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## Bounding the symbol length in the Galois cohomology of function fields of $p$ -adic curves

Venapally Suresh

*Dedicated to my teacher Professor R. Parimala on her 60th birthday*

**Abstract.** Let  $K$  be a function field of a  $p$ -adic curve and  $l$  a prime not equal to  $p$ . Assume that  $K$  contains a primitive  $l^{\text{th}}$  root of unity. We show that every element in the  $l$ -torsion subgroup of the Brauer group of  $K$  is a tensor product of two cyclic algebras over  $K$ .

**Mathematics Subject Classification (2000).** Galois cohomology, central simple algebras, cyclic algebras, symbols.

**Keywords.** Primary 12G05; Secondary 11R58, 11R34.

### Introduction

Let  $k$  be a field and  $l$  a prime number not equal to the characteristic of  $k$ . Let  $\mu_l$  be the group of  $l^{\text{th}}$  roots of unity and  $\mu_l(m)$  the tensor product of  $m$  copies of  $\mu_l$ . For  $n \geq 0$ , let  $H^n(k, \mu_l)$  denote the  $n^{\text{th}}$  Galois cohomology group with coefficients in  $\mu_l$ . Let  $k^* = k \setminus \{0\}$ . We have an isomorphism  $k^*/k^{*l} \rightarrow H^1(k, \mu_l)$ . For  $a \in k^*$ , let  $(a)$  denote its image in  $H^1(k, \mu_l)$ . For  $a_1, \dots, a_m \in k^*$ , the cup product gives an element  $(a_1) \cdot (a_2) \dots (a_m) \in H^n(k, \mu_l(m))$ , which we call a *symbol*.

Assume that  $k$  contains a primitive  $l^{\text{th}}$  root of unity. Fix a primitive  $l^{\text{th}}$  root of unity  $\zeta \in k$ . Then we have isomorphisms  $\mu_l \rightarrow \mu_l(m)$  of Galois groups. Hence we have isomorphisms  $H^n(k, \mu_l(m)) \rightarrow H^n(k, \mu_l)$ . A *symbol* in  $H^n(k, \mu_l)$  is simply the image of a symbol under this map.

A classical theorem of Merkurjev ([M]) asserts that every element in  $H^2(k, \mu_2)$  is a sum of symbols. A deep result of Merkurjev and Suslin ([MS]) says that every element in  $H^2(k, \mu_l)$  is a sum of symbols. By a theorem of Voevodsky ([V]), every element in  $H^n(k, \mu_2)$  is a sum of symbols. Suppose that  $k$  is a  $p$ -adic field. Local class field theory tells us that every element in  $H^2(k, \mu_l)$  is a symbol and  $H^n(k, \mu_l) = 0$  for  $n \geq 3$ . If  $k$  is a global field, then the global class field theory asserts that every element in  $H^n(k, \mu_l)$  is a symbol.

**Question 1.** Do there exist integers  $N_l(n)(k)$  such that every element in  $H^n(k, \mu_l)$  is a sum of at most  $N_l(n)(k)$  symbols?

Of course, the answer to the above question is negative in general. It can be shown that for  $K = k(X_1, \dots, X_n, \dots)$ , there is no such  $N_l(n)(K)$  for  $n \geq 2$ . However we can restrict to some special fields. It is well-known that if  $N_l(n)(k)$  exist for  $k$ , then  $N_l(n)(k((t)))$  exist. We ask the following

**Question 2.** Suppose that  $N_l(n)(k)$  exist for some field  $k$ . Do they exist for  $k(t)$ ?

This is an open question. However we can restrict to fields of arithmetic interest. For example we consider the  $p$ -adic fields. The most important result in this direction is the following

**Theorem** (Saltman, [S1], (cf. [S2])). *Let  $k$  be a  $p$ -adic field and  $K/k(t)$  be a finite extension. Suppose that  $l \neq p$ . If  $A$  is a central simple algebra over  $K$  representing an element in  $H^2(K, \mu_l)$ , then  $\text{ind}(A)$  divides  $l^2$ .*

Let  $K$  be as in the above theorem. Suppose  $p \neq 2$ . Let  $\alpha \in H^2(K, \mu_2)$  and  $A$  a central simple algebra over  $K$  representing  $\alpha$ . Then by the above theorem, we have  $\text{ind}(A) = 1, 2, 4$ . If  $\text{ind}(A) = 1$ , then  $\alpha$  is a trivial element. If  $\text{ind}(A) = 2$ , then it is well known that  $\alpha$  is a symbol. Assume that  $\text{ind}(A) = 4$ . By a classical theorem of Albert ([A]),  $\alpha$  is a sum of two symbols. For  $H^3(K, \mu_l)$ , we have the following

**Theorem** ([PS2], 3.5, (cf. [PS1], 3.9)). *Let  $k$  be a  $p$ -adic field and  $K/k(t)$  be a finite extension. Suppose that  $l \neq p$ . Every element in  $H^3(K, \mu_l)$  is a symbol.*

Let  $k$  and  $K$  be as above. The field  $K$  is of cohomological dimension 3 and  $H^n(K, \mu_l) = 0$  for  $n \geq 4$ . By the above theorem,  $N_l(3)(K) = 1$  and the only case where  $N_l(n)(K)$  is to be determined is for  $n = 2$ . It is known that  $N_l(2)(K) \geq 2$  (cf. [S1], Appendix). In this article we prove the following

**Theorem.** *Let  $k$  be a  $p$ -adic field and  $K/k(t)$  be a finite extension. Suppose that  $l \neq p$ . Every element in  $H^2(K, \mu_l)$  is a sum of at most two symbols; in other words,  $N_l(2)(K) = 2$ .*

## 1. Some preliminaries

In this section we recall a few basic facts about Galois cohomology groups and divisors on arithmetic surfaces. We refer the reader to ([C]), ([Li1]), ([Li2]) and ([Se]).

