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Dynamics on the unit disk: Short geodesics and simple cycles

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1. Introduction

In this paper we show that rotation cycles on S^1 for a proper holomorphic map $f: \Delta \rightarrow \Delta$ share several of the analytic, geometric and topological features of simple closed geodesics on a compact hyperbolic surface.

Dynamics on the unit disk. Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For $d > 1$ let $\mathcal{B}_d \cong \Delta^{(d-1)}$ denote the space of all proper holomorphic maps $f: \Delta \rightarrow \Delta$ of the form

$$f(z) = z \prod_1^{d-1} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right),$$

$|a_i| < 1$. Every degree d holomorphic map $g: \Delta \rightarrow \Delta$ with a fixed point in the disk can be put into the form above, by normalizing so its fixed point is $z = 0$.

The maps $f \in \mathcal{B}_d$ have the property that $f|_{S^1}$ is measure-preserving and $|f'| > 1$ on the circle. Moreover, there is a unique *marking* homeomorphism $\phi_f: S^1 \rightarrow S^1$

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that varies continuously with f , conjugates f to $p_d(z) = z^d$, and satisfies $\phi_f(z) = z$ when $f = p_d$. We define the *length* on f of a periodic cycle C for p_d by

$$L(C, f) = \log |(f^q)'(z)|, \quad (1.1)$$

where $q = |C|$ and $\phi_f(z) \in C$.

The *degree* of a cycle C is the least $e > 0$ such that $p_d|_C$ extends to a covering map of the circle of degree e . We say C is *simple* if $\deg(p_d|_C) = 1$; equivalently, if $p_d|_C$ preserves its cyclic ordering. A finite collection of cycles C_i is *binding* if $\deg(\bigcup C_i) = d$ and if $\bigcup C_i$ is not renormalizable (§7).

In this paper we establish four main results.

Theorem 1.1. *Any cycle with $L(C, f) < \log 2$ is simple. All such cycles C_i have the same rotation number, and $p_d|_{\bigcup C_i}$ preserves the cyclic ordering of $\bigcup C_i$.*

Theorem 1.2. *Every $f \in \mathcal{B}_d$ has a simple cycle C with $L(C, f) = O(d)$.*

Theorem 1.3. *Let $(C_i)_1^n$ be a binding collection of cycles. Then for any $M > 0$, the set of $f \in \mathcal{B}_d$ with $\sum_1^n L(C_i, f) \leq M$ has compact closure in the moduli space of all rational maps of degree d .*

Theorem 1.4. *The closure $E \subset S^1$ of the simple cycles for a given $f \in \mathcal{B}_d$ has Hausdorff dimension zero.*

See Theorems 4.1, 5.8, 7.1 and 2.2 below.

Hyperbolic surfaces. The results above echo the following fundamental facts about compact hyperbolic surfaces X of genus $g > 1$:

- (1) The closed geodesics on X of length less than $\log(3 + 2\sqrt{2})$ are simple and disjoint.
- (2) There exists a simple closed geodesic on X with length $O(\log g)$.
- (3) If $(\gamma_i)_1^n$ is a binding collection of closed curves, then the locus in Teichmüller space \mathcal{T}_g where $\sum L(\gamma_i, X) \leq M$ is compact for any $M > 0$.¹
- (4) The union of the simple geodesics on $X = \Delta/\Gamma$ is a closed set of Hausdorff dimension one.

See [Bus, §4, §5], [Ker, Lemma 3.1] and [BS] for proofs. Thus simple cycles behave in many ways like simple closed geodesics.

¹ A collection of closed curves is *binding* if their geodesic representatives cut X into disks.

Rotation numbers and slopes. Next we formulate a more direct connection between short cycles and short geodesics. Suppose $f \in \mathcal{B}_d$ satisfies $\alpha = f'(0) = \exp(2\pi i \tau) \neq 0$. The action of $\langle f \rangle$ on Δ (with the orbit of $z = 0$ removed) determines a natural *quotient torus*, isomorphic to

$$X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) \cong \mathbb{C}^*/\alpha^{\mathbb{Z}}.$$

Let $L(p/q, X_\tau)$ denote the length of a closed geodesic on X_τ in the homotopy class $(-p, q)$, for the flat metric of area one. The slope $p/q \bmod 1$ which minimizes $L(p/q, X_\tau)$ depends only on $f'(0) \in \Delta^*$. The regions $T(p/q) \subset \Delta^*$ where a given slope is shortest rest on the corresponding roots of unity, and form a tiling of Δ^* (see Figure 1).

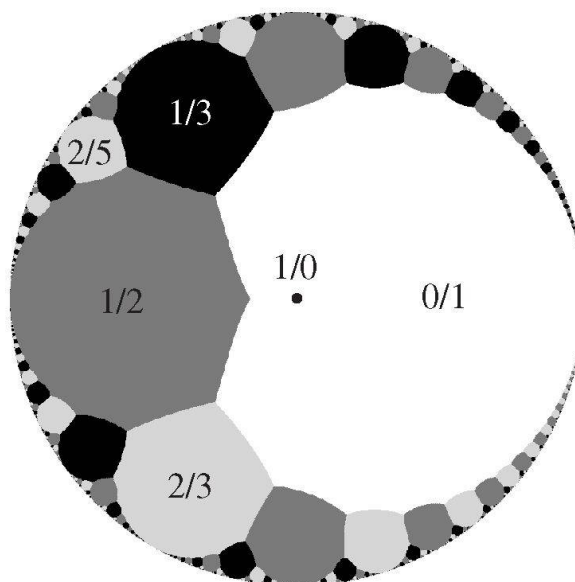


Figure 1. Tiling of Δ^* according to the slope of the shortest loop on the torus $\mathbb{C}^*/\alpha^{\mathbb{Z}}$.

In §6 we will show:

Theorem 1.5. *For any $f \in \mathcal{B}_d$ with $f'(0) \in T(p/q)$, there is a nonempty collection of compatible simple cycles C_i with rotation number p/q such that*

$$\frac{1}{L(p/q, X_\tau)^2} \leq \sum \frac{\pi}{L(C_i, f)} \leq \frac{1}{L(p/q, X_\tau)^2} + O(d),$$

and all other cycles satisfy $L(C, f) > \epsilon_d > 0$.

(Compatibility is defined in §2.) This result implies Theorem 1.2 and gives an alternate proof of Theorem 1.1 (with $\log 2$ replaced by ϵ_d); it also yields:

Corollary 1.6. *If a sequence $f_n \in \mathcal{B}_d$ satisfies $L(C, f_n) \rightarrow 0$, then $f'_n(0) \rightarrow \exp(2\pi i p/q)$ where p/q is the rotation number of C .*

On the other hand, we will see in §3:

Proposition 1.7. *If $f_n \in \mathcal{B}_d$ and $f'_n(0) \rightarrow \exp(2\pi i \theta)$ where θ is irrational, then $L(C, f_n) \rightarrow \infty$ for every cycle C .*

Thus the cycles of moderate length guaranteed by Theorem 1.2 may be forced to have very large periods.

Petals. The proof of Theorem 1.5 is illustrated in Figure 2. Consider a map $f \in \mathcal{B}_2$ with $f'(0) = \exp(2\pi i \tau) \in T(1/3)$, $\tau = 1/3 + i/10$. The dark petals shown in the figure form the preimage $\tilde{A} \subset \Delta$ of an annulus A in the homotopy class $[3\tau - 1]$ on the quotient torus for the attracting fixed point at $z = 0$. Any two adjacent rectangles within a petal give a fundamental domain for the action of f . The three largest petals join $z = 0$ to the repelling cycle on S^1 labeled by $C = (1/7, 2/7, 4/7)$. Thus a copy of A embeds in the quotient torus for the repelling cycle as well; by the method of extremal length (§5), this gives an upper bound for $L(C, f)$ in terms of $L(1/3, X_\tau)$. (The lower bound comes from the holomorphic Lefschetz fixed-point theorem.)



Figure 2. Petals joining $z = 0$ to the $(1, 2, 4)/7$ cycle on S^1 .

Rational maps. Here is a related result from §5 for general rational maps $f: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. Let $L(f) = \inf \log |\beta|$, where β ranges over the multipliers of all repelling and indifferent periodic cycles for f .

Theorem 1.8. *If $f_n \in \text{Rat}_d$ and $L(f_n) \rightarrow \infty$, then the maps f_n have fixed points z_n with $f'(z_n) \rightarrow 0$.*

Questions. We conclude with some natural questions suggested by the analogy with hyperbolic surfaces.

- (1) Let C be a simple cycle. Is the function $L(C, f)$ free of critical points in \mathcal{B}_d ?
- (2) Let (C_i) be a binding collection of cycles. Does $\sum L(C_i, f)$ achieve its minimum at a unique point $f \in \mathcal{B}_d$?
- (3) Let \mathcal{QB}_d denote the space rational maps of the form

$$f(z) = z \prod_{i=1}^{d-1} \left(\frac{z - a_i}{1 - \bar{b}_i z} \right)$$

such that $\prod |a_i| < 1$, $\prod |b_i| < 1$, and $J(f)$ is a Jordan curve. Each $f \in \mathcal{QB}_d$ can be regarded as a *marked quasiblaschke product*, obtained by gluing together a pair of maps $f_1, f_2 \in \mathcal{B}_d$ using their markings on S^1 .

