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# A product formula for valuations on manifolds with applications to the integral geometry of the quaternionic line

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**Abstract.** The Alesker–Poincaré pairing for smooth valuations on manifolds is expressed in terms of the Rumin differential operator acting on the cosphere-bundle. It is shown that the derivation operator, the signature operator and the Laplace operator acting on smooth valuations are formally self-adjoint with respect to this pairing. As an application, the product structure of the space of  $SU(2)$ - and translation invariant valuations on the quaternionic line is described. The principal kinematic formula on the quaternionic line  $\mathbb{H}$  is stated and proved.

**Mathematics Subject Classification (2000).** 53C65.

**Keywords.** Valuations on manifolds, kinematic formulas, integral geometry, Alesker–Poincaré pairing.

## 1. Smooth valuations on manifolds

Let  $M$  be a smooth manifold of dimension  $n$ . For simplicity, we suppose that  $M$  is oriented, although the whole theory works in the non-oriented case as well. Following Alesker, we set  $\mathcal{P}(M)$  to be the set of compact submanifolds with corners.

**Definition 1.1.** A valuation on  $M$  is a real valued map  $\mu$  on  $\mathcal{P}(M)$  which is additive in the following sense: whenever  $X, Y, X \cap Y$  and  $X \cup Y$  belong to  $\mathcal{P}(M)$ , then

$$\mu(X \cup Y) + \mu(X \cap Y) = \mu(X) + \mu(Y).$$

A set  $X \in \mathcal{P}(M)$  admits a conormal cycle  $\text{cnc}(X)$ , which is a compactly supported Legendrian cycle on the cosphere bundle  $S^*M$ . Sometimes it will be convenient to think of  $S^*M$  as the set of pairs  $(p, P)$  with  $p \in M$  and  $P \subset T_p M$  an oriented hyperplane, at other places it is better to think of it as the set of pairs  $(p, [\xi])$  where  $p \in M$  and  $\xi \in T_p^*M \setminus \{0\}$  and the brackets denote the equivalence class for the relation  $\xi_1 \sim \xi_2 \iff \xi_1 = \lambda \xi_2, \lambda > 0$ .

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A valuation  $\mu$  on  $M$  is called *smooth* if there exist an  $(n-1)$ -form  $\omega \in \Omega^{n-1}(S^*M)$  and an  $n$ -form  $\phi \in \Omega^n(M)$  such that

$$\mu(X) = \text{cnc}(X)(\omega) + \int_X \phi, \quad X \in \mathcal{P}(M). \quad (1)$$

If  $\mu$  can be expressed in the form (1), we say that  $\mu$  is *represented* by  $(\omega, \phi)$ . The space of smooth valuations on  $M$  is denoted by  $\mathcal{V}^\infty(M)$ . It is a Fréchet space (see [6], Section 3.2 for the definition of the topology). If  $M = V$  is a vector space, the subspace of translation invariant smooth valuations will be denoted by  $\text{Val}^{\text{sm}}(V)$ .

Let  $N$  be another oriented  $n$ -dimensional smooth manifold and  $\rho: N \rightarrow M$  an orientation preserving immersion. Then  $\rho$  induces a map  $\tilde{\rho}: S^*N \rightarrow S^*M$ , sending  $(p, P)$  to  $(\rho(p), T_p\rho(P))$ . It clearly satisfies  $\pi \circ \tilde{\rho} = \rho \circ \pi$ .

The valuation  $\rho^*\mu$  on  $N$  such that

$$\rho^*\mu(X) := \mu(\rho(X)), \quad X \in \mathcal{P}(N)$$

is again smooth. If  $\mu$  is represented by  $(\omega, \phi)$ , then  $\rho^*\mu$  is represented by  $(\tilde{\rho}^*\omega, \rho^*\phi)$ . This follows from the fact that  $\text{cnc}(\rho(X)) = \tilde{\rho}_* \text{cnc}(X)$ . Note also that  $\rho^{-1} = (\tilde{\rho})^{-1}$  if  $\rho$  is a diffeomorphism.

We will use some results of [11], which we would like to recall. The cosphere bundle  $S^*M$  is a contact manifold of dimension  $2n - 1$  with a global contact form  $\alpha$  ( $\alpha$  is not unique, but this will play no role here). The projection from  $S^*M$  to  $M$  will be denoted by  $\pi$ , it induces a linear map  $\pi_*$  (fiber integration) on the level of forms.

Given an  $(n-1)$ -form  $\omega$  on  $S^*M$ , there exists a unique vertical form  $\alpha \wedge \xi$  such that  $d(\omega + \alpha \wedge \xi)$  is vertical (i.e. a multiple of  $\alpha$ ). The Rumin differential operator  $D$  is defined as  $D\omega := d(\omega + \alpha \wedge \xi)$  [18]. The following theorem was proved in [11].

**Theorem 1.2.** *Let  $\omega \in \Omega^{n-1}(S^*M)$ ,  $\phi \in \Omega^n(M)$  and define the smooth valuation  $\mu$  by (1). Then  $\mu = 0$  if and only if*

- (1)  $D\omega + \pi^*\phi = 0$ , and
- (2)  $\pi_*\omega = 0$  for all  $p \in M$ .

Moreover, if  $D\omega + \pi^*\phi = 0$ , then  $\mu$  is a multiple of the Euler characteristic  $\chi$ .

The support of a smooth valuation  $\mu$  is defined as

$$\text{spt } \mu := M \setminus \{p \in M : \text{there exists } U \subset M \text{ open such that } \mu|_U = 0\}.$$

The subspace of  $\mathcal{V}^\infty(M)$  consisting of compactly supported valuations will be denoted by  $\mathcal{V}_c^\infty(M)$ .

Let  $f: \mathcal{V}_c^\infty(M) \rightarrow \mathbb{R}$  denote the integration functional [7]. If  $\mu$  has compact support, then  $f\mu := \mu(X)$ , where  $X \in \mathcal{P}(M)$  is an  $n$ -dimensional manifold with

boundary containing  $\text{spt } \mu$  in its interior. It is clear that, if  $\mu$  is represented by  $(\omega, \phi)$  with compact supports, then

$$\int \mu = \int_M \phi = [\phi] \in H_c^n(M) = \mathbb{R}.$$

Before stating our main theorem we have to recall two other constructions of Alesker.

The first one is the Euler–Verdier involution  $\sigma: \mathcal{V}^\infty(M) \rightarrow \mathcal{V}^\infty(M)$  [6]. Let  $s: S^*M \rightarrow S^*M$  be the natural involution on  $S^*M$ , sending  $(p, P)$  to  $(p, \bar{P})$ , where  $\bar{P}$  is the hyperplane  $P$  with the reversed orientation. If a valuation  $\mu \in \mathcal{V}^\infty(M)$  is represented by the pair  $(\omega, \phi)$ , then  $\sigma\mu$  is defined as the valuation which is represented by the pair  $((-1)^n s^*\omega, (-1)^n \phi)$ .

