $f_n \in \mathcal{B}_d$  satisfies  $f'_n(0) \rightarrow \exp(2\pi i\theta)$  where  $\theta$  is irrational, then  $L(C, f_n) \rightarrow \infty$  for every cycle  $C$ .*

*Proof.* Suppose to the contrary that  $L(C, f_n)$  is bounded for some cycle  $C$ . Let  $C_n \subset S^1$  be the corresponding periodic cycle for  $f_n$ . Pass to a subsequence such that  $f_n \rightarrow (F, S) \in \partial\mathcal{B}_d$  and  $C_n \rightarrow D \subset S^1$  in the Hausdorff topology. Then  $F(z) = \exp(2\pi i\theta)z$  and by Corollary 3.3 we have  $F(D) = D$ , contradicting the irrationality of  $\theta$ . □

**Variants.** Here are two useful variants of the results above:

**Proposition 3.5.** *For any proper holomorphic map  $f: \mathbb{H} \rightarrow \mathbb{H}$  and  $x \in \mathbb{R}$ , we have*

$$\sup_y |f'(x + iy)| \leq f'(x).$$

**Proposition 3.6.** *Assume  $f_n \in \text{Rat}_d$  converges to  $(F, S) \in \overline{\text{Rat}}_d$ ,  $z_n \rightarrow z$ , and  $\|Df_n(z_n)\| = O(1)$  in the spherical metric on  $\hat{\mathbb{C}}$ . Then we have*

$$f_n(z_n) \rightarrow F(z)$$

*provided  $z_n$  belongs to a circle  $T_n$  with  $f_n^{-1}(T_n) = T_n$ , and  $\inf_n \text{diam}(T_n) > 0$ .*

**Proofs.** The first result follows directly from the representation  $f(z) = a_0z + b_0 + \sum_{i=1}^{d-1} a_i/(b_i - z)$  with  $a_i > 0$  and  $b_i \in \mathbb{R}$ , and the second follows by the same argument as Corollary 3.3. □

The maps  $f_n(z) = 1/(1 + nz^2)$  satisfy  $f'_n(0) = 0$  and  $\lim f_n(0) = 1 \neq F(0) = 0$ ; thus some extra hypothesis is needed to interchange limits as in Proposition 3.6.

**Moduli space of rational maps.** Let  $\text{MRat}_d = \text{Rat}_d / \text{Aut}(\widehat{\mathbb{C}})$  denote the moduli space of holomorphic conjugacy classes of rational maps of degree  $d > 1$ . A pair of Blaschke products are conjugate iff they are related by the modular group or by  $z \mapsto 1/\bar{z}$ ; thus we have an inclusion

$$\mathcal{B}_d / (\text{Mod}_d \ltimes \mathbb{Z}/2) \hookrightarrow \text{MRat}_d.$$

The next result shows this inclusion is almost proper.

**Theorem 3.7.** *If  $f_n \rightarrow (F, S) \in \partial \mathcal{B}_d$  but  $[f_n]$  remains bounded in  $\text{MRat}_d$ , then  $F(z) = z$  and  $\text{supp } S$  is a single point. In particular, we have  $f'_n(0) \rightarrow 1$ .*

*Proof.* Pass to a subsequence such  $[f_n] \rightarrow [g] \in \text{MRat}_d$  and  $f_n \rightarrow (F, S) \in \partial \mathcal{B}_d$ . Then there are conjugates  $h_n = A_n f_n A_n^{-1} \rightarrow g$ . Since  $f_n$  diverges in  $\mathcal{B}_d$ ,  $A_n \rightarrow \infty$  in  $\text{Aut}(\widehat{\mathbb{C}})$ . On the other hand, the measures of maximal entropy satisfy  $\mu(h_n) \rightarrow \mu(g)$  and  $\mu(f_n) \rightarrow \mu(F, S)$ , by [D, Theorem 0.1] (see also [Mc5]). Since  $\mu(g)$  is nonatomic, this implies  $\mu(F, S) = \lim A_n^*(\mu(h_n))$  is supported at a single point. But  $\text{supp } \mu(F, S)$  is  $F$ -invariant and includes  $\text{supp } S$ ; thus  $F(z) = z$  and  $\text{supp } S = \{s\}$  is itself a single point.  $\square$

**Example.** The sequence  $f_n(z) = z(z + a_n)/(1 + a_n z)$ , with  $a_n = 1 - 1/n$ , is divergent in  $\mathcal{B}_2$  but convergent in  $\text{MRat}_2$ . To see this, normalize so the origin is a critical point instead of a fixed point; then  $f_n(z)$  is conjugate to  $h_n(z) = (z^2 + b_n)/(1 + b_n z^2)$ , and  $b_n = a_n/(2 + a_n) \rightarrow 1/3$  as  $a_n \rightarrow 1$ .

