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On the Weinstein conjecture in higher dimensions

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Abstract. The existence of a "Plastikstufe" for a contact structure implies the Weinstein conjecture for all supporting contact forms.

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1. Introduction and main result

A one-form λ on an odd-dimensional manifold M^{2n-1} is called a contact form, provided $\lambda \wedge d\lambda^{n-1}$ is a volume-form. Associated to a contact form λ we have the Reeb vector field X defined by

$$i_X\lambda = 1 \quad \text{and} \quad i_X d\lambda = 0$$

and the contact structure $\xi = \ker(\lambda)$. In 1978, A. Weinstein, [21], motivated by a result of P. Rabinowitz, [16], and one of his own results, [20], made the following conjecture:

A Reeb vector field on a closed manifold M^{2n-1} admits a periodic orbit.

The first break-through on this conjecture was obtained by C. Viterbo, [19], showing that compact energy surfaces in $\mathbb{R}^{2n}$ of contact-type have periodic orbits. Extending Gromov's theory of pseudoholomorphic curves, [3], to symplectized contact manifolds, H. Hofer, [4], related the Weinstein conjecture to the existence of certain pseudoholomorphic curves. He showed that in dimension three the Weinstein conjectures holds in many cases. In particular, he showed that Reeb vector fields associated to over-twisted contact structures admit periodic orbits. Recently the Weinstein conjecture in dimension three was completely settled by C. Taubes, [17, 18], who exploited relationships between Seiberg–Witten–Floer homology, [12], and embedded contact

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homology, [11], in order to construct holomorphic curves in the symplectized contact manifold out of nontrivial Seiberg–Witten–Floer homology classes. For more references on the Weinstein conjecture see [6].

In this note we show that many Reeb vector fields on higher dimensional closed manifolds have periodic orbits generalizing the main result from [4]. Our existence result is closely connected to the interesting attempt by K. Niederkrüger [13] to generalize the three-dimensional notion of an overtwisted contact structure. He introduced the concept of a *Plastikstufe* which currently seems to be the most compelling generalisation given recent further developments by F. Presas, [15], and K. Niederkrüger and O. van Koert, [14].

Let us denote by (M, ξ) a pair consisting of a closed manifold M of dimension $2n-1$ and a co-oriented contact structure ξ. We denote by $\mathbb{D}^2$ the closed unit disk in $\mathbb{C}$ with coordinates $x + iy$.

Definition 1.1. We say that (M, ξ) contains a *Plastikstufe* with singular set S provided M admits a closed submanifold S of dimension $n-2$ and an embedding $\iota\colon \mathbb{D}^2 \times S \to M$ with $\iota(\{0\} \times S) = S$ having the following properties:

(1) There exists a contact form λ_{PS} inducing ξ so that the one-form $\beta := \iota^*\lambda_{\mathrm{PS}}$ satisfies $\beta \wedge d\beta = 0$ and moreover $\beta \neq 0$ on $(\mathbb{D}^2 \setminus \{0\}) \times S$. Near $\{0\} \times S$ the form β is given by $\beta = x\,dy - y\,dx$ and the pull-back of β to $\partial\mathbb{D}^2 \times S$ vanishes.

(2) The complement of $\{0\} \times S$ in $(\mathbb{D}^2 \setminus \partial\mathbb{D}^2) \times S$ is smoothly foliated by β via an S^1-family of leaves diffeomorphic to $(0, 1) \times S$, where one of the ends converges to the singular set $\{0\} \times S$ and the other is asymptotic to the leaf $\partial\mathbb{D}^2 \times S$.

The set $\mathcal{PS}(S) = \iota(\mathbb{D}^2 \times S)$ is called the *Plastikstufe*.

Let us observe that the existence of a *Plastikstufe* for a given contact structure involves the existence of a certain inducing contact form. This is different from the three-dimensional case where an over-twisted disk is defined only in terms of the contact structure and does not require the existence of a particular contact form. In the following we shall call a closed co-oriented contact manifold (M, ξ) PS-overtwisted provided there exists a contact form λ_{PS} inducing ξ containing a *Plastikstufe*. Recently Niederkrüger and van Koert showed that every odd-dimensional sphere S^{2n-1} with $n \geq 3$ has a contact structure admitting a *Plastikstufe*. If now (M^{2n-1}, ξ) is a co-oriented contact manifold then a connected sum with a PS-overtwisted sphere admits by standard arguments a contact structure which is PS-overtwisted. In particular, any closed manifold of dimension $2n-1$ admitting a co-oriented contact structure also admits a PS-overtwisted contact structure. Our main result is the following theorem.