The second construction is the Alesker–Fu product [9], which is a bilinear map

$$\mathcal{V}^\infty(M) \times \mathcal{V}^\infty(M) \rightarrow \mathcal{V}^\infty(M), \quad (\mu_1, \mu_2) \mapsto \mu_1 \cdot \mu_2.$$

We refer to [9] for its construction. It is characterized by the following properties:

- (1) “ $\cdot$ ” is continuous and linear in both variables;
- (2) if  $\rho: N \rightarrow M$  is a diffeomorphism and  $\mu_1, \mu_2 \in \mathcal{V}^\infty(M)$ , then

$$\rho^*(\mu_1 \cdot \mu_2) = \rho^*\mu_1 \cdot \rho^*\mu_2;$$

- (3) if  $m_1, m_2$  are smooth measures on an  $n$ -dimensional vector space  $V$ ,  $A_1, A_2 \in \mathcal{K}(V)$  convex bodies with strictly convex smooth boundary and if  $\mu_i \in \mathcal{V}^\infty(V)$ ,  $i = 1, 2$  is defined by

$$\mu_i(K) = m_i(K + A_i), \quad K \in \mathcal{K}(V), \quad (2)$$

then

$$\mu_1 \cdot \mu_2(K) = m_1 \times m_2(\Delta(K) + A_1 \times A_2),$$

where  $\Delta: V \rightarrow V \times V$  is the diagonal embedding.

Our first main theorem is the following relation between Alesker–Fu product, integration functional, Euler–Verdier involution and Rumin differential.

**Theorem 1.3.** *Let  $\mu_1 \in \mathcal{V}^\infty(M)$  be represented by  $(\omega_1, \phi_1)$ ; let  $\mu_2 \in \mathcal{V}_c^\infty(M)$  be represented by  $(\omega_2, \phi_2)$ . Then*

$$\int \mu_1 \cdot \sigma\mu_2 = (-1)^n \int_{S^*M} \omega_1 \wedge (D\omega_2 + \pi^*\phi_2) + \int_M \phi_1 \wedge \pi_*\omega_2. \quad (3)$$

Let us call the pairing

$$\begin{aligned} \mathcal{V}^\infty(M) \times \mathcal{V}_c^\infty(M) &\rightarrow \mathbb{R}, \\ (\mu_1, \mu_2) &\mapsto \int \mu_1 \cdot \mu_2 =: \langle \mu_1, \mu_2 \rangle \end{aligned} \quad (4)$$

the *Alesker–Poincaré pairing*. Note that Theorem 1.3 is equivalent to

$$\langle \mu_1, \mu_2 \rangle = \int_{S^*M} \omega_1 \wedge s^*(D\omega_2 + \pi^*\phi_2) + \int_M \phi_1 \wedge \pi_*\omega_2. \quad (5)$$

From Theorem 1.3 and from the fact that the Poincaré pairings on  $M$  and  $S^*M$  are perfect, we get the following corollary (which was first proved by Alesker).

**Corollary 1.4** ([7], Theorem 6.1.1). *The Alesker–Poincaré pairing (4) is a perfect pairing.*

Some more operators on  $\mathcal{V}^\infty(M)$  were introduced in [11]. For this, we suppose that  $M$  is a Riemannian manifold. Then  $S^*M$  admits an induced metric, the *Sasaki metric* [20].

The first operator is the derivation operator  $\Lambda$  (which was denoted by  $\mathcal{L}$  in [11]). The metric on  $S^*M$  provides a canonical choice of  $\alpha$ , namely  $\alpha|_{(p, [\xi])} := \frac{1}{\|\xi\|} \pi^*\xi$  for all  $(p, [\xi]) \in S^*M$ . Let  $T$  be the Reeb vector field on  $S^*M$  (i.e.  $\alpha(T) = 1$  and  $\mathcal{L}_T\alpha = 0$ ).

If the smooth valuation  $\mu$  is represented by  $(\omega, \phi)$ , then  $\Lambda\mu$  is by definition the valuation which is represented by  $(\mathcal{L}_T\omega + i_T\pi^*\phi, 0)$ .

Let us recall the definitions of the signature operator  $\mathcal{S}$  and the Laplace operator  $\Delta$ . Let  $*$  be the Hodge star acting on  $\Omega^*(S^*M)$ . Let  $\mu \in \mathcal{V}^\infty(M)$  be represented by  $(\omega, \phi)$ . Then  $\mathcal{S}\mu$  is defined as the valuation which is represented by  $(*(D\omega + \pi^*\phi), 0)$ .

The Laplace operator  $\Delta$  is defined as  $\Delta := (-1)^n \mathcal{S}^2$ .

Our second main theorem shows that these operators fit well into Alesker’s theory. In fact, they are formally self-adjoint with respect to the Alesker–Poincaré pairing.

**Theorem 1.5.** *For valuations  $\mu_1 \in \mathcal{V}^\infty(M)$  and  $\mu_2 \in \mathcal{V}_c^\infty(M)$ , the following equations hold:*

$$\langle \Lambda\mu_1, \mu_2 \rangle = \langle \mu_1, \Lambda\mu_2 \rangle, \quad (6)$$

$$\langle \mathcal{S}\mu_1, \mu_2 \rangle = \langle \mu_1, \mathcal{S}\mu_2 \rangle, \quad (7)$$

$$\langle \Delta\mu_1, \mu_2 \rangle = \langle \mu_1, \Delta\mu_2 \rangle. \quad (8)$$

We will apply these theorems in the study of the integral geometry of  $SU(2)$ . This group acts on the quaternionic line  $\mathbb{H}$ . In this setting, it is more natural to work with

the space  $\mathcal{K}(\mathbb{H})$  of convex sets instead of manifolds with corners. By Proposition 2.6. of [8], there is no loss of generality in doing so.

It was shown by Alesker [3] that the space of  $SU(2)$ -invariant and translation invariant valuations on the quaternionic line  $\mathbb{H}$  is of dimension 10. For each purely complex number  $u$  of norm 1, let  $I_u$  be the complex structure given by multiplication from the right with  $u$  and  $\mathbb{C}\mathbb{P}_u^1$  the corresponding Grassmannian of complex lines (with its unique  $SU(2)$ -invariant Haar measure). Alesker defined a valuation  $Z_u$  by

$$Z_u(K) := \int_{\mathbb{C}\mathbb{P}_u^1} \text{vol}(\pi_L(K)) dL, \quad K \in \mathcal{K}(\mathbb{H}). \quad (9)$$

He showed that  $Z_i, Z_j, Z_k, Z_{\frac{i+j}{\sqrt{2}}}, Z_{\frac{i-j}{\sqrt{2}}}, Z_{\frac{j-i}{\sqrt{2}}}$ , together with Euler characteristic  $\chi$ , the volume  $\text{vol}$  and the intrinsic volumes  $\text{vol}_1, \text{vol}_3$  form a basis of  $\text{Val}^{SU(2)}$ . Following a suggestion of Fu, we will state the kinematic formula using a more symmetric choice. Noting that  $Z_u = Z_{-u}$  for all  $u \in S^2$ , the 12 vertices  $\pm u_i, i = 1, \dots, 6$  of an icosahedron on  $S^2$  define 6 valuations  $Z_{u_i}, i = 1, \dots, 6$ .