#### 4. The thin part of $f(z)$

Let us define the *thin part* of  $f \in \mathcal{B}_d$  by

$$S_{\text{thin}}^1(f) = \{z \in S^1 : |f'(z)| < 2\}.$$

In this section we will show:

**Theorem 4.1.** *For any  $f \in \mathcal{B}_d$ , the map  $f|_{S_{\text{thin}}^1(f)}$  extends to a degree one homeomorphism of the circle.*

**Corollary 4.2.** *All cycles of  $f$  with  $L(C, f) < \log 2$  are simple and compatible.*

**Visual angles.** The derivative of

$$f(z) = z \prod_1^{d-1} \left( \frac{z - a_i}{1 - \bar{a}_i z} \right)$$

can be conveniently analyzed using the *hyperbolic visual angle*, defined for  $a, z \in \bar{\Delta}$  by

$$\alpha(z, a) = 2 \arg(z - a) - \arg(z).$$

This is the angle at  $a$  of the hyperbolic geodesic  $\overline{az}$ . For  $z \in S^1$  we have  $\arg(1 - \bar{a}z) = \arg(z) - \arg(z - a)$ , and thus

$$\arg(f(z)) = \arg(z) + \sum_1^{d-1} \alpha(z, a_i). \quad (4.1)$$

(Note this simplifies to  $\arg(f(z)) = 2 \arg(z - a_1)$  when  $d = 2$ .) Letting  $\theta = \arg(z)$  and  $\dot{\alpha} = d\alpha/d\theta$ , we then obtain

$$|f'(z)| = 1 + \sum_1^{d-1} \dot{\alpha}(z, a_i) \quad (4.2)$$

for  $z \in S^1$ .

**The visual density.** The *visual density*  $\dot{\alpha}(z, a)$  is essentially the Poisson kernel; for  $a = r \geq 0$  it is given by

$$\dot{\alpha}(z, r) = \frac{1 - r^2}{1 + r^2 - 2r \cos \theta}, \quad (4.3)$$

where  $\theta = \arg z$ . Geometrically,  $(\dot{\alpha}(z, a)/2\pi) d\theta$  is the hitting measure on the circle for a random hyperbolic geodesic starting at  $a$ .

For fixed  $z \in S^1$ , the level sets of  $\dot{\alpha}(z, a)$  are horocycles resting on  $z$ . Thus

$$J(a) = \{z \in S^1 : \dot{\alpha}(z, a) < 1\}$$

is the large arc cut off by the chord perpendicular to  $\overline{0a}$ . This follows from the fact that the horocycle resting on one of the endpoints of  $J(a)$  and passing through 0 also passes through  $a$  (see Figure 5).

**Proposition 4.3.** *The visual density  $\dot{\alpha}(z, a)|_{J(a)}$  is strictly convex, and decreases as  $a$  moves radially towards the circle. In other words, we have*

$$\ddot{\alpha}(z, a) > 0 \quad \text{and} \quad \frac{d}{ds} \dot{\alpha}(z, sa) \Big|_{s=1} < 0$$

for all  $z \in J(a)$ .

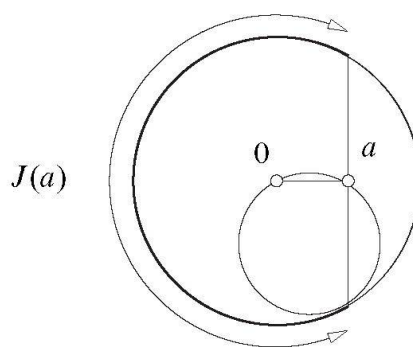


Figure 5. The arc  $J(a)$  where  $\dot{\alpha}(z, a) < 1$ .

*Proof.* To verify convexity, consider the case where  $a = r \in [0, 1)$ . By (4.3), in this case we have  $\dot{\alpha} = (1 - r^2)/u$  where  $u = 1 + r^2 - 2r \cos \theta$ . We may assume  $\theta \in (0, \pi)$ . Cross-multiplying and differentiating, we obtain

$$\begin{aligned}\dot{\alpha}u &= 1 - r^2, \\ \ddot{\alpha}u + \dot{\alpha}(2r \sin \theta) &= 0, \quad \text{and} \\ \ddot{\alpha}u + \ddot{\alpha}(4r \sin \theta) + \dot{\alpha}(2r \cos \theta) &= 0.\end{aligned}$$

Since  $r, u$  and  $\sin \theta$  are all positive, we have  $\dot{\alpha} > 0$  and  $\ddot{\alpha} < 0$ . Comparing the last two equations, we find the sign of  $\ddot{\alpha}$  is the same as the sign of the determinant

$$D = \det \begin{pmatrix} 2r \sin \theta & u \\ 2r \cos \theta & 4r \sin \theta \end{pmatrix} = 8r^2 \sin^2 \theta - 2ru \cos \theta.$$

We claim  $D > 0$  when  $z \in J(r)$ , i.e. when  $u = |z - r|^2 > 1 - r^2$ . The claim is evident if  $\cos \theta$  is negative, so assume  $\theta \in (0, \pi/2)$ ; then

$$u = |z - r|^2 \leq |z - 1|^2 \leq 2(\operatorname{Im} z)^2 = 2 \sin^2 \theta.$$