Theorem. *Let (M, ξ) be a closed PS-overtwisted contact manifold. Then every Reeb vector field associated to a contact form λ inducing ξ has a contractible periodic orbit.*

Remark 1.2. In [13] Niederkrüger shows that a PS-overtwisted contact structure does not have a semi-positive symplectic filling. We noticed that some of his idea combined with ideas from [4] lead to the above theorem. We also observed that the limitation to semi-positive fillings is not necessary and can be removed using polyfolds [5]. This will be discussed in a forthcoming paper.

2. Background

All material in this section is taken from [13].

2.1. Local normal form. Let (M, λ) contain a Plastikstufe $\mathcal{PS}(S)$. In [13, Section 3.1] it is proved that there exist constants $\varepsilon, C > 0$ and an open set V in the symplectic manifold $\big((-\varepsilon, 0] \times M, d(e^s \lambda)\big)$ such that $\{0\} \times S \subset V$ and V is symplectomorphic to the subset

$$U := \left\{ \big((z_1, z_2), (q, p)\big) \in \mathbb{C}^2 \times T^*S \;\middle|\; \begin{array}{l} -C < \operatorname{Re}(z_1) \leq 0, \\ -C < \operatorname{Im}(z_1) < C, \\ \operatorname{Re}(z_1) + \frac{1}{4}|z_2|^2 + \frac{1}{2}||p||^2 \leq 0 \end{array} \right\} \tag{2.1}$$

of $\mathbb{C}^2 \times T^*S$ which carries its natural symplectic structure. Moreover, $M \cap V$ corresponds to equality in the last equation and $\mathcal{PS}(S) \cap V$ to equality and $\operatorname{Im}(z_1) = 0$, $p = 0$.

2.2. Bishop family. The local model U contains a natural $(n-1)$-dimensional *Bishop family* given by

$$\begin{aligned} u_{t_0,q_0} \colon \mathbb{D}^2 &\longrightarrow \mathbb{C}^2 \times T^*S, \\ z &\longmapsto \big((-t_0, 2\sqrt{t_0}\, z), (q_0, 0)\big), \end{aligned} \tag{2.2}$$

where $0 \leq t_0 < C$ is a real parameter and $q_0 \in S$. The maps $u_{(t_0,q_0)}$ are $(i \times j)$-holomorphic, where j denotes the natural almost complex structure on T^*S induced by the Levi-Civita connection of a Riemannian metric on S. Moreover, they have boundary on the set corresponding to $\mathcal{PS}(S)$.

We denote by J the almost complex structure on V obtained by pulling back the almost complex structure $i \times j$ from $\mathbb{C}^2 \times T^*S$. Then we can pull back the Bishop family to holomorphic maps (denoted by the same symbols)

$$u_{t_0,q_0} \colon \mathbb{D}^2 \longrightarrow V \subset (-\varepsilon, 0] \times M. \tag{2.3}$$

2.3. Uniqueness results for holomorphic disks. We extend the almost complex structure J from the set V to a compatible almost complex structure on $\big(W := (-\infty, 0] \times M, d(e^s\lambda)\big)$. We introduce the notation

$$\widehat{\mathcal{PS}}(S) = \mathcal{PS}(S) \setminus (\partial\,\mathcal{PS}(S) \cup S) \tag{2.4}$$

and remark that $\widehat{\mathcal{PS}}(S)$ is totally real with respect to J. The following proposition is taken from [13, Proposition 7].

Proposition 2.1. *Let $u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to \big(W, \widehat{\mathcal{PS}}(S)\big)$ be a J-holomorphic disk which is simple. Moreover, we assume that $u(S^1) \subset \mathcal{PS}(S)$ bounds a disk in $\mathcal{PS}(S)$ and*

$$\mathrm{image}(u) \cap V \neq \emptyset. \tag{2.5}$$

Then, up to an element in $\mathrm{Aut}(\mathbb{D}^2)$, we have

$$u = u_{t_0,q_0}, \tag{2.6}$$

that is, after reparametrization, the holomorphic disk u is a member of the Bishop family.