Does there exist an $\epsilon_d > 0$ such for all $f \in \mathcal{QB}_d$, all cycles of length shorter than ϵ_d are simple?

- (4) Suppose the cycles (C_1, C_2) are binding. Does the set of $f \in \mathcal{QB}_d$ with $L(C_1, f_1) + L(C_2, f_2) \leq M$ have compact closure in the moduli space of all rational maps of degree d ?

The analogous questions for hyperbolic surfaces and quasifuchsian groups are known to have positive answers [Ker, §3], [Ot], [Th, Theorem 4.4].

Notes and references. This paper is a sequel to [Mc4] and [Mc5] in which we construct a Weil–Petersson metric on \mathcal{B}_d and an embedding of \mathcal{B}_d into the space of invariant measures for $p_d(z) = z^d$.

Simple cycles in degree two play a central role in the combinatorics of the Mandelbrot set [DH], [Ke], and are studied for higher degree in [Gol] and [GM]. Extremal length arguments similar to those we use in §5 are well-known both in the theory of Kleinian groups [Bers, Theorem 3], [Th, Proposition 1.3], [Mc1, §6.3], [Pet1], [Mil2] and rational maps [Pom], [Lev], [Hub], [Pet2]. The quotient Riemann surface of a general rational map is discussed in [McS]; other aspects of the dictionary between rational maps and Kleinian groups are presented in [Mc2]. See [PL] for a related discussion of spinning degenerations of the quotient torus.

2. Simple cycles

In this section we discuss the combinatorics of periodic cycles for the map $p_d(t) = d \cdot t \bmod 1$, and prove the closure of the simple cycles has Hausdorff dimension zero.

Degree and rotation number. Let $S^1 = \mathbb{R}/\mathbb{Z}$. Given $a \neq b \in S^1$, let $[a, b] \subset S^1$ denote the unique subinterval that is positively oriented from a to b . We write $a < c < b$ if $c \in [a, b]$. The length of an interval is denoted $|I|$.

Let $f: S^1 \rightarrow S^1$ be a topological covering map of degree $d > 0$, and suppose $f(X) = X$. The *degree* of $f|X$, denoted $\deg(f|X)$, is the least $e > 0$ such that $f|X$ extends to a topological covering $g: S^1 \rightarrow S^1$ of degree e .

Note that $\deg(f|X) = 1$ iff f preserves the cyclic ordering of X , in which case $f|X$ also has a well-defined *rotation number* $\rho(f|X) \in S^1$. If X is finite then $\rho(f|X) = p/q$ is rational and the orbits of $f|X$ have size q .

Example. Suppose $X = \{x_0, x_1, \dots, x_n = x_0\}$ in increasing cyclic order, and $f|X$ is a permutation; then we have

$$\deg(f|X) = \sum_0^{n-1} |[f(x_i), f(x_{i+1})]|.$$

Indeed, an extension of $f|X$ of minimal degree is obtain by mapping $[x_i, x_{i+1}]$ homeomorphically to $[f(x_i), f(x_{i+1})]$. The degree is thus a variant of the number of *descents* of a permutation (see e.g. [St, §1.3]).

The model map and its modular group. Now fix $d > 1$, and let $p_d(t) = d \cdot t \bmod 1$. Any expanding map $f: S^1 \rightarrow S^1$ of degree d is topologically conjugate to p_d [Sh].

The *modular group* $\text{Mod}_d \subset \text{Aut}(S^1)$ is the cyclic group of rotations generated by $t \mapsto 1/(d-1) + t \bmod 1$; it coincides with the group of (degree one) topological automorphisms of p_d . Note that Mod_d acts transitively on the fixed points of p_d .

Simple cycles. A finite set $C \subset S^1$ is a *cycle* of degree d if $p_d|C$ is a transitive permutation. As in §1, we say a cycle is *simple* if $\deg(p_d|C) = 1$. Simple cycles (C_1, \dots, C_m) are *compatible* if $\deg(p_d|\bigcup C_i) = 1$.

It is elementary to see:

Proposition 2.1. *The simple cycles (C_1, \dots, C_m) are compatible iff they are pairwise compatible.*

We let \mathcal{C}_d denote the set of all cycles of degree d , and $\mathcal{C}_d(p/q) \subset \mathcal{C}_d$ the simple cycles with rotation number p/q .

Portraits of fixed points. The *fixed-point portrait* [Gol] of a simple cycle $C \in \mathcal{C}_d(p/q)$ is the monotone increasing function

$$\sigma: \{1, \dots, d-2\} \rightarrow \{0, 1, \dots, q\}$$

given by

$$\sigma(j) = |C \cap [0, j/(d-1))|.$$

This invariant specifies how C is interleaved between the fixed points of p_d , which are all of the form $j/(d-1) \bmod 1$.

Basic properties. The following results are immediate from [Gol] (see especially Lemma 2 and Theorem 7).

- (1) A simple cycle $C \in \mathcal{C}_d(p/q)$ is uniquely determined by its fixed-point portrait $\sigma(j)$, and all possible monotone increasing functions $\sigma(j)$ arise.
- (2) The number of simple cycles of degree d and rotation number p/q is $\binom{d+q-2}{q}$.
- (3) The number of cycles of period q grows like d^q , while the number of simple cycles is $O(q^{d-1})$; so most cycles are not simple.
- (4) Cycles $C_1, C_2 \in \mathcal{C}_d(p/q)$ are compatible iff their fixed-point portraits satisfy

$$\sigma_1(j) \leq \sigma_2(j) \leq \sigma_1(j) + 1$$

for $0 \leq j \leq d-2$, or the same with σ_1 and σ_2 reversed.

- (5) Every maximal collection of compatible cycles has cardinality $d-1$.

From portraits to cycles. A simple cycle $C \in \mathcal{C}_d$ can be reconstructed explicitly from its rotation number p/q and its fixed-point portrait σ as follows. Let τ be the ‘transpose’ of σ , namely the monotone function $\tau: \{0, 1, \dots, q-1\} \rightarrow \{0, 1, \dots, d-1\}$ given by

$$\tau(i) = |\{j : \sigma(j) \leq i\}|, \quad (2.1)$$

and let

$$\tau'(i) = \tau(i) + \begin{cases} 0 & \text{if } 0 \leq i < q-p, \text{ and} \\ 1 & \text{otherwise,} \end{cases}$$

where i is taken $\bmod q$. Then the periodic point given by $t = 0.\tau'(0)\tau'(p)\tau'(2p)\dots$ in base d generates C ; indeed, t is the ‘first point’ in the cycle C .

Examples. To simplify notation, let $(p_1/q, \dots, p_m/q) = (p_1, \dots, p_m)/q$, and let $\sigma = n_1 \dots n_{d-1}$ denote the function with values $\sigma(j) = n_j$.

Degree $d = 2$. In the quadratic case, σ is trivial and hence there is a unique simple cycle $C(p/q)$ for each possible rotation number; e.g.

$$C(1/2) = (1, 2)/3,$$

$$C(1/3) = (1, 2, 4)/7,$$

$$C(2/5) = (5, 10, 20, 9, 18)/31.$$

The only cycle of period ≤ 4 which is not simple is $C = (1, 2, 4, 3)/5$. For period 5 there are two such, namely C and $-C$ where $C = (3, 6, 12, 24, 17)/31$. Any two distinct quadratic simple cycles are incompatible.

Degree $d = 3$. In the cubic case p_d has two fixed points, 0 and $1/2$, and three cycles of period two, given by

$$C(1/2, 0) = (5, 7)/8,$$

$$C(1/2, 1) = (1, 3)/4$$

and

$$C(1/2, 2) = (1, 3)/8.$$

The first and last are incompatible, while the other pairs are compatible. In general there are $q + 1$ cubic simple cycles with rotation number p/q , whose fixed-point portraits are given by $\sigma(1) = 0, 1, \dots, q$. Only the pairs with adjacent values of $\sigma(1)$ are compatible.

Degree $d = 4$. In the quartic case there are six cycles in $\mathcal{C}_4(1/2)$, generated by $t = p/15$ with $p = 1, 2, 3, 6, 7$ and 11. The compatibility relation between these cycles is shown in Figure 3. The four visible triangles give the four distinct triples of compatible simple cycles with rotation number $1/2$. Note that the modular group $\text{Mod}_4 \cong \mathbb{Z}/3$ acts by rotations on this diagram.

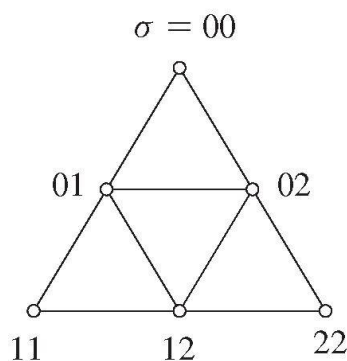


Figure 3. Compatibility of degree 4 cycles of the form $C(1/2, \sigma)$.