We also have  $\cos \theta = \operatorname{Re}(z) < r$  for  $z \in J(r)$ , and thus:

$$D \geq 4r^2u - 2r^2u > 0.$$

The proof of the density decreasing property is straightforward.  $\square$

**Properties of the thin part of  $f$ .** We can now show that  $f|_{S_{\text{thin}}^1(f)}$  acts like a rotation. We first observe:

**Proposition 4.4.** *For any  $f \in \mathcal{B}_d$ ,*

- (i) *the map  $f|_{S_{\text{thin}}^1(f)}$  is injective,*
- (ii) *we have  $S_{\text{thin}}^1(f) \subset \bigcap J(a_i)$ ,*

- (iii)  $S_{\text{thin}}^1(f)$  consists of at most  $(d - 1)$  disjoint open intervals, and
- (iv)  $S_{\text{thin}}^1(f)$  increases as the zeros  $a_i$  of  $f$  move radially towards the circle.

*Proof.* If  $f(x_1) = f(x_2)$  for two distinct points in  $S_{\text{thin}}^1(f)$ , then  $|f'(x_1)| + |f'(x_2)| > 1/2 + 1/2 = 1$ , which violates the measure-preserving property (3.2) of  $f$ ; thus  $f|_{S_{\text{thin}}^1(f)}$  is injective. Equation (4.2) implies (ii). Since  $\bigcup (S^1 - J(a_i))$  has at most  $(d - 1)$  components, so does  $I = \bigcap J(a_i)$ . By Proposition 4.3,  $|f'(z)|$  is locally convex on  $I$ ; thus the intersection of  $S_{\text{thin}}^1(f)$  with any component of  $I$  is connected, and (iii) follows. The density decreasing property stated in Proposition 4.3 implies (iv).  $\square$

*Proof of Theorem 4.1.* By moving the points  $(a_i)$  radially to the circle, we obtain a smooth 1-parameter family of maps  $f_t \in \bar{\mathcal{B}}_d$ ,  $t \in [0, 1]$ , with  $f_0 = f$  and  $f_1 = (F, S)$ . Since  $\deg(S) = d - 1$ , we have  $\deg(F) = 1$ . Proposition 4.4 implies that  $f_t|_{T_t} = S_{\text{thin}}^1(f_t)$  is injective,  $T_s \subset T_t$  when  $s < t$ , and  $\text{supp } S \cap T_t = \emptyset$ . Thus for any three distinct points  $x_i \in S_{\text{thin}}^1(f)$ , the triple  $(f_t(x_1), f_t(x_2), f_t(x_3))$  moves by isotopy as  $t$  increases from 0 to 1, and converges to  $F(x_1), F(x_2), F(x_3)$  as  $t \rightarrow 1$ . Since  $F$  is a rotation, it preserves the cyclic ordering of the points  $(x_i)$ , so the same is true of  $f$ . Consequently  $f$  extends from  $S_{\text{thin}}^1(f)$  to an orientation-preserving homeomorphism of the circle.  $\square$

## 5. Bounds on repelling cycles

In this section we show that every  $f \in \mathcal{B}_d$  has a simple cycle with  $L(C, f) = O(d)$ , and obtain related results for general rational maps.

**Moduli and tori.** We begin by summarizing some well-known facts about extremal length on tori.

Any point  $\tau \in \mathbb{H}$  determines a complex torus

$$X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$$

with a flat metric inherited from the plane, and a distinguished basis  $\langle 1, \tau \rangle$  for its fundamental group. Factoring the covering map  $\mathbb{C} \rightarrow X_\tau$  through the map  $\xi: \mathbb{C} \rightarrow \mathbb{C}^* \cong \mathbb{C}/\mathbb{Z}$  given by  $\xi(z) = \exp(2\pi iz)$ , we have

$$X_\tau = \mathbb{C}^*/\alpha^\mathbb{Z}$$

where  $\alpha = \xi(\tau)$  satisfies  $0 < |\alpha| < 1$ . The same construction can be made when  $-\tau \in \mathbb{H}$ ; then  $|\alpha| > 1$ .

Given a slope  $p/q \in \mathbb{Q} \cup \{\infty\}$ , let  $\gamma_{p/q} \subset X_\tau$  denote the simple closed geodesic obtained as the projection of the line  $\mathbb{R} \cdot (\tau - p/q)$  from  $\mathbb{C}$  to  $X_\tau$ . Its preimage  $\tilde{\gamma}_{p/q}$

in the intermediate cover  $\mathbb{C}^*$  consists of  $q$  arcs joining 0 to  $\infty$ , cyclically permuted with rotation number  $p/q$  by  $z \mapsto \alpha z$ .

Any annulus  $A$  is conformally equivalent to a right cylinder, which is unique up to scale. The ratio  $\text{mod}(A) = h/c$  between the height and circumference of this cylinder is the *modulus* of  $A$ .