We endow  $SU(2)$  with its Haar measure and the semidirect product  $\overline{SU(2)} = SU(2) \ltimes \mathbb{H}$  with the product measure. Let  $\text{vol}_k$  denote the  $k$ -dimensional intrinsic volume [15].

**Theorem 1.6** (Principal kinematic formula for  $SU(2)$ ). *Let  $K, L \in \mathcal{K}(\mathbb{H})$ . Then*

$$\begin{aligned} \int_{\overline{SU(2)}} \chi(K \cap \bar{g}L) d\bar{g} &= \chi(K) \text{vol}(L) + \frac{4}{3\pi} \text{vol}_1(K) \text{vol}_3(L) \\ &+ \frac{17}{4} \sum_{i=1}^6 Z_{u_i}(K) Z_{u_i}(L) - \frac{3}{4} \sum_{1 \leq i \neq j \leq 6} Z_{u_i}(K) Z_{u_j}(L) \\ &+ \frac{4}{3\pi} \text{vol}_3(K) \text{vol}_1(L) + \text{vol}(K) \chi(L). \end{aligned}$$

This theorem implies and generalizes the Poincaré formulas of Tasaki [19], (which contained an error in some constant) as we will explain in the last section.

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## 2. The Alesker–Poincaré pairing in terms of forms

In order to prove Theorem 1.3 and Theorem 1.4, we will need three lemmas which are of independent interest.

**Lemma 2.1** (Partition of unity for valuations, [7], Proposition 6.2.1). *Let  $M = \bigcup_i U_i$  be a locally finite open cover of  $M$ . Then there exist valuations  $\mu_i \in \mathcal{V}^\infty(M)$  such that  $\text{spt } \mu_i \subset U_i$  and*

$$\sum_i \mu_i = \chi.$$

*Proof.* Let  $1 = \sum_i f_i$  a partition of unity subordinate to  $M = \bigcup_i U_i$ . We represent  $\chi$  by  $(\omega, \phi)$  and let  $\mu_i$  be the valuation represented by  $(\pi^* f_i \wedge \omega, f_i \phi)$ .  $\square$

By inspecting the proof of Theorem 1.2 (which uses a local variational argument), one gets the following lemma.

**Lemma 2.2.** *Let  $\omega \in \Omega^{n-1}(S^*M)$ ,  $\phi \in \Omega^n(M)$  and define the smooth valuation  $\mu$  by (1). Then*

$$\text{spt } D\omega + \pi^*\phi \subset \pi^{-1}(\text{spt } \mu) \quad \text{and} \quad \text{spt } \pi_*\omega \subset \text{spt } \mu.$$

**Lemma 2.3.** *Let  $\mu \in \mathcal{V}^\infty(M)$  be compactly supported. Then  $\mu$  can be represented by a pair  $(\omega, \phi) \in \Omega^{n-1}(S^*M) \times \Omega^n(M)$  of compactly supported forms.*

*Proof.* We suppose  $M$  is non-compact (otherwise the statement is trivial). Let  $\mu$  be represented by a pair  $(\omega', \phi')$ . Then  $D\omega' + \pi^*\phi'$  is compactly supported. Since  $H_c^n(S^*M) = \mathbb{R}$ , there exists a compactly supported form  $\phi \in \Omega_c^n(M)$  such that

$$[D\omega' + \pi^*\phi'] = [\pi^*\phi] \in H_c^n(S^*M).$$

In other words, there is a compactly supported form  $\omega \in H_c^{n-1}(S^*M)$  such that  $d\omega = D\omega = D\omega' + \pi^*\phi' - \pi^*\phi$ . By Theorem 1.2, the pair  $(\omega, \phi)$  represents  $\mu$  up to a multiple of  $\chi$ . Since the valuation represented by  $(\omega, \phi)$  and the valuation  $\mu$  are both compactly supported, whereas  $\chi$  is not, they have to be the same.  $\square$

*Proof of Theorem 1.3.* Note first that the right hand side of (3) is well defined: since  $\mu_2$  is compactly supported, the same holds true for  $D\omega_2 + \pi^*\phi_2$  and  $\pi_*\omega_2$  by Lemma 2.2.

Next, both sides of (3) are linear in  $\mu_1$  and  $\mu_2$ . Using Lemma 2.1, we may therefore assume that the supports of  $\mu_1$  and  $\mu_2$  are contained in the support of a coordinate chart. Since the Alesker–Fu product, the Euler–Verdier involution and the integration functional are natural with respect to diffeomorphisms, it suffices to prove (3) in the case where  $M = V$  is a real vector space of dimension  $n$ .

Let us first suppose that  $\mu_1$  and  $\sigma\mu_2$  are of the type (2). We thus have  $\mu_1(K) = m_1(K + A_1)$  and  $\sigma\mu_2(K) = m_2(K + A_2)$  with smooth measures  $m_1, m_2$  and smooth convex bodies  $A_1, A_2$  with strictly convex boundary.

The left hand side of (3) is given by

$$\begin{aligned}
 \int \mu_1 \cdot \sigma \mu_2 &= m_1 \times m_2(\Delta(V) + A_1 \times A_2) \\
 &= \int_V m_1((\Delta V + A_1 \times A_2) \cap V \times \{x\}) dm_2(x) \\
 &= \int_V m_1(x - A_2 + A_1) dm_2(x) \tag{10} \\
 &= \int_V \mu_1(x - A_2) dm_2(x) \\
 &= \int_V \text{cnc}(x - A_2)(\omega_1) dm_2(x) + \int_V \left( \int_{x-A_2} \phi_1 \right) dm_2(x).
 \end{aligned}$$

Let  $A \in \mathcal{K}(V)$  be smooth with strictly convex boundary. Its support function is defined by

$$\begin{aligned}
 h_A: V^* &\rightarrow \mathbb{R}, \\
 \xi &\mapsto \sup_{x \in A} \xi(x).
 \end{aligned}$$

Note that  $h_A$  is homogeneous of degree 1 and that  $h_{-A}(\xi) = h_A(-\xi)$ .

Define the map  $G_A: S^*V \rightarrow S^*V, (x, [\xi]) \mapsto (x + d_\xi h_A, [\xi])$  (since  $h_A$  is homogeneous of degree 1,  $d_\xi h_A \in V^{**} = V$  only depends on  $[\xi]$ ).  $G_A$  is an orientation preserving diffeomorphism of  $S^*V$ .

