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**Autor:** Nabutovsky, Alexander / Rotman, Regina  
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## The length of the second shortest geodesic

Alexander Nabutovsky and Regina Rotman

**Abstract.** According to the classical result of J.P. Serre ([S]) any two points on a closed Riemannian manifold can be connected by infinitely many geodesics. The length of a shortest of them trivially does not exceed the diameter  $d$  of the manifold. But how long are the shortest remaining geodesics? In this paper we prove that any two points on a closed  $n$ -dimensional Riemannian manifold can be connected by two distinct geodesics of length  $\leq 2qd \leq 2nd$ , where  $q$  is the smallest value of  $i$  such that the  $i$ th homotopy group of the manifold is non-trivial.

**Mathematics Subject Classification (2000).** 53C23, 53C22, 58E10.

**Keywords.** Length of geodesics, length functional, curvature-free inequalities in Riemannian geometry

### 1. Main result

Here is the main result of the present paper:

**Theorem 1.1.** *Let  $M^n$  be a closed  $n$ -dimensional Riemannian manifold, let  $d$  denote the diameter of  $M^n$ , and let  $q = \min_i \{\pi_i(M^n) \neq 0\}$ . Then for each pair of points  $x, y \in M^n$  there exist at least two distinct geodesics connecting  $x, y$  of length not exceeding  $2qd$  ( $\leq 2nd$ ).*

Observe that if  $x = y$ , then the trivial geodesic is the shortest geodesic connecting  $x$  and  $y$ . In this case our theorem asserts the existence of a geodesic loop of length  $\leq 2d$  based at an arbitrary point  $x$  of  $M^n$ . This result is the main result of the paper [R] of one of the authors. Theorem 1.1 can be viewed as a generalization of this result. Our proof of Theorem 1.1 is heavily reliant on methods of [R]. We would like also to refer the reader to the foundational paper of M. Gromov [G] that was a source of motivation for us and inspired some of the techniques of the present paper.

## 2. Filling of cages

Let us begin by introducing the following definitions and notations.

**Definition 2.1.** Let  $x, y$  be two points in  $M^n$ . An  $m$ -cage  $c$  based at  $x, y$  is a collection of  $m$  paths  $c_1, \dots, c_m$  from  $x$  to  $y$ . (For every  $i$   $c_i$  is a continuous map of  $[0, 1]$  into  $M^n$ .) The space  $C_{x,y,m}$  of all  $m$ -cages based at  $x$  and  $y$  can be identified with the  $m$ th power of the space of paths from  $x$  to  $y$ . For every  $L$  let  $C_{x,y,m}^L$  denote the space of all  $m$ -cages based at  $x, y$  such that the length of each of the  $m$  paths forming the cage is at most  $L$ . Further, let  $C_{x,y,m}^{L,\bar{L}}$  denote the space of all  $m$ -cages  $c = (c_1, \dots, c_m)$  based at  $x, y$  such that the length of  $c_1$  does not exceed  $\bar{L}$ , and the length of  $c_i$  for every  $i = 2, 3, \dots, m$  does not exceed  $L$ .

Let  $\sigma^m = [v_0, v_1, \dots, v_m]$  be the standard  $m$ -dimensional simplex with edges of length one. (Here  $v_0, v_1, \dots, v_m$  are its vertices.) As usual, we use the notation  $C(\sigma^m, M^n)$  for the space of continuous maps from  $\sigma^m$  to  $M^n$ . Of course, this space can be identified with the space of continuous maps of the  $m$ -dimensional ball into  $M^n$ .

**Definition 2.2.** Let  $x, y$  be two points in  $M^n$ ,  $L, \bar{L}$  two positive numbers, and  $N$  a positive integer. A *coherent  $N$ -filling* of  $m$ -cages based at  $x, y$  from  $C_{x,y,m}^{L,\bar{L}}$  is a collection of continuous maps  $\phi_m: C_{x,y,m}^{L,\bar{L}} \rightarrow C(\sigma^m, M^n)$  for all  $m = 1, 2, \dots, N$  with the following properties:

1. For every  $m$  and  $m$ -cage  $c$  the map  $\phi_m(c): \sigma^m \rightarrow M^n$  maps the  $(m-1)$ -dimensional face  $[v_1, \dots, v_m]$  of  $\sigma^m$  into  $y$ .
2. For every  $m$  and every  $m$ -cage  $c = (c_1, \dots, c_m)$  the map  $\phi_m(c)$  maps each of  $m$  one-dimensional simplices  $[v_0, v_i]$  by the map  $c_i$ . (Here we identify  $[v_0, v_i]$  with  $[0, 1]$ .) In particular,  $v_0$  is mapped into  $x$ , and for every 1-cage  $c$  we have  $\phi_1(c) = c$ .
3. (Coherence) For every  $m = 2, 3, \dots, N$ , every  $m$ -cage  $c$  and every  $i = 1, 2, \dots, m$  the restriction of  $\phi_m$  to the  $(m-1)$ -dimensional face  $[v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_m]$  of  $\sigma_m$  coincides with  $\phi_{m-1}(c_{(i)})$ , where  $c_{(i)}$  denotes the  $(m-1)$ -cage  $(c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_m)$ .

To explain the meaning of conditions 1 and 2 collapse  $[v_1, \dots, v_m]$  into a point. Identify this point with the North pole and  $v_0$  with the South pole of the ball  $D^m$ . Then  $m$  edges  $[v_0, v_i]$  become  $m$  meridians on the sphere bounding this ball. We can view  $m$ -cages as maps from this collection of  $m$  meridians into  $M^n$ . The meaning of conditions 1 and 2 is that one can regard a coherent  $N$ -filling of an  $m$ -cage as an

extension of this map to a map of the whole  $m$ -ball. This extension must depend continuously on the  $m$ -cage. The meaning of the coherence condition is that extensions in different dimensions are compatible.

**Proposition 2.3.** *Let  $L, \bar{L}$  be positive real numbers such that  $\bar{L} \geq L$ , and  $N$  a positive integer. Let  $x, y, z$  be any three points of  $M^n$  such that the distance between any two of them does not exceed  $L$ . Assume that there exists exactly one geodesic between  $x$  and  $y$  of length  $\leq \bar{L} + (2N - 3)L$ . (If  $x = y$ , then this geodesic is the constant geodesic.) Then there exists a coherent  $N$ -filling of  $m$ -cages based at  $x, z$  from  $C_{x,z,m}^{L,\bar{L}}$ .*

*Proof.* We present a proof by induction on  $N$ . Its base corresponds to the case  $N = 1$ . In this case  $\phi_1(c) = c$ . (Recall that each 1-cage is, by definition, a path in  $M^n$ , i.e. a continuous map of  $\sigma^1 = [0, 1]$  into  $M^n$ .) The proof of the induction step is based on the following lemma:

**Lemma 2.4.** *Let  $\bar{L}, L$  be positive numbers. Assume that  $x, y, z$  are three points in  $M^n$  such that all distances between them do not exceed  $L$ . Assume that there exists only one geodesic between  $x$  and  $y$  of length  $\leq \max\{\bar{L}, L\} + L$ . Then any two paths  $\gamma_1, \gamma_2$  starting at  $x$  and ending at  $z$  such that the length of  $\gamma_1$  is  $\leq \bar{L}$  and the length of  $\gamma_2$  is  $\leq L$  can be connected by a path homotopy that passes only through paths of length  $\leq \bar{L} + 2L$ . This path homotopy depends continuously on  $\gamma_1$  and  $\gamma_2$ .*

*Proof.* Let  $\sigma$  be the unique shortest geodesic from  $x$  to  $y$ ,  $\tau$  be one of the shortest geodesics from  $z$  to  $y$  (see Figure 1). Every path from  $x$  to  $y$  of length  $\leq 2L$  or  $\leq \bar{L} + L$  can be connected to  $\sigma$  by a length non-increasing homotopy. (Otherwise, there will be