The maximum modulus of an annulus  $A \subset X_\tau$  homotopic to  $\gamma_{p/q}$  is given by

$$\text{mod}(p/q, X_\tau) = \frac{\text{area}(X_\tau)}{L(\gamma_{p/q}, X_\tau)^2} = \frac{|\text{Im } \tau|}{|q\tau - p|^2} \quad (5.1)$$

(assuming  $\gcd(p, q) = 1$ ). This maximum is realized by taking  $A = X_\tau \setminus \gamma_{p/q}$ . The set of  $\tau \in \mathbb{H}$  with  $\text{mod}(p/q, X_\tau) \geq m$  is a horoball of diameter  $1/(mq^2)$  resting on the real axis at  $p/q$ . For  $p/q = 1/0$  we have

$$\text{mod}(\infty, X_\tau) = |\text{Im } \tau|.$$

The *intersection inequality*

$$\text{mod}(p/q, X_\tau) \text{mod}(r/s, X_\tau) \leq \left( \det \begin{pmatrix} p & q \\ r & s \end{pmatrix} \right)^{-2} \quad (5.2)$$

is easily verified by considering the determinant of the lattice  $\mathbb{Z}(q\tau - p) \oplus \mathbb{Z}(s\tau - r)$ . This inequality implies:

*There is at most one slope with  $\text{mod}(p/q, X_\tau) > 1$ .*

On the other hand we have:

**Proposition 5.1.** *For any  $\tau \in \mathbb{H}$ , there exists a slope  $p/q \in \mathbb{Q} \cup \{\infty\}$  such that*

$$\text{mod}(p/q, X_\tau) \geq \sqrt{3}/2.$$

*Proof.* Since the statement is invariant under the action of  $\text{SL}_2(\mathbb{Z})$  on  $\mathbb{H}$ , it suffices to verify it when  $\tau$  lies in the fundamental domain  $|\tau| \geq 1$ ,  $|\text{Re } \tau| \leq 1/2$ ; and in this case, we have  $\text{mod}(\infty, X_\tau) = |\text{Im } \tau| \geq \sqrt{3}/2$ .  $\square$

**Rational maps.** Now let  $f: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  be a rational map of degree  $d > 1$ . If  $z \in \hat{\mathbb{C}}$  is a point of period  $q$ , its *multiplier* is given by  $\beta = (f^q)'(z)$ . The *grand orbit* of  $z$  is the set  $\bigcup_{i,j \geq 0} f^{-i} \circ f^j(z)$ .

Suppose  $f$  has a fixed point at  $z = 0$  and a periodic point  $w \neq 0$  with period  $q$ . We say  $w$  has *rotation number*  $p/q$  relative to  $z = 0$  if there are arcs  $(\delta_i)_0^{q-1} \subset \hat{\mathbb{C}}$  joining  $z = 0$  to  $f^i(w)$ , meeting only at  $z = 0$ , which are cyclically permuted by  $f$  with rotation number  $p/q$ .

**Theorem 5.2.** *Let  $f$  be a rational map with an attracting fixed-point at  $z = 0$ , with multiplier*

$$\alpha = f'(0) = \exp(2\pi i \tau) \neq 0.$$

*Let  $e$  be the number of grand orbits of critical points in the immediate basin  $\Omega$  of  $z = 0$ . Then for each  $p/q \in \mathbb{Q}$ , there exists a repelling or parabolic periodic point  $w \in \partial\Omega$  such that*

- (1) *the rotation number of  $w$  relative to  $z = 0$  is  $p/q$ ; and*
- (2) *its multiplier has the form  $\beta = (f^q)'(w) = \exp(-2\pi i \sigma)$ , where  $\sigma = 0$  or*

$$\frac{\operatorname{Im} \sigma}{|\sigma|^2} \geq \frac{\operatorname{mod}(p/q, X_\tau)}{e}. \quad (5.3)$$

*In particular, we have*

$$|\beta| \leq \left( \exp \left( \frac{2\pi}{\operatorname{mod}(p/q, X_\tau)} \right) \right)^e. \quad (5.4)$$

*Proof.* Let  $\Omega^*$  denote the immediate basin of  $z = 0$  with the grand orbits of all critical points in  $\Omega$  and of  $z = 0$  deleted. Then  $f: \Omega^* \rightarrow \Omega^*$  is a covering map. Moreover, the holomorphic linearizing map

$$\phi(z) = \lim \alpha^{-n} f^n(z)$$

is defined for all  $z \in \Omega^*$ , and satisfies  $\phi(f(z)) = \alpha \phi(z)$ . Consequently  $\phi$  descends to an inclusion of the space of grand orbits  $Y = \Omega^*/\langle f \rangle$  into the torus  $X_\tau = \mathbb{C}^*/\alpha^{\mathbb{Z}}$ , making the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{\phi} & \mathbb{C}^* \\ \downarrow & & \downarrow \\ \Omega/\langle f \rangle = Y & \hookrightarrow & X_\tau = \mathbb{C}^*/\alpha^{\mathbb{Z}} \end{array}$$

commute. By assumption we have  $|Y - X_\tau| = e$ .

