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# $\mathrm{SL}_n(\mathbb{Z}[t])$ is not $\mathrm{FP}_{n-1}$

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**Abstract.** We prove the result from the title using the geometry of Euclidean buildings.

**Mathematics Subject Classification (2000).** 20F65.

**Keywords.** Euclidean buildings, finiteness properties, geometric group theory.

## 1. Introduction

Little is known about the finiteness properties of  $\mathrm{SL}_n(\mathbb{Z}[t])$  for arbitrary  $n$ .

In 1959 Nagao proved that if  $k$  is a field then  $\mathrm{SL}_2(k[t])$  is a free product with amalgamation [Na]. It follows from his description that  $\mathrm{SL}_2(\mathbb{Z}[t])$  and its abelianization are not finitely generated.

In 1977 Suslin proved that when  $n \geq 3$ ,  $\mathrm{SL}_n(\mathbb{Z}[t])$  is finitely generated by elementary matrices [Su]. It follows that  $H_1(\mathrm{SL}_n(\mathbb{Z}[t]), \mathbb{Z})$  is trivial when  $n \geq 3$ .

More recent, Krstić and McCool proved in [Kr-Mc] that  $\mathrm{SL}_3(\mathbb{Z}[t])$  is not finitely presented.

In this paper we provide a generalization of the results of Nagao and Krstić–McCool mentioned above for the groups  $\mathrm{SL}_n(\mathbb{Z}[t])$ .

**Theorem 1.** *If  $n \geq 2$ , then  $\mathrm{SL}_n(\mathbb{Z}[t])$  is not of type  $\mathrm{FP}_{n-1}$ .*

Recall that a group  $\Gamma$  is of type  $\mathrm{FP}_m$  if there exists a projective resolution of  $\mathbb{Z}$  as the trivial  $\mathbb{Z}\Gamma$  module

$$P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$$

where each  $P_i$  is a finitely generated, projective  $\mathbb{Z}\Gamma$  module.

In particular, Theorem 1 implies that there is no  $K(\mathrm{SL}_n(\mathbb{Z}[t]), 1)$  with finite  $(n-1)$ -skeleton, where  $K(G, 1)$  is the Eilenberg–Mac Lane space for  $G$ .

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**1.1. Outline of the paper.** The general outline of this paper is modelled on the proofs in [Bu-Wo1] and [Bu-Wo2], though some important modifications have to be made to carry out the proof in this setting.

As in [Bu-Wo1] and [Bu-Wo2], our approach is to apply Brown’s filtration criterion [Br1]. Here we will examine the action of  $\mathrm{SL}_n(\mathbb{Z}[t])$  on the locally infinite Euclidean building for  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ . In Section 2 we will show that the infinite groups that arise as cell stabilizers for this action are of type  $\mathrm{FP}_m$  for all  $m$ , which is a technical condition that is needed for our application of Brown’s criterion.

In Section 3 we will demonstrate the existence of a family of diagonal matrices that will imply the existence of a “nice” isometrically embedded codimension 1 Euclidean space in the building for  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ . In [Bu-Wo1] analogous families of diagonal matrices were constructed using some standard results from the theory of algebraic groups over locally compact fields. Because  $\mathbb{Q}((t^{-1}))$  is not locally compact, our treatment in Section 3 is quite a bit more hands on.

Section 4 contains the main body of our proof. We use translates of portions of the codimension 1 Euclidean subspace found in Section 3 to construct spheres in the Euclidean building for  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$  (also of codimension 1). These spheres will lie “near” an orbit of  $\mathrm{SL}_n(\mathbb{Z}[t])$ , but will be nonzero in the homology of cells “not as near” the same  $\mathrm{SL}_n(\mathbb{Z}[t])$  orbit. Theorem 1 will then follow from Brown’s criterion.

**1.2. Background material.** Our proof relies heavily on the geometry of the Euclidean and spherical buildings for  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ . A good source of information for the former topic is Chapter 6 of [Br2]. For the latter, we recommend Chapter 5 of [Ti].

## 2. Stabilizers

**Lemma 2.** *If  $X$  is the Euclidean building for  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ , then the  $\mathrm{SL}_n(\mathbb{Z}[t])$  stabilizers of cells in  $X$  are  $\mathrm{FP}_m$  for all  $m$ .*

*Proof.* Let  $x_0 \in X$  be the vertex stabilized by  $\mathrm{SL}_n(\mathbb{Q}[[t^{-1}]])$ . We denote a diagonal matrix in  $\mathrm{GL}_n(\mathbb{Q}((t^{-1})))$  with entries  $s_1, s_2, \dots, s_n \in \mathbb{Q}((t^{-1}))^\times$  by  $D(s_1, s_2, \dots, s_n)$ , and we let  $\mathfrak{S} \subseteq X$  be the sector based at  $x_0$  and containing vertices of the form  $D(t^{m_1}, t^{m_2}, \dots, t^{m_n})x_0$  where each  $m_i \in \mathbb{Z}$  and  $m_1 \geq m_2 \geq \dots \geq m_n$ .

The sector  $\mathfrak{S}$  is a fundamental domain for the action of  $\mathrm{SL}_n(\mathbb{Q}[t])$  on  $X$  (see [So]). In particular, for any vertex  $z \in X$ , there is some  $h'_z \in \mathrm{SL}_n(\mathbb{Q}[t])$  and some integers  $m_1 \geq m_2 \geq \dots \geq m_n$  with  $z = h'_z D_z(t^{m_1}, t^{m_2}, \dots, t^{m_n})x_0$ . We let  $h_z = h'_z D_z(t^{m_1}, t^{m_2}, \dots, t^{m_n})$ .

