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Freeness of conic-line arrangements in $\mathbb{P}^2$

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Abstract. Let $\mathcal{C} = \bigcup_{i=1}^{n} C_i \subseteq \mathbb{P}^2$ be a collection of smooth rational plane curves. We prove that the addition–deletion operation used in the study of hyperplane arrangements has an extension which works for a large class of arrangements of smooth rational curves, giving an inductive tool for understanding the freeness of the module $\Omega^1(\mathcal{C})$ of logarithmic differential forms with pole along $\mathcal{C}$. We also show that the analog of Terao's conjecture (freeness of $\Omega^1(\mathcal{C})$ is combinatorially determined if $\mathcal{C}$ is a union of lines) is false in this setting.

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1. Introduction

One of the fundamental objects associated to a hyperplane arrangement $\mathcal{A} \subseteq \mathbb{P}_{\mathbb{K}}(V)$ is the module $\Omega^1(\mathcal{A})$ of logarithmic one-forms with pole along the arrangement or (dually) the module $D(\mathcal{A})$ of derivations tangent to the arrangement. Both are graded $S = \mathrm{Sym}(V^*)$ modules; $D(\mathcal{A}) \subseteq \mathrm{Der}_{\mathbb{K}}(S)$ is defined via:

Definition 1.1. $D(\mathcal{A}) = \{\theta \mid \theta(l_i) \in \langle l_i \rangle \text{ for all } l_i \text{ such that } V(l_i) \in \mathcal{A}\}$.

Over a field of characteristic zero, $D(\mathcal{A}) \simeq E \oplus D_0(\mathcal{A})$, where E is the Euler derivation and $D_0(\mathcal{A})$ corresponds to the module of syzygies on the Jacobian ideal of the defining polynomial of $\mathcal{A}$. When $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, an elegant theorem of Terao relates the freeness of the module $D(\mathcal{A})$ to the Poincaré polynomial of $V \setminus \mathcal{A}$. In this note, we restrict to $\mathbb{P}^2$, but broaden the class of curves which make up the arrangement. In particular, suppose

$$\mathcal{C} = \bigcup_{i=1}^{n} C_i,$$

where each C_i is a smooth rational plane curve; call such a collection a conic-line (CL) arrangement.

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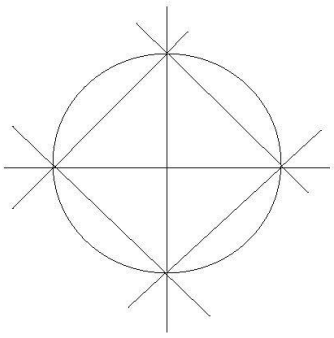
Example 1.2. For the CL arrangement below, $D(\mathcal{C}) \simeq S(-1) \oplus S(-2) \oplus S(-5)$.



1.1. Line arrangements. We begin with some facts about hyperplane arrangements; for more information see Orlik and Terao [5]. A hyperplane arrangement $\mathcal{A}$ is a finite collection of codimension one linear subspaces of a fixed vector space V. $\mathcal{A}$ is *central* if each hyperplane contains the origin of V. The intersection lattice $L_{\mathcal{A}}$ of $\mathcal{A}$ consists of the intersections of the elements of $\mathcal{A}$; the rank of $x \in L_{\mathcal{A}}$ is simply the codimension of x. V is the lattice element $\hat{0}$; the rank one elements are the hyperplanes themselves. $\mathcal{A}$ is called *essential* if rank $L_{\mathcal{A}} = \dim V$.

Definition 1.3. The Möbius function $\mu\colon L_{\mathcal{A}} \to \mathbb{Z}$ is defined by

$$\mu(\hat{0}) = 1,$$
$$\mu(t) = -\sum_{s<t} \mu(s), \quad \text{if } \hat{0} < t.$$

We now restrict to the case that V is complex. A foundational result is that the Poincaré polynomial of $X = V \setminus \mathcal{A}$ is purely combinatorial; in particular

$$P(X,t) = \sum_{x \in L_{\mathcal{A}}} \mu(x) \cdot (-t)^{\operatorname{rank}(x)}.$$

An arrangement $\mathcal{A}$ is *free* if $D(\mathcal{A}) \simeq \oplus S(-a_i)$; the a_i are called the *exponents* of $\mathcal{A}$. Terao's famous theorem [11] states that if $D(\mathcal{A}) \simeq \oplus S(-a_i)$, then $P(X,t) = \prod(1 + a_i t)$. If $\mathcal{A} \subseteq \mathbb{C}^3$ is central, then $\mathcal{A}$ also defines a set of lines in $\mathbb{P}^2$, and obviously $X = \mathbb{C}^3 \setminus \mathcal{A} \simeq \mathbb{C}^* \times \widetilde{X}$, where $\widetilde{X}$ is the complement of the corresponding arrangement of lines in $\mathbb{P}^2$. Hence

$$P(\widetilde{X},t) = 1 + (n-1)t + \Big(\sum_{\substack{x \in L_{\mathcal{A}} \\ \operatorname{rank}(x)=2}} \mu(x) - n + 1 \Big) t^2$$

It follows from Terao's theorem that if $D_0(\mathcal{A}) \simeq S(-a) \oplus S(-b)$, then $P(\widetilde{X},t) = (1+at)(1+bt)$. This can be generalized [9] to line arrangements which are not free, using the Chern polynomial. The motivating question of this paper is: *what happens if the arrangement of lines is replaced with a CL arrangement?*

1.2. Rational curve arrangements. In [2], Cogolludo-Agustín studies the complement of an arrangement of rational curves in $\mathbb{P}^2$, where the individual curves can have singularities, and can meet non-transversally. The main result is that the cohomology ring of the complement to a rational curve arrangement is generated by logarithmic 1- and 2-forms and its structure depends on a finite number of invariants of the curve. One fact is that if $\widetilde{X}$ is the complement of an arrangement of n irreducible curves in $\mathbb{P}^2$, then

$$h^1(\widetilde{X}, \mathbb{C}) = n - 1,$$

$$h^2(\widetilde{X}, \mathbb{C}) = 1 + \sum_{p \in \mathrm{Sing}(\mathcal{C})} (r_p - 1) - \sum_{1}^{n} (\chi(\hat{C}_i) - 1),$$

where r_p is the number of branches passing through p, and $\hat{C}_i$ is the normalization of C_i. Since we are assuming that all the C_i are smooth and rational, we have that

$$h^2(\widetilde{X}, \mathbb{C}) = \sum_{p \in L_2(\mathcal{C})} (r_p - 1) - |\mathcal{C}| + 1,$$

where the intersection poset $L(\mathcal{C})$ is defined precisely as for a linear arrangement (typically, $L(\mathcal{C})$ is only a poset, not a lattice).

1.3. Milnor and Tjurina numbers. A crucial distinction between line and curve arrangements, even in our simple setting, is the difference between the Milnor and Tjurina numbers at a singularity. Let $C = V(f)$ be a reduced (but not necessarily irreducible) curve in $\mathbb{C}^2$, let $(0, 0) \in C$, and let $\mathbb{C}\{x, y\}$ denote the ring of convergent power series.

Definition 1.4. The Milnor number of C at $(0, 0)$ is

$$\mu_{(0,0)}(C) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\} \Big/ \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle.$$

To define μ_p for an arbitrary point p, we translate so that p is the origin.

Definition 1.5. The Tjurina number of C at $(0, 0)$ is

$$\tau_{(0,0)}(C) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\} \Big/ \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \right\rangle.$$

Definition 1.6. A singularity is quasihomogeneous iff there exists a holomorphic change of variables so the defining equation becomes weighted homogeneous; $f(x, y) = \sum c_{ij} x^i y^j$ is weighted homogeneous if there exist rational numbers α, β such that $\sum c_{ij} x^{i \cdot \alpha} y^{j \cdot \beta}$ is homogeneous.

In [6], Reiffen proved that if $f(x, y)$ is a convergent power series with isolated singularity at the origin, then $f(x, y)$ is in the ideal generated by the partial derivatives if and only if f is quasihomogeneous (see [8] for a generalization).

