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Diameter pinching in almost positive Ricci curvature

Erwann Aubry

Abstract. In this paper we prove a diameter sphere theorem and its corresponding λ_1 sphere theorem under L^p control of the curvature. They are generalizations of some results due to S. Ilias [8].

Mathematics Subject Classification (2000). 53C20.

Keywords. Ricci curvature, comparison theorems, integral bounds on the curvature, sphere theorems.

1. Introduction

Let (M^n, g) be a complete manifold with Ricci curvature $\text{Ric} \geq n - 1$. Then (M^n, g) satisfies the following classical results (the proofs can be found in [13] for instance):

- $\text{Diam}(M^n, g) \leq \pi$ (S. Myers) with equality iff $(M^n, g) = (\mathbb{S}^n, \text{can})$ (S. Cheng),
- $\lambda_1(M^n, g) \geq n$ (A. Lichnerowicz) with equality iff $(M^n, g) = (\mathbb{S}^n, \text{can})$ (M. Obata),

where Diam is the diameter and λ_1 is the first positive eigenvalue.

Studying the properties of the sphere kept by manifolds with $\text{Ric} \geq n - 1$ and almost extremal diameter or λ_1 , S. Ilias proved in [8] the following results:

Theorem 1.1 (S. Ilias). *For any $A > 0$, there exists $\epsilon(A, n) > 0$ such that any n -manifold with $\text{Ric} \geq n - 1$, sectional curvature $\sigma \leq A$ and $\lambda_1 \leq n + \epsilon$ is homeomorphic to \mathbb{S}^n .*

Theorem 1.2 (S. Ilias). *For any $A > 0$, there exists $\epsilon(A, n) > 0$ such that any n -manifold with $\text{Ric} \geq n - 1$, $\sigma \leq A$ and $\text{Diam}(M) \geq \pi - \epsilon$ is homeomorphic to \mathbb{S}^n .*

Remark 1.3. C. Croke proves in [7] that for n -manifolds with $\text{Ric} \geq n - 1$, $\lambda_1(M)$ close to n implies $\text{Diam}(M)$ close to π . The converse is proved in [8] (using a spectral inequality due to S. Cheng [6]).

Remark 1.4. For $n \geq 4$, M. Anderson [1] and Y. Otsu [10] construct sequences of complete metrics g_i with $\text{Ric}(g_i) \geq n - 1$, $\lambda_1(g_i) \rightarrow n$ and $\text{Diam}(g_i) \rightarrow \pi$ on manifolds that are not homotopy to S^n (more precisely, Otsu shows that if $n \geq 5$, these manifolds can have infinitely many different fundamental groups).

Remark 1.5. The two results of S. Ilias have been improved by G. Perelman in [11], where the assumption $\sigma \leq A$ is replaced by $\sigma \geq -A$ (note that under the Ilias's assumptions $\sigma \leq A$ and $\text{Ric} \geq n - 1$ we have $|\sigma| \leq (n - 2)A$).

Subsequently, we denote $\underline{\text{Ric}}(x)$ the lowest eigenvalue of the Ricci tensor and $\bar{\sigma}(x)$ the maximal sectional curvature at x . In [4], we prove the following generalization of the theorems of Myers and Lichnerowicz:

Theorem 1.6. For any $p > n/2$, there exists $C(p, n)$ such that if (M^n, g) is a complete manifold with $\int_M (\underline{\text{Ric}} - (n - 1))_-^p < \frac{\text{Vol } M}{C(p, n)}$, then M is compact, has finite fundamental group and satisfies

$$\begin{aligned} \text{Diam}(M) &\leq \pi \left[1 + C(p, n) \left(\frac{\rho_p}{\text{Vol } M} \right)^{\frac{1}{10}} \right], \\ \lambda_1(M) &\geq n \left[1 - C(p, n) \left(\frac{\rho_p}{\text{Vol } M} \right)^{\frac{1}{p}} \right], \end{aligned}$$

where $\rho_p = \int_M (\underline{\text{Ric}} - (n - 1))_-^p$ and $x_- = \max(0, -x)$.