Let  $k$  be a field and  $l$  a prime number not equal to the characteristic of  $k$ . Assume that  $k$  contains a primitive  $l^{\text{th}}$  root of unity. Let  $\zeta \in k$  be a primitive  $l^{\text{th}}$  root of unity. Let  $\mu_l$  be the group of  $l^{\text{th}}$  roots of unity. Since  $k$  contains a primitive  $l^{\text{th}}$  root of unity, the absolute Galois group of  $k$  acts trivially on  $\mu_l$ . For  $m \geq 1$ , let  $\mu_l(m)$  denote the tensor product of  $m$  copies of  $\mu_l$ . By fixing a primitive  $l^{\text{th}}$  root of unity  $\zeta$  in  $k$ , we have isomorphisms of Galois modules  $\mu_l(m) \rightarrow \mu_l$ . Throughout this paper we fix a primitive  $l^{\text{th}}$  root of unity and identify  $\mu_l(m)$  with  $\mu_l$ .

Let  $H^n(k, A)$  be the  $n^{\text{th}}$  Galois cohomology group of the absolute Galois group  $\Gamma$  of  $k$  with values in a discrete  $\Gamma$ -module  $A$ . The identification of  $\mu_l(m)$  with  $\mu_l$  gives an identification of  $H^n(k, \mu_l(m))$  with  $H^n(k, \mu_l)$ . In the rest of this paper we use this identification.

Let  $k^* = k \setminus \{0\}$ . For  $a, b, c \in k^*$  we have the following relations in  $H^2(k, \mu_l)$ .

- (1)  $(a) \cdot (bc) = (a) \cdot (b) + (a) \cdot (c)$ ;
- (2)  $(a) \cdot (b) = -((b) \cdot (a))$ ;
- (3)  $(a) \cdot (b^l) = 0$ ;
- (4)  $(a) \cdot (-a) = 0$ .

If  $l \geq 3$ , we have  $(a) \cdot (a) = (a) \cdot ((-1)^l a) = (a) \cdot (-a) = 0$ .

Let  $K$  be a field and  $l$  a prime number not equal to the characteristic of  $K$ . Let  $v$  be a discrete valuation of  $K$ . The residue field of  $v$  is denoted by  $\kappa(v)$ . Suppose  $\text{char}(\kappa(v)) \neq l$ . Then there is a residue homomorphism  $\partial_v: H^n(K, \mu_l(m)) \rightarrow H^{n-1}(\kappa(v), \mu_l(m-1))$ . Let  $\alpha \in H^n(K, \mu_l(m))$ . We say that  $\alpha$  is *unramified* at  $v$  if  $\partial_v(\alpha) = 0$ ; otherwise it is said to be *ramified* at  $v$ .

Let  $\mathcal{X}$  be a regular integral scheme of dimension  $d$ , with function field  $K$ . Let  $\mathcal{X}^1$  be the set of points of  $\mathcal{X}$  of codimension 1. A point  $x \in \mathcal{X}^1$  gives rise to a discrete valuation  $v_x$  on  $K$ . The residue field of this discrete valuation ring is denoted by  $\kappa(x)$ . The corresponding residue homomorphism is denoted by  $\partial_x$ . We say that an element  $\zeta \in H^n(K, \mu_l(m))$  is *unramified* at  $x$  if  $\partial_x(\zeta) = 0$ ; otherwise it is said to be *ramified* at  $x$ . We define the ramification divisor  $\text{ram}_{\mathcal{X}}(\zeta) = \sum x$  as  $x$  runs over points in  $\mathcal{X}^1$  where  $\zeta$  is ramified. Suppose  $C$  is an irreducible subscheme of  $\mathcal{X}$  of codimension 1. Then the generic point  $x$  of  $C$  belongs to  $\mathcal{X}^1$  and we set  $\partial_C = \partial_x$ . If  $\alpha \in H^n(K, \mu_l(m))$  is unramified at  $x$ , then we say that  $\alpha$  is *unramified* at  $C$ . We say that  $\alpha$  is *unramified* on  $\mathcal{X}$  if it is unramified at every point of  $\mathcal{X}^1$ .