For a given  $p/q \in \mathbb{Q}$ , the geodesics parallel to  $\gamma_{p/q}$  passing through the punctures of  $Y$  cut it into  $\leq e$  parallel annuli, one of which satisfies

$$\operatorname{mod}(A) \geq \operatorname{mod}(p/q, X_\tau)/e. \quad (5.5)$$

Let  $\delta \subset A$  be the core curve of  $A$ , and  $\delta_0 \subset \Omega^*$  one of its lifts which is incident to  $z = 0$ . Let  $\delta_i = f^i(\delta_0)$ . By construction, the arc  $\delta_0$  is invariant under  $f^q$ , and  $f^q|_{\delta_0}$  is a bounded translation in the hyperbolic metric on  $\Omega^*$ . Consequently  $\delta_0$  must join



$z = 0$  to another fixed point  $w$  of  $f^q$  in  $\partial\Omega$ . By the Snail Lemma [Mil1, Lem. 16.2],  $w$  is repelling or parabolic.

We have seen that the preimage of  $\gamma_{p/q}$  on  $\mathbb{C}^*$  consists of  $q$  arcs, cyclically permuted with rotation number  $p/q$  by  $z \mapsto \alpha z$ . Since  $\phi$  is a homeomorphism near  $z = 0$ , the arcs  $\delta_0, \dots, \delta_{q-1}$  are also cyclically permuted with rotation number  $p/q$  by  $f$ . In particular  $w$  has rotation number  $p/q$  relative to  $z = 0$ .

Now suppose  $w$  is repelling, with multiplier  $\beta$ . Choose an injective branch of  $f^{-q}$  defined on a punctured neighborhood  $U^*$  of  $w$  such that  $f^{-q}: U^* \rightarrow U^*$  and

$$Z = U^*/\langle f^{-q} \rangle \cong \mathbb{C}^*/\beta^{\mathbb{Z}} = X_\sigma,$$

where  $\sigma = \log(\beta/2\pi i)$ . There is a unique choice of the logarithm such that the invariant arc  $\delta_0 \cap U^*$  descends to a loop isotopic to  $\gamma_0$  on  $X_\sigma$ .

By construction,  $A \subset Y$  is covered by a strip  $A_0 \subset \Omega^*$  which retracts to  $\delta_0$ , and hence we have an inclusion

$$A \cong A_0/\langle f^q \rangle \hookrightarrow Z \cong X_\sigma$$

in the same homotopy class as  $\gamma_0$ . This implies

$$\text{mod}(0, X_\sigma) \geq \text{mod}(A),$$

and the bound (5.3) follows from equations (5.1) and (5.5).  $\square$

**Corollary 5.3.** *If  $f \in \text{Rat}_d$  has an attracting fixed point with multiplier satisfying*

$$|\alpha| > \exp(-\pi\sqrt{3}) = 0.0043\dots,$$

*then it also has a repelling or parabolic cycle with multiplier satisfying*

$$|\beta| \leq \exp(4\pi/\sqrt{3})^{2d-2} \leq 1416^{2d-2}.$$

*Proof.* The lower bound on  $|\alpha|$  implies  $\text{Im}(\tau) = \text{mod}(\infty, X_\tau) < \sqrt{3}/2$ , where  $\tau = (\log \alpha)/2\pi i$ . Hence  $\text{mod}(p/q, X_\tau) \geq \sqrt{3}/2$  for some  $p/q \in \mathbb{Q}$ , by Proposition 5.1. Now apply equation (5.4) and note that  $e \leq 2d - 2$ .  $\square$

**Corollary 5.4.** *If a map  $f \in \text{Rat}_d$  has an attracting fixed point with multiplier  $\alpha$ , then it also has a repelling or parabolic cycle with multiplier satisfying*

$$|\beta| \leq (\exp(4\pi/\sqrt{3})/|\alpha|)^{2d-2}.$$

*Proof.* Choose  $\tau = (\log \alpha)/2\pi i = x + iy$  with  $x \in [-1/2, 1/2]$ . The previous corollary shows the desired bound holds when  $y < \sqrt{3}/2$ . For  $y \geq \sqrt{3}/2$  we have

$$m = \text{mod}(0, X_\tau)^{-1} \leq \frac{x^2 + y^2}{y} \leq \frac{1}{2\sqrt{3}} + y < \frac{2}{\sqrt{3}} + y,$$

which implies  $\exp(2\pi/m) \leq \exp(4\pi/\sqrt{3})/|\alpha|$ ; thus by (5.4) the desired bound holds in this case as well.  $\square$

**The bottom of the spectrum.** Here is a qualitative consequence of the preceding corollary.

Let the *spectrum*  $S(f) \subset \mathbb{C}$  be the set of all multipliers  $\beta$  that arise from periodic points of  $f \in \text{Rat}_d$ , and let

$$L(f) = \inf \{ \log |\beta| : \beta \in S(f) \text{ and } |\beta| \geq 1 \}.$$

By the fixed-point formula for rational maps [Mil1, Theorem 12.4], the multipliers of  $f$  at its fixed points satisfy

$$\sum \frac{1}{\mu_j - 1} = 1, \quad (5.6)$$

provided no  $\mu_j = 1$ ; in particular,  $|\mu_j| \leq d + 1$  for some  $j$ . Thus if  $f$  has no attracting fixed points, it satisfies

$$L(f) \leq \log(d + 1).$$

Combining this observation with Corollary 5.4, we obtain:

**Corollary 5.5.** *Let  $f_n \in \text{Rat}_d$  be a sequence of rational maps with  $L(f_n) \rightarrow \infty$ . Then the maps  $f_n$  have fixed points with multipliers  $\alpha_n \rightarrow 0$ .*

**Examples.** It is easy to see that  $f_n(z) = z^2 + n^2$  satisfies  $L(f_n) \rightarrow \infty$  as  $n \rightarrow \infty$ , since its Julia set lies close to  $\pm n$ . Of course  $f_n$  has a fixed point at infinity with multiplier  $\alpha_n = 0$ .