As noted earlier, for a line arrangement with defining polynomial F, $D_0(\mathcal{A})$ consists of the syzygies on the Jacobian ideal J_F of F. If $V(F) \subseteq \mathbb{P}^2$ is a reduced curve, then after a change of coordinates, we may assume that $V(F)$ has no singularities on the line $z = 0$. Dehomogenizing so that $f(x, y) = F(X, Y, 1)$ yields

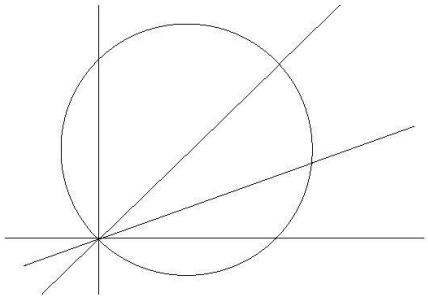
$$\deg(J_F) = \dim_{\mathbb{C}} \mathbb{C}[x, y] \Big/ \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \right\rangle = \sum_{p \in \mathrm{Sing}(C)} \tau_p(f).$$

It follows that if all the singular points are quasihomogeneous, then

$$\deg(J_F) = \sum_{p \in \mathrm{Sing}(V(f))} \mu_p(f).$$

For a line arrangement, the singularities are always quasihomogeneous, but this is not the case for CL arrangements:

Example 1.7. Let $\mathcal{C} = V(xy(x - y)(x - 2y)(x^2 - xz + y^2 - yz))$ be as below:



$\mathcal{C}$ has five singular points, all ordinary. When p is an ordinary singularity and C has n distinct branches at p, then $\mu_p(C) = (n - 1)^2$, so the sum of the Milnor numbers is 20. However, $\deg(J) = 19$; at $(0 : 0 : 1)$ we have $\mu = 16$ but $\tau = 15$.

1.4. Criteria for freeness. The first criterion for the freeness of $D(\mathcal{A})$ is the following:

Proposition 1.8 (Saito, [7]). *$\mathcal{A}$ is free exactly when there exist $n + 1$ elements*

$$\theta_i = \sum_{j=0}^{n} f_{ij} \frac{\partial}{\partial x_j} \in D(\mathcal{A})$$

such that the determinant of the matrix $[f_{ij}]$ is a nonzero constant multiple of the defining polynomial of $\mathcal{A}$.

Saito's criterion holds for an arrangement of reduced hypersurfaces $\mathcal{C} \subseteq \mathbb{P}^n$; let $\mathcal{C} = V(F)$ where $F = f_1 \dots f_d$, and $\gcd(f_i, f_j) = 1$. By induction,

$$\theta(f_1 \dots f_d) = f_1\theta(f_2 \dots f_d) + f_2 \dots f_d\theta(f_1) \in \langle f_1 \dots f_d \rangle,$$

so we have

$$\begin{aligned} D(\mathcal{C}) &= \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(f_i) \in \langle f_i \rangle, i = 1, \dots, d\} \\ &= \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(F) \in \langle F \rangle\}. \end{aligned}$$

Any arrangement of (reduced) hypersurfaces will have a singular locus of codimension two. As for a linear arrangement, $D(\mathcal{C}) \simeq E \oplus D_0(\mathcal{C})$, with $D_0(\mathcal{C}) = \mathrm{syz}(J_F)$, so freeness is equivalent to $\mathrm{pdim}(S/J_F) = 2$ (so also equivalent to J_F Cohen–Macaulay). By the Hilbert–Burch theorem ([3]), any codimension two Cohen–Macaulay ideal I with $m + 1$ generators is generated by the maximal minors of an $m \times (m + 1)$ matrix M, whose columns generate the module of first syzygies on I. So when $I = J_F$, appending a column vector $[x_0, \dots, x_n]$ to M and taking the determinant yields a multiple of F, by Euler's formula. Saito's criterion is most useful when an explicit set of candidates for the generating set of $\mathrm{syz}(J_F)$ is known.

There are two other fundamental tools that can be used to prove that a line arrangement is free. The first method is based on an inductive operation known as deletion-restriction: given an arrangement $\mathcal{A}$ and a choice of hyperplane $H \in \mathcal{A}$, set

$$\mathcal{A}' = \mathcal{A} \setminus H \quad \text{and} \quad \mathcal{A}'' = \mathcal{A}|_H.$$

The collection $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ is called a *triple*, and a triple yields (see Proposition 4.45 of [5]) a left exact sequence

$$0 \longrightarrow D(\mathcal{A}')(-1) \xrightarrow{\cdot H} D(\mathcal{A}) \longrightarrow D(\mathcal{A}'').$$

For a triple with $\mathcal{A} \subseteq \mathbb{P}^2$, more is true (see [10]): after pruning the Euler derivations and sheafifying, there is an exact sequence

$$0 \longrightarrow \mathcal{D}_0{}'(-1) \longrightarrow \mathcal{D}_0 \longrightarrow i_*\mathcal{D}_0{}'' \longrightarrow 0, \tag{1}$$

where $i\colon H \hookrightarrow \mathbb{P}^2$; $i_*\mathcal{D}_0{}'' \simeq \mathcal{O}_H(1 - |\mathcal{A}''|)$. In [12], Terao showed that freeness of a triple is related via:

Theorem 1.9 (Addition–Deletion). *Let $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ be a triple. Then any two of the following imply the third:*

- $D(\mathcal{A}) \simeq \bigoplus_{i=1}^{n} S(-b_i)$;
- $D(\mathcal{A}') \simeq S(-b_n + 1) \bigoplus_{i=1}^{n-1} S(-b_i)$;
- $D(\mathcal{A}'') \simeq \bigoplus_{i=1}^{n-1} S(-b_i)$.

Theorem 1.9 applies in general, not just to arrangements in $\mathbb{P}^2$. A smooth conic is intrinsically a $\mathbb{P}^1$, so it is natural to ask if CL arrangements which admit a short

exact sequence similar to (1) have an addition–deletion theorem; we tackle this in the next two sections.

A second criterion for freeness is special to the case of line arrangements; to state it we need to define freeness for *multiarrangements*. A multiarrangement $(\mathcal{A}, \boldsymbol{m})$ is an arrangement together with a multiplicity m_i for each hyperplane. The module of derivations consists of θ such that $l_i^{m_i} | \theta(l_i)$. As shown by Ziegler in [15], freeness of multiarrangements is not combinatorial; for recent progress see [13].

Theorem 1.10 (Yoshinaga's multiarrangement criterion, [16]). *$\mathcal{A} \subseteq \mathbb{P}^2$ is free iff $\pi(\mathcal{A}, t) = (1+t)(1+at)(1+bt)$ and for all $H \in \mathcal{A}$ the multiarrangement $\mathcal{A}|_H$ has minimal generators in degree a and b.*

The main results of this paper (Theorems 2.5 and 3.4) show that an addition–deletion construction holds for CL arrangements with quasihomogeneous singularities; the freeness of Example 1.2 is explained by our results. As one application, we show that a free CL arrangement, when restricted to different lines, can yield multiarrangements with different exponents; hence any version of Theorem 1.10 for CL arrangements will be quite subtle. An addition–deletion theorem for multiarrangements has recently been proven by Abe–Terao–Wakefield in [1]; our results are the first (to our knowledge) to give an inductive criterion for freeness for nonlinear arrangements.

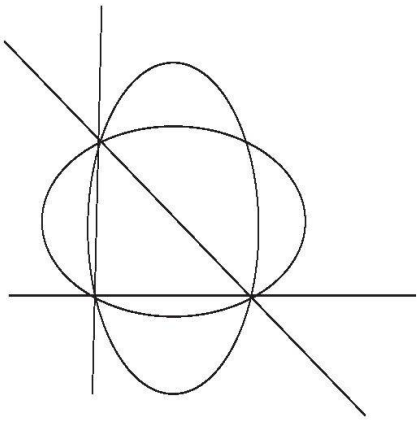
2. Addition–deletion for a line

Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ be a triple of CL arrangements in $\mathbb{P}^2$, where $\mathcal{C}' = \mathcal{C} \setminus \{L\}, \mathcal{C}'' = \mathcal{C}|_L$ and $L \in \mathcal{C}$ *is a line*. We begin by examining some examples:

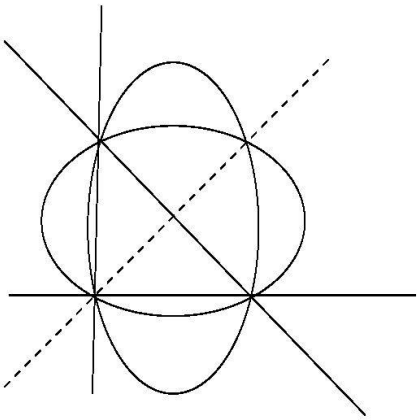
Example 2.1. Let $\mathcal{C}'$ be the union of

$$\begin{aligned} C_1 &= x^2 - xz + 5y^2 - 5yz = 0, \\ C_2 &= x^2 + 2y^2 - xz - 2yz = 0, \\ L_1 &= x = 0, \\ L_2 &= y = 0, \\ L_3 &= x + y - z = 0. \end{aligned}$$

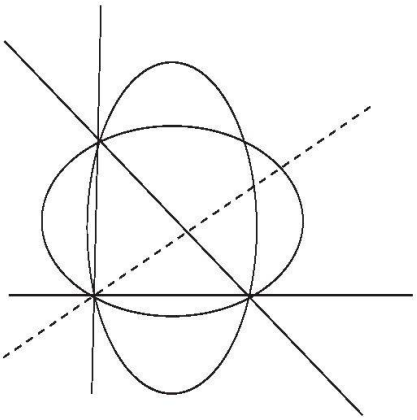
$D(\mathcal{C}')$ is free with exponents $\{1, 2, 4\}$, and the degree of the Jacobian ideal is 28, which is equal to the sum of the Milnor numbers at the intersection points. Therefore at each singular point $\tau = \mu$. If we restrict to any line, the corresponding multiarrangement has two points of multiplicity 3, and it follows from [13] that the exponents are $\{3, 3\}$. Hence the obvious generalization of Yoshinaga's criterion does not hold.