For any  $N \in \mathbb{N}$ , let  $W_N$  be the  $(N + 1)$ -dimensional vector space

$$W_N = \{ p(t) \in \mathbb{C}[t] \mid \deg(p(t)) \leq N \}$$

which is endowed with the obvious  $\mathbb{Q}$ –structure. If  $N_1, \dots, N_{n^2}$  in  $\mathbb{N}$  are arbitrary then let

$$\mathbf{G}_{\{N_1, \dots, N_{n^2}\}} = \{\mathbf{x} \in \prod_{i=1}^{n^2} W_{N_i} \mid \det(\mathbf{x}) = 1\}$$

where  $\det(\mathbf{x})$  is a polynomial in the coordinates of  $\mathbf{x}$ . To be more precise this is obtained from the usual determinant function when one considers the usual  $n \times n$  matrix presentation of  $\mathbf{x}$ , and calculates the determinant in  $\mathrm{Mat}_n(\mathbb{C}[t])$ .

For our choice of vertex  $z \in X$  above, the stabilizer of  $z$  in  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$  equals  $h_z \mathrm{SL}_n(\mathbb{Q}[[t^{-1}]]) h_z^{-1}$ . And with our fixed choice of  $h_z$ , there clearly exist some  $N_i^z \in \mathbb{N}$  such that the stabilizer of the vertex  $z$  in  $\mathrm{SL}_n(\mathbb{Q}[t])$  is  $\mathbf{G}_{\{N_1^z, \dots, N_{n^2}^z\}}(\mathbb{Q})$ . Furthermore, conditions on  $N_i^z$  force a group structure on  $\mathbf{G}_z = \mathbf{G}_{\{N_1^z, \dots, N_{n^2}^z\}}$ . Therefore, the stabilizer of  $z$  in  $\mathrm{SL}_n(\mathbb{Q}[t])$  is the  $\mathbb{Q}$ –points of the affine  $\mathbb{Q}$ –group  $\mathbf{G}_z$ , and the stabilizer of  $z$  in  $\mathrm{SL}_n(\mathbb{Z}[t])$  is  $\mathbf{G}_z(\mathbb{Z})$ .

The action of  $\mathrm{SL}_n(\mathbb{Q}[t])$  on  $X$  is type preserving, so if  $\sigma \subset \mathfrak{S}$  is a simplex with vertices  $z_1, z_2, \dots, z_m$ , then the stabilizer of  $\sigma$  in  $\mathrm{SL}_n(\mathbb{Z}[t])$  is simply

$$(\mathbf{G}_{z_1} \cap \dots \cap \mathbf{G}_{z_m})(\mathbb{Z}).$$

That is, the stabilizer of  $\sigma$  in  $\mathrm{SL}_n(\mathbb{Z}[t])$  is an arithmetic group, and Borel–Serre proved that any such group is  $\mathrm{FP}_m$  for all  $m$  [Bo-Se].  $\square$

### 3. Polynomial points of tori

This section is devoted exclusively to a proof of the following

**Proposition 3.** *There is a group  $A \leq \mathrm{SL}_n(\mathbb{Z}[t])$  such that the following holds:*

- (i)  $A \cong \mathbb{Z}^{n-1}$ .
- (ii) *There is some  $g \in \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$  such that  $gAg^{-1}$  is a group of diagonal matrices.*
- (iii) *No nontrivial element of  $A$  fixes a point in the Euclidean building for  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ .*

The proof of this proposition is modelled on a classical approach to finding diagonalizable subgroups of  $\mathrm{SL}_n(\mathbb{Z})$ . The proof will take a few steps.

**3.1. A polynomial over  $\mathbb{Z}[t]$  with roots in  $\mathbb{Q}((t^{-1}))$ .** Let  $\{p_1, p_2, p_3, \dots\} = \{2, 3, 5, \dots\}$  be the sequence of prime numbers. Let  $q_1 = 1$ . For  $2 \leq i \leq n$ , let  $q_i = p_{i-1} + 1$ .



Let  $f(x) \in \mathbb{Z}[t][x]$  be the polynomial given by

$$f(x) = \left[ \prod_{i=1}^n (x + q_i t) \right] - 1.$$

It will be clear by the conclusion of our proof that  $f(x)$  is irreducible over  $\mathbb{Q}(t)$ , but we will not need to use this directly.

**Lemma 4.** *There is some  $\alpha \in \mathbb{Q}((t^{-1}))$  such that  $f(\alpha) = 0$ .*

*Proof.* We want to show that there are  $c_i \in \mathbb{Q}$  such that if  $\alpha = \sum_{i=0}^{\infty} c_i t^{1-in}$  then  $f(\alpha) = 0$ .

To begin let  $c_0 = -1$ . We will define the remaining  $c_i$  recursively. Define  $c_{i,k}$  by  $\alpha + q_k t = \sum_{i=0}^{\infty} c_{i,k} t^{1-in}$ . Thus,  $c_{i,k} = c_i$  when  $i \geq 1$ , each  $c_{0,k}$  is contained in  $\mathbb{Q}$ , and  $c_{0,1} = 0$ .

