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$\mathrm{SL}_n(\mathbb{Z}[t])$ is not FP_{n-1}

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Abstract. We prove the result from the title using the geometry of Euclidean buildings.

Mathematics Subject Classification (2000). 20F65.

Keywords. Euclidean buildings, finiteness properties, geometric group theory.

1. Introduction

Little is known about the finiteness properties of $\mathrm{SL}_n(\mathbb{Z}[t])$ for arbitrary n .

In 1959 Nagao proved that if k is a field then $\mathrm{SL}_2(k[t])$ is a free product with amalgamation [Na]. It follows from his description that $\mathrm{SL}_2(\mathbb{Z}[t])$ and its abelianization are not finitely generated.

In 1977 Suslin proved that when $n \geq 3$, $\mathrm{SL}_n(\mathbb{Z}[t])$ is finitely generated by elementary matrices [Su]. It follows that $H_1(\mathrm{SL}_n(\mathbb{Z}[t]), \mathbb{Z})$ is trivial when $n \geq 3$.

More recent, Krstić and McCool proved in [Kr-Mc] that $\mathrm{SL}_3(\mathbb{Z}[t])$ is not finitely presented.

In this paper we provide a generalization of the results of Nagao and Krstić–McCool mentioned above for the groups $\mathrm{SL}_n(\mathbb{Z}[t])$.

Theorem 1. *If $n \geq 2$, then $\mathrm{SL}_n(\mathbb{Z}[t])$ is not of type FP_{n-1} .*

Recall that a group Γ is of type FP_m if there exists a projective resolution of \mathbb{Z} as the trivial $\mathbb{Z}\Gamma$ module

$$P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$$

where each P_i is a finitely generated, projective $\mathbb{Z}\Gamma$ module.

In particular, Theorem 1 implies that there is no $K(\mathrm{SL}_n(\mathbb{Z}[t]), 1)$ with finite $(n-1)$ -skeleton, where $K(G, 1)$ is the Eilenberg–Mac Lane space for G .

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1.1. Outline of the paper. The general outline of this paper is modelled on the proofs in [Bu-Wo1] and [Bu-Wo2], though some important modifications have to be made to carry out the proof in this setting.

As in [Bu-Wo1] and [Bu-Wo2], our approach is to apply Brown’s filtration criterion [Br1]. Here we will examine the action of $\mathrm{SL}_n(\mathbb{Z}[t])$ on the locally infinite Euclidean building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. In Section 2 we will show that the infinite groups that arise as cell stabilizers for this action are of type FP_m for all m , which is a technical condition that is needed for our application of Brown’s criterion.

In Section 3 we will demonstrate the existence of a family of diagonal matrices that will imply the existence of a “nice” isometrically embedded codimension 1 Euclidean space in the building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. In [Bu-Wo1] analogous families of diagonal matrices were constructed using some standard results from the theory of algebraic groups over locally compact fields. Because $\mathbb{Q}((t^{-1}))$ is not locally compact, our treatment in Section 3 is quite a bit more hands on.

Section 4 contains the main body of our proof. We use translates of portions of the codimension 1 Euclidean subspace found in Section 3 to construct spheres in the Euclidean building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ (also of codimension 1). These spheres will lie “near” an orbit of $\mathrm{SL}_n(\mathbb{Z}[t])$, but will be nonzero in the homology of cells “not as near” the same $\mathrm{SL}_n(\mathbb{Z}[t])$ orbit. Theorem 1 will then follow from Brown’s criterion.

1.2. Background material. Our proof relies heavily on the geometry of the Euclidean and spherical buildings for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. A good source of information for the former topic is Chapter 6 of [Br2]. For the latter, we recommend Chapter 5 of [Ti].

2. Stabilizers

Lemma 2. *If X is the Euclidean building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$, then the $\mathrm{SL}_n(\mathbb{Z}[t])$ stabilizers of cells in X are FP_m for all m .*

Proof. Let $x_0 \in X$ be the vertex stabilized by $\mathrm{SL}_n(\mathbb{Q}[[t^{-1}]])$. We denote a diagonal matrix in $\mathrm{GL}_n(\mathbb{Q}((t^{-1})))$ with entries $s_1, s_2, \dots, s_n \in \mathbb{Q}((t^{-1}))^\times$ by $D(s_1, s_2, \dots, s_n)$, and we let $\mathfrak{S} \subseteq X$ be the sector based at x_0 and containing vertices of the form $D(t^{m_1}, t^{m_2}, \dots, t^{m_n})x_0$ where each $m_i \in \mathbb{Z}$ and $m_1 \geq m_2 \geq \dots \geq m_n$.

The sector \mathfrak{S} is a fundamental domain for the action of $\mathrm{SL}_n(\mathbb{Q}[t])$ on X (see [So]). In particular, for any vertex $z \in X$, there is some $h'_z \in \mathrm{SL}_n(\mathbb{Q}[t])$ and some integers $m_1 \geq m_2 \geq \dots \geq m_n$ with $z = h'_z D_z(t^{m_1}, t^{m_2}, \dots, t^{m_n})x_0$. We let $h_z = h'_z D_z(t^{m_1}, t^{m_2}, \dots, t^{m_n})$.