In general $\mathcal{C}_d(p/q)$ can be identified with the vertices of the q -fold barycentric subdivision of a $(d - 2)$ -simplex, with the top-dimensional cells corresponding to maximal collections of compatible cycles.

Sample computation in degree $d = 5$. To compute $C(3/7, 013)$, we first use equation (2.1) to compute the ‘transpose’ $\tau = 1223333$ of $\sigma = 013$. Note that the graphs of σ and τ , shown in white and black in Figure 4, fit together to form a rectangle. Evaluating $\tau' = 1223444$ along the sequence $ip \bmod q, i = 0, 1, 2, \dots$ we obtain the base 5 expansion $t = 0.\overline{1342424}_5 = 6966/19531$ for a generator of C .

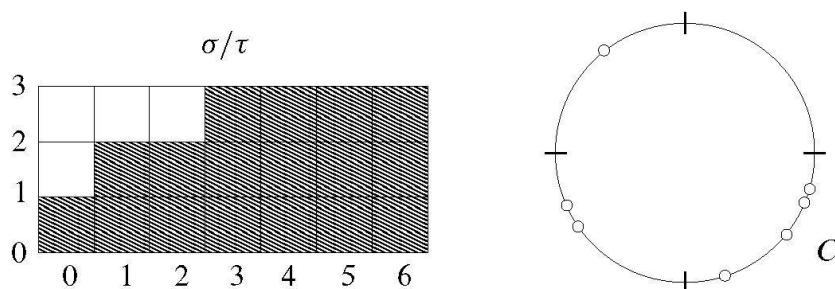


Figure 4. The degree 5 simple cycle with rotation number $3/7$ and $\sigma = 013$.

The cycle C , along with the four fixed points of p_5 , is drawn at the right in Figure 4. Note that $\sigma = 013$ gives the running total of the number of points of C in the first three quadrants.

Comparison with simple geodesics. The simple cycles for $p_d|S^1$ behave in many ways like simple closed geodesics on a compact hyperbolic surface $X = \Delta/\Gamma$ of genus g , with compatible cycles corresponding to disjoint geodesics. For example, every maximal collection of disjoint simple closed curves on X has $3g - 3$ elements, just as every maximal collection of compatible cycles for p_d has $d - 1$ elements.

It is also known that the endpoints of lifts of simple geodesics lie in a closed set $E \subset S^1$ of Hausdorff dimension zero [BS]. The analogous statement for simple cycles is:

Theorem 2.2. *The closure E of the union of all simple cycles $C \subset S^1$ of degree d has Hausdorff dimension zero.*

Proof. Let us say a finite set $P \subset S^1$ is a *precycle* if it is the forward orbit of preperiodic point $x \in S^1$ under p_d . We say P is *simple*, with rotation number p/q , if $p_d|P$ extends to a continuous, monotone increasing map $f: S^1 \rightarrow S^1$ with rotation number p/q . Then $q \leq n$ and the periodic part C of P is a simple cycle.

Let $\mathcal{P}_d(n, p/q)$ denote the set of all simple precycles of length n and rotation number p/q . The argument that shows $|\mathcal{C}_d(p/q)| = O(q^{d-2})$ can be adapted to show that $|\mathcal{P}_d(n, p/q)| = O(n^{d-2})$ as well.

Now fix $N > 0$. We claim that every $x \in E$ lies within distance $O(d^{-N})$ of a simple precycle P with $|P| \leq N$. To find this precycle, simply increase x continuously until two of the points among $x, f(x), \dots, f^N(x)$ coincide. This requires moving x only slightly, since $|(f^N)'(x)| = d^N$.

Thus E is contained in a neighborhood of diameter $O(d^{-N})$ of the union E_N of all simple precycles with $|P| \leq N$. Since $|E_N| = O(N^{d+2})$ grows only like a polynomial in N , this implies $\dim(E) = 0$. \square

Proof of Theorem 1.4. The Hölder continuous conjugacy ϕ_f between f and p_d preserves sets of Hausdorff dimension zero. \square

Remark: Invariant measures. The basic properties of simple cycles can also be developed using the correspondence between invariant measures and covering relations established in [Mc5]. For example, any union $D = \bigcup C_i$ of compatible cycles in $\mathcal{C}_d(p/q)$ arises as the support of an invariant measure ν for $p_d|S^1$. Invariant measures, in turn, correspond bijectively to covering relations (F, S) of degree d . In the case at hand, $F(t) = t + p/q \bmod 1$ and S is a divisor on S^1 of degree $d - 1$. By perturbing S so its points have multiplicity one, we obtain a nearby invariant measure ν' whose support $D' \supset D$ is a maximal union of exactly $(d - 1)$ compatible cycles (property (5) above).

The compactification of the space of Blaschke products by covering relations (F, S) is discussed in the following section.

Question. Is there a useful notion of intersection number for a pair of cycles?

3. Blaschke products

This section presents basic facts about marked Blaschke products, their derivatives and their images in the moduli space of all rational maps. See [Mc5] for related background material.

Blaschke products. Identify $S^1 = \mathbb{R}/\mathbb{Z}$ with the unit circle in the complex plane, using the coordinate $z = \exp(2\pi i t)$. Let $\Delta = \{z : |z| < 1\}$ be the unit disk, and $\Delta^{(n)}$ its n -fold symmetric product.

Given $d > 1$, let $\mathcal{B}_d \cong \Delta^{(d-1)}$ denote the space of Blaschke products $f : \Delta \rightarrow \Delta$ of the form

$$f(z) = z \prod_1^{d-1} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)$$

with $a_i \in \Delta$. Note that f extends to a rational map on the whole Riemann sphere, and $f|S^1$ is a covering map of degree d .

A proper holomorphic map $g : \Delta \rightarrow \Delta$ of degree $d > 1$ is conjugate to some $f \in \mathcal{B}_d$ iff g has a fixed point.

Derivatives and measure. By logarithmic differentiation, any $f \in \mathcal{B}_d$ satisfies

$$|f'(z)| = 1 + \sum_1^{d-1} \frac{1 - |a_i|^2}{|z - a_i|^2} \quad (3.1)$$

for $z \in S^1$. In particular, $f|S^1$ is expanding.

More importantly, $f|S^1$ preserves normalized Lebesgue measure λ on the circle; equivalently, $f_*(dz/z) = dz/z$, as can be verified by residue considerations. This means

$$\sum_{f(w)=z} |f'(w)|^{-1} = 1 \quad (3.2)$$

for any $z \in S^1$.

Markings. All $f \in \mathcal{B}_d$ are topologically conjugate to the model mapping $p_d(z) = z^d$. A *marking* for f the choice of one such conjugacy, i.e. the choice of a degree one homeomorphism $\phi: S^1 \rightarrow S^1$ such that

$$f(z) = \phi^{-1} \circ p_d \circ \phi(z).$$

There is a unique marking ϕ_f which varies continuously in f and satisfies $\phi_f(z) = z$ when $f = p_d$. Thus \mathcal{B}_d can be regarded as the space of *marked Blaschke products*.

The modular group $\text{Mod}_d \cong \mathbb{Z}/(d-1)$ acts on \mathcal{B}_d by $(a_i) \mapsto (\zeta a_i)$ where $\zeta^{d-1} = 1$. Its orbits correspond to different markings of the same map. Thus $f_1, f_2 \in \mathcal{B}_d$ are conformally conjugate on Δ iff they are in the same orbit of the modular group.

Lengths. The canonical marking allows one to label the cycles of f by the cycles of p_d . We define the *length on f* of a cycle $C \in \mathcal{C}_d$ of period q by

$$L(C, f) = \log |(f^q)'(z)|$$

for any $z \in S^1$ with $\phi_f(z) \in C$.

Limits of lower degree. The space of Blaschke products has a natural compactification $\bar{\mathcal{B}}_d \cong \bar{\Delta}^{(d-1)}$, whose boundary points (a_i) can be interpreted as pairs (F, S) consisting of a Blaschke product

$$F(z) = z \prod_{|a_i| < 1} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right) \cdot \prod_{|a_i| = 1} (-a_i)$$

and a divisor of *sources*

$$S = \sum_{|a_i| = 1} 1 \cdot a_i \in \text{Div}(S^1),$$

satisfying $\deg F + \deg S = d$. It is easy to see:

Proposition 3.1. *A sequence $f_n \in \mathcal{B}_d$ converges to $(F, S) \in \partial \mathcal{B}_d$ iff*

- (i) $f_n(z) \rightarrow F(z)$ uniformly on compact subsets of $\hat{\mathbb{C}} - \text{supp } S$, and
- (ii) the zeros $Z(f_n)$ converge to $Z(F) + S$ as divisors on $\hat{\mathbb{C}}$.

More generally, the space Rat_d of degree d rational maps $f: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ has a compactification $\overline{\text{Rat}}_d \cong \mathbb{P}^{2d+1}$, whose boundary points (F, S) are pairs consisting of a rational map F and an effective divisor $S \in \text{Div}(\hat{\mathbb{C}})$ with $\deg(F) + \deg(S) = d$. We have $f_n \rightarrow (F, S)$ in $\overline{\text{Rat}}_d$ iff their graphs satisfy

$$\text{gr}(f_n) \rightarrow \text{gr}(F) + S \times \hat{\mathbb{C}}$$

as divisors of degree $(1, d)$ on $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ (cf. [D, §1]).