Parabolics must be included in the definition of  $L(f)$  to obtain Corollary 5.5. In fact, if we let  $L^*(f) = \inf \{ \log |\beta| : \beta \in S(f), |\beta| > 1 \}$ , then  $f_n(z) = z - 1/z + n$  satisfies  $L^*(f_n) \rightarrow \infty$  even though  $f_n$  has no attracting fixed point. (The map  $f_n(z)$  behaves like the Hecke group  $\langle z \mapsto -1/z, z \mapsto z + n \rangle$ ; cf. [Mc3, Theorem 6.2].)

**Question.** Does Corollary 5.5 remain true if only parabolic and repelling multipliers are included in the definition of  $L(f)$ ?

**Blaschke products.** We now return to the setting of a proper map  $f: \Delta \rightarrow \Delta$  fixing  $z = 0$ . In this case formula (5.6) implies:

**Proposition 5.6.** *The multipliers  $(\lambda_i)_1^{d-1}$  of  $f \in \mathcal{B}_d$  at its fixed points on the circle satisfy*

$$\sum_1^{d-1} \frac{1}{\lambda_i - 1} = \frac{1 - |\alpha|^2}{|1 - \alpha|^2},$$

where  $\alpha = f'(0)$ .

**Corollary 5.7.** *If  $|\alpha| < 1/2$ , then  $f$  has a repelling fixed point with multiplier satisfying  $1 < \beta \leq 1 + (d - 1)/3$ .*

**Theorem 5.8.** *Every  $f \in \mathcal{B}_d$  has a simple cycle with  $L(C, f) = O(d)$ .*

*Proof.* Combine Corollaries 5.4 and 5.7. □

## 6. Short cycles and short geodesics

In this section we use the fixed-point formula for rational maps to obtain the following more detailed connection between the short cycles for  $f$  and the short geodesics on its quotient torus.

**Theorem 6.1.** *Given  $f \in \mathcal{B}_d$  with  $f'(0) = \exp(2\pi i \tau)$ , choose  $p/q \in \mathbb{Q}$  to maximize  $\text{mod}(p/q, X_\tau)$ . Then there exist compatible simple cycles  $C_i$  with rotation number  $p/q$ , such that*

(1) *their lengths satisfy*

$$\text{mod}(p/q, X_\tau) \leq \pi \sum L(C_i, f)^{-1} \leq \text{mod}(p/q, X_\tau) + O(d); \quad (6.1)$$

(2) *all other cycles satisfy  $L(C, f) > \epsilon_d > 0$ ; and*

(3) *for any  $r > 0$ , the multipliers of  $f^r$  at its repelling fixed points satisfy*

$$\frac{1}{r} \sum' \frac{1}{\lambda_j - 1} = O(d), \quad (6.2)$$

*where the prime indicates that fixed points in  $\bigcup C_i$  are excluded.*

In qualitative terms, the construction shows:

**Corollary 6.2.** *All cycles with  $L(C, f) < \epsilon_d$  arise from short geodesics on the quotient torus for  $f$ .*

**Tiling of  $\Delta^*$ .** The slope  $p/q \bmod 1$  appearing in the theorem above depends only on  $\alpha = f'(0) \in \Delta^*$ . Figure 1 of the Introduction shows the regions  $T(p/q) \subset \Delta^*$  where a given slope maximizes the value of  $\text{mod}(p/q, X_\tau) = \text{mod}(p/q, \mathbb{C}/\alpha\mathbb{Z})$ .

This picture is nothing more than the image, under the covering map  $\xi: \mathbb{H} \rightarrow \Delta^*$  given by  $\xi(\tau) = \exp(2\pi i \tau)$ , of the tiling of  $\mathbb{H}$  by  $\text{SL}_2(\mathbb{Z})$  translates of the Dirichlet region

$$F = \{\tau \in \mathbb{H} : |\tau - n| \geq 1 \ \forall n \in \mathbb{Z}\}$$

for the cusp  $\tau = \infty$ . The tile  $T(\infty) = \xi(F)$  lies in a ball of radius  $\exp(-\pi\sqrt{3}) \approx 1/230$  about the origin. In this tile the short curve is  $\gamma_\infty \subset X_\tau$ , which lifts to a loop around  $z = 0$  rather than a path connecting  $z = 0$  to a periodic point. Thus the length of  $\gamma_\infty$  can go to zero without any cycle getting short.

