

<b>Zeitschrift:</b>	Commentarii Mathematici Helvetici
<b>Herausgeber:</b>	Schweizerische Mathematische Gesellschaft
<b>Band:</b>	84 (2009)
<b>Artikel:</b>	A product formula for valuations on manifolds with applications to the integral geometry of the quaternionic line
<b>Autor:</b>	Bernig, Andreas
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-99108">https://doi.org/10.5169/seals-99108</a>

#### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 02.02.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# A product formula for valuations on manifolds with applications to the integral geometry of the quaternionic line

Andreas Bernig\*

**Abstract.** The Alesker–Poincaré pairing for smooth valuations on manifolds is expressed in terms of the Rumin differential operator acting on the cosphere-bundle. It is shown that the derivation operator, the signature operator and the Laplace operator acting on smooth valuations are formally self-adjoint with respect to this pairing. As an application, the product structure of the space of  $SU(2)$ - and translation invariant valuations on the quaternionic line is described. The principal kinematic formula on the quaternionic line  $\mathbb{H}$  is stated and proved.

**Mathematics Subject Classification (2000).** 53C65.

**Keywords.** Valuations on manifolds, kinematic formulas, integral geometry, Alesker–Poincaré pairing.

## 1. Smooth valuations on manifolds

Let  $M$  be a smooth manifold of dimension  $n$ . For simplicity, we suppose that  $M$  is oriented, although the whole theory works in the non-oriented case as well. Following Alesker, we set  $\mathcal{P}(M)$  to be the set of compact submanifolds with corners.

**Definition 1.1.** A valuation on  $M$  is a real valued map  $\mu$  on  $\mathcal{P}(M)$  which is additive in the following sense: whenever  $X, Y, X \cap Y$  and  $X \cup Y$  belong to  $\mathcal{P}(M)$ , then

$$\mu(X \cup Y) + \mu(X \cap Y) = \mu(X) + \mu(Y).$$

A set  $X \in \mathcal{P}(M)$  admits a conormal cycle  $\text{cnc}(X)$ , which is a compactly supported Legendrian cycle on the cosphere bundle  $S^*M$ . Sometimes it will be convenient to think of  $S^*M$  as the set of pairs  $(p, P)$  with  $p \in M$  and  $P \subset T_p M$  an oriented hyperplane, at other places it is better to think of it as the set of pairs  $(p, [\xi])$  where  $p \in M$  and  $\xi \in T_p^* M \setminus \{0\}$  and the brackets denote the equivalence class for the relation  $\xi_1 \sim \xi_2 \iff \xi_1 = \lambda \xi_2, \lambda > 0$ .

---

\*Supported by the Schweizerischer Nationalfonds grants PP002-114715/1 and SNF 200020-113199.

A valuation  $\mu$  on  $M$  is called *smooth* if there exist an  $(n-1)$ -form  $\omega \in \Omega^{n-1}(S^*M)$  and an  $n$ -form  $\phi \in \Omega^n(M)$  such that

$$\mu(X) = \text{cnc}(X)(\omega) + \int_X \phi, \quad X \in \mathcal{P}(X). \quad (1)$$

If  $\mu$  can be expressed in the form (1), we say that  $\mu$  is *represented* by  $(\omega, \phi)$ . The space of smooth valuations on  $M$  is denoted by  $\mathcal{V}^\infty(M)$ . It is a Fréchet space (see [6], Section 3.2 for the definition of the topology). If  $M = V$  is a vector space, the subspace of translation invariant smooth valuations will be denoted by  $\text{Val}^{\text{sm}}(V)$ .

Let  $N$  be another oriented  $n$ -dimensional smooth manifold and  $\rho: N \rightarrow M$  an orientation preserving immersion. Then  $\rho$  induces a map  $\tilde{\rho}: S^*N \rightarrow S^*M$ , sending  $(p, P)$  to  $(\rho(p), T_p\rho(P))$ . It clearly satisfies  $\pi \circ \tilde{\rho} = \rho \circ \pi$ .

The valuation  $\rho^*\mu$  on  $N$  such that

$$\rho^*\mu(X) := \mu(\rho(X)), \quad X \in \mathcal{P}(N)$$

is again smooth. If  $\mu$  is represented by  $(\omega, \phi)$ , then  $\rho^*\mu$  is represented by  $(\tilde{\rho}^*\omega, \rho^*\phi)$ . This follows from the fact that  $\text{cnc}(\rho(X)) = \tilde{\rho}_* \text{cnc}(X)$ . Note also that  $\tilde{\rho}^{-1} = (\tilde{\rho})^{-1}$  if  $\rho$  is a diffeomorphism.

We will use some results of [11], which we would like to recall. The cosphere bundle  $S^*M$  is a contact manifold of dimension  $2n - 1$  with a global contact form  $\alpha$  ( $\alpha$  is not unique, but this will play no role here). The projection from  $S^*M$  to  $M$  will be denoted by  $\pi$ , it induces a linear map  $\pi_*$  (fiber integration) on the level of forms.

Given an  $(n-1)$ -form  $\omega$  on  $S^*M$ , there exists a unique vertical form  $\alpha \wedge \xi$  such that  $d(\omega + \alpha \wedge \xi)$  is vertical (i.e. a multiple of  $\alpha$ ). The Rumin differential operator  $D$  is defined as  $D\omega := d(\omega + \alpha \wedge \xi)$  [18]. The following theorem was proved in [11].

**Theorem 1.2.** *Let  $\omega \in \Omega^{n-1}(S^*M)$ ,  $\phi \in \Omega^n(M)$  and define the smooth valuation  $\mu$  by (1). Then  $\mu = 0$  if and only if*

- (1)  $D\omega + \pi^*\phi = 0$ , and
- (2)  $\pi_*\omega = 0$  for all  $p \in M$ .

Moreover, if  $D\omega + \pi^*\phi = 0$ , then  $\mu$  is a multiple of the Euler characteristic  $\chi$ .

The support of a smooth valuation  $\mu$  is defined as

$$\text{spt } \mu := M \setminus \{p \in M : \text{there exists } p \in U \subset M \text{ open such that } \mu|_U = 0\}.$$

The subspace of  $\mathcal{V}^\infty(M)$  consisting of compactly supported valuations will be denoted by  $\mathcal{V}_c^\infty(M)$ .

Let  $\int: \mathcal{V}_c^\infty(M) \rightarrow \mathbb{R}$  denote the integration functional [7]. If  $\mu$  has compact support, then  $\int \mu := \mu(X)$ , where  $X \in \mathcal{P}(M)$  is an  $n$ -dimensional manifold with

boundary containing  $\text{spt } \mu$  in its interior. It is clear that, if  $\mu$  is represented by  $(\omega, \phi)$  with compact supports, then

$$\int \mu = \int_M \phi = [\phi] \in H_c^n(M) = \mathbb{R}.$$

Before stating our main theorem we have to recall two other constructions of Alesker.

The first one is the Euler–Verdier involution  $\sigma: \mathcal{V}^\infty(M) \rightarrow \mathcal{V}^\infty(M)$  [6]. Let  $s: S^*M \rightarrow S^*M$  be the natural involution on  $S^*M$ , sending  $(p, P)$  to  $(p, \bar{P})$ , where  $\bar{P}$  is the hyperplane  $P$  with the reversed orientation. If a valuation  $\mu \in \mathcal{V}^\infty(M)$  is represented by the pair  $(\omega, \phi)$ , then  $\sigma\mu$  is defined as the valuation which is represented by the pair  $((-1)^n s^* \omega, (-1)^n \phi)$ .