For any $N \in \mathbb{N}$, let W_N be the $(N + 1)$ -dimensional vector space

$$W_N = \{ p(t) \in \mathbb{C}[t] \mid \deg(p(t)) \leq N \}$$

which is endowed with the obvious \mathbb{Q} -structure. If N_1, \dots, N_{n^2} in \mathbb{N} are arbitrary then let

$$\mathbf{G}_{\{N_1, \dots, N_{n^2}\}} = \{\mathbf{x} \in \prod_{i=1}^{n^2} W_{N_i} \mid \det(\mathbf{x}) = 1\}$$

where $\det(\mathbf{x})$ is a polynomial in the coordinates of \mathbf{x} . To be more precise this is obtained from the usual determinant function when one considers the usual $n \times n$ matrix presentation of \mathbf{x} , and calculates the determinant in $\mathrm{Mat}_n(\mathbb{C}[t])$.

For our choice of vertex $z \in X$ above, the stabilizer of z in $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ equals $h_z \mathrm{SL}_n(\mathbb{Q}[[t^{-1}]]) h_z^{-1}$. And with our fixed choice of h_z , there clearly exist some $N_i^z \in \mathbb{N}$ such that the stabilizer of the vertex z in $\mathrm{SL}_n(\mathbb{Q}[t])$ is $\mathbf{G}_{\{N_1^z, \dots, N_{n^2}^z\}}(\mathbb{Q})$. Furthermore, conditions on N_i^z force a group structure on $\mathbf{G}_z = \mathbf{G}_{\{N_1^z, \dots, N_{n^2}^z\}}$. Therefore, the stabilizer of z in $\mathrm{SL}_n(\mathbb{Q}[t])$ is the \mathbb{Q} -points of the affine \mathbb{Q} -group \mathbf{G}_z , and the stabilizer of z in $\mathrm{SL}_n(\mathbb{Z}[t])$ is $\mathbf{G}_z(\mathbb{Z})$.

The action of $\mathrm{SL}_n(\mathbb{Q}[t])$ on X is type preserving, so if $\sigma \subset \mathfrak{S}$ is a simplex with vertices z_1, z_2, \dots, z_m , then the stabilizer of σ in $\mathrm{SL}_n(\mathbb{Z}[t])$ is simply

$$(\mathbf{G}_{z_1} \cap \dots \cap \mathbf{G}_{z_m})(\mathbb{Z}).$$

That is, the stabilizer of σ in $\mathrm{SL}_n(\mathbb{Z}[t])$ is an arithmetic group, and Borel–Serre proved that any such group is FP_m for all m [Bo-Se]. \square

3. Polynomial points of tori

This section is devoted exclusively to a proof of the following

Proposition 3. *There is a group $A \leq \mathrm{SL}_n(\mathbb{Z}[t])$ such that the following holds:*

- (i) $A \cong \mathbb{Z}^{n-1}$.
- (ii) *There is some $g \in \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ such that gAg^{-1} is a group of diagonal matrices.*
- (iii) *No nontrivial element of A fixes a point in the Euclidean building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$.*

The proof of this proposition is modelled on a classical approach to finding diagonalizable subgroups of $\mathrm{SL}_n(\mathbb{Z})$. The proof will take a few steps.

3.1. A polynomial over $\mathbb{Z}[t]$ with roots in $\mathbb{Q}((t^{-1}))$. Let $\{p_1, p_2, p_3, \dots\} = \{2, 3, 5, \dots\}$ be the sequence of prime numbers. Let $q_1 = 1$. For $2 \leq i \leq n$, let $q_i = p_{i-1} + 1$.

Let $f(x) \in \mathbb{Z}[t][x]$ be the polynomial given by

$$f(x) = \left[\prod_{i=1}^n (x + q_i t) \right] - 1.$$

It will be clear by the conclusion of our proof that $f(x)$ is irreducible over $\mathbb{Q}(t)$, but we will not need to use this directly.

Lemma 4. *There is some $\alpha \in \mathbb{Q}((t^{-1}))$ such that $f(\alpha) = 0$.*

Proof. We want to show that there are $c_i \in \mathbb{Q}$ such that if $\alpha = \sum_{i=0}^{\infty} c_i t^{1-in}$ then $f(\alpha) = 0$.

To begin let $c_0 = -1$. We will define the remaining c_i recursively. Define $c_{i,k}$ by $\alpha + q_k t = \sum_{i=0}^{\infty} c_{i,k} t^{1-in}$. Thus, $c_{i,k} = c_i$ when $i \geq 1$, each $c_{0,k}$ is contained in \mathbb{Q} , and $c_{0,1} = 0$.