Remark 1.7. It follows from [4] that the constant $C(p, n)$ is computable, that if $\int_M (\underline{\text{Ric}} - (n - 1))_-^p$ is finite (for $p > n/2$), then $\text{Vol } M$ is finite, and that we cannot bound the diameter or the first non zero eigenvalue under the assumption $\rho_p \leq \frac{1}{C(p, n)}$ or $\rho_{\frac{n}{2}}$ small (see [4]).

In this paper we prove the following extensions of Ilias's stability results.

Theorem 1.8. Let $n \geq 2$ be an integer, $A > 0$ and $p > n$ be some reals. There exists a positive constant $C(p, n, A)$ such that any complete n -manifold which satisfies

$$\int_M (\underline{\text{Ric}} - (n - 1))_-^p < C(p, n, A) \text{Vol } M, \quad \int_M \bar{\sigma}_+^p < A \text{Vol } M$$

and

$$\text{Diam}(M) \geq \pi(1 - C(p, n, A))$$

is homeomorphic to S^n (where $x_+ = \max(0, x)$).

Theorem 1.9. Let $n \geq 2$ be an integer, $A > 0$ and $p > n$ be some reals. There exists a positive constant $C(p, n, A)$ such that any complete n -manifold which satisfies

$$\int_M (\underline{\text{Ric}} - (n - 1))_-^p < C(p, n, A) \text{Vol } M, \quad \int_M \bar{\sigma}_+^p < A \text{Vol } M$$

and

$$\lambda_1(M) \leq n(1 + C(p, n, A))$$

is homeomorphic to \mathbb{S}^n .

Remark 1.10. By the Hölder inequality, the two curvature assumptions of Theorem 1.9 can be replaced by

$$\int_M (\underline{\text{Ric}} - (n-1))_- < C(p, n, A) \text{Vol } M \quad \text{and} \quad \int_M \sigma^p < A \text{Vol } M,$$

where $\sigma(x)$ is an upper bound for the absolute value of the sectional curvatures at x .

2. Comparison results in almost positive Ricci curvature

Subsequently we denote $B(x, r)$ (resp. $S(x, r)$) the geodesic ball (resp. sphere) with center x and radius r and $L_k(r)$ (resp. $A_k(r)$) the volume of a geodesic sphere (resp. ball) of radius r in $(\mathbb{S}^n, \frac{1}{k}g)$. Besides Theorem 1.6, we will need the following comparison results for manifolds of almost positive Ricci curvature (see [4] for a proof).

Proposition 2.1. *For any $n \geq 2$ and $p > n/2$ ($p \geq 1$ if $n = 2$) there exists a constant $C(p, n)$ such that for any complete Riemannian n -manifold (M^n, g) with $\eta^{10} = \frac{\bar{\rho}_p}{\text{Vol } M} \leq \frac{1}{C(p, n)}$, we have*

$$\left(\frac{\text{Vol}_{n-1} S(x, R)}{L_{1-\eta}(R)} \right)^{\frac{1}{2p-1}} - \left(\frac{\text{Vol}_{n-1} S(x, r)}{L_{1-\eta}(r)} \right)^{\frac{1}{2p-1}} \leq C(p, n) \eta^2 (R-r)^{\frac{2p-n}{2p-1}},$$

$$\frac{\text{Vol } B(x, r)}{\text{Vol } B(x, R)} \geq (1 - C(p, n)\eta) \frac{A_1(r)}{A_1(R)},$$

$$\text{Vol}_{n-1} S(x, R) \leq (1 + \eta^2) L_{1-\eta}(R),$$

$$\text{Vol } B(x, R) \leq (1 + \eta) A_1(R)$$

for all $x \in M$ and all radii $0 \leq r \leq R$.