Radial bounds on $f'(z)$. The following elementary observation is useful for studying limits as above.

Proposition 3.2. *For any proper holomorphic map $f: \Delta \rightarrow \Delta$ and $\zeta \in S^1$, we have*

$$\sup_{r \in [0,1]} |f'(r\zeta)| \leq 4|f'(\zeta)|.$$

Note that we do not require that $f(0) = 0$. This bound is sharp, as can be seen by considering $f(z) = (z + a)/(1 + az)$ as $a \rightarrow 1-$.

Proof. We can write

$$f(z) = e^{i\theta} \prod_1^d M_i(z), \quad (3.3)$$

where $M_i(z) = (z - a_i)/(1 - \bar{a}_i z)$ and $a_i \in \Delta$. Composing with a rotation, we can also assume that $\zeta = 1$. For $r \in [0, 1]$ we have

$$\left| \frac{M'_i(r)}{M'_i(1)} \right| = \frac{|1 - a_i|^2}{|1 - ra_i|^2},$$

and therefore

$$|M'_i(r)| \leq 4|M'_i(1)|,$$

since the distance from 1 to a_i is never more than twice the distance from 1 to ra_i . Differentiating the product (3.3) and using the fact that $|\prod_{j \neq i} M_j(r)| \leq 1$, we obtain:

$$|f'(r)| \leq \sum |M'_i(r)| \leq 4 \sum |M'_i(1)| = 4|f'(1)|.$$

□

Corollary 3.3. *If $f_n \rightarrow (F, S) \in \overline{\mathcal{B}}_d$, $z_n \in S^1$, $z_n \rightarrow z$ and $|f'_n(z_n)| = O(1)$, then $\lim f_n(z_n) = F(z)$.*

Proof. Suppose $\sup |f'_n(z_n)| = M$; then for any $r < 1$ we have

$$\begin{aligned} \limsup |f_n(z_n) - F(z)| &\leq \limsup |f_n(rz_n) - F(z)| + 4M(1-r) \\ &= |F(rz) - F(z)| + 4M(1-r); \end{aligned}$$

now let $r \rightarrow 1$. □

Irrational rotations. As a sample application, we prove the following result stated in the Introduction:

Corollary 3.4. *If $f_n \in \mathcal{B}_d$ satisfies $f'_n(0) \rightarrow \exp(2\pi i\theta)$ where θ is irrational, then $L(C, f_n) \rightarrow \infty$ for every cycle C .*

Proof. Suppose to the contrary that $L(C, f_n)$ is bounded for some cycle C . Let $C_n \subset S^1$ be the corresponding periodic cycle for f_n . Pass to a subsequence such that $f_n \rightarrow (F, S) \in \partial\mathcal{B}_d$ and $C_n \rightarrow D \subset S^1$ in the Hausdorff topology. Then $F(z) = \exp(2\pi i\theta)z$ and by Corollary 3.3 we have $F(D) = D$, contradicting the irrationality of θ . □

Variants. Here are two useful variants of the results above:

Proposition 3.5. *For any proper holomorphic map $f: \mathbb{H} \rightarrow \mathbb{H}$ and $x \in \mathbb{R}$, we have*

$$\sup_y |f'(x + iy)| \leq f'(x).$$

Proposition 3.6. *Assume $f_n \in \text{Rat}_d$ converges to $(F, S) \in \overline{\text{Rat}}_d$, $z_n \rightarrow z$, and $\|Df_n(z_n)\| = O(1)$ in the spherical metric on $\hat{\mathbb{C}}$. Then we have*

$$f_n(z_n) \rightarrow F(z)$$

provided z_n belongs to a circle T_n with $f_n^{-1}(T_n) = T_n$, and $\inf_n \text{diam}(T_n) > 0$.

Proofs. The first result follows directly from the representation $f(z) = a_0z + b_0 + \sum_{i=1}^{d-1} a_i/(b_i - z)$ with $a_i > 0$ and $b_i \in \mathbb{R}$, and the second follows by the same argument as Corollary 3.3. □

The maps $f_n(z) = 1/(1 + nz^2)$ satisfy $f'_n(0) = 0$ and $\lim f_n(0) = 1 \neq F(0) = 0$; thus some extra hypothesis is needed to interchange limits as in Proposition 3.6.

Moduli space of rational maps. Let $\text{MRat}_d = \text{Rat}_d / \text{Aut}(\widehat{\mathbb{C}})$ denote the moduli space of holomorphic conjugacy classes of rational maps of degree $d > 1$. A pair of Blaschke products are conjugate iff they are related by the modular group or by $z \mapsto 1/\bar{z}$; thus we have an inclusion

$$\mathcal{B}_d / (\text{Mod}_d \ltimes \mathbb{Z}/2) \hookrightarrow \text{MRat}_d.$$

The next result shows this inclusion is almost proper.

Theorem 3.7. *If $f_n \rightarrow (F, S) \in \partial \mathcal{B}_d$ but $[f_n]$ remains bounded in MRat_d , then $F(z) = z$ and $\text{supp } S$ is a single point. In particular, we have $f'_n(0) \rightarrow 1$.*

Proof. Pass to a subsequence such $[f_n] \rightarrow [g] \in \text{MRat}_d$ and $f_n \rightarrow (F, S) \in \partial \mathcal{B}_d$. Then there are conjugates $h_n = A_n f_n A_n^{-1} \rightarrow g$. Since f_n diverges in \mathcal{B}_d , $A_n \rightarrow \infty$ in $\text{Aut}(\widehat{\mathbb{C}})$. On the other hand, the measures of maximal entropy satisfy $\mu(h_n) \rightarrow \mu(g)$ and $\mu(f_n) \rightarrow \mu(F, S)$, by [D, Theorem 0.1] (see also [Mc5]). Since $\mu(g)$ is nonatomic, this implies $\mu(F, S) = \lim A_n^*(\mu(h_n))$ is supported at a single point. But $\text{supp } \mu(F, S)$ is F -invariant and includes $\text{supp } S$; thus $F(z) = z$ and $\text{supp } S = \{s\}$ is itself a single point. \square

Example. The sequence $f_n(z) = z(z + a_n)/(1 + a_n z)$, with $a_n = 1 - 1/n$, is divergent in \mathcal{B}_2 but convergent in MRat_2 . To see this, normalize so the origin is a critical point instead of a fixed point; then $f_n(z)$ is conjugate to $h_n(z) = (z^2 + b_n)/(1 + b_n z^2)$, and $b_n = a_n/(2 + a_n) \rightarrow 1/3$ as $a_n \rightarrow 1$.

4. The thin part of $f(z)$

Let us define the *thin part* of $f \in \mathcal{B}_d$ by

$$S_{\text{thin}}^1(f) = \{z \in S^1 : |f'(z)| < 2\}.$$

In this section we will show:

Theorem 4.1. *For any $f \in \mathcal{B}_d$, the map $f|_{S_{\text{thin}}^1(f)}$ extends to a degree one homeomorphism of the circle.*

Corollary 4.2. *All cycles of f with $L(C, f) < \log 2$ are simple and compatible.*

Visual angles. The derivative of

$$f(z) = z \prod_1^{d-1} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)$$

can be conveniently analyzed using the *hyperbolic visual angle*, defined for $a, z \in \bar{\Delta}$ by

$$\alpha(z, a) = 2 \arg(z - a) - \arg(z).$$

This is the angle at a of the hyperbolic geodesic \overline{az} . For $z \in S^1$ we have $\arg(1 - \bar{a}z) = \arg(z) - \arg(z - a)$, and thus

$$\arg(f(z)) = \arg(z) + \sum_1^{d-1} \alpha(z, a_i). \quad (4.1)$$

(Note this simplifies to $\arg(f(z)) = 2 \arg(z - a_1)$ when $d = 2$.) Letting $\theta = \arg(z)$ and $\dot{\alpha} = d\alpha/d\theta$, we then obtain

$$|f'(z)| = 1 + \sum_1^{d-1} \dot{\alpha}(z, a_i) \quad (4.2)$$

for $z \in S^1$.

The visual density. The *visual density* $\dot{\alpha}(z, a)$ is essentially the Poisson kernel; for $a = r \geq 0$ it is given by

$$\dot{\alpha}(z, r) = \frac{1 - r^2}{1 + r^2 - 2r \cos \theta}, \quad (4.3)$$

where $\theta = \arg z$. Geometrically, $(\dot{\alpha}(z, a)/2\pi) d\theta$ is the hitting measure on the circle for a random hyperbolic geodesic starting at a .

For fixed $z \in S^1$, the level sets of $\dot{\alpha}(z, a)$ are horocycles resting on z . Thus

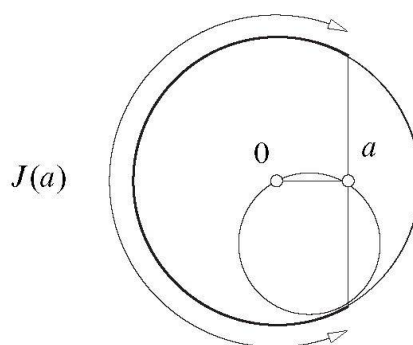
$$J(a) = \{z \in S^1 : \dot{\alpha}(z, a) < 1\}$$

is the large arc cut off by the chord perpendicular to $\overline{0a}$. This follows from the fact that the horocycle resting on one of the endpoints of $J(a)$ and passing through 0 also passes through a (see Figure 5).