The second construction is the Alesker–Fu product [9], which is a bilinear map

$$\mathcal{V}^\infty(M) \times \mathcal{V}^\infty(M) \rightarrow \mathcal{V}^\infty(M), \quad (\mu_1, \mu_2) \mapsto \mu_1 \cdot \mu_2.$$

We refer to [9] for its construction. It is characterized by the following properties:

- (1) “.” is continuous and linear in both variables;
- (2) if  $\rho: N \rightarrow M$  is a diffeomorphism and  $\mu_1, \mu_2 \in \mathcal{V}^\infty(M)$ , then

$$\rho^*(\mu_1 \cdot \mu_2) = \rho^* \mu_1 \cdot \rho^* \mu_2;$$

- (3) if  $m_1, m_2$  are smooth measures on an  $n$ -dimensional vector space  $V$ ,  $A_1, A_2 \in \mathcal{K}(V)$  convex bodies with strictly convex smooth boundary and if  $\mu_i \in \mathcal{V}^\infty(V)$ ,  $i = 1, 2$  is defined by

$$\mu_i(K) = m_i(K + A_i), \quad K \in \mathcal{K}(V), \quad (2)$$

then

$$\mu_1 \cdot \mu_2(K) = m_1 \times m_2(\Delta(K) + A_1 \times A_2),$$

where  $\Delta: V \rightarrow V \times V$  is the diagonal embedding.

Our first main theorem is the following relation between Alesker–Fu product, integration functional, Euler–Verdier involution and Rumin differential.

**Theorem 1.3.** *Let  $\mu_1 \in \mathcal{V}^\infty(M)$  be represented by  $(\omega_1, \phi_1)$ ; let  $\mu_2 \in \mathcal{V}_c^\infty(M)$  be represented by  $(\omega_2, \phi_2)$ . Then*

$$\int \mu_1 \cdot \sigma \mu_2 = (-1)^n \int_{S^*M} \omega_1 \wedge (D\omega_2 + \pi^* \phi_2) + \int_M \phi_1 \wedge \pi_* \omega_2. \quad (3)$$

Let us call the pairing

$$\begin{aligned} \mathcal{V}^\infty(M) \times \mathcal{V}_c^\infty(M) &\rightarrow \mathbb{R}, \\ (\mu_1, \mu_2) &\mapsto \int \mu_1 \cdot \mu_2 =: \langle \mu_1, \mu_2 \rangle \end{aligned} \tag{4}$$

the *Alesker–Poincaré pairing*. Note that Theorem 1.3 is equivalent to

$$\langle \mu_1, \mu_2 \rangle = \int_{S^*M} \omega_1 \wedge s^*(D\omega_2 + \pi^*\phi_2) + \int_M \phi_1 \wedge \pi_*\omega_2. \tag{5}$$

From Theorem 1.3 and from the fact that the Poincaré pairings on  $M$  and  $S^*M$  are perfect, we get the following corollary (which was first proved by Alesker).

**Corollary 1.4** ([7], Theorem 6.1.1). *The Alesker–Poincaré pairing (4) is a perfect pairing.*

Some more operators on  $\mathcal{V}^\infty(M)$  were introduced in [11]. For this, we suppose that  $M$  is a Riemannian manifold. Then  $S^*M$  admits an induced metric, the *Sasaki metric* [20].

The first operator is the derivation operator  $\Lambda$  (which was denoted by  $\mathfrak{L}$  in [11]). The metric on  $S^*M$  provides a canonical choice of  $\alpha$ , namely  $\alpha|_{(p,[\xi])} := \frac{1}{\|\xi\|} \pi^*\xi$  for all  $(p, [\xi]) \in S^*M$ . Let  $T$  be the Reeb vector field on  $S^*M$  (i.e.  $\alpha(T) = 1$  and  $\mathcal{L}_T\alpha = 0$ ).

If the smooth valuation  $\mu$  is represented by  $(\omega, \phi)$ , then  $\Lambda\mu$  is by definition the valuation which is represented by  $(\mathcal{L}_T\omega + i_T\pi^*\phi, 0)$ .

Let us recall the definitions of the signature operator  $\mathcal{S}$  and the Laplace operator  $\Delta$ . Let  $*$  be the Hodge star acting on  $\Omega^*(S^*M)$ . Let  $\mu \in \mathcal{V}^\infty(M)$  be represented by  $(\omega, \phi)$ . Then  $\mathcal{S}\mu$  is defined as the valuation which is represented by  $(*(D\omega + \pi^*\phi), 0)$ .

The Laplace operator  $\Delta$  is defined as  $\Delta := (-1)^n \mathcal{S}^2$ .

Our second main theorem shows that these operators fit well into Alesker’s theory. In fact, they are formally self-adjoint with respect to the Alesker–Poincaré pairing.

**Theorem 1.5.** *For valuations  $\mu_1 \in \mathcal{V}^\infty(M)$  and  $\mu_2 \in \mathcal{V}_c^\infty(M)$ , the following equations hold:*

$$\langle \Lambda\mu_1, \mu_2 \rangle = \langle \mu_1, \Lambda\mu_2 \rangle, \tag{6}$$

$$\langle \mathcal{S}\mu_1, \mu_2 \rangle = \langle \mu_1, \mathcal{S}\mu_2 \rangle, \tag{7}$$

$$\langle \Delta\mu_1, \mu_2 \rangle = \langle \mu_1, \Delta\mu_2 \rangle. \tag{8}$$

We will apply these theorems in the study of the integral geometry of  $SU(2)$ . This group acts on the quaternionic line  $\mathbb{H}$ . In this setting, it is more natural to work with

the space  $\mathcal{K}(\mathbb{H})$  of convex sets instead of manifolds with corners. By Proposition 2.6. of [8], there is no loss of generality in doing so.

It was shown by Alesker [3] that the space of  $SU(2)$ -invariant and translation invariant valuations on the quaternionic line  $\mathbb{H}$  is of dimension 10. For each purely complex number  $u$  of norm 1, let  $I_u$  be the complex structure given by multiplication from the right with  $u$  and  $\mathbb{CP}_u^1$  the corresponding Grassmannian of complex lines (with its unique  $SU(2)$ -invariant Haar measure). Alesker defined a valuation  $Z_u$  by

$$Z_u(K) := \int_{\mathbb{CP}_u^1} \text{vol}(\pi_L(K)) dL, \quad K \in \mathcal{K}(\mathbb{H}). \quad (9)$$

He showed that  $Z_i, Z_j, Z_k, Z_{\frac{i+j}{\sqrt{2}}}, Z_{\frac{i+j}{\sqrt{2}}}, Z_{\frac{i+j}{\sqrt{2}}}$ , together with Euler characteristic  $\chi$ , the volume  $\text{vol}$  and the intrinsic volumes  $\text{vol}_1, \text{vol}_3$  form a basis of  $\text{Val}^{SU(2)}$ . Following a suggestion of Fu, we will state the kinematic formula using a more symmetric choice. Noting that  $Z_u = Z_{-u}$  for all  $u \in S^2$ , the 12 vertices  $\pm u_i, i = 1, \dots, 6$  of an icosahedron on  $S^2$  define 6 valuations  $Z_{u_i}, i = 1, \dots, 6$ .

We endow  $SU(2)$  with its Haar measure and the semidirect product  $\overline{SU(2)} = SU(2) \ltimes \mathbb{H}$  with the product measure. Let  $\text{vol}_k$  denote the  $k$ -dimensional intrinsic volume [15].