For any $n \geq 2$ and $p > n/2$ there exists a constant $C(p, n)$ such that if (M^n, g) is a complete n -manifold with $\bar{\rho}_p \leq \frac{1}{C(p, n)}$, then $\|u\|_{\frac{2n}{n-2}} \leq \text{Diam}(M)C(p, n)\|du\|_2 + \|u\|_2$, for any $u \in H^{1,2}(M)$. In the case $n = 2$, we have $\|u\|_4 \leq \text{Diam}(M)C\|du\|_2 + \|u\|_2$ if $\bar{\rho}_1 \leq \frac{1}{C}$.

Similar estimates are proved in [12] under the assumption that M is compact and $\text{Diam}(M)^{2p} \frac{\bar{\rho}_p}{\text{Vol } M} \leq \frac{1}{C(p, n)}$.

3. Theorem 1.9 implies Theorem 1.8

Proposition 3.1. *Let $n \geq 2$ and $p > n/2$. There exists $C(p, n) > 0$ such that if (M^n, g) is a complete n -manifold with $\eta^{10} = \bar{\rho}_p \leq \frac{1}{C(p, n)}$ and $\text{Diam}(M) \geq \pi - \frac{1}{C(p, n)}$, then we have*

$$\lambda_1(M) \leq n + C(p, n)[\eta + (\text{Diam}(M) - \pi)_-].$$

The main tool to prove this proposition is the following lemma:

Lemma 3.2. *Let $n \geq 2$ and $p > n/2$ ($p \geq 1$ if $n = 2$) and $\bar{x}_0 \in \mathbb{S}^n$. There exists a constant $C(p, n)$ such that if (M^n, g) is a complete n -manifold with $\eta^{10} = \bar{\rho}_p \leq \frac{1}{C(p, n)}$, then there exists $x_0 \in M$ such that for any C^1 -function $u : [0, 2\pi] \rightarrow \mathbb{R}$ we have*

$$\begin{aligned} & \left| \frac{1}{\text{Vol } M} \int_M u \circ d_M(x_0, \cdot) dv_g - \frac{1}{\text{Vol } \mathbb{S}^n} \int_{\mathbb{S}^n} u \circ d_{\mathbb{S}^n}(\bar{x}_0, \cdot) dv_{\mathbb{S}^n} \right| \\ & \leq \|u'\|_\infty C(p, n)[\eta + (\text{Diam}(M) - \pi)_-]. \end{aligned}$$

Proof. Let $(x_0, y_0) \in M^2$ such that $d = \text{Diam}(M) = d(x_0, y_0)$. The functions A, L, A_1 and L_1 are defined in Proposition 2.1 and prolonged by 0 to \mathbb{R} (note that the diameter of M can be greater than π). The function $r \rightarrow u(r)A(r)$ is continuous and has right differential on \mathbb{R} equal to $u'A + uL$. We infer the equalities

$$\begin{aligned} u(d) \text{Vol } M &= \int_0^d u(r)L(r) dr + \int_0^d u'(r)A(r) dr, \\ u(\pi) \text{Vol } \mathbb{S}^n &= \int_0^\pi u(r)L_1(r) dr + \int_0^\pi u'(r)A_1(r) dr \end{aligned}$$

which imply

$$\begin{aligned} & \left| \frac{1}{\text{Vol } M} \int_M u \circ d_M(x_0, x) dv_g - \frac{1}{\text{Vol } \mathbb{S}^n} \int_{\mathbb{S}^n} u \circ d_{\mathbb{S}^n}(\bar{x}_0, x) dv_{\mathbb{S}^n} \right| \\ &= \left| \int_0^d \frac{u(r)L(r)}{\text{Vol } M} dr - \int_0^\pi \frac{u(r)L_1(r)}{\text{Vol } \mathbb{S}^n} dr \right| \\ &= \left| u(d) - u(\pi) + \int_0^\pi \frac{u'A_1}{\text{Vol } \mathbb{S}^n} - \int_0^d \frac{u'A}{\text{Vol } M} \right| \\ &= \left| \int_0^d u' \left(\frac{A_1}{\text{Vol } \mathbb{S}^n} - \frac{A}{\text{Vol } M} \right) + \int_d^\pi u' \left(\frac{A_1}{\text{Vol } \mathbb{S}^n} - 1 \right) \right| \\ &\leq \|u'\|_\infty \left(\int_0^d \left| \frac{A_1}{\text{Vol } \mathbb{S}^n} - \frac{A}{\text{Vol } M} \right| dr + |\pi - d| \right). \end{aligned}$$