That  $\alpha$  is a root of  $f$  is equivalent to

$$1 = \prod_{k=1}^n (\alpha + q_k t) = \prod_{k=1}^n \left( \sum_{i=0}^{\infty} c_{i,k} t^{1-in} \right) = \sum_{i=0}^{\infty} \left( \sum_{\sum_{k=1}^n i_k = i} \left( \prod_{k=1}^n c_{i_k, k} \right) \right) t^{n(1-i)}.$$

Our task is to find  $c_m$ 's so that the above is satisfied.

Note that for the above equation to hold we must have

$$0 \cdot t^n = \sum_{\sum_{k=1}^n i_k = 0} \left( \prod_{k=1}^n c_{i_k, k} \right) t^{n(1-0)}.$$

That is,

$$0 = \prod_{k=1}^n c_{0,k}$$

which is an equation we know is satisfied because  $c_{0,1} = 0$ . Now assume that we have determined  $c_0, c_1, \dots, c_{m-1} \in \mathbb{Q}$ . We will find  $c_m \in \mathbb{Q}$ .

Notice that the first coefficient in our Laurent series expansion above which involves  $c_m$  is the coefficient for the  $t^{-nm}$  term. This follows from the fact that each  $i_k$  is nonnegative.

Since

$$\sum_{\sum_{k=1}^n i_k = m} \left( \prod_{k=1}^n c_{i_k, k} \right)$$

is the coefficient of the  $t^{-nm}$  term in the expansion of 1, we have

$$0 = \sum_{\sum_{k=1}^n i_k = m} \left( \prod_{k=1}^n c_{i_k, k} \right).$$

The above equation is linear over  $\mathbb{Q}$  in the single variable  $c_m$  and the coefficient of  $c_m$  is nonzero. Indeed,  $\sum_{k=1}^n i_k = m$ , each  $i_k \geq 0$ , and  $c_0, \dots, c_{m-1} \in \mathbb{Q}$  are assumed to be known quantities. Thus,  $c_m \in \mathbb{Q}$ .  $\square$

**3.2. Matrices representing ring multiplication.** By Lemma 4 we have that the field  $\mathbb{Q}(t)(\alpha) \leq \mathbb{Q}((t^{-1}))$  is an extension of  $\mathbb{Q}(t)$  of degree  $d$  where  $d \leq n$ . It follows that  $\mathbb{Z}[t][\alpha]$  is a free  $\mathbb{Z}[t]$ -module of rank  $d$  with basis  $\{1, \alpha, \alpha^2, \dots, \alpha^{d-1}\}$ .

For any  $y \in \mathbb{Z}[t][\alpha]$ , the action of  $y$  on  $\mathbb{Q}(t)(\alpha)$  by multiplication is a linear transformation that stabilizes  $\mathbb{Z}[t][\alpha]$ . Thus, we have a representation of  $\mathbb{Z}[t][\alpha]$  into the ring of  $d \times d$  matrices with entries in  $\mathbb{Z}[t]$ . We embed the ring of  $d \times d$  matrices with entries in  $\mathbb{Z}[t]$  into the upper left corner of the ring of  $n \times n$  matrices with entries in  $\mathbb{Z}[t]$ .

By Lemma 4

$$\prod_{i=1}^n (\alpha + q_i t) = 1$$

so each of the following matrices are invertible:

$$\alpha + q_1 t, \alpha + q_2 t, \dots, \alpha + q_n t.$$

(We will be blurring the distinction between the elements of  $\mathbb{Z}[t][\alpha]$  and the matrices that represent them.)

For  $1 \leq i \leq n-1$ , we let  $a_i = \alpha + q_{i+1} t$ . Since  $a_i$  is invertible, it is an element of  $\mathrm{GL}_n(\mathbb{Z}[t])$ , and hence has determinant  $\pm 1$ . By replacing each  $a_i$  with its square, we may assume that  $a_i \in \mathrm{SL}_n(\mathbb{Z}[t])$  for all  $i$ . We let  $A = \langle a_1, \dots, a_{n-1} \rangle$  so that  $A$  is clearly abelian as it is a representation of multiplication in an integral domain. This group  $A$  will satisfy Proposition 3.

**3.3.  $A$  is free abelian on the  $a_i$ .** To prove part (i) of Proposition 3 we have to show that if there are  $m_i \in \mathbb{Z}$  with

$$\prod_{i=1}^{n-1} a_i^{m_i} = 1$$

then each  $m_i = 0$ . But the first nonzero term in the Laurent series expansion for  $\alpha$  is  $-t$ , which implies that the first nonzero term in the Laurent series expansion for each  $a_i$  is  $-t + q_{i+1} t = p_i t$ . Hence, the first nonzero term of

$$\prod_{i=1}^{n-1} a_i^{m_i} = 1$$

is

$$\prod_{i=1}^{n-1} (p_i t)^{m_i} = t^0.$$

Thus

$$\prod_{i=1}^{n-1} p_i^{m_i} = 1$$

and it follows by the uniqueness of prime factorization that  $m_i = 0$  for all  $i$  as desired.

Thus, part (i) of Proposition 3 is proved.

**3.4.  $A$  is diagonalizable.** Recall that  $\alpha$  is a  $d \times d$  matrix with entries in  $\mathbb{Z}[t]$  where  $d$  is the degree of the minimal polynomial of  $\alpha$  over  $\mathbb{Q}(t)$ . Let that minimal polynomial be  $q(x)$ . Because the characteristic of  $\mathbb{Q}(t)$  equals 0,  $q(x)$  has distinct roots in  $\mathbb{Q}(t)(\alpha)$ .