That α is a root of f is equivalent to

$$1 = \prod_{k=1}^n (\alpha + q_k t) = \prod_{k=1}^n \left(\sum_{i=0}^{\infty} c_{i,k} t^{1-in} \right) = \sum_{i=0}^{\infty} \left(\sum_{\sum_{k=1}^n i_k = i} \left(\prod_{k=1}^n c_{i_k, k} \right) \right) t^{n(1-i)}.$$

Our task is to find c_m 's so that the above is satisfied.

Note that for the above equation to hold we must have

$$0 \cdot t^n = \sum_{\sum_{k=1}^n i_k = 0} \left(\prod_{k=1}^n c_{i_k, k} \right) t^{n(1-0)}.$$

That is,

$$0 = \prod_{k=1}^n c_{0,k}$$

which is an equation we know is satisfied because $c_{0,1} = 0$. Now assume that we have determined $c_0, c_1, \dots, c_{m-1} \in \mathbb{Q}$. We will find $c_m \in \mathbb{Q}$.

Notice that the first coefficient in our Laurent series expansion above which involves c_m is the coefficient for the t^{-nm} term. This follows from the fact that each i_k is nonnegative.

Since

$$\sum_{\sum_{k=1}^n i_k = m} \left(\prod_{k=1}^n c_{i_k, k} \right)$$

is the coefficient of the t^{-nm} term in the expansion of 1, we have

$$0 = \sum_{\sum_{k=1}^n i_k = m} \left(\prod_{k=1}^n c_{i_k, k} \right).$$

The above equation is linear over \mathbb{Q} in the single variable c_m and the coefficient of c_m is nonzero. Indeed, $\sum_{k=1}^n i_k = m$, each $i_k \geq 0$, and $c_0, \dots, c_{m-1} \in \mathbb{Q}$ are assumed to be known quantities. Thus, $c_m \in \mathbb{Q}$. \square

3.2. Matrices representing ring multiplication. By Lemma 4 we have that the field $\mathbb{Q}(t)(\alpha) \leq \mathbb{Q}((t^{-1}))$ is an extension of $\mathbb{Q}(t)$ of degree d where $d \leq n$. It follows that $\mathbb{Z}[t][\alpha]$ is a free $\mathbb{Z}[t]$ -module of rank d with basis $\{1, \alpha, \alpha^2, \dots, \alpha^{d-1}\}$.

For any $y \in \mathbb{Z}[t][\alpha]$, the action of y on $\mathbb{Q}(t)(\alpha)$ by multiplication is a linear transformation that stabilizes $\mathbb{Z}[t][\alpha]$. Thus, we have a representation of $\mathbb{Z}[t][\alpha]$ into the ring of $d \times d$ matrices with entries in $\mathbb{Z}[t]$. We embed the ring of $d \times d$ matrices with entries in $\mathbb{Z}[t]$ into the upper left corner of the ring of $n \times n$ matrices with entries in $\mathbb{Z}[t]$.

By Lemma 4

$$\prod_{i=1}^n (\alpha + q_i t) = 1$$

so each of the following matrices are invertible:

$$\alpha + q_1 t, \alpha + q_2 t, \dots, \alpha + q_n t.$$

(We will be blurring the distinction between the elements of $\mathbb{Z}[t][\alpha]$ and the matrices that represent them.)

For $1 \leq i \leq n-1$, we let $a_i = \alpha + q_{i+1} t$. Since a_i is invertible, it is an element of $\mathrm{GL}_n(\mathbb{Z}[t])$, and hence has determinant ± 1 . By replacing each a_i with its square, we may assume that $a_i \in \mathrm{SL}_n(\mathbb{Z}[t])$ for all i . We let $A = \langle a_1, \dots, a_{n-1} \rangle$ so that A is clearly abelian as it is a representation of multiplication in an integral domain. This group A will satisfy Proposition 3.

3.3. A is free abelian on the a_i . To prove part (i) of Proposition 3 we have to show that if there are $m_i \in \mathbb{Z}$ with

$$\prod_{i=1}^{n-1} a_i^{m_i} = 1$$

then each $m_i = 0$. But the first nonzero term in the Laurent series expansion for α is $-t$, which implies that the first nonzero term in the Laurent series expansion for each a_i is $-t + q_{i+1} t = p_i t$. Hence, the first nonzero term of

$$\prod_{i=1}^{n-1} a_i^{m_i} = 1$$

is

$$\prod_{i=1}^{n-1} (p_i t)^{m_i} = t^0.$$

Thus

$$\prod_{i=1}^{n-1} p_i^{m_i} = 1$$

and it follows by the uniqueness of prime factorization that $m_i = 0$ for all i as desired.

Thus, part (i) of Proposition 3 is proved.

3.4. A is diagonalizable. Recall that α is a $d \times d$ matrix with entries in $\mathbb{Z}[t]$ where d is the degree of the minimal polynomial of α over $\mathbb{Q}(t)$. Let that minimal polynomial be $q(x)$. Because the characteristic of $\mathbb{Q}(t)$ equals 0, $q(x)$ has distinct roots in $\mathbb{Q}(t)(\alpha)$.

