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# $NK_0$ and $NK_1$ of the groups $C_4$ and $D_4$

## Addendum to "Lower algebraic $K$-theory of hyperbolic 3-simplex reflection groups" by J.-F. Lafont and I. J. Ortiz

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**Abstract.** In this addendum to [LO] we explicitly compute the Bass Nil-groups $NK_i(\mathbb{Z}[C_4])$ for $i = 0, 1$ and $NK_0(\mathbb{Z}[D_4])$. We also show that $NK_1(\mathbb{Z}[D_4])$ is not trivial. Here $C_4$ denotes the cyclic group of order 4 and $D_4$ is the dihedral group of order 8.

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In [LO], Lafont and Ortiz computed the lower algebraic $K$-theory of the integral group ring of all 32 hyperbolic 3-simplex reflection groups (see [LO, Tables 6–7]). For 25 of these integral group rings, their computation was completely explicit. For the remaining 7 examples, the expression for some of the $K$-groups involved the Bass Nil-groups $NK_0$ and $NK_1$ associated to $D_4$ (the dihedral group of order 8).

In [L05], Lück computed the lower algebraic $K$-theory of the integral group ring of the semi-direct product of the three-dimensional discrete Heisenberg group by $C_4$ (the cyclic group of order 4). These computations involved the Bass Nil-groups $NK_0$ and $NK_1$ associated to $C_4$ (see [L05, Corollary 3.9]).

In this addendum we compute the Bass Nil-groups

$$NK_n(\mathbb{Z}G) = \ker\{K_n(\mathbb{Z}G[x]) \xrightarrow{x\mapsto 0} K_n(\mathbb{Z}G)\},$$

where $G$ is $D_2$, $C_4$ or $D_4$, and $n = 0, 1$. We will use these calculations to complement the calculations of [L05] and [LO] in 1.5 and 2.9 below.

Our calculation will keep track of the additional structure on the groups $NK_n(A)$ given by the Verschiebung and Frobenius operators, $V_m$ and $F_m$, as well as the continuous module structure over the ring $W(\mathbb{Z})$ of big Witt vectors; the additive group of $W(\mathbb{Z})$ is the abelian group $(1 + x\mathbb{Z}[[x]])^\times$. (See [We80] for more details.) In fact, it is a module over the slightly larger Cartier algebra consisting of row-and-column finite sums $\sum V_m[a_{mn}]F_n$, where $V_m$ and $F_m$ are the Verschiebung and Frobenius operators, and the $[a]$ are the homotheties operators for $a \in \mathbb{Z}$; see [DW93] as well as Remarks 1.2.1 and 2.4 below. Some of the identities satisfied by these operators include: $V_mV_n = V_{mn}$, $F_mF_n = F_{mn}$, $F_mV_m = m$, $[a]V_m = V_m[a^m]$ and $F_m[a] = [a^m]F_m$.

It is convenient to write $V$ for the continuous $W(\mathbb{F}_2)$-module $x\mathbb{F}_2[x]$, which, as an abelian group, is just a countable direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2$ on generators $x^i$, $i > 0$. The module structure on $V$ is determined by: $V_m(x^n) = x^{mn}$; $[a]x^n = a^nx^n$; $F_m(x^n) = 0$ if $(m,n) = 1$ $(m > 1)$ and $F_d(x^n) = d\, x^{n/d}$ when $d \mid n$.

## 1. The groups $C_2$, $D_2$ and $C_4$

For the cyclic group $C_2 = \langle\sigma\rangle$ of order two, consider the Rim square:

$$\begin{tikzcd} \mathbb{Z}[C_2] \arrow[r, "\sigma\mapsto+1"] \arrow[d, "\sigma\mapsto-1"'] & \mathbb{Z} \arrow[d, "q"] \\ \mathbb{Z} \arrow[r, "q"] & \mathbb{F}_2 \end{tikzcd} \quad \text{or, equivalently,} \quad \begin{tikzcd} \mathbb{Z}[C_2] \arrow[r, "\sigma\mapsto(1,-1)"] \arrow[d] & \mathbb{Z}\times\mathbb{Z} \arrow[d, "q\times q"] \\ \mathbb{F}_2 \arrow[r, "\Delta"] & \mathbb{F}_2\times\mathbb{F}_2 \end{tikzcd} \qquad (1)$$

from which we immediately get $NK_0(\mathbb{Z}[C_2]) = NK_1(\mathbb{Z}[C_2]) = 0$ as in [Bas68, XII.10.6] and [Mi71, 6.4]. From Guin–Loday–Keune [GL80], [Keu81], the double relative group $NK_2(\mathbb{Z}[C_2], \sigma+1, \sigma-1)$ is isomorphic to $V$, with the Dennis–Stein symbol $\langle x^n(\sigma-1), \sigma+1\rangle$ corresponding to $x^n \in V$. We also have a diagram

$$\begin{tikzcd} & NK_2(\mathbb{Z}[C_2],\sigma+1,\sigma-1) \cong V \arrow[d, "\cong"] & & \\ 0 = NK_3(\mathbb{Z}) \arrow[r] & NK_2(\mathbb{Z}[C_2],\sigma+1) \arrow[r, "\cong"] \arrow[d] & NK_2(\mathbb{Z}[C_2]) \arrow[r] \arrow[d] & NK_2(\mathbb{Z}) = 0 \\ & 0 = NK_2(\mathbb{Z},2) \arrow[r] & NK_2(\mathbb{Z}) = 0. & \end{tikzcd}$$