It is easy to show ([10], [12]) that for  $X \in \mathcal{K}(V)$

$$\text{cnc}(X + A) = (G_A)_* \text{cnc}(X). \tag{11}$$

We next compute that for all  $(x, [\xi]) \in S^*V$

$$\begin{aligned}
 G_A \circ s(x, [\xi]) &= G_A(x, [-\xi]) = (x + d_{-\xi} h_A, [-\xi]) = (x - d_\xi h_{-A}, [-\xi]) \\
 &= s(x - d_\xi h_{-A}, [\xi]) = s \circ G_{-A}^{-1}(x, [\xi]).
 \end{aligned} \tag{12}$$

Let  $\kappa_2 \in \Omega^n(V)$  be the form representing the measure  $m_2$ . The first term in (10) is equal to

$$\begin{aligned}
 \int_V \text{cnc}(x - A_2)(\omega_1) dm_2(x) &= \int_V \text{cnc}(\{x\})(G_{-A_2}^* \omega_1) dm_2(x) \\
 &= \int_V \pi_*(G_{-A_2}^* \omega_1) \wedge \kappa_2 \\
 &= \int_{S^*V} G_{-A_2}^* \omega_1 \wedge \pi^* \kappa_2 \\
 &= \int_{S^*V} \omega_1 \wedge (G_{-A_2}^{-1})^* \pi^* \kappa_2.
 \end{aligned} \tag{13}$$

By (11) we have  $(-1)^n Ds^* \omega_2 + (-1)^n \pi^* \phi_2 = G_A^* \pi^* \kappa_2$ . Applying  $s^*$  to both sides and using (12), we get

$$(-1)^n (D\omega_2 + \pi^* \phi_2) = s^* G_{A_2}^* \pi^* \kappa_2 = (G_{-A_2}^{-1})^* \pi^* \kappa_2.$$

Hence (13) equals  $(-1)^n \int_{S^*V} \omega_1 \wedge (D\omega_2 + \pi^* \phi_2)$ , which is the first term in (3).

By Fubini's theorem, the second term in (10) equals

$$\int_V \left( \int_{x-A_2} \phi_1 \right) dm_2(x) = \int_V m_2(y + A_2) \phi_1(y) = \int_V \sigma \mu_2(\{y\}) \phi_1(y).$$

For  $y \in V$ , we have  $s_* \text{cnc}(\{y\}) = (-1)^n \text{cnc}(\{y\})$ , since the antipodal map on  $S^{n-1}$  is orientation preserving precisely if  $n$  is even. Hence

$$\sigma \mu_2(\{y\}) = \pi_* \omega_2(y).$$

The second term in (10) thus equals  $\int_V \phi_1 \wedge \pi_* \omega_2$ , which corresponds to the second term in (3).

This finishes the proof in the case where  $\mu_1$  and  $\sigma \mu_2$  are of type (2). By linearity of both sides, (3) holds true for linear combinations of such valuations. Given arbitrary  $\mu_1 \in \mathcal{V}^\infty(M)$  and  $\mu_2 \in \mathcal{V}_c^\infty(M)$ , we find sequences  $\mu_1^j \in \mathcal{V}^\infty(M)$  and  $\mu_2^j \in \mathcal{V}_c^\infty(M)$  such that  $\mu_1^j \rightarrow \mu_1$  and  $\mu_2^j \rightarrow \mu_2$  and such that  $\mu_1^j$  and  $\sigma \mu_2^j$  are linear combinations of valuations of type (2) (compare [5] and [6]).

By definition of the topology on  $\mathcal{V}^\infty(M)$  (see Section 3.2 of [6]) and the open mapping theorem, there are sequences  $(\omega_1^j, \phi_1^j)$  and  $(\omega_2^j, \phi_2^j)$  representing  $\mu_1^j, \mu_2^j$  and converging to  $(\omega_1, \phi_1), (\omega_2, \phi_2)$  in the  $C^\infty$ -topology. By what we have proved,

$$\int \mu_1^j \cdot \sigma \mu_2^j = (-1)^n \int_{S^*M} \omega_1^j \wedge (D\omega_2^j + \pi^* \phi_2^j) + \int_M \phi_1^j \wedge \pi_* \omega_2^j$$

for all  $j$ . Letting  $j$  tend to infinity, Equation (3) follows.  $\square$

### 3. Self-adjointness of natural operators

*Proof of Theorem 1.5.* Note first the following equation:

$$\langle \sigma \mu_1, \mu_2 \rangle = (-1)^n \langle \mu_1, \sigma \mu_2 \rangle. \quad (14)$$

This equation is immediate from (5) and the fact that  $s: S^*M \rightarrow S^*M$  preserves orientation if and only if  $n$  is even.

Let  $\mu_i$  be represented by  $(\omega_i, \phi_i)$ . By Lemma 2.3 we may suppose that  $\omega_2$  and  $\phi_2$  are compactly supported.

$\Lambda\mu_i$  is represented by  $\xi_i := i_T(D\omega_i + \pi^*\phi_i)$ . Since  $D\omega_i + \pi^*\phi_i = \alpha \wedge \xi_i$ , we get

$$\begin{aligned} \langle \Lambda\mu_1, \sigma\mu_2 \rangle &= (-1)^n \int_{S^*M} \xi_1 \wedge (D\omega_2 + \pi^*\phi_2) \\ &= (-1)^n \int_{S^*M} \xi_1 \wedge \alpha \wedge \xi_2 \\ &= - \int_{S^*M} \xi_2 \wedge (D\omega_1 + \pi^*\phi_1) \\ &= (-1)^{n+1} \int_{S^*M} \omega_1 \wedge D\xi_2 - \int_M \phi_1 \wedge \pi_*\xi_2 \\ &= -\langle \mu_1, \sigma\Lambda\mu_2 \rangle. \end{aligned} \tag{15}$$

Since  $D$  and  $s^*$  commute and since  $i_T \circ s^* = -s^* \circ i_T$ , it is easily checked that

$$\Lambda \circ \sigma = -\sigma \circ \Lambda. \tag{16}$$

Now (6) follows from (15) and (16).

Let us next prove (7) ((8) is an immediate consequence).