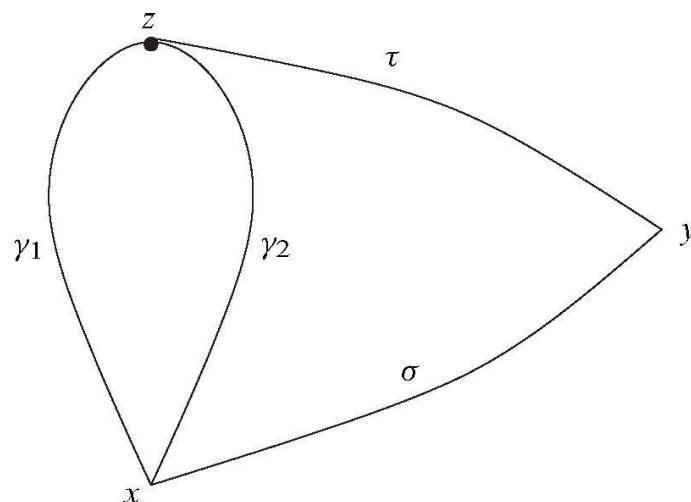


Figure 1

a second geodesic of length  $\leq \max\{\bar{L}, L\} + L$ .) Moreover, we can choose a specific length non-increasing homotopy, e.g. the Birkhoff curve-shortening process with fixed endpoints. (See [C] for a detailed description of the Birkhoff curve-shortening process.) This homotopy continuously depends on the initial path. In particular, this homotopy can be used to deform  $\gamma_i * \tau$  to  $\sigma$ , ( $i = 1, 2$ ), as well as to deform  $\sigma$  back to  $\gamma_i * \tau$ .

Let  $\tau^{-1}$  denote the path  $\tau$  traversed in the opposite direction. One can construct the desired path homotopy from  $\gamma_1$  to  $\gamma_2$  as follows:  $\gamma_1 \rightarrow \gamma_1 * \tau * \tau^{-1} \rightarrow \sigma * \tau^{-1} \rightarrow \gamma_2 * \tau * \tau^{-1} \rightarrow \gamma_2$ . Here arrows denote homotopies. Note that all homotopies depend continuously on  $\gamma_1$  and  $\gamma_2$ , and that for each of these homotopies the length of paths during the homotopy does not exceed the maximum of lengths of paths at its beginning and its end.  $\square$

Now assume we have constructed the maps  $\phi_1, \dots, \phi_{m-1}$ . We will next construct  $\phi_m$ . Let  $c = (c_1, c_2, \dots, c_m)$  be an  $m$ -net. We need to map  $\sigma^m = [s_0, \dots, s_m]$  to  $M^n$ . Because of the coherence condition a map  $\psi_m(c)$  defined as the restriction of  $\phi_m$  to  $\partial\sigma^m$  into  $M^n$  is already prescribed. By virtue of the induction assumption  $\psi_m$  is a continuous function of  $c$ . We need only to contract the (map of the) sphere  $\psi_m(c)$  to a point so that the contracting homotopy depends continuously on  $c$ . To achieve this goal note that according to Lemma 2.4 there exists a path homotopy between  $c_1$  and  $c_2$  that passes through paths  $c_t$ ,  $t \in [1, 2]$ , of length  $\leq \tilde{L} = \bar{L} + 2L$  only. (Here we use the fact that  $(2m - 3)L \geq L$  for every  $m \geq 2$ . So, the assumption of Lemma 2.4 about the non-existence of a second short geodesic between  $x$  and  $y$  follows from a similar assumption in Proposition 2.3.) Consider a 1-parametric family of  $m$ -cages  $c^{(t)} = (c_t, c_2, \dots, c_m)$ . So,  $c^{(1)} = c$  and  $c^{(2)} = (c_2, c_2, c_3, \dots, c_m)$ . Note that  $c^{(t)} \in C_{x,y,m}^{L,\tilde{L}}$  for every  $t$ . By virtue of the induction assumption there exists a coherent filling of all  $(m - 1)$ -subcages of  $c^{(t)}$  obtained by removal of one of  $m$  strands  $c_i^{(t)}$ , and for every  $t$  the resulting  $m$  maps of  $(m - 1)$ -dimensional simplices can be “glued” to each other into a map  $\psi_m(c^{(t)}): \partial\sigma^m \rightarrow M^n$ . Of course, it is important here that  $\tilde{L} + (2(m - 1) - 3)L = \bar{L} + (2m - 3)L$ , and so the required assumption about the non-existence of a second geodesic between  $x$  and  $y$  of length  $\leq \tilde{L} + (2(m - 1) - 3)L$  holds. Thus, one obtains a homotopy  $\psi_m(c^{(t)})$  between  $\psi_m(c)$  and  $\psi_m(c^{(2)})$ .