Thus we obtain:

**Theorem 1.1.** *$NK_2(\mathbb{Z}[C_2]) \cong V$ with $\langle x^n(\sigma-1), \sigma+1\rangle$ corresponding to $x^n \in V$.*

We now turn to the group $D_2 = C_2 \times C_2$. First we need a calculation. Let $\Phi(V)$ denote the subgroup (and Cartier submodule) $x^2\mathbb{F}_2[x^2]$ of $V$, and write $\Omega_R$ for the Kähler differentials of $R$, so that $\Omega_{\mathbb{F}_2[x]} \cong \mathbb{F}_2[x]dx$. By abuse, we will write $\mathbb{F}_2[\varepsilon]$ for the 2-dimensional algebra $\mathbb{F}_2[\varepsilon]/(\varepsilon^2)$.

**Lemma 1.2.** *The map* $q\colon \mathbb{Z}[C_2] \to \mathbb{F}_2[C_2] \cong \mathbb{F}_2[\varepsilon]$ *in* (1) *induces an exact sequence*

$$0 \longrightarrow \Phi(V) \longrightarrow NK_2(\mathbb{Z}[C_2]) \xrightarrow{q} NK_2(\mathbb{F}_2[\varepsilon]) \xrightarrow{D} \Omega_{\mathbb{F}_2[x]} \longrightarrow 0.$$

*Proof.* Van der Kallen computed $NK_2(\mathbb{F}_2[\varepsilon])$ in [vdK71, Exemple 3]: there is a split short exact sequence

$$0 \longrightarrow V/\Phi(V) \xrightarrow{F} NK_2(\mathbb{F}_2[\varepsilon]) \xrightarrow{D} \Omega_{\mathbb{F}_2[x]} \longrightarrow 0, \tag{2}$$

where $F(x^n) = \langle x^n\varepsilon, \varepsilon\rangle$ and $D(\langle f\varepsilon, g + g'\varepsilon\rangle) = f\,dg$. The map $NK_2(\mathbb{Z}[C_2]) \xrightarrow{q} NK_2(\mathbb{F}_2[\varepsilon])$ sends $\langle x^n(\sigma+1), \sigma-1\rangle$ to $F(x^n) = \langle x^n\varepsilon, \varepsilon\rangle$. By Theorem 1.1 and (2), this map has kernel $\Phi(V)$ and image $F(V/\Phi(V))$. □

**Remark 1.2.1.** Although $\Omega_{\mathbb{F}_2[x]}$ is isomorphic to $V$ as an abelian group, it has a different $W(\mathbb{F}_2)$-module structure. This is determined by the formulas in $\Omega_{\mathbb{F}_2[x]}$:

$$V_m(x^{n-1}\,dx) = mx^{mn-1}dx, \quad F_m(x^{n-1}dx) = \begin{cases} x^{n/m-1}dx & \text{if } m \mid n, \\ 0 & \text{else.} \end{cases}$$

Grunewald has pointed out that since $\Omega_{\mathbb{F}_2[x]}$ is not finitely generated as a module over the $\mathbb{F}_2$-Cartier algebra (of row-and-column finite sums $\sum V_m[a_{mn}]F_n$), or over the subalgebra $W(\mathbb{F}_2)$), neither are $NK_2(\mathbb{F}_2[\varepsilon])$ or (by 1.3 below) $NK_1(\mathbb{Z}[D_2])$.

**Theorem 1.3.** *For* $D_2 = C_2 \times C_2$, $NK_0(\mathbb{Z}[D_2]) \cong V$, $NK_1(\mathbb{Z}[D_2]) \cong \Omega_{\mathbb{F}_2[x]}$ *and the image of the map* $NK_2(\mathbb{Z}[D_2]) \to NK_2(\mathbb{Z}[C_2])^2 \cong V^2$ *is* $\Phi(V) \times V$.