By Lemma 2.3 we may suppose that  $\omega_2$  and  $\phi_2$  have compact support. Then

$$\begin{aligned} \langle \mu_1, \sigma\mathcal{S}\mu_2 \rangle &= (-1)^n \int_{S^*M} \omega_1 \wedge D * (D\omega_2 + \pi^*\phi_2) \\ &\quad + \int_M \phi_1 \wedge \pi_* * (D\omega_2 + \pi^*\phi_2) \\ &= \int_{S^*M} (D\omega_1 + \pi^*\phi_1) \wedge * (D\omega_2 + \pi^*\phi_2) \\ &= \int_{S^*M} * (D\omega_1 + \pi^*\phi_1) \wedge (D\omega_2 + \pi^*\phi_2) \\ &= \int_{S^*M} (-1)^n s^* * (D\omega_1 + \pi^*\phi_1) \wedge s^* (D\omega_2 + \pi^*\phi_2) \\ &= \langle \sigma\mathcal{S}\mu_1, \mu_2 \rangle. \end{aligned} \tag{17}$$

Since  $s$  changes the orientation of  $S^*M$  by  $(-1)^n$ , we get  $s^* \circ * = (-1)^n * \circ s^*$  on  $\Omega^*(S^*V)$ . It follows that  $\sigma \circ \mathcal{S} = (-1)^n \mathcal{S} \circ \sigma$ . Therefore (7) follows from (14) and (17).  $\square$

Alesker defined the space  $\mathcal{V}^{-\infty}(M)$  of *generalized valuations* on  $M$  by

$$\mathcal{V}^{-\infty}(M) := (\mathcal{V}_c^\infty(M))^*,$$

where the star means the topological dual. This space is endowed with the weak topology. By the perfectness of the Alesker–Poincaré pairing, there is a natural dense embedding  $\mathcal{V}^\infty(M) \hookrightarrow \mathcal{V}^{-\infty}(M)$ .

**Corollary 3.1.** *Let  $M$  be a Riemannian manifold. Each of the operators  $\Lambda$ ,  $\mathcal{S}$ ,  $\Delta$  acting on  $\mathcal{V}^\infty(M)$  admits a unique continuous extension to  $\mathcal{V}^{-\infty}(M)$ .*

*Proof.* Uniqueness of the extension is clear, since  $\mathcal{V}^\infty(M)$  is dense in  $\mathcal{V}^{-\infty}(M)$ . We let  $\Lambda$  act on  $\mathcal{V}^{-\infty}$  by  $\Lambda\xi(\mu) := \xi(\Lambda\mu)$ . By Theorem 1.5, this is consistent with the embedding of  $\mathcal{V}^\infty(M)$  into  $\mathcal{V}^{-\infty}(M)$  and we are done. The cases of  $\mathcal{S}$  and  $\Delta$  are similar.  $\square$

#### 4. The translation invariant case

From now on,  $V$  will denote an oriented  $n$ -dimensional real vector space. We will consider valuations on the space  $\mathcal{K}(V)$  of compact convex sets (i.e. convex valuations).

A convex valuation  $\mu$  on  $V$  is called translation invariant, if  $\mu(x + K) = \mu(K)$  for all  $K \in \mathcal{K}(V)$  and all  $x \in V$ .

A translation invariant convex valuation  $\mu$  is said to be of degree  $k$  if  $\mu(tK) = t^k \mu(K)$  for  $t > 0$  and  $K \in \mathcal{K}(V)$ . By  $\text{Val}_k(V)$  we denote the space of translation invariant convex valuations of degree  $k$ . A valuation  $\mu$  is even if  $\mu(-K) = \mu(K)$  and odd if  $\mu(-K) = -\mu(K)$ , the corresponding spaces will be denoted by a superscript  $+$  or  $-$ .

In [16] it is shown that the space of translation invariant valuations can be written as a direct sum

$$\text{Val}(V) = \bigoplus_{k=0}^n \text{Val}_k(V).$$

Each space  $\text{Val}_k(V)$  splits further as  $\text{Val}_k(V) = \text{Val}_k^+(V) \oplus \text{Val}_k^-(V)$ .

The spaces  $\text{Val}_0(V)$  and  $\text{Val}_n(V)$  are both 1-dimensional (generated by  $\chi$  and a Lebesgue measure respectively). For  $\mu \in \text{Val}(V)$ , we denote by  $\mu_n$  its component of degree  $n$ .

Let us prove the following version of Theorem 1.3 in the translation invariant case.

**Theorem 4.1.** *Let  $\mu_1, \mu_2 \in \text{Val}^{\text{sm}}(V)$  be represented by translation invariant forms  $(\omega_1, \phi_1)$ ,  $(\omega_2, \phi_2)$  respectively. Then  $(\mu_1 \cdot \sigma\mu_2)_n$  is represented by the  $n$ -form*

$$(-1)^n \pi_*(\omega_1 \wedge (D\omega_2 + \pi^*\phi_2)) + \phi_1 \wedge \pi_*\omega_2 \in \Omega^n(V).$$

*Proof.* The proof is similar to that of Theorem 1.3. Fix a Euclidean metric on  $V$ . For  $R > 0$ , let  $B_R$  denote the ball of radius  $R$ , centered at the origin. Let us suppose that

$\mu_1(K) = \text{vol}(K + A_1)$  and  $\sigma\mu_2(K) = \text{vol}(K + A_2)$  for all  $K \in \mathcal{K}(V)$ . Then

$$\begin{aligned} \mu_1 \cdot \sigma\mu_2(B_R) &= \text{vol}_{2n}(\Delta(B_R) + A_1 \times A_2) \\ &= \int_{B_R} \text{vol}(x - A_2 + A_1) dx + o(R^n) \\ &= \int_{B_R} \mu_1(x - A_2) + o(R^n) \\ &= \int_{B_R} \text{cnc}(x - A_2)(\omega_1) dx + \int_{B_R} \int_{x-A_2} \phi_1 dx + o(R^n). \end{aligned}$$

The first term is given by

$$\begin{aligned} \int_{B_R} \text{cnc}(x - A_2)(\omega_1) dx &= \int_{B_R} \pi_*(G_{-A_2}^* \omega_1) dx + o(R^n) \\ &= \int_{B_R \times S^*(V)} G_{-A_2}^* \omega_1 \wedge \pi^*(dx) + o(R^n) \\ &= \int_{G_{-A_2}(B_R \times S^*(V))} \omega_1 \wedge (G_{-A_2}^{-1})^* \pi^* dx_2 + o(R^n) \\ &= (-1)^n \int_{B_R \times S^*(V)} \omega_1 \wedge (D\omega_2 + \pi^* \phi_2) + o(R^n) \\ &= (-1)^n \int_{B_R} \pi_*(\omega_1 \wedge (D\omega_2 + \pi^* \phi_2)) + o(R^n). \end{aligned}$$

The second term yields

$$\begin{aligned} \int_{B_R} \int_{x-A_2} \phi_1 dx &= \int_V \text{vol}((y + A_2) \cap B_R) \phi_1(y) \\ &= \int_{B_R} \text{vol}(y + A_2) \phi_1(y) + o(R^n) \\ &= \int_{B_R} \mu_2(\{y\}) \phi_1(y) + o(R^n) \\ &= \int_{B_R} \phi_1 \wedge \pi_* \omega_2 + o(R^n). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} (\mu_1 \cdot \sigma\mu_2)_n &= \lim_{R \rightarrow \infty} \frac{1}{R^n} \mu_1 \cdot \sigma\mu_2(B_R) \\ &= \lim_{R \rightarrow \infty} \frac{1}{R^n} \int_{B_R} (-1)^n \pi_*(\omega_1 \wedge (D\omega_2 + \pi^* \phi_2)) + \phi_1 \wedge \pi_* \omega_2. \end{aligned}$$