Example 2.2. Let $L_4 = \{x - y = 0\}$ and let $\mathcal{C}_1 = \mathcal{C}' \cup L_4$. The degree of the Jacobian ideal is 39, which is equal to the sum of Milnor numbers at the points. It will follow from our results that $D(\mathcal{C}_1)$ is free with exponents $\{1, 2, 5\}$.



Example 2.3. Let $L_4 = \{x - 2y = 0\}$ and let $\mathcal{C}_2 = \mathcal{C}' \cup L_4$. Then $\mathcal{C}_2$ is free with exponents $\{1, 3, 4\}$. The degree of the Jacobian ideal is 37, whereas the sum of the Milnor numbers is 38; the singularity at $(0 : 0 : 1)$ has $\tau = 15$ and $\mu = 16$.



For CL arrangements similar to $\mathcal{C}_1$, there is an addition–deletion theorem:

Definition 2.4. A triple $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ of CL arrangements is called quasihomogeneous if $\tau = \mu$ at each singular point of $\mathcal{C}'$ and $\mathcal{C}$.

Theorem 2.5. *Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ be a quasihomogeneous triple with $|L \cap \mathcal{C}| = |\mathcal{C}''| = k$. The following are equivalent:*

(1) *$\mathcal{C}'$ is free with exponents $\{1, k-1, a\}$.*

(2) *$\mathcal{C}$ is free with exponents $\{1, k-1, a+1\}$.*

Examples 2.1 and 2.2 illustrate the theorem; before giving the proof of Theorem 2.5, we need some preliminaries.

Lemma 2.6. *Let $L = \{x = 0\}$. Then the maps $p\colon D(\mathcal{C}') \to D(\mathcal{C})$, $p(\theta) = x\theta$ and $q\colon D(\mathcal{C}) \to D(\mathcal{C}'')$, $q(a\partial_x + b\partial_y + c\partial_z) = b(0,y,z)\partial_y + c(0,y,z)\partial_z$ are well defined and yield an exact sequence*

$$0 \longrightarrow D(\mathcal{C}') \longrightarrow D(\mathcal{C}) \longrightarrow D(\mathcal{C}'').$$

Proof. Let $f = xf'$ be the defining polynomial of $\mathcal{C}$, where f' is the defining polynomial of $\mathcal{C}'$. Then the defining polynomial of $\mathcal{C}''$ is $f'' = Rad(f'|_{x=0})$. If $\theta' \in D(\mathcal{C}')$, then $\theta'(f') = Pf'$ for some $P \in S$;

$$p(\theta')(f) = x\theta'(xf') = x(f'\theta'(x) + x\theta'(f')) \in \langle f \rangle.$$

So p is well defined and injective. Let $\theta = a\partial_x + b\partial_y + c\partial_z \in D(\mathcal{C})$. Then $\theta(x) = a \in \langle x \rangle$, so $a = xa'$. If $\theta \in \ker(q)$, then $b = xb'$ and $c = xc'$, hence $\theta = x\theta'$, where $\theta' = a'\partial_x + b'\partial_y + c'\partial_z$. Because $\theta \in D(\mathcal{C})$, $\theta(f') = x\theta'(f') \in \langle f' \rangle$. Since x and f' are relatively prime, we get that $\theta'(f') \in \langle f' \rangle$, which implies that $\theta \in \mathrm{Im}(p)$.

It remains to show is that q is well defined. For suitable $u_i, v_i \in \mathbb{C}$ and $m_i \in \mathbb{Z}$ we have that

$$f'|_{x=0} = \prod_i (u_i y + v_i z)^{m_i}, \text{ so } f'' = \prod_i (u_i y + v_i z).$$

Let L' be a line in $\mathcal{C}'$ defined by the vanishing of $t_i x + u_i y + v_i z = 0$ for some i and $t_i \in \mathbb{C}$, and let $\theta = a\partial_x + b\partial_y + c\partial_z \in D(\mathcal{C})$. Then $\theta(L') \in \langle L' \rangle$, so evaluating at $x = 0$ and using the earlier observation that $a = xa'$, we find $(b(0,y,z)\partial_y + c(0,y,z)\partial_z)(u_i y + v_i z) \in \langle u_i y + v_i z \rangle$.

Now suppose C is a conic in $\mathcal{C}'$; after a change of coordinates we may assume C intersects $L = \{x = 0\}$ in the points $(0:0:1)$ and $(0:u:v)$. Then

$C = xA + y(vy - uz)$ and $C|_{x=0} = y(vy - uz)$, where A is some linear form. We have

$$\theta(C) = a(A + x\partial_x(A)) + x(b\partial_y(A) + c\partial_z(A)) + b\partial_y(y(vy - uz)) + c\partial_z(y(vy - uz)) \in \langle C \rangle.$$

Evaluating at $x = 0$ and again using that $a = xa'$ we find

$$(b(0, y, z)\partial_y + c(0, y, z)\partial_z)(y(vy - uz)) \in \langle y(vy - uz) \rangle.$$

Since y and $vy - uz$ are relatively prime we obtain

$$(b(0, y, z)\partial_y + c(0, y, z)\partial_z)(y) \in \langle y \rangle,$$
$$(b(0, y, z)\partial_y + c(0, y, z)\partial_z)(vy - uz) \in \langle vy - uz \rangle.$$

This shows that for each factor $u_i y + v_i z$ of f'',

$$(b(0, y, z)\partial_y + c(0, y, z)\partial_z)(u_i y + v_i z) \in \langle u_i y + v_i z \rangle,$$

so the map q is well defined. It follows that $D_0(\mathcal{C}'') = \mathbb{C}[y, z](-(k - 1))$, where $k = |L \cap \mathcal{C}'| = \deg(f'')$. A similar argument works if C is tangent to L. □

Lemma 2.7. *Let X and Y be two reduced plane curves with no common component, meeting at a point p. Then*

$$\mu_p(X \cup Y) = \mu_p(X) + \mu_p(Y) + 2(X \cdot Y)_p - 1,$$

where $(X \cdot Y)_p$ is the intersection number of X and Y at p.

Proof. See [14], Theorem 6.5.1; the point is that the Milnor fiber is a connected curve, and the result follows from using the additivity of the Euler characteristic and the interpretation of μ_p as the first Betti number of the Milnor fiber. □

Proposition 2.8. *Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ be a quasihomogeneous triple. Then*

$$0 \longrightarrow \mathcal{D}_0'(-1) \longrightarrow \mathcal{D}_0 \longrightarrow i_* \mathcal{D}_0''$$

is also right exact.

Proof. It follows from Lemma 2.6 that quotienting by the Euler derivation and sheafifying yields the left exact sequence above; so it will suffice to show that $HP(D_0, t) = HP(D_0'(-1), t) + HP(i_* D_0'', t)$, where $HP(-, t)$ denotes the Hilbert polynomial. For a CL arrangement $\mathcal{C}$ with m lines and n conics, let $d = 2n + m - 1$. We have an exact sequence:

$$0 \longrightarrow D_0(\mathcal{C}) \longrightarrow S^3 \longrightarrow S(d) \longrightarrow S(d)/J \longrightarrow 0,$$

where $S = \mathbb{K}[x, y, z]$ and J is the Jacobian ideal of the defining polynomial of $\mathcal{C}$. Since

$$HP(D_0, t) = 3\binom{t+2}{2} - \binom{t+2+d}{2} + \deg(J),$$

$$HP(D_0'(-1), t) = 3\binom{t+1}{2} - \binom{t+d}{2} + \deg(J'),$$

we find that

$$HP(D_0, t) - HP(D_0'(-1), t) = \deg(J) - \deg(J') + t - 2d + 2.$$

By the assumption that $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ is a quasihomogeneous triple,

$$\deg(J) = \sum_{p \in \mathrm{Sing}(C)} \mu_p(C) \quad \text{and} \quad \deg(J') = \sum_{p \in \mathrm{Sing}(C')} \mu_p(C').$$

Let α be the sum of Milnor numbers of points off L, so

$$\deg(J) = \alpha + \sum_{p \in L \cap C'} \mu_p(C).$$

Since $\mu_p(L) = 0$, by Lemma 2.7, the above is

$$\alpha + \sum_{p \in L \cap C'} (\mu_p(C') + 2(L \cdot C')_p - 1).$$

As $\deg(J') = \alpha + \sum_{p \in L \cap C'} \mu_p(C')$ and $| L \cap C' | = k$, we obtain

$$\deg(J) - \deg(J') = 2 \sum_{p \in L \cap C'} (L \cdot C')_p - k.$$

By Bezout's theorem,

$$\sum_{p \in L \cap C'} (L \cdot C') = d,$$

so $\deg(J) - \deg(J') = 2d - k$, hence

$$HP(D_0, t) - HP(D_0'(-1), t) = t + 2 - k = t + 1 - (|\mathcal{C}''| - 1).$$

Since $i_* \mathcal{D}_0'' = \mathcal{O}_L(1 - |\mathcal{C}''|)$, this yields the result. □

Definition 2.9. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ is m-regular iff $H^i \mathcal{F}(m - i) = 0$ for every $i \geq 1$. The smallest number m such that $\mathcal{F}$ is m-regular is denoted by $\mathrm{reg}(\mathcal{F})$.