3. Proof of the theorem

By assumption there exists a contact form λ_{PS} on M containing a Plastikstufe. Let λ be another contact form inducing the same contact structure.

We assume by contradiction that there exists no contractible closed Reeb orbit for λ.

3.1. The set-up. We choose a function $f\colon M \to \mathbb{R}$ such that $\lambda = f\lambda_{\mathrm{PS}}$. Since multiplying λ with a non-zero constant does not change its Reeb orbits (up to reparametrization) we may assume without loss of generality that the function f takes only values in $(0, 1)$. Then we can choose a smooth family of functions $f_s\colon M \to (0, \infty)$ for $s \in [-1, -\varepsilon]$ satisfying

$$f_s = \begin{cases} 1 & \text{near } s = -\varepsilon, \\ f & \text{near } s = -1, \end{cases} \quad \text{and moreover} \quad \frac{\partial f_s}{\partial s} \geq 0. \tag{3.1}$$

This gives rise to a smooth family $\lambda_s = f_s\lambda_{\mathrm{PS}}$ of contact forms which we extend by λ_{PS} for $s \geq -\varepsilon$ and by λ for $s \leq -1$. On $W = (-\infty, 0] \times M$ we choose an exact symplectic form Ω which satisfies

$$\Omega = \begin{cases} d(e^s\lambda_{\mathrm{PS}}) & \text{on } [-\varepsilon, 0] \times M, \\ d(e^s\lambda) & \text{on } (-\infty, -1] \times M. \end{cases} \tag{3.2}$$

This is possible due to our choice of the family f_s. This has been used in the literature many times, see for instance [7]. We modify the almost complex structure J from above to a compatible almost complex structure J on (W, Ω) by requiring that on $(-\infty, -2] \times M$ the almost complex structure is adapted to the negative part of the symplectization of λ, in the sense of [1]. On V it remains as defined in the previous section. In particular, (W, Ω) still contains the Bishop family u_{t_0,q_0}. We denote the relative homotopy class given by the Bishop disks by $a \in \pi_2(W, \widehat{\mathcal{PS}}(S))$ and set

$$\mathcal{M}(J) := \{u : (\mathbb{D}^2, \partial\mathbb{D}^2) \longrightarrow (W, \widehat{\mathcal{PS}}(S)) \mid \bar\partial_J u = 0,\ [u] = a,\ \mathrm{lk}(u, S) = 1\}, \tag{3.3}$$

$$\widehat{\mathcal{M}}(J) := \mathcal{M}(J)/\mathrm{Aut}(\mathbb{D}^2), \tag{3.4}$$

where $\mathrm{lk}(u, S)$ is the linking number of $u(S^1)$ in $\mathcal{PS}(S)$ with the set S. This is defined as follows. By definition $\widehat{\mathcal{PS}}(S)$ is foliated by an S^1-family of Legendrian submanifolds, thus there exists a natural map $\theta\colon \widehat{\mathcal{PS}}(S) \to S^1$. We set $\mathrm{lk}(u, S) := \deg(\theta \circ u|_{S^1})$.

3.2. The proof. We need the following three facts established in Propositions 8–10 in [13].

(1) The Maslov index of a equals $\mu_{\mathrm{Maslov}}(a) = 2$,

(2) the almost complex structure J is regular at members of the Bishop family,

(3) the energy of all elements in $\mathcal{M}(J)$ is uniformly bounded.

The totally real submanifold $\widehat{\mathcal{PS}}(S)$ is non-compact. Since $\partial\,\mathcal{PS}(S)$ is a closed leaf of the characteristic foliation the maximum principle implies that no holomorphic maps intersect $\partial\,\mathcal{PS}(S)$ at an interior point. According to Proposition 2.1 near S the only holomorphic disks are members of the Bishop family. Therefore, the non-compactness of $\widehat{\mathcal{PS}}(S)$ poses no problem. Moreover, due to the energy bounds and the specific structure of the almost complex structure J on the end of W we can apply the ideas of the SFT-compactness theorem [1]. Since we assumed that there exists no contractible closed Reeb orbits bubbling-off cannot occur in the interior. Therefore, the only non-compactness of the moduli space $\widehat{\mathcal{M}}(J)$ comes from bubbling-off of holomorphic disks having boundary on $\widehat{\mathcal{PS}}(S)$. The next proposition is taken from [13, Proposition 11] and shows that there exists no bubbling-off of holomorphic disks.