By Proposition 2.1 we have, for all $r \leq d$:

$$\begin{aligned} (1 - C(p, n)\eta) \frac{A_1(r)}{\text{Vol } \mathbb{S}^n} &\leq \frac{A(r)}{\text{Vol } M} \leq 1 - \frac{\text{Vol } B(y_0, d - r)}{\text{Vol } M} \\ &\leq 1 - (1 - C(p, n)\eta) \frac{A_1(d - r)}{\text{Vol } \mathbb{S}^n} \\ &\leq \frac{A_1(r + \pi - d)}{\text{Vol } \mathbb{S}^n} + C(p, n)\eta. \end{aligned}$$

Hence $\left| \frac{A(r)}{\text{Vol } M} - \frac{A_1(r)}{\text{Vol } \mathbb{S}^n} \right| \leq C(p, n)\eta + \frac{(A_1(r) - A_1(r + \pi - d))_-}{\text{Vol } \mathbb{S}^n}$. An easy computation gives $\left\| \frac{(A_1(\cdot) - A_1(\cdot + h))_-}{\text{Vol } \mathbb{S}^n} \right\|_\infty \leq C(n)(-h)_-$, and by Proposition 2.1 we get:

$$\begin{aligned} \left| \frac{1}{\text{Vol } M} \int_M u \circ d_M(x_0, x) dv_g - \frac{1}{\text{Vol } \mathbb{S}^n} \int_{\mathbb{S}^n} u \circ d_{\mathbb{S}^n}(\bar{x}_0, x) dv_{\mathbb{S}^n} \right| \\ \leq \|u'\|_\infty C(p, n)[\eta + (d - \pi)_-]. \end{aligned} \quad \square$$

We now finish the proof of Proposition 3.1.

Proof. Lemma 3.2 applied to $u = \sin^2$, $u = \cos^2$ and $u = \cos$ gives

$$\begin{aligned} \left| \int_M \frac{\sin^2 d_M(x_0, \cdot)}{\text{Vol } M} - \int_{\mathbb{S}^n} \frac{\sin^2 d_{\mathbb{S}^n}(\bar{x}_0, \cdot)}{\text{Vol } \mathbb{S}^n} \right| &\leq C(p, n)(\eta + (d - \pi)_-) \leq 1, \\ \left| \int_M \frac{\cos^2 d_M(x_0, \cdot)}{\text{Vol } M} - \int_{\mathbb{S}^n} \frac{\cos^2 d_{\mathbb{S}^n}(\bar{x}_0, \cdot)}{\text{Vol } \mathbb{S}^n} \right| &\leq C(p, n)(\eta + (d - \pi)_-) \leq 1, \\ \left| \int_M \frac{\cos d_M(x_0, \cdot)}{\text{Vol } M} - \int_{\mathbb{S}^n} \frac{\cos d_{\mathbb{S}^n}(\bar{x}_0, \cdot)}{\text{Vol } \mathbb{S}^n} \right| &\leq C(p, n)(\eta + (d - \pi)_-) \leq 1. \end{aligned}$$