Lemma 2.10. *For a quasihomogeneous triple with* $|\mathcal{C}''| = k$,

$$\operatorname{reg}(\mathcal{D}_0) \leq \max\{\operatorname{reg}(\mathcal{D}_0') + 1, k-1\}.$$

Proof. Immediate from Proposition 2.8 (see [10]). □

Lemma 2.11. *If* $\mathcal{D}_0' = \mathcal{O}_{\mathbb{P}^2}(1-k) \oplus \mathcal{O}_{\mathbb{P}^2}(-a)$, *then there is an exact sequence of* S*-modules*

$$0 \longrightarrow D_0'(-1) \longrightarrow D_0 \longrightarrow D_0'' \longrightarrow 0.$$

Proof. For all t, $H^1(\mathcal{D}_0'(t-1)) = 0$, so the long exact sequence in cohomology arising from Proposition 2.8 and the vanishing of $H^1(\mathcal{D}_0'(t))$ yield an exact sequence:

$$0 \longrightarrow \bigoplus_t H^0(\mathcal{D}_0'(-1)(t)) \longrightarrow \bigoplus_t H^0(\mathcal{D}_0(t)) \longrightarrow \bigoplus_t H^0(\mathcal{D}_0''(t)) \longrightarrow 0.$$

Theorem A.4.1 of [3] relates a graded module to its sheaf and local cohomology (at the maximal ideal $\mathfrak{m}$) modules:

$$0 \longrightarrow H^0_{\mathfrak{m}}(D_0) \longrightarrow D_0 \longrightarrow \bigoplus_t H^0(\mathcal{D}_0(t)) \longrightarrow H^1_{\mathfrak{m}}(D_0) \longrightarrow 0.$$

This is true also for $D_0'(-1)$ and D_0''. By [3], A.4.3, $H^0_{\mathfrak{m}}(M) = H^1_{\mathfrak{m}}(M) = 0$ if $\operatorname{depth}(M) \geq 2$. Lemma 2.1 of [4] gives the desired bound on depth for the modules of derivations, which concludes the proof. □

The next two lemmas prove the two implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) of Theorem 2.5. In what follows, $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ is a quasihomogeneous triple, with L a line and $|L \cap \mathcal{C}| = |\mathcal{C}''| = k$.

Lemma 2.12. *If* $\mathcal{C}'$ *is free with* $\exp(\mathcal{C}') = \{1, k-1, a\}$, *then* $\mathcal{C}$ *is free with* $\exp(\mathcal{C}) = \{1, k-1, a+1\}$.

Proof. First, if $S = \mathbb{K}[x_1, \ldots, x_\ell]$, and $S(-i)$ is a free graded S-module with generator in degree i, then the Hilbert series satisfies

$$HS(S(-i), t) = \frac{t^i}{(1-t)^\ell}.$$

If $\mathcal{C}'$ is free with exponents $\{1, k-1, a\}$, then $D_0'(-1) \simeq S(-k) \oplus S(-1-a)$. It follows from the proof of Proposition 2.8 that $HS(D_0'', t) = \frac{t^{k-1}}{(1-t)^2}$. So by Lemma 2.11 and the additivity of Hilbert series on an exact sequence,

$$HS(D_0) = \frac{t^k + t^{a+1}}{(1-t)^3} + \frac{t^{k-1}}{(1-t)^2} = \frac{t^{a+1} + t^{k-1}}{(1-t)^3}.$$

Since $D_0' \simeq S(-k+1) \oplus S(-a)$, $\operatorname{reg}(D_0') = \max\{k-1, a\}$. By Lemma 2.10, if $a \geq k-1$, then $\operatorname{reg}(D_0) \leq a+1$; and if $a \leq k-2$, then $\operatorname{reg}(D_0) \leq k$. If $a \leq k-2$, then a free resolution for D_0 is of the form

$$0 \longleftarrow D_0 \longleftarrow S(-k+1) \oplus S(-a-1) \oplus S(-b)^d \longleftarrow S(-b)^d \longleftarrow 0.$$

From regularity constraints, b must be at most k. As this is a minimal free resolution, and it is impossible to have a syzygy on a single generator, the only situation which can actually arise occurs when $b = k$:

$$0 \longleftarrow D_0 \longleftarrow S(-k+1) \oplus S(-a-1) \oplus S(-k)^d \longleftarrow S(-k)^d \longleftarrow 0.$$

Let t_1, t_2 be two independent derivations in D_0 of degrees $\deg(t_1) = a+1$ and $\deg(t_2) = k-1$; our computation of the Hilbert series, combined with the fact that $\operatorname{pdim}(D_0) \leq 1$ means such derivations must exist. Let E, t_1', t_2' be a basis for D' with $\deg(t_1') = a$ and $\deg(t_2') = k-1$, and E the Euler derivation.

Now note that $t_1' \in D_0' \setminus D_0$, for otherwise in D_0 there would be an element of degree a. So $t_1'(x) \notin \langle x \rangle$. Since $D \subset D'$, then $t_1 = f_1 E + x t_1'$ and $t_2 = f_2 E + u t_2' + f t_1'$, where u is a constant, $\deg(f) = k-1-a$, $\deg(f_1) = a$ and $\deg(f_2) = k-2$. For a resolution as above, $g t_1 = L t_2$, where L is a linear form and $\deg(g) = k-(a+1)$. Hence

$$(g f_1 - L f_2) E + (g x - L f) t_1' + (-L u) t_2' = 0,$$

and since E, t_1' and t_2' form a basis, we find that u vanishes and $gx = Lf$. But $(t_2 - f_2 E)(x) \in \langle x \rangle$ and $t_1'(x) \notin \langle x \rangle$. Since $u = 0$, x must divide f, and so $g = Lg'$ for some g'. Since $g t_1 = L t_2$, we obtain $t_2 = g' t_1$, a contradiction. If $a \geq k-1$, simply switch the roles of a and k above. □

Lemma 2.13. *If $\mathcal{C}$ is free with $\exp(\mathcal{C}) = \{1, k-1, a+1\}$, then $\mathcal{C}'$ is free with $\exp(\mathcal{C}') = \{1, k-1, a\}$.*

Proof. In order to obtain an appropriate vanishing, we need to dualize. Apply $\operatorname{Hom}(-, \mathcal{O}_{\mathbb{P}^2})$ to the exact sequence

$$0 \longrightarrow \mathcal{D}_0'(-1) \longrightarrow \mathcal{D}_0 \longrightarrow i_* \mathcal{D}_0'' \longrightarrow 0.$$

The vanishing of $\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^1}(t), \mathcal{O}_{\mathbb{P}^2})$ and $\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{D}_0, \mathcal{O}_{\mathbb{P}^2})$ yields an exact sequence

$$0 \longrightarrow \mathcal{D}_0^{\vee} \longrightarrow \mathcal{D}_0'^{\vee}(1) \longrightarrow \operatorname{Ext}^1_S(\mathcal{O}_L(1-k), \mathcal{O}_{\mathbb{P}^2}) \longrightarrow 0.$$

The free $\mathcal{O}_{\mathbb{P}^2}$ resolution for $\mathcal{O}_L(1-k)$ is

$$0 \longrightarrow O_{\mathbb{P}^2}(-k) \longrightarrow O_{\mathbb{P}^2}(1-k) \longrightarrow \mathcal{O}_L(1-k) \longrightarrow 0,$$

so $\mathrm{Ext}^1_S(\mathcal{O}_L(1-k), \mathcal{O}_{\mathbb{P}^2}) \simeq \mathcal{O}_L(k)$. Since $\mathcal{D}_0^\vee = \mathcal{O}_{\mathbb{P}^2}(k-1) \oplus \mathcal{O}_{\mathbb{P}^2}(a+1)$, combining this with the long exact sequence in cohomology yields a regularity bound

$$\mathrm{reg}(\mathcal{D}_0'^\vee) \leq \max\{\mathrm{reg}(\mathcal{D}_0^\vee) + 1, 1-k\},$$

and the exact sequence of S-modules

$$0 \longrightarrow D_0^\vee(-1) \longrightarrow D_0'^\vee \longrightarrow S(k-1)/L \longrightarrow 0,$$

with $D_0^\vee = S(k-1) \oplus S(a+1)$. So

$$HS(D_0'^\vee) = \frac{t^{-a} + t^{1-k}}{(1-t)^3}.$$

An argument as in the proof of Lemma 2.12 shows that $D_0'^\vee = S(a) \oplus S(k-1)$, hence $D(\mathcal{C}')$ is free with exponents $\{1, k-1, a\}$. □

Corollary 2.14. *A free CL arrangement, when restricted to a line, can yield different multiarrangements.*

Proof. In Example 2.2, add the line $L = \{x - \alpha y + (\alpha - 1)z = 0\}$, where $\alpha \notin \{0, 1, -5, -2, \infty\}$. Then L passes through $(1:1:1)$, and the choices for α ensure that L is not tangent to any conic, and misses all singularities save $(1:1:1)$. The new arrangement is quasihomogeneous, and L meets $\mathcal{C}_1$ in six points. By Theorem 2.5, the new arrangement is free with exponents $\{1, 3, 5\}$.