Let $Q(x)$ be the characteristic polynomial of the matrix α . The polynomial Q also has degree d and leading coefficient ± 1 with $Q(\alpha) = 0$. Therefore, $q = \pm Q$. Hence, Q has distinct roots in $\mathbb{Q}(t)(\alpha)$ which implies that α is diagonalizable over $\mathbb{Q}(t)(\alpha) \leq \mathbb{Q}((t^{-1}))$. That is to say that there is some $g \in \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ such that $g\alpha g^{-1}$ is diagonal.

Because every element of $\mathbb{Z}[t][\alpha]$ is a linear combination of powers of α , we have that $g(\mathbb{Z}[t][\alpha])g^{-1}$ is a set of diagonal matrices. In particular, we have proved part (ii) of Proposition 3.

3.5. A has trivial stabilizers. To prove part (iii) of Proposition 3 we begin with the following

Lemma 5. *If $\Gamma \leq \mathrm{SL}_n(\mathbb{Q}[t])$ is bounded under the valuation for $\mathbb{Q}((t^{-1}))$, then the eigenvalues for any $\gamma \in \Gamma$ lie in $\overline{\mathbb{Q}}$.*

Proof. We let X be the Euclidean building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. By the Bruhat–Tits fixed point theorem, $\Gamma z = z$ for some $z \in X$.

Let $x_0 \in X$ be the vertex stabilized by $\mathrm{SL}_n(\mathbb{Q}[[t^{-1}]])$. We denote a diagonal matrix in $\mathrm{GL}_n(\mathbb{Q}((t^{-1})))$ with entries $s_1, s_2, \dots, s_n \in \mathbb{Q}((t^{-1}))^\times$ by $D(s_1, s_2, \dots, s_n)$, and we let $\mathfrak{S} \subseteq X$ be the sector based at x_0 and containing vertices of the form $D(t^{m_1}, t^{m_2}, \dots, t^{m_n})x_0$ where each $m_i \in \mathbb{Z}$ and $m_1 \geq m_2 \geq \dots \geq m_n$.

The sector \mathfrak{S} is a fundamental domain for the action of $\mathrm{SL}_n(\mathbb{Q}[t])$ on X [So] which implies that there is some $h \in \mathrm{SL}_n(\mathbb{Q}[t])$ with $hz \in \mathfrak{S}$.

Clearly we have $(h\Gamma h^{-1})hz = hz$, and since eigenvalues of $h\Gamma h^{-1}$ are the same as those for Γ , we may assume that Γ fixes a vertex $z \in \mathfrak{S}$.

Fix $m_1, \dots, m_n \in \mathbb{Z}$, $m_1 \geq \dots \geq m_n \geq 0$, such that $z = D(t^{m_1}, \dots, t^{m_n})x_0$. Without loss of generality, there is a partition of n – say $\{k_1, \dots, k_\ell\}$ – such that

$$\{m_1, \dots, m_n\} = \{q_1, \dots, q_1, q_2, \dots, q_2, \dots, q_\ell, \dots, q_\ell\}$$

where each q_i occurs exactly k_i times and

$$q_1 > q_2 > \cdots > q_\ell.$$

We have that $D(t^{m_1}, \dots, t^{m_n})^{-1} \Gamma D(t^{m_1}, \dots, t^{m_n}) x_0 = x_0$. That gives us, $D(t^{m_1}, \dots, t^{m_n})^{-1} \Gamma D(t^{m_1}, \dots, t^{m_n}) \subset \mathrm{SL}_n(\mathbb{Q}[[t^{-1}]])$. Furthermore, a trivial calculation of resulting valuation restrictions for the entries of

$$D(t^{m_1}, \dots, t^{m_n}) \mathrm{SL}_n(\mathbb{Q}[[t^{-1}]]) D(t^{m_1}, \dots, t^{m_n})^{-1}$$

shows that Γ is contained in a subgroup of $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ that is isomorphic to

$$\prod_{i=1}^{\ell} \mathrm{SL}_{k_i}(\mathbb{Q}) \ltimes U$$

where $U \leq \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ is a group of upper-triangular unipotent matrices.

The lemma is proved. \square

Our proof of Proposition 3 will conclude by proving

Lemma 6. *No nontrivial element of A fixes a point in the Euclidean building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$.*

Proof. Suppose $a \in A$ fixes a point in the building. We will show that $a = 1$. Let $F(x) \in \mathbb{Z}[t][x]$ be the characteristic polynomial for $a \in \mathrm{SL}_n(\mathbb{Z}[t])$. Then

$$F(x) = \pm \prod_{i=1}^n (x - \beta_i)$$

where each $\beta_i \in \mathbb{Q}((t^{-1}))$ is an eigenvalue of a . By the previous lemma, each $\beta_i \in \overline{\mathbb{Q}}$. Hence, each $\beta_i \in \mathbb{Q} = \overline{\mathbb{Q}} \cap \mathbb{Q}((t^{-1}))$. It follows that $F(x) \in \mathbb{Z}[x]$ so that each β_i is an algebraic integer contained in \mathbb{Q} . We conclude that each β_i is contained in \mathbb{Z} .