Proposition 4.3. *The visual density $\dot{\alpha}(z, a)|_{J(a)}$ is strictly convex, and decreases as a moves radially towards the circle. In other words, we have*

$$\ddot{\alpha}(z, a) > 0 \quad \text{and} \quad \frac{d}{ds} \dot{\alpha}(z, sa) \Big|_{s=1} < 0$$

for all $z \in J(a)$.

Figure 5. The arc $J(a)$ where $\dot{\alpha}(z, a) < 1$.

Proof. To verify convexity, consider the case where $a = r \in [0, 1)$. By (4.3), in this case we have $\dot{\alpha} = (1 - r^2)/u$ where $u = 1 + r^2 - 2r \cos \theta$. We may assume $\theta \in (0, \pi)$. Cross-multiplying and differentiating, we obtain

$$\begin{aligned}\dot{\alpha}u &= 1 - r^2, \\ \ddot{\alpha}u + \dot{\alpha}(2r \sin \theta) &= 0, \quad \text{and} \\ \ddot{\alpha}u + \ddot{\alpha}(4r \sin \theta) + \dot{\alpha}(2r \cos \theta) &= 0.\end{aligned}$$

Since r, u and $\sin \theta$ are all positive, we have $\dot{\alpha} > 0$ and $\ddot{\alpha} < 0$. Comparing the last two equations, we find the sign of $\ddot{\alpha}$ is the same as the sign of the determinant

$$D = \det \begin{pmatrix} 2r \sin \theta & u \\ 2r \cos \theta & 4r \sin \theta \end{pmatrix} = 8r^2 \sin^2 \theta - 2ru \cos \theta.$$

We claim $D > 0$ when $z \in J(r)$, i.e. when $u = |z - r|^2 > 1 - r^2$. The claim is evident if $\cos \theta$ is negative, so assume $\theta \in (0, \pi/2)$; then

$$u = |z - r|^2 \leq |z - 1|^2 \leq 2(\operatorname{Im} z)^2 = 2 \sin^2 \theta.$$

We also have $\cos \theta = \operatorname{Re}(z) < r$ for $z \in J(r)$, and thus:

$$D \geq 4r^2u - 2r^2u > 0.$$

The proof of the density decreasing property is straightforward. \square

Properties of the thin part of f . We can now show that $f|_{S_{\text{thin}}^1(f)}$ acts like a rotation. We first observe:

Proposition 4.4. *For any $f \in \mathcal{B}_d$,*

- (i) *the map $f|_{S_{\text{thin}}^1(f)}$ is injective,*
- (ii) *we have $S_{\text{thin}}^1(f) \subset \bigcap J(a_i)$,*

- (iii) $S_{\text{thin}}^1(f)$ consists of at most $(d - 1)$ disjoint open intervals, and
- (iv) $S_{\text{thin}}^1(f)$ increases as the zeros a_i of f move radially towards the circle.

Proof. If $f(x_1) = f(x_2)$ for two distinct points in $S_{\text{thin}}^1(f)$, then $|f'(x_1)| + |f'(x_2)| > 1/2 + 1/2 = 1$, which violates the measure-preserving property (3.2) of f ; thus $f|_{S_{\text{thin}}^1(f)}$ is injective. Equation (4.2) implies (ii). Since $\bigcup (S^1 - J(a_i))$ has at most $(d - 1)$ components, so does $I = \bigcap J(a_i)$. By Proposition 4.3, $|f'(z)|$ is locally convex on I ; thus the intersection of $S_{\text{thin}}^1(f)$ with any component of I is connected, and (iii) follows. The density decreasing property stated in Proposition 4.3 implies (iv). \square

Proof of Theorem 4.1. By moving the points (a_i) radially to the circle, we obtain a smooth 1-parameter family of maps $f_t \in \bar{\mathcal{B}}_d$, $t \in [0, 1]$, with $f_0 = f$ and $f_1 = (F, S)$. Since $\deg(S) = d - 1$, we have $\deg(F) = 1$. Proposition 4.4 implies that $f_t|_{T_t} = S_{\text{thin}}^1(f_t)$ is injective, $T_s \subset T_t$ when $s < t$, and $\text{supp } S \cap T_t = \emptyset$. Thus for any three distinct points $x_i \in S_{\text{thin}}^1(f)$, the triple $(f_t(x_1), f_t(x_2), f_t(x_3))$ moves by isotopy as t increases from 0 to 1, and converges to $F(x_1), F(x_2), F(x_3)$ as $t \rightarrow 1$. Since F is a rotation, it preserves the cyclic ordering of the points (x_i) , so the same is true of f . Consequently f extends from $S_{\text{thin}}^1(f)$ to an orientation-preserving homeomorphism of the circle. \square

5. Bounds on repelling cycles

In this section we show that every $f \in \mathcal{B}_d$ has a simple cycle with $L(C, f) = O(d)$, and obtain related results for general rational maps.

Moduli and tori. We begin by summarizing some well-known facts about extremal length on tori.

Any point $\tau \in \mathbb{H}$ determines a complex torus

$$X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$$

with a flat metric inherited from the plane, and a distinguished basis $\langle 1, \tau \rangle$ for its fundamental group. Factoring the covering map $\mathbb{C} \rightarrow X_\tau$ through the map $\xi: \mathbb{C} \rightarrow \mathbb{C}^* \cong \mathbb{C}/\mathbb{Z}$ given by $\xi(z) = \exp(2\pi iz)$, we have

$$X_\tau = \mathbb{C}^*/\alpha^\mathbb{Z}$$

where $\alpha = \xi(\tau)$ satisfies $0 < |\alpha| < 1$. The same construction can be made when $-\tau \in \mathbb{H}$; then $|\alpha| > 1$.

Given a slope $p/q \in \mathbb{Q} \cup \{\infty\}$, let $\gamma_{p/q} \subset X_\tau$ denote the simple closed geodesic obtained as the projection of the line $\mathbb{R} \cdot (\tau - p/q)$ from \mathbb{C} to X_τ . Its preimage $\tilde{\gamma}_{p/q}$

in the intermediate cover \mathbb{C}^* consists of q arcs joining 0 to ∞ , cyclically permuted with rotation number p/q by $z \mapsto \alpha z$.

Any annulus A is conformally equivalent to a right cylinder, which is unique up to scale. The ratio $\text{mod}(A) = h/c$ between the height and circumference of this cylinder is the *modulus* of A .

The maximum modulus of an annulus $A \subset X_\tau$ homotopic to $\gamma_{p/q}$ is given by

$$\text{mod}(p/q, X_\tau) = \frac{\text{area}(X_\tau)}{L(\gamma_{p/q}, X_\tau)^2} = \frac{|\text{Im } \tau|}{|q\tau - p|^2} \quad (5.1)$$

(assuming $\gcd(p, q) = 1$). This maximum is realized by taking $A = X_\tau \setminus \gamma_{p/q}$. The set of $\tau \in \mathbb{H}$ with $\text{mod}(p/q, X_\tau) \geq m$ is a horoball of diameter $1/(mq^2)$ resting on the real axis at p/q . For $p/q = 1/0$ we have

$$\text{mod}(\infty, X_\tau) = |\text{Im } \tau|.$$

The *intersection inequality*

$$\text{mod}(p/q, X_\tau) \text{mod}(r/s, X_\tau) \leq \left(\det \begin{pmatrix} p & q \\ r & s \end{pmatrix} \right)^{-2} \quad (5.2)$$

is easily verified by considering the determinant of the lattice $\mathbb{Z}(q\tau - p) \oplus \mathbb{Z}(s\tau - r)$. This inequality implies:

There is at most one slope with $\text{mod}(p/q, X_\tau) > 1$.

On the other hand we have:

Proposition 5.1. *For any $\tau \in \mathbb{H}$, there exists a slope $p/q \in \mathbb{Q} \cup \{\infty\}$ such that*

$$\text{mod}(p/q, X_\tau) \geq \sqrt{3}/2.$$

Proof. Since the statement is invariant under the action of $\text{SL}_2(\mathbb{Z})$ on \mathbb{H} , it suffices to verify it when τ lies in the fundamental domain $|\tau| \geq 1$, $|\text{Re } \tau| \leq 1/2$; and in this case, we have $\text{mod}(\infty, X_\tau) = \text{Im } \tau \geq \sqrt{3}/2$. \square

Rational maps. Now let $f: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational map of degree $d > 1$. If $z \in \hat{\mathbb{C}}$ is a point of period q , its *multiplier* is given by $\beta = (f^q)'(z)$. The *grand orbit* of z is the set $\bigcup_{i,j \geq 0} f^{-i} \circ f^j(z)$.