Each remaining tile  $T(p/q)$  contains a horocycle  $H$  resting on the root of unity  $\exp(2\pi i p/q) \in S^1$ . Within a still smaller horocycle  $H' \subset H$ ,  $\gamma_{p/q}$  becomes very short, and hence  $f$  has a very short cycle with rotation number  $p/q$ .

**Moduli and multipliers.** We begin the proof of Theorem 6.1 by connecting Diophantine properties of  $\alpha \in \Delta^*$  to lengths of geodesics on  $\mathbb{C}^*/\alpha^{\mathbb{Z}}$ .

**Lemma 6.3.** *For any  $\alpha = \exp(2\pi i \tau) \in \Delta^*$  and  $q > 0$ , we have*

$$\sup_p \frac{\text{mod}(p/q, X_\tau)}{\gcd(p, q)^2} = \frac{\pi}{q} \frac{1 - |\alpha^q|^2}{|1 - \alpha^q|^2} + O(1).$$

*Proof.* First consider the case  $q = 1$ , and assume  $\tau$  is chosen so  $|\text{Re } \tau| \leq 1/2$ . Then we have  $2\pi i \tau \approx 1 - \alpha$  when either side is small, and hence

$$\sup_p \text{mod}(p, X_\tau) = \frac{\text{Im } \tau}{|\tau|^2} = \pi \frac{1 - |\alpha|^2}{|1 - \alpha|^2} + O(1).$$

The general case follows using the fact that

$$\frac{\text{mod}(p/q, X_\tau)}{\gcd(p, q)^2} = \frac{\text{mod}(p, X_{q\tau})}{q}. \quad \square$$

*Proof of Theorem 6.1.* Choose  $p$  so that  $\text{mod}(p/q, X_\tau)$  is maximized. As in Theorem 5.2, by cutting the torus  $X_\tau$  open along  $e \leq d - 1$  geodesics parallel to  $\gamma_{p/q}$  we obtain annuli  $A_1, \dots, A_e \subset Y$  with

$$\text{mod}(p/q, X_\tau) = \sum \text{mod}(A_i).$$

Each annulus  $A_i$ , when lifted to the unit disk, connects  $z = 0$  to a simple cycle  $C_i$  for  $f$  with rotation number  $p/q$  and multiplier  $\beta_i > 1$ .

The lifts of the annuli  $A_i$  are disjoint, so the cycles  $C_i$  are compatible. Assume for the moment they are also distinct. Since two copies of  $A_i$  embed in the quotient torus  $\mathbb{C}^*/\beta_i^{\mathbb{Z}}$  (one for the inside of the disk and one for the outside), we have

$$2\text{mod}(A_i) \leq \frac{2\pi}{\log \beta_i} = \frac{2\pi}{L(C_i, f)}.$$

The combination of these inequalities yields:

$$\text{mod}(p/q, X_\tau) \leq \pi \sum L(C_i, f)^{-1}.$$

This lower bound also holds when the cycles are not distinct; then we simply have more annuli  $A_i$  embedded in a given torus  $\mathbb{C}^*/\beta_j^{\mathbb{Z}}$ .

For the upper bound, let  $(\lambda_j)$  denote the multipliers of the repelling fixed points of  $f^q$ . Note that each cycle  $C_i$  contributes  $q$  fixed points, each with multiplier  $\beta_i$ . Combining Proposition 5.6 and Lemma 6.3, we obtain:

$$\begin{aligned} \frac{1}{q} \sum \frac{1}{\lambda_j - 1} &= \frac{1}{q} \sum' \frac{1}{\lambda_j - 1} + \sum \frac{1}{\beta_i - 1} = \frac{1 - |\alpha^q|^2}{q|1 - \alpha^q|^2} \\ &= \pi^{-1} \bmod(p/q, X_\tau) + O(1). \end{aligned}$$

(Again, the prime indicates fixed points in  $\bigcup C_i$  are excluded.) Since the cycles  $C_i$  are compatible, there are no more than  $d - 1$  of them, and hence

$$\sum \frac{1}{\beta_i - 1} = \sum \left( \frac{1}{\log \beta_i} + O(1) \right) = \left( \sum L(C_i)^{-1} \right) + O(d).$$

This yields the upper bound in (6.1); and it also implies

$$\frac{1}{q} \sum' \frac{1}{\lambda_j - 1} = O(d).$$

That is, equation (6.2) holds for  $r = q$ .

To obtain (6.2) for other values of  $r$ , recall that by (5.2) we have  $\bmod(s/r, X_\tau) < 1$  whenever  $s/r \neq p/q$ . Thus if  $q$  does not divide  $r$ , Lemma 6.3 implies

$$\frac{1}{r} \sum' (\lambda_j - 1)^{-1} \leq \frac{1 - |\alpha^r|^2}{r|1 - \alpha^r|^2} \leq \sup_s \bmod(s/r, X_\tau) + O(1) = O(1);$$

while for  $r = nq$  we obtain

$$\frac{1}{r} \sum' \frac{1}{\lambda_j - 1} + \frac{q}{r} \sum \frac{1}{\beta_i^n - 1} = \frac{1 - |\alpha^r|^2}{r|1 - \alpha^r|^2} = \frac{\bmod(p/q, X_\tau)}{\pi n^2} + O(1),$$

which again implies (6.2), since (6.1) gives

$$\frac{q}{r} \sum \frac{1}{\beta_i^n - 1} = \frac{1}{n} \left( \sum \frac{1}{nL(C_i, f)} + O(1) \right) = \frac{\bmod(p/q, X_\tau)}{\pi n^2} + O(d).$$