It remains to show that  $\psi_m(c^{(2)})$  is canonically and, therefore, continuously contractible. (Here we are concerned about the continuity of the contracting homotopy as a function of  $c$ .) Note that the boundary of  $\sigma^m$  consists of  $(m + 1)$  simplices of dimension  $(m - 1)$ . The maps  $\phi_{m-1}$  and, thereby,  $\psi_m$  map two of these faces, namely, faces corresponding to two copies of the  $(m - 1)$ -cage  $(c_2, c_3, \dots, c_m)$  in an identical way. Together these two cells form a “folded” map of  $S^{m-1}$  to  $M^n$  that factors through the projection of  $S^{m-1}$  to the disc  $D^{m-1}$ . This map is obviously canonically contractible. In order to construct a homotopy of  $\psi_m(c^{(2)})$  to this “folded” map we

need to “eliminate” the remaining  $(m - 1)$  maps of  $(m - 1)$ -dimensional faces of  $\sigma^m$ . But one of these maps is constant, and the remaining  $(m - 2)$  maps correspond to  $(m - 1)$ -cages of the form  $(c_2, c_2, \dots)$ . Therefore each of these maps is similarly “folded” and can be connected by a canonical homotopy (over its image) to a map of the corresponding face which is a composition of the projection of the considered face to one of its codimension one faces and  $\phi_{m-2}((c_2, c_3, \dots, c_{i-1}, c_{i+1}, \dots, c_m))$  for an appropriate  $i$ . These homotopies eliminate the remaining  $m - 1$  faces, as desired.  $\square$

### 3. Filling of $(m, \varepsilon)$ -umbrellas

Let  $\sigma^{m-1} = [s_1, \dots, s_m]$  denote the standard  $(m - 1)$ -dimensional simplex such that the lengths of all of its edges are equal to 1. Let  $s_*$  denote the center of  $\sigma^{m-1}$ .

**Definition 3.1.** An  $(m, r)$ -umbrella based at  $x, y$  consists of a singular  $(m - 1)$ -simplex  $\rho: \sigma^{m-1} \rightarrow M^n$ , a point  $x \in M^n$  and  $m$  paths in  $M^n$  connecting  $x$  with images of the vertices of  $\sigma^{m-1}$  under  $\rho$  so that  $y = \rho(s_*)$ , the image of  $\rho$  is contained in a metric ball of radius  $r$  in  $M^n$  centered at  $y$ , and the length of the image of every straight line segment in  $\sigma^{m-1}$  under  $\rho$  is less than  $r$ .

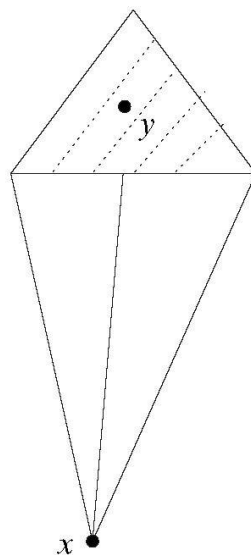


Figure 2.  $(m, r)$ -umbrella.

This notion generalizes the notion of  $m$ -cages that can be considered as  $(m, 0)$ -umbrellas (with a constant  $\rho$ ). The goal of this section is to generalize the notion of coherent filling for  $(m, r)$ -umbrellas and to extend Theorem 1.1 to  $(m, \varepsilon)$ -umbrellas for small positive  $\varepsilon$ . Denote the space of all  $(m, r)$ -umbrellas based at  $x, y$  by  $U_{m,r,x,y}$ .



and its subspace formed by all umbrellas where the length of the paths connecting  $x$  with the first vertex of  $\rho$  does not exceed  $\bar{L}$ , and the lengths of all paths connecting  $x$  with the remaining  $m - 1$  vertices of the singular simplex  $\rho$  do not exceed  $L$  by  $U_{m,r,x,y}^{L,\bar{L}}$ . Each umbrella  $u$  can be represented as  $(c_1, \dots, c_m, \rho)$ , where  $c_i$  are continuous paths from  $x$  to the vertices of the singular simplex  $\rho$ . It is obvious that 1-umbrellas based at  $x, y$  are merely continuous paths starting at  $x$  and ending at  $y$ .