Recall, that a has determinant 1, and that the determinant of a can be expressed as $\prod_{i=1}^n \beta_i$. Hence, each β_i is a unit in \mathbb{Z} , so each eigenvalue $\beta_i = \pm 1$. It follows – by the diagonalizability of a – that a is a finite order element of $A \cong \mathbb{Z}^{n-1}$. That is, $a = 1$. \square

We have completed our proof of Proposition 3.

4. Body of the proof

Let $P \leq \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ be the subgroup where each of the first $n - 1$ entries along the bottom row equal 0. Let $R_u(P) \leq P$ be the subgroup of elements that contain a $(n - 1) \times (n - 1)$ copy of the identity matrix in the upper left corner. Thus $R_u(P) \cong \mathbb{Q}((t^{-1}))^{n-1}$ with the operation of vector addition.

Let $L \leq P$ be the copy of $\mathrm{SL}_{n-1}(\mathbb{Q}((t^{-1})))$ in the upper left corner of $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. We apply Proposition 3 to L (notice that the n in the proposition is now an $n - 1$) to derive a subgroup $A \leq L$ that is isomorphic to \mathbb{Z}^{n-2} . By the same proposition, there is a matrix $g \in L$ such that gAg^{-1} is diagonal.

Let $b \in \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ be the diagonal matrix given in the notation from the proofs of Lemmas 2 and 5 as $D(t, t, \dots, t, t^{-(n-1)})$. Note that $b \in P$ commutes with L , and therefore, with A . Thus the Zariski closure of the group generated by b and A determines an apartment in X , namely $g^{-1}\mathcal{A}$ where \mathcal{A} is the apartment corresponding to the diagonal subgroup of $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$.

4.1. Actions on $g^{-1}\mathcal{A}$. If $x_* \in g^{-1}\mathcal{A}$, then it follows from Proposition 3 that the convex hull of the orbit of x_* under A is an $(n - 2)$ -dimensional affine space that we will name V_{x_*} . Furthermore, the orbit Ax_* forms a lattice in the space V_{x_*} .

We let $g^{-1}\mathcal{A}(\infty)$ be the visual boundary of $g^{-1}\mathcal{A}$ in the Tits boundary of X . Recall that the Tits boundary of X is isomorphic to the spherical building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. The definition of visual boundary used above is the standard definition from CAT(0) geometry.

The visual boundary of V_{x_*} is clearly an equatorial sphere in $g^{-1}\mathcal{A}(\infty)$. Precisely, we let P^- be the transpose of P . Then P and P^- are opposite vertices in $g^{-1}\mathcal{A}(\infty)$. It follows that there is a unique sphere in $g^{-1}\mathcal{A}(\infty)$ that is realized by all points equidistant to P and P^- . We call this sphere S_{P, P^-} .

Lemma 7. *The visual boundary of V_{x_*} equals S_{P, P^-} .*

Proof. Since $g \in P \cap P^-$, it suffices to prove that gV_{x_*} is the sphere in the boundary of \mathcal{A} that is determined by the vertices P and P^- .

Note that gV_{x_*} is a finite Hausdorff distance from any orbit of a point in \mathcal{A} under the action of the diagonal subgroup of L . The result follows by observing that the inverse transpose map on $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ stabilizes diagonal matrices while interchanging P and P^- . \square

We let R_1, R_2, \dots, R_{n-1} be the standard root subgroups of $R_u(P)$. Recall that associated to each R_i there is a closed geodesic hemisphere $H_i \subseteq \mathcal{A}(\infty)$ such that any nontrivial element of R_i fixes H_i pointwise and translates any point in the open hemisphere $\mathcal{A}(\infty) - H_i$ outside of $\mathcal{A}(\infty)$. Note that ∂H_i is a codimension 1 geodesic sphere in $\mathcal{A}(\infty)$.

We let $M \subseteq g^{-1}\mathcal{A}(\infty)$ be the union of chambers in $g^{-1}\mathcal{A}(\infty)$ that contain the vertex P . There is also an equivalent geometric description of M :

Lemma 8. *The union of chambers $M \subseteq g^{-1}\mathcal{A}(\infty)$ can be realized as an $(n-2)$ -simplex. Furthermore,*

$$M = \bigcap_{i=1}^{n-1} g^{-1}H_i$$

and, when M is realized as a single simplex, each of the $n-1$ faces of M is contained in a unique equatorial sphere $g^{-1}\partial H_i = \partial g^{-1}H_i$.