*Proof.* We tensor (1) with $\mathbb{Z}[C_2]$. Since $\mathbb{F}_2[C_2] \cong \mathbb{F}_2[\varepsilon]$, $\varepsilon^2 = 0$, and $NK_1(\mathbb{F}_2[C_2]) \cong (1 + x\varepsilon\mathbb{F}_2[x])^\times \cong V$, then the Mayer–Vietoris sequence in [Mi71, Theorem 6.4] for the $NK$-functor,

$$\begin{aligned} NK_2(\mathbb{Z}[D_2]) &\longrightarrow (NK_2(\mathbb{Z}[C_2]))^2 \xrightarrow{q\times q} NK_2(\mathbb{F}_2[\varepsilon]) \longrightarrow NK_1(\mathbb{Z}[D_2]) \\ &\longrightarrow (NK_1(\mathbb{Z}[C_2])^2 \longrightarrow NK_1(\mathbb{F}_2[\varepsilon]) \xrightarrow{\cong} NK_0(\mathbb{Z}[D_2]) \longrightarrow NK_0(\mathbb{Z}[C_2]), \end{aligned} \tag{3}$$

quickly gives $NK_0(\mathbb{Z}[D_2]) \cong NK_1(\mathbb{F}_2[\varepsilon]) \cong V$. By Lemma 1.2, the initial portion of (3) yields the calculation of $NK_1(\mathbb{Z}[D_2])$ and the asserted surjection $NK_2(\mathbb{Z}[D_2]) \twoheadrightarrow \Phi(V) \times V$. □

**Remark.** The kernel $K$ of the map $NK_2(\mathbb{Z}[D_2]) \to V^2$ in Theorem 1.3 has a subgroup generated by the double relative group $NK_2(\mathbb{Z}[D_2], \sigma_1+1, \sigma_1-1)$, which is isomorphic to $\mathbb{F}_2[\varepsilon] \otimes V$ on the symbols $\langle x^n(a+b\sigma_2)(\sigma_1+1), \sigma_2-1\rangle$, where $\sigma_1, \sigma_2$ are the generators of $D_2 = C_2 \times C_2$. The quotient of $K$ by this subgroup is generated by the image of $NK_3(\mathbb{F}_2[\varepsilon])$, a group which I do not know.

The analysis for the cyclic group $C_4$ of order 4 on generator $\sigma$ is similar, using the Rim square

$$\begin{array}{ccc} \mathbb{Z}[C_4] & \xrightarrow{\sigma\mapsto i} & \mathbb{Z}[i] \\ {\scriptstyle \sigma^2\mapsto 1}\downarrow & & \downarrow{\scriptstyle i\mapsto 1+\varepsilon} \\ \mathbb{Z}[C_2] & \xrightarrow{q} & \mathbb{F}_2[\varepsilon]\,. \end{array} \tag{4}$$

**Theorem 1.4.** *$NK_1(\mathbb{Z}[C_4]) \cong \Omega_{F_2[x]}$ and $NK_0(\mathbb{Z}[C_4]) \cong V$.*

*Proof.* Since $\mathbb{Z}[i]$ is a regular ring, the Mayer–Vietoris sequence for (4) reduces to

$$NK_2(\mathbb{Z}[C_4]) \xrightarrow{p_2} NK_2(\mathbb{Z}[C_2]) \xrightarrow{q} NK_2(\mathbb{F}_2[\varepsilon]) \longrightarrow NK_1(\mathbb{Z}[C_4])$$
$$\longrightarrow NK_1(\mathbb{Z}[C_2]) \to NK_1(\mathbb{F}_2[\varepsilon]) \xrightarrow{\cong} NK_0(\mathbb{Z}[C_4]) \longrightarrow NK_0(\mathbb{Z}[C_2]).$$

The isomorphism marked in this sequence follows from Theorem 1.1. By Lemma 1.2, the image of the first map $p_2$ is $\Phi(V)$ and the cokernel of the map $q$ is $\Omega_{\mathbb{F}_2[\varepsilon]}$. $\square$

**Remark.** The proof provides a surjection $NK_2(\mathbb{Z}[C_4]) \xrightarrow{p_2} \Phi(V)$. The kernel of $p_2$ contains the image $E$ of the double relative group $NK_2(\mathbb{Z}[C_4], \sigma^2+1, \sigma^2-1)$, which is isomorphic to $\mathbb{F}_2[\varepsilon] \otimes V$ on symbols $\langle \sigma^2+1, x^n(\sigma^2-1)\rangle$. The quotient $\ker(p_2)/E$ is generated by the image of $NK_3(\mathbb{F}_2[\varepsilon])$, which I do not know.

Here is an application of this calculation. Let Hei denote the *three-dimensional discrete Heisenberg group*, which is the subgroup of $GL(3,\mathbb{Z})$ consisting of upper triangular integral matrices with ones along the diagonal. Consider the action of the cyclic group $C_4$ given by

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -z & y-xz \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Combining Theorem 1.4 with [L05, 3.9], the lower $K$-theory of the group Hei $\rtimes C_4$ is given in the following proposition.