Let  $Q(x)$  be the characteristic polynomial of the matrix  $\alpha$ . The polynomial  $Q$  also has degree  $d$  and leading coefficient  $\pm 1$  with  $Q(\alpha) = 0$ . Therefore,  $q = \pm Q$ . Hence,  $Q$  has distinct roots in  $\mathbb{Q}(t)(\alpha)$  which implies that  $\alpha$  is diagonalizable over  $\mathbb{Q}(t)(\alpha) \leq \mathbb{Q}((t^{-1}))$ . That is to say that there is some  $g \in \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$  such that  $g\alpha g^{-1}$  is diagonal.

Because every element of  $\mathbb{Z}[t][\alpha]$  is a linear combination of powers of  $\alpha$ , we have that  $g(\mathbb{Z}[t][\alpha])g^{-1}$  is a set of diagonal matrices. In particular, we have proved part (ii) of Proposition 3.

**3.5.  $A$  has trivial stabilizers.** To prove part (iii) of Proposition 3 we begin with the following

**Lemma 5.** *If  $\Gamma \leq \mathrm{SL}_n(\mathbb{Q}[t])$  is bounded under the valuation for  $\mathbb{Q}((t^{-1}))$ , then the eigenvalues for any  $\gamma \in \Gamma$  lie in  $\overline{\mathbb{Q}}$ .*

*Proof.* We let  $X$  be the Euclidean building for  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ . By the Bruhat–Tits fixed point theorem,  $\Gamma z = z$  for some  $z \in X$ .

Let  $x_0 \in X$  be the vertex stabilized by  $\mathrm{SL}_n(\mathbb{Q}[[t^{-1}]])$ . We denote a diagonal matrix in  $\mathrm{GL}_n(\mathbb{Q}((t^{-1})))$  with entries  $s_1, s_2, \dots, s_n \in \mathbb{Q}((t^{-1}))^\times$  by  $D(s_1, s_2, \dots, s_n)$ , and we let  $\mathfrak{S} \subseteq X$  be the sector based at  $x_0$  and containing vertices of the form  $D(t^{m_1}, t^{m_2}, \dots, t^{m_n})x_0$  where each  $m_i \in \mathbb{Z}$  and  $m_1 \geq m_2 \geq \dots \geq m_n$ .

The sector  $\mathfrak{S}$  is a fundamental domain for the action of  $\mathrm{SL}_n(\mathbb{Q}[t])$  on  $X$  [So] which implies that there is some  $h \in \mathrm{SL}_n(\mathbb{Q}[t])$  with  $hz \in \mathfrak{S}$ .

Clearly we have  $(h\Gamma h^{-1})hz = hz$ , and since eigenvalues of  $h\Gamma h^{-1}$  are the same as those for  $\Gamma$ , we may assume that  $\Gamma$  fixes a vertex  $z \in \mathfrak{S}$ .

Fix  $m_1, \dots, m_n \in \mathbb{Z}$ ,  $m_1 \geq \dots \geq m_n \geq 0$ , such that  $z = D(t^{m_1}, \dots, t^{m_n})x_0$ . Without loss of generality, there is a partition of  $n$  – say  $\{k_1, \dots, k_\ell\}$  – such that

$$\{m_1, \dots, m_n\} = \{q_1, \dots, q_1, q_2, \dots, q_2, \dots, q_\ell, \dots, q_\ell\}$$

where each  $q_i$  occurs exactly  $k_i$  times and

$$q_1 > q_2 > \cdots > q_\ell.$$

We have that  $D(t^{m_1}, \dots, t^{m_n})^{-1} \Gamma D(t^{m_1}, \dots, t^{m_n}) x_0 = x_0$ . That gives us,  $D(t^{m_1}, \dots, t^{m_n})^{-1} \Gamma D(t^{m_1}, \dots, t^{m_n}) \subset \mathrm{SL}_n(\mathbb{Q}[[t^{-1}]])$ . Furthermore, a trivial calculation of resulting valuation restrictions for the entries of

$$D(t^{m_1}, \dots, t^{m_n}) \mathrm{SL}_n(\mathbb{Q}[[t^{-1}]]) D(t^{m_1}, \dots, t^{m_n})^{-1}$$

shows that  $\Gamma$  is contained in a subgroup of  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$  that is isomorphic to

$$\prod_{i=1}^{\ell} \mathrm{SL}_{k_i}(\mathbb{Q}) \ltimes U$$

where  $U \leq \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$  is a group of upper-triangular unipotent matrices.

The lemma is proved.  $\square$

Our proof of Proposition 3 will conclude by proving

**Lemma 6.** *No nontrivial element of  $A$  fixes a point in the Euclidean building for  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ .*

*Proof.* Suppose  $a \in A$  fixes a point in the building. We will show that  $a = 1$ . Let  $F(x) \in \mathbb{Z}[t][x]$  be the characteristic polynomial for  $a \in \mathrm{SL}_n(\mathbb{Z}[t])$ . Then

$$F(x) = \pm \prod_{i=1}^n (x - \beta_i)$$

where each  $\beta_i \in \mathbb{Q}((t^{-1}))$  is an eigenvalue of  $a$ . By the previous lemma, each  $\beta_i \in \overline{\mathbb{Q}}$ . Hence, each  $\beta_i \in \mathbb{Q} = \overline{\mathbb{Q}} \cap \mathbb{Q}((t^{-1}))$ . It follows that  $F(x) \in \mathbb{Z}[x]$  so that each  $\beta_i$  is an algebraic integer contained in  $\mathbb{Q}$ . We conclude that each  $\beta_i$  is contained in  $\mathbb{Z}$ .