**Theorem 1.6** (Principal kinematic formula for  $SU(2)$ ). *Let  $K, L \in \mathcal{K}(\mathbb{H})$ . Then*

$$\begin{aligned} \int_{\overline{SU(2)}} \chi(K \cap \bar{g}L) d\bar{g} &= \chi(K) \text{vol}(L) + \frac{4}{3\pi} \text{vol}_1(K) \text{vol}_3(L) \\ &\quad + \frac{17}{4} \sum_{i=1}^6 Z_{u_i}(K) Z_{u_i}(L) - \frac{3}{4} \sum_{1 \leq i \neq j \leq 6} Z_{u_i}(K) Z_{u_j}(L) \\ &\quad + \frac{4}{3\pi} \text{vol}_3(K) \text{vol}_1(L) + \text{vol}(K) \chi(L). \end{aligned}$$

This theorem implies and generalizes the Poincaré formulas of Tasaki [19], (which contained an error in some constant) as we will explain in the last section.

**1.0.1. Acknowledgements.** I wish to thank Joseph Fu for illuminating discussions, in particular for the beautiful idea of using the vertices of an icosahedron as basis for  $\text{Val}_2^{SU(2)}$ .

## 2. The Alesker–Poincaré pairing in terms of forms

In order to prove Theorem 1.3 and Theorem 1.4, we will need three lemmas which are of independent interest.

**Lemma 2.1** (Partition of unity for valuations, [7], Proposition 6.2.1). *Let  $M = \bigcup_i U_i$  be a locally finite open cover of  $M$ . Then there exist valuations  $\mu_i \in \mathcal{V}^\infty(M)$  such that  $\text{spt } \mu_i \subset U_i$  and*

$$\sum_i \mu_i = \chi.$$

*Proof.* Let  $1 = \sum_i f_i$  a partition of unity subordinate to  $M = \bigcup_i U_i$ . We represent  $\chi$  by  $(\omega, \phi)$  and let  $\mu_i$  be the valuation represented by  $(\pi^* f_i \wedge \omega, f_i \phi)$ .  $\square$

By inspecting the proof of Theorem 1.2 (which uses a local variational argument), one gets the following lemma.

**Lemma 2.2.** *Let  $\omega \in \Omega^{n-1}(S^* M)$ ,  $\phi \in \Omega^n(M)$  and define the smooth valuation  $\mu$  by (1). Then*

$$\text{spt } D\omega + \pi^*\phi \subset \pi^{-1}(\text{spt } \mu) \quad \text{and} \quad \text{spt } \pi_*\omega \subset \text{spt } \mu.$$

**Lemma 2.3.** *Let  $\mu \in \mathcal{V}^\infty(M)$  be compactly supported. Then  $\mu$  can be represented by a pair  $(\omega, \phi) \in \Omega^{n-1}(S^* M) \times \Omega^n(M)$  of compactly supported forms.*

*Proof.* We suppose  $M$  is non-compact (otherwise the statement is trivial). Let  $\mu$  be represented by a pair  $(\omega', \phi')$ . Then  $D\omega' + \pi^*\phi'$  is compactly supported. Since  $H_c^n(S^* M) = \mathbb{R}$ , there exists a compactly supported form  $\phi \in \Omega_c^n(M)$  such that

$$[D\omega' + \pi^*\phi'] = [\pi^*\phi] \in H_c^n(S^* M).$$

In other words, there is a compactly supported form  $\omega \in H_c^{n-1}(S^* M)$  such that  $d\omega = D\omega = D\omega' + \pi^*\phi' - \pi^*\phi$ . By Theorem 1.2, the pair  $(\omega, \phi)$  represents  $\mu$  up to a multiple of  $\chi$ . Since the valuation represented by  $(\omega, \phi)$  and the valuation  $\mu$  are both compactly supported, whereas  $\chi$  is not, they have to be the same.  $\square$

*Proof of Theorem 1.3.* Note first that the right hand side of (3) is well defined: since  $\mu_2$  is compactly supported, the same holds true for  $D\omega_2 + \pi^*\phi_2$  and  $\pi_*\omega_2$  by Lemma 2.2.

Next, both sides of (3) are linear in  $\mu_1$  and  $\mu_2$ . Using Lemma 2.1, we may therefore assume that the supports of  $\mu_1$  and  $\mu_2$  are contained in the support of a coordinate chart. Since the Alesker–Fu product, the Euler–Verdier involution and the integration functional are natural with respect to diffeomorphisms, it suffices to prove (3) in the case where  $M = V$  is a real vector space of dimension  $n$ .

Let us first suppose that  $\mu_1$  and  $\sigma\mu_2$  are of the type (2). We thus have  $\mu_1(K) = m_1(K + A_1)$  and  $\sigma\mu_2(K) = m_2(K + A_2)$  with smooth measures  $m_1, m_2$  and smooth convex bodies  $A_1, A_2$  with strictly convex boundary.

The left hand side of (3) is given by

$$\begin{aligned}
\int \mu_1 \cdot \sigma \mu_2 &= m_1 \times m_2(\Delta(V) + A_1 \times A_2) \\
&= \int_V m_1((\Delta V + A_1 \times A_2) \cap V \times \{x\}) dm_2(x) \\
&= \int_V m_1(x - A_2 + A_1) dm_2(x) \\
&= \int_V \mu_1(x - A_2) dm_2(x) \\
&= \int_V \text{cnc}(x - A_2)(\omega_1) dm_2(x) + \int_V \left( \int_{x-A_2} \phi_1 \right) dm_2(x).
\end{aligned} \tag{10}$$

Let  $A \in \mathcal{K}(V)$  be smooth with strictly convex boundary. Its support function is defined by

$$\begin{aligned}
h_A: V^* &\rightarrow \mathbb{R}, \\
\xi &\mapsto \sup_{x \in A} \xi(x).
\end{aligned}$$

Note that  $h_A$  is homogeneous of degree 1 and that  $h_{-A}(\xi) = h_A(-\xi)$ .

Define the map  $G_A: S^*V \rightarrow S^*V$ ,  $(x, [\xi]) \mapsto (x + d_\xi h_A, [\xi])$  (since  $h_A$  is homogeneous of degree 1,  $d_\xi h_A \in V^{**} = V$  only depends on  $[\xi]$ ).  $G_A$  is an orientation preserving diffeomorphism of  $S^*V$ .

It is easy to show ([10], [12]) that for  $X \in \mathcal{K}(V)$

$$\text{cnc}(X + A) = (G_A)_* \text{cnc}(X). \tag{11}$$

We next compute that for all  $(x, [\xi]) \in S^*V$

$$\begin{aligned}
G_A \circ s(x, [\xi]) &= G_A(x, [-\xi]) = (x + d_{-\xi} h_A, [-\xi]) = (x - d_\xi h_{-A}, [-\xi]) \\
&= s(x - d_\xi h_{-A}, [\xi]) = s \circ G_{-A}^{-1}(x, [\xi]).
\end{aligned} \tag{12}$$

Let  $\kappa_2 \in \Omega^n(V)$  be the form representing the measure  $m_2$ . The first term in (10) is equal to

$$\begin{aligned}
\int_V \text{cnc}(x - A_2)(\omega_1) dm_2(x) &= \int_V \text{cnc}(\{x\})(G_{-A_2}^* \omega_1) dm_2(x) \\
&= \int_V \pi_*(G_{-A_2}^* \omega_1) \wedge \kappa_2 \\
&= \int_{S^*V} G_{-A_2}^* \omega_1 \wedge \pi^* \kappa_2 \\
&= \int_{S^*V} \omega_1 \wedge (G_{-A_2}^{-1})^* \pi^* \kappa_2.
\end{aligned} \tag{13}$$

By (11) we have  $(-1)^n Ds^* \omega_2 + (-1)^n \pi^* \phi_2 = G_A^* \pi^* \kappa_2$ . Applying  $s^*$  to both sides and using (12), we get

$$(-1)^n (D\omega_2 + \pi^* \phi_2) = s^* G_{A_2}^* \pi^* \kappa_2 = (G_{-A_2}^{-1})^* \pi^* \kappa_2.$$

Hence (13) equals  $(-1)^n \int_{S^* V} \omega_1 \wedge (D\omega_2 + \pi^* \phi_2)$ , which is the first term in (3).