This finishes the proof of Theorem 4.1 in the case where  $\mu_1$  and  $\sigma\mu_2$  are of type  $K \mapsto \text{vol}(K+A)$ . Using linearity of both sides, it also holds for linear combinations of such valuations. Since they are dense in  $\text{Val}(V)$  (by Alesker's solution of McMullen's conjecture [1]), Theorem 4.1 is true in general.  $\square$

Let us next suppose that  $V$  is endowed with a Euclidean product. We can identify  $\text{Val}_n(V)$  with  $\mathbb{R}$  by sending  $\text{vol}$  to 1. We get a symmetric bilinear form (called Alesker pairing)

$$\begin{aligned} \text{Val}^{\text{sm}}(V) \times \text{Val}^{\text{sm}}(V) &\rightarrow \mathbb{R}, \\ (\mu_1, \mu_2) &\mapsto \langle \mu_1, \mu_2 \rangle := (\mu_1 \cdot \mu_2)_n. \end{aligned}$$

**Corollary 4.2.** *For valuations  $\mu_1, \mu_2 \in \text{Val}^{\text{sm}}(V)$ , the following equations hold:*

$$\begin{aligned} \langle \Lambda \mu_1, \mu_2 \rangle &= \langle \mu_1, \Lambda \mu_2 \rangle, \\ \langle \mathcal{S} \mu_1, \mu_2 \rangle &= \langle \mu_1, \mathcal{S} \mu_2 \rangle, \\ \langle \Delta \mu_1, \mu_2 \rangle &= \langle \mu_1, \Delta \mu_2 \rangle. \end{aligned}$$

*Proof.* Analogous to the proof of Theorem 1.5.  $\square$

## 5. Kinematic formulas and Poincaré formulas

**5.1. Kinematic formulas.** In this section, we suppose that  $G$  is a subgroup of  $O(V)$  acting transitively on the unit sphere. By a result of Alesker [3], the space of translation invariant and  $G$ -invariant valuations  $\text{Val}^G$  is a finite-dimensional vector space.

Let  $\phi_1, \dots, \phi_N$  a basis of  $\text{Val}^G$ . Suppose we have a kinematic formula

$$\int_{\bar{G}} \chi(K \cap \bar{g}L) d\bar{g} = \sum_{i,j=1}^N c_{i,j} \phi_i(K) \phi_j(L).$$

Here and in the following,  $G$  is endowed with its Haar measure and  $\bar{G} := G \times V$  with the product measure.

Set

$$k_G(\chi) := \sum_{i,j=1}^N c_{i,j} \phi_i \otimes \phi_j \in \text{Val}^G \otimes \text{Val}^G = \mathbf{Hom}(\text{Val}^G, \text{Val}^{G*}).$$

The Alesker pairing induces a bijective map

$$\mathbf{PD} \in \mathbf{Hom}(\text{Val}^G, \text{Val}^{G*}).$$

Fu [13] showed that these two maps are inverse to each other:

$$k_G(\chi) = \mathbf{PD}^{-1}. \tag{18}$$

For further use, we give another interpretation of (18). Let  $G$  be as above. The scalar product on the finite-dimensional space  $\text{Val}^G$  induces a scalar product on  $\text{Val}^{G*}$  such that  $\mathbf{PD}$  is an isometry.

Given  $K \in \mathcal{K}(V)$ , let  $\mu_K \in \text{Val}^{G*}$  be defined by

$$\mu_K(\mu) = \mu(K), \quad \mu \in \text{Val}^G.$$

**Proposition 5.1** (Principal kinematic formula). *Let  $G$  be a subgroup of  $O(V)$  acting transitively on the unit sphere. Then for  $K, L \in \mathcal{K}(V)$*

$$\int_{\bar{G}} \chi(K \cap \bar{g}L) d\bar{g} = \langle \mu_K, \mu_L \rangle.$$

*Proof.* Let  $\phi_1, \dots, \phi_N$  be a basis of  $\text{Val}^G$ . Set  $g_{ij} := \langle \phi_i, \phi_j \rangle$ ,  $i, j = 1, \dots, N$ . Let us denote by  $(g^{ij})_{i,j=1,\dots,N}$  the inverse matrix. Then

$$\int_{\bar{G}} \chi(K \cap \bar{g}L) d\bar{g} = \sum_{i,j} g^{ij} \phi_i(K) \phi_j(L) = \sum_{i,j} g^{ij} \mu_K(\phi_i) \mu_L(\phi_j) = \langle \mu_K, \mu_L \rangle. \quad \square$$

**5.2. Klain functions.** Let us suppose additionally that  $-1 \in G$ , which implies that  $\text{Val}^G \subset \text{Val}^+$ .

For  $0 \leq k \leq n$ , the action of  $G$  on  $V$  induces an action on the Grassmannian  $\text{Gr}_k(V)$ . We set  $\mathcal{P}_k := \text{Gr}_k(V)/G$  for the quotient space. Given  $u \in \mathcal{P}_k$ , the space of  $k$ -planes contained in  $u$  admits a unique  $G$ -invariant probability measure and we define  $Z_u \in \text{Val}^G$  by

$$Z_u(K) := \int_{L \in u} \text{vol}(\pi_L K) dL, \quad K \in \mathcal{K}(V).$$

Recall that the Klain function of an even, translation invariant valuation  $\mu$  of degree  $k$  on a Euclidean vector space  $V$  is the function  $\text{Kl}_\mu: \text{Gr}_k(V) \rightarrow \mathbb{R}$  such that the restriction of  $\mu$  to  $L \in \text{Gr}_k(V)$  is given by  $\text{Kl}_\mu(L)$  times the Lebesgue measure. An even, translation invariant valuation is uniquely determined by its Klain function [14]. If  $M$  is a compact  $k$ -dimensional submanifold (possibly with boundaries or corners), then

$$\mu(M) = \int_M \text{Kl}_\mu(T_p M) dp.$$

Alesker proved the existence of a duality operator (or Fourier transform)  $\mathbb{F}$  on  $\text{Val}^{+,sm}$  such that  $\text{Kl}_{\mathbb{F}\mu} = \text{Kl}_\mu \circ \perp$  for all  $\mu \in \text{Val}^{+,sm}$ .  $\mathbb{F}$  is formally self-adjoint with respect to the Alesker pairing.

**Proposition 5.2.** *Let  $u, v \in \mathcal{P}_k$  and  $L \in v$ . Then*

$$\text{Kl}_{Z_u}(L) = \langle \mathbb{F} Z_u, Z_v \rangle. \quad (19)$$

*Proof.* Immediate from Lemma 2.2. of [12].  $\square$

**Lemma 5.3.** *There are finitely many elements  $u_1, \dots, u_N$  such that  $Z_{u_i}, i = 1, \dots, N$  is a basis of  $\text{Val}_k^G$ .*

*Proof.* Let  $\phi_1, \dots, \phi_N$  be a basis of  $\text{Val}_k^G$ . Let  $m_i$  be the push-forward of a Crofton measure for  $\phi_i$  on  $\text{Gr}_k(V)$  under the projection  $\text{Gr}_k(V) \rightarrow \mathcal{P}_k$ .