**Proposition 1.5.** *We have*

$$\mathrm{Wh}_n(\mathrm{Hei} \rtimes C_4) = \begin{cases} \bigoplus_\infty \mathbb{Z}/2, & n = 0,1, \\ 0, & n \le -1. \end{cases}$$

## 2. The dihedral group $D_4$

Before moving on to the group ring of $D_4$, we need some facts about the double relative groups $K_1(A, B, I)$ when $A \to B$ is an injection. These groups were described in [GW83, 0.2] as follows:

$$K_1(A, B, I) \cong (B/A) \otimes (I/I^2)/\{b \otimes cz + c \otimes zb - bc \otimes z\} \quad (b, c \in B, z \in I). \tag{5}$$

Moreover, by [GW83, 3.12 and 4.1], the map $K_1(A, B, I) \to K_1(A, I)$ sends the class of $b \otimes z$ to the class of the matrix $\left(\begin{smallmatrix} 1-zb & z \\ -bzb & 1+bz \end{smallmatrix}\right)$.

**Lemma 2.1.** *Suppose that $A \to B$ is a ring homomorphism mapping an ideal $I$ of $A$ isomorphically onto an ideal of $B$. Then the double relative group satisfies*

$$K_1(A[x], B[x], I[x]) \cong K_1(A, B, I) \otimes \mathbb{Z}[x],$$

*and*

$$NK_1(A, B, I) \cong K_1(A, B, I) \otimes x\mathbb{Z}[x].$$

*Proof.* Because $x$ is central in $B[x]$, the formulas are immediate from (5). □

We will be specifically interested in the twisted group ring $A = \mathbb{Z}[i] \rtimes C_2$, where $C_2 = \langle \tau \rangle$ acts on $\mathbb{Z}[i]$ by $\tau i \tau^{-1} = -i$. It injects into the matrix ring $B = M_2(\mathbb{Z})$ by the map $\phi\colon A \to B$ defined by $\phi(i) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\phi(\tau) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The ideal $I = (2, 1+\tau)A$ maps isomorphically to $2B$, and $A/I \cong \mathbb{F}_2[\varepsilon_1]$, where $\varepsilon_1 = 1 + i$ and $\varepsilon_1^2 = 0$. Hence we have the following cartesian square:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & M_2(\mathbb{Z}) \\ {\scriptstyle \bmod I}\big\downarrow & & \big\downarrow{\scriptstyle \bmod 2B} \\ \mathbb{F}_2[\varepsilon_1] & \longrightarrow & M_2(\mathbb{F}_2). \end{array} \tag{6}$$

To calculate $NK_1(A)$, we use the following double relative calculation.

**Lemma 2.2.** *The double relative group $K_1(A, B, 2B)$ of (6) is isomorphic to $\mathbb{F}_2$, and $NK_1(A, B, 2B) \cong V$. The map $V \cong NK_1(A, B, 2B) \to NK_1(A)$ sends $x^n \in V$ to the class of the unit $1 + x^n i(1+\tau)$ of $A[x]$.*

*Proof.* Since $\dim(B/A) = 2$ and $\dim(I/I^2) = 4$, the group $(B/A) \otimes (I/I^2)$ has 8 generators and 64 relations; a basis of $B/A$ is $\{e_{11}, e_{12}\}$ and the $2e_{ij}$ span $I/I^2$. By inspection of the relations in (5) we see that the map $B/I \otimes I/I^2 \to \mathbb{F}_2$ sending $e_{ij} \otimes 2e_{kl}$ to $\delta_{il} + \delta_{jk}$ sends $A/I \otimes I/I^2$ and all the relations in (5) to zero, and sends $e_{11} \otimes 2e_{12}$ to 1. Thus it induces a surjection $K_1(A, B, I) \to \mathbb{F}_2$. We claim that this is an isomorphism.

The relations for $(b, c, z) = (e_{11}, e_{11}, 2e_{11})$, $(e_{11}, e_{11}, 2e_{22})$, $(e_{11}, e_{12}, 2e_{12})$ and $(e_{11}, e_{21}, 2e_{21})$ in (5) yield the relations

$$0 = e_{11} \otimes 2e_{11} = e_{11} \otimes 2e_{22} = e_{12} \otimes 2e_{12} = e_{21} \otimes 2e_{21}.$$

The relations for $(e_{12}, e_{11}, 2e_{11})$, $(b, c, z) = (e_{11}, e_{21}, 2e_{11})$ and $(e_{12}, e_{12}, 2e_{21})$ in (5) yield the relations

$$0 = e_{11} \otimes 2e_{12} - e_{12} \otimes 2e_{11} = e_{21} \otimes 2e_{11} - e_{11} \otimes 2e_{21} = e_{12} \otimes 2e_{22} - e_{12} \otimes 2e_{11}.$$

This verifies the claim, proving that $K_1(A, B, 2B) \cong \mathbb{F}_2$.