Let  $k$  be a  $p$ -adic field and  $K$  the function field of a smooth projective geometrically integral curve  $X$  over  $k$ . By the resolution of singularities for surfaces (cf. [Li1] and [Li2]), there exists a regular projective model  $\mathcal{X}$  of  $X$  over the ring of integers  $\mathcal{O}_k$  of  $k$ . We call such an  $\mathcal{X}$  a *regular projective model* of  $K$ . Since the generic fibre  $X$  of  $\mathcal{X}$  is geometrically integral, it follows that the special fibre  $\bar{\mathcal{X}}$  is connected. Further if  $D$  is a divisor on  $\mathcal{X}$ , there exists a proper birational morphism  $\mathcal{X}' \rightarrow \mathcal{X}$  such that the total transform of  $D$  on  $\mathcal{X}'$  is a divisor with normal crossings (cf. [Sh], Theorem, p. 38 and Remark 2, p. 43). We use this result throughout this paper without further

reference. If  $P \in \mathcal{X}$  is a closed point and  $f \in K$  is a unit at  $P$ , then we denote the image of  $f$  in the residue field at  $P$  by  $f(P)$ .

Let  $k$  be a  $p$ -adic field and  $K$  the function field of a smooth projective geometrically integral curve over  $k$ . Let  $l$  be a prime number not equal to  $p$ . Assume that  $k$  contains a primitive  $l^{\text{th}}$  root of unity. Let  $\alpha \in H^2(K, \mu_l)$ . Let  $\mathcal{X}$  be a regular projective model of  $K$  such that  $\text{ram}_{\mathcal{X}}(\alpha) = C + E$ , where  $C$  and  $E$  are regular curves with normal crossings. We have the following

**Theorem 1.1** (Saltman [S1]). *Let  $K, \alpha, \mathcal{X}, C$  and  $E$  be as above and  $P \in C \cup E$ . Let  $R$  be the local ring at  $P$ . Let  $\pi, \delta \in R$  be local equations of  $C$  and  $E$  respectively at  $P$ .*

- (1) *If  $P \in C \setminus E$  (or  $E \setminus C$ ), then  $\alpha = \alpha' + (\pi) \cdot (u)$  (or  $\alpha = \alpha' + (\delta) \cdot (u)$ ) for some unit  $u \in R, \alpha' \in H^2(K, \mu_l)$  unramified on  $R$ .*
- (2) *If  $P \in C \cap E$ , then either  $\alpha = \alpha' + (\pi) \cdot (u) + (\delta) \cdot (v)$  or  $\alpha = \alpha' + (\pi) \cdot (u\delta^i)$  for some units  $u, v \in R, \alpha' \in H^2(K, \mu_l)$  unramified on  $R$ .*

Let  $P \in C \cap E$ . Suppose that  $\alpha = \alpha' + (\pi) \cdot (u) + (\delta) \cdot (v)$  for some units  $u, v \in R, \alpha' \in H^2(K, \mu_l)$  unramified on  $R$  and  $\pi, \delta$  are local equations of  $C$  and  $E$  respectively. Then  $u(P) = \partial_C(\alpha)(P)$  and  $v(P) = \partial_E(\alpha)(P)$ . Note that  $u(P)$  and  $v(P)$  are uniquely defined modulo  $l^{\text{th}}$  powers. Following Saltman ([S3], §2), we say that  $P$  is a *hot point* of  $\alpha$  if  $u(P)$  and  $v(P)$  do not generate the same subgroup of  $\kappa(P)^*/\kappa(P)^{*l}$ .

We have the following

**Theorem 1.2** (Saltman ([S3], 5.2)). *Let  $k, K, \alpha, \mathcal{X}$  be as above. Then  $\alpha$  is a symbol if and only if there are no hot points of  $\alpha$ .*

## 2. The main theorem

Let  $k$  be a  $p$ -adic field and  $K/k(t)$  be a finite extension. Let  $l \geq 3$  be a prime number not equal to  $p$ . Assume that  $k$  contains a primitive  $l^{\text{th}}$  root of unity. Let  $\beta \in H^2(K, \mu_l)$  and  $\mathcal{X}$  a regular proper model of  $K$ . Let  $\phi: \mathcal{X}' \rightarrow \mathcal{X}$  be a blow-up such that  $\mathcal{X}'$  is a regular proper model of  $K$  and  $\text{ram}_{\mathcal{X}'}(\beta) = C' + E'$ , where  $C'$  and  $E'$  are two regular curves with normal crossings (cf. §1 or [S1], Proof of 2.1). Let  $Q \in C' \cap E'$ . Let  $C'_1 \subset C'$  and  $E'_1 \subset E'$  be the irreducible curves containing  $Q$ . Let  $R' = \mathcal{O}_{\mathcal{X}', Q}$  be the regular local ring at  $Q$  and  $m_Q$  its maximal ideal. We have  $m_Q = (\pi', \delta')$ , where  $\pi'$  and  $\delta'$  are local equations of  $C'_1$  and  $E'_1$  at  $Q$  respectively. Let  $\nu_{C'_1}$  and  $\nu_{E'_1}$  be the discrete valuations on  $K$  at  $C'_1$  and  $E'_1$  respectively.

Let  $P = \phi(Q)$ . Let  $R$  be the regular local ring at  $P$  and  $m_P$  its maximal ideal. We have the induced homomorphism  $\phi^*: R \rightarrow R'$  of local rings, which is injective. Let  $\pi, \delta \in R$  be such that  $m_P = (\pi, \delta)$ .