By  $G$ -invariance of  $\phi_i$ , we get

$$\phi_i(K) = \int_{\mathcal{P}_k} \int_{L \in u} \text{vol}(\pi_L K) dL dm_i(u).$$

Now, for sufficiently close approximations of the  $m_i$  by discrete measures  $\sum_{j=1}^{k_i} c_{i,j} \delta_{u_{i,j}}$  with  $u_{i,j} \in \mathcal{P}_k, c_{i,j} \in \mathbb{R}$ , the valuations  $\sum_j c_{i,j} Z_{u_{i,j}}$  form a basis of  $\text{Val}_k^G$ . Hence  $\{Z_{u_{i,j}}, i = 1, \dots, N, j = 1, \dots, k_i\}$  is a finite generating set of  $\text{Val}_k^G$ , from which we can extract a finite basis.  $\square$

**5.3. Poincaré formulas.** Poincaré formulas for  $G$  are special cases of the principal kinematic formula for  $G$ , when  $K$  and  $L$  are replaced by smooth compact submanifolds  $M_1$  and  $M_2$  (possibly with boundary) of complementary dimension (note that  $M_1, M_2 \in \mathcal{P}(V)$ , so there is no problem in evaluating a valuation in  $M_1$  and  $M_2$ ). Then the right hand side of the principal kinematic formula is the “average number” of intersections of  $M_1$  and  $\bar{g}M_2$ .

**Proposition 5.4** (General Poincaré formula). *Let  $M_1, M_2$  be smooth compact submanifolds, possibly with boundaries, of complementary dimensions  $k$  and  $n - k$ . Then*

$$\int_{\bar{G}} \#(M_1 \cap \bar{g}M_2) d\bar{g} = \int_{M_1 \times M_2} \alpha(T_p M_1, T_q M_2) dpdq$$

with

$$\begin{aligned} \alpha: \mathcal{P}_k \times \mathcal{P}_{n-k} &\rightarrow \mathbb{R}, \\ (u, v) &\mapsto \langle Z_u, Z_v \rangle. \end{aligned}$$

*Proof.* Let  $u_1, \dots, u_N$  be such that  $Z_{u_i}, i = 1, \dots, N$  is a basis of  $\text{Val}_k^G$ . Let  $v_1, \dots, v_N$  be such that  $Z_{v_j}, j = 1, \dots, N$  is a basis of  $\text{Val}_{n-k}^G$  (note that the dimensions of these two spaces agree by Theorem 1.2.2 in [2]). Setting  $g_{ij} := \langle Z_{u_i}, Z_{v_j} \rangle$

and  $(g^{ij})$  for the inverse matrix, the principal kinematic formula implies that for all  $M_1$  and  $M_2$  as above

$$\begin{aligned} \int_{\bar{G}} \#(M_1 \cap \bar{g}M_2) d\bar{g} &= \sum_{i,j} g^{ij} Z_{u_i}(M_1) Z_{v_j}(M_2) \\ &= \int_{M_1 \times M_2} \sum_{i,j} g^{i,j} \text{Kl}_{Z_{u_i}}(T_p M_1) \text{Kl}_{Z_{v_j}}(T_q M_2) dpdq. \end{aligned}$$

This shows that

$$\alpha(u, v) = \sum_{i,j} g^{ij} \text{Kl}_{Z_{u_i}}(u) \text{Kl}_{Z_{v_j}}(v) = \langle Z_u, Z_v \rangle;$$

where the last equation follows from (19) and the self-adjointness of  $\mathbb{F}$ . □

## 6. Kinematic formulas for $SU(2)$

We apply the results of the preceding section to the special case  $G = SU(2)$  acting on the quaternionic line

$$\mathbb{H} = \{x_1 + x_2i + x_3j + x_4k : (x_1, x_2, x_3, x_4) \in \mathbb{R}^4\}.$$

Since this action is transitive on the unit sphere,  $\text{Val}^{SU(2)}$  is finite dimensional and  $\text{Val}_k^{SU(2)}$  is one-dimensional except for the case  $k = 2$ . The quotient space  $\mathcal{P}_2 := \text{Gr}_2(\mathbb{H})/SU(2)$  is the two-dimensional projective space  $\mathbb{R}\mathbb{P}^2 = S^2/\{\pm 1\}$  ([19]). Following Tasaki, we denote by  $(\omega_1(L) : \omega_2(L) : \omega_3(L)) \in \mathbb{R}\mathbb{P}^2$  the class of  $L \in \text{Gr}_2(\mathbb{H})$ .

A canonical representative in the preimage of  $(a : b : c) \in \mathbb{R}\mathbb{P}^2$  is given by the 2-plane spanned by 1 and  $ai + bj + ck$ .

If  $u = (a, b, c) \in S^2$ , then the planes in  $u$  are the complex lines for the complex structure  $I_u$  which is defined by multiplication by  $u$  from the right on  $\mathbb{H}$ . We will therefore write  $\mathbb{C}\mathbb{P}_u$  instead of  $u$ . Note that  $\mathbb{C}\mathbb{P}_u = \mathbb{C}\mathbb{P}_{-u}$ .

The following  $SU(2)$ -invariant and translation invariant valuations of degree 2 were introduced by Alesker [3].

**Definition 6.1.** Given  $u \in \mathbb{R}\mathbb{P}^2$ , set

$$Z_u(K) := \int_{\mathbb{C}\mathbb{P}_u} \text{vol}(\pi_L(K)) dL, \quad K \in \mathcal{K}(\mathbb{H}).$$

*Proof of Theorem 1.6.* From Lemma 5.3 we infer that there is a finite number of points  $u_1, \dots, u_N \in \mathbb{R}\mathbb{P}^2$  such that  $Z_{u_i}, i = 1, \dots, N$  form a basis of  $\text{Val}_2^{\text{SU}(2)}$ . Alesker showed that  $N = 6$  and that  $Z_i, Z_j, Z_k, Z_{\frac{i+j}{\sqrt{2}}}, Z_{\frac{i+k}{\sqrt{2}}}, Z_{\frac{j+k}{\sqrt{2}}}$  is such a basis.

Our aim is to compute the product  $\langle Z_u, Z_v \rangle$  for  $u, v \in \mathbb{R}\mathbb{P}^2$ . We will achieve it by first expressing each  $Z_u$  as a smooth valuation represented by some 3-form  $\omega_u \in \Omega^3(S^*\mathbb{H})$  and then applying Theorem 4.1.

Since the metric induces a diffeomorphism between  $S^*\mathbb{H}$  and  $S\mathbb{H}$ , we may as well work with the latter space. The image of the conormal cycle of a compact convex set  $K$  under this diffeomorphism is the normal cycle  $\text{nc}(K)$ .