By Fubini's theorem, the second term in (10) equals

$$\int_V \left( \int_{x-A_2} \phi_1 \right) dm_2(x) = \int_V m_2(y + A_2) \phi_1(y) = \int_V \sigma \mu_2(\{y\}) \phi_1(y).$$

For  $y \in V$ , we have  $s_* \text{cnc}(\{y\}) = (-1)^n \text{cnc}(\{y\})$ , since the antipodal map on  $S^{n-1}$  is orientation preserving precisely if  $n$  is even. Hence

$$\sigma \mu_2(\{y\}) = \pi_* \omega_2(y).$$

The second term in (10) thus equals  $\int_V \phi_1 \wedge \pi_* \omega_2$ , which corresponds to the second term in (3).

This finishes the proof in the case where  $\mu_1$  and  $\sigma \mu_2$  are of type (2). By linearity of both sides, (3) holds true for linear combinations of such valuations. Given arbitrary  $\mu_1 \in \mathcal{V}^\infty(M)$  and  $\mu_2 \in \mathcal{V}_c^\infty(M)$ , we find sequences  $\mu_1^j \in \mathcal{V}^\infty(M)$  and  $\mu_2^j \in \mathcal{V}_c^\infty(M)$  such that  $\mu_1^j \rightarrow \mu_1$  and  $\mu_2^j \rightarrow \mu_2$  and such that  $\mu_1^j$  and  $\sigma \mu_2^j$  are linear combinations of valuations of type (2) (compare [5] and [6]).

By definition of the topology on  $\mathcal{V}^\infty(M)$  (see Section 3.2 of [6]) and the open mapping theorem, there are sequences  $(\omega_1^j, \phi_1^j)$  and  $(\omega_2^j, \phi_2^j)$  representing  $\mu_1^j, \mu_2^j$  and converging to  $(\omega_1, \phi_1), (\omega_2, \phi_2)$  in the  $C^\infty$ -topology. By what we have proved,

$$\int \mu_1^j \cdot \sigma \mu_2^j = (-1)^n \int_{S^* M} \omega_1^j \wedge (D\omega_2^j + \pi^* \phi_2^j) + \int_M \phi_1^j \wedge \pi_* \omega_2^j$$

for all  $j$ . Letting  $j$  tend to infinity, Equation (3) follows.  $\square$

### 3. Self-adjointness of natural operators

*Proof of Theorem 1.5.* Note first the following equation:

$$\langle \sigma \mu_1, \mu_2 \rangle = (-1)^n \langle \mu_1, \sigma \mu_2 \rangle. \quad (14)$$

This equation is immediate from (5) and the fact that  $s: S^* M \rightarrow S^* M$  preserves orientation if and only if  $n$  is even.

Let  $\mu_i$  be represented by  $(\omega_i, \phi_i)$ . By Lemma 2.3 we may suppose that  $\omega_2$  and  $\phi_2$  are compactly supported.

$\Lambda \mu_i$  is represented by  $\xi_i := i_T(D\omega_i + \pi^*\phi_i)$ . Since  $D\omega_i + \pi^*\phi_i = \alpha \wedge \xi_i$ , we get

$$\begin{aligned}
 \langle \Lambda \mu_1, \sigma \mu_2 \rangle &= (-1)^n \int_{S^* M} \xi_1 \wedge (D\omega_2 + \pi^*\phi_2) \\
 &= (-1)^n \int_{S^* M} \xi_1 \wedge \alpha \wedge \xi_2 \\
 &= - \int_{S^* M} \xi_2 \wedge (D\omega_1 + \pi^*\phi_1) \\
 &= (-1)^{n+1} \int_{S^* M} \omega_1 \wedge D\xi_2 - \int_M \phi_1 \wedge \pi_* \xi_2 \\
 &= -\langle \mu_1, \sigma \Lambda \mu_2 \rangle.
 \end{aligned} \tag{15}$$

Since  $D$  and  $s^*$  commute and since  $i_T \circ s^* = -s^* \circ i_T$ , it is easily checked that

$$\Lambda \circ \sigma = -\sigma \circ \Lambda. \tag{16}$$

Now (6) follows from (15) and (16).

Let us next prove (7) ((8) is an immediate consequence).

By Lemma 2.3 we may suppose that  $\omega_2$  and  $\phi_2$  have compact support. Then

$$\begin{aligned}
 \langle \mu_1, \sigma \mathcal{S} \mu_2 \rangle &= (-1)^n \int_{S^* M} \omega_1 \wedge D * (D\omega_2 + \pi^*\phi_2) \\
 &\quad + \int_M \phi_1 \wedge \pi_* * (D\omega_2 + \pi^*\phi_2) \\
 &= \int_{S^* M} (D\omega_1 + \pi^*\phi_1) \wedge * (D\omega_2 + \pi^*\phi_2) \\
 &= \int_{S^* M} * (D\omega_1 + \pi^*\phi_1) \wedge (D\omega_2 + \pi^*\phi_2) \\
 &= \int_{S^* M} (-1)^n s^* * (D\omega_1 + \pi^*\phi_1) \wedge s^* (D\omega_2 + \pi^*\phi_2) \\
 &= \langle \sigma \mathcal{S} \mu_1, \mu_2 \rangle.
 \end{aligned} \tag{17}$$

Since  $s$  changes the orientation of  $S^* M$  by  $(-1)^n$ , we get  $s^* \circ * = (-1)^n * \circ s^*$  on  $\Omega^*(S^* V)$ . It follows that  $\sigma \circ \mathcal{S} = (-1)^n \mathcal{S} \circ \sigma$ . Therefore (7) follows from (14) and (17).  $\square$

Alesker defined the space  $\mathcal{V}^{-\infty}(M)$  of *generalized valuations* on  $M$  by

$$\mathcal{V}^{-\infty}(M) := (\mathcal{V}_c^\infty(M))^*,$$

where the star means the topological dual. This space is endowed with the weak topology. By the perfectness of the Alesker–Poincaré pairing, there is a natural dense embedding  $\mathcal{V}^\infty(M) \hookrightarrow \mathcal{V}^{-\infty}(M)$ .

**Corollary 3.1.** *Let  $M$  be a Riemannian manifold. Each of the operators  $\Lambda, \mathcal{S}, \Delta$  acting on  $\mathcal{V}^\infty(M)$  admits a unique continuous extension to  $\mathcal{V}^{-\infty}(M)$ .*

*Proof.* Uniqueness of the extension is clear, since  $\mathcal{V}^\infty(M)$  is dense in  $\mathcal{V}^{-\infty}(M)$ . We let  $\Lambda$  act on  $\mathcal{V}^{-\infty}$  by  $\Lambda\xi(\mu) := \xi(\Lambda\mu)$ . By Theorem 1.5, this is consistent with the embedding of  $\mathcal{V}^\infty(M)$  into  $\mathcal{V}^{-\infty}(M)$  and we are done. The cases of  $\mathcal{S}$  and  $\Delta$  are similar.  $\square$

#### 4. The translation invariant case

From now on,  $V$  will denote an oriented  $n$ -dimensional real vector space. We will consider valuations on the space  $\mathcal{K}(V)$  of compact convex sets (i.e. convex valuations).