Recall, that  $a$  has determinant 1, and that the determinant of  $a$  can be expressed as  $\prod_{i=1}^n \beta_i$ . Hence, each  $\beta_i$  is a unit in  $\mathbb{Z}$ , so each eigenvalue  $\beta_i = \pm 1$ . It follows – by the diagonalizability of  $a$  – that  $a$  is a finite order element of  $A \cong \mathbb{Z}^{n-1}$ . That is,  $a = 1$ .  $\square$

We have completed our proof of Proposition 3.



#### 4. Body of the proof

Let  $P \leq \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$  be the subgroup where each of the first  $n - 1$  entries along the bottom row equal 0. Let  $R_u(P) \leq P$  be the subgroup of elements that contain a  $(n - 1) \times (n - 1)$  copy of the identity matrix in the upper left corner. Thus  $R_u(P) \cong \mathbb{Q}((t^{-1}))^{n-1}$  with the operation of vector addition.

Let  $L \leq P$  be the copy of  $\mathrm{SL}_{n-1}(\mathbb{Q}((t^{-1})))$  in the upper left corner of  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ . We apply Proposition 3 to  $L$  (notice that the  $n$  in the proposition is now an  $n - 1$ ) to derive a subgroup  $A \leq L$  that is isomorphic to  $\mathbb{Z}^{n-2}$ . By the same proposition, there is a matrix  $g \in L$  such that  $gAg^{-1}$  is diagonal.

Let  $b \in \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$  be the diagonal matrix given in the notation from the proofs of Lemmas 2 and 5 as  $D(t, t, \dots, t, t^{-(n-1)})$ . Note that  $b \in P$  commutes with  $L$ , and therefore, with  $A$ . Thus the Zariski closure of the group generated by  $b$  and  $A$  determines an apartment in  $X$ , namely  $g^{-1}\mathcal{A}$  where  $\mathcal{A}$  is the apartment corresponding to the diagonal subgroup of  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ .

**4.1. Actions on  $g^{-1}\mathcal{A}$ .** If  $x_* \in g^{-1}\mathcal{A}$ , then it follows from Proposition 3 that the convex hull of the orbit of  $x_*$  under  $A$  is an  $(n - 2)$ -dimensional affine space that we will name  $V_{x_*}$ . Furthermore, the orbit  $Ax_*$  forms a lattice in the space  $V_{x_*}$ .

We let  $g^{-1}\mathcal{A}(\infty)$  be the visual boundary of  $g^{-1}\mathcal{A}$  in the Tits boundary of  $X$ . Recall that the Tits boundary of  $X$  is isomorphic to the spherical building for  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ . The definition of visual boundary used above is the standard definition from CAT(0) geometry.

The visual boundary of  $V_{x_*}$  is clearly an equatorial sphere in  $g^{-1}\mathcal{A}(\infty)$ . Precisely, we let  $P^-$  be the transpose of  $P$ . Then  $P$  and  $P^-$  are opposite vertices in  $g^{-1}\mathcal{A}(\infty)$ . It follows that there is a unique sphere in  $g^{-1}\mathcal{A}(\infty)$  that is realized by all points equidistant to  $P$  and  $P^-$ . We call this sphere  $S_{P, P^-}$ .

**Lemma 7.** *The visual boundary of  $V_{x_*}$  equals  $S_{P, P^-}$ .*

*Proof.* Since  $g \in P \cap P^-$ , it suffices to prove that  $gV_{x_*}$  is the sphere in the boundary of  $\mathcal{A}$  that is determined by the vertices  $P$  and  $P^-$ .

Note that  $gV_{x_*}$  is a finite Hausdorff distance from any orbit of a point in  $\mathcal{A}$  under the action of the diagonal subgroup of  $L$ . The result follows by observing that the inverse transpose map on  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$  stabilizes diagonal matrices while interchanging  $P$  and  $P^-$ .  $\square$

We let  $R_1, R_2, \dots, R_{n-1}$  be the standard root subgroups of  $R_u(P)$ . Recall that associated to each  $R_i$  there is a closed geodesic hemisphere  $H_i \subseteq \mathcal{A}(\infty)$  such that any nontrivial element of  $R_i$  fixes  $H_i$  pointwise and translates any point in the open hemisphere  $\mathcal{A}(\infty) - H_i$  outside of  $\mathcal{A}(\infty)$ . Note that  $\partial H_i$  is a codimension 1 geodesic sphere in  $\mathcal{A}(\infty)$ .

We let  $M \subseteq g^{-1}\mathcal{A}(\infty)$  be the union of chambers in  $g^{-1}\mathcal{A}(\infty)$  that contain the vertex  $P$ . There is also an equivalent geometric description of  $M$ :

**Lemma 8.** *The union of chambers  $M \subseteq g^{-1}\mathcal{A}(\infty)$  can be realized as an  $(n-2)$ -simplex. Furthermore,*

$$M = \bigcap_{i=1}^{n-1} g^{-1}H_i$$

and, when  $M$  is realized as a single simplex, each of the  $n-1$  faces of  $M$  is contained in a unique equatorial sphere  $g^{-1}\partial H_i = \partial g^{-1}H_i$ .