Hence, if we set $f = \cos d_M(x_0, \cdot)$, we get

$$\begin{aligned} \left| \|\nabla f\|_2^2 - \frac{n}{n+1} \right| &\leq C(p, n)(\eta + (d - \pi)_-) \leq 1, \\ \left| \|f\|_2^2 - \frac{1}{n+1} \right| &\leq C(p, n)(\eta + (d - \pi)_-) \leq 1, \\ \left| \frac{1}{\text{Vol } M} \int_M f \right| &\leq C(p, n)(\eta + (d - \pi)_-) \leq 1, \end{aligned}$$

which readily implies that

$$\lambda_1(M) \leq \frac{\|\nabla(f - \bar{f})\|_2^2}{\|f - \bar{f}\|_2} \leq n(1 + C(p, n)(\eta + (d - \pi)_-)),$$

where we have set $\bar{f} = \frac{1}{\text{Vol } M} \int_M f$. □

Remark 3.3. The same technique as in [5] can be used to prove that manifolds with almost positive Ricci curvature and λ_1 is close to n have a diameter close to π (see [12]).

4. Proof of Theorem 1.9

4.1. Fiber bundle E . Let E be the fiber bundle $TM \oplus \mathbb{R}e \rightarrow M$ endowed with the following scalar product and linear connection:

$$\begin{aligned} \langle X + fe, Y + he \rangle_E &= g(X, Y) + fh, \\ D_Z^E(X + fe) &= D_Z^M X + fZ + (df(Z) - g(Z, X)) \cdot e, \end{aligned}$$

where D^M is the Levi-Civita connection of the metric g on M . We denote by p the orthogonal projection of E on TM , $\text{Ric}'(S) = \text{Ric}_M(p(S)) - (n - 1)p(S)$ and $\Delta_{\text{sph}} = \bar{\Delta}^E + \text{Ric}'$.

The following lemma is proved in [3]:

Lemma 4.1. *If $f : M \rightarrow \mathbb{R}$ satisfies $\Delta f = \lambda f$, then $S_f = \nabla f + fe$ satisfies $\Delta_{\text{sph}}(S_f) = (\lambda - n)(\nabla f - fe)$ and $\langle D_X^E S_f, X \rangle = Ddf(X, X) + fg(X, X)$.*

Note also that we have

$$R_{(Z,Y)}^E(X + fe) = R^M(Z, Y)X - (g(Y, X)Z - g(Z, X)Y).$$

4.2. Bound on the Hessian of the first eigenfunction. To prove Theorem 1.9 we need an L^∞ bound on the Hessian of the first eigenfunction. For that purpose, we will modify the proof of Theorem 2.4 in [2] (whose proof would give us only a bound on $\|DS_f\|_{n+\epsilon} / \|S_f\|_\infty$ for a given $\epsilon = \epsilon(p, n)$). In our case we really need to perform a Moser iteration.

Proposition 4.2. *Let $n \geq 2$ and $\infty \geq p > n/2$. There exists a constant $C(p, n)$ such that if (M^n, g) is any manifold with $\bar{\rho}_p \leq \frac{1}{C(p,n)}$ and $\lambda_1 \leq n + \frac{1}{C(p,n)}$, then for $f : M \rightarrow \mathbb{R}$ such that $\Delta f = \lambda_1 f$ we have*

$$\frac{\|D^E S_f\|_\infty}{\|S_f\|_\infty} \leq C(p, n)(\lambda_1 + \|R\|_{2p})^\gamma (|\lambda_1 - n| + \bar{\rho}_p)^{\frac{1}{2(1+\gamma)}},$$

where $S_f = \nabla f + f \cdot e$ and $\gamma = \frac{pn}{2p-n}$.

To prove Proposition 4.2 we need a commutation lemma (see [2]):

Lemma 4.3. *For any section $S \in \Gamma(E)$ we have*

$$\begin{aligned} & \frac{1}{2} \Delta (|DS|^2) + |D^2S|^2 \\ & \leq \langle D^*R^E S, DS \rangle + \underline{\text{Ric}}^- |DS|^2 + \langle D\bar{\Delta}S, DS \rangle + \|R^E\| \cdot |DS|^2, \end{aligned}$$

where $\|R^E\|$ is the norm of the linear map $R^E: \wedge^2 T_m M \rightarrow \wedge^2 E_m^*$ defined by $R^E(u \wedge v)(T, S) = \langle R^E(u, v)T, S \rangle$.