Proof. Let $M' \subseteq \mathcal{A}(\infty)$ be the union of chambers in $\mathcal{A}(\infty)$ containing the vertex P . Since $M = g^{-1}M'$, it suffices to prove that M' is an $(n-2)$ -simplex with $M' = \bigcap_{i=1}^{n-1} H_i$ and with each face of M' contained in a unique ∂H_i .

For any nonempty, proper subset $I \subseteq \{1, 2, \dots, n\}$, we let V_I be the $|I|$ -dimensional vector subspace of $\mathbb{Q}((t^{-1}))^n$ spanned by the coordinates given by I , and we let P_I be the stabilizer of V_I in $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. For example, $P = P_{\{1, 2, \dots, n-1\}}$.

Recall that the vertices of $\mathcal{A}(\infty)$ are given by the parabolic groups P_I , that edges connect P_I and $P_{I'}$ exactly when $I \subseteq I'$ or $I' \subseteq I$, and that the remaining simplicial description of $\mathcal{A}(\infty)$ is given by the condition that $\mathcal{A}(\infty)$ is a flag complex.

We let \mathcal{V} be the set of vertices in $\mathcal{A}(\infty)$ of the form P_J where $\emptyset \neq J \subseteq \{1, 2, \dots, n-1\}$. Note that M' is exactly the set of vertices \mathcal{V} together with the simplices described by the incidence relations inherited from $\mathcal{A}(\infty)$. Thus, M' is easily seen to be isomorphic to a barycentric subdivision of an abstract $(n-2)$ -simplex. Indeed, if \bar{M}' is the abstract simplex on vertices $P_{\{1\}}, P_{\{2\}}, \dots, P_{\{n-1\}}$, then a simplex of dimension k in \bar{M}' corresponds to a unique $P_J \in \mathcal{V}$ with $|J| = k+1$. So we have that M' can be topologically realized as an $(n-2)$ -simplex.

Let F_i be a face of the simplex \bar{M}' . Then there is some $1 \leq i \leq n-1$ such that the set of vertices of F_i is exactly $\{P_{\{1\}}, P_{\{2\}}, \dots, P_{\{n-1\}}\} - P_{\{i\}}$.

Note that $R_i V_I = V_I$ exactly when $n \in I$ implies $i \in I$. It follows that R_i fixes M' pointwise, and thus $M' \subseteq H_i$ for all $1 \leq i \leq n-1$. Furthermore, if $P_I \in H_i$ for all $1 \leq i \leq n-1$, then $R_i P_I = P_I$ for all i so that $n \in I$ implies $i \in I$ for all $1 \leq i \leq n-1$. As I must be a proper subset of $\{1, 2, \dots, n\}$, we have $P_I \in \mathcal{V}$, so that $M' = \bigcap_{i=1}^{n-1} H_i$.

All that remains to be verified for this lemma is that $F_i \subseteq \partial H_i$. For this fact, recall that F_i is comprised of $(n-3)$ -simplices in $\mathcal{A}(\infty)$ whose vertices are given by P_J where $J \subseteq \{1, 2, \dots, n-1\} - \{i\}$. Hence, if $\sigma \subseteq \mathcal{A}(\infty)$ is an $(n-3)$ simplex of $\mathcal{A}(\infty)$ with $\sigma \subseteq F_i$, then σ is a face of exactly 2 chambers in $\mathcal{A}(\infty)$: \mathbb{C}_P and $\mathbb{C}_{P_{J'}}$ where \mathbb{C}_P contains P and thus $\mathbb{C}_P \subseteq M'$, and $\mathbb{C}_{P_{J'}}$ contains $P_{J'}$ where $J' = \{1, 2, \dots, n\} - \{i\}$ and thus $\mathbb{C}_{P_{J'}} \not\subseteq M'$. Furthermore, $\sigma = \mathbb{C}_P \cap \mathbb{C}_{P_{J'}}$.

Since $R_i V_{J'} \neq V_{J'}$, it follows that $\mathbb{C}_{P_{J'}}$ is not fixed by R_i . Since \mathbb{C}_{P_J} is fixed by R_i we have that $\sigma = \mathbb{C}_P \cap \mathbb{C}_{P_{J'}} \subseteq \partial H_i$. Therefore, $F_i \subseteq \partial H_i$. \square

For any vertex $y \in X$, we let $C_y \subseteq X$ be the union of sectors based at y and limiting to a chamber in M . Thus, C_y is a cone. Note also that because any chamber in $g^{-1}\mathcal{A}(\infty)$ has diameter less than $\pi/2$, it follows that $M \cap S_{P,P^-} = \emptyset$. Therefore, if we choose x_* , $y \in g^{-1}\mathcal{A}$ such that x_* is closer to P than y , then $C_y \subseteq g^{-1}\mathcal{A}$ and $V_{x_*} \cap C_y$ is a simplex of dimension $n - 2$.