Finally note that equation (6.2) implies  $L(C, f) > \epsilon_d \asymp 1/d > 0$ , since any cycle  $C$  of period  $r$  and multiplier  $\beta$ , not among the  $C_i$ , contributes  $1/(\beta - 1)$  to the sum  $(1/r) \sum' (\lambda_j - 1)^{-1}$ .  $\square$

## 7. Binding and renormalization

We conclude by proving the following compactness result.

**Theorem 7.1.** *Let  $(C_i)_1^n$  be a binding set of cycles of degree  $d$ . Then for any  $M > 0$ , the set of  $f \in \mathcal{B}_d$  such that  $\sum_i L(C_i, f) \leq M$  has compact closure in  $\text{MRat}_d$ .*

**Corollary 7.2.** *The set of  $f \in \mathcal{B}_d$  such that  $\sum_i L(C_i, f) \leq M$  and  $|f'(0) - 1| \geq 1/M$  is compact.*

*Proof.* By Theorem 3.7, the only way a sequence  $f_n$  can diverge in  $\mathcal{B}_d$  but remain bounded in  $\text{MRat}_d$  is if  $f'_n(0) \rightarrow 1$ .  $\square$

**Definitions.** Sets  $A, B \subset S^1$  are *unlinked* if they lie in disjoint connected sets; equivalently, if their convex hulls in the unit disk are disjoint. A map  $f: X \rightarrow X$  with  $X \subset S^1$  is *renormalizable* if there is a nontrivial partition of  $X$  into disjoint, unlinked subsets  $X_1, \dots, X_n$ , such that every  $f(X_i)$  lies in some  $X_j$ .

We say a collection of degree  $d$  cycles  $C_1, \dots, C_m$  is *binding* if  $\deg(p_d| \bigcup C_i) = d$  and  $p_d| \bigcup C_i$  is not renormalizable.

*Proof of Theorem 7.1.* Suppose to the contrary that we have a sequence  $f_n \in \mathcal{B}_d$  with  $\sum_i L(C_i, f_n) \leq M$  that is divergent in moduli space. Let

$$D_n = \phi_{f_n}^{-1}\left(\bigcup C_i\right) \subset S^1$$

be the finite  $f_n$ -invariant set corresponding to the binding cycles. Since  $f_n|S^1$  is expanding, we have  $|f'_n| \leq e^M$  on  $D_n$ .

Next we conjugate the entire picture by an affine transformation depending on  $n$ , so that  $0 \in D_n$  and  $\text{diam}(D_n) = 1$ . Then  $S^1$  goes over to a circle  $T_n \supset D_n$  invariant by  $f_n$ , and we still have  $|f'_n|_{D_n} \leq e^M$ .

Pass to a subsequence such that  $f_n \rightarrow (F, S) \in \overline{\text{Rat}}_d$ . Since  $f_n$  diverges in  $\text{MRat}_d$ , we have  $\deg(F) < d$ . Passing to a further subsequence, we can find a finite set  $D$  containing zero and a circle  $T \subset \widehat{\mathbb{C}}$  such that  $D_n \rightarrow D$  and  $T_n \rightarrow T$  in the Hausdorff topology. Note that  $|D| > 1$  since  $\text{diam } D = 1$ .

By Proposition 3.6, the map  $f_n|D_n$  converges to  $F|D$ . But if  $|D| = |\bigcup C_i|$ , the map  $F|(D \subset T)$  is combinatorially the same as  $p_d|(\bigcup C_i \subset S^1)$ , contradicting our assumption that  $\deg(p_d| \bigcup C_i) = d$ . Similarly, if  $|D| < |\bigcup C_i|$ , then the collapse of  $D_n$  to  $D$  provides an invariant partition for  $\bigcup C_i$ , contradicting our assumption that  $p_d| \bigcup C_i$  is not renormalizable.  $\square$



**Examples.** The single cycle  $C = (3, 6, 12, 24, 17)/31$  in degree 2 is already binding, as is any cycle of prime order with  $\deg(p_d|C) = d$ .

The first renormalizable cycle in degree 2 is  $C = (1, 2, 4, 3)/5$ . Although  $\deg(p_2|C) = 2$ ,  $L(C, f_n)$  remains bounded as  $f_n \in \mathcal{B}_2$  diverges along the sequence specified by  $f'_n(0) = -1 + 1/n$ . Indeed,  $f_n^2$  can be renormalized so that  $C$  converges to the cycle of period 2 for  $G(z) = z - 1/z$  [Ep]; and thus  $L(C, f_n) \rightarrow \log 9$ . For more details, see [Mc6, §14].

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Curtis T. McMullen, Mathematics Department, Harvard University, 1 Oxford St, Cambridge, MA 02138-2901, U.S.A.