Restrict this new arrangement to the line $L_3 = \{x + y - z = 0\}$. After a change of coordinates, we obtain a multiarrangement with defining polynomial

$$x^3y^3(x-y)(\alpha x - y).$$

This is exactly Ziegler's example from [15]: $\alpha = -1$ gives exponents $\{3, 5\}$, and for $\alpha \neq -1$, the exponents are $\{4, 4\}$. □

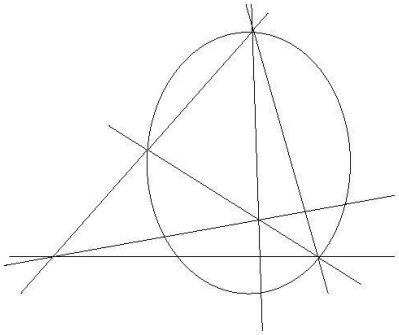
3. Addition–deletion for a conic

Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ be a triple of CL arrangements in $\mathbb{P}^2$, where C is a conic in $\mathcal{C}$, and $\mathcal{C}' = \mathcal{C} \setminus \{C\}$, $\mathcal{C}'' = \mathcal{C}'|_C$. We begin with some examples.

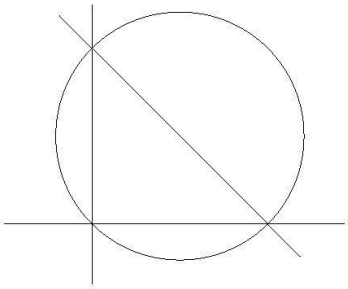
Example 3.1. Suppose $\mathcal{C}$ is as in Example 2.2, so $\mathcal{C}$ has quasihomogeneous singularities, and is free with exponents $\{1, 2, 5\}$. If we delete one of the conics, the resulting arrangement $\mathcal{C}'$ is free and quasihomogeneous, with exponents $\{1, 2, 3\}$.

When k is odd, the situation is more complicated:

Example 3.2. Let $\mathcal{C}'$ be the braid arrangement $A_3 = V(xyz(x-z)(y-z)(x+y-z))$, and $\mathcal{C} = \mathcal{C}' \cup C$, where the conic $C = V(xy+7xz+13yz)$. $\mathcal{C}'$ is a free arrangement with exponents $\{1,2,3\}$, and $|\mathcal{C}''| = 7$. $\mathcal{C}$ is also quasihomogeneous, but not free.



Example 3.3. Let $\mathcal{C}$ be the quasihomogeneous CL arrangement with defining polynomial $(x^2 - xz + 2y^2 - 2yz)xy(x+y-z)$. $D(\mathcal{C})$ is free with exponents $\{1,2,2\}$. Deleting the conic yields a free line arrangement with exponents $\{1,1,1\}$.



Theorem 3.4. *Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ be a quasihomogeneous triple, with $|C \cap \mathcal{C}'| = |\mathcal{C}''| = k$. If $k = 2m$ then the following are equivalent:*

(1) *$\mathcal{C}'$ is free with $\exp(\mathcal{C}') = \{1, m, a\}$;*

(2) *$\mathcal{C}$ is free with $\exp(\mathcal{C}) = \{1, m, a+2\}$.*

If $k = 2m+1$ then:

(1) $\exp(\mathcal{C}') = \{1,m,m\} \iff \exp(\mathcal{C}) = \{1, m+1, m+1\}$;

(2) *if* $\exp(\mathcal{C}') = \{1,m,a\}$ *with* $a \neq m$ *then* $\mathcal{C}$ *is not free;*

(3) *if* $\exp(\mathcal{C}) = \{1, m+1, a+1\}$ *with* $a \neq m$ *then* $\mathcal{C}'$ *is not free.*

We begin with some preliminaries. After an appropriate change of coordinates, we may suppose that $C = \{y^2 - xz = 0\}$. Let i be the composition of the maps

$$\mathbb{P}^1 \xrightarrow{v} C \hookrightarrow \mathbb{P}^2,$$

where $v(s:t) = (s^2 : st : t^2)$, and let ψ be the composite map

$$S = \mathbb{K}[x, y, z] \xrightarrow{\phi} \mathbb{K}[s^2, st, t^2] \hookrightarrow \mathbb{K}[s, t],$$

where $\phi(x) = s^2, \phi(y) = st, \phi(z) = t^2$.

Let $\theta = a_1\partial_x + a_2\partial_y + a_3\partial_z \in D(\mathcal{C})$ be a derivation. Then $\theta(C) \in \langle C \rangle$, which means $-za_1 + 2ya_2 - xa_3 = (y^2 - xz)P$ for some $P \in S$. Via the map ψ this translates into

$$t^2\psi(a_1) - 2st\psi(a_2) + s^2\psi(a_3) = 0.$$

So there exist $Q_1, Q_2 \in \mathbb{K}[s, t]$ such that

$$\begin{aligned} \psi(a_1) &= sQ_1, \\ \psi(a_2) &= \frac{tQ_1 + sQ_2}{2}, \\ \psi(a_3) &= tQ_2. \end{aligned}$$

If $\psi : S \to A$ is a ring map and M is an A-module, let M_ψ denote the S-module obtained by restriction of scalars.

Proposition 3.5. *There is an exact sequence of S-modules*

$$0 \longrightarrow D(\mathcal{C}')(-2) \xrightarrow{\cdot C} D(\mathcal{C}) \xrightarrow{\rho} D(\mathcal{A}'')_\psi,$$

where

$$\rho(a_1\partial_x + a_2\partial_y + a_3\partial_z) = Q_1\partial_s + Q_2\partial_t,$$

for every $a_1\partial_x + a_2\partial_y + a_3\partial_z \in D(\mathcal{C})$ and Q_1, Q_2 are defined as above; and $\mathcal{A}''$ is the arrangement of the reduced points $i^{-1}(C \cap \mathcal{C}')$ in $\mathbb{P}^1$.

Proof. It is easy to check that ρ is a homomorphism. For exactness, note:

$$\begin{aligned} \theta = a_1\partial_x + a_2\partial_y + a_3\partial_z \in \ker(\rho) &\iff Q_1 = 0 \text{ and } Q_2 = 0 \\ &\iff \psi(a_1) = \psi(a_2) = \psi(a_3) = 0 \\ &\iff a_1, a_2, a_3 \in \langle y^2 - xz \rangle \\ &\iff \theta = C\theta' \text{ with } \theta' \in D(\mathcal{C}'). \end{aligned}$$

It remains to show that the image of ρ is in $D(\mathcal{A}'')_\psi$. Suppose $\alpha x + \beta y + \gamma z = 0$ is a line of $\mathcal{C}$. Let $\theta = a_1\partial_x + a_2\partial_y + a_3\partial_z \in D(\mathcal{C})$. Then

$$\alpha a_1 + \beta a_2 + \gamma a_3 = (\alpha x + \beta y + \gamma z)P_1$$

for some $P_1 \in S$. Therefore

$$\alpha\psi(a_1) + \beta\psi(a_2) + \gamma\psi(a_3) = (\alpha s^2 + \beta st + \gamma t^2)\psi(P_1),$$

which implies

$$(2\alpha s + \beta t)Q_1 + (\beta s + 2\gamma t)Q_2 = 2(\alpha s^2 + \beta st + \gamma t^2)\psi(P_1).$$

This means that $(Q_1\partial_s + Q_2\partial_t)(\alpha s^2 + \beta st + \gamma t^2) \in (\alpha s^2 + \beta st + \gamma t^2)\mathbb{K}[s,t]$. Since $\alpha s^2 + \beta st + \gamma t^2$ is the defining polynomial of the two points $i^{-1}(\{\alpha x + \beta y + \gamma z = 0\} \cap C)$ in $\mathbb{P}^1$, we get that $Q_1\partial_s + Q_2\partial_t$ is a derivation on the arrangement of these two points.