We will set on a fixed choice of y before x_* , and we will choose y to satisfy the below

Lemma 9. *There is some $y \in g^{-1}\mathcal{A}$ such that the $\mathbb{Q}[[t^{-1}]]$ -points of $R_u(P)$ fix C_y pointwise.*

Proof. Let x_0 be the point in X stabilized by $\mathrm{SL}_n(\mathbb{Q}[[t^{-1}]])$. Recall that $R_u(P)M = M$ so that the $\mathbb{Q}[[t^{-1}]]$ -points of $R_u(P)$ fix C_{x_0} pointwise.

Because $M \subseteq g^{-1}\mathcal{A}(\infty)$, there is a $y \in C_{x_0} \cap g^{-1}\mathcal{A}$. Any such y satisfies the lemma. \square

Choose e such that with $x_* = e$ as above and with y as in Lemma 9, there exists a fundamental domain D_e for the action of A on V_e that is contained in C_y . The choice of e can be made by travelling arbitrarily far from y along a geodesic ray in $g^{-1}\mathcal{A}$ that limits to P .

By the choice of D_e we have that

$$AD_e = V_e$$

and that the $\mathbb{Q}[[t^{-1}]]$ -points of $R_u(P)$ fix D_e .

4.2. The filtration. We let

$$X_0 = \mathrm{SL}_n(\mathbb{Z}[t])D_e$$

and for any $i \in \mathbb{N}$ we choose an $\mathrm{SL}_n(\mathbb{Z}[t])$ -invariant and cocompact space $X_i \subseteq X$ somewhat arbitrarily to satisfy the inclusions

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq \bigcup_{i=1}^{\infty} X_i = X.$$

In our present context, Brown's criterion takes on the following form [Br1].

Brown's Filtration Criterion. *By Lemma 2, the group $\mathrm{SL}_n(\mathbb{Z}[t])$ is not of type FP_{n-1} if for any $i \in \mathbb{N}$, there exists some class in the homology group $\tilde{H}_{n-2}(X_0, \mathbb{Z})$ which is nonzero in $\tilde{H}_{n-2}(X_i, \mathbb{Z})$.*

4.3. Translation to P moves away from filtration sets. The following is essentially Mahler's compactness criterion.

Lemma 10. *Given any $i \in \mathbb{N}$, there is some $k \in \mathbb{N}$ such that $b^k e \notin X_i$.*

Proof. The lemma follows from showing that the sequence

$$\{\mathrm{SL}_n(\mathbb{Z}[t])b^k e\}_k \subseteq \mathrm{SL}_n(\mathbb{Z}[t]) \setminus X$$

is unbounded.

Since stabilizers of points in X are bounded subgroups of $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$, the claim above follows from showing that the sequence

$$\{\mathrm{SL}_n(\mathbb{Z}[t])b^k\}_k \subseteq \mathrm{SL}_n(\mathbb{Z}[t]) \setminus \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$$

is unbounded.

But bounded sets in $\mathrm{SL}_n(\mathbb{Z}[t]) \setminus \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ do not contain sequences of elements $\{\mathrm{SL}_n(\mathbb{Z}[t])g_\ell\}_\ell$ such that $1 \in \overline{g_\ell^{-1}(\mathrm{SL}_n(\mathbb{Z}[t]) - \{1\})g_\ell}$. And clearly b^k 's contract some root groups to 1. Thus none of the sequences above is bounded. \square

4.4. Applying Brown's criterion. As is described by Brown's criterion, we will prove Theorem 1 by fixing X_i and finding an $(n-2)$ -cycle in X_0 that is nontrivial in the homology of X_i .

Recall that we denote the standard root subgroups of $R_u(P)$ by R_1, \dots, R_{n-1} . Each group $g^{-1}R_jg$ determines a family of parallel walls in $g^{-1}\mathcal{A}$. By Lemma 8, each face of the cone C_y is contained in a wall of one of these families.

Choose $r_j \in g^{-1}R_jg$ for all j such that $b^k e$ is contained in the wall determined by r_j where k is determined by i as in Lemma 10. In particular, $r_j b^k e = b^k e$.

The intersection of the fixed point sets in $g^{-1}\mathcal{A}$ of the elements r_1, \dots, r_{n-1} determine a cone that we name Z . Note that Z is contained in $-$ and is a finite Hausdorff distance from $-$ the cone C_y .

Let $Z^- \subseteq g^{-1}\mathcal{A}$ be the closure of the set of points in $g^{-1}\mathcal{A}$ that are fixed by none of the r_j . The set Z^- is a cone based at $b^k e$, containing y , and asymptotically containing the vertex P^- .

As the walls of Z^- are parallel to those of Z – and hence of C_y , we have that $Z^- \cap V_e$ is an $(n-2)$ -dimensional simplex. We will name this simplex σ .

The component of $Z^- - V_e$ that contains $b^k e$ is an $(n-1)$ -simplex that has σ as a face. Call this $(n-1)$ simplex Y .