Finally, the map $NK_1(A, B, 2B) \to NK_1(A, I)$ sends the class of $x^n e_{11} \otimes 2e_{12}$ to the class of the matrix $\begin{pmatrix} 1-2x^n e_{12}e_{11} & 2x^n e_{12} \\ -x^{2n} e_{11}e_{12}e_{11} & 1+2x^n e_{11}e_{12} \end{pmatrix} = \begin{pmatrix} 1 & x^n i(1+\tau) \\ 0 & 1+x^n i(1+\tau) \end{pmatrix}$, which is the class of $1 + x^n i(1+\tau)$ in $NK_1(A)$. □

**Remark.** The elements $u = i + \tau$ and $v = i(1+\tau)$ of $A$ satisfy $u^2 = v^2 = 0$, and are distinct in $A/2A = \mathbb{F}_2[i, \tau]$. Hence the units $(1+x^m u)(1+x^n v)$ of $A[x]$ generate a subgroup of $NK_1(A)$ isomorphic to $V^2$, which injects into $NK_1(\mathbb{F}_2[i, \tau]) \cong V^3$. (Since $\mathbb{F}_2[i, \tau] = \mathbb{F}_2[u, v]/(u^2, v^2)$, the other copy of $V$ in $NK_1(\mathbb{F}_2[i, \tau])$ is the subgroup generated by all $(1 + x^n uv)$.)

**Proposition 2.3.** *$NK_0(A) = 0$ and $NK_1(A) \cong V^2$ on the units $(1+x^m u)(1+x^n v)$.*

*The maps $A \to \mathbb{F}_2[i, \tau] \cong \mathbb{F}_2[\varepsilon_1, \varepsilon_2]$ and $\Omega_{\mathbb{F}_2[x]} \xrightarrow{\delta} NK_2(\mathbb{F}_2[\varepsilon_1, \varepsilon_2])$ sending $x^n$ to $\langle x^{n-1}\varepsilon_1\varepsilon_2, x\rangle$ induce a surjection $NK_2(A) \times \Omega_{\mathbb{F}_2[x]} \to NK_2(\mathbb{F}_2[\varepsilon_1, \varepsilon_2])$.*

*Proof.* Consider the Mayer–Vietoris sequence of the square (6). Since $B = M_2(\mathbb{Z})$ and $B/I = M_2(\mathbb{F}_2)$ are regular rings, $NK_n(B) = NK_n(B/I) = 0$ and hence $NK_n(B, I) = 0$ for all $n$. We immediately get that $NK_n(A, B, I) \cong NK_n(A, I)$, that the Mayer–Vietoris sequence reduces to $NK_0(A) \cong NK_0(A/I) = 0$, and that there is an exact sequence

$$NK_2(A) \to NK_2(\mathbb{F}_2[\varepsilon_1]) \to NK_1(A, B, I) \to NK_1(A) \to NK_1(\mathbb{F}_2[\varepsilon_1]) \to 0.$$

By Lemma 2.2 and the remark preceding it, this yields the calculation of $NK_1(A)$.

Now $\pi\colon A \to \mathbb{F}_2[\varepsilon_1, \varepsilon_2]$ satisfies $\pi(u) = \varepsilon_1 + \varepsilon_2$, $\pi(v) = \varepsilon_1 + \varepsilon_1\varepsilon_2$ and $\pi(uv) = \varepsilon_1\varepsilon_2$, so we may write $\mathbb{F}_2[\varepsilon_1, \varepsilon_2] \cong \mathbb{F}_2[\bar{u}, \bar{v}]/(\bar{u}^2, \bar{v}^2)$. By [vdK71], the group $NK_2(\mathbb{F}_2[\bar{u}, \bar{v}])$ is isomorphic to the direct sum of $NK_2(\mathbb{F}_2[\bar{u}])$, $NK_2(\mathbb{F}_2[\bar{v}])$ and a group with the following generators:

$$\langle x^n\bar{u}, \bar{v}\rangle, \quad \langle x^n\bar{u}\bar{v}, \bar{u}\rangle, \quad \langle x^n\bar{u}\bar{v}, \bar{v}\rangle \quad \text{and} \quad \langle x^{n-1}\bar{u}\bar{v}, x\rangle.$$

Since $u^2 = v^2 = 0$ in $A$, all these symbols lift to Dennis–Stein symbols in $NK_2(A)$ except possibly the symbols $\langle x^{n-1}\bar{u}\bar{v}, x\rangle$. But these symbols are hit by the image of $\Omega_{\mathbb{F}_2[x]}$ under $\delta$. $\square$

**Remark.** $\delta\colon \Omega_{\mathbb{F}_2[x]} \to NK_2(\mathbb{F}_2[\varepsilon_1, \varepsilon_2])$ is a homomorphism by the Dennis–Stein identity $\langle f, x\rangle\langle g, x\rangle = \langle f + g - fgx, x\rangle$ with $fg = 0$; see [GL80, p. 184]. It is a morphism of $\mathbb{F}_2$-Cartier modules since $V_m\langle x^{n-1}\varepsilon_1\varepsilon_2, x\rangle = m\langle x^{mn-1}\varepsilon_1\varepsilon_2, x\rangle = \delta(V_m(x^n))$ and (by [St80, 2.1])

$$F_m\langle x^{n-1}\varepsilon_1\varepsilon_2, x\rangle = \begin{cases} \langle x^{n/m-1}\varepsilon_1\varepsilon_2, x\rangle, & m \mid n \\ r\langle x^{n-1}(\varepsilon_1\varepsilon_2)^m, x\rangle - s\langle x^n(\varepsilon_1\varepsilon_2)^{m-1}, \varepsilon_1\varepsilon_2\rangle = 0, & rm + sn = 1. \end{cases}$$

Our analysis of $D_4$ will involve the units of the ring $\mathbb{Z}/4[x][C_2]$.