Remark 4.4. This lemma is valid for any Riemannian fiber bundle $(E, D, \langle \cdot, \cdot \rangle)$.

We now give the proof of Proposition 4.2.

Proof of Proposition 4.2. We set $u = \sqrt{|DS|^2 + \epsilon^2}$. We have

$$\begin{aligned} u \Delta u &= \frac{1}{2} \Delta(u^2) + |du|^2 = \frac{1}{2} \Delta(u^2) + \frac{|\langle D^2S, DS \rangle|^2}{|DS|^2 + \epsilon^2} \\ &\leq \frac{1}{2} \Delta(|DS|^2) + |D^2S|^2. \end{aligned}$$

Hence, by Lemma 4.3

$$\begin{aligned} \int_M |d(u^k)|^2 &\leq \frac{k^2}{2k-1} \int_M \left(\frac{1}{2} \Delta |DS|^2 + |D^2S|^2 \right) u^{2(k-1)} \\ &\leq \frac{k^2}{2k-1} \left(\int_M \underline{\text{Ric}}^- u^{2k} + \int_M \langle D\bar{\Delta}S, DS \rangle u^{2(k-1)} \right. \\ &\quad \left. + \int_M \langle D^*R^E S, DS \rangle u^{2(k-1)} + \int_M \|R^E\| u^{2k} \right). \end{aligned}$$

We now apply the divergence theorem to the form $u^{2(k-1)} \langle \bar{\Delta}S, D\bullet S \rangle$, and get for any $k \geq 1$:

$$\begin{aligned} & \int_M \langle D\bar{\Delta}S, DS \rangle u^{2(k-1)} \\ &= \int_M |\bar{\Delta}S|^2 u^{2(k-1)} - 2(k-1) \sum_i \int_M \langle \bar{\Delta}S, DS(i) \rangle du(i) \cdot u^{2k-3} \\ &\leq \int_M |\bar{\Delta}S|^2 u^{2(k-1)} + 2(k-1) \int_M |\bar{\Delta}S| |du| u^{2(k-1)} \\ &\leq \frac{k-1}{2} \int_M |du|^2 u^{2(k-1)} + (2k-1) \int_M |\bar{\Delta}S|^2 u^{2(k-1)}. \end{aligned}$$

We do the same with the form $u^{2(k-1)}(tr_{1,3}(\langle R_{(\bullet,\bullet)}^E S, D_\bullet S \rangle))$ and get

$$\begin{aligned} & \int_M \langle D^* R^E S, DS \rangle u^{2(k-1)} \\ &= \int_M \frac{1}{2} |R^E S|^2 u^{2(k-1)} + 2(k-1) \sum_{i,j} \int_M \langle R^E(i,j) S, D_j S \rangle du(i) u^{2k-3} \\ &\leq \frac{k-1}{2} \int_M |du|^2 u^{2(k-1)} + (2k-1) \int_M |R^E S|^2 u^{2(k-1)}, \end{aligned}$$

where we have used $\sum_{i,j} \langle R^E S(i,j), D^2 S(i,j) \rangle = \frac{1}{2} |R^E S|^2$.

Since $\int_M |du|^2 u^{2(k-1)} = \frac{1}{k^2} \int_M |d(u^k)|^2$, the three last inequalities give, for any $k \geq 1$,

$$\begin{aligned} \|d(u^k)\|_2^2 &\leq k \left(\int_M \underline{\text{Ric}}^- u^{2k} + \int_M \|R^E\| u^{2k} \right) \\ &\quad + k(2k-1) \left(\int_M \|R^E S\|^2 u^{2k-2} + \int_M \|\bar{\Delta} S\|^2 u^{2k-2} \right) \\ &\leq 4k^2 \left(B_1 \|u\|_{\frac{2kp}{p-1}}^{\frac{2k}{p-1}} + B_2 \|S\|_\infty^2 \|u\|_{\frac{2(k-1)p}{p-1}}^{2(k-1)} \right), \end{aligned}$$