**Definition 3.2.** Let  $N$  be a positive integer and  $L, \bar{L}$  positive real numbers. A coherent  $N$ -filling of  $(m, r)$ -umbrellas based at  $x, y$  from  $U_{m,r,x,y}^{L,\bar{L}}$  is a family of continuous maps  $\phi_m: U_{m,r,x,y}^{L,\bar{L}} \rightarrow C(\sigma^m, M^n)$  for  $m = 1, 2, \dots, N$  such that for every  $(m, r)$ -umbrella  $u = (c_1, c_2, \dots, c_m, \rho) \in U_{m,r,x,y}^{L,\bar{L}}$  the following conditions hold:

1. The restriction of  $\phi_m(u)$  to the  $(m - 1)$ -dimensional face  $[s_1, s_2, \dots, s_m]$  coincides with  $\rho$ ;
2. The restrictions of  $\phi_m(u)$  to 1-dimensional simplices  $[s_0 s_i]$  coincide with  $c_i$  for  $i = 1, 2, \dots, m$ . In particular,  $\phi_1(u) = u$  for all 1-umbrellas  $u$ ;
3. (Coherence) For every  $i = 1, 2, \dots, m$  the restriction of  $\phi_m(u)$  to  $(m - 1)$ -dimensional simplex  $[s_0, s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_m]$  coincides with  $\phi_{m-1}(u_i)$ , where  $u_i = (c_1, c_2, \dots, c_{i-1}, c_{i+1}, \dots, c_m, \rho_i)$ , and  $\rho_i$  is the restriction of  $\rho$  to the  $(m - 2)$ -dimensional face of the standard simplex  $\sigma^{m-1}$  obtained by exclusion of the  $i$ th vertex.

The notion of  $(m, \varepsilon)$ -umbrellas can be regarded as a generalization of the notion of  $m$ -cages, where one of the endpoints is being “enlarged” into a small simplex. (If this simplex degenerates into a point, the umbrella becomes a cage.) The next proposition asserts that there exists a generalization of the process of filling of cages described in the proof of Proposition 2.3 to  $(m, \varepsilon)$ -umbrellas for small  $\varepsilon$ . The idea of the proof of this generalization is very simple: One can just shrink the small simplex  $\rho$  in the definition of umbrellas over itself to a point, thus, obtaining a cage, which then can be filled as in the proof of Proposition 2.3.

**Proposition 3.3.** Let  $L, \bar{L}$  be positive numbers such that  $\bar{L} \geq L$ . Let  $x, y, z$  be any three points in a closed Riemannian manifold  $M^n$  such that the distance between any two of these three points does not exceed  $L$ . Let  $N > 1$  be an integer. Then for every  $0 < \varepsilon < \frac{\text{inj} M^n}{2}$ , where  $\text{inj} M^n$  is the injectivity radius of  $M^n$ , the following assertion holds: Provided there exists exactly one geodesic between  $x$  and  $y$  of length  $\leq \bar{L} + (2N - 3)L + (2N - 2)\varepsilon$ , then there exists a coherent  $N$ -filling of  $(m, \varepsilon)$ -umbrellas based at  $x, z$  from  $U_{m,\varepsilon,x,z}^{L,\bar{L}}$ .

*Proof.* The proof is inductive on  $N$ . It follows the same pattern as the proof of Proposition 2.3. To prove the base of induction we define  $\phi_1(u) = u$  for every 1-umbrella  $u$ . Assume that the theorem holds for  $N = m - 1$ , ( $m > 1$ ). To prove

the theorem for  $N = m$  note that conditions 1 and 3 imply that we have no choice in construction of  $\psi_m(u) = \phi_m(u)|_{\partial\sigma^m}$ : One of  $(m + 1)$  faces of  $\sigma^m$  of dimension  $(m - 1)$  must be mapped using the mapping  $\rho$ , whereas the remaining  $m$  faces should be mapped using  $\phi_{m-1}(u_i)$ . Using a part of the induction assumption we can conclude that  $\psi_m$  is a continuous function of  $u$ .