**Example 2.4.** *Consider the modular group ring $B = \mathbb{Z}/4[C_2] = \mathbb{Z}/4[e]/(e^2 - 2e)$, with $e = 1 - \tau$. The ideals $2eB$ of $B$ and $eB/2eB$ of $B/2eB$ are isomorphic to $\mathbb{F}_2$, so both $NK_1(B, 2e)$ and $NK_1(B/2e, e)$ are isomorphic to $V$ and the group $NK_1(B, e)$, identified with the abelian group $(1 + xeB[x])^\times$, is a nontrivial extension:*

$$0 \to V \to NK_1(B, e) \to V \to 0.$$

*As an abelian group, $NK_1(B, e)$ is the direct sum of a countably infinite free $\mathbb{Z}/4$-module on the $(1 + ex^m)$ $(m = 1, 2, \dots)$ and a countably infinite free $\mathbb{Z}/2$-module on the $(1 + 2ex^{2i-1})$ $(i = 1, 2, \dots)$. As a module over the $\mathbb{Z}/4$-Cartier algebra (generated by the operators $V_m$, $F_m$ and homothety [2]), $NK_1(B, e)$ is cyclic on generator $u = 1 + ex$; $V_m(u) = 1 + ex^m$ and $V_m[2](u) = 1 + 2ex^m$.*

Finally, we are in position to analyze $NK_0(\mathbb{Z}[D_4])$. The sharp exponent 4 for $NK_0(\mathbb{Z}[D_4])$ in Theorem 2.5 is a slight improvement on the bound in [CP02]. It is convenient to write $D_4$ as the semidirect product of $C_4$ (on $\sigma$) with the cyclic group $C_2 = \{1, \tau\}$, with relation $\tau\sigma\tau = \sigma^{-1}$.

**Theorem 2.5.** *The group $NK_0(\mathbb{Z}[D_4])$ is isomorphic to the cyclic Cartier module $NK_1(\mathbb{Z}/4[C_2], 1 - \tau)$, described in Example 2.4. As a group, it is the direct sum of a countably infinite free $\mathbb{Z}/4$-module and a countably infinite free $\mathbb{Z}/2$-module.*

*Proof.* We can map $\mathbb{Z}[D_4]$ to the twisted ring $A = \mathbb{Z}[i] \rtimes C_2$ occurring in (6) above, sending $\sigma$ to $i$. Combining this with the natural surjection onto the subring $\mathbb{Z}[D_2]$ of $\mathbb{Z}[C_2] \times \mathbb{Z}[C_2]$, we get a ring map $\mathbb{Z}[D_4] \to A \times \mathbb{Z}[C_2] \times \mathbb{Z}[C_2]$. The ideal $I = (4, 2 - 2\sigma, \sigma^2 - 1)\mathbb{Z}[D_4]$ has $B_0 = \mathbb{Z}[D_4]/I = \mathbb{Z}/4[D_2]/(2 - 2\sigma)$, and is isomorphic to the ideal $2A\times(4)\times(4)$ of $A\times\mathbb{Z}[C_2]\times\mathbb{Z}[C_2]$. Consider the following cartesian square:

$$\begin{tikzcd} \mathbb{Z}[D_4] \arrow[r, "\sigma\mapsto(i,1,-1)", "\tau\mapsto(\tau,\tau,\tau)"'] \arrow[d] & A\times\mathbb{Z}[C_2]\times\mathbb{Z}[C_2] \arrow[d, "\pi"] \\ B_0=\mathbb{Z}/4[D_2]/(2-2\sigma) \arrow[r, "q=(q_0,q_+,q_-)"] & \mathbb{F}_2[D_2]\times B\times B. \end{tikzcd} \tag{7}$$

The kernel of the split surjection $q_+\colon B_0\to B=\mathbb{Z}/4[C_2]$ is the 2-dimensional ideal $J=(1-\sigma)B_0$. This implies that $NK_1(B_0)=NK_1(B)\oplus NK_1(B_0,J)$. Because $NK_1(\mathbb{Z}[C_2])=NK_0(\mathbb{Z}[C_2])=NK_0(A)=0$ (by 2.3), the Mayer–Vietoris sequence associated to (7) ends

$$NK_1(A)\times NK_1(B_0,J)\xrightarrow{\eta} NK_1(\mathbb{F}_2[D_2])\times NK_1(B)\to NK_0(\mathbb{Z}[D_4])\to 0. \tag{8}$$