A convex valuation  $\mu$  on  $V$  is called translation invariant, if  $\mu(x + K) = \mu(K)$  for all  $K \in \mathcal{K}(V)$  and all  $x \in V$ .

A translation invariant convex valuation  $\mu$  is said to be of degree  $k$  if  $\mu(tK) = t^k \mu(K)$  for  $t > 0$  and  $K \in \mathcal{K}(V)$ . By  $\text{Val}_k(V)$  we denote the space of translation invariant convex valuations of degree  $k$ . A valuation  $\mu$  is even if  $\mu(-K) = \mu(K)$  and odd if  $\mu(-K) = -\mu(K)$ , the corresponding spaces will be denoted by a superscript  $+$  or  $-$ .

In [16] it is shown that the space of translation invariant valuations can be written as a direct sum

$$\text{Val}(V) = \bigoplus_{k=0}^n \text{Val}_k(V).$$

Each space  $\text{Val}_k(V)$  splits further as  $\text{Val}_k(V) = \text{Val}_k^+(V) \oplus \text{Val}_k^-(V)$ .

The spaces  $\text{Val}_0(V)$  and  $\text{Val}_n(V)$  are both 1-dimensional (generated by  $\chi$  and a Lebesgue measure respectively). For  $\mu \in \text{Val}(V)$ , we denote by  $\mu_n$  its component of degree  $n$ .

Let us prove the following version of Theorem 1.3 in the translation invariant case.

**Theorem 4.1.** *Let  $\mu_1, \mu_2 \in \text{Val}^{\text{sm}}(V)$  be represented by translation invariant forms  $(\omega_1, \phi_1), (\omega_2, \phi_2)$  respectively. Then  $(\mu_1 \cdot \sigma \mu_2)_n$  is represented by the  $n$ -form*

$$(-1)^n \pi_*(\omega_1 \wedge (D\omega_2 + \pi^* \phi_2)) + \phi_1 \wedge \pi_* \omega_2 \in \Omega^n(V).$$

*Proof.* The proof is similar to that of Theorem 1.3. Fix a Euclidean metric on  $V$ . For  $R > 0$ , let  $B_R$  denote the ball of radius  $R$ , centered at the origin. Let us suppose that

$\mu_1(K) = \text{vol}(K + A_1)$  and  $\sigma\mu_2(K) = \text{vol}(K + A_2)$  for all  $K \in \mathcal{K}(V)$ . Then

$$\begin{aligned}\mu_1 \cdot \sigma\mu_2(B_R) &= \text{vol}_{2n}(\Delta(B_R) + A_1 \times A_2) \\ &= \int_{B_R} \text{vol}(x - A_2 + A_1)dx + o(R^n) \\ &= \int_{B_R} \mu_1(x - A_2) + o(R^n) \\ &= \int_{B_R} \text{cnc}(x - A_2)(\omega_1)dx + \int_{B_R} \int_{x-A_2} \phi_1 dx + o(R^n).\end{aligned}$$

The first term is given by

$$\begin{aligned}\int_{B_R} \text{cnc}(x - A_2)(\omega_1)dx &= \int_{B_R} \pi_*(G_{-A_2}^* \omega_1)dx + o(R^n) \\ &= \int_{B_R \times S^*(V)} G_{-A_2}^* \omega_1 \wedge \pi^*(dx) + o(R^n) \\ &= \int_{G_{-A_2}(B_R \times S^*(V))} \omega_1 \wedge (G_{-A_2}^{-1})^* \pi^* dx_2 + o(R^n) \\ &= (-1)^n \int_{B_R \times S^*(V)} \omega_1 \wedge (D\omega_2 + \pi^* \phi_2) + o(R^n) \\ &= (-1)^n \int_{B_R} \pi_*(\omega_1 \wedge (D\omega_2 + \pi^* \phi_2)) + o(R^n).\end{aligned}$$

The second term yields

$$\begin{aligned}\int_{B_R} \int_{x-A_2} \phi_1 dx &= \int_V \text{vol}((y + A_2) \cap B_R) \phi_1(y) \\ &= \int_{B_R} \text{vol}(y + A_2) \phi_1(y) + o(R^n) \\ &= \int_{B_R} \mu_2(\{y\}) \phi_1(y) + o(R^n) \\ &= \int_{B_R} \phi_1 \wedge \pi_* \omega_2 + o(R^n).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}(\mu_1 \cdot \sigma\mu_2)_n &= \lim_{R \rightarrow \infty} \frac{1}{R^n} \mu_1 \cdot \sigma\mu_2(B_R) \\ &= \lim_{R \rightarrow \infty} \frac{1}{R^n} \int_{B_R} (-1)^n \pi_*(\omega_1 \wedge (D\omega_2 + \pi^* \phi_2)) + \phi_1 \wedge \pi_* \omega_2.\end{aligned}$$

This finishes the proof of Theorem 4.1 in the case where  $\mu_1$  and  $\sigma\mu_2$  are of type  $K \mapsto \text{vol}(K+A)$ . Using linearity of both sides, it also holds for linear combinations of such valuations. Since they are dense in  $\text{Val}(V)$  (by Alesker's solution of McMullen's conjecture [1]), Theorem 4.1 is true in general.  $\square$

Let us next suppose that  $V$  is endowed with a Euclidean product. We can identify  $\text{Val}_n(V)$  with  $\mathbb{R}$  by sending  $\text{vol}$  to 1. We get a symmetric bilinear form (called Alesker pairing)

$$\begin{aligned} \text{Val}^{\text{sm}}(V) \times \text{Val}^{\text{sm}}(V) &\rightarrow \mathbb{R}, \\ (\mu_1, \mu_2) &\mapsto \langle \mu_1, \mu_2 \rangle := (\mu_1 \cdot \mu_2)_n. \end{aligned}$$

**Corollary 4.2.** *For valuations  $\mu_1, \mu_2 \in \text{Val}^{\text{sm}}(V)$ , the following equations hold:*

$$\begin{aligned} \langle \Lambda\mu_1, \mu_2 \rangle &= \langle \mu_1, \Lambda\mu_2 \rangle, \\ \langle \mathcal{S}\mu_1, \mu_2 \rangle &= \langle \mu_1, \mathcal{S}\mu_2 \rangle, \\ \langle \Delta\mu_1, \mu_2 \rangle &= \langle \mu_1, \Delta\mu_2 \rangle. \end{aligned}$$

*Proof.* Analogous to the proof of Theorem 1.5.  $\square$

## 5. Kinematic formulas and Poincaré formulas

**5.1. Kinematic formulas.** In this section, we suppose that  $G$  is a subgroup of  $\text{O}(V)$  acting transitively on the unit sphere. By a result of Alesker [3], the space of translation invariant and  $G$ -invariant valuations  $\text{Val}^G$  is a finite-dimensional vector space.

Let  $\phi_1, \dots, \phi_N$  a basis of  $\text{Val}^G$ . Suppose we have a kinematic formula

$$\int_{\bar{G}} \chi(K \cap \bar{g}L) d\bar{g} = \sum_{i,j=1}^N c_{i,j} \phi_i(K) \phi_j(L).$$

Here and in the following,  $G$  is endowed with its Haar measure and  $\bar{G} := G \ltimes V$  with the product measure.