**Lemma 2.1.** *Suppose that  $\beta = \beta' + (f') \cdot (g')$  for some  $f', g' \in K$  and  $\beta'$  unramified on  $R'$ . Then  $Q$  is not a hot point of  $\beta$ .*

*Proof.* Since  $\beta'$  is unramified on  $R'$ , the ramification data of  $\beta$  on  $R'$  is same as that of  $(f') \cdot (g')$ . Since  $(f') \cdot (g')$ , being a symbol, has no hot points ([S3], cf. 1.2),  $Q$  is not a hot point of  $\beta$ .  $\square$

**Lemma 2.2.** *Suppose that  $\beta = \beta' + (\delta) \cdot (gv) + (f) \cdot (g)$ , where  $\beta'$  is unramified on  $R'$ ,  $f \in R$  is not divisible by  $\delta$  and  $v, g \in R$  are units with  $g(P) = v(P)$ . Then  $Q$  is not a hot point of  $\beta$ .*

*Proof.* We have  $m_Q = (\pi', \delta')$  and  $\beta$  has ramification on  $R'$  only at  $\pi'$  and  $\delta'$ . Since  $R/m_P \hookrightarrow R'/m_Q$ , we have  $g(Q) = g(P) = v(P) = v(Q)$ . If  $C'_1$  (or  $E'_1$ ) is the strict transform of a curve on  $\mathcal{X}$ , then either  $\delta$  is a local equation of  $C'_1$  or  $v_{C'_1}(\delta) = 0$ . In fact, if  $C'_1$  is the strict transform of  $C_1$  on  $\mathcal{X}$ , then  $v_{C_1}(\delta) = v_{C'_1}(\delta)$  and  $\delta$  itself being a prime in  $R$ , the assertion follows.

Suppose that  $C'_1$  and  $E'_1$  are strict transforms of two irreducible curves on  $\mathcal{X}$ . If  $\delta$  is not a local equation for either  $C'_1$  or  $E'_1$ , we claim that  $(\delta) \cdot (gv)$  is unramified on  $R'$ . In fact, since  $g$  and  $v$  are units in  $R$ ,  $(\delta) \cdot (gv)$  is unramified on  $R$  except possibly at  $\delta$ . Since  $f$  is not divisible by  $\delta$ ,  $(f) \cdot (g)$  is unramified at  $\delta$ . Since  $\beta$  is ramified on  $R'$  only at  $\pi'$  and  $\delta'$  and  $\delta$  is not one of them,  $(\delta) \cdot (gv)$  is unramified on  $R'$ . By (2.1),  $Q$  is not a hot point of  $\beta$ . Assume that  $\delta$  is a local equation for one of them, say  $C'_1$ . Since  $\delta$  does not divide  $f$ , we have  $\partial_{C'_1}(\beta) = \overline{v}g$  and  $\partial_{E'_1}(\beta) = \tilde{g}^{v_{E'_1}(f)}$ , where  $\overline{\phantom{x}}$  denotes the image in the residue field of  $C'_1$  and  $\tilde{\phantom{x}}$  denotes the image in the residue field of  $E'_1$ . Since  $\beta$  is ramified at  $E'_1$ ,  $v_{E'_1}(f)$  is not a multiple of  $l$ . We have  $\partial_{C'_1}(\beta)(Q) = v(Q)g(Q) = g(Q)^2$  and  $\partial_{E'_1}(\beta)(Q) = g(Q)^{v_{E'_1}(f)}$ . Since  $l \neq 2$  and  $v_{E'_1}(f)$  is not a multiple of  $l$ ,  $g(Q)^2$  and  $g(Q)^{v_{E'_1}(f)}$  generate the same subgroup modulo  $l^{\text{th}}$  powers. Hence  $Q$  is not a hot point of  $\beta$ .

Suppose that  $C'_1$  is a strict transform of an irreducible curve on  $\mathcal{X}$  and  $E'_1$  is an exceptional curve on  $\mathcal{X}'$ . We have  $\partial_{E'_1}(\beta) = (\tilde{g}v)^{v_{E'_1}(\delta)} \tilde{g}^{v_{E'_1}(f)}$ . Since  $E'_1$  is an exceptional fibre in  $\mathcal{X}'$ , the residue field of  $R$  is contained in the residue field at  $E'_1$ . Hence  $\partial_{E'_1}(\beta) = (g(P)v(P))^{v_{E'_1}(\delta)} g(P)^{v_{E'_1}(f)} = g(P)^{2v_{E'_1}(\delta) + v_{E'_1}(f)}$ . Since  $\beta$  is ramified at  $E'_1$ ,  $2v_{E'_1}(\delta) + v_{E'_1}(f)$  is not a multiple of  $l$ . Suppose  $\delta$  is a local equation of  $C'_1$  at  $Q$ . Since  $\delta$  does not divide  $f$  and  $v_{C'_1}(\delta) = 1$ , we have  $\partial_{C'_1}(\beta) = \overline{g}v$ . Thus  $\partial_{C'_1}(\beta)(Q) = g(P)v(P) = g(P)^2$ . Since  $l \neq 2$  and  $2v_{E'_1}(\delta) + v_{E'_1}(f)$  is not a multiple of  $l$ , the subgroups generated by  $g(Q)^2$  and  $g(P)^{2v_{E'_1}(\delta) + v_{E'_1}(f)}$  are equal modulo  $l^{\text{th}}$  powers. Hence  $Q$  is not a hot point of  $\beta$ . Suppose  $\delta$  is not a local equation of  $C'_1$  at  $Q$ . We have  $\partial_{C'_1}(\beta) = \tilde{g}^{v_{C'_1}(f)}$ . Since  $\beta$  is ramified at  $C'_1$ ,  $v_{C'_1}(f)$  is not a multiple of  $l$ . Thus as above  $Q$  is not a hot point of  $\beta$ .