It remains only to contract  $\psi_m(u)$  by a homotopy that continuously depends on  $u$ . The idea is to eliminate the simplex  $\rho$  by contracting it over its image and then to proceed as in the proof of Proposition 2.3.

Recall that  $s_*$  denotes the center of  $\sigma^{m-1}$ . Fix a contraction  $h_t$  of  $\sigma^{m-1} = [s_1, s_2, \dots, s_m]$  to  $\{s_*\}$ , ( $h_0$  is the identity map,  $h_1(\sigma^{m-1}) = \{s_*\}$ ), such that all points of  $\sigma^{m-1}$  move to  $s_*$  along straight lines with a constant speed. This will provide us with a homotopy of umbrellas: If  $u = (c_1, \dots, c_m, \rho)$ , then  $H_t(u)$  is defined as  $(c_{1t}, \dots, c_{mt}, \rho \circ h_t)$ , where  $c_{it}$  is the join of  $c_i$  with  $\rho([s_i h_t(s_i)])$  for every  $i$ . If  $u$  is an  $(m, \varepsilon)$ -umbrella, the length of  $c_{it}$  does not exceed the sum of the length of  $c_i$  and  $\varepsilon$ . For every  $t \in [0, 1]$  we can consider  $\psi_m(H_t(u))$ . The composition  $\psi_m \circ H_t$  will constitute the first stage in a homotopy contracting  $\psi_m(u)$ .

It remains to contract  $\psi_m(H_1(u))$ . Note that  $H_1(u)$  looks like an  $m$ -cage since its  $(m - 1)$ -dimensional simplex is constant. Therefore we can contract the resulting  $(m - 1)$ -dimensional sphere repeating the corresponding step in the proof of Proposition 2.3 almost verbatim.

Namely, we use Lemma 2.4 to construct a path homotopy  $c_{1t}$ , ( $t \in [1, 2]$ ), between  $c_{11} = h_1(c_1)$  and  $c_{21} = h_1(c_2)$  such that it passes only through paths of length  $\leq \bar{L} + 2L + 3\varepsilon$ . Let  $u_t = (c_t, c_{21}, \dots, c_{m1}, \rho \circ h_1)$ . The next stage of our homotopy contracting  $\psi_m(u)$  will consist of  $(m - 1)$ -dimensional spheres  $\psi_m(u_t)$ ,  $t \in [1, 2]$ .

Finally, note that  $u_2 = (c_{21}, c_{21}, \dots, c_{m1}, \rho \circ h_1)$ , so that  $\phi_m(u_2)$  will be a “folded”  $(m - 1)$ -dimensional sphere that can be canonically contracted over itself exactly as this had been done in the proof of Proposition 2.3.  $\square$

#### 4. Proof of Theorem 1.1

We are going to prove the theorem by contradiction. Assume that there exists exactly one geodesic between  $x$  and  $y$  of length  $\leq 2qd$ . Therefore there exists  $\delta > 0$  such that there exists exactly one geodesic between  $x$  and  $y$  of length  $\leq 2qd + \delta$ . (Indeed, otherwise there will be a sequence of geodesics between  $x$  and  $y$  with lengths strictly decreasing to  $2qd$ . The Ascoli–Arzela theorem implies that a subsequence of this sequence converges to a geodesic between  $x$  and  $y$  of length  $2qd \geq 2d > d$ . Therefore, this geodesic cannot be minimizing and, therefore, is the second geodesic between  $x$  and  $y$  of length  $\leq 2qd$ , which contradicts to our assumption.) Let  $\varepsilon = \min\{\frac{\delta}{2n}, \frac{\text{inj}(M^n)}{2}\}$ , where  $\text{inj}(M^n)$  denotes the injectivity radius of  $M^n$ . Let  $f: S^q \rightarrow M^n$  be a non-contractible map of the  $q$ -dimensional sphere into  $M^n$ .