*Proof.* Let  $M' \subseteq \mathcal{A}(\infty)$  be the union of chambers in  $\mathcal{A}(\infty)$  containing the vertex  $P$ . Since  $M = g^{-1}M'$ , it suffices to prove that  $M'$  is an  $(n-2)$ -simplex with  $M' = \bigcap_{i=1}^{n-1} H_i$  and with each face of  $M'$  contained in a unique  $\partial H_i$ .

For any nonempty, proper subset  $I \subseteq \{1, 2, \dots, n\}$ , we let  $V_I$  be the  $|I|$ -dimensional vector subspace of  $\mathbb{Q}((t^{-1}))^n$  spanned by the coordinates given by  $I$ , and we let  $P_I$  be the stabilizer of  $V_I$  in  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ . For example,  $P = P_{\{1, 2, \dots, n-1\}}$ .

Recall that the vertices of  $\mathcal{A}(\infty)$  are given by the parabolic groups  $P_I$ , that edges connect  $P_I$  and  $P_{I'}$  exactly when  $I \subseteq I'$  or  $I' \subseteq I$ , and that the remaining simplicial description of  $\mathcal{A}(\infty)$  is given by the condition that  $\mathcal{A}(\infty)$  is a flag complex.

We let  $\mathcal{V}$  be the set of vertices in  $\mathcal{A}(\infty)$  of the form  $P_J$  where  $\emptyset \neq J \subseteq \{1, 2, \dots, n-1\}$ . Note that  $M'$  is exactly the set of vertices  $\mathcal{V}$  together with the simplices described by the incidence relations inherited from  $\mathcal{A}(\infty)$ . Thus,  $M'$  is easily seen to be isomorphic to a barycentric subdivision of an abstract  $(n-2)$ -simplex. Indeed, if  $\bar{M}'$  is the abstract simplex on vertices  $P_{\{1\}}, P_{\{2\}}, \dots, P_{\{n-1\}}$ , then a simplex of dimension  $k$  in  $\bar{M}'$  corresponds to a unique  $P_J \in \mathcal{V}$  with  $|J| = k+1$ . So we have that  $M'$  can be topologically realized as an  $(n-2)$ -simplex.

Let  $F_i$  be a face of the simplex  $\bar{M}'$ . Then there is some  $1 \leq i \leq n-1$  such that the set of vertices of  $F_i$  is exactly  $\{P_{\{1\}}, P_{\{2\}}, \dots, P_{\{n-1\}}\} - P_{\{i\}}$ .

Note that  $R_i V_I = V_I$  exactly when  $n \in I$  implies  $i \in I$ . It follows that  $R_i$  fixes  $M'$  pointwise, and thus  $M' \subseteq H_i$  for all  $1 \leq i \leq n-1$ . Furthermore, if  $P_I \in H_i$  for all  $1 \leq i \leq n-1$ , then  $R_i P_I = P_I$  for all  $i$  so that  $n \in I$  implies  $i \in I$  for all  $1 \leq i \leq n-1$ . As  $I$  must be a proper subset of  $\{1, 2, \dots, n\}$ , we have  $P_I \in \mathcal{V}$ , so that  $M' = \bigcap_{i=1}^{n-1} H_i$ .

All that remains to be verified for this lemma is that  $F_i \subseteq \partial H_i$ . For this fact, recall that  $F_i$  is comprised of  $(n-3)$ -simplices in  $\mathcal{A}(\infty)$  whose vertices are given by  $P_J$  where  $J \subseteq \{1, 2, \dots, n-1\} - \{i\}$ . Hence, if  $\sigma \subseteq \mathcal{A}(\infty)$  is an  $(n-3)$ -simplex of  $\mathcal{A}(\infty)$  with  $\sigma \subseteq F_i$ , then  $\sigma$  is a face of exactly 2 chambers in  $\mathcal{A}(\infty)$ :  $\mathbb{C}_P$  and  $\mathbb{C}_{P_{J'}}$  where  $\mathbb{C}_P$  contains  $P$  and thus  $\mathbb{C}_P \subseteq M'$ , and  $\mathbb{C}_{P_{J'}}$  contains  $P_{J'}$  where  $J' = \{1, 2, \dots, n\} - \{i\}$  and thus  $\mathbb{C}_{P_{J'}} \not\subseteq M'$ . Furthermore,  $\sigma = \mathbb{C}_P \cap \mathbb{C}_{P_{J'}}$ .

Since  $R_i V_{J'} \neq V_{J'}$ , it follows that  $\mathbb{C}_{P_{J'}}$  is not fixed by  $R_i$ . Since  $\mathbb{C}_{P_J}$  is fixed by  $R_i$  we have that  $\sigma = \mathbb{C}_P \cap \mathbb{C}_{P_{J'}} \subseteq \partial H_i$ . Therefore,  $F_i \subseteq \partial H_i$ .  $\square$

For any vertex  $y \in X$ , we let  $C_y \subseteq X$  be the union of sectors based at  $y$  and limiting to a chamber in  $M$ . Thus,  $C_y$  is a cone. Note also that because any chamber in  $g^{-1}\mathcal{A}(\infty)$  has diameter less than  $\pi/2$ , it follows that  $M \cap S_{P,P^-} = \emptyset$ . Therefore, if we choose  $x_*$ ,  $y \in g^{-1}\mathcal{A}$  such that  $x_*$  is closer to  $P$  than  $y$ , then  $C_y \subseteq g^{-1}\mathcal{A}$  and  $V_{x_*} \cap C_y$  is a simplex of dimension  $n - 2$ .