Suppose $C' = \{u_0x^2 + u_1xy + u_2xz + u_3y^2 + u_4yz + u_5z^2 = 0\}$ is a conic in the CL arrangement $\mathcal{C}$. Let $\theta = a_1\partial_x + a_2\partial_y + a_3\partial_z \in D(\mathcal{C})$. Computations as above show that

$$\begin{aligned}(Q_1\partial_s + Q_2\partial_t)(u_0s^4 + u_1s^3t + (u_2+u_3)s^2t^2 + u_4st^3 + u_5t^4)\\ \in (u_0s^4 + u_1s^3t + (u_2+u_3)s^2t^2 + u_4st^3 + u_5t^4)\mathbb{K}[s,t].\end{aligned}$$

Since $u_0s^4 + u_1s^3t + (u_2+u_3)s^2t^2 + u_4st^3 + u_5t^4$ is the defining polynomial of the four points $i^{-1}(C' \cap C)$ in $\mathbb{P}^1$, we get that $Q_1\partial_s + Q_2\partial_t$ is a derivation on the arrangement of these four points. Similar arguments work in the case of tangencies. □

Let $\theta = a_1\partial_x + a_2\partial_y + a_3\partial_z \in D(\mathcal{C})_d$ such that $\rho(\theta) = s\partial_s + t\partial_t$. Then $a_1, a_2, a_3 \in S_d$ with $\psi(a_1) = s^2, \psi(a_2) = st, \psi(a_3) = t^2$. Thus $d = 1$ and θ is the Euler derivation in $D(\mathcal{C})$. So quotienting by the Euler derivations yields an exact sequence:

$$0 \longrightarrow D_0'(-2) \xrightarrow{\cdot C} D_0 \xrightarrow{\rho} (D(\mathcal{A}'')_0)_\psi.$$

Since $|\mathcal{A}''| = k$, after sheafifying, $\mathcal{D}_0(\mathcal{A}'') = \mathcal{O}_{\mathbb{P}^1}(-k)$, and hence the sheafification of $D_0(\mathcal{A}'')_\psi$ is $i_*\mathcal{O}_{\mathbb{P}^1}(-k)$.

Lemma 3.6. *$HP(i_*\mathcal{O}_{\mathbb{P}^1}(-k), t) = 2t + 1 - k$.*

Proof. Case 1: $k = 2m$. Let E be the divisor of the reduced k points $i^{-1}(C \cap \mathcal{C}')$. Then the ideal sheaf $\mathcal{I}_E = \langle f \rangle$, where $f \in \mathbb{K}[s,t]$ of degree $k = 2m$. There exists $g \in S_m$, unique modulo $(y^2 - xz)$, such that $g(s^2, st, t^2) = f$. Clearly $y^2 - xz$ cannot divide g, otherwise $g(s^2, st, t^2) = 0 = f$, so the ideal of the reduced k points on C is $\langle y^2 - xz, g\rangle$. Hence $i_*\mathcal{I}_E = \langle \bar{g}\rangle$ as an ideal of $S/\langle y^2 - xz\rangle$. As an S-module, it has free resolution

$$0 \longrightarrow S(-2-m) \xrightarrow{\cdot C} S(-m) \longrightarrow \langle \bar{g}\rangle \longrightarrow 0,$$

which yields

$$HP(i_*\mathcal{O}_{\mathbb{P}^1}(-2m),t) = \binom{t+2-m}{2} - \binom{t-m}{2} = 2t+1-2m.$$

Case 2: $k = 2m+1$. Let E be the divisor of the reduced k points $i^{-1}(C \cap \mathcal{C}')$. Then the ideal sheaf $\mathcal{I}_E = \langle f \rangle$, where $f \in K[s,t]$ of degree $k = 2m+1$. Let $L_1, L_2 \in \mathbb{K}[s,t]_1$ be two independent linear forms which do not divide f, and let $f_i = L_i f$. Since $\langle L_1 f, L_2 f \rangle = \langle L_1, L_2 \rangle \cap \langle f \rangle$, then $\langle f_1, f_2 \rangle$ defines the same ideal sheaf on $\mathbb{P}^1$ as $\langle f \rangle$. So $\mathcal{I}_E = \langle f_1, f_2 \rangle$.

Both f_1 and f_2 are of even degree $2m+2$. So there exist $g_1, g_2 \in S = \mathbb{K}[x,y,z]$ of degree $m+1$ such that $g_i(s^2, st, t^2) = f_i, i = 1,2$. Next we show that $J = \langle y^2 - xz, g_1, g_2 \rangle$ is the ideal of the reduced points $C \cap \mathcal{C}'$ on C. To see this, note that if $p \in C \cap \mathcal{C}'$, then $f_i(i^{-1}(p)) = 0, i = 1,2$. So $g_i(p) = 0, i = 1,2$, and hence $g_i \in J, i = 1,2$. Clearly $y^2 - xz$ does not divide g_i, otherwise f_i is identically zero. Also, suppose $g_2 = \lambda g_1 + P(y^2 - xz)$, where λ is a constant. Then $f_2 = \lambda f_1$, i.e. $L_2 = \lambda L_1$; a contradiction. So J is the ideal of $2m+1$ points on the conic $y^2 - xz = 0$. By the Hilbert–Burch theorem, such an ideal is minimally generated by the 2×2 minors of

$$\begin{bmatrix} x & y \\ y & z \\ \alpha & \beta \end{bmatrix}$$

where both α and β have degree m. So indeed $\langle y^2 - xz, g_1, g_2 \rangle = J$, and $i_*\mathcal{I}_E = \langle \bar{g_1}, \bar{g_2} \rangle \subseteq S/\langle y^2 - xz \rangle$. As an S-module it has free resolution

$$0 \longrightarrow S^2(-2-m) \xrightarrow{\begin{bmatrix} x & y \\ y & z \end{bmatrix}} S^2(-1-m) \longrightarrow \langle \bar{g_1}, \bar{g_2} \rangle \longrightarrow 0,$$

so for the odd case we find that

$$\begin{aligned} HP(i_*\mathcal{O}_{\mathbb{P}^1}(-2m-1),t) &= 2\binom{t+1-m}{2} - 2\binom{t-m}{2} \\ &= 2t - 2m = 2t+1-(2m+1). \end{aligned}$$ □

Proposition 3.7. *For a quasihomogeneous triple* $(\mathcal{C}', \mathcal{C}, \mathcal{C}'' = \mathcal{C}'|_C)$, *the sequence*

$$0 \longrightarrow \mathcal{D}_0'(-2) \xrightarrow{\cdot C} \mathcal{D}_0 \longrightarrow i_*\mathcal{O}_L(-k) \longrightarrow 0$$

is exact, where $i\colon L \xrightarrow{[s^2:st:t^2]} \mathbb{P}^2$.

Proof. We have $HP(D_0,t)-HP(D_0'(-2),t) = 2t-4d+9+(\deg(J_f)-\deg(J_{f'}))$, where $d+1$ is the degree of the defining polynomial f of $\mathcal{C}$ and f' is the defining polynomial of $\mathcal{C}'$. Since $(\mathcal{C}',\mathcal{C},\mathcal{C}'')$ is a quasihomogeneous triple, Bezout's theorem and Lemma 2.7 imply that $\deg(J_f)-\deg(J_{f'}) = 4d-4-k$ and hence

$$HP(D_0,t) - HP(D_0'(-2),t) = 2t+1-k.$$

By Lemma 3.6, this is exactly the Hilbert polynomial of the sheaf $i_*\mathcal{O}_{\mathbb{P}^1}(-k)$ associated to $D_0(\mathcal{C}'')_\psi$. □

Lemma 3.8. *For a quasihomogeneous triple such that $\mathcal{C}'$ is free with exponents $\{1,m,a\}$,*

$$0 \longrightarrow D_0'(-2) \longrightarrow D_0 \longrightarrow D_0(\mathcal{A}'')_\psi \longrightarrow 0$$

is exact.

Proof. As we have seen,

$$\bigoplus_t H^0((i_*\mathcal{O}_L(-k))(t)) = \bigoplus_t H^0(\mathcal{O}_{\mathbb{P}^1}(2t-k)) = D_0(\mathcal{A}'')_\psi.$$

With the assumption on $\mathcal{C}'$, $H^1(\mathcal{D}_0'(t-2))$ vanishes for all t, and exactness follows as in the proof of Lemma 2.11. □

Theorem 3.4 will follow from the next two lemmas.

Lemma 3.9. *Let $(\mathcal{C}',\mathcal{C},\mathcal{C}'')$ be a quasihomogeneous triple, with $|C\cap\mathcal{C}'| = |\mathcal{C}''| = k$. If $\mathcal{C}'$ is free with exponents $\{1,m,a\}$, then the following holds.*

(1) *If $k=2m$ then $\mathcal{C}$ is free with* $\exp(\mathcal{C}) = \{1,m,a+2\}$.