Suppose f has a fixed point at $z = 0$ and a periodic point $w \neq 0$ with period q . We say w has *rotation number* p/q relative to $z = 0$ if there are arcs $(\delta_i)_0^{q-1} \subset \hat{\mathbb{C}}$ joining $z = 0$ to $f^i(w)$, meeting only at $z = 0$, which are cyclically permuted by f with rotation number p/q .

Theorem 5.2. *Let f be a rational map with an attracting fixed-point at $z = 0$, with multiplier*

$$\alpha = f'(0) = \exp(2\pi i \tau) \neq 0.$$

Let e be the number of grand orbits of critical points in the immediate basin Ω of $z = 0$. Then for each $p/q \in \mathbb{Q}$, there exists a repelling or parabolic periodic point $w \in \partial\Omega$ such that

- (1) *the rotation number of w relative to $z = 0$ is p/q ; and*
- (2) *its multiplier has the form $\beta = (f^q)'(w) = \exp(-2\pi i \sigma)$, where $\sigma = 0$ or*

$$\frac{\operatorname{Im} \sigma}{|\sigma|^2} \geq \frac{\operatorname{mod}(p/q, X_\tau)}{e}. \quad (5.3)$$

In particular, we have

$$|\beta| \leq \left(\exp \left(\frac{2\pi}{\operatorname{mod}(p/q, X_\tau)} \right) \right)^e. \quad (5.4)$$

Proof. Let Ω^* denote the immediate basin of $z = 0$ with the grand orbits of all critical points in Ω and of $z = 0$ deleted. Then $f: \Omega^* \rightarrow \Omega^*$ is a covering map. Moreover, the holomorphic linearizing map

$$\phi(z) = \lim \alpha^{-n} f^n(z)$$

is defined for all $z \in \Omega^*$, and satisfies $\phi(f(z)) = \alpha \phi(z)$. Consequently ϕ descends to an inclusion of the space of grand orbits $Y = \Omega^* / \langle f \rangle$ into the torus $X_\tau = \mathbb{C}^* / \alpha^\mathbb{Z}$, making the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{\phi} & \mathbb{C}^* \\ \downarrow & & \downarrow \\ \Omega / \langle f \rangle = Y & \hookrightarrow & X_\tau = \mathbb{C}^* / \alpha^\mathbb{Z} \end{array}$$

commute. By assumption we have $|Y - X_\tau| = e$.

For a given $p/q \in \mathbb{Q}$, the geodesics parallel to $\gamma_{p/q}$ passing through the punctures of Y cut it into $\leq e$ parallel annuli, one of which satisfies

$$\operatorname{mod}(A) \geq \operatorname{mod}(p/q, X_\tau)/e. \quad (5.5)$$

Let $\delta \subset A$ be the core curve of A , and $\delta_0 \subset \Omega^*$ one of its lifts which is incident to $z = 0$. Let $\delta_i = f^i(\delta_0)$. By construction, the arc δ_0 is invariant under f^q , and $f^q|_{\delta_0}$ is a bounded translation in the hyperbolic metric on Ω^* . Consequently δ_0 must join

$z = 0$ to another fixed point w of f^q in $\partial\Omega$. By the Snail Lemma [Mil1, Lem. 16.2], w is repelling or parabolic.

We have seen that the preimage of $\gamma_{p/q}$ on \mathbb{C}^* consists of q arcs, cyclically permuted with rotation number p/q by $z \mapsto \alpha z$. Since ϕ is a homeomorphism near $z = 0$, the arcs $\delta_0, \dots, \delta_{q-1}$ are also cyclically permuted with rotation number p/q by f . In particular w has rotation number p/q relative to $z = 0$.

Now suppose w is repelling, with multiplier β . Choose an injective branch of f^{-q} defined on a punctured neighborhood U^* of w such that $f^{-q}: U^* \rightarrow U^*$ and

$$Z = U^*/\langle f^{-q} \rangle \cong \mathbb{C}^*/\beta^{\mathbb{Z}} = X_\sigma,$$

where $\sigma = \log(\beta/2\pi i)$. There is a unique choice of the logarithm such that the invariant arc $\delta_0 \cap U^*$ descends to a loop isotopic to γ_0 on X_σ .

By construction, $A \subset Y$ is covered by a strip $A_0 \subset \Omega^*$ which retracts to δ_0 , and hence we have an inclusion

$$A \cong A_0/\langle f^q \rangle \hookrightarrow Z \cong X_\sigma$$

in the same homotopy class as γ_0 . This implies

$$\text{mod}(0, X_\sigma) \geq \text{mod}(A),$$

and the bound (5.3) follows from equations (5.1) and (5.5). \square

Corollary 5.3. *If $f \in \text{Rat}_d$ has an attracting fixed point with multiplier satisfying*

$$|\alpha| > \exp(-\pi\sqrt{3}) = 0.0043\dots,$$

then it also has a repelling or parabolic cycle with multiplier satisfying

$$|\beta| \leq \exp(4\pi/\sqrt{3})^{2d-2} \leq 1416^{2d-2}.$$

Proof. The lower bound on $|\alpha|$ implies $\text{Im}(\tau) = \text{mod}(\infty, X_\tau) < \sqrt{3}/2$, where $\tau = (\log \alpha)/2\pi i$. Hence $\text{mod}(p/q, X_\tau) \geq \sqrt{3}/2$ for some $p/q \in \mathbb{Q}$, by Proposition 5.1. Now apply equation (5.4) and note that $e \leq 2d - 2$. \square

Corollary 5.4. *If a map $f \in \text{Rat}_d$ has an attracting fixed point with multiplier α , then it also has a repelling or parabolic cycle with multiplier satisfying*

$$|\beta| \leq (\exp(4\pi/\sqrt{3})/|\alpha|)^{2d-2}.$$

Proof. Choose $\tau = (\log \alpha)/2\pi i = x + iy$ with $x \in [-1/2, 1/2]$. The previous corollary shows the desired bound holds when $y < \sqrt{3}/2$. For $y \geq \sqrt{3}/2$ we have

$$m = \text{mod}(0, X_\tau)^{-1} \leq \frac{x^2 + y^2}{y} \leq \frac{1}{2\sqrt{3}} + y < \frac{2}{\sqrt{3}} + y,$$

which implies $\exp(2\pi/m) \leq \exp(4\pi/\sqrt{3})/|\alpha|$; thus by (5.4) the desired bound holds in this case as well. \square

The bottom of the spectrum. Here is a qualitative consequence of the preceding corollary.

Let the *spectrum* $S(f) \subset \mathbb{C}$ be the set of all multipliers β that arise from periodic points of $f \in \text{Rat}_d$, and let

$$L(f) = \inf \{ \log |\beta| : \beta \in S(f) \text{ and } |\beta| \geq 1 \}.$$

By the fixed-point formula for rational maps [Mil1, Theorem 12.4], the multipliers of f at its fixed points satisfy

$$\sum \frac{1}{\mu_j - 1} = 1, \quad (5.6)$$

provided no $\mu_j = 1$; in particular, $|\mu_j| \leq d + 1$ for some j . Thus if f has no attracting fixed points, it satisfies

$$L(f) \leq \log(d + 1).$$

Combining this observation with Corollary 5.4, we obtain:

Corollary 5.5. *Let $f_n \in \text{Rat}_d$ be a sequence of rational maps with $L(f_n) \rightarrow \infty$. Then the maps f_n have fixed points with multipliers $\alpha_n \rightarrow 0$.*

Examples. It is easy to see that $f_n(z) = z^2 + n^2$ satisfies $L(f_n) \rightarrow \infty$ as $n \rightarrow \infty$, since its Julia set lies close to $\pm n$. Of course f_n has a fixed point at infinity with multiplier $\alpha_n = 0$.

Parabolics must be included in the definition of $L(f)$ to obtain Corollary 5.5. In fact, if we let $L^*(f) = \inf \{ \log |\beta| : \beta \in S(f), |\beta| > 1 \}$, then $f_n(z) = z - 1/z + n$ satisfies $L^*(f_n) \rightarrow \infty$ even though f_n has no attracting fixed point. (The map $f_n(z)$ behaves like the Hecke group $\langle z \mapsto -1/z, z \mapsto z + n \rangle$; cf. [Mc3, Theorem 6.2].)

Question. Does Corollary 5.5 remain true if only parabolic and repelling multipliers are included in the definition of $L(f)$?

Blaschke products. We now return to the setting of a proper map $f: \Delta \rightarrow \Delta$ fixing $z = 0$. In this case formula (5.6) implies:

Proposition 5.6. *The multipliers $(\lambda_i)_1^{d-1}$ of $f \in \mathcal{B}_d$ at its fixed points on the circle satisfy*

$$\sum_1^{d-1} \frac{1}{\lambda_i - 1} = \frac{1 - |\alpha|^2}{|1 - \alpha|^2},$$

where $\alpha = f'(0)$.

Corollary 5.7. *If $|\alpha| < 1/2$, then f has a repelling fixed point with multiplier satisfying $1 < \beta \leq 1 + (d - 1)/3$.*

Theorem 5.8. *Every $f \in \mathcal{B}_d$ has a simple cycle with $L(C, f) = O(d)$.*

Proof. Combine Corollaries 5.4 and 5.7. □

6. Short cycles and short geodesics

In this section we use the fixed-point formula for rational maps to obtain the following more detailed connection between the short cycles for f and the short geodesics on its quotient torus.