The displayed map $\eta$ is given by the matrix $\left(\begin{smallmatrix}\pi & 0\\ q_0 & q_-\end{smallmatrix}\right)$. It is easy to see that $NK_1(B_0,J)$ is isomorphic to $V^2$ on the terms $(1+(1-\sigma)x^m)$ and $(1+(1-\sigma)\tau x^n)$. An elementary calculation using the isomorphism $NK_1(A)\cong V^2$ of 2.3 shows that $\eta$ is an injection, sending the module $NK_1(A)\times NK_1(B_0,J)\cong V^4$ isomorphically onto the subgroup $NK_1(\mathbb{F}_2[D_2])\times NK_1(\mathbb{Z}/4)$. Since $NK_1(B)=NK_1(\mathbb{Z}/4)\oplus NK_1(B,eB)$, $e=1-\tau$, it follows that the induced map $NK_1(B,e)\to NK_0(\mathbb{Z}[D_4])$ is an isomorphism. □

To begin the calculation of $NK_1(\mathbb{Z}[D_4])$, we extend the Mayer–Vietoris sequence (8) associated to (7) to the left. This is possible by the following observation: since $B_0$ maps onto each of the three ring factors on the lower right of (7), the presentation (5) shows that the double relative $K_1$ obstruction vanishes. Because $\eta$ is an injection in (8), the continuation of the Mayer–Vietoris sequence yields the exact sequence

$$(*)\xrightarrow{\left(\begin{smallmatrix}\pi & 0\\ q_0 & q_-\end{smallmatrix}\right)} NK_2(\mathbb{F}_2[D_2])\times NK_2(B)\to NK_1(\mathbb{Z}[D_4])\to 0, \tag{9}$$

where $(*)$ denotes $NK_2(A)\times NK_2(\mathbb{Z}[C_2])^2\times NK_2(B_0,J)$.

**Definition 2.6.** The map $\cup[x]\colon NK_0(\mathbb{Z}[D_4])\to NK_1(\mathbb{Z}[D_4])$ is obtained by composing the isomorphism $NK_1(B,e)\cong NK_0(\mathbb{Z}[D_4])$ of Theorem 2.5 with the canonical map $NK_2(B,e)\to NK_2(B)\to NK_1(\mathbb{Z}[D_4])$ of (9).

**Remark.** There is also a canonical map $NK_1(B,e)\to NK_2(B,e)$ sending the unit $1-aex$ to $\langle ae,x\rangle$; the composition with $NK_2(B,e)\subset K_2(B[x,x^{-1}],e)$ is given by $1-aex\mapsto\{1-aex,x\}$ (multiplication by the class of $x$ in $K_1(\mathbb{Z}[x,x^{-1}])$).

The analogous maps from $V \cong NK_1(B, 2e)$ and $V \cong NK_1(B/2eB, e)$ to $\Omega_{\mathbb{F}_2[x]} \cong NK_2(B, 2e)$ and $\Omega_{\mathbb{F}_2[x]} \cong NK_2(B/2eB, e)$ are compatible with the divided power map $[d]\colon V \to \Omega_{\mathbb{F}_2[x]}$ sending $x^n$ to $x^{n-1}\,dx$. Note that $[d]$ is an isomorphism of abelian groups but is not a morphism of $\mathbb{F}_2$-Cartier modules.

**Theorem 2.7.** *The map $\cup[x]\colon NK_0(\mathbb{Z}[D_4]) \to NK_1(\mathbb{Z}[D_4])$ in Definition 2.6 is a surjection. Hence the group $NK_1(\mathbb{Z}[D_4])$ has exponent 2 or 4, and there is a commutative diagram whose rows are exact:*

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, "\cong"', "{[d]}"] & NK_0(\mathbb{Z}[D_4]) \arrow[r] \arrow[d, "\cup[x]"] & V \arrow[r] \arrow[d, "\cong"', "{[d]}"] & 0 \\
 & \Omega_{\mathbb{F}_2[x]} \arrow[r] & NK_1(\mathbb{Z}[D_4]) \arrow[r] & \Omega_{\mathbb{F}_2[x]} \arrow[r] & 0.
\end{tikzcd}$$

*Proof.* A diagram chase on (9) shows that $NK_1(\mathbb{Z}[D_4])$ is an extension of the cokernel of $NK_2(\mathbb{Z}[C_2]) \times NK_2(B_0, J) \to NK_2(B)$ by a quotient of the cokernel of $NK_2(A) \to NK_2(\mathbb{F}_2[D_2])$. These cokernels are both $\Omega_{\mathbb{F}_2[x]}$, by Proposition 2.3 and Lemma 2.8 below, yielding the bottom row of the theorem. The map $\cup[x]$ sends the element corresponding to $1 - x^n ae \in NK_1(B, e)$ to the element corresponding to $\langle x^{n-1}ae, x\rangle \in NK_2(B, e)$, so the diagram in the theorem commutes by inspection. □