Set

$$k_G(\chi) := \sum_{i,j=1}^N c_{i,j} \phi_i \otimes \phi_j \in \text{Val}^G \otimes \text{Val}^G = \mathbf{Hom}(\text{Val}^G, \text{Val}^{G*}).$$

The Alesker pairing induces a bijective map

$$\mathbf{PD} \in \mathbf{Hom}(\text{Val}^G, \text{Val}^{G*}).$$

Fu [13] showed that these two maps are inverse to each other:

$$k_G(\chi) = \mathbf{PD}^{-1}. \quad (18)$$

For further use, we give another interpretation of (18). Let  $G$  be as above. The scalar product on the finite-dimensional space  $\text{Val}^G$  induces a scalar product on  $\text{Val}^{G*}$  such that  $\mathbf{PD}$  is an isometry.

Given  $K \in \mathcal{K}(V)$ , let  $\mu_K \in \text{Val}^{G*}$  be defined by

$$\mu_K(\mu) = \mu(K), \quad \mu \in \text{Val}^G.$$

**Proposition 5.1** (Principal kinematic formula). *Let  $G$  be a subgroup of  $\text{O}(V)$  acting transitively on the unit sphere. Then for  $K, L \in \mathcal{K}(V)$*

$$\int_{\bar{G}} \chi(K \cap \bar{g}L) d\bar{g} = \langle \mu_K, \mu_L \rangle.$$

*Proof.* Let  $\phi_1, \dots, \phi_N$  be a basis of  $\text{Val}^G$ . Set  $g_{ij} := \langle \phi_i, \phi_j \rangle$ ,  $i, j = 1, \dots, N$ . Let us denote by  $(g^{ij})_{i,j=1,\dots,N}$  the inverse matrix. Then

$$\int_{\bar{G}} \chi(K \cap \bar{g}L) d\bar{g} = \sum_{i,j} g^{ij} \phi_i(K) \phi_j(L) = \sum_{i,j} g^{ij} \mu_K(\phi_i) \mu_L(\phi_j) = \langle \mu_K, \mu_L \rangle. \quad \square$$

**5.2. Klain functions.** Let us suppose additionally that  $-1 \in G$ , which implies that  $\text{Val}^G \subset \text{Val}^+$ .

For  $0 \leq k \leq n$ , the action of  $G$  on  $V$  induces an action on the Grassmannian  $\text{Gr}_k(V)$ . We set  $\mathcal{P}_k := \text{Gr}_k(V)/G$  for the quotient space. Given  $u \in \mathcal{P}_k$ , the space of  $k$ -planes contained in  $u$  admits a unique  $G$ -invariant probability measure and we define  $Z_u \in \text{Val}^G$  by

$$Z_u(K) := \int_{L \in u} \text{vol}(\pi_L K) dL, \quad K \in \mathcal{K}(V).$$

Recall that the Klain function of an even, translation invariant valuation  $\mu$  of degree  $k$  on a Euclidean vector space  $V$  is the function  $\text{Kl}_\mu : \text{Gr}_k(V) \rightarrow \mathbb{R}$  such that the restriction of  $\mu$  to  $L \in \text{Gr}_k(V)$  is given by  $\text{Kl}_\mu(L)$  times the Lebesgue measure. An even, translation invariant valuation is uniquely determined by its Klain function [14]. If  $M$  is a compact  $k$ -dimensional submanifold (possibly with boundaries or corners), then

$$\mu(M) = \int_M \text{Kl}_\mu(T_p M) dp.$$

Alesker proved the existence of a duality operator (or Fourier transform)  $\mathbb{F}$  on  $\text{Val}^{+,sm}$  such that  $\text{Kl}_{\mathbb{F}\mu} = \text{Kl}_\mu \circ \perp$  for all  $\mu \in \text{Val}^{+,sm}$ .  $\mathbb{F}$  is formally self-adjoint with respect to the Alesker pairing.

**Proposition 5.2.** *Let  $u, v \in \mathcal{P}_k$  and  $L \in v$ . Then*

$$\text{Kl}_{Z_u}(L) = \langle \mathbb{F}Z_u, Z_v \rangle. \quad (19)$$

*Proof.* Immediate from Lemma 2.2. of [12].  $\square$

**Lemma 5.3.** *There are finitely many elements  $u_1, \dots, u_N$  such that  $Z_{u_i}$ ,  $i = 1, \dots, N$  is a basis of  $\text{Val}_k^G$ .*

*Proof.* Let  $\phi_1, \dots, \phi_N$  be a basis of  $\text{Val}_k^G$ . Let  $m_i$  be the push-forward of a Crofton measure for  $\phi_i$  on  $\text{Gr}_k(V)$  under the projection  $\text{Gr}_k(V) \rightarrow \mathcal{P}_k$ .

By  $G$ -invariance of  $\phi_i$ , we get

$$\phi_i(K) = \int_{\mathcal{P}_k} \int_{L \in u} \text{vol}(\pi_L K) dL dm_i(u).$$

Now, for sufficiently close approximations of the  $m_i$  by discrete measures  $\sum_{j=1}^{k_i} c_{i,j} \delta_{u_{i,j}}$  with  $u_{i,j} \in \mathcal{P}_k$ ,  $c_{i,j} \in \mathbb{R}$ , the valuations  $\sum_j c_{i,j} Z_{u_{i,j}}$  form a basis of  $\text{Val}_k^G$ . Hence  $\{Z_{u_{i,j}}, i = 1, \dots, N, j = 1, \dots, k_i\}$  is a finite generating set of  $\text{Val}_k^G$ , from which we can extract a finite basis.  $\square$

**5.3. Poincaré formulas.** Poincaré formulas for  $G$  are special cases of the principal kinematic formula for  $G$ , when  $K$  and  $L$  are replaced by smooth compact submanifolds  $M_1$  and  $M_2$  (possibly with boundary) of complementary dimension (note that  $M_1, M_2 \in \mathcal{P}(V)$ , so there is no problem in evaluating a valuation in  $M_1$  and  $M_2$ ). Then the right hand side of the principal kinematic formula is the “average number” of intersections of  $M_1$  and  $\bar{g}M_2$ .

**Proposition 5.4** (General Poincaré formula). *Let  $M_1, M_2$  be smooth compact submanifolds, possibly with boundaries, of complementary dimensions  $k$  and  $n - k$ . Then*

$$\int_{\bar{G}} \#(M_1 \cap \bar{g}M_2) d\bar{g} = \int_{M_1 \times M_2} \alpha(T_p M_1, T_q M_2) dp dq$$

with

$$\begin{aligned} \alpha: \mathcal{P}_k \times \mathcal{P}_{n-k} &\rightarrow \mathbb{R}, \\ (u, v) &\mapsto \langle Z_u, Z_v \rangle. \end{aligned}$$

*Proof.* Let  $u_1, \dots, u_N$  be such that  $Z_{u_i}$ ,  $i = 1, \dots, N$  is a basis of  $\text{Val}_k^G$ . Let  $v_1, \dots, v_N$  be such that  $Z_{v_j}$ ,  $j = 1, \dots, N$  is a basis of  $\text{Val}_{n-k}^G$  (note that the dimensions of these two spaces agree by Theorem 1.2.2 in [2]). Setting  $g_{ij} := \langle Z_{u_i}, Z_{v_j} \rangle$

and  $(g^{ij})$  for the inverse matrix, the principal kinematic formula implies that for all  $M_1$  and  $M_2$  as above

$$\begin{aligned} \int_{\bar{G}} \#(M_1 \cap \bar{g}M_2) d\bar{g} &= \sum_{i,j} g^{ij} Z_{u_i}(M_1) Z_{v_j}(M_2) \\ &= \int_{M_1 \times M_2} \sum_{i,j} g^{i,j} \text{Kl}_{Z_{u_i}}(T_p M_1) \text{Kl}_{Z_{v_j}}(T_q M_2) dp dq. \end{aligned}$$