The case  $E'_1$  is a strict transform of an irreducible curve on  $\mathcal{X}$  and  $C'_1$  is an exceptional curve in  $\mathcal{X}'$  follows on similar lines.

Suppose that both  $C'_1$  and  $E'_1$  are exceptional curves in  $\mathcal{X}'$ . Then as above we have  $\partial_{C'_1}(\beta) = g(P)^{2\nu_{C'_1}(\delta) + \nu_{C'_1}(f)}$  and  $\partial_{E'_1}(\beta) = g(P)^{2\nu_{E'_1}(\delta) + \nu_{E'_1}(f)}$ . Since  $\beta$  is ramified at  $C'_1$  and  $E'_1$ ,  $2\nu_{C'_1}(\delta) + \nu_{C'_1}(f)$  and  $2\nu_{E'_1}(\delta) + \nu_{E'_1}(f)$  are not multiples of  $l$ . In particular, the subgroups generated by  $g(P)^{2\nu_{C'_1}(\delta) + \nu_{C'_1}(f)}$  and  $g(P)^{2\nu_{E'_1}(\delta) + \nu_{E'_1}(f)}$  are equal modulo the  $l^{\text{th}}$  powers. Thus  $Q$  is not a hot point of  $\beta$ .  $\square$

**Lemma 2.3.** *Suppose that  $\beta = \beta' + (\pi) \cdot (u) + (\delta) \cdot (v)$ , where  $\beta'$  unramified on  $R'$  and  $u, v \in R$  units with  $u(P) = v(P)$ . Then  $Q$  is not a hot point of  $\beta$ .*

*Proof.* Since  $\beta$  is ramified at  $C'_1$ , either  $\nu_{C'_1}(\pi)$  or  $\nu_{C'_1}(\delta)$  is not divisible by  $l$ . In particular their sum  $\nu_{C'_1}(\pi\delta)$  is non-zero. We have

$$\partial_{C'_1}(\beta)(Q) = u(P)^{\nu_{C'_1}(\pi)} v(P)^{\nu_{C'_1}(\delta)} = u(P)^{\nu_{C'_1}(\pi\delta)}.$$

Suppose that  $\nu_{C'_1}(\pi\delta)$  is a multiple of  $l$ . Since  $\nu_{C'_1}(\pi\delta)$  is non-zero,  $C'_1$  is an exceptional curve. As in the proof of (2.2), we see that  $\partial_{C'_1}(\beta) = u(P)^{\nu_{C'_1}(\pi\delta)} = 1$ . Which is a contradiction, as  $\beta$  is ramified at  $C'_1$ . Hence  $\nu_{C'_1}(\pi\delta)$  is not a multiple of  $l$ . Similarly, we have  $\partial_{E'_1}(\beta)(Q) = u(P)^{\nu_{E'_1}(\pi\delta)}$  and  $\nu_{E'_1}(\pi\delta)$  is not a multiple of  $l$ . Hence  $u(P)^{\nu_{C'_1}(\pi\delta)}$  and  $u(P)^{\nu_{E'_1}(\pi\delta)}$  generate the same subgroup of  $\kappa(P)^*$  modulo  $\kappa(P)^{*l}$  and  $Q$  is not a hot point of  $\beta$ .  $\square$

**Theorem 2.4.** *Let  $k$  be a  $p$ -adic field and  $K/k(t)$  be a finite extension. Let  $l$  be a prime number not equal to  $p$ . Suppose that  $k$  contains a primitive  $l^{\text{th}}$  root of unity. Then every element in  $H^2(K, \mu_l)$  is a sum of at most two symbols.*

*Proof.* If  $l = 2$ , then, as we mentioned before, by ([A]),  $\alpha$  is a sum of at most two symbols. Assume that  $l \geq 3$ . Let  $\alpha \in H^2(K, \mu_l)$ . Let  $\mathcal{X}$  be a regular proper model of  $K$  such that  $\text{ram}_{\mathcal{X}}(\alpha) = C + E$ , where  $C$  and  $E$  are regular curves with normal crossings.

Let  $P \in C \cup E$  be a closed point of  $\mathcal{X}$ . Let  $R_P$  be the regular local ring at  $P$  on  $\mathcal{X}$  and  $m_P$  be its maximal ideal.