We will set on a fixed choice of  $y$  before  $x_*$ , and we will choose  $y$  to satisfy the below

**Lemma 9.** *There is some  $y \in g^{-1}\mathcal{A}$  such that the  $\mathbb{Q}[[t^{-1}]]$ -points of  $R_u(P)$  fix  $C_y$  pointwise.*

*Proof.* Let  $x_0$  be the point in  $X$  stabilized by  $\mathrm{SL}_n(\mathbb{Q}[[t^{-1}]])$ . Recall that  $R_u(P)M = M$  so that the  $\mathbb{Q}[[t^{-1}]]$ -points of  $R_u(P)$  fix  $C_{x_0}$  pointwise.

Because  $M \subseteq g^{-1}\mathcal{A}(\infty)$ , there is a  $y \in C_{x_0} \cap g^{-1}\mathcal{A}$ . Any such  $y$  satisfies the lemma.  $\square$

Choose  $e$  such that with  $x_* = e$  as above and with  $y$  as in Lemma 9, there exists a fundamental domain  $D_e$  for the action of  $A$  on  $V_e$  that is contained in  $C_y$ . The choice of  $e$  can be made by travelling arbitrarily far from  $y$  along a geodesic ray in  $g^{-1}\mathcal{A}$  that limits to  $P$ .

By the choice of  $D_e$  we have that

$$AD_e = V_e$$

and that the  $\mathbb{Q}[[t^{-1}]]$ -points of  $R_u(P)$  fix  $D_e$ .

**4.2. The filtration.** We let

$$X_0 = \mathrm{SL}_n(\mathbb{Z}[t])D_e$$

and for any  $i \in \mathbb{N}$  we choose an  $\mathrm{SL}_n(\mathbb{Z}[t])$ -invariant and cocompact space  $X_i \subseteq X$  somewhat arbitrarily to satisfy the inclusions

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq \bigcup_{i=1}^{\infty} X_i = X.$$

In our present context, Brown's criterion takes on the following form [Br1].

**Brown's Filtration Criterion.** *By Lemma 2, the group  $\mathrm{SL}_n(\mathbb{Z}[t])$  is not of type  $\mathrm{FP}_{n-1}$  if for any  $i \in \mathbb{N}$ , there exists some class in the homology group  $\tilde{H}_{n-2}(X_0, \mathbb{Z})$  which is nonzero in  $\tilde{H}_{n-2}(X_i, \mathbb{Z})$ .*



**4.3. Translation to  $P$  moves away from filtration sets.** The following is essentially Mahler's compactness criterion.

**Lemma 10.** *Given any  $i \in \mathbb{N}$ , there is some  $k \in \mathbb{N}$  such that  $b^k e \notin X_i$ .*

*Proof.* The lemma follows from showing that the sequence

$$\{\mathrm{SL}_n(\mathbb{Z}[t])b^k e\}_k \subseteq \mathrm{SL}_n(\mathbb{Z}[t]) \setminus X$$

is unbounded.

Since stabilizers of points in  $X$  are bounded subgroups of  $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ , the claim above follows from showing that the sequence

$$\{\mathrm{SL}_n(\mathbb{Z}[t])b^k\}_k \subseteq \mathrm{SL}_n(\mathbb{Z}[t]) \setminus \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$$

is unbounded.

But bounded sets in  $\mathrm{SL}_n(\mathbb{Z}[t]) \setminus \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$  do not contain sequences of elements  $\{\mathrm{SL}_n(\mathbb{Z}[t])g_\ell\}_\ell$  such that  $1 \in \overline{g_\ell^{-1}(\mathrm{SL}_n(\mathbb{Z}[t]) - \{1\})g_\ell}$ . And clearly  $b^k$ 's contract some root groups to 1. Thus none of the sequences above is bounded.  $\square$

**4.4. Applying Brown's criterion.** As is described by Brown's criterion, we will prove Theorem 1 by fixing  $X_i$  and finding an  $(n-2)$ -cycle in  $X_0$  that is nontrivial in the homology of  $X_i$ .

Recall that we denote the standard root subgroups of  $R_u(P)$  by  $R_1, \dots, R_{n-1}$ . Each group  $g^{-1}R_jg$  determines a family of parallel walls in  $g^{-1}\mathcal{A}$ . By Lemma 8, each face of the cone  $C_y$  is contained in a wall of one of these families.

Choose  $r_j \in g^{-1}R_jg$  for all  $j$  such that  $b^k e$  is contained in the wall determined by  $r_j$  where  $k$  is determined by  $i$  as in Lemma 10. In particular,  $r_j b^k e = b^k e$ .

The intersection of the fixed point sets in  $g^{-1}\mathcal{A}$  of the elements  $r_1, \dots, r_{n-1}$  determine a cone that we name  $Z$ . Note that  $Z$  is contained in  $-$  and is a finite Hausdorff distance from  $-$  the cone  $C_y$ .

Let  $Z^- \subseteq g^{-1}\mathcal{A}$  be the closure of the set of points in  $g^{-1}\mathcal{A}$  that are fixed by none of the  $r_j$ . The set  $Z^-$  is a cone based at  $b^k e$ , containing  $y$ , and asymptotically containing the vertex  $P^-$ .

As the walls of  $Z^-$  are parallel to those of  $Z$  – and hence of  $C_y$ , we have that  $Z^- \cap V_e$  is an  $(n-2)$ -dimensional simplex. We will name this simplex  $\sigma$ .

The component of  $Z^- - V_e$  that contains  $b^k e$  is an  $(n-1)$ -simplex that has  $\sigma$  as a face. Call this  $(n-1)$  simplex  $Y$ .