Proposition 3.1. *Given a sequence $(u_n) \subset \widehat{\mathcal{M}}(J)$ there exists a subsequence either converging to an element in $\widehat{\mathcal{M}}(J)$ or to a point in S.*

The latter case occurs if a family of Bishop disks shrinks to a point in S. We remark that in the former case the limit is simple.

Proposition 3.2. *For a compatible almost complex structure J, which is generic on the subset $\big((-2,0]\times M\big)\setminus V$ of (W,Ω), the moduli space $\mathcal{M}(J)$ is a smooth manifold of dimension*

$$\dim \mathcal{M}(J) = n+2. \tag{3.5}$$

Proof. We pick $u \in \mathcal{M}(J)$. In case that $\mathrm{image}(u) \cap V \neq \emptyset$ we conclude from Proposition 2.1 that u is a member of the Bishop family. In particular, $\mathrm{image}(u) \subset V$. Moreover, J is already regular for members in the Bishop family.

If $\mathrm{image}(u)\cap V = \emptyset$ then u has to pass through the region $\big((-2,0]\times M\big)\setminus V$. Since all the disks are simple a generic J will be regular, see for example [2]. The dimension formula follows from the fact that $\mu_{\mathrm{Maslov}}(a) = 2$ and $\dim \widehat{\mathcal{PS}}(S) = n$. □

We consider the evaluation map

$$\begin{aligned} \mathrm{ev}\colon \widehat{\mathcal{M}}(J)_{S^1} := \mathcal{M}(J) \times_{\mathrm{Aut}(\mathbb{D}^2)} S^1 &\longrightarrow \widehat{\mathcal{PS}}(S) \subset M, \\ [u,t] &\longmapsto u(e^{2\pi i t}), \end{aligned} \tag{3.6}$$

defined on the smooth manifolds $\widehat{\mathcal{M}}(J)_{S^1}$ of dimension $\dim \widehat{\mathcal{M}}(J)_{S^1} = n$. The following proposition is proved analogously to the transversality result above.

Proposition 3.3. *For a generic J as in the previous proposition the evaluation map is smooth.*

To derive the contradiction to the assumption that λ has no closed Reeb orbits we make the following

Definition 3.4. For a point $p = \iota(z,s) \in \iota(\mathbb{D}^2\times S) = \mathcal{PS}(S)$ we define the distance of p to S by $d(p,S) = |z|$ and set for $\delta > 0$ sufficiently small

$$\widehat{\mathcal{M}}(J)^{\delta}_{S^1} := \big\{[u,t] \in \widehat{\mathcal{M}}(J)_{S^1} \mid d(\mathrm{ev}([u,t]),S) \geq \delta\big\}. \tag{3.7}$$

Then we have by construction of the Bishop family

$$\mathrm{ev}\big(\partial\widehat{\mathcal{M}}(J)^{\delta}_{S^1}\big) = \iota(S^1_\delta \times S), \tag{3.8}$$

where $S^1_\delta = \{z \in \mathbb{D}^2 \mid |z| = \delta\}$. We conclude that

$$\big[\mathrm{ev}\big(\partial\widehat{\mathcal{M}}(J)^{\delta}_{S^1}\big)\big] \in \mathrm{H}_{n-1}(\widehat{\mathcal{PS}}(S),\mathbb{Z}/2)$$

is the generator. On the other hand the set $\mathrm{ev}\big(\partial\widehat{\mathcal{M}}(J)^{\delta}_{S^1}\big)$ is clearly the boundary of the compact, see Proposition 3.1, manifold $\mathrm{ev}\big(\widehat{\mathcal{M}}(J)^{\delta}_{S^1}\big)$. This implies, that

$$\big[\mathrm{ev}\big(\partial\widehat{\mathcal{M}}(J)^{\delta}_{S^1}\big)\big] = 0 \in \mathrm{H}_{n-1}(\widehat{\mathcal{PS}}(S),\mathbb{Z}/2).$$

This contradictions concludes the proof of the theorem.

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