Theorem 6.1. *Given $f \in \mathcal{B}_d$ with $f'(0) = \exp(2\pi i \tau)$, choose $p/q \in \mathbb{Q}$ to maximize $\text{mod}(p/q, X_\tau)$. Then there exist compatible simple cycles C_i with rotation number p/q , such that*

(1) *their lengths satisfy*

$$\text{mod}(p/q, X_\tau) \leq \pi \sum L(C_i, f)^{-1} \leq \text{mod}(p/q, X_\tau) + O(d); \quad (6.1)$$

(2) *all other cycles satisfy $L(C, f) > \epsilon_d > 0$; and*

(3) *for any $r > 0$, the multipliers of f^r at its repelling fixed points satisfy*

$$\frac{1}{r} \sum' \frac{1}{\lambda_j - 1} = O(d), \quad (6.2)$$

where the prime indicates that fixed points in $\bigcup C_i$ are excluded.

In qualitative terms, the construction shows:

Corollary 6.2. *All cycles with $L(C, f) < \epsilon_d$ arise from short geodesics on the quotient torus for f .*

Tiling of Δ^* . The slope $p/q \bmod 1$ appearing in the theorem above depends only on $\alpha = f'(0) \in \Delta^*$. Figure 1 of the Introduction shows the regions $T(p/q) \subset \Delta^*$ where a given slope maximizes the value of $\text{mod}(p/q, X_\tau) = \text{mod}(p/q, \mathbb{C}/\alpha\mathbb{Z})$.

This picture is nothing more than the image, under the covering map $\xi: \mathbb{H} \rightarrow \Delta^*$ given by $\xi(\tau) = \exp(2\pi i \tau)$, of the tiling of \mathbb{H} by $\text{SL}_2(\mathbb{Z})$ translates of the Dirichlet region

$$F = \{\tau \in \mathbb{H} : |\tau - n| \geq 1 \ \forall n \in \mathbb{Z}\}$$

for the cusp $\tau = \infty$. The tile $T(\infty) = \xi(F)$ lies in a ball of radius $\exp(-\pi\sqrt{3}) \approx 1/230$ about the origin. In this tile the short curve is $\gamma_\infty \subset X_\tau$, which lifts to a loop around $z = 0$ rather than a path connecting $z = 0$ to a periodic point. Thus the length of γ_∞ can go to zero without any cycle getting short.

Each remaining tile $T(p/q)$ contains a horocycle H resting on the root of unity $\exp(2\pi i p/q) \in S^1$. Within a still smaller horocycle $H' \subset H$, $\gamma_{p/q}$ becomes very short, and hence f has a very short cycle with rotation number p/q .

Moduli and multipliers. We begin the proof of Theorem 6.1 by connecting Diophantine properties of $\alpha \in \Delta^*$ to lengths of geodesics on $\mathbb{C}^*/\alpha^{\mathbb{Z}}$.

Lemma 6.3. *For any $\alpha = \exp(2\pi i \tau) \in \Delta^*$ and $q > 0$, we have*

$$\sup_p \frac{\text{mod}(p/q, X_\tau)}{\gcd(p, q)^2} = \frac{\pi}{q} \frac{1 - |\alpha^q|^2}{|1 - \alpha^q|^2} + O(1).$$

Proof. First consider the case $q = 1$, and assume τ is chosen so $|\text{Re } \tau| \leq 1/2$. Then we have $2\pi i \tau \approx 1 - \alpha$ when either side is small, and hence

$$\sup_p \text{mod}(p, X_\tau) = \frac{\text{Im } \tau}{|\tau|^2} = \pi \frac{1 - |\alpha|^2}{|1 - \alpha|^2} + O(1).$$

The general case follows using the fact that

$$\frac{\text{mod}(p/q, X_\tau)}{\gcd(p, q)^2} = \frac{\text{mod}(p, X_{q\tau})}{q}. \quad \square$$

Proof of Theorem 6.1. Choose p so that $\text{mod}(p/q, X_\tau)$ is maximized. As in Theorem 5.2, by cutting the torus X_τ open along $e \leq d - 1$ geodesics parallel to $\gamma_{p/q}$ we obtain annuli $A_1, \dots, A_e \subset Y$ with

$$\text{mod}(p/q, X_\tau) = \sum \text{mod}(A_i).$$

Each annulus A_i , when lifted to the unit disk, connects $z = 0$ to a simple cycle C_i for f with rotation number p/q and multiplier $\beta_i > 1$.

The lifts of the annuli A_i are disjoint, so the cycles C_i are compatible. Assume for the moment they are also distinct. Since two copies of A_i embed in the quotient torus $\mathbb{C}^*/\beta_i^{\mathbb{Z}}$ (one for the inside of the disk and one for the outside), we have

$$2\text{mod}(A_i) \leq \frac{2\pi}{\log \beta_i} = \frac{2\pi}{L(C_i, f)}.$$

The combination of these inequalities yields:

$$\text{mod}(p/q, X_\tau) \leq \pi \sum L(C_i, f)^{-1}.$$

This lower bound also holds when the cycles are not distinct; then we simply have more annuli A_i embedded in a given torus $\mathbb{C}^*/\beta_j^{\mathbb{Z}}$.

For the upper bound, let (λ_j) denote the multipliers of the repelling fixed points of f^q . Note that each cycle C_i contributes q fixed points, each with multiplier β_i . Combining Proposition 5.6 and Lemma 6.3, we obtain:

$$\begin{aligned} \frac{1}{q} \sum \frac{1}{\lambda_j - 1} &= \frac{1}{q} \sum' \frac{1}{\lambda_j - 1} + \sum \frac{1}{\beta_i - 1} = \frac{1 - |\alpha^q|^2}{q|1 - \alpha^q|^2} \\ &= \pi^{-1} \bmod(p/q, X_\tau) + O(1). \end{aligned}$$

(Again, the prime indicates fixed points in $\bigcup C_i$ are excluded.) Since the cycles C_i are compatible, there are no more than $d - 1$ of them, and hence

$$\sum \frac{1}{\beta_i - 1} = \sum \left(\frac{1}{\log \beta_i} + O(1) \right) = \left(\sum L(C_i)^{-1} \right) + O(d).$$

This yields the upper bound in (6.1); and it also implies

$$\frac{1}{q} \sum' \frac{1}{\lambda_j - 1} = O(d).$$

That is, equation (6.2) holds for $r = q$.

To obtain (6.2) for other values of r , recall that by (5.2) we have $\bmod(s/r, X_\tau) < 1$ whenever $s/r \neq p/q$. Thus if q does not divide r , Lemma 6.3 implies

$$\frac{1}{r} \sum' (\lambda_j - 1)^{-1} \leq \frac{1 - |\alpha^r|^2}{r|1 - \alpha^r|^2} \leq \sup_s \bmod(s/r, X_\tau) + O(1) = O(1);$$

while for $r = nq$ we obtain

$$\frac{1}{r} \sum' \frac{1}{\lambda_j - 1} + \frac{q}{r} \sum \frac{1}{\beta_i^n - 1} = \frac{1 - |\alpha^r|^2}{r|1 - \alpha^r|^2} = \frac{\bmod(p/q, X_\tau)}{\pi n^2} + O(1),$$

which again implies (6.2), since (6.1) gives

$$\frac{q}{r} \sum \frac{1}{\beta_i^n - 1} = \frac{1}{n} \left(\sum \frac{1}{nL(C_i, f)} + O(1) \right) = \frac{\bmod(p/q, X_\tau)}{\pi n^2} + O(d).$$

Finally note that equation (6.2) implies $L(C, f) > \epsilon_d \asymp 1/d > 0$, since any cycle C of period r and multiplier β , not among the C_i , contributes $1/(\beta - 1)$ to the sum $(1/r) \sum' (\lambda_j - 1)^{-1}$. \square

7. Binding and renormalization

We conclude by proving the following compactness result.

Theorem 7.1. *Let $(C_i)_1^n$ be a binding set of cycles of degree d . Then for any $M > 0$, the set of $f \in \mathcal{B}_d$ such that $\sum_i L(C_i, f) \leq M$ has compact closure in MRat_d .*

Corollary 7.2. *The set of $f \in \mathcal{B}_d$ such that $\sum_i L(C_i, f) \leq M$ and $|f'(0) - 1| \geq 1/M$ is compact.*

Proof. By Theorem 3.7, the only way a sequence f_n can diverge in \mathcal{B}_d but remain bounded in MRat_d is if $f'_n(0) \rightarrow 1$. \square

Definitions. Sets $A, B \subset S^1$ are *unlinked* if they lie in disjoint connected sets; equivalently, if their convex hulls in the unit disk are disjoint. A map $f: X \rightarrow X$ with $X \subset S^1$ is *renormalizable* if there is a nontrivial partition of X into disjoint, unlinked subsets X_1, \dots, X_n , such that every $f(X_i)$ lies in some X_j .