(2) *If $k=2m+1$ and $a=m$, then $\mathcal{C}$ is free with* $\exp(\mathcal{C}) = \{1,m+1,m+1\}$.

(3) *If $k=2m+1$ and $a\neq m$, then $\mathcal{C}$ is not free.*

Proof. It follows from the computations in the proof of Lemma 3.6 that

- if $k=2m+1$, then $HS(D_0(\mathcal{A}'')_\psi,t) = \frac{2t^{m+1}}{(1-t)^2}$;
- if $k=2m$, then $HS(D_0(\mathcal{A}'')_\psi,t) = \frac{t^m(1+t)}{(1-t)^2}$.

Combining these results yields the Hilbert series of D_0.

Case 1: $k=2m$. By Lemma 3.8,

$$HS(D_0,t) = \frac{t^m+t^{a+2}}{(1-t)^3}.$$

Since $\operatorname{pdim}(D_0)\leq 1$, this means that there exist minimal generators $\theta,\eta\in D_0$ with $\deg(\theta)=m$ and $\deg(\eta)=a+2$. Suppose $\{E,\theta_1,\theta_2\}$ basis for D' with E the Euler derivation and $\deg(\theta_1)=m, \deg(\theta_2)=a$. We now use that $D\subset D'$.

- $m < a$. Since $\theta \in D_0$, $\theta = fE + c\theta_1$ for some $c \in \mathbb{C}^*$. Then $\{E, \theta, \theta_2\}$ is a basis for D', so by Saito's criterion $\{E, \theta, C\theta_2\}$ is a basis for D.
- $m = a$. Then $\theta = fE + c_1\theta_1 + c_2\theta_2$, where c_1, c_2 constants, not both zero. If $c_2 \neq 0$, then $\{E, \theta_1, \theta\}$ is a basis for D', so by Saito's criterion $\{E, C\theta_1, \theta\}$ is a basis for D.
- $m = a + 1$. Then $\theta = fE + c_1\theta_1 + L_2\theta_2$, where c is a constant and L_2 is a linear form, not both zero. If $c_1 = 0$ then $L_2\theta_2(C) \in \langle C \rangle$. Since C is irreducible, then $\theta_2(C) \in \langle C \rangle$, and so $\theta_2 \in D_0$ is of degree $a < m, a + 2$. This is inconsistent with the Hilbert series of D_0. So $c_1 \neq 0$, and so $\{E, \theta, \theta_2\}$ is a basis for D', and again by Saito's criterion $\{E, \theta, C\theta_2\}$ is a basis for D.
- $m = a + 2$. Then $\theta = f_1E + c_1\theta_1 + g_1\theta_2$, where c_1 is a constant and g_1 is a quadratic form, not both zero and $\eta = f_2E + c_2\theta_1 + g_2\theta_2$, where c_2 is a constant and g_2 is a quadratic form, not both zero. If $c_1 = c_2 = 0$, then either $g_i = c_i'C, c_i' \neq 0, i = 1, 2$ or $\theta_2 \in D_0$, a contradiction (because $\deg(\theta_2) = a$). Therefore $c_2'\theta - c_1'\eta \in D_0 \cap ES = \{0\}$, contradicting the fact that θ, η are minimal generators of D_0. So if $c_2 \neq 0$, then $\{E, \eta, \theta_2\}$ is a basis for D', and so by Saito's criterion $\{E, \eta, C\theta_2\}$ is a basis for D.
- $m > a + 2$. Then $\theta = f_1E + c_1\theta_1 + g_1\theta_2$, where c_1 is a constant and g_1 is a polynomial, not both zero and $\eta = f_2E + c_2\theta_1 + g_2\theta_2$, where c_2 is a constant and g_2 is a quadratic form, not both zero. If $c_1 = c_2 = 0$, then $g_1 = Cg_1', g_1' \neq 0$ and $g_2 = c_2'C, c_2'$ nonzero constant, and the argument used above yields a contradiction. So $c_1 \neq 0$ or $c_2 \neq 0$. Applying Saito's criterion yields the desired result.

Case 2: $k = 2m + 1, m = a$. By Lemma 3.8,

$$HS(D_0, t) = \frac{2t^{m+1}}{(1-t)^3}.$$

This implies there exist degree $m + 1$ minimal generators $\theta, \mu \in D_0$. Suppose $\{E, \theta_1, \theta_2\}$ is a basis for D' where E is the Euler derivation and $\deg(\theta_1) = m$, $\deg(\theta_2) = m$. So $\theta = f_1E + L_1\theta_1 + K_1\theta_2$ and $\mu = f_2E + L_2\theta_1 + K_2\theta_2$, where L_1, L_2, K_1, K_2 are linear forms, and for any $i = 1, 2$, L_i, K_i cannot be simultaneously zero. Hence $L_2\theta - L_1\mu - (L_2f_1 - L_1f_2)E = (L_2K_1 - L_1K_2)\theta_2 \in D(\mathcal{C})$. But θ_2 is in $D(\mathcal{C}')$ and $\theta_2(C) \notin \langle C \rangle$, else $(D_0)_m$ is nonzero, which is inconsistent with the Hilbert series. Hence $L_2K_1 - L_1K_2 = cC$, where c is a constant.

If $c = 0$, then $L_1 = uK_1, L_2 = uK_2, u \neq 0$ or $L_1 = vL_2, K_1 = vK_2, v \neq 0$, where u, v are constants, and that $K_2f_1 = K_1f_2$ and $L_2f_1 = L_1f_2$. If $L_1 = uK_1$, $L_2 = uK_2, u \neq 0$, and $K_1 \neq ct \cdot K_2$ we get $\theta = K_1(gE + u\theta_1 + \theta_2)$. Since $K_1 \neq 0$ (else $L_1 = 0$) then $\theta(C) \in \langle C \rangle$ implies $(gE + u\theta_1 + \theta_2)(C) \in \langle C \rangle$, yielding a degree m derivation in $D(\mathcal{C})$, a contradiction. If $K_1 = ct \cdot K_2$, then θ

and μ are not minimal generators, also a contradiction. If $c \neq 0$, then we find $\det[E, \theta, \mu] = cC \det[E, \theta_1, \theta_2]$, and Saito's criterion shows that $\{E, \theta, \mu\}$ is a basis for $D(\mathcal{C})$.

Case 3: $k = 2m+1, m \neq a$. By Lemma 3.8,

$$HS(D_0, t) = \frac{t^{a+2} + 2t^{m+1} - t^{m+2}}{(1-t)^3}.$$

Since $m \neq a$, there is no cancellation in the numerator, hence D_0 cannot be free. □

Lemma 3.10. *Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ be a quasihomogeneous triple, with $|C \cap \mathcal{C}'| = |\mathcal{C}''| = k$. If $\mathcal{C}$ is free, then*

(1) *If $k = 2m$ and $\exp(\mathcal{C}) = \{1, m, a+2\}$, then $\mathcal{C}'$ is free with $\exp(\mathcal{C}') = \{1, m, a\}$.*

(2) *If $k = 2m+1$ and $\exp(\mathcal{C}) = \{1, m+1, m+1\}$, then $\mathcal{C}'$ is free with $\exp(\mathcal{C}') = \{1, m, m\}$.*

(3) *If $k = 2m+1$ and $\exp(\mathcal{C}) = \{1, m+1, a+1\}$ with $a \neq m$, then $\mathcal{C}'$ is not free.*

Proof. As in Lemma 2.13, apply $\mathrm{Hom}(-, \mathcal{O}_{\mathbb{P}^2})$ to the exact sequence

$$0 \longrightarrow \mathcal{D}_0'(-2) \xrightarrow{\cdot C} \mathcal{D}_0 \longrightarrow i_*\mathcal{O}_L(-k) \longrightarrow 0.$$

Since $i_*\mathcal{O}_L(-k)$ is supported on the conic C, $\mathrm{Hom}(i_*\mathcal{O}_L(-k), \mathcal{O}_{\mathbb{P}^2}) = 0$. The assumption that D_0 is free implies that $\mathrm{Ext}^1_S(\mathcal{D}_0, \mathcal{O}_{\mathbb{P}^2}) = 0$. This yields an exact sequence:

$$0 \longrightarrow \mathcal{D}_0^\vee \longrightarrow \mathcal{D}_0'^\vee(2) \longrightarrow \mathrm{Ext}^1_S(i_*\mathcal{O}_L(-k), \mathcal{O}_{\mathbb{P}^2}) \longrightarrow 0.$$

As D_0 free with known exponents, so also is $D_0^\vee$, and the Hilbert series is known. The proof of Lemma 3.6 provides a free resolution of $i_*\mathcal{J}_E$, which allows us to compute $\mathrm{Ext}^1_S(i_*\mathcal{J}_E, S)$. Combining everything yields the Hilbert series of $D_0'^\vee$, and the result follows as in the previous analysis. □

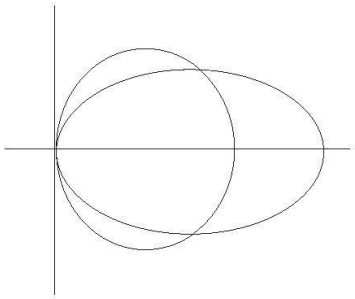
4. Freeness of CL arrangements is not combinatorial

We close with a pair of examples which show that in the CL case, Terao's conjecture that freeness is a combinatorial invariant of an arrangement is false.