For any $\ell \in \mathbb{N}$, there are exactly 2^{n-1} possible subsets of the set $\{r_1^\ell, \dots, r_{n-1}^\ell\}$. For each such subset S_ℓ , we let

$$Y_{S_\ell} = \left(\prod_{g \in S_\ell} g \right) Y$$

and

$$\sigma_{S_\ell} = \left(\prod_{g \in S_\ell} g \right) \sigma.$$

Notice that the product of group elements in the equations above are well-defined regardless of the order of the multiplication since $R_u(P)$ is abelian. In the degenerate cases, $\prod_{g \in \emptyset} g = 1$, so $Y_\emptyset = Y$ and $\sigma_\emptyset = \sigma$.

For any $\ell \in \mathbb{N}$, we let $Y_\ell = \bigcup_{S_\ell} Y_{S_\ell}$. Because the wall in $g^{-1}\mathcal{A}$ determined by r_j^ℓ is the same as the wall determined by r_j , the space Y_ℓ is a closed ball containing $b^k e$ whose boundary sphere is $\bigcup_{S_\ell} \sigma_{S_\ell}$. Indeed the simplicial decomposition of Y_ℓ described above is isomorphic to the simplicial decomposition of the unit ball in \mathbb{R}^{n-1} that is given by the $n-1$ hyperplanes defined by setting a coordinate equal to 0.

Let $\omega_\ell = \bigcup_{S_\ell} \sigma_{S_\ell}$. Thus $\omega_\ell = \partial Y_\ell$. Furthermore, the building X is $(n-1)$ -dimensional and contractible, so any $(n-1)$ -chain with boundary equal to ω_ℓ must contain Y_ℓ and thus $b^k e$. That is for all $\ell \in \mathbb{N}$

$$[\omega_\ell] \neq 0 \in \tilde{H}_{n-2}(X - b^k e, \mathbb{Z})$$

If we can show that $\omega_\ell \subseteq X_0$ for some choice of ℓ , then we will have proved our main theorem by application of Brown's criterion since we would have

$$[\omega_\ell] \neq 0 \in \tilde{H}_{n-2}(X_i, \mathbb{Z})$$

by Lemma 10.

Lemma 11. *There exists some $\ell \in \mathbb{N}$ such that $\omega_\ell \subseteq X_0$.*

Proof. For any $u \in R_u(P)$ there is a decomposition $u = u' u''$ where the entries of $u' \in R_u(P)$ are contained in $\mathbb{Q}[t]$ and the entries of $u'' \in R_u(P)$ are contained in $\mathbb{Q}[[t^{-1}]]$.

For any $a \in A$ and $u \in R_u(P)$ there is a power $\ell(a, u) \in \mathbb{N}$ such that

$$(a^{-1} u^{\ell(a, u)} a)' = ((a^{-1} u a)')^{\ell(a, u)} \in \mathrm{SL}_n(\mathbb{Z}[t]).$$

(For the above equality recall that $A \leq L$ normalizes $R_u(P)$ and the group operation on $R_u(P)$ is vector addition.)

There are only finitely many $a \in A$ such that $a D_e \cap \sigma \neq \emptyset$ (or equivalently, such that $a D_e \cap Z^- \neq \emptyset$). Call this finite set $\mathcal{D} \subseteq A$.

At this point we fix

$$\ell = \prod_{a \in \mathcal{D}} \prod_{i=1}^{n-1} \ell(a, r_i).$$

Thus,

$$\left[a^{-1} \left(\prod_{g \in S_\ell} g \right) a \right]' \in \mathrm{SL}_n(\mathbb{Z}[t])$$

for any $a \in \mathcal{D}$ and any $S_\ell \subseteq \{r_i^\ell\}_{i=1}^{n-1}$.

Because $\omega_\ell = \bigcup_{S_\ell} \sigma_{S_\ell}$ and $\sigma_{S_\ell} = (\prod_{g \in S_\ell} g)\sigma = (\prod_{g \in S_\ell} g)(AD_e \cap Z^-)$, we can finish our proof of this lemma by showing

$$\left(\prod_{g \in S_\ell} g \right) a D_e \subseteq X_0$$

for each $a \in \mathcal{D} \subseteq A \leq \mathrm{SL}_n(\mathbb{Z}[t])$ and each $S_\ell \subseteq \{r_i^\ell\}_{i=1}^{n-1}$. For this, recall that the $\mathbb{Q}[[t^{-1}]]$ -points of $R_u(P)$ fix D_e and thus

$$\begin{aligned} \left(\prod_{g \in S_\ell} g \right) a D_e &= a \left[a^{-1} \left(\prod_{g \in S_\ell} g \right) a \right] D_e \\ &= a \left[a^{-1} \left(\prod_{g \in S_\ell} g \right) a \right]' \left[a^{-1} \left(\prod_{g \in S_\ell} g \right) a \right]'' D_e \\ &= a \left[a^{-1} \left(\prod_{g \in S_\ell} g \right) a \right]' D_e \\ &\subseteq \mathrm{SL}_n(\mathbb{Z}[t]) D_e \\ &= X_0 \end{aligned}$$

□

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