We say a collection of degree d cycles C_1, \dots, C_m is *binding* if $\deg(p_d| \bigcup C_i) = d$ and $p_d| \bigcup C_i$ is not renormalizable.

Proof of Theorem 7.1. Suppose to the contrary that we have a sequence $f_n \in \mathcal{B}_d$ with $\sum_i L(C_i, f_n) \leq M$ that is divergent in moduli space. Let

$$D_n = \phi_{f_n}^{-1}\left(\bigcup C_i\right) \subset S^1$$

be the finite f_n -invariant set corresponding to the binding cycles. Since $f_n|S^1$ is expanding, we have $|f'_n| \leq e^M$ on D_n .

Next we conjugate the entire picture by an affine transformation depending on n , so that $0 \in D_n$ and $\text{diam}(D_n) = 1$. Then S^1 goes over to a circle $T_n \supset D_n$ invariant by f_n , and we still have $|f'_n|_{D_n} \leq e^M$.

Pass to a subsequence such that $f_n \rightarrow (F, S) \in \overline{\text{Rat}}_d$. Since f_n diverges in MRat_d , we have $\deg(F) < d$. Passing to a further subsequence, we can find a finite set D containing zero and a circle $T \subset \widehat{\mathbb{C}}$ such that $D_n \rightarrow D$ and $T_n \rightarrow T$ in the Hausdorff topology. Note that $|D| > 1$ since $\text{diam } D = 1$.

By Proposition 3.6, the map $f_n|D_n$ converges to $F|D$. But if $|D| = |\bigcup C_i|$, the map $F|(D \subset T)$ is combinatorially the same as $p_d|(\bigcup C_i \subset S^1)$, contradicting our assumption that $\deg(p_d| \bigcup C_i) = d$. Similarly, if $|D| < |\bigcup C_i|$, then the collapse of D_n to D provides an invariant partition for $\bigcup C_i$, contradicting our assumption that $p_d| \bigcup C_i$ is not renormalizable. \square

Examples. The single cycle $C = (3, 6, 12, 24, 17)/31$ in degree 2 is already binding, as is any cycle of prime order with $\deg(p_d|C) = d$.

The first renormalizable cycle in degree 2 is $C = (1, 2, 4, 3)/5$. Although $\deg(p_2|C) = 2$, $L(C, f_n)$ remains bounded as $f_n \in \mathcal{B}_2$ diverges along the sequence specified by $f'_n(0) = -1 + 1/n$. Indeed, f_n^2 can be renormalized so that C converges to the cycle of period 2 for $G(z) = z - 1/z$ [Ep]; and thus $L(C, f_n) \rightarrow \log 9$. For more details, see [Mc6, §14].

References

- [Bers] L. Bers, On boundaries of Teichmüller spaces and on Kleinian groups. I. *Ann. of Math.* **91** (1970), 570–600. [Zbl 0197.06001](#) [MR 0297992](#)
- [BS] J. S. Birman and C. Series, Geodesics with bounded intersection number on surfaces are sparsely distributed. *Topology* **24** (1985), 217–225. [Zbl 0568.57006](#) [MR 0793185](#)
- [Bus] P. Buser, *Geometry and Spectra of Compact Riemann Surfaces*. Progr. Math. 106, Birkhäuser, Boston, Mass., 1992. [Zbl 0770.53001](#) [MR 1183224](#)
- [D] L. DeMarco, Iteration at the boundary of the space of rational maps. *Duke Math. J.* **130** (2005), 169–197. [Zbl 05004325](#) [MR 2176550](#) [MR 0762431](#) [MR 0812271](#)
- [DH] A. Douady and J. Hubbard, *Étude dynamique des polynômes complexes*. Pub. Math. d’Orsay 84–2, 85–4, Université de Paris-Sud, Département de Mathématiques, Orsay, 1984, 1985. [Zbl 0552.30018](#) [Zbl 0571.30026](#) [MR 0762431](#) [MR 0812271](#)
- [Ep] A. L. Epstein, Bounded hyperbolic components of quadratic rational maps. *Ergodic Theory Dynam. Systems* **20** (2000), 727–748. [Zbl 0963.37041](#) [MR 1764925](#)
- [Gol] L. Goldberg, Fixed points of polynomial maps I. Rotation subsets of the circles. *Ann. Sci. École Norm. Sup.* **25** (1992), 679–685. [Zbl 0771.30027](#) [MR 1198093](#)
- [GM] L. R. Goldberg and J. Milnor, Fixed points of polynomial maps II: Fixed point portraits. *Ann. Sci. École Norm. Sup.* **26** (1993), 51–98. [Zbl 0771.30028](#) [MR 1209913](#)
- [Hub] J. H. Hubbard, Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. In *Topological Methods in Modern Mathematics*, ed. by L. R. Goldberg and A. V. Phillips, Publish or Perish, Inc., 1993, 467–511. [Zbl 0797.58049](#) [MR 1215974](#)
- [Ke] K. Keller, *Invariant Factors, Julia Equivalences and the (Abstract) Mandelbrot Set*, Lecture Notes in Math. 1732, Springer-Verlag, Berlin 2000. [Zbl 0970.37032](#) [MR 1761576](#)
- [Ker] S. Kerckhoff, The Nielsen realization problem. *Ann. of Math.* **117** (1983), 235–265. [Zbl 0528.57008](#) [MR 0690845](#)
- [Lev] G. M. Levin, On Pommerenke’s inequality for the eigenvalues of fixed points. *Colloq. Math.* **62** (1991), 167–177. [Zbl 0742.30026](#) [MR 1114630](#)
- [Mc1] C. McMullen, Iteration on Teichmüller space. *Invent. Math.* **99** (1990), 425–454. [Zbl 0695.57012](#) [MR 1031909](#)
- [Mc2] C. McMullen, The classification of conformal dynamical systems. In *Current Developments in Mathematics, 1995*, International Press, Cambridge, Mass., 1995, 323–360. [Zbl 0908.30028](#) [MR 1474980](#)

- [Mc3] C. McMullen, Hausdorff dimension and conformal dynamics III: Computation of dimension. *Amer. J. Math.* **120** (1998), 691–721. [Zbl 0953.30026](#) [MR 1637951](#)
- [Mc4] C. McMullen, Thermodynamics, dimension and the Weil-Petersson metric. *Invent. Math.* **173** (2008), 365–425. [Zbl 1156.30035](#) [MR 2415311](#)
- [Mc5] C. McMullen, A compactification of the space of expanding maps on the circle. *Geom. Funct. Anal.* **18** (2008), 2101–2119. [Zbl 05603850](#) [MR 2491699](#)
- [Mc6] C. McMullen, Ribbon \mathbb{R} -trees and holomorphic dynamics on the unit disk. *J. Topol.* **2** (2009), 23–76. [Zbl 05549240](#) [MR 2499437](#)
- [McS] C. McMullen and D. Sullivan, Quasiconformal homeomorphisms and dynamics III: The Teichmüller space of a holomorphic dynamical system. *Adv. Math.* **135** (1998), 351–395. [Zbl 0926.30028](#) [MR 1620850](#)
- [Mil1] J. Milnor, *Dynamics in One Complex Variable: Introductory Lectures*. Vieweg, Wiesbaden 1999. [Zbl 0946.30013](#) [MR 1721240](#)
- [Mil2] J. Milnor, Local connectivity of Julia sets: expository lectures. In *The Mandelbrot Set, Theme and Variations*, ed. by Tan Lei, Cambridge University Press, Cambridge 2000, 67–116. [Zbl 1107.37305](#) [MR 1765085](#)
- [Ot] J.-P. Otal, Sur le nouage des géodésiques dans les variétés hyperboliques. *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995), 847–852. [Zbl 0840.57008](#) [MR 1326694](#)
- [Pet1] C. L. Petersen, A PLY inequality for Kleinian groups. *Ann. Acad. Sci. Fenn.* **18** (1993), 23–29. [Zbl 0783.30033](#) [MR 1207892](#)
- [Pet2] C. L. Petersen, No elliptic limits for quadratic maps. *Ergodic Theory Dynam. Systems* **19** (1999), 127–141. [Zbl 0921.30019](#) [MR 1676926](#)
- [PL] K. Pilgrim and T. Lei, Spinning deformations of rational maps. *Conform. Geom. Dyn.* **8** (2004), 52–86. [Zbl 1084.37039](#) [MR 2060378](#)
- [Pom] Ch. Pommerenke, On conformal mapping and iteration of rational functions. *Complex Variables Theory Appl.* **5** (1986), 117–126. [MR 0846481](#)
- [Sh] M. Shub, Endomorphisms of compact differentiable manifolds. *Amer. J. Math.* **91** (1969), 175–199. [Zbl 0201.56305](#) [MR 0240824](#)
- [St] R. P. Stanley, *Enumerative Combinatorics*. Vol. I, Wadsworth Brooks/Cole Math. Ser., Wadsworth & Brooks, Monterey, CA, 1986. [Zbl 0608.05001](#) [MR 0847717](#)
- [Th] W. P. Thurston, Hyperbolic structures on 3-manifolds II: Surface groups and 3-manifolds which fiber over the circle. Preprint, 1986.

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