**Lemma 2.8.** *The cokernel of the map $NK_2(\mathbb{Z}[C_2]) \times NK_2(B_0, J) \to NK_2(B)$ in (9) is $\Omega_{\mathbb{F}_2[x]}$, on symbols $\langle x^{n-1}e, x\rangle$.*

*Proof.* The kernel of the map $q_-\colon B_0 \to B$ is the ideal $J' = (1+\sigma)B_0$. Because $J \cap J' = 0$ in $B_0$, the double relative group $NK_2(B_0, J, J')$ is isomorphic to $\mathbb{F}_2[C_2][x]$ on symbols $\langle x^m(1+\sigma), (1-\sigma)\rangle$ and $\langle x^m\tau(1+\sigma), (1-\sigma)\rangle$ by [GL80], [Keu81]. Since $J \cong 2B$, we have an exact sequence

$$\mathbb{F}_2[C_2][x] \to NK_2(B_0, J) \xrightarrow{q_-} NK_2(B, 2B) \to 0. \tag{10}$$

Combining this with the ideal sequence for $2B \subset B$ shows that the cokernel of $NK_2(B_0, J) \to NK_2(B)$ is $NK_2(B/2B)$. Since $B/2B \cong \mathbb{F}_2[C_2]$, the lemma now follows from Lemma 1.2. □

Inserting the calculations of Theorems 2.5 and 2.7 into Tables 6–7 in [LO], we obtain the following result.

**Theorem 2.9.** *Let $\Gamma$ be one of the following hyperbolic 3-simplex reflection groups: $[(3,4,3,6)]$, $[4,3^{[3]}]$, $[4,3,6]$, $[(3^3,4)]$, $[4,3,5]$, $[(3,4)^{[2]}]$, $[(3,4,3,5)]$. Then the lower algebraic $K$-theory of the groups $\Gamma$ is given by the following table:*

| $\Gamma$ | $K_{-1} \neq 0$ | $\widetilde{K}_0 \neq 0$ | $\mathrm{Wh} \neq 0$ |
|---|---|---|---|
| $[(3,4,3,6)]$ | $\mathbb{Z}^3$ | $(\mathbb{Z}/4)^2 \oplus \mathrm{Nil}_0$ | $\mathrm{Nil}_1$ |
| $[4,3^{[3]}]$ | $\mathbb{Z}^3$ | $(\mathbb{Z}/4)^2 \oplus \mathrm{Nil}_0 \oplus \bigoplus_\infty \mathbb{Z}/2$ | $\mathrm{Nil}_1 \oplus \bigoplus_\infty \mathbb{Z}/2$ |
| $[4,3,6]$ | $\mathbb{Z}^4$ | $(\mathbb{Z}/4)^2 \oplus \mathrm{Nil}_0 \oplus \bigoplus_\infty \mathbb{Z}/2$ | $\mathrm{Nil}_1 \oplus \bigoplus_\infty \mathbb{Z}/2$ |
| $[(3^3,4)]$ | $\mathbb{Z}^2$ | $(\mathbb{Z}/4)^2 \oplus \mathrm{Nil}_0 \oplus \bigoplus_\infty \mathbb{Z}/2$ | $\mathrm{Nil}_1 \oplus \bigoplus_\infty \mathbb{Z}/2$ |
| $[4,3,5]$ | $\mathbb{Z}^4$ | $(\mathbb{Z}/4)^2 \oplus \mathrm{Nil}_0 \oplus \bigoplus_\infty \mathbb{Z}/2$ | $\mathbb{Z}^3 \oplus \mathrm{Nil}_1 \oplus \bigoplus_\infty \mathbb{Z}/2$ |
| $[(3,4)^{[2]}]$ | $\mathbb{Z}^4$ | $(\mathbb{Z}/4)^4 \oplus 2\mathrm{Nil}_0 \oplus \bigoplus_\infty \mathbb{Z}/2$ | $2\mathrm{Nil}_1 \oplus \bigoplus_\infty \mathbb{Z}/2$ |
| $[(3,4,3,5)]$ | $\mathbb{Z}^6$ | $(\mathbb{Z}/4)^2 \oplus \mathrm{Nil}_0 \oplus \bigoplus_\infty \mathbb{Z}/2$ | $\mathbb{Z}^3 \oplus \mathrm{Nil}_1 \oplus \bigoplus_\infty \mathbb{Z}/2$ |

*In this table,* $\mathrm{Nil}_0 = NK_0(\mathbb{Z}[D_4])$ *is the direct sum of a countably infinite free* $\mathbb{Z}/4$*-module and a countably infinite free* $\mathbb{Z}/2$*-module, and* $\mathrm{Nil}_1 = NK_1(\mathbb{Z}[D_4])$ *is a countably infinite torsion group of exponent* 2 *or* 4*.*

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