This shows that

$$\alpha(u, v) = \sum_{i,j} g^{ij} \text{Kl}_{Z_{u_i}}(u) \text{Kl}_{Z_{v_j}}(v) = \langle Z_u, Z_v \rangle;$$

where the last equation follows from (19) and the self-adjointness of  $\mathbb{F}$ .  $\square$

## 6. Kinematic formulas for $\text{SU}(2)$

We apply the results of the preceding section to the special case  $G = \text{SU}(2)$  acting on the quaternionic line

$$\mathbb{H} = \{x_1 + x_2i + x_3j + x_4k : (x_1, x_2, x_3, x_4) \in \mathbb{R}^4\}.$$

Since this action is transitive on the unit sphere,  $\text{Val}^{\text{SU}(2)}$  is finite dimensional and  $\text{Val}_k^{\text{SU}(2)}$  is one-dimensional except for the case  $k = 2$ . The quotient space  $\mathcal{P}_2 := \text{Gr}_2(\mathbb{H})/\text{SU}(2)$  is the two-dimensional projective space  $\mathbb{RP}^2 = S^2/\{\pm 1\}$  ([19]). Following Tasaki, we denote by  $(\omega_1(L) : \omega_2(L) : \omega_3(L)) \in \mathbb{RP}^2$  the class of  $L \in \text{Gr}_2(\mathbb{H})$ .

A canonical representative in the preimage of  $(a : b : c) \in \mathbb{RP}^2$  is given by the 2-plane spanned by 1 and  $ai + bj + ck$ .

If  $u = (a, b, c) \in S^2$ , then the planes in  $u$  are the complex lines for the complex structure  $I_u$  which is defined by multiplication by  $u$  from the right on  $\mathbb{H}$ . We will therefore write  $\mathbb{CP}_u$  instead of  $u$ . Note that  $\mathbb{CP}_u = \mathbb{CP}_{-u}$ .

The following  $\text{SU}(2)$ -invariant and translation invariant valuations of degree 2 were introduced by Alesker [3].

**Definition 6.1.** Given  $u \in \mathbb{RP}^2$ , set

$$Z_u(K) := \int_{\mathbb{CP}_u} \text{vol}(\pi_L(K)) dL, \quad K \in \mathcal{K}(\mathbb{H}).$$

*Proof of Theorem 1.6.* From Lemma 5.3 we infer that there is a finite number of points  $u_1, \dots, u_N \in \mathbb{RP}^2$  such that  $Z_{u_i}$ ,  $i = 1, \dots, N$  form a basis of  $\text{Val}_2^{\text{SU}(2)}$ . Alesker showed that  $N = 6$  and that  $Z_i, Z_j, Z_k, Z_{\frac{i+j}{\sqrt{2}}}, Z_{\frac{i+k}{\sqrt{2}}}, Z_{\frac{j+k}{\sqrt{2}}}$  is such a basis.

Our aim is to compute the product  $\langle Z_u, Z_v \rangle$  for  $u, v \in \mathbb{RP}^2$ . We will achieve it by first expressing each  $Z_u$  as a smooth valuation represented by some 3-form  $\omega_u \in \Omega^3(S^* \mathbb{H})$  and then applying Theorem 4.1.

Since the metric induces a diffeomorphism between  $S^* \mathbb{H}$  and  $S\mathbb{H}$ , we may as well work with the latter space. The image of the conormal cycle of a compact convex set  $K$  under this diffeomorphism is the normal cycle  $\text{nc}(K)$ .

Let us introduce several differential forms on  $S\mathbb{H}$ , depending on the choice of the complex structure  $I_u$ . We follow the notation of [17].

Let  $\alpha, \beta, \gamma$  be 1-forms on  $S\mathbb{H}$  which, at a point  $(x, v) \in S\mathbb{H}$ , equal

$$\begin{aligned}\alpha(w) &= \langle v, d\pi(w) \rangle, & w \in T_{(x,v)}S\mathbb{H}, \\ \beta_u(w) &= \langle v, I_u d\pi(w) \rangle, & w \in T_{(x,v)}S\mathbb{H}, \\ \gamma_u(w) &= \langle v, I_u d\pi_2(w) \rangle, & w \in T_{(x,v)}S\mathbb{H}.\end{aligned}$$

Note that  $\alpha$  is the canonical 1-form (in particular independent of  $u$ ), whereas  $\beta_u$  and  $\gamma_u$  depend on  $u$ .

Let  $\Omega$  be the pull-back of the symplectic form on  $(\mathbb{H}, I_u)$  to  $S\mathbb{H}$ , i.e.

$$\Omega_u(w_1, w_2) := \langle d\pi(w_1), I_u d\pi(w_2) \rangle, \quad w_1, w_2 \in T_{(x,v)}S\mathbb{H}.$$

*Claim.*  $Z_u$  is represented by the 3-form

$$\omega_u := \frac{1}{8\pi} \beta_u \wedge d\beta_u + \frac{1}{4\pi} \gamma_u \wedge \Omega_u.$$

Since  $\omega_u$  is  $\text{U}(2)$ - and translation invariant and has bidegree  $(2, 1)$  (with respect to the product decomposition  $S\mathbb{H} = \mathbb{H} \times S(\mathbb{H})$ ), it represents some  $\text{U}(2)$ -invariant, translation invariant valuation  $\mu_u$  of degree 2. Here  $\text{U}(2)$  is the unitary group for the complex structure  $I_u$ .

Now the space of valuations with these properties is of dimension 2 [2]. It is thus enough to show that the valuation  $Z_u$  and the valuation  $\mu_u$  agree on the unit ball  $B$  as well as on a complex disk  $D_u$ .

It is clear that  $Z_u(B) = \omega_2 = \pi$ . It was shown by Fu (compare Equation (37) in [13],) that  $Z_u(D_u) = \frac{\pi}{2}$ .

By [11], the derivation of a smooth translation invariant valuation  $\mu$  on a finite-dimensional Euclidean vector space is given by

$$\Lambda \mu(K) = \frac{d}{dt} \Big|_{t=0} \mu(K + tB).$$

It follows that, if  $\mu$  is of degree  $k$ , then  $\Lambda\mu(B) = k\mu(B)$ .

It is easily checked that  $\mathcal{L}_T\beta = \gamma$ ,  $\mathcal{L}_T\gamma = 0$  and  $\mathcal{L}_T^2\Omega = d\gamma$ , so that

$$\mathcal{L}_T^2\omega_u = \frac{1}{2\pi}\gamma \wedge d\gamma.$$

Note that  $\gamma \wedge d\gamma$  is twice the volume form on  $S^3$ , hence  $\Lambda^2\mu_u = 2\pi\chi$ . It follows that

$$\mu_u(B) = \frac{1}{2}\Lambda^2\mu_u(B) = \pi.$$

The restriction of  $\beta_u$  to the normal cycle of the complex disc  $D_u$  clearly vanishes.  $\gamma_u$  is the length element of the fibers of  $\pi: \text{nc}(D_u) \rightarrow D_u$  (which are circles), whereas  $\Omega_u$  is the (pull-back of) the volume form on  $D_u$ . It follows that  $\omega_u(D_u) = \frac{\pi}{2}$ . The claim is proved.