where we have set

$$\begin{aligned} B_1 &= \|\underline{\text{Ric}}^-\|_p + \|R^E\|_p \leq C(n) (\|R^M\|_{2p}^2 + \lambda_1^2) = B^2, \\ B_2 &= \frac{\|\bar{\Delta} S\|_{2p}^2}{\|S\|_\infty^2} + \frac{\|R^E S\|_{2p}^2}{\|S\|_\infty^2} \leq \frac{\|\Delta_{\text{sph}} S\|_{2p}^2}{\|S\|_\infty^2} + \|\text{Ric}'\|_{2p}^2 + \|R^E\|_{2p}^2 \\ &\leq C(n) (\lambda_1^2 + \|R^M\|_{2p}^2) = B^2. \end{aligned}$$

By the Sobolev inequality given by Proposition 2.1 we get

$$\|DS\|_{\frac{2kn}{n-2}}^k \leq \|DS\|_{2k}^k + C(p,n) Bk \sqrt{\|DS\|_{\frac{2kp}{p-1}}^{\frac{2k}{p-1}} + \|S\|_\infty^2 \|DS\|_{\frac{2(k-1)p}{p-1}}^{\frac{2(k-1)}{p-1}}},$$

and by $\|DS\|_{2k} \leq \|DS\|_{\frac{2kp}{p-1}} \leq \|DS\|_\infty^{1/k} \|DS\|_{\frac{2(k-1)p}{p-1}}^{(1-1/k)}$ we have

$$\begin{aligned} & \left(\frac{\|DS\|_{\frac{2kn}{n-2}}}{\|DS\|_\infty} \right)^{\frac{2kn}{n-2}} \\ & \leq \left[1 + BkC(p,n) \left(1 + \frac{\|S\|_\infty^2}{\|DS\|_\infty^2} \right) \right]^{\frac{2n}{n-2}} \left(\frac{\|DS\|_{\frac{2(k-1)p}{p-1}}}{\|DS\|_\infty} \right)^{\nu \frac{2(k-1)p}{p-1}}, \end{aligned}$$

where $\nu = \frac{n(p-1)}{2p(n-2)} > 1$. We set $k = \frac{a_n(p-1)}{2p} + 1$, where $(a_n)_n$ is the sequence defined by $a_0 = \frac{2p}{p-1}$ and $a_{n+1} = \nu a_n + \frac{2n}{n-2}$. Then we get

$$\begin{aligned} & \left(\frac{\|DS\|_{a_{n+1}}}{\|DS\|_\infty} \right)^{\frac{a_{n+1}}{\nu^{n+1}}} \\ & \leq \left[1 + a_n C(p, n) B \left(1 + \frac{\|S\|_\infty^2}{\|DS\|_\infty^2} \right) \right]^{\frac{2n}{(n-2)\nu^{n+1}}} \left(\frac{\|DS\|_{a_n}}{\|DS\|_\infty} \right)^{\frac{a_n}{\nu^n}}. \end{aligned}$$

Hence

$$\begin{aligned} 1 &= \lim_{n \rightarrow +\infty} \left(\frac{\|DS\|_{a_n}}{\|DS\|_\infty} \right)^{\frac{a_n}{\nu^n}} \\ &\leq \prod_{i=1}^\infty \left(1 + C(p, n) a_i B \left(1 + \frac{\|S\|_\infty^2}{\|DS\|_\infty^2} \right) \right)^{\frac{2n}{(n-2)\nu^i}} \left(\frac{\|DS\|_{a_0}}{\|DS\|_\infty} \right)^{a_0}. \end{aligned}$$