Let  $T$  be a finite set of closed points of  $\mathcal{X}$  containing  $C \cap E$  and at least one closed point from each irreducible curve in  $C$  and  $E$ . Let  $A$  be the semi-local ring at  $T$  on  $\mathcal{X}$ . Let  $\pi_1, \dots, \pi_r, \delta_1, \dots, \delta_s \in A$  be prime elements corresponding to irreducible curves in  $C$  and  $E$  respectively. Let  $f_1 = \pi_1 \dots \pi_r \delta_1 \dots \delta_s \in A$ . Let  $P \in C \cap E$ . Then  $P \in C_i \cap E_j$  for unique irreducible curves  $C_i$  in  $C$  and  $E_j$  in  $E$ . Then  $\pi = \pi_i$  and  $\delta = \delta_j$  are local equations of  $C$  and  $E$  at  $P$ . We have  $\alpha = \alpha' + (\pi) \cdot (u_P) + (\delta) \cdot (v_P)$



or  $\alpha = \alpha' + (\pi) \cdot (u_P \delta^i)$  for some units,  $u_P, v_P \in R$  and  $\alpha'$  unramified on  $R$  ([S1], cf. 1.1). By the choice of  $f_1$ , we have  $f_1 = \pi \delta w_P$  for some  $w_P \in A$  which is a unit at  $P$ . Let  $u \in A$  be such that  $u(P) = w_P(P)^{-1} u_P(P)$  for all  $P \in C \cap E$ . Let  $f = f_1 u \in A$ . Then, we have  $(f) = C + E + F$ , where  $F$  is a divisor on  $\mathcal{X}$  which avoids  $C, E$  and all the points of  $C \cap E$ . Further, for each  $P \in C_i \cap E_j$ , we have  $f = \pi_i \delta_j w_{ij}$  for some  $w_{ij} \in A$  such that  $w_{ij}(P) = u_P(P)$ .

By a similar argument, choose  $g \in K$  satisfying

- (1)  $(g) = C + G$ , where  $G$  is a divisor on  $\mathcal{X}$  which avoids  $C, E, F$  and also avoids the points of  $C \cap E, C \cap F, E \cap F$ ;
- (2) if  $P \in E \cap F$  and  $\alpha = \alpha' + (\delta) \cdot (v)$  for some unit  $v \in R_P$  and  $\alpha'$  is unramified at  $P$ , then  $g(P) = v(P)$ .

Since  $C \cap E \cap F = \emptyset$ , such a  $g$  exists.

We claim that  $\beta = \alpha + (f) \cdot (g)$  is a symbol.

Let  $\phi: \mathcal{X}' \rightarrow \mathcal{X}$  be a blow up of  $\mathcal{X}$  such that  $\mathcal{X}'$  is a regular proper model of  $K$  and  $\text{ram}_{\mathcal{X}'}(\beta) = C' + E'$ , where  $C'$  and  $E'$  are regular curves with normal crossings.

To show that  $\beta$  is a symbol, it is enough to show that  $\beta$  has no hot points ([S3], cf. 1.2). Let  $Q \in C' \cap E'$ . Let  $P = \phi(Q)$ . Then  $P$  is a closed point of  $\mathcal{X}$ ,  $R = \mathcal{O}_{\mathcal{X}, P} \subset \mathcal{O}_{\mathcal{X}', Q} = R'$  and the maximal ideal  $m_P$  of  $R$  is contained in the maximal ideal  $m_Q$  of  $R'$ . Let  $m_Q = (\pi', \delta')$ , with  $\pi'$  and  $\delta'$  be local equations of  $C'$  and  $E'$  at  $Q$  respectively.

Suppose that  $P \notin C \cup E$ . Then  $\alpha$  is unramified at  $P$  and hence unramified at  $Q$ . By (2.1),  $Q$  is not a hot point of  $\beta$ .

Assume that  $P \in C \cup E$ .

Suppose that  $P \in C \cap E$ . Let  $\pi$  and  $\delta$  be local equations of  $C$  and  $E$  at  $P$  respectively. Then  $m_P = (\pi, \delta)$ . By the choice of  $f$  and  $g$ , we have  $f = \pi \delta w_1$  and  $g = \pi w_2$  for some units  $w_1, w_2 \in R$ . In particular,  $\beta$  is ramified on  $R$  only at  $\pi$  and  $\delta$ . Suppose that  $\alpha = \alpha' + (\pi) \cdot (u) + (\delta) \cdot (v)$  for some units  $u, v \in R$  and  $\alpha'$  unramified on  $R$ . We have

$$\begin{aligned}
 \beta &= \alpha + (f) \cdot (g) \\
 &= \alpha' + (\pi) \cdot (u) + (\delta) \cdot (v) + (\pi \delta w_1) \cdot (\pi w_2) \\
 &= \alpha' + (\pi) \cdot (u) + (\delta w_1) \cdot (v) + (w_1^{-1}) \cdot (v) + (\pi) \cdot (\pi w_2) + (\delta w_1) \cdot (\pi w_2) \\
 &= \alpha' + (w_1^{-1}) \cdot (v) + (\pi) \cdot (u \pi w_2) + (\delta w_1) \cdot (\pi w_2 v) \\
 &= \alpha' + (w_1^{-1}) \cdot (v) + (\pi) \cdot (u w_2) + (\delta w_1) \cdot (\pi w_2 v) \\
 &= \alpha' + (w_1^{-1}) \cdot (v) + (\pi w_2 v) \cdot (u w_2) + (w_2^{-1} v^{-1}) \cdot (u w_2) + (\delta w_1) \cdot (\pi w_2 v) \\
 &= \alpha' + (w_1^{-1}) \cdot (v) + (w_2^{-1} v^{-1}) \cdot (u w_2) + (\pi w_2 v) \cdot (u w_2 \delta^{-1} w_1^{-1}).
 \end{aligned}$$

Since  $\alpha' + (w_1^{-1}) \cdot (v) + (w_2^{-1} v^{-1}) \cdot (u w_2)$  is unramified on  $R$ , by (2.1),  $Q$  is not a hot point of  $\beta$ .