Let us introduce several differential forms on  $S\mathbb{H}$ , depending on the choice of the complex structure  $I_u$ . We follow the notation of [17].

Let  $\alpha, \beta, \gamma$  be 1-forms on  $S\mathbb{H}$  which, at a point  $(x, v) \in S\mathbb{H}$ , equal

$$\begin{aligned}\alpha(w) &= \langle v, d\pi(w) \rangle, & w \in T_{(x,v)}S\mathbb{H}, \\ \beta_u(w) &= \langle v, I_u d\pi(w) \rangle, & w \in T_{(x,v)}S\mathbb{H}, \\ \gamma_u(w) &= \langle v, I_u d\pi_2(w) \rangle, & w \in T_{(x,v)}S\mathbb{H}.\end{aligned}$$

Note that  $\alpha$  is the canonical 1-form (in particular independent of  $u$ ), whereas  $\beta_u$  and  $\gamma_u$  depend on  $u$ .

Let  $\Omega$  be the pull-back of the symplectic form on  $(\mathbb{H}, I_u)$  to  $S\mathbb{H}$ , i.e.

$$\Omega_u(w_1, w_2) := \langle d\pi(w_1), I_u d\pi(w_2) \rangle, \quad w_1, w_2 \in T_{(x,v)}S\mathbb{H}.$$

*Claim.*  $Z_u$  is represented by the 3-form

$$\omega_u := \frac{1}{8\pi} \beta_u \wedge d\beta_u + \frac{1}{4\pi} \gamma_u \wedge \Omega_u.$$

Since  $\omega_u$  is  $U(2)$ - and translation invariant and has bidegree  $(2, 1)$  (with respect to the product decomposition  $S\mathbb{H} = \mathbb{H} \times S(\mathbb{H})$ ), it represents some  $U(2)$ -invariant, translation invariant valuation  $\mu_u$  of degree 2. Here  $U(2)$  is the unitary group for the complex structure  $I_u$ .

Now the space of valuations with these properties is of dimension 2 [2]. It is thus enough to show that the valuation  $Z_u$  and the valuation  $\mu_u$  agree on the unit ball  $B$  as well as on a complex disk  $D_u$ .

It is clear that  $Z_u(B) = \omega_2 = \pi$ . It was shown by Fu (compare Equation (37) in [13],) that  $Z_u(D_u) = \frac{\pi}{2}$ .

By [11], the derivation of a smooth translation invariant valuation  $\mu$  on a finite-dimensional Euclidean vector space is given by

$$\Lambda\mu(K) = \frac{d}{dt} \Big|_{t=0} \mu(K + tB).$$

It follows that, if  $\mu$  is of degree  $k$ , then  $\Lambda\mu(B) = k\mu(B)$ .

It is easily checked that  $\mathcal{L}_T\beta = \gamma$ ,  $\mathcal{L}_T\gamma = 0$  and  $\mathcal{L}_T^2\Omega = d\gamma$ , so that

$$\mathcal{L}_T^2\omega_u = \frac{1}{2\pi}\gamma \wedge d\gamma.$$

Note that  $\gamma \wedge d\gamma$  is twice the volume form on  $S^3$ , hence  $\Lambda^2\mu_u = 2\pi\chi$ . It follows that

$$\mu_u(B) = \frac{1}{2}\Lambda^2\mu_u(B) = \pi.$$

The restriction of  $\beta_u$  to the normal cycle of the complex disc  $D_u$  clearly vanishes.  $\gamma_u$  is the length element of the fibers of  $\pi: \text{nc}(D_u) \rightarrow D_u$  (which are circles), whereas  $\Omega_u$  is the (pull-back of) the volume form on  $D_u$ . It follows that  $\omega_u(D_u) = \frac{\pi}{2}$ . The claim is proved.

Next, the Rumin operator is easily computed as

$$D\omega_u = d\left(\omega_u + \frac{1}{8\pi}\alpha \wedge \beta_u \wedge \gamma_u - \frac{1}{8\pi}\alpha \wedge \Omega_u\right) = \frac{1}{2\pi}\alpha \wedge \beta_u \wedge d\gamma_u.$$

From Theorem 4.1 we infer that  $\mu_u \cdot \mu_v$  is represented by the 4-form

$$\frac{1}{16\pi^2}\pi_*((\beta_u \wedge d\beta_u + 2\gamma_u \wedge \Omega_u) \wedge \alpha \wedge \beta_v \wedge d\gamma_v) \in \Omega^4(\mathbb{H}).$$

If  $u = (a : b : c)$  and  $v = (\tilde{a} : \tilde{b} : \tilde{c})$ , then

$$\begin{aligned} &(\beta_u \wedge d\beta_u + 2\gamma_u \wedge \Omega_u) \wedge \alpha \wedge \beta_v \wedge d\gamma_v \\ &= 2((a\tilde{b} - \tilde{a}b)^2 + (a\tilde{c} - \tilde{a}c)^2 + (b\tilde{c} - \tilde{b}c)^2 + 2(a\tilde{a} + b\tilde{b} + c\tilde{c})^2)d \text{vol}_{S\mathbb{H}} \\ &= 2(1 + (a\tilde{a} + b\tilde{b} + c\tilde{c})^2)d \text{vol}_{S\mathbb{H}}. \end{aligned}$$

It follows that

$$\langle Z_u, Z_v \rangle = \frac{1}{4}(1 + (a\tilde{a} + b\tilde{b} + c\tilde{c})^2). \tag{20}$$

Let  $\pm u_i, i = 1, \dots, 6$  be the 12 vertices of an icosahedron  $I$  on  $S^2$ . They induce 6 valuations  $Z_{u_i}, i = 1, \dots, 6$ . Since the edge length  $a$  of  $I$  satisfies  $\cos a = \frac{\sqrt{5}}{5}$ , (20) implies that

$$\langle Z_{u_i}, Z_{u_j} \rangle = \begin{cases} \frac{1}{2}, & i = j, \\ \frac{3}{10}, & i \neq j. \end{cases} \tag{21}$$

Theorem 1.6 follows easily from (21) and (18). □

The general Poincaré formula (Proposition 5.4) implies the following (corrected version of the) Poincaré formula on the quaternionic line.

**Corollary 6.2** (Poincaré formula for  $SU(2)$ , [19]). *Let  $M_1, M_2 \subset \mathbb{H}$  be compact smooth 2-dimensional submanifolds. Then*

$$\int_{\overline{SU(2)}} \#(M_1 \cap \bar{g}M_2) d\bar{g} = \frac{1}{4} \int_{M_1 \times M_2} (1 + A(T_p M_1, T_q M_2)) dpdq$$

with

$$\begin{aligned} A(T_p M_1, T_q M_2) = & (\omega_1(T_p M_1)\omega_1(T_q M_2) \\ & + \omega_2(T_p M_1)\omega_2(T_q M_2) + \omega_3(T_p M_1)\omega_3(T_q M_2))^2. \end{aligned}$$

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