For any  $\ell \in \mathbb{N}$ , there are exactly  $2^{n-1}$  possible subsets of the set  $\{r_1^\ell, \dots, r_{n-1}^\ell\}$ . For each such subset  $S_\ell$ , we let

$$Y_{S_\ell} = \left( \prod_{g \in S_\ell} g \right) Y$$



and

$$\sigma_{S_\ell} = \left( \prod_{g \in S_\ell} g \right) \sigma.$$

Notice that the product of group elements in the equations above are well-defined regardless of the order of the multiplication since  $R_u(P)$  is abelian. In the degenerate cases,  $\prod_{g \in \emptyset} g = 1$ , so  $Y_\emptyset = Y$  and  $\sigma_\emptyset = \sigma$ .

For any  $\ell \in \mathbb{N}$ , we let  $Y_\ell = \bigcup_{S_\ell} Y_{S_\ell}$ . Because the wall in  $g^{-1}\mathcal{A}$  determined by  $r_j^\ell$  is the same as the wall determined by  $r_j$ , the space  $Y_\ell$  is a closed ball containing  $b^k e$  whose boundary sphere is  $\bigcup_{S_\ell} \sigma_{S_\ell}$ . Indeed the simplicial decomposition of  $Y_\ell$  described above is isomorphic to the simplicial decomposition of the unit ball in  $\mathbb{R}^{n-1}$  that is given by the  $n-1$  hyperplanes defined by setting a coordinate equal to 0.

Let  $\omega_\ell = \bigcup_{S_\ell} \sigma_{S_\ell}$ . Thus  $\omega_\ell = \partial Y_\ell$ . Furthermore, the building  $X$  is  $(n-1)$ -dimensional and contractible, so any  $(n-1)$ -chain with boundary equal to  $\omega_\ell$  must contain  $Y_\ell$  and thus  $b^k e$ . That is for all  $\ell \in \mathbb{N}$

$$[\omega_\ell] \neq 0 \in \tilde{H}_{n-2}(X - b^k e, \mathbb{Z})$$

If we can show that  $\omega_\ell \subseteq X_0$  for some choice of  $\ell$ , then we will have proved our main theorem by application of Brown's criterion since we would have

$$[\omega_\ell] \neq 0 \in \tilde{H}_{n-2}(X_i, \mathbb{Z})$$

by Lemma 10.

**Lemma 11.** *There exists some  $\ell \in \mathbb{N}$  such that  $\omega_\ell \subseteq X_0$ .*

*Proof.* For any  $u \in R_u(P)$  there is a decomposition  $u = u' u''$  where the entries of  $u' \in R_u(P)$  are contained in  $\mathbb{Q}[t]$  and the entries of  $u'' \in R_u(P)$  are contained in  $\mathbb{Q}[[t^{-1}]]$ .

For any  $a \in A$  and  $u \in R_u(P)$  there is a power  $\ell(a, u) \in \mathbb{N}$  such that

$$(a^{-1} u^{\ell(a, u)} a)' = ((a^{-1} u a)')^{\ell(a, u)} \in \mathrm{SL}_n(\mathbb{Z}[t]).$$

(For the above equality recall that  $A \leq L$  normalizes  $R_u(P)$  and the group operation on  $R_u(P)$  is vector addition.)

There are only finitely many  $a \in A$  such that  $a D_e \cap \sigma \neq \emptyset$  (or equivalently, such that  $a D_e \cap Z^- \neq \emptyset$ ). Call this finite set  $\mathcal{D} \subseteq A$ .

At this point we fix

$$\ell = \prod_{a \in \mathcal{D}} \prod_{i=1}^{n-1} \ell(a, r_i).$$

Thus,

$$\left[ a^{-1} \left( \prod_{g \in S_\ell} g \right) a \right]' \in \mathrm{SL}_n(\mathbb{Z}[t])$$

for any  $a \in \mathcal{D}$  and any  $S_\ell \subseteq \{r_i^\ell\}_{i=1}^{n-1}$ .

Because  $\omega_\ell = \bigcup_{S_\ell} \sigma_{S_\ell}$  and  $\sigma_{S_\ell} = (\prod_{g \in S_\ell} g)\sigma = (\prod_{g \in S_\ell} g)(AD_e \cap Z^-)$ , we can finish our proof of this lemma by showing

$$\left( \prod_{g \in S_\ell} g \right) a D_e \subseteq X_0$$

for each  $a \in \mathcal{D} \subseteq A \leq \mathrm{SL}_n(\mathbb{Z}[t])$  and each  $S_\ell \subseteq \{r_i^\ell\}_{i=1}^{n-1}$ . For this, recall that the  $\mathbb{Q}[[t^{-1}]]$ -points of  $R_u(P)$  fix  $D_e$  and thus

$$\begin{aligned} \left( \prod_{g \in S_\ell} g \right) a D_e &= a \left[ a^{-1} \left( \prod_{g \in S_\ell} g \right) a \right] D_e \\ &= a \left[ a^{-1} \left( \prod_{g \in S_\ell} g \right) a \right]' \left[ a^{-1} \left( \prod_{g \in S_\ell} g \right) a \right]'' D_e \\ &= a \left[ a^{-1} \left( \prod_{g \in S_\ell} g \right) a \right]' D_e \\ &\subseteq \mathrm{SL}_n(\mathbb{Z}[t]) D_e \\ &= X_0 \end{aligned}$$

□

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