The Hölder inequality $\|DS\|_{a_0} \leq \|DS\|_2^{1-\frac{1}{p}} \|DS\|_\infty^{\frac{1}{p}}$, gives

$$\|DS\|_\infty \leq \prod_{i=1}^\infty \left(1 + C(p, n) a_i B \left(1 + \frac{\|S\|_\infty^2}{\|DS\|_\infty^2} \right) \right)^{\frac{n}{(n-2)\nu^i}} \|DS\|_2. \quad (*)$$

If $\|DS\|_\infty \geq \|S\|_\infty$, then inequality (*) gives

$$\begin{aligned} \|DS\|_\infty &\leq \prod_{i=1}^\infty \left(1 + C(p, n) a_i B \right)^{\frac{n}{(n-2)\nu^i}} \|DS\|_2 \\ &\leq C(p, n) (\lambda_1 + \|R\|_{2p})^{\frac{pn}{2p-n}} \|DS\|_2. \end{aligned}$$

If $\|DS\|_\infty \leq \|S\|_\infty$, then inequality (*) gives

$$\frac{\|DS\|_\infty}{\|DS\|_2} \leq \left(\frac{\|S\|_\infty}{\|DS\|_\infty} \right)^{\frac{2pn}{2p-n}} \prod_{i=1}^\infty \left(1 + C(p, n) a_i B \right)^{\frac{n}{(n-2)\nu^i}},$$

hence

$$\frac{\|DS\|_\infty}{\|S\|_\infty} \leq C(p, n) (\lambda_1 + \|R\|_{2p})^{\frac{pn}{2p-n}} \left(\frac{\|DS\|_2}{\|S\|_\infty} \right)^{\frac{2p-n}{2p-n+2pn}}.$$

At this stage note that, by Lemma 4.1 we have

$$\begin{aligned} \|DS\|_2^2 &= \langle \bar{\Delta}_{\text{sph}}(S), S \rangle_{L^2} - \langle \text{Ric}'(S), S \rangle_{L^2} \\ &\leq |\lambda_1 - n| \|S\|_2^2 + \int_M \frac{(\text{Ric} - (n-1))^-}{\text{Vol } M} |S|^2 \leq (|\lambda_1 - n| + \bar{\rho}_p) \|S\|_\infty^2. \end{aligned}$$

Since we have $\frac{2p-n}{2p-n+2pn} \leq 1$, we get the result. □

4.3. Critical points of the first eigenfunction. By Proposition 4.2, the section $S_f = \nabla f + fe$ of E satisfies $\|D^E S_f\|_\infty \leq C(p, n, A)(|\lambda_1 - n| + \bar{\rho}_p)^{\frac{1}{1+\nu}} \|S_f\|_\infty$. Since we can suppose the pinching on $|\lambda_1 - n|$ and $\bar{\rho}_p$ small enough to have

$$C(p, n, A)(|\lambda_1 - n| + \bar{\rho}_p)^{\frac{1}{1+\nu}} \leq 1/4,$$

the previous inequality and Theorem 1.6 give

$$\begin{aligned} \inf |S_f| &\geq [1 - C(p, n, A)(|\lambda_1 - n| + \bar{\rho}_p)^{\frac{1}{1+\nu}}] \|S_f\|_\infty \\ &> C(p, n, A)(|\lambda_1 - n| + \bar{\rho}_p)^{\frac{1}{1+\nu}} \|S_f\|_\infty \geq \|D^E S_f\|_\infty. \end{aligned}$$

We infer that if x_0 is a critical point of f , then by Lemma 4.1 we have

$$|Ddf_{x_0}(X, X) + f(x_0)| = |\langle D_X^E S_f, X \rangle_E| \leq \|D^E S_f\|_\infty < |S_f(x_0)| = |f(x_0)|$$

for any unit vector X of $T_{x_0}M$. Hence we have $-|f(x_0)| - f(x_0) < Ddf_{x_0}(X, X) < |f(x_0)| - f(x_0)$ for any critical point x_0 of f . So the only critical points of f are non degenerate global extrema, which implies that M is homeomorphic to S^n by Reeb's theorem.

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