We are going to extend  $f$  to a map of the  $(q + 1)$ -dimensional disc  $D^{q+1}$  thereby reaching the desired contradiction. First, choose a fine smooth triangulation of  $S^q$  such that for every singular simplex  $\tau: \sigma^i \rightarrow S^{q+1}$ , ( $i \in \{1, \dots, q + 1\}$ ), the image under  $f \circ \tau$  of  $\sigma^i$  is contained in an  $\varepsilon$ -ball centered at the image of the center of  $\sigma^i$  under  $f \circ \tau$ , and the length of the image of every straight line segment in  $\sigma^i$  under  $f \circ \tau$  is less than  $\varepsilon$ .

Triangulate  $D^{q+1}$  as the cone over the chosen triangulation of  $S^{q+1}$ . Extend  $f$  to the 1-skeleton of the triangulation of  $D^{q+1}$  by mapping the center of  $D^{q+1}$  to  $x$ , and every new 1-dimensional simplices into a minimal geodesic between the images of the endpoints of the 1-simplex. (Here one can choose any minimal geodesic, if there is more than one.)

We are going to continue the extension process inductively. Assume that we have already extended  $f$  to the  $i$ -skeleton of the triangulation of  $D^{q+1}$ . In order to extend it to the  $(i + 1)$ -skeleton observe that every new  $(i + 1)$ -dimensional simplex is a cone over a  $i$ -dimensional simplex  $\tau$  of the chosen triangulation of  $S^q$ . Consider a  $(i + 1, \varepsilon)$ -umbrella based at  $x$  and the image of the center of  $\tau$  under  $f$ , such that  $\rho = f \circ \tau$ . Take  $\bar{L} = L = d$ . Apply Proposition 3.3 to fill this umbrella. The coherence condition implies that the resulting map of the  $(i + 1)$ -dimensional simplex of the triangulation of  $D^{q+1}$  extends maps of its faces constructed on the previous steps of the induction.

Once  $f$  is extended to the  $(q + 1)$ -skeleton of  $D^{q+1}$ , the extension process becomes complete, and we obtain the desired contradiction.  $\square$

## 5. Concluding remarks

In [NR1] we made the following conjecture:

**Conjecture 5.1.** There exists a function  $f(n, k)$  such that for every positive integer  $k$ , every closed Riemannian manifold  $M^n$  and every pair of points  $x, y \in M^n$  there exist  $k$  distinct geodesics between  $x$  and  $y$  in  $M^n$  of length  $\leq f(n, k)d$ , where  $d$  denotes the diameter of  $M^n$ .

In fact, we made even a stronger conjecture that there exist  $k$  distinct geodesics of length  $\leq kd$ . This stronger conjecture holds for round spheres and for all closed Riemannian manifolds with infinite torsion-free fundamental groups. Yet, F. Balacheff, C. Croke and M. Katz recently constructed Riemannian metrics on  $S^2$  arbitrarily close to round metrics such that the length of the shortest geodesic loop at every point is strictly greater than  $2d$  ([BCK]). Thus, this stronger conjecture is not true even when  $n = k = 2$  and  $x = y$ .

In the present paper we proved our conjecture for  $k = 2$  for an arbitrary  $M^n$  and arbitrary  $x, y \in M^n$ . Thus, we demonstrated that one can take  $f(n, 2) = 2n$ .

Our paper [NR2] contains another result in this direction: If  $n = 2$  and  $M^n$  is diffeomorphic to  $S^2$ , then for every  $k$  and every pair of points  $x, y$  in the Riemannian manifold there exist  $k$  distinct geodesics between  $x$  and  $y$  of length  $\leq (4k^2 - 2k - 1)d$ . Therefore, one can take  $f(2, k) = 4k^2 - 2k - 1$ .

Our most recent result in this direction establishes the conjecture for all Riemannian manifolds homotopy equivalent to the product of  $S^2$  and an arbitrary closed manifold. In this case for every pair of points  $x, y$  there exist at least  $k$  distinct geodesics between  $x$  and  $y$  of length  $\leq 20k!d$  (see [NR3]).

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Alexander Nabutovsky and Regina Rotman, Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 2E4, Canada

E-mail: alex@math.toronto.edu; rina@math.toronto.edu