Suppose that  $\alpha = \alpha' + (\pi) \cdot (u\delta^i)$  for some units,  $u, v \in R$  and  $\alpha'$  unramified on  $R$ . Then we have

$$\begin{aligned}\beta &= \alpha + (f) \cdot (g) \\ &= \alpha' + (\pi) \cdot (u\delta^i) + (\pi\delta w_1) \cdot (\pi w_2) \\ &= \alpha' + (\pi) \cdot (u\delta^i) + (\delta w_1 w_2^{-1}) \cdot (\pi w_2) \\ &= \alpha' + (\pi) \cdot (u\delta^i (\delta w_1 w_2^{-1})^{-1}) + (\delta w_1 w_2^{-1}) \cdot (w_2) \\ &= \alpha' + (\pi) \cdot (\delta^{i-1} u w_1^{-1} w_2) + (\delta w_1 w_2^{-1}) \cdot (w_2).\end{aligned}$$

If  $i = 1$ , then  $\beta = \alpha' + (\pi) \cdot (u w_1^{-1} w_2) + (\delta w_1 w_2^{-1}) \cdot (w_2)$ . Since, by the choice of  $f$ ,  $u(P) = w_1(P)$ , by (2.3),  $Q$  is not a hot point of  $\beta$ . Assume that  $i > 1$ . Then  $1 \leq i-1 < l-1$ . Let  $i'$  be the inverse of  $1-i$  modulo  $l$ . We have

$$\begin{aligned}\beta &= \alpha' + (\pi) \cdot (\delta^{i-1} u w_1^{-1} w_2) + (\delta w_1 w_2^{-1}) \cdot (w_2) \\ &= \alpha' + (\delta^{1-i} u^{-1} w_1 w_2^{-1}) \cdot (\pi) + (\delta w_1 w_2^{-1}) \cdot (w_2) \\ &= \alpha' + ((\delta(u^{-1} w_1 w_2^{-1})^{i'})^{1-i}) \cdot (\pi) + (\delta(u^{-1} w_1 w_2^{-1})^{i'}) \cdot (w_2) \\ &\quad + ((u^{-1} w_1 w_2^{-1})^{-i'}) \cdot (w_2) + (w_1 w_2^{-1}) \cdot (w_2) \\ &= \alpha' + ((u^{-1} w_1 w_2^{-1})^{-i'}) \cdot (w_2) + (w_1 w_2^{-1}) \cdot (w_2) \\ &\quad + ((\delta(u^{-1} w_1 w_2^{-1})^{i'})^{1-i}) \cdot (\pi^{1-i} w_2).\end{aligned}$$

Since  $\alpha' + ((u^{-1} w_1 w_2^{-1})^{-i'}) \cdot (w_2) + (w_1 w_2^{-1}) \cdot (w_2)$  is unramified on  $R$ , by (2.1),  $Q$  is not a hot point of  $\beta$ .

Suppose that  $P \in C \setminus E$ . We have  $\alpha = \alpha' + (\pi) \cdot (u)$  for some unit  $u$  in  $R$  and  $\alpha'$  unramified on  $R$ . We also have  $f = \pi f_1$  for some  $f_1 \in R$  which is not divisible by  $\pi$ . We have

$$\begin{aligned}\beta &= \alpha + (f) \cdot (g) \\ &= \alpha' + (\pi) \cdot (u) + (\pi f_1) \cdot (g) \\ &= \alpha' + (f_1^{-1}) \cdot (u) + (\pi f_1) \cdot (u) + (\pi f_1) \cdot (g) \\ &= \alpha' + (f_1^{-1}) \cdot (u) + (\pi f_1) \cdot (gu).\end{aligned}$$

If  $f_1$  is a unit in  $R$ , then  $\alpha' + (f_1^{-1}) \cdot (u)$  is unramified on  $R$ , by (2.1),  $Q$  is not a hot point of  $\beta$ . Assume that  $f_1$  is not a unit in  $R$ . Then  $P \in C \cap F$  and  $g = \pi g_1$  for some unit  $g_1 \in R$ . We have

$$\begin{aligned}\beta &= \alpha + (f) \cdot (g) \\ &= \alpha' + (\pi) \cdot (u) + (\pi f_1) \cdot (\pi g_1) \\ &= \alpha' + (\pi g_1) \cdot (u) + (g_1^{-1}) \cdot (u) + (\pi f_1) \cdot (\pi g_1) \\ &= \alpha' + (g_1^{-1}) \cdot (u) + (\pi g_1) \cdot (u(\pi f_1)^{-1}).\end{aligned}$$

Since  $\alpha' + (g_1^{-1}) \cdot (u)$  is unramified on  $R$ , by (2.1),  $Q$  is not a hot point of  $\beta$ .