Example 4.1. Let $\mathcal{C}_1$ be given by

$$\begin{aligned} C_1 &= \{y^2 + xz = 0\}, \\ C_2 &= \{y^2 + x^2 + 2xz = 0\}, \\ L_1 &= \{x = 0\}. \end{aligned}$$

L_1 is tangent to both C_1 and C_2 at the point $P = (0 : 0 : 1)$; C_1 and C_2 have two other points in common. Adding the line $L = \{y = 0\}$ passing through P to $\mathcal{C}_1$ yields a quasihomogeneous, free CL arrangement $\mathcal{C}$, with $D_0(\mathcal{C}) \simeq S(-2) \oplus S(-3)$:



The line $L' = \{x - 13y = 0\}$ passes through P, and misses the other singularities of $\mathcal{C}_1$. The CL arrangement $\mathcal{C}' = L' \cup \mathcal{C}_1$ is combinatorially identical to $\mathcal{C}$, but $\mathcal{C}'$ is not quasihomogeneous, and not free:

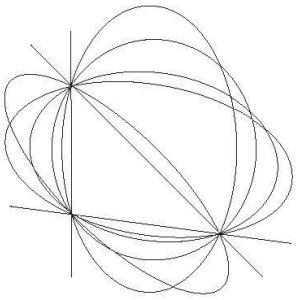
$$0 \longrightarrow S(-9) \longrightarrow S(-8)^3 \longrightarrow D_0(\mathcal{C}') \longrightarrow 0.$$

Even for CL arrangements with ordinary singularities, there are counterexamples:

Example 4.2. Let $\mathcal{A}$ be the union of the five smooth conics:

$$\begin{aligned}
C_1 &= (x - 3z)^2 + (y - 4z)^2 - 25z^2 = 0,\\
C_2 &= (x - 4z)^2 + (y - 3z)^2 - 25z^2 = 0,\\
C_3 &= (x + 3z)^2 + (y - 4z)^2 - 25z^2 = 0,\\
C_4 &= (x + 4z)^2 + (y - 3z)^2 - 25z^2 = 0,\\
C_5 &= (x - 5z)^2 + y^2 - 25z^2 = 0.
\end{aligned}$$

$\mathcal{A}$ has 13 singular points, all ordinary. At 10 of these points only two branches of $\mathcal{A}$ meet, while at the points $(0 : 0 : 1), (1 : i : 0), (1 : -i : 0)$, all five conics meet. The Milnor and Tjurina numbers agree at all singularities except $(0 : 0 : 1)$, where $\tau = 15$ and $\mu = 16$. Adding lines L_1, L_2, L_3 connecting $(0 : 0 : 1), (1 : i : 0), (1 : -i : 0)$ yields a free CL arrangement $\mathcal{C}$, with $D_0(\mathcal{C}) = S(-6)^2$.



Next, let $\mathcal{A}'$ be the union of the following five smooth conics:

$$\begin{aligned} C_1 &= x^2 + 8y^2 + 21xy - xz - 8yz = 0, \\ C_2 &= x^2 + 5y^2 + 13xy - xz - 5yz = 0, \\ C_3 &= x^2 + 9y^2 - 4xy - xz - 9yz = 0, \\ C_4 &= x^2 + 11y^2 + xy - xz - 11yz = 0, \\ C_5 &= x^2 + 17y^2 - 5xy - xz - 17yz = 0. \end{aligned}$$

$\mathcal{A}'$ is combinatorially identical to $\mathcal{A}$, but at the points $(0:0:1), (1:0:1), (0:1:1)$ where all the branches meet, $\tau = 15$ and $\mu = 16$. Adding the lines connecting these three points yields a CL arrangement $\mathcal{C}'$ which is combinatorially identical to $\mathcal{C}$ but *not free*; the free resolution of $D_0(\mathcal{C}')$ is:

$$0 \longrightarrow S(-8)^2 \longrightarrow S(-7)^4 \longrightarrow D_0(\mathcal{C}') \longrightarrow 0.$$

As was pointed out by the referee, the complements of arrangements $\mathcal{A}$ and $\mathcal{A}'$ are homeomorphic (via a Cremona transformation centered on the three multiple intersection points) to the complements of a pair of line arrangements consisting of eight lines in general position. The moduli space of such objects is connected, so the complements are rigidly isotopic, hence homeomorphic. So freeness is also not a topological invariant.

Concluding remarks

(1) As noted in §1.2, for the complement X of a CL arrangement in $\mathbb{P}^2$ the Betti numbers $h^1(X)$ and $h^2(X)$ depend only on the combinatorics, and so if X is quasihomogeneous and free, there is a version of Terao's theorem, which we leave for the interested reader.

(2) In the examples above, the Jacobian ideals are of different degrees, so are not even members of the same Hilbert scheme. Do there exist CL arrangements with isomorphic intersection poset *and* singularities which are locally isomorphic, one free and one nonfree? Do there exist counterexamples where all singularities are quasihomogeneous?

(3) As shown by Example 2.3, quasihomogenity is not a necessary condition for freeness of CL arrangements. However, without this assumption, the sequences in Propositions 2.8 and 3.8 may not be exact, which means that any form of addition–deletion will require hypotheses on higher cohomology.

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References

[1] T. Abe, H. Terao, M. Wakefield, The Euler multiplicity and addition-deletion theorems for multiarrangements. *J. London Math. Soc.* 77 (2008), 335–348. Zbl 1140.52015 MR 2400395

[2] J. I. Cogolludo-Agustín, Topological invariants of the complement to arrangements of rational plane curves. Mem. Amer. Math. Soc. 159 (2002), no. 756. Zbl 1038.32025 MR 1921584

[3] D. Eisenbud, *Commutative Algebra with a view towards Algebraic Geometry.* Grad. Texts in Math. 150, Springer-Verlag, New York 1995. Zbl 0819.13001 MR 1322960

[4] M. Mustaţă, H. Schenck, The module of logarithmic p-forms of a locally free arrangement. *J. Algebra* **241** (2001), 699–719. Zbl 1047.14007 MR 1843320

[5] P. Orlik, H. Terao, *Arrangements of hyperplanes*. Grundlehren Math. Wiss. 300, Springer-Verlag, Berlin 1992. Zbl 0757.55001 MR 1217488

[6] H. Reiffen, Das Lemma von Poincaré für holomorphe Differentialformen auf komplexen Räumen. *Math. Z.* **101** (1967), 269–284. Zbl 0164.09401 MR 0223599

[7] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 265–291. Zbl 0496.32007 MR 0586450

[8] K. Saito, Quasihomogene isolierte Singularitäten von Hyperflächen. *Invent. Math.* **14** (1971), 123–142. Zbl 0224.32011 MR 0294699

[9] H. Schenck, A rank two vector bundle associated to a three arrangement, and its Chern polynomial. *Adv. Math.* **149** (2000), 214–229. Zbl 0977.52029 MR 1742707

[10] H. Schenck, Elementary modifications and line configurations in $\mathbb{P}^2$. *Comment. Math. Helv.* **78** (2003), 447–462. Zbl 1033.52018 MR 1998388

[11] H. Terao, Generalized exponents of a free arrangement of hyperplanes and Shepard-Todd-Brieskorn formula. *Invent. Math.* **63** (1981), 159–179. Zbl 0437.51002 MR MR0608532

[12] H. Terao, Arrangements of hyperplanes and their freeness I. *J. Fac. Sci. Univ. Tokyo* **27** (1980), 293–312. Zbl 0509.14006 MR 0586451

[13] M. Wakefield, S. Yuzvinsky, Derivations of an effective divisor on the complex projective line. *Trans. Amer. Math. Soc.* **359** (2007), 4389–4403. Zbl 1121.52036 MR 2309190

[14] C. T. C. Wall, *Singular points of plane curves*. Cambridge University Press, Cambridge 2004. Zbl 1057.14001 MR 2107253

[15] G. Ziegler, Multiarrangements of hyperplanes and their freeness. In *Singularities*, Contemp. Math. 90, Amer. Math. Soc., Providence, RI, 1989, 345–359. Zbl 0678.51010 MR 1000610

[16] M. Yoshinaga, Characterization of a free arrangement and conjecture of Edelman and Reiner. *Invent. Math.* **157** (2004), 449–454. Zbl 1113.52039 MR 2077250

[17] M. Yoshinaga, On the freeness of 3-arrangements. *Bull. London Math. Soc.* **37** (2005), 126–134. Zbl 1071.52019 MR 2105827

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