Next, the Rumin operator is easily computed as

$$D\omega_u = d\left(\omega_u + \frac{1}{8\pi}\alpha \wedge \beta_u \wedge \gamma_u - \frac{1}{8\pi}\alpha \wedge \Omega_u\right) = \frac{1}{2\pi}\alpha \wedge \beta_u \wedge d\gamma_u.$$

From Theorem 4.1 we infer that  $\mu_u \cdot \mu_v$  is represented by the 4-form

$$\frac{1}{16\pi^2}\pi_*((\beta_u \wedge d\beta_u + 2\gamma_u \wedge \Omega_u) \wedge \alpha \wedge \beta_v \wedge d\gamma_v) \in \Omega^4(\mathbb{H}).$$

If  $u = (a : b : c)$  and  $v = (\tilde{a} : \tilde{b} : \tilde{c})$ , then

$$\begin{aligned} & (\beta_u \wedge d\beta_u + 2\gamma_u \wedge \Omega_u) \wedge \alpha \wedge \beta_v \wedge d\gamma_v \\ &= 2((a\tilde{b} - \tilde{a}b)^2 + (a\tilde{c} - \tilde{a}c)^2 + (b\tilde{c} - \tilde{b}c)^2 + 2(a\tilde{a} + b\tilde{b} + c\tilde{c})^2)d \text{ vol}_{S\mathbb{H}} \\ &= 2(1 + (a\tilde{a} + b\tilde{b} + c\tilde{c})^2)d \text{ vol}_{S\mathbb{H}}. \end{aligned}$$

It follows that

$$\langle Z_u, Z_v \rangle = \frac{1}{4}(1 + (a\tilde{a} + b\tilde{b} + c\tilde{c})^2). \quad (20)$$

Let  $\pm u_i$ ,  $i = 1, \dots, 6$  be the 12 vertices of an icosahedron  $I$  on  $S^2$ . They induce 6 valuations  $Z_{u_i}$ ,  $i = 1, \dots, 6$ . Since the edge length  $a$  of  $I$  satisfies  $\cos a = \frac{\sqrt{5}}{5}$ , (20) implies that

$$\langle Z_{u_i}, Z_{u_j} \rangle = \begin{cases} \frac{1}{2}, & i = j, \\ \frac{3}{10}, & i \neq j. \end{cases} \quad (21)$$

Theorem 1.6 follows easily from (21) and (18).  $\square$

The general Poincaré formula (Proposition 5.4) implies the following (corrected version of the) Poincaré formula on the quaternionic line.

**Corollary 6.2** (Poincaré formula for  $SU(2)$ , [19]). *Let  $M_1, M_2 \subset \mathbb{H}$  be compact smooth 2-dimensional submanifolds. Then*

$$\int_{SU(2)} \#(M_1 \cap \bar{g}M_2) d\bar{g} = \frac{1}{4} \int_{M_1 \times M_2} (1 + A(T_p M_1, T_q M_2)) dp dq$$

with

$$\begin{aligned} A(T_p M_1, T_q M_2) = & (\omega_1(T_p M_1) \omega_1(T_q M_2) \\ & + \omega_2(T_p M_1) \omega_2(T_q M_2) + \omega_3(T_p M_1) \omega_3(T_q M_2))^2. \end{aligned}$$

## References

- [1] S. Alesker, Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture. *Geom. Funct. Anal.* **11** (2001), 244–272. [Zbl 0995.52001](#) [MR 1837364](#)
- [2] S. Alesker, Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. *J. Differential Geom.* **63** (2003), 63–95. [Zbl 1073.52004](#) [MR 2015260](#)
- [3] S. Alesker,  $SU(2)$ -invariant valuations. In *Geometric aspects of functional analysis*, Lecture Notes in Math. 1850, Springer-Verlag, Berlin 2004, 21–29. [Zbl 1064.52010](#) [MR 2087147](#)
- [4] S. Alesker, The multiplicative structure on polynomial valuations. *Geom. Funct. Anal.* **14** (2004), 1–26. [Zbl 1072.52011](#) [MR 2053598](#)
- [5] S. Alesker, Theory of valuations on manifolds, I. Linear Spaces. *Israel J. Math.* **156** (2006), 311–339. [Zbl 1132.52017](#) [MR 2282381](#)
- [6] S. Alesker, Theory of valuations on manifolds, II. *Adv. Math.* **207** (2006), 420–454. [Zbl 1117.52016](#) [MR 2264077](#)
- [7] S. Alesker, Theory of valuations on manifolds IV. New properties of the multiplicative structure. In *Geometric aspects of functional analysis*, Lecture Notes in Math. 1910, Springer-Verlag, Berlin 2007, 1–44. [Zbl 1127.52016](#) [MR 2347038](#)
- [8] S. Alesker, Theory of valuations on manifolds: a survey. *Geom. Funct. Anal.* **17** (2007), 1321–1341. [Zbl 1132.52018](#) [MR 2373020](#)
- [9] S. Alesker, J. H. G. Fu, Theory of valuations on manifolds III. Multiplicative structure in the general case. *Trans. Amer. Math. Soc.* **360** (2008), 1951–1981. [Zbl 1130.52008](#) [MR 2366970](#)
- [10] A. Bernig, Support functions, projections and Minkowski addition of Legendrian cycles. *Indiana U. Math. J.* **55** (2006), 443–464. [Zbl 1125.53063](#) [MR 2225441](#)

- [11] A. Bernig, L. Bröcker, Valuations on manifolds and Rumin cohomology. *J. Differential Geom.* **75** (2007), 433–457. [Zbl 1117.58005](#) [MR 2301452](#)
- [12] A. Bernig, J. H. G. Fu, Convolution of convex valuations. *Geom. Dedicata* **123** (2006), 153–169. [Zbl 1117.53054](#) [MR 2299731](#)
- [13] J. H. G. Fu, Structure of the unitary valuation algebra. *J. Differential Geom.* **72** (2006), 509–533. [Zbl 1096.52003](#) [MR 2219942](#)
- [14] D. Klain, Even valuations on convex bodies. *Trans. Amer. Math. Soc.* **352** (2000), 71–93. [Zbl 0940.52002](#) [MR 1487620](#)
- [15] D. Klain, G.-C. Rota, *Introduction to Geometric probability*. Lezione Lincee, Cambridge University Press, Cambridge 1997. [Zbl 0896.60004](#) [MR 1608265](#)
- [16] P. McMullen, Valuations and Euler-type relations on certain classes of convex polytopes. *Proc. London Math. Soc.* **35** (1977), 113–135. [Zbl 0353.52001](#) [MR 0448239](#)
- [17] H. Park, Kinematic formulas for the real subspaces of complex space forms of dimension 2 and 3. PhD thesis, University of Georgia 2002.
- [18] M. Rumin, Formes différentielles sur les variétés de contact. *J. Differential Geom.* **39** (1994), 281–330. [Zbl 0973.53524](#) [MR 1267892](#)
- [19] H. Tasaki, Integral geometry under the action of the first symplectic group. *Arch. Math.* **80** (2003), 106–112. [Zbl 1035.53107](#) [MR 1968293](#)
- [20] K. Yano, S. Ishihara, *Tangent and cotangent bundles*. Pure and Appl. Math. 16, Marcel Dekker, New York 1973. [Zbl 0262.53024](#) [MR 0350650](#)

Received November 7, 2006

Andreas Bernig, Département de Mathématiques, Chemin du Musée 23, 1700 Fribourg, Switzerland  
E-mail: [andreas.bernig@unifr.ch](mailto:andreas.bernig@unifr.ch)