Suppose that  $P \in E \setminus C$ . Then  $\alpha = \alpha' + (\delta) \cdot (v)$  for some unit  $v \in R$  and  $f = \delta f_1$  for some  $f_1 \in R$  which is not divisible by  $\delta$ . Suppose that  $f_1$  is a unit in  $R$ . Then, as above,  $Q$  is not a hot point of  $\beta$ . Assume that  $f_1$  is not a unit in  $R$ . Then  $P \in E \cap F$  and  $g$  is a unit in  $R$ . We have

$$\begin{aligned}\beta &= \alpha + (f) \cdot (g) \\ &= \alpha' + (\delta) \cdot (v) + (\delta f_1) \cdot (g) \\ &= \alpha' + (\delta) \cdot (vg) + (f_1) \cdot (g).\end{aligned}$$

Since  $\alpha'$  is unramified on  $R$  and by the choice of  $g$ ,  $g(P) = v(P)$ , by (2.2),  $Q$  is not a hot point of  $\beta$ .

By ([S3], cf. 1.2),  $\beta$  is symbol. Thus  $\alpha = (f) \cdot (g) - \beta$  is a sum of at most two symbols.  $\square$

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## References

- [A] A. A. Albert, Normal division algebras of degree four over an algebraic field. *Trans. Amer. Math. Soc.* **34** (1931), 363–372. [Zbl 0004.10002](#) [MR 1501642](#)
- [Ar] J. K. Arason, Cohomologische Invarianten quadratischer Formen. *J. Algebra* **36** (1975), 448–491. [Zbl 0314.12104](#) [MR 0389761](#)
- [AEJ] J. K. Arason, R. Elman, and B. Jacob, Fields of cohomological 2-dimension three. *Math. Ann.* **274** (1986), 649–657. [Zbl 0576.12025](#) [MR 0848510](#)
- [C] J.-L. Colliot-Thélène, Birational invariants, purity, and the Gersten conjecture. In *K-theory and algebraic geometry: connections with quadratic forms and division algebras*, Proc. Sympos. Pure Math. 58, Part 1, Amer. Math. Soc., Providence, RI, 1995, 1–64. [Zbl 0834.14009](#) [MR 1327280](#)
- [Li1] J. Lipman, Introduction to resolution of singularities. In *Algebraic geometry*, Proc. Sympos. Pure Math. 29, Amer. Math. Soc., Providence, R.I., 1975, 187–230. [Zbl 0306.14007](#) [MR 0389901](#)
- [Li2] Lipman, J., Desingularization of two-dimensional schemes. *Ann. of Math.* **107** (1978), 151–207. [Zbl 0349.14004](#) [MR 0491722](#)
- [M] A. S. Merkurjev, On the norm residue symbol of degree 2. *Dokl. Akad. Nauk. SSSR* **261** (1981), 542–547; English transl. *Soviet Math. Dokl.* **24** (1981), no. 3, 546–551. [Zbl 0496.16020](#) [MR 0638926](#)
- [MS] A. S. Merkurjev and A. A. Suslin,  $K$ -cohomology of Severi-Brauer varieties and the norm residue homomorphism. *Dokl. Akad. Nauk SSSR* **264** (1982), 555–559; English transl. *Soviet Math. Dokl.* **25** (1982), no. 3, 690–693. [Zbl 0525.18007](#) [MR 0659762](#)

- [PS1] R. Parimala and V. Suresh, Isotropy of quadratic forms over function fields in one variable over  $p$ -adic fields. *Inst. Hautes Études Sci. Publ. Math.* **88** (1998), 129–150. [Zbl 0972.11020](#) [MR 1733328](#)
- [PS2] R. Parimala and V. Suresh, The  $u$ -invariant of the function fields of  $p$ -adic curves. Preprint 2007. [arXiv:0708.3128v1](#)
- [S1] D. J. Saltman, Division Algebras over  $p$ -adic curves. *J. Ramanujan Math. Soc.* **12** (1997), 25–47. [Zbl 0902.16021](#) [MR 1462850](#)
- [S2] D. J. Saltman, Correction to “Division algebras over  $p$ -adic curves”. *J. Ramanujan Math. Soc.* **13** (1998), 125–130. [Zbl 0920.16008](#) [MR 1666362](#)
- [S3] D. J. Saltman, Cyclic algebras over  $p$ -adic curves. *J. Algebra* **314** (2007), 817–843. [Zbl 1129.16014](#) [MR 2344586](#)
- [Se] J.-P. Serre, Cohomologie Galoisienne: progrès et problèmes. *Astérisque* **227** (1995), Exp. No. 783, 229–257. [Zbl 0837.12003](#) [MR 1321649](#)
- [Sh] I. R. Shafarevich, *Lectures on minimal models and birational transformations of two dimensional schemes*. Tata Inst. Fund. Res. Lect. Math. Phys. 37, Tata Institute of Fundamental Research, Bombay 1966. [Zbl 164.51704](#) [MR 0217068](#)
- [V] V. Voevodsky, Motivic cohomology with  $\mathbb{Z}/2$ -coefficients. *Inst. Hautes Études Sci. Publ. Math.* **98** (2003), 59–104. [Zbl 1057.14028](#) [